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Fuzzy closure structures as formal concepts

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Abstract

Galois connections seem to be ubiquitous in mathematics. They have been used to model solutions for both pure and application-oriented problems. Throughout the paper, the general framework is a complete fuzzy lattice over a complete residuated lattice. The existence of three fuzzy Galois connections (two antitone and one isotone) between three specific ordered sets is proved in this paper. The most interesting part is that fuzzy closure systems, fuzzy closure operators and strong fuzzy closure relations are formal concepts of these fuzzy Galois connections.

Keywords: Closure system; Galois connection; Fuzzy lattice

1. Introduction

Galois connections appear in several mathematical theories and in plenty of instances in the theory of relations [23]. Therefore, studying these structures is worthwhile. It is well-known that the derivation operators of Formal Concept Analysis form a Galois connection [14]. Therefore, the research on Galois connections complements that on FCA. The extension of the notion of Galois connection to the fuzzy framework was introduced by Bělohlávek [1]. A pointwise equality of the preorder relations is the substitute for the so-called Galois condition, which is an “if and only if” in the crisp case. This extension to the fuzzy framework provided a way to study Fuzzy Formal Concept Analysis.

Besides Galois connections and formal concepts, the main notion in this paper are closure structures. Closure systems, also called Moore families, were introduced by E. H. Moore in 1910 [18]. They play a major role in computer science and both pure and applied mathematics [11]. There is extensive literature on the extension of this concept to the fuzzy framework, a non-exhaustive list would be the following [2,7,13,17,19,21]. However, the definition used in this paper will be the one introduced in [21] that extends closure systems as meet-subsemilattices in the framework of complete fuzzy lattices. The counterpart of closure systems, called closure operators, have also been extended to the fuzzy setting, and the extension appears to be standard to most authors. Fuzzy closure operators were defined in [2,6] and they appear naturally in different areas of fuzzy logic and its applications.

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In [21], the framework of the paper was a complete $\mathbb{L}$-fuzzy lattice (A, ρ), where the infimum and the supremum are denoted by $\sqcap$ and $\sqcup$, respectively and $\mathbb{L}$ is a complete residuated lattice. Lattice type fuzzy orders were originally introduced by Bělohlávek [5]. There are other definitions of lattice type orders in the literature, e.g., [24], but this is also the case concerning fuzzy orders. The notion of fuzzy order is scattered through the literature; many distinct definitions have appeared since the original one by Zadeh, but there is no exhaustive analysis or comparison among them. The one used in this paper follows the spirit of Bodenhofer [8].

The topic in [21] was the search for a definition of fuzzy closure system. In that, there were two mappings that turned fuzzy closure operators into fuzzy closure systems and vice versa. These mappings are defined as follows, let $\Phi \in L^A$ be a fuzzy closure system and $\mathtt{c}\colon A \to A$ be a fuzzy closure operator. Then, $\mathtt{c}_\Phi\colon A \to A$ is defined as $\mathtt{c}_\Phi(a) = \bigsqcap(a^\rho \otimes \Phi)$ and $\Phi_{\mathtt{c}}(a) = \rho(\mathtt{c}(a), a)$. In [20], fuzzy closure operators are extended to the relational framework. The problem of finding an appropriate definition of fuzzy closure relation is tackled by extending inflationarity, isotonicity and idempotency to the relational framework. However, the problem is not considered solved if there is no one-to-one relation with fuzzy closure systems. For this bijective correspondence to hold some additional conditions, such as extensionality and minimality must be required. These fuzzy relations were called strong fuzzy closure relations and have several equivalent definitions, such as being the extensional hull of a fuzzy closure operator or being a fuzzy function in the sense of [12] whose core is a fuzzy closure operator. The mappings that relate strong fuzzy closure relations and fuzzy closure systems in that paper are the following ones. Let $\Phi \in L^A$ be a fuzzy closure system and $\kappa\colon A \to A$ be a strong fuzzy closure relation. Then, $\kappa_\Phi\colon A \times A \to L$ is defined as $\kappa_\Phi(a, b) = \bigsqcap(a^\rho \otimes \Phi) \approx b$, and $\Phi_\kappa(a) = \rho_\propto(a^\kappa, a)$.

In this paper, we elaborate on these mappings since their domains and codomains do not need to be the sets of fuzzy closure structures. Actually, these mappings are well-defined for all fuzzy sets, all functions and all fuzzy relations on A. The main goal of the paper is studying whether these mappings defined in the most general domains and codomains form fuzzy Galois connections. The outline of the paper is as follows. First, a section of preliminaries recalls the main results already in the literature that will be used throughout the paper. The next sectionstudies the nature of those “bigger” sets, namely the set of all fuzzy sets with the subsethood order relation, the set of all isotone mappings on A with a pointwise fuzzy relation and the set of total and isotone fuzzy relations with a preorder relation called $\tilde{\rho}$. The following section proves that the first pair of mappings above indeed form a fuzzy Galois connection. In addition, fuzzy closure structures are fixed points of this Galois connection, even though there are fixed points which are not formed by closure structures. The core of Section 5 is proving that the second pair of mappings defined above, the one relating fuzzy sets and fuzzy relations indeed forms a fuzzy Galois connection. In fact, fuzzy closure systems and strong fuzzy closure relations are again fixed points, but there are fixed points which are not closure structures. The following section studies the fuzzy isotone Galois connection that relates functions and fuzzy relations on A formed by the pair $(-^1, -^\approx)$, where the first mapping is taking the one cut of the relation and the second one is taking the extensional hull of the mapping as a crisp relation. Furthermore, the fixed points of these adjunction are studied and some results on the commutativity of the diagram formed by the six mappings are given. Last, there is a section of conclusions and further work where the results are discussed and some hints of future research lines are shown.

2. Preliminaries

Binary $\mathbb{L}$-relations (binary fuzzy relations) on a set U can be thought of as $\mathbb{L}$-sets on the universe $U \times U$. That is, a binary $\mathbb{L}$-relation on U is a mapping $\rho \in L^{U\times U}$ assigning to each $x, y \in U$ a truth degree $\rho(x, y) \in L$ (a degree to which x and y are related by ρ).

Definition 1. Given a fuzzy poset $\mathbb{A} = (A, \rho)$, the *symmetric kernel relation* is defined as $\approx\colon A \times A \to L$ where $(x \approx y) = \rho(x, y) \otimes \rho(y, x)$ for all $x, y \in A$.

For ρ being a binary $\mathbb{L}$-relation in U, we say that

- ρ is *reflexive* if $\rho(x, x) = 1$ for all $x \in U$.
- ρ is *symmetric* if $\rho(x, y) = \rho(y, x)$ for all $x, y \in U$.
- ρ is *antisymmetric* if $\rho(x, y) \otimes \rho(y, x) = 1$ implies $x = y$ for all $x, y \in U$.
- ρ is *transitive* if $\rho(x, y) \otimes \rho(y, z) \le \rho(x, z)$ for all $x, y, z \in U$.

Notice that this definition of antisymmetry differs from the original one by Zadeh. There is a wide variety of distinct definitions in the literature. The one used in this paper follows the idea of Bodenhofer [8, Section 5] where the equality relation $\approx$ is defined by the preorder ρ, which matches Definition 1.

Definition 2. Given a non-empty set A and a binary $\mathbb{L}$-relation ρ on A, the pair $\mathbb{A} = (A, \rho)$ is said to be a

- *fuzzy preposet* if ρ is a *fuzzy preorder*, i.e. if ρ is reflexive and transitive;
- *fuzzy poset* if ρ is a *fuzzy order*, i.e. if ρ is reflexive, antisymmetric and transitive.

A typical example of fuzzy poset is (L^U, S).

The notion of extensionality was introduced in the very beginning of the study of fuzzy sets. It has also been called compatibility (with respect to the similarity relation) in the literature.

Definition 3. A fuzzy set $X \in L^A$ is said to be extensional with respect to $\approx$ if it satisfies $X(x) \otimes (x \approx y) \leq X(y)$, for all $x, y \in A$.

The general framework throughout the paper is going to be a complete residuated lattice $\mathbb{L} = (L, \wedge, \vee, \otimes, \to, 0, 1)$, for the properties of complete residuated lattices we refer the reader to [4, Chapter 2].

Given a fuzzy relation μ between A and B, i.e., a crisp mapping $\mu\colon A \times B \to L$, and $a \in A$, the *afterset* a^μ is the fuzzy set $a^\mu\colon B \to L$ given by $a^\mu(b) = \mu(a, b)$. A fuzzy relation μ is said to be *total* if, for all $a \in A$, the aftersets a^μ are normal fuzzy sets, i.e., there exists $x \in A$ such that $a^\kappa(x) = 1$. The composition of two relations $\rho_1\colon A \times B \to L$ and $\rho_2\colon B \times C \to L$ is defined as $(\rho_1 \circ \rho_2)(x, y) = \bigvee_{z \in B}(\rho_1(x, z) \otimes \rho_2(z, y))$. The so-called *full fuzzy powering* $\rho_\propto$ is a fuzzy relation between two powersets that has been used in previous works [9,10]. Its direct extension to fuzzy powersets is as follows: for all $X, Y \in L^A$,

$$\rho_\propto(X, Y) = \bigwedge_{x, y \in A} (X(x) \otimes Y(y)) \to \rho(x, y).$$

A fuzzy set $X \in L^A$ is said to be a *clique* if $\rho_\propto(X, X) = 1$. The relation $\rho_\propto$ is not a preorder in general. However, it satisfies a sort of transitivity.

Theorem 4. *Let $X, Y, Z \in L^A$. If Y is normal then,*

$$\rho_\propto(X, Y) \otimes \rho_\propto(Y, Z) \leq \rho_\propto(X, Z)$$

The definition of infimum, supremum and lattice-like fuzzy orders used throughout the paper is the standard one in the fuzzy framework, originally introduced by Bělohlávek in [5], we write it out to ease the reading of the paper.

Definition 5. Let $\mathbb{A} = (A, \rho)$ be a fuzzy poset and $X \in L^A$. An element $a \in A$ is said to be *infimum* (resp. *supremum*) of X if the following conditions hold:

1. $\rho_\propto(a, X) = 1$ (resp. $\rho_\propto(X, a) = 1$).
2. $\rho_\propto(x, X) \leq \rho(x, a)$ (resp. $\rho_\propto(X, x) \leq \rho(a, x)$), for all $x \in A$.

Hereinafter, suprema and infima in A will be denoted by $\sqcup$ and $\sqcap$, respectively. As an straightforward consequence we have that, if $a = \bigsqcap X$, then $X \subseteq a^\rho$.

Theorem 6. *An element $a \in A$ is infimum (resp. supremum) of $X \in L^A$ if and only if, for all $x \in A$,*

$$\rho(x, a) = \rho_\propto(x, X) \qquad \textit{(resp. } \rho(a, x) = \rho_\propto(X, x)\textit{)}.$$

Definition 7. Let (A, ρ) be a fuzzy poset. The couple (A, ρ) is said to be a complete fuzzy lattice if $\bigsqcap X$ and $\bigsqcup X$ exist for all $X \in L^A$.

The last definition was originally introduced by Bělohlávek under the name completely lattice $\mathbb{L}$-ordered set, and has also been used by Zhang and Fan [26] under the name $\mathbb{L}$-fuzzy complete lattice or Konečny [16] under the name fuzzy complete lattice.

Fuzzy closure operators and systems were first introduced in the fuzzy framework by Bělohlávek in [3]. The definition of fuzzy closure operator used in this paper is the original, used also in [2–4], i.e., a mapping $c\colon A \to A$ that is inflationary, isotone and idempotent. On the other hand, we will consider fuzzy closure systems on arbitrary complete fuzzy lattices, not necessarily on the powerset, as defined in [21], where they are extensional hulls of crisp sets which are closure systems.

Proposition 8. *Let $\mathbb{A} = (A, \rho)$ be a complete fuzzy lattice. Let $Y \in L^A$ be a normal clique and $x_0 \in A$. For all $y_0 \in \mathrm{Core}(Y)$ it holds*

$$\rho_\propto(x_0, Y) = \rho(x_0, y_0).$$

Proof. Let $Y \in L^A$ be a normal clique and $y_0 \in \mathrm{Core}(Y)$, $x_0 \in A$. Then, by taking $y = y_0$,

$$\rho_\propto(x_0, Y) = \bigwedge_{y\in A} Y(y) \to \rho(x_0, y) \le Y(y_0) \to \rho(x_0, y_0) = \rho(x_0, y_0).$$

Conversely,

$$\begin{aligned}\rho(x_0, y_0) &= 1 \to \rho(x_0, y_0)\\ &\overset{\text{(i)}}{\le} \rho(y_0, y) \to \rho(x_0, y_0) \otimes \rho(y_0, y)\\ &\overset{\text{(ii)}}{\le} Y(y_0) \otimes Y(y) \to \rho(x_0, y)\\ &= Y(y) \to \rho(x_0, y)\end{aligned}$$

where the inequality (i) holds by (2.47) in [4] and the inequality (ii) holds by (2.43), (2.44) in [4], transitivity of ρ and X being a clique.

Thus, $\rho_\propto(x_0, y_0) \le \bigwedge_{y\in A} Y(y) \to \rho(x_0, y) = \rho_\propto(x_0, Y)$.

Therefore, $\rho_\propto(x_0, Y) = \rho(x_0, y_0)$. □

Remark 1. Analogously to last result, if X is a normal clique, $X(x_0) = 1$ and $y_0 \in A$ we can prove $\rho_\propto(X, y_0) = \rho(x_0, y_0)$. A detailed proof can be found in [20].

Fuzzy Galois connections are a main concept in this paper as well. Let us recall the definition.

Definition 9 *([25]).* Let $\mathbb{A} = \langle A, \rho_A\rangle$ and $\mathbb{B} = \langle B, \rho_B\rangle$ be fuzzy posets, $f\colon A \to B$ and $g\colon B \to A$ be two mappings. The pair (f, g) is called an *isotone fuzzy Galois connection* or *fuzzy adjunction between* $\mathbb{A}$ *and* $\mathbb{B}$, denoted by $(f, g)\colon \mathbb{A} \rightleftharpoons \mathbb{B}$, if

$$\rho_A(g(b), a) = \rho_B(b, f(a)) \quad \text{for all } a \in A \text{ and } b \in B.$$

Definition 10 *([25]).* Let $\mathbb{A} = \langle A, \rho_A\rangle$ and $\mathbb{B} = \langle B, \rho_B\rangle$ be fuzzy posets, $f\colon A \to B$ and $g\colon B \to A$ be two mappings. The pair (f, g) is called a *fuzzy Galois connection between* $\mathbb{A}$ *and* $\mathbb{B}$, denoted by $(f, g)\colon \mathbb{A} \rightharpoonup\!\!\!\!\leftharpoonup \mathbb{B}$, if

$$\rho_A(a, g(b)) = \rho_B(b, f(a)) \quad \text{for all } a \in A \text{ and } b \in B.$$

A *fixed point*, also called *fixed pair* or *formal concept*, of a fuzzy Galois connection (f, g) is a couple $(a, b) \in A \times B$ such that $f(a) = b$ and $g(b) = a$.

Throughout the paper, the couple $(A^A, \tilde{\rho})$ consists of the set of (crisp) mappings on A and the pointwise $\mathbb{L}$-order defined as

$$\tilde{\rho}(f_1, f_2) = \bigwedge_{x\in A} \rho(f_1(x), f_2(x)) \text{ for all } f_1, f_2 \in A^A.$$

$$(L^A, S) \xrightarrow{(\widehat{\mathtt{c}}, \widetilde{\Psi})} (\mathrm{Isot}(A^A), \tilde{\rho})$$

$$(L^A, S) \xrightarrow{(\kappa, \widehat{\Psi})} (\mathrm{IsotTot}(L^{A\times A}), \hat{\rho})$$

$$(\mathrm{IsotTot}(L^{A\times A}), \hat{\rho}) \xrightarrow{(-^{1}, -^{\approx})} (\mathrm{Isot}(A^A), \tilde{\rho})$$

Fig. 1. Antitone/isotone Galois connections.

Among the mappings in $(A^A, \tilde{\rho})$, the isotone (or "order preserving") ones play an important role because they reflect the idea of homomorphism between $\mathbb{L}$-posets. We denote the set of isotone mappings on (A, ρ) as $(\mathrm{Isot}(A^A), \tilde{\rho})$.

In [22], it was proved that there is a fuzzy Galois connection between the sets (L^A, S) and $(\mathrm{Isot}(A^A), \tilde{\rho})$. This discussion was done in the framework of complete Heyting algebras.

Theorem 11 *([22]). Let* $\hat{\mathtt{c}} \colon (L^A, S) \to (\mathrm{Isot}(A^A), \tilde{\rho})$ *defined as* $\hat{\mathtt{c}}(\Phi)(a) = \prod(a^{\rho} \cap \Phi)$ *and* $\widetilde{\Psi} \colon (\mathrm{Isot}(A^A), \tilde{\rho}) \to (L^A, S)$ *given by* $\widetilde{\Psi}(f)(a) = \rho(f(a), a)$*. Then, the couple* $(\widehat{\mathtt{c}}, \widetilde{\Psi})$ *is a Galois connection between* (L^A, S) *and* $(\mathrm{Isot}(A^A), \tilde{\rho})$*.*

The following theorem characterizes the formal concepts of this fuzzy Galois connection as a couple formed by a fuzzy closure system and a fuzzy closure operator.

Theorem 12 *([22]). The following statements are equivalent:*

1. *The couple* (Φ, f) *is a fixed point.*
2. *The fuzzy set* Φ *is fuzzy closure system and* $f = \widehat{\mathtt{c}}(\Phi)$*.*
3. *The isotone mapping* f *is a fuzzy closure operator and* $\Phi = \widetilde{\Psi}(f)$*.*

Notice that the proof uses strongly the restriction of being in a Heyting algebra. However, in the general residuated lattice case, where, as explored in [21], the natural construction of $\hat{\mathtt{c}}(\Phi) = \mathtt{c}_{\Phi}$ is $x \mapsto \prod(x^{\rho} \otimes \Phi)$, the analogous result does not hold. A counterexample of last theorem in the general residuated lattice case was also presented in [22].

3. Framework of the proposal

In this paper, we study fuzzy closure structures as formal concepts of certain Galois connections. The fuzzy Galois connections taken into consideration will ideally be the couples of mappings defined in [20,21], that is, the mappings that transformed fuzzy closure systems into fuzzy closure operators, fuzzy closure systems into fuzzy closure relations and vice versa.

Hereinafter, the couple (A, ρ) is a complete fuzzy lattice. In order to extend the set of functions to the relational framework, we can consider the couple $(L^{A\times A}, \hat{\rho})$, where $\hat{\rho}(\kappa_1, \kappa_2) = \bigwedge_{a\in A} \rho_{\propto}(a^{\kappa_1}, a^{\kappa_2})$, for all $\kappa_1, \kappa_2 \colon A \times A \to L$. As well as in the case of mappings, the isotone ones are the ones of interest to us. As a matter of fact, in order to link this point of view with the one in [20], where the relationship between fuzzy closure relations and fuzzy closure systems is considered, we will use only the total and isotone relations. This set will be denoted by $(\mathrm{IsotTot}(L^{A\times A}), \hat{\rho})$.

The main goal of the paper is to study the antitone/isotone fuzzy Galois connections in Fig. 1 and study the relationship between their fixed points and fuzzy closure structures.

In this section, we study the ordered structures present in Fig. 1. It is well-known that (L^A, S) is a complete fuzzy lattice, so we omit the proof.

Proposition 13. *The couple* $(A^A, \tilde{\rho})$ *is a complete fuzzy lattice and* $(\mathrm{Isot}(A^A), \tilde{\rho})$ *is a complete fuzzy sublattice.*

Proof. First of all, let us check that $\tilde{\rho}$ is a fuzzy order. Reflexivity and antisymmetry are trivial since the relation is defined pointwise. Let us check transitivity, let $f_1, f_2, f_3 \in A^A$,

$$\begin{aligned}\tilde{\rho}(f_1, f_2) \otimes \tilde{\rho}(f_2, f_3) &= \bigwedge_{x\in A} \rho(f_1(x), f_2(x)) \otimes \bigwedge_{y\in A} \rho(f_2(y), f_3(y))\\ &\overset{(*)}{\leq} \bigwedge_{x\in A}\bigwedge_{y\in A} (\rho(f_1(x), f_2(x)) \otimes \rho(f_2(y), f_3(y)))\\ &\leq \bigwedge_{x\in A} (\rho(f_1(x), f_2(x)) \otimes \rho(f_2(x), f_3(x))) \leq \bigwedge_{x\in A} \rho(f_1(x), f_3(x)) = \tilde{\rho}(f_1, f_3),\end{aligned}$$

where $(*)$ holds due to (2.52) in [4].

Let $F\colon A^A \to L$. Consider $X_F\colon A \to L^A$ defined by

$$(X_F(x))(y) = \bigvee_{\substack{f\in A^A\\ f(x)=y}} F(f).$$

We claim that the infimum and the supremum of F in $(A^A, \tilde{\rho})$ are

$$\left(\bigsqcap F\right)(x) = \bigsqcap X_F(x) \quad \text{and} \quad \left(\bigsqcup F\right)(x) = \bigsqcup X_F(x)$$

we denote $\bigsqcap(X_F(x))$ by $\hat{F}(x)$. Then, for all $x \in A$,

$$F(f) \leq \bigvee_{\substack{g\in A^A\\ g(x)=f(x)}} F(g) = (X_F(x))(f(x)) \leq \rho\left(\hat{F}(x), f(x)\right) \tag{1}$$

Therefore, we get $F(f) \leq \bigwedge_{x\in A} \rho\left(\hat{F}(x), f(x)\right) = \tilde{\rho}(\hat{F}, f)$, i.e., $\hat{F}$ is a lower bound of F.

We now prove that it is the biggest lower bound, that is,

$$\bigwedge_{g\in A^A} (F(g) \to \tilde{\rho}\,(f, g)) \leq \tilde{\rho}(f, \hat{F})$$

Starting from the right hand side we get

$$\begin{aligned}\tilde{\rho}(f, \hat{F}) &= \bigwedge_{a\in A} \rho(f(a), \hat{F}(a)) = \bigwedge_{a\in A} \rho\left(f(a), \bigsqcap(X_F(a))\right)\\ &\overset{\text{(i)}}{=} \bigwedge_{a\in A}\bigwedge_{b\in A} (X_F(a))(b) \to \rho\,(f(a), b)) = \bigwedge_{a,b\in A}\left(\bigvee_{\substack{g\in A^A\\ g(a)=b}} F(g) \to \rho(f(a), b)\right)\\ &\overset{\text{(ii)}}{=} \bigwedge_{a,b\in A}\bigwedge_{\substack{g\in A^A\\ g(a)=b}} (F(g) \to \rho\,(f(a), b)) \geq \bigwedge_{a\in A}\bigwedge_{g\in A^A} (F(g) \to \rho\,(f(a), g(a)))\\ &\overset{\text{(iii)}}{=} \bigwedge_{g\in A^A} (F(g) \to \bigwedge_{a\in A} \rho\,(f(a), g(a))) = \bigwedge_{g\in A^A} (F(g) \to \tilde{\rho}\,(f, g))\,,\end{aligned}$$

where (i) holds due to Theorem 6, and (ii), (iii) are due to (2.53) and (2.51) in [4], respectively. Thus, $\hat{F}$ is the infimum of F. The proof for $\bigsqcup(X_F(x))$ being the supremum of F is analogous.

To prove $(\mathrm{Isot}(A^A), \tilde{\rho})$ is a sublattice let $F \in L^{\mathrm{Isot}(\mathrm{A^A})}$, then we get

$$\rho(\hat{F}(a), \hat{F}(b)) = \rho\left(\hat{F}(a), \bigsqcap(X_F(b))\right)$$

$$\overset{\text{(i)}}{=} \bigwedge_{x\in A} X_F(b)(x) \to \rho(\hat{F}(a), x) = \bigwedge_{x\in A} \left(\bigvee_{\substack{g\in \mathrm{Isot}(A^A)\\ g(b)=x}} F(g) \to \rho(\hat{F}(a), x) \right)$$

$$\overset{\text{(ii)}}{=} \bigwedge_{x\in A} \bigwedge_{\substack{g\in \mathrm{Isot}(A^A)\\ g(b)=x}} (F(g) \to \rho(\hat{F}(a), x))$$

$$\overset{(1)}{\geq} \bigwedge_{x\in A} \bigwedge_{\substack{g\in \mathrm{Isot}(A^A)\\ g(b)=x}} (\rho(\hat{F}(a), g(a)) \to \rho(\hat{F}(a), x))$$

$$\geq \bigwedge_{g\in \mathrm{Isot}(A^A)} (\rho(\hat{F}(a), g(a)) \to \rho(\hat{F}(a), g(b)))$$

$$\overset{\text{(iii)}}{\geq} \bigwedge_{g\in \mathrm{Isot}(A^A)} \rho(g(a), g(b)) \geq \rho(a, b),$$

where (i) is due to Theorem 6, (ii) is due to (2.52) in [4] and (iii) is by transitivity.

The proof for $\bigsqcup X_F$ being isotone as well is analogous. □

Proposition 14. *The couple* $(\mathrm{IsotTot}(L^{A\times A}), \hat{\rho})$ *is a fuzzy preordered set. In addition,* a^μ *is a normal clique for all* $a \in A$, $\mu \in \mathrm{IsotTot}(L^{A\times A})$.

Proof. Let $\kappa_1, \kappa_2, \kappa_3 \in \mathrm{IsotTot}(L^{A\times A})$. For reflexivity, by isotonicity of κ_1 and reflexivity of ρ we get,

$$\hat{\rho}(\kappa_1, \kappa_1) = \bigwedge_{a\in A} \rho_\propto(a^{\kappa_1}, a^{\kappa_1}) \geq \bigwedge_{a\in A} \rho(a, a) = 1.$$

Let us prove now transitivity:

$$\hat{\rho}(\kappa_1, \kappa_2) \otimes \hat{\rho}(\kappa_2, \kappa_3) = \bigwedge_{a\in A} \rho_\propto(a^{\kappa_1}, a^{\kappa_2}) \otimes \bigwedge_{b\in A} \rho_\propto(b^{\kappa_2}, b^{\kappa_3})$$

$$\overset{\text{(i)}}{\leq} \bigwedge_{a\in A} \bigwedge_{b\in A} (\rho_\propto(a^{\kappa_1}, a^{\kappa_2}) \otimes \rho_\propto(b^{\kappa_2}, b^{\kappa_3})) \leq \bigwedge_{a\in A} (\rho_\propto(a^{\kappa_1}, a^{\kappa_2}) \otimes \rho_\propto(a^{\kappa_2}, a^{\kappa_3}))$$

$$\overset{\text{(ii)}}{\leq} \bigwedge_{a\in A} \rho_\propto(a^{\kappa_1}, a^{\kappa_3}) = \hat{\rho}(\kappa_1, \kappa_3),$$

where (i) is due to (2.53) in [4] and (ii) is due to Theorem 4.

Let $\mu \in \mathrm{IsotTot}(L^{A\times A})$, $a \in A$, then a^μ is a normal clique. By μ being total we get a^μ is normal. By isotony we get,

$$1 = \rho(a, a) \leq \rho_\propto(a^\mu, a^\mu)$$

Thus, a^μ is a normal clique. □

In general, $(\mathrm{IsotTot}(L^{A\times A}), \hat{\rho})$ is not a fuzzy poset. This is shown in the following example.

Example 1. Consider the Gödel unit interval as the underlying algebra of truth values and the complete fuzzy lattice (A, ρ) where $A = \{a, b\}$ and

ρ	a	b
a	1	1
b	0.8	1

Consider the following two total fuzzy relations κ_1 and κ_2.

κ_1	a	b
a	1	0.3
b	0.4	1

κ_2	a	b
a	1	0.1
b	0.1	1

Both relations are isotone as $\rho_\propto(x^{\kappa_i}, y^{\kappa_i}) = 1$ for all $i \in \{1, 2\}$ and $x, y \in A$.

Likewise,

$$\begin{aligned}\hat{\rho}(\kappa_1, \kappa_2) &= \rho_\propto(a^{\kappa_1}, a^{\kappa_2}) \wedge \rho_\propto(b^{\kappa_1}, b^{\kappa_2}) \\ &= \bigwedge_{x,y\in A} (a^{\kappa_1}(x) \wedge a^{\kappa_2}(y) \to \rho(x, y)) \wedge \bigwedge_{x,y\in A} (b^{\kappa_1}(x) \wedge b^{\kappa_2}(y) \to \rho(x, y))\end{aligned}$$

The only case where $\rho(x, y) \neq 1$ is $x = b$ and $y = a$. Hence,

$$\begin{aligned}\hat{\rho}(\kappa_1, \kappa_2) &= (a^{\kappa_1}(b) \wedge a^{\kappa_2}(a) \to \rho(b, a)) \wedge (b^{\kappa_1}(b) \wedge b^{\kappa_2}(a) \to \rho(b, a)) \\ &= (0.3 \wedge 1 \to 0.8) \wedge (1 \wedge 0.1 \to 0.8) = 1.\end{aligned}$$

Similarly for $\hat{\rho}(\kappa_2, \kappa_1)$. Nevertheless, $\kappa_1 \neq \kappa_2$, thus $\hat{\rho}$ is not antisymmetric.

4. The fuzzy Galois connection between L-sets and isotone mappings

In this section, the results in [22] are extended to the general case of complete residuated lattices.

Theorem 15. *Let* $\hat{\mathtt{c}}\colon (L^A, S) \to (\mathrm{Isot}(A^A), \tilde{\rho})$ *defined as* $\hat{\mathtt{c}}(\Phi)(a) = \mathtt{c}_\Phi(a) = \bigsqcap(a^\rho \otimes \Phi)$ *and* $\widetilde{\Psi}\colon (\mathrm{Isot}(A^A), \tilde{\rho}) \to (L^A, S)$ *given by* $\widetilde{\Psi}(f)(a) = \Psi_f(a) = \rho(f(a), a)$. *Then, the couple* $(\widehat{\mathtt{c}}, \widetilde{\Psi})$ *is a fuzzy Galois connection between* (L^A, S) *and* $(\mathrm{Isot}(A^A), \tilde{\rho})$.

Proof. First, we need to prove that the mappings are well-defined. The only thing to check is that $\widehat{\mathtt{c}}(\Phi)$ is an isotone mapping for all $\Phi \in L^A$.

Let $x, y \in A$, we have that

$$\begin{aligned}\rho(x, y) &\leq \bigwedge_{z\in A}(\rho(y, z) \to \rho(x, z)) && \text{by transitivity of } \rho \\ &\leq \bigwedge_{z\in A}((y^\rho \otimes \Phi)(z) \to (x^\rho \otimes \Phi)(z)) && \text{by (2.47) in [4]} \\ &\leq \bigwedge_{z\in A}((y^\rho \otimes \Phi)(z) \to \rho(\mathtt{c}_\Phi(x), z)) && \text{by Definition 5 and (2.43) in [4]} \\ &= \rho(\mathtt{c}_\Phi(x), \mathtt{c}_\Phi(y)) && \text{by Theorem 6.}\end{aligned}$$

Hence, $\widehat{\mathtt{c}}(\Phi)$ is isotone.

Now, to prove that the couple $(\widehat{\mathtt{c}}, \widetilde{\Psi})$ is a fuzzy Galois connection, consider a fuzzy set $\Phi \in L^A$ and an isotone mapping $f \in (\mathrm{Isot}(A^A), \tilde{\rho})$, we claim $\bigwedge_{a,x\in A}\big((a^\rho \otimes \Phi)(x) \to \rho(f(a), x)\big) = \bigwedge_{a\in A}\big(\Phi(a) \to \rho(f(a), a)\big)$. Therefore,

$$\begin{aligned}\tilde{\rho}(f, \mathtt{c}(\Phi)) &= \bigwedge_{a\in A} \rho(f(a), \mathtt{c}_\Phi(a)) \\ &= \bigwedge_{a,x\in A} \big((a^\rho \otimes \Phi)(x) \to \rho(f(a), x)\big) && \text{by Theorem 6} \\ &= \bigwedge_{a\in A} \big(\Phi(a) \to \rho(f(a), a)\big) && \text{by claim} \\ &= S(\Phi, \Psi(f))\end{aligned}$$

To prove the claim,

$$\bigwedge_{a,x\in A}\big((a^\rho\otimes\Phi)(x)\to\rho(f(a),x)\big)\le\bigwedge_{a\in A}\big((a^\rho\otimes\Phi)(a)\to\rho(f(a),a)\big)$$
$$=\bigwedge_{a\in A}\big(\Phi(a)\to\rho(f(a),a)\big)$$

Conversely, for all $x\in A$

$$\bigwedge_{a\in A}\big(\Phi(a)\to\rho(f(a),a)\big)\overset{\text{(i)}}{\le}\bigwedge_{a\in A}\big((x^\rho\otimes\Phi)(a)\to\rho(f(a),a)\otimes\rho(x,a)\big)$$
$$\overset{\text{(ii)}}{\le}\bigwedge_{a\in A}\big((x^\rho\otimes\Phi)(a)\to\rho(f(x),f(a))\otimes\rho(f(a),a)\big)$$
$$\le\bigwedge_{a\in A}\big((x^\rho\otimes\Phi)(a)\to\rho(f(x),a)\big),$$

where (i) is due to (2.47) in [4] and (ii) is by isotonicity of f.

Therefore, $\bigwedge_{a\in A}\big(\Phi(a)\to\rho(f(a),a)\big)=\bigwedge_{a,x\in A}\big((a^\rho\otimes\Phi)(x)\to\rho(f(a),x)\big)$. □

As stated in the preliminaries section, the fixed points of the fuzzy Galois connection defined above are not fuzzy closure systems and fuzzy closure operators in general. However, the next result shows that the formal concepts of the Galois connection are in some sense related to those notions.

Lemma 16.

1. *For all $\Phi\in L^A$, the mapping $\widehat{\mathtt{c}}(\Phi)$ is inflationary.*
2. *For all $f\in(\mathrm{Isot}(A^A),\tilde\rho)$, the fuzzy set $\widetilde\Psi(f)$ is extensional wrt $\approx$.*

Proof. First, given $\Phi\in L^A$ and $a\in A$, by Theorem 6, we have that

$$\rho(a,\mathtt{c}_\Phi(a))=\rho(a,\textstyle\bigsqcap(a^\rho\otimes\Phi))=\displaystyle\bigwedge_{x\in A}\big((a^\rho\otimes\Phi)(x)\to\rho(a,x)\big)=1$$

On the other hand, given $f\in(\mathrm{Isot}(A^A),\tilde\rho)$, for all $x\in A$, we have that

$$\Psi(f)(z)\otimes\rho(z,x)\otimes\rho(x,z)\le\rho(f(z),x)\otimes\rho(x,z)$$
$$\overset{\text{(i)}}{\le}\rho(f(z),x)\otimes\rho(f(x),f(z))\le\rho(f(x),x)=\Psi(f)(x)$$

where (i) holds by the isotonicity of f. □

The next result shows that one of the implications in Theorem 12 still holds.

Theorem 17.

1. *Let Φ be a fuzzy closure system. Then, $(\Phi,\widehat{\mathtt{c}}(\Phi))$ is a fixed point of the Galois connection $(\widehat{\mathtt{c}},\widetilde\Psi)$.*
2. *Let $\mathtt{c}$ be a fuzzy closure operator. Then, $(\widetilde\Psi(\mathtt{c}),\mathtt{c})$ is a fixed point of the Galois connection $(\widehat{\mathtt{c}},\widetilde\Psi)$.*

Proof. Let Φ be a fuzzy closure system, that is, according to [21], $\min(a^\rho\otimes\Phi)$ exists, for all $a\in A$ and Φ is extensional. Since $(\hat{\mathtt{c}},\widetilde\Psi)$ is a fuzzy Galois connection, $\widetilde\Psi\circ\widehat{\mathtt{c}}$ is inflationary by [15], then $\Phi\subseteq\widetilde\Psi(\widehat{\mathtt{c}}(\Phi))$. For the converse inclusion, let $a\in A$ and $m=\min(a^\rho\otimes\Phi)=\widehat{\mathtt{c}}(a)$,

$$\begin{aligned}\widetilde\Psi(\widehat{\mathtt{c}}(\Phi))(a)&=\rho\,(m,a)\\&=\Phi(m)\otimes\rho(m,a)\otimes\rho(a,m)&&\text{by } m \text{ being a minimum}\\&\le\Phi(a)&&\text{by extensionality of }\Phi.\end{aligned}$$

For the second item, let $\mathrm{c}\colon A \to A$ be a closure operator. Since $(\hat{\mathrm{c}}, \widetilde{\Psi})$ is a fuzzy Galois connection, again by [15], we have $\widetilde{\rho}(\mathrm{c}, \widehat{\mathrm{c}}(\widetilde{\Psi}(\mathrm{c}))) = 1$. For the converse inclusion,

$$\widehat{\mathrm{c}}(\widetilde{\Psi}(\mathrm{c}))(a) = \bigsqcap(a^\rho \otimes \widetilde{\Psi}(\mathrm{c})) = m.$$

The element m verifies

$$(a^\rho \otimes \widetilde{\Psi}(\mathrm{c}))(x) = \rho(a,x) \otimes \rho(\mathrm{c}(x), x) \le \rho(m,x), \text{ for all } x \in A.$$

Thus, taking $x = \mathrm{c}(a)$, we get

$$1 = \rho(a, \mathrm{c}(a)) \otimes \rho(\mathrm{c}(\mathrm{c}(a)), \mathrm{c}(a)) \le \rho(m, \mathrm{c}(a)),$$

where the first equality holds due to c being inflationary and idempotent.

Therefore, $\widehat{\mathrm{c}}(\widetilde{\Psi}(\mathrm{c})) = \mathrm{c}$ and c is a fixed point of the fuzzy Galois connection. □

Let us recall that Theorem 12 proved an equivalence between the following three statements in a Heyting algebra.

1. The couple (Φ, c) is a fixed point of the fuzzy Galois connection $(\widehat{\mathrm{c}}, \widetilde{\Psi})$.
2. The fuzzy set Φ is fuzzy closure system and $\mathrm{c} = \widehat{\mathrm{c}}(\Phi)$.
3. The isotone mapping c is a fuzzy closure operator and $\Phi = \widetilde{\Psi}(\mathrm{c})$.

Last theorem proved that, in the general case, 2 implies 1 and 3 implies 1. In addition, by Corollary 19 in [20], we have that 2 and 3 are also equivalent. However, the following example shows that 1 does not imply 2 nor 3.

Example 2. Let $\mathbb{L} = (\{0, 0.5, 1\}, \wedge, \vee, \otimes, \to, 0, 1)$ be a Łukasiewicz residuated lattice with three values, and (A, ρ) be the fuzzy lattice with $A = \{\bot, a, b, c, d, e, \top\}$ and the fuzzy relation $\rho\colon A \times A \to L$ is described by the following table:

ρ	$\bot$	a	b	c	d	e	$\top$
$\bot$	1	1	1	1	1	1	1
a	0.5	1	0.5	1	1	1	1
b	0.5	0.5	1	1	1	1	1
c	0.5	0.5	0.5	1	1	1	1
d	0	0.5	0	0.5	1	0.5	1
e	0	0	0.5	0.5	0.5	1	1
$\top$	0	0	0	0.5	0.5	0.5	1

For the fuzzy set $\Phi = \{a/1, b/0.5\}$, the mapping $\mathrm{c}_\Phi(x) = \bigsqcap(x^\rho \otimes \Phi) = f(x)$, which is isotone and inflationary, is $f(\bot) = f(a) = a$; $f(b) = c$, $f(c) = f(d) = d$ and $f(e) = f(\top) = \top$. However, f is not a closure operator since it is not idempotent, $\rho(f(f(b)), f(b)) = \rho(d,c) = 0.5 \neq 1$.

5. The fuzzy Galois connection between L-sets and isotone total L-relations

Recall that a fuzzy relation $\kappa\colon A \times A \to L$ is said to be extensional if it is extensional when considered as a fuzzy subset of $A \times A$ with the similarity relation defined pointwise as

$$(a_1, b_1) \approx_{A\times A} (a_2, b_2) = (a_1 \approx a_2) \otimes (b_1 \approx b_2).$$

It is easy to see that, due to the reflexivity of $\approx$, extensionality in $A \times A$ is equivalent to satisfying left and right extensionality:

$$\kappa(a_1, b) \otimes (a_1 \approx a_2) \le \kappa(a_2, b), \qquad \text{for all } a_1, a_2, b \in A.$$
$$\kappa(a, b_1) \otimes (b_1 \approx b_2) \le \kappa(a, b_2), \qquad \text{for all } a, b_1, b_2 \in A.$$

The use of this binary fuzzy equivalence is a common approach which can be seen, for example, in [12, Remark 3.3].

Next result proves that the mapping $\kappa\colon (L^A, S) \to \mathrm{IsotTot}(L^{A\times A})$ is well-defined. Moreover, every fuzzy relation in the image of κ is inflationary and extensional.

Proposition 18. *Let $\Phi \in L^A$ be a fuzzy set, then $\kappa(\Phi) \in L^{A\times A}$, defined as $\kappa(\Phi)(a,b) = \kappa_\Phi(a,b) = (\bigsqcap(a^\rho \otimes \Phi) \approx b)$, is total, inflationary, isotone and extensional wrt $\approx_{A\times A}$.*

Proof. Throughout the proof, we will use the notation $\mathrm{c}_\Phi(a) = \bigsqcap(a^\rho \otimes \Phi)$ as in Section 4.

The fuzzy relation κ_Φ is total because $\mathrm{c}_\Phi(a) \in A$ is an element of the afterset (a^{κ_Φ}) with degree 1, for all $a \in A$:

$$\kappa_\Phi\,(a, \mathrm{c}_\Phi(a)) = (\mathrm{c}_\Phi(a) \approx \mathrm{c}_\Phi(a)) = 1.$$

On the other hand, given $a \in A$, it holds

$$\begin{aligned}
\rho_\propto(a, a^{\kappa_\Phi}) &= \bigwedge_{x\in A} \kappa_\Phi(a,x) \to \rho(a,x) \\
&= \bigwedge_{x\in A} (\mathrm{c}_\Phi(a) \approx x) \to \rho(a,x) \\
&\geq \bigwedge_{x\in A} \rho\,(\mathrm{c}_\Phi(a), x) \to \rho(a,x) \\
&\overset{\text{(i)}}{\geq} \rho(a, \mathrm{c}_\Phi(a)) \overset{\text{(ii)}}{=} 1,
\end{aligned}$$

where (i) is by adjointness and transitivity and (ii) is due to the first part of Lemma 16.

Therefore, κ_Φ is inflationary. Next, let $a, b \in A$,

$$\begin{aligned}
\rho_\propto(a^{\kappa_\Phi}, b^{\kappa_\Phi}) &= \bigwedge_{x,y\in A} \left(a^{\kappa_\Phi}(x) \otimes b^{\kappa_\Phi}(y) \to \rho(x,y)\right) \\
&= \bigwedge_{x,y\in A} ((\mathrm{c}_\Phi(a) \approx x) \otimes (\mathrm{c}_\Phi(b) \approx y) \to \rho(x,y)) \\
&= \bigwedge_{x,y\in A} ((x \approx \mathrm{c}_\Phi(a)) \otimes (\mathrm{c}_\Phi(b) \approx y) \to \rho(x,y)) \\
&\overset{\text{(i)}}{\geq} \bigwedge_{x,y\in A} (\rho\,(\mathrm{c}_\Phi(a), \mathrm{c}_\Phi(b)) \to \rho(x,y)) \to \rho(x,y) \\
&\overset{\text{(ii)}}{\geq} \bigwedge_{x,y\in A} \rho\,(\mathrm{c}_\Phi(a), \mathrm{c}_\Phi(b)) \\
&= \rho\,(\mathrm{c}_\Phi(a), \mathrm{c}_\Phi(b)) \geq \rho(a,b),
\end{aligned}$$

where (i) is a consequence of

$$(x \approx \mathrm{c}_\Phi(a)) \otimes \rho(\mathrm{c}_\Phi(a), \mathrm{c}_\Phi(b)) \otimes (\mathrm{c}_\Phi(b) \approx y) \leq \rho(x,y)$$

being true by transitivity, and applying adjointness

$$(x \approx \mathrm{c}_\Phi(a)) \otimes (\mathrm{c}_\Phi(b) \approx y) \leq \rho(\mathrm{c}_\Phi(a), \mathrm{c}_\Phi(b)) \to \rho(x,y),$$

item (ii) is the direct use of property (2.26) in [4] and the last inequality is due to c_Φ being isotone according to Theorem 15. Hence, κ_Φ is isotone.

Finally, let us prove that κ_Φ is extensional for all $\Phi \in L^A$:

$$\begin{aligned}
&\kappa_\Phi(a,b) \otimes (a \approx x) \otimes (b \approx y) \\
&\quad= \left(\bigsqcap(a^\rho \otimes \Phi) \approx b\right) \otimes (a \approx x) \otimes (b \approx y) \\
&\quad\overset{\text{sym.}}{=} (x \approx a) \otimes \left(\bigsqcap(a^\rho \otimes \Phi) \approx b\right) \otimes (b \approx y) \\
&\quad\overset{\text{isot.}\mathrm{c}_\Phi}{\leq} \left(\bigsqcap(x^\rho \otimes \Phi) \approx \bigsqcap(a^\rho \otimes \Phi)\right) \otimes \left(\left(\bigsqcap(a^\rho \otimes \Phi)\right) \approx b\right) \otimes (b \approx y) \\
&\quad\overset{\text{trans.}}{\leq} \left(\bigsqcap(x^\rho \otimes \Phi) \approx y\right) = \kappa_\Phi(x,y). \quad \square
\end{aligned}$$

As κ is well-defined, we can go one step further and show that $(\kappa, \widehat{\Psi})$ is a fuzzy Galois connection between (L^A, S) and $(\mathrm{IsotTot}(L^{A\times A}), \hat{\rho})$. This is proved in the next theorem.

Theorem 19. *Let* $\kappa\colon (L^A, S) \to (\mathrm{IsotTot}(L^{A\times A}), \hat{\rho})$ *defined as* $\kappa(\Phi)(a,b) = \kappa_\Phi(a,b) = (\bigsqcap(a^\rho \otimes \Phi) \approx b)$ *and* $\widehat{\Psi}\colon (\mathrm{IsotTot}(L^{A\times A}), \hat{\rho}) \to (L^A, S)$ *given by* $\widehat{\Psi}(\mu)(a) = \Phi_\mu(a) = \rho_\propto(a^\mu, a)$. *Then, the couple* $(\kappa, \widehat{\Psi})$ *is a fuzzy Galois connection between* (L^A, S) *and* $(\mathrm{IsotTot}(L^{A\times A}), \hat{\rho})$.

Proof. Let $\mu \in \mathrm{IsotTot}(L^{A\times A})$ and $\Phi \in (L^A, S)$. First of all, observe that κ is well defined as a consequence of Proposition 18. Let us show now an equivalent expression for $\hat{\rho}(\mu, \kappa_\Phi)$ which will be used during the proof.

$$\begin{aligned}
\hat{\rho}(\mu, \kappa_\Phi) &= \bigwedge_{a\in A} \rho_\propto(a^\mu, a^{\kappa_\Phi}) \\
&= \bigwedge_{a,x\in A} \Big(\mu(a,x) \to \rho(x, \bigsqcap(a^\rho \otimes \Phi))\Big) && \text{by Proposition 8} \\
&= \bigwedge_{a,x\in A} \left(\mu(a,x) \to \bigwedge_{y\in A} \big((a^\rho \otimes \Phi)(y) \to \rho(x,y)\big)\right) && \text{by Theorem 6} \\
&= \bigwedge_{a,x,y\in A} \big((\mu(a,x) \otimes (a^\rho \otimes \Phi)(y)) \to \rho(x,y)\big) && \text{by (2.51),(2.33) in [4]} \\
&= \bigwedge_{a,x,y\in A} ((\mu(a,x) \otimes \rho(a,y) \otimes \Phi(y)) \to \rho(x,y)).
\end{aligned}$$

Notice that taking $y = a$ we get

$$\begin{aligned}
\hat{\rho}(\mu, \kappa_\Phi) &\le \bigwedge_{a,x\in A} ((\mu(a,x) \otimes \rho(a,a) \otimes \Phi(a)) \to \rho(x,a)) \\
&= \bigwedge_{a\in A} \left(\Phi(a) \to \bigwedge_{x\in A} (\mu(a,x) \to \rho(x,a))\right) && \text{by (2.51),(2.33) in [4]} \\
&= \bigwedge_{a\in A} (\Phi(a) \to \Phi_\mu(a)) = S(\Phi, \Phi_\mu)
\end{aligned}$$

Conversely,

$$\begin{aligned}
S(\Phi, \Phi_\mu) &= \bigwedge_{a\in A} \big(\Phi(a) \to \Phi_\mu(a)\big) = \bigwedge_{y\in A} \big(\Phi(y) \to \rho_\propto(y^\mu, y)\big) \\
&\le \bigwedge_{y\in A} \big((a^\rho \otimes \Phi)(y) \to \rho(a,y) \otimes \rho_\propto(y^\mu, y)\big) && \text{by (2.47) in [4]} \\
&\le \bigwedge_{y\in A} \big((a^\rho \otimes \Phi)(y) \to \rho_\propto(a^\mu, y^\mu) \otimes \rho_\propto(y^\mu, y)\big) && \text{by isotonicity} \\
&\le \bigwedge_{y\in A} \big((a^\rho \otimes \Phi)(y) \to \rho_\propto(a^\mu, y)\big) && \text{by Theorem 4} \\
&= \bigwedge_{x,y\in A} ((\rho(a,y) \otimes \Phi(y) \otimes \mu(a,x)) \to \rho(x,y)) && \text{by (2.51),(2.33) in [4]} \\
&= \hat{\rho}(\mu, \kappa_\Phi).
\end{aligned}$$

Thus, $S(\Phi, \Phi_\mu) = \hat{\rho}(\mu, \kappa_\Phi)$, and these mappings form a fuzzy Galois connection. □

Analogous to previous sections, fuzzy closure systems and strong fuzzy closure relations are fixed points of this fuzzy Galois connection.

Theorem 20.

1. *Let Φ be a fuzzy closure system. Then, $(\Phi, \kappa(\Phi))$ is a fixed point of the Galois connection $(\kappa, \widehat{\Psi})$.*
2. *Let μ be a strong fuzzy closure relation. Then, $(\widehat{\Psi}(\mu), \mu)$ is a fixed point of the Galois connection $(\kappa, \widehat{\Psi})$.*

Proof. Let Φ be a fuzzy closure system, that is, according to [21], $\min(a^\rho \otimes \Phi)$ exists, for all $a \in A$ and Φ is extensional. Since $(\kappa, \widehat{\Psi})$ is a fuzzy Galois connection, $\widehat{\Psi} \circ \kappa$ is inflationary by [15], then $\Phi \subseteq \widehat{\Psi}(\kappa(\Phi))$. For the converse inclusion, let $a \in A$ and $m = \min(a^\rho \otimes \Phi) = \widehat{c}(a)$,

$$\widehat{\Psi}(\kappa(\Phi))(a) = \rho_\propto(a^{\kappa\Phi}, a) = \bigwedge_{x \in A}\left(\left(\bigsqcap(a^\rho \otimes \Phi) \approx x\right) \to \rho(x, a)\right)$$
$$\overset{\text{(i)}}{\le} \rho(m, a) \overset{\text{(ii)}}{=} \Phi(m) \otimes \rho(a, m) \otimes \rho(m, a) \overset{\text{(iii)}}{\le} \Phi(a),$$

where (i) holds by taking $x = m$ and (ii) holds by applying that m is the minimum of $(a^\rho \otimes \Phi)$ and (iii) holds by extensionality. Therefore, $\Phi = \widehat{\Psi}\kappa(\Phi)$.

For the second item, let $\mu\colon A \times A \to L$ be a strong fuzzy closure relation. Consider the composition,

$$\kappa(\widehat{\Psi}(\mu))(a, b) = \bigsqcap(a^\rho \otimes \widehat{\Psi}(\mu)) \approx b.$$

Since μ is a strong fuzzy closure relation, there exists a closure operator c such that $\mu(a, b) = (c(a) \approx b)$, [20]. Furthermore, this closure operator is defined by $c(a) = \mathrm{Core}(a^\mu)$. Then, it suffices to show $a^* \in \mathrm{Core}(a^\mu)$ is exactly $\bigsqcap(a^\rho \otimes \widehat{\Psi}(\mu))$.

By definition of infimum, $m = \bigsqcap(a^\rho \otimes \widehat{\Psi}(\mu))$ satisfies,

$$(a^\rho \otimes \widehat{\Psi}(\mu))(x) = \rho(a, x) \otimes \rho_\propto(x^\mu, x) \le \rho(m, x) \qquad \text{for all } x \in A.$$
$$\bigwedge_{y \in A} \rho(a, y) \otimes \rho_\propto(y^\mu, y) \to \rho(x, y) \le \rho(x, m) \qquad \text{for all } x \in A.$$

Using Proposition 8, inflationarity and idempotency of μ we have that, $\rho_\propto(a, a^\mu) = \rho(a, a^*) = 1$ and

$$\begin{aligned}
1 &= \rho_\propto(a^{\mu\circ\mu}, a^\mu)\\
&= \bigwedge_{x,y \in A} (a^{\mu\circ\mu}(x)) \otimes a^\mu(y) \to \rho(x, y)\\
&= \bigwedge_{x,y,z \in A} (a^\mu(z) \otimes z^\mu(x) \otimes a^\mu(y)) \to \rho(x, y) && \text{by (2.52) in [4]}\\
&\le \bigwedge_{x \in A} (a^\mu(a^*) \otimes (a^*)^\mu(x) \otimes a^\mu(a^*)) \to \rho(x, a^*) && \text{by taking } y = z = a^*\\
&= \bigwedge_{x \in A} (a^*)^\mu(x) \to \rho(x, a^*) = \rho_\propto(a^{*^\mu}, a^*).
\end{aligned}$$

Hence, using the first inequality with $x = a^*$ we have that $\rho(m, a^*) = 1$.

In addition, using the second inequality we get

$$\begin{aligned}
\rho(a^*, m) &\ge \bigwedge_{y \in A} \rho(a, y) \otimes \rho_\propto(y^\mu, y) \to \rho(a^*, y)\\
&\ge \bigwedge_{y \in A} \rho_\propto(a^\mu, y^\mu) \otimes \rho_\propto(y^\mu, y) \to \rho(a^*, y)\\
&\ge \bigwedge_{y \in A} \rho_\propto(a^\mu, y) \to \rho(a^*, y)\\
&= \bigwedge_{y \in A} \rho(a^*, y) \to \rho(a^*, y) = 1
\end{aligned}$$

Thus, $\rho(a^*, m) = 1$ and $a^* = m$. Therefore $\mathrm{Core}(a^\mu) = \mathrm{Core}(a^{\kappa\widehat{\Psi}(\mu)})$ and, since they are vague descriptions of their core, we also have $\mu = \kappa\widehat{\Psi}(\mu)$. □

Last theorem shows that closure structures are fixed points of the fuzzy Galois connection. However, there are fixed points that are not formed by closure structures. This is illustrated in the next example.

Example 3. Let A be the complete fuzzy lattice from Example 2. For the fuzzy set $\Phi = \{{}^{a}/_{1}, {}^{b}/_{0.5}\}$, the fuzzy relation $\kappa(\Phi)(x, y) = \left(\bigsqcap(x^\rho \otimes \Phi) \approx y\right)$, which is total, isotone, extensional and inflationary. We will show that it is not idempotent. It is only a matter of calculation to reach the explicit expression of $\kappa(\Phi)$.

$\kappa(\Phi)$	$\bot$	a	b	c	d	e	$\top$
$\bot$	0.5	1	0	0.5	0.5	0	0
a	0.5	1	0	0.5	0.5	0	0
b	0.5	0.5	0.5	1	0.5	0.5	0.5
c	0	0.5	0	0.5	1	0.5	0.5
d	0	0.5	0	0.5	1	0.5	0.5
e	0	0	0	0.5	0.5	0.5	1
$\top$	0	0	0	0.5	0.5	0.5	1

Consider now the following, by taking $y = z = c$ and $x = d$ we get

$$\begin{aligned}\rho_\propto(b^{\kappa\circ\kappa}, b^\kappa) &= \bigwedge_{x,y\in A} (b^{\kappa\circ\kappa}(x) \otimes b^\kappa(y)) \to \rho(x, y)\\ &= \bigwedge_{x,y,z\in A} (b^\kappa(z) \otimes z^\kappa(x) \otimes b^\kappa(y)) \to \rho(x, y)\\ &\le \bigwedge_{x\in A} c^\kappa(x) \to \rho(x, c) \le c^\kappa(d) \to \rho(d, c) = 0.5 < 1.\end{aligned}$$

Therefore, even though $\kappa(\Phi)$ is a fixed point of the fuzzy Galois connection, it is not a strong fuzzy closure relation.

6. The fuzzy adjunction between isotone total L-relations and isotone mappings

In this section, we focus on the fuzzy adjunction in Fig. 1 formed by the 1-cut mapping, or core, and the extensional hull mapping. These mappings are isotone and form a fuzzy adjunction. Proving this statement is the goal of the section. Additionally, there is some discussion on the fixed points of this fuzzy adjunction.

The following theorem proves that these mappings are indeed well-defined and form a fuzzy adjunction.

Theorem 21. *Let* $-^1\colon (\mathrm{IsotTot}(L^{A\times A}), \hat\rho) \to (\mathrm{Isot}(A^A), \tilde\rho)$ *defined as* $\mu \mapsto \mu^1 = \{(a, b) \in A \times A \mid \mu(a, b) = 1\}$ *and* $-^\approx\colon (\mathrm{Isot}(A^A), \tilde\rho) \to (\mathrm{IsotTot}(L^{A\times A}), \hat\rho)$ *defined as* $f \mapsto f^\approx(a, b) = (f(a) \approx b)$. *Then, the couple* $(-^1, -^\approx)$ *is a fuzzy adjunction between* $(\mathrm{IsotTot}(L^{A\times A}), \hat\rho)$ *and* $(\mathrm{Isot}(A^A), \tilde\rho)$.

Proof. First, we prove that the mappings above are well-defined.

Let $\mu \in L^{A\times A}$ be total and isotone, it is shown below that μ^1 is a crisp function from A to A, i.e., $\mathrm{Core}(a^\mu)$ is a singleton, for all $a \in A$: if $\mu(a, b_1) = \mu(a, b_2) = 1$, then

$$\begin{aligned}&1 = \rho(a, a) \le \rho_\propto(a^\mu, a^\mu) = \bigwedge_{x,y\in A} ((a^\mu(x) \otimes a^\mu(y)) \to \rho(x, y)) \le\\ &((a^\mu(b_1) \otimes a^\mu(b_2)) \to \rho(b_1, b_2)) \wedge ((a^\mu(b_2) \otimes a^\mu(b_1)) \to \rho(b_2, b_1)) =\\ &\rho(b_1, b_2) \wedge \rho(b_2, b_1).\end{aligned}$$

Hence $b_1 = b_2$, by antisymmetry.

Since μ^1 is a mapping, we will denote it with the standard function notation. Next, we show that μ^1 is isotone, that is, for any $\mu \in \mathrm{IsotTot}(L^{A\times A})$ the mapping $\mu^1\colon A \to A$ defined by $\mu^1(a) = \mathrm{Core}(a^\mu)$ for all $a \in A$, is isotone:

$$\rho(a,b) \le \rho_\propto(a^\mu, b^\mu) = \bigwedge_{x,y\in A} ((a^\mu(x) \otimes b^\mu(y)) \to \rho(x,y)) \le$$
$$(a^\mu(\mu^1(a)) \otimes b^\mu(\mu^1(b))) \to \rho(\mu^1(a), \mu^1(b)) = \rho(\mu^1(a), \mu^1(b)).$$

On the other hand, let $f \in \mathrm{Isot}(A^A)$, we recall that $f^\approx \colon A \times A \to L$ is defined by

$$f^\approx(a,b) = (f(a) \approx b) = \rho(f(a), b) \otimes \rho(b, f(a)).$$

First observe that $a^{f^\approx}$ is normal for all $a \in A$, as $f(a) \in \mathrm{Core}(a^{f^\approx})$ by reflexivity. In addition, $f^\approx$ is an isotone relation:

$$\rho(a,b) \otimes f^\approx(a,x) \otimes f^\approx(b,y) \le \rho(f(a), f(b)) \otimes \rho(x, f(a)) \otimes \rho(f(b), y) \le \rho(x,y)$$

for all $a, b, x, y \in A$. Hence,

$$\rho(a,b) \le \bigwedge_{x,y\in A} (f^\approx(a,x) \otimes f^\approx(b,y) \to \rho(x,y)) = \rho_\propto(a^{f^\approx}, b^{f^\approx})$$

Now it is proved that the couple $(-^1, -^\approx)$ is a fuzzy adjunction, that is,

$$\tilde{\rho}(f, \mu^1) = \hat{\rho}(f^\approx, \mu).$$

On the one hand,

$$\begin{aligned}
\hat{\rho}(f^\approx, \mu) &= \bigwedge_{a\in A} \rho_\propto(a^{f^\approx}, a^\mu) \\
&= \bigwedge_{a,x,y\in A} ((a^{f^\approx}(x) \otimes a^\mu(y)) \to \rho(x,y)) \\
&\le \bigwedge_{a\in A} \big((a^{f^\approx}(f(a)) \otimes a^\mu(\mu^1(a))) \to \rho(f(a), \mu^1(a))\big) \\
&= \bigwedge_{a\in A} \rho(f(a), \mu^1(a)) = \tilde{\rho}(f, \mu^1).
\end{aligned}$$

On the other hand, for all $a, x, y \in A$,

$$\begin{aligned}
\rho(f(a), \mu^1(a)) \otimes (f(a) \approx x) \otimes a^\mu(y) &\le \rho(x, \mu^1(a)) \otimes a^\mu(y) \\
&= \rho(x, \mu^1(a)) \otimes a^\mu(\mu^1(a)) \otimes a^\mu(y) \\
&\overset{(*)}{\le} \rho(x, \mu^1(a)) \otimes (\mu^1(a) \approx y) \\
&\le \rho(x,y),
\end{aligned}$$

where $(*)$ holds due to Proposition 14.

Hence,

$$\rho(f(a), \mu^1(a)) \le (a^{f^\approx}(x) \otimes a^\mu(y)) \to \rho(x,y)$$

and, as a consequence,

$$\tilde{\rho}(f, \mu^1) \le \hat{\rho}(f^\approx, \mu). \quad \square$$

As in previous sections, the answer to whether fuzzy closure operators and strong fuzzy closure systems are fixed points of the fuzzy adjunction is affirmative. This is proved in the following proposition.

Proposition 22.

1. *Let* c *be a fuzzy closure operator. Then,* $(\mathrm{c}^\approx, \mathrm{c})$ *is a fixed point of the fuzzy adjunction* $(-^1, -^\approx)$.
2. *Let* μ *be a strong fuzzy closure relation. Then,* (μ, μ^1) *is a fixed point of the fuzzy adjunction* $(-^1, -^\approx)$.

Proof. Let $\mathrm{c}\colon A \to A$ be a fuzzy closure operator. Then, we define the fuzzy relation $\mathrm{c}^{\approx}(a,b) = (\mathrm{c}(a) \approx b)$. This fuzzy relation satisfies $(\mathrm{c}^{\approx})^1(a,b) = \{(a,b) \in A \times A \mid (\mathrm{c}(a) \approx b) = 1\} = \{(a, \mathrm{c}(a)) \mid a \in A\} = \mathrm{c}$. Therefore, $(\mathrm{c}^{\approx}, \mathrm{c})$ is a fixed point of $(-^1, -^{\approx})$.

For the second item, let $\mu\colon A \times A \to L$ be a strong fuzzy closure relation. By one of the characterizations of strong fuzzy closure relation [20, Proposition 20] there exists a closure operator $\mathrm{c}\colon A \to A$ such that $\mu(a,b) = (\mathrm{c}(a) \approx b)$. Then, as in the previous item $\mu^1 = \mathrm{c}$ and $(\mu^1)^{\approx}(a,b) = (\mu^1(a) \approx b) = (\mathrm{c}(a) \approx b) = \mu(a,b)$. Hence, (μ, μ^1) is a fixed point of $(-^1, -^{\approx})$. □

The converse does not hold. There are fixed points of the fuzzy adjunction which are not closure structures. An example of such a case is given below.

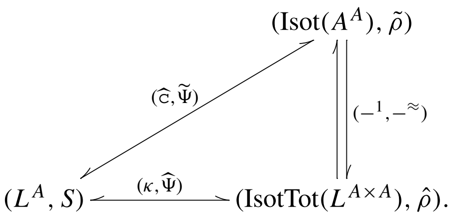
Example 4. Let A be the complete fuzzy lattice from Example 2. Consider the isotone function $f(x) = \bot$, for all $x \in A$. It is clear that f is not a closure operator because it is not inflationary, e.g., $\rho(\top, f(\top)) = 0$. Nevertheless, $f^{\approx}(x,y) = (f(x) \approx y) = (\bot \approx y)$ and $(f^{\approx})^1(x) = \mathrm{Core}(x^{f^{\approx}}) = \bot = f(x)$, for all $x \in A$. Hence, $(f^{\approx}, f)$ is indeed a fixed point of $(-^1, -^{\approx})$ but f is not a fuzzy closure operator. The fuzzy relation $f^{\approx}$ is not a strong fuzzy closure relation either since it is not inflationary, $\rho_{\propto}(\top, \top^{f^{\approx}}) = \rho(\top, \bot) = 0 \neq 1$.

Remark 2. Notice that last example does not depend on the expression of the particular f. For any isotone mapping $g\colon A \to A$ we have $g^{\approx}(x,y) = (g(x) \approx y)$ and $(g^{\approx})^1(x) = \mathrm{Core}(x^{g^{\approx}}) = \{y \in A \mid (g(x) \approx y) = 1\} = g(x)$. Therefore, the composition $-^1 \circ -^{\approx}$ is the identity mapping.

However, $-^{\approx} \circ -^1$ is not the identity mapping since $\mathrm{Im}(-^{\approx})$ is formed by extensional fuzzy relations and there are fuzzy relations in $\mathrm{IsotTot}(L^{A\times A})$ which are not extensional.

7. Looking for commutative diagrams

In this section we wonder whether the fuzzy Galois connections in Fig. 1 describe a commutative diagram such as the following.



The next Proposition shows some positive results.

Proposition 23.

1. *Let* $\Phi \in L^A$, *then* $(\widehat{\mathrm{c}}(\Phi))^{\approx} = \kappa(\Phi)$ *and* $\kappa(\Phi)^1 = \widehat{\mathrm{c}}(\Phi)$.
2. *Let* $f \in \mathrm{Isot}(A^A)$, *then* $\widehat{\Psi}(f^{\approx}) = \widetilde{\Psi}(f)$.
3. *Let* $\mu \in \mathrm{IsotTot}(L^{A\times A})$, *then* $\widetilde{\Psi}(\mu^1) = \widehat{\Psi}(\mu)$.

Proof. Let $\Phi \in L^A$, then we need to prove $(\widehat{\mathrm{c}}(\Phi))^{\approx} = \kappa(\Phi)$ and $\kappa(\Phi)^1 = \widehat{\mathrm{c}}(\Phi)$. These both equalities hold trivially since just applying definition, for all $a, b \in A$, we get

$$(\widehat{\mathrm{c}}(\Phi))^{\approx}(a,b) = (\widehat{\mathrm{c}}(\Phi)(a) \approx b) = \left(\bigsqcap (a^{\rho} \otimes \Phi) \approx b\right) = \kappa(\Phi)(a,b) \text{ and}$$

$$\kappa(\Phi)^1(a) = \mathrm{Core}(a^{\kappa(\Phi)}) = \mathrm{Core}\left(\bigsqcap (a^{\rho} \otimes \Phi) \approx -\right) = \bigsqcap (a^{\rho} \otimes \Phi) = \widehat{\mathrm{c}}(\Phi)(a).$$

Let $f \in \mathrm{Isot}(A^A)$, then we need to prove $\widehat{\Psi}(f^{\approx}) = \widetilde{\Psi}(f)$. This is a direct consequence of using Remark 1 and $\mathrm{Core}(f^{\approx}) = f$ to get,

$$\widehat{\Psi}(f^{\approx})(a) = \rho_{\propto}(a^{f^{\approx}}, a) = \rho(f(a), a) = \widetilde{\Psi}(f)(a)$$

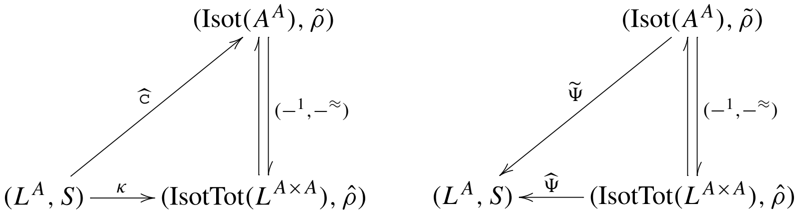
for all $a \in A$.

Lastly, let $\mu \in \mathrm{IsotTot}(L^{A\times A})$, we need to prove $\widetilde{\Psi}(\mu^1) = \widehat{\Psi}(\mu)$. For all $a \in A$, by Proposition 14, a^{μ} is a normal clique and using again Remark 1 we get

$$\widetilde{\Psi}(\mu^1)(a) = \rho(\mu^1(a), a) = \rho(\mathrm{Core}(a^{\mu}), a) = \rho_{\propto}(a^{\mu}, a) = \widehat{\Psi}(\mu)(a).$$

This concludes the proof. □

Thus, the above proposition ensures that the following diagrams commute.



The following example shows that the equalities with the other compositions, i.e. $\kappa\widetilde{\Psi} = -^{\approx}$ and $\widehat{\mathrm{c}}\widehat{\Psi} = -^1$, do not hold in general.

Example 5. Let A be the complete fuzzy lattice from Example 2. Consider the isotone function $f(x) = \bot$ for all $x \in A$. This mapping is trivially isotone. Then, on the one hand, $f^{\approx}(x, y) = (f(x) \approx y) = \rho(y, \bot)$, for all $x, y \in A$.

On the other hand $\kappa(\widetilde{\Psi}(f))(x, y) = \bigsqcap(x^{\rho} \otimes \widetilde{\Psi}(f)) \approx y$. But notice that $\widetilde{\Psi}(f)(z) = \rho(f(z), z) = \rho(\bot, z) = 1$ for all $z \in A$. Hence, $\widetilde{\Psi}(f) = A$ and $\kappa(\widetilde{\Psi}(f))(x, y) = (\bigsqcap(x^{\rho} \otimes \widetilde{\Psi}(f)) \approx y) = (\bigsqcap(x^{\rho}) \approx y) = (x \approx y)$. These relations are clearly distinct since $f^{\approx}(\top, \top) = \rho(\top, \bot) = 0$ and $\kappa\widetilde{\Psi}(f)(\top, \top) = (\top \approx \top) = 1$.

A similar example can be given for fuzzy relations, again on the basis of Example 2. Consider $\mu\colon A \times A \to L$ defined by $\mu(x, a) = 1$ for all $x \in A$ and 0 otherwise. It is clear that μ is isotone and total. Since μ is crisp, we have that $\mu^1 = \mu$. However, $\widehat{\mathrm{c}}(\widehat{\Psi}(\mu))(x) = \bigsqcap(x^{\rho} \otimes \widehat{\Psi}(\mu))$. It is an easy exercise to check that $(\widehat{\mathrm{c}}(\widehat{\Psi}(\mu)))(\top) = \top \neq a = \mu^1(\top)$. Therefore, $(\widehat{\mathrm{c}}(\widehat{\Psi}(\mu))) \neq \mu^1$.

For the sake of completeness, we examine in which cases the equalities $\kappa \circ \widetilde{\Psi} = -^{\approx}$ and $\widehat{\mathrm{c}} \circ \widehat{\Psi} = -^1$ hold.

Corollary 24.

1. *Let $f \in \mathrm{Isot}(A^A)$, then $(\kappa \circ \widetilde{\Psi})(f) = f^{\approx}$ if and only if there exists $X \in L^A$ such that $(X, f^{\approx})$ is a fixed point of $(\kappa, \widehat{\Psi})$.*
2. *Let $\mu \in \mathrm{IsotTot}(L^{A\times A})$, then $(\widehat{\mathrm{c}} \circ \widehat{\Psi})(\mu) = \mu^1$ if and only if there exists $Y \in L^A$ such that (Y, μ^1) is a fixed point of $(\widehat{\mathrm{c}}, \widetilde{\Psi})$.*

Proof. Let $f \in \mathrm{Isot}(A^A)$ be such that $\kappa(\widetilde{\Psi}(f)) = f^{\approx}$. By the compositions proved in Proposition 23, we have that $(\kappa \circ \widehat{\Psi})(f^{\approx}) = \kappa(\widetilde{\Psi}(f)) = f^{\approx}$. Thus, $(\widehat{\Psi}(f^{\approx}), f^{\approx})$ is a fixed point of $(\kappa, \widehat{\Psi})$.

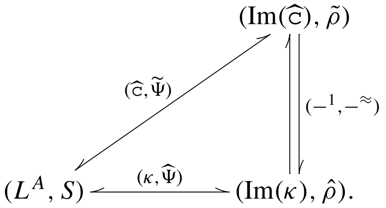
Conversely, let $(X, f^{\approx})$ be a fixed point of $(\kappa, \widehat{\Psi})$. Then, by Proposition 23, we have that $\kappa(\widetilde{\Psi}(f)) = \kappa(\widehat{\Psi}(f^{\approx})) = f^{\approx}$.

Similarly, let $\mu \in \mathrm{IsotTot}(L^{A\times A})$ such that $\widehat{\mathrm{c}}(\widehat{\Psi}(\mu)) = \mu^1$. Then we have, using the compositions in Proposition 23, that $\widehat{\mathrm{c}}(\widetilde{\Psi}(\mu^1)) = \widehat{\mathrm{c}}(\widehat{\Psi}(\mu)) = \mu^1$. Therefore, $(\widetilde{\Psi}(\mu^1), \mu^1)$ is a fixed point of $(\widehat{\mathrm{c}}, \widetilde{\Psi})$.

Conversely, let (Y, μ^1) be a fixed point of $(\widehat{\mathrm{c}}, \widetilde{\Psi})$. Then, by Proposition 23, we have that $\widehat{\mathrm{c}}(\widehat{\Psi}(\mu)) = \widehat{\mathrm{c}}(\widetilde{\Psi}(\mu^1)) = \mu^1$. Hence, $(\widehat{\mathrm{c}} \circ \widehat{\Psi})(\mu) = \mu^1$. □

This hints the following result.

Theorem 25. *The following diagram is commutative*



Proof. First of all, we have to prove that if f is an isotone mapping such that $f \in \mathrm{Im}(\widehat{\mathrm{c}})$, then $f^{\approx}$ is an isotone and total fuzzy relation such that $f^{\approx} \in \mathrm{Im}(\kappa)$. Since $f \in \mathrm{Im}(\widehat{\mathrm{c}})$, there is a fuzzy set $X \in L^A$ such that $\widehat{\mathrm{c}}(X) = f$. Hence, by Proposition 23, $f^{\approx} = (\widehat{\mathrm{c}}(X))^{\approx} = \kappa(X)$. Therefore, $f^{\approx} \in \mathrm{Im}(\kappa)$ and $-^{\approx}$ is well-defined. As a consequence, and by $(\kappa, \widehat{\Psi})$ being a Galois connection, we have that $(\widehat{\Psi}(f^{\approx}), f^{\approx})$ is a fixed point of $(\kappa, \widehat{\Psi})$, and by Corollary 24, we get $(\kappa \circ \widetilde{\Psi})(f) = f^{\approx}$.

The converse can be proved similarly. □

8. Conclusions and further work

This paper continues the line of work which initiated in [21], where fuzzy closure systems were introduced in the framework of complete fuzzy lattices. The mappings that take fuzzy closure systems to fuzzy closure operators and vice versa are studied in a more general setting and are proved to form a fuzzy Galois connection. Similarly, the mappings used in [20] that relate fuzzy closure systems to strong fuzzy closure relations form a fuzzy Galois connection between the lattice of fuzzy sets (L^A, S) and the set of total isotone fuzzy relations with the relation $\hat{\rho}$. In order to finish this study, we consider the 1-cut and the extensional hull as mappings from the isotone mappings in A to the total isotone fuzzy relations in A with their preorder relations. These mappings form a fuzzy adjuntion. In addition, fuzzy closure systems, fuzzy closure operators and strong fuzzy closure relations are *formal concepts* of the fuzzy Galois connections studied in the paper.

As a prospect of future work, since we know the images of the mappings introduced in [21] are not closure operators and fuzzy closure systems in general, this analysis can be continued and study the nature of its fixed points and examine whether they are interesting for solving some problems as some sort of pre-closure structures.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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