J. Math. Anal. Appl. 526 (2023) 127287

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

journal homepage: www.elsevier.com/locate/jmaa

Regular Articles

Cesàro-type operators associated with Borel measures on the unit disc acting on some Hilbert spaces of analytic functions ☆

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A R T I C L E I N F O

Article history:
Received 23 September 2022
Available online 3 April 2023
Submitted by D. Khavinson

Keywords:
Hardy spaces
Weighted Bergman spaces
Mean Lipschitz spaces
Cesàro-type operators
Hilbert-Schmidt operators

A B S T R A C T

Given a complex Borel measure μ on the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, we consider the Cesàro-type operator $\mathcal{C}_\mu$ defined on the space $\mathrm{Hol}(\mathbb{D})$ of all analytic functions in $\mathbb{D}$ as follows:
If $f \in \mathrm{Hol}(\mathbb{D})$, $f(z) = \sum_{n=0}^{\infty} a_n z^n$ $(z \in \mathbb{D})$, then $\mathcal{C}_\mu(f)(z) = \sum_{n=0}^{\infty} \mu_n \left(\sum_{k=0}^{n} a_k\right) z^n$, $(z \in \mathbb{D})$, where, for $n \geq 0$, μ_n denotes the n-th moment of the measure μ, that is, $\mu_n = \int_{\mathbb{D}} w^n \, d\mu(w)$.
We study the action of the operators C_μ on some Hilbert spaces of analytic function in $\mathbb{D}$, namely, the Hardy space H^2 and the weighted Bergman spaces A^2_α $(\alpha > -1)$. Among other results, we prove that, if we set $F_\mu(z) = \sum_{n=0}^{\infty} \mu_n z^n$ $(z \in \mathbb{D})$, then $\mathcal{C}_\mu$ is bounded on H^2 or on A^2_α if and only if F_μ belongs to the mean Lipschitz space $\Lambda^2_{1/2}$. We prove also that $\mathcal{C}_\mu$ is a Hilbert-Schmidt operator on H^2 if and only if F_μ belongs to the Dirichlet space $\mathcal{D}$, and that $\mathcal{C}_\mu$ is a Hilbert-Schmidt operator on A^2_α if and only if F_μ belongs to the Dirichlet-type space $\mathcal{D}^2_{-1-\alpha}$.

1. Introduction and main results

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ denote the open unit disc in the complex plane $\mathbb{C}$ and let $\mathrm{Hol}(\mathbb{D})$ be the space of all analytic functions in $\mathbb{D}$. Also, dA will denote the area measure on $\mathbb{D}$, normalized so that the area of $\mathbb{D}$ is 1. Thus $dA(z) = \frac{1}{\pi}\, dx\, dy = \frac{1}{\pi}\, r\, dr\, d\theta$.

☆ This research is supported in part by a grant from "El Ministerio de Economía y Competitividad" Spain (PGC2018-096166-B-I00) and by grants from la Junta de Andalucía (FQM-210 and UMA18-FEDERJA-002).
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https://doi.org/10.1016/j.jmaa.2023.127287
0022-247X/

For $0 \leq r < 1$ and f analytic in $\mathbb{D}$ we set

$$M_p(r,f) = \left(\frac{1}{2\pi} \int_0^{2\pi} \left| f(re^{i\theta}) \right|^p d\theta \right)^{1/p}, \quad 0 < p < \infty,$$
$$M_\infty(r,f) = \max_{|z|=r} |g(z)|.$$

For $0 < p \leq \infty$ the Hardy space H^p consists of those functions f, analytic in $\mathbb{D}$, for which

$$||f||_{H^p} \stackrel{\text{def}}{=} \sup_{0<r<1} M_p(r,f) < \infty.$$

We refer to [8] for the theory of Hardy spaces.

For $0 < p < \infty$ and $\alpha > -1$ the weighted Bergman space A^p_α consists of those $f \in \mathrm{Hol}(\mathbb{D})$ such that

$$\|f\|_{A^p_\alpha} \stackrel{\text{def}}{=} \left((\alpha+1) \int_{\mathbb{D}} (1-|z|^2)^\alpha |f(z)|^p \, dA(z) \right)^{1/p} < \infty.$$

The unweighted Bergman space A^p_0 is simply denoted by A^p. We refer to [9,18,27] for the notation and results about Bergman spaces.

The space $BMOA$ consists of those functions $f \in H^1$ whose boundary values have bounded mean oscillation on $\partial\mathbb{D}$. The Bloch space $\mathcal{B}$ is the space of those $f \in \mathrm{Hol}(\mathbb{D})$ such that

$$\|f\|_{\mathcal{B}} \stackrel{\text{def}}{=} |f(0)| + \sup_{z\in\mathbb{D}} (1-|z|^2)|f'(z)| < \infty.$$

We mention [12] and [2] for the theory these spaces.

Given $1 \leq p \leq \infty$ and $0 < \alpha \leq 1$, the mean Lipschitz space Λ^p_α consists of those functions f analytic in $\mathbb{D}$ having a non-tangential limit almost everywhere for which $\omega_p(\delta, f) = O(\delta^\alpha)$, as $\delta \to 0$. Here, $\omega_p(\cdot, f)$ denotes the modulus of continuity of order p of the boundary values $f(e^{i\theta})$ of f. We write Λ_α instead of Λ^∞_α. This is the usual Lipschitz space of order α.

A classical result of Hardy and Littlewood [16] (see also Chapter 5 of [8]) asserts that for $1 \leq p \leq \infty$ and $0 < \alpha \leq 1$, we have that $\Lambda^p_\alpha \subset H^p$ and

$$\Lambda^p_\alpha = \{f \text{ analytic in } \mathbb{D}\text{: } M_p(r,f') = \mathrm{O}\left(\frac{1}{(1-r)^{1-\alpha}}\right), \quad \text{as } r \to 1\}.$$

Of special interest are the spaces $\Lambda^p_{1/p}$ since they lie in the border of continuity. If $1 < p < \infty$ and $1/p < \alpha \leq 1$, then Λ^p_α is contained in the disc algebra. On the other hand, the function f given by $f(z) = \log \frac{1}{1-z}$ $(z \in \mathbb{D})$ is an unbounded function which lies in $\Lambda^p_{1/p}$ for any $p \in (1, \infty)$. We have [5,4]

$$\Lambda^p_{1/p} \subset BMOA, \quad 1 < p < \infty.$$

The space of those $f \in \mathrm{Hol}(\mathbb{D})$ such that

$$M_p(r,f') = \mathrm{o}\left(\frac{1}{(1-r)^{1-\alpha}}\right), \quad \text{as } r \to 1,$$

is denoted by λ^p_α.

The Cesàro operator $\mathcal{C}$ is defined over the space of all complex sequences as follows: If $(a) = \{a_k\}_{k=0}^{\infty}$ is a sequence of complex numbers then

$$\mathcal{C}\left((a)\right) = \left\{ \frac{1}{n+1} \sum_{k=0}^{n} a_k \right\}_{n=0}^{\infty}.$$

The operator $\mathcal{C}$ is known to be bounded from ℓ^p to ℓ^p for $1 < p \le \infty$. This was proved by Hardy [14] and Landau [20] (see also [17, Theorem 326, p. 239]).

Identifying any given function $f \in \mathrm{Hol}(\mathbb{D})$ with the sequence $\{a_k\}_{k=0}^{\infty}$ of its Taylor coefficients, the Cesàro operator $\mathcal{C}$ becomes a linear operator from $\mathrm{Hol}(\mathbb{D})$ into itself as follows:

If $f \in \mathrm{Hol}(\mathbb{D})$, $f(z) = \sum_{k=0}^{\infty} a_k z^k$ $(z \in \mathbb{D})$, then

$$\mathcal{C}(f)(z) = \sum_{n=0}^{\infty} \left(\frac{1}{n+1} \sum_{k=0}^{n} a_k \right) z^n, \quad z \in \mathbb{D}.$$

The Cesàro operator is bounded on H^p for $0 < p < \infty$. For $1 < p < \infty$, this follows from a result of Hardy on Fourier series [15] together with the M. Riesz's theorem on the conjugate function [8, Theorem 4.1]. Siskakis [23] used semigroups of composition operators to give an alternative proof of this result and to extend it to $p = 1$. A direct proof of the boundedness on H^1 was given by Siskakis in [24]. Miao [22] dealt with the case $0 < p < 1$. Stempak [25] gave a proof valid for $0 < p \le 2$ and Andersen [1] provided another proof valid for all $p < \infty$.

Blasco [3] has recently obtained a number of interesting new results on the Cesàro operator acting on Hardy spaces and on some other related spaces such as $BMOA$, the Bloch space, and the spaces $\Lambda^p_{1/p}$ $(1 < p < \infty)$.

Recently, the authors have considered in [11] a natural generalization of the Cesàro operator acting on spaces of analytic functions in $\mathbb{D}$. For a positive and finite Borel measure μ on the radius $[0, 1)$ the operator $\mathcal{C}_\mu$ is defined on the space $\mathrm{Hol}(\mathbb{D})$ as follows:

If $f \in \mathrm{Hol}(\mathbb{D})$, $f(z) = \sum_{n=0}^{\infty} a_n z^n$ $(z \in \mathbb{D})$, $\mathcal{C}_\mu(f)$ is defined by

$$\mathcal{C}_\mu(f)(z) = \sum_{n=0}^{\infty} \mu_n \left(\sum_{k=0}^{n} a_k \right) z^n = \int_{[0,1)} \frac{f(tz)}{1 - tz} \, d\mu(t), \quad z \in \mathbb{D},$$

where, for $n = 0, 1, 2, \ldots$, μ_n denotes the n-th moment of μ, $\mu_n = \int_{[0,1)} t^n \, d\mu(t)$. When μ is the Lebesgue measure on $[0, 1)$, the operator $\mathcal{C}_\mu$ reduces to the classical Cesàro operator $\mathcal{C}$. Among other results, it is proved in [11] that the following conditions are equivalent:

(i) μ is a Carleson measure, that is, $\mu(t) \le C(1 - t)$ $(0 < t < 1)$.
(ii) $\mu_n = \mathrm{O}\left(\frac{1}{n}\right)$.
(iii) $1 \le p < \infty$ and $\mathcal{C}_\mu$ is bounded from H^p into itself.
(iv) $1 < p < \infty$, $\alpha > -1$, and $\mathcal{C}_\mu$ is bounded from A^p_α into itself.

Blasco [3] has generalized the definition of the operators $\mathcal{C}_\mu$ by dealing with complex Borel measures on $[0, 1)$ and he has extended results of [11] to this more general setting.

In this paper we shall deal with complex Borel measures on $\mathbb{D}$, not necessarily supported on $[0, 1)$. Just as above, if μ is a complex Borel measure on $\mathbb{D}$ and $n \ge 0$, we set

$$\mu_n = \int_{\mathbb{D}} w^n \, d\mu(w)$$

and we define the operator $\mathcal{C}_\mu : \mathrm{Hol}(\mathbb{D}) \to \mathrm{Hol}(\mathbb{D})$ as follows:

If $f \in \mathrm{Hol}(\mathbb{D})$, $f(z) = \sum_{n=0}^{\infty} a_n z^n$ $(z \in \mathbb{D})$, $\mathcal{C}_\mu(f)$ is defined by

$$\mathcal{C}_\mu(f)(z) = \sum_{n=0}^{\infty} \mu_n \left(\sum_{k=0}^{n} a_k \right) z^n = \int_{\mathbb{D}} \frac{f(wz)}{1-wz}\, d\mu(w), \quad z \in \mathbb{D}.$$

It is natural to look for a characterization of those complex Borel measures μ on $\mathbb{D}$ for which the operator $\mathcal{C}_\mu$ is bounded on the Hardy space H^p or on the weighted Bergman space A^p_α. In this paper we solve this question in the case $p = 2$, that is, in the case when we are dealing with Hilbert spaces. Our main results are included in the following theorem.

Theorem 1. *Suppose that $\alpha > -1$ and let μ be a complex Borel measure on $\mathbb{D}$. Set*

$$\mu_n = \int_{\mathbb{D}} w^n \, d\mu(w), \quad n \geq 0,$$

and

$$F_\mu(z) = \sum_{n=0}^{\infty} \mu_n z^n, \quad z \in \mathbb{D}.$$

The following conditions are equivalent:

(i) *The operator $\mathcal{C}_\mu$ is bounded from A^2_α into itself.*
(ii) *The operator $\mathcal{C}_\mu$ is bounded from H^2 into itself.*
(iii) $F_\mu \in \Lambda^2_{1/2}$.

In Section 3 we characterize the measures μ for which $\mathcal{C}_\mu$ is a compact operator from H^2 into itself and, also, those for which $\mathcal{C}_\mu$ is Hilbert-Schmidt on H^2 and on the Bergman space A^2_α.

2. Proofs and some further results

Before embarking into the proofs of our results, let us remark that if μ is a finite positive Borel measure on $[0, 1)$ then the sequence $\{\mu_n\}$ is a decreasing sequence of non-negative numbers and then it is known that $F_\mu \in \Lambda^2_{1/2}$ if and only if $\mu_n = \mathrm{O}\left(\frac{1}{n}\right)$ (see, e.g. [13, Lemma 3.1] or [21, Lemma 2]). Hence our results here are consistent with those in [11].

Let us start with the results involving the Bergman spaces A^2_α. The implication (iii) ⇒ (i) in Theorem 1 is a particular case of the following result.

Proposition 2. *Suppose that $\alpha > -1$ and $1 < p < \infty$. If $F_\mu \in \Lambda^p_{1/p}$ then $\mathcal{C}_\mu$ is bounded from A^p_α into itself.*

Before we get into the proof, let us recall that if f and g are two analytic functions in the unit disc,

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad g(z) = \sum_{n=0}^{\infty} b_n z^n, \quad z \in \mathbb{D},$$

the convolution $f \star g$ of f and g is defined by

$$f \star g(z) = \sum_{n=0}^{\infty} a_n b_n z^n, \quad z \in \mathbb{D}.$$

Proof of Proposition 2. Suppose $F_\mu \in \Lambda^p_{1/p}$.

Arguing as in [11, p. 21-22] we see that Theorem 4 of [10] implies that

$$F_\mu \text{ is a coefficient multiplier from } A_\alpha^{p/(p+1)} \text{ into } A^p_\alpha. \tag{2.1}$$

Take $f \in A^p_\alpha$. We have to show that $\mathcal{C}_\mu(f) \in A^p_\alpha$.

Let g be defined by

$$g(z) = \frac{f(z)}{1-z}, \quad z \in \mathbb{D}.$$

Just as in [11, p. 6], we have that $\mathcal{C}_\mu(f)$ is the convolution of F_μ and g,

$$\mathcal{C}_\mu(f) = F_\mu \star g.$$

Since $1/(1-z) \in A^1_\alpha$, using Theorem C of [26] (see also Theorem C of [11]), we see that $g \in A_\alpha^{p/(p+1)}$. Then (2.1) yields $\mathcal{C}_\mu(f) = F_\mu \star g \in A^p_\alpha$. $\square$

Proof of the implication (i) ⇒ (iii). Suppose $\mathcal{C}_\mu$ is a bounded operator from A^2_α into itself.

For $0 < b < 1$, set

$$f_b(z) = \frac{(1-b)^{1/2}}{(1-bz)^{1+\frac{\alpha+1}{2}}} = \sum_{k=0}^\infty a_{k,b} z^k, \quad z \in \mathbb{D}.$$

Using [27, Lemma 3.10], we see that $f_b \in A^2_\alpha$ and

$$\|f_b\|^2_{A^2_\alpha} \asymp 1.$$

Then we have that

$$1 \gtrsim \|\mathcal{C}_\mu(f_b)\|^2_{A^2_\alpha}. \tag{2.2}$$

Also,

$$a_{k,b} \asymp (1-b)^{1/2} k^{(\alpha+1)/2} b^k. \tag{2.3}$$

For every $N \in \mathbb{N}$, we have

$$C_\mu(f_b)(z) = \sum_{n=0}^\infty \mu_n \left(\sum_{k=0}^n a_{k,b}\right) z^n, \quad z \in \mathbb{D}.$$

Then, using (2.2) and (2.3), we obtain

$$\begin{aligned} 1 &\gtrsim \|\mathcal{C}_\mu(f_b)\|^2_{A^2_\alpha} \\ &\asymp \sum_{n=0}^\infty \frac{1}{(n+1)^{\alpha+1}} |\mu_n|^2 \left|\sum_{k=0}^n a_{k,b}\right|^2 \\ &\asymp (1-b) \sum_{n=0}^\infty \frac{|\mu_n|^2}{(n+1)^{\alpha+1}} \left(\sum_{k=0}^n k^{(\alpha+1)/2} b^k\right)^2 \end{aligned}$$

$$\geq (1-b)\sum_{n=0}^{N}\frac{|\mu_n|^2}{(n+1)^{\alpha+1}}\left(\sum_{k=0}^{n}k^{(\alpha+1)/2}b^k\right)^2.$$

Taking $b = 1 - \frac{1}{N}$, we obtain

$$1 \gtrsim \frac{1}{N}\sum_{n=0}^{N}\frac{|\mu_n|^2}{(n+1)^{\alpha+1}}\left(n^{\frac{\alpha+1}{2}+1}\right)^2 \asymp \frac{1}{N}\sum_{n=0}^{N}n^2|\mu_n|^2.$$

Consequently, we have that $\sum_{n=0}^{N} n^2|\mu_n|^2 = \mathrm{O}(N)$. Now a standard argument using summation by parts shows that this is equivalent to saying that $F_\mu \in \Lambda^2_{1/2}$. □

Let us turn now to prove our results regarding the Hardy space H^2.

Proof of the implication (iii) ⇒ (ii). Suppose that $F_\mu \in \Lambda^2_{1/2}$. It is well known (see e.g. [4, Theorem 3.1]) that this is equivalent to

$$\sum_{k=2^n-1}^{2^{n+1}-2}(k+1)^2|\mu_k|^2 = \mathrm{O}(2^n). \tag{2.4}$$

Take $f \in H^2$, $f(z) = \sum_{n=0}^{\infty} a_n z^n$ $(z \in \mathbb{D})$. Set

$$f_1(z) = \sum_{n=0}^{\infty}|a_n|z^n, \quad z \in \mathbb{D}.$$

We have that $f_1 \in H^2$ and $\|f_1\|_{H^2} = \|f\|_{H^2}$.

Now,

$$\begin{aligned}
\|\mathcal{C}_\mu(f)\|^2_{H^2} &\leq \sum_{k=0}^{\infty}|\mu_k|^2\left(\sum_{j=0}^{k}|a_j|\right)^2 \\
&= \sum_{k=0}^{\infty}|\mu_k|^2\left(\sum_{j=0}^{k}\frac{|a_j|}{j+k+1}(j+k+1)\right)^2 \\
&\leq \sum_{k=0}^{\infty}(2k+1)^2|\mu_k|^2\left(\sum_{j=0}^{k}\frac{|a_j|}{j+k+1}\right)^2 \\
&\leq 4\sum_{k=0}^{\infty}(k+1)^2|\mu_k|^2\left(\sum_{j=0}^{\infty}\frac{|a_j|}{j+k+1}\right)^2 \\
&\leq 4\sum_{n=0}^{\infty}\left(\sum_{k=2^n-1}^{2^{n+1}-2}(k+1)^2|\mu_k|^2\left(\sum_{j=0}^{\infty}\frac{|a_j|}{j+2^n}\right)^2\right) \\
&\leq 4\sum_{n=0}^{\infty}\left(\sum_{j=0}^{\infty}\frac{|a_j|}{j+2^n}\right)^2\left(\sum_{k=2^n-1}^{2^{n+1}-2}(k+1)^2|\mu_k|^2\right).
\end{aligned}$$

Using (2.4) we obtain

$$\|\mathcal{C}_\mu(f)\|_{H^2}^2 \lesssim \sum_{n=0}^{\infty} 2^n \left(\sum_{j=0}^{\infty} \frac{|a_j|}{j+2^n} \right)^2. \tag{2.5}$$

Now,

$$\begin{aligned} \sum_{n=0}^{\infty} 2^n \left(\sum_{j=0}^{\infty} \frac{|a_j|}{j+2^n} \right)^2 &\lesssim \sum_{n=0}^{\infty} \sum_{k=2^n-1}^{2^{n+1}-2} \left(\sum_{j=0}^{\infty} \frac{|a_j|}{k+j+1} \right)^2 \\ &= \sum_{n=0}^{\infty} \left(\sum_{j=0}^{\infty} \frac{|a_j|}{n+j+1} \right)^2. \end{aligned} \tag{2.6}$$

Recall now that the Hilbert operator $\mathcal{H}$ is formally defined on the space $\mathrm{Hol}(\mathbb{D})$ as follows: If $\varphi \in \mathrm{Hol}(\mathbb{D})$, $\varphi(z) = \sum_{n=0}^{\infty} \alpha_n z^n$ $(z \in \mathbb{D})$, then

$$\mathcal{H}(f)(z) = \sum_{n=0}^{\infty} \left(\sum_{j=0}^{\infty} \frac{\alpha_j}{n+j+1} \right) z^n,$$

whenever the right hand side makes sense and defines and analytic function in $\mathbb{D}$.

Hardy's inequality [8, p. 48] guarantees that the operator $\mathcal{H}$ is well defined in H^1 and Hilbert's inequality (see also [8, p. 48]) implies that $\mathcal{H}$ is bounded from H^2 into itself and that

$$\|\mathcal{H}(\varphi)\|_{H^2} \le \pi \|\varphi\|_{H^2}, \quad \varphi \in H^2. \tag{2.7}$$

Actually, the Hilbert operator is bounded from H^p into itself for all $p \in (1, \infty)$ [6] and the norm of $\mathcal{H}$ as an operator from H^p into itself was computed in [7].

Notice that

$$\mathcal{H}(f_1)(z) = \sum_{n=0}^{\infty} \left(\sum_{j=0}^{\infty} \frac{|a_j|}{n+j+1} \right) z^n, \quad z \in \mathbb{D}.$$

Hence,

$$\sum_{n=0}^{\infty} \left(\sum_{j=0}^{\infty} \frac{|a_j|}{n+j+1} \right)^2 = \|\mathcal{H}(f_1)\|_{H^2}^2.$$

Using this, (2.7), (2.6), and (2.5), we see that

$$\|\mathcal{C}_\mu(f)\|_{H^2}^2 \lesssim \|f_1\|_{H^2}^2 = \|f\|_{H^2}^2. \quad \square$$

Proof of the implication (ii) ⇒ (iii). Suppose that $\mathcal{C}_\mu$ is a bounded operator on H^2. For $0 < a < 1$, set

$$f_a(z) = \frac{(1-a^2)^{1/2}}{1-az} = (1-a^2)^{1/2} \sum_{n=0}^{\infty} a^n z^n, \quad z \in \mathbb{D}.$$

We have that, for all $a \in (0,1)$, $f_a \in H^2$ and $\|f_a\|_{H^2} = 1$. Consequently, there exists $A > 0$ such that

$$\|\mathcal{C}_\mu(f_a)\|_{H^2}^2 \le A, \quad 0 < a < 1. \tag{2.8}$$

Since

$$C_\mu(f_a)(z) \;=\; (1-a^2)^{1/2}\sum_{n=0}^{\infty}\mu_n\left(\sum_{k=0}^{n}a^k\right)z^n, \quad z\in\mathbb{D},$$

(2.8) implies that, for $N\in\mathbb{N}$,

$$A \geq \|\mathcal{C}_\mu(f_a)\|_{H^2}^2 = (1-a^2)\sum_{n=0}^{\infty}|\mu_n|^2\left(\sum_{k=0}^{n}a^k\right)^2 \geq (1-a)\sum_{n=0}^{N}|\mu_n|^2\left(\sum_{k=0}^{n}a^k\right)^2.$$

Taking $a=1-\frac{1}{N}$, we obtain

$$\frac{1}{N}\sum_{n=0}^{N}n^2|\mu_n|^2 = \mathrm{O}(1)$$

or, equivalently,

$$\sum_{n=0}^{N}n^2|\mu_n|^2 = \mathrm{O}(N).$$

As mentioned above, this is equivalent to saying that $F_\mu\in\Lambda^2_{1/2}$. $\square$

It is possible to give a direct proof of the implication (ii) ⇒ (i). Indeed, this implication follows trivially from Proposition 3.

Proposition 3. *Let μ be a complex Borel measure on $\mathbb{D}$ and suppose that $\mathcal{C}_\mu$ is a bounded operator from H^2 into itself. Then there exists $C>0$ such that*

$$M_2\left(r,\mathcal{C}_\mu(f)\right) \;\leq\; CM_2(r,f), \quad 0<r<1,$$

for every $f\in\mathrm{Hol}(\mathbb{D})$.

Proof. Say that $\|\mathcal{C}_\mu(g)\|_{H^2}\leq C\|g\|_{H^2}$, for all $g\in H^2$.

Take $f\in\mathrm{Hol}(\mathbb{D})$, $f(z)=\sum_{n=0}^{\infty}a_nz^n$ $(z\in\mathbb{D})$. Set

$$\varphi(z)=\sum_{n=0}^{\infty}|a_n|z^n, \quad z\in\mathbb{D},$$

and, for $0<r<1$,

$$\varphi_r(z)=\varphi(rz)=\sum_{n=0}^{\infty}|a_n|r^nz^n, \quad z\in\mathbb{D}.$$

We have

$$M_2\left(r,\mathcal{C}_\mu(f)\right)^2 \leq M_2\left(r,\mathcal{C}_\mu(\varphi)\right)^2 = \sum_{n=0}^{\infty}|\mu_n|^2\left(\sum_{k=0}^{n}|a_k|\right)^2 r^{2n}$$

$$\leq \sum_{n=0}^{\infty} |\mu_n|^2 \left(\sum_{k=0}^{n} |a_k| r^k \right)^2 = \|\mathcal{C}_\mu(\varphi_r)\|_{H^2}^2$$
$$\leq C^2 \|\varphi_r\|_{H^2}^2 = C^2 M_2(r,f)^2. \quad \square$$

3. Compactness

Theorem 4. *Let μ be a complex Borel measure on $\mathbb{D}$ and $\alpha > -1$. Set*

$$\mu_n = \int_{\mathbb{D}} w^n \, d\mu(w), \quad n \geq 0,$$

and

$$F_\mu(z) = \sum_{n=0}^{\infty} \mu_n z^n, \quad z \in \mathbb{D}.$$

The following conditions are equivalent:

(i) $F_\mu \in \lambda^2_{1/2}$.
(ii) *The operator $\mathcal{C}_\mu$ is a compact operator from H^2 into itself.*

Proof of the implication (i) ⇒ (ii). Suppose that $F_\mu \in \lambda^2_{1/2}$. Then

$$\sum_{k=2^n-1}^{2^{n+1}-2} (k+1)^2 |\mu_k|^2 = \mathrm{o}(2^n), \quad \text{as } n \to \infty. \tag{3.1}$$

Take a sequence $\{f_m\}_{m=1}^{\infty} \subset H^2$ such that

$$\sup \|f_m\|_{H^2} < \infty \text{ and } \{f_m\} \xrightarrow[m\to\infty]{} 0, \text{ uniformly in compact subsets of } \mathbb{D}.$$

We have to prove that $\|\mathcal{C}_\mu(f_m)\|_{H^2} \xrightarrow[m\to\infty]{} 0$.

Say that

$$f_m(z) = \sum_{j=0}^{\infty} a_j^{(m)} z^j, \quad z \in \mathbb{D}, \quad j = 1, 2, 3, \dots.$$

Set

$$g_m(z) = \sum_{j=0}^{\infty} |a_j^{(m)}| z^j, \quad z \in \mathbb{D}, \quad j = 1, 2, 3, \dots.$$

Since $\|f_m\|_{H^2} = \|g_m\|_{H^2}$ and the Hilbert operator $\mathcal{H}$ is bounded on H^2, there exists $M > 0$ such that

$$\|\mathcal{H}(g_m)\|_{H^2}^2 \leq M, \quad \text{for all } m. \tag{3.2}$$

Take $\varepsilon > 0$. Use (3.1) to pick $N \in \mathbb{N}$ such that

$$\sum_{k=2^n-1}^{2^{n+1}-2}(k+1)^2|\mu_k|^2 \le \frac{\varepsilon}{2M}2^n, \quad \text{for all } n \ge N. \tag{3.3}$$

We have

$$\begin{aligned}\|\mathcal{C}_\mu(f_m)\|_{H^2}^2 &= \sum_{k=0}^{2^N-2}|\mu_k|^2\left(\sum_{j=0}^{k}|a_j^{(m)}|\right)^2 + \sum_{k=2^N-1}^{\infty}|\mu_k|^2\left(\sum_{j=0}^{k}|a_j^{(m)}|\right)^2 \\ &= I + II.\end{aligned}$$

Since $f_m \xrightarrow[m\to\infty]{} 0$, uniformly in compact subsets of $\mathbb{D}$, it follows that $a_j^{(m)} \xrightarrow[m\to\infty]{} 0$ for every j. Then there exists $m_0 \in \mathbb{N}$ such that $I < \frac{\varepsilon}{2}$ for every $m \ge m_0$.

Arguing as in the proof of the implication (iii) ⇒ (ii) of Theorem 1 and using (3.3), we obtain, for all m,

$$\begin{aligned} II &\lesssim \sum_{k=2^N-1}^{\infty}(k+1)^2|\mu_k|^2\left(\sum_{j=0}^{k}\frac{|a_j^{(m)}|}{j+k+1}\right)^2 \\ &\lesssim \sum_{n=N}^{\infty}\sum_{k=2^n-1}^{2^{n+1}-2}(k+1)^2|\mu_k|^2\left(\sum_{j=0}^{\infty}\frac{|a_j^{(m)}|}{j+2^n+1}\right)^2 \\ &\lesssim \frac{\varepsilon}{2M}\sum_{n=N}^{\infty}2^n\left(\sum_{j=0}^{\infty}\frac{|a_j^{(m)}|}{j+2^n+1}\right)^2 \\ &\lesssim \frac{\varepsilon}{2M}\sum_{n=0}^{\infty}2^n\left(\sum_{j=0}^{\infty}\frac{|a_j^{(m)}|}{j+2^n+1}\right)^2 \\ &\lesssim \frac{\varepsilon}{2M}\sum_{k=0}^{\infty}\left(\sum_{j=0}^{\infty}\frac{|a_j^{(m)}|}{j+k+1}\right)^2 \\ &= \frac{\varepsilon}{2M}\|\mathcal{H}(g_m)\|_{H^2}^2.\end{aligned}$$

Using (3.2) we obtain that $II \le \frac{\varepsilon}{2}$ for all m. Consequently, if $m \ge m_0$ then $\|\mathcal{C}_\mu(f_m)\|_{H^2}^2 < \varepsilon$. □

Proof of the implication (ii) ⇒ (i). Suppose that $\mathcal{C}_\mu$ is a compact operator from H^2 into itself. As in the proof of Theorem 1, for $0 < a < 1$, set

$$f_a(z) = \frac{(1-a^2)^{1/2}}{1-az} = (1-a^2)^{1/2}\sum_{n=0}^{\infty}a^n z^n, \quad z \in \mathbb{D}.$$

We have that, for all $a \in (0,1)$, $f_a \in H^2$ and $\|f_a\|_{H^2} = 1$. Also

$$\lim_{a\to 1} f_a(z) = 0, \quad \text{uniformly in compact subsets of } \mathbb{D}.$$

Then it follows that

$$\|\mathcal{C}_\mu(f_a)\|_{H^2}^2 \to 0, \quad \text{as } a \to 1. \tag{3.4}$$

In the course of the proof of the implication (ii) ⇒ (iii) in Theorem 1, we proved that

$$\|\mathcal{C}_\mu(f_a)\|_{H^2}^2 \geq (1-a)\sum_{n=0}^{N}|\mu_n|^2\left(\sum_{k=0}^{n}a^k\right)^2, \quad 0<a<1, \quad N\geq 2.$$

Taking $a = 1-\frac{1}{N}$ and using (3.4), we obtain that $\sum_{n=0}^{N} n^2|\mu_n|^2 = \mathrm{o}(N)$. This is equivalent to saying that $F_\mu \in \lambda^2_{1/2}$. □

It is natural to conjecture that $\mathcal{C}_\mu$ is compact on A^2_α ($\alpha > -1$) if and only if $F_\mu \in \lambda^2_{1/2}$. The fact that if $\mathcal{C}_\mu$ is compact on A^2_α then $F_\mu \in \lambda^2_{1/2}$ can be proved with an argument similar to the one used to prove the corresponding result for H^2. We do not know whether or not the other implication is true. Let us remark that one of the ingredients used to prove this implication in the case of H^2 is the fact that the Hilbert operator is bounded in H^2. This is not true on the spaces A^2_α with $\alpha \geq 0$ [19, p. 243]. In fact, the Hilbert operator is not even defined on these spaces [7].

Finally, we characterize the measures μ for which $\mathcal{C}_\mu$ is a Hilbert-Schmidt operator on the Hilbert spaces we have been working on.

Let us recall that if H is a Hilbert space, a linear operator $T : H \to H$ is said to be a Hilbert-Schmidt operator if $\sum_{i\in I}\|T(e_i)\|^2 < \infty$ for some (equivalently, for all) orthonormal basis $\{e_i\}_{i\in I}$ of H. Let us recall also that if $f \in \mathrm{Hol}(\mathbb{D})$, $f(z) = \sum_{n=0}^{\infty} a_n z^n$ ($z \in \mathbb{D}$), the Dirichlet integral $\mathcal{D}(f)$ of f is defined by

$$\mathcal{D}(f) = \int_{\mathbb{D}} |f'(z)|^2\, dA(z) = \sum_{n=0}^{\infty} n|a_n|^2.$$

The Dirichlet space $\mathcal{D}$ consists of those $f \in \mathrm{Hol}(\mathbb{D})$ whose Dirichlet integral is finite,

$$f \in \mathcal{D}, \text{ if and only if } f \in \mathrm{Hol}(\mathbb{D}) \text{ and } \mathcal{D}(f) < \infty.$$

Theorem 5. *Let μ be a complex Borel measure on $\mathbb{D}$. The following conditions are equivalent.*

(i) $F_\mu \in \mathcal{D}$.
(ii) *The operator $\mathcal{C}_\mu$ is Hilbert-Schmidt on H^2.*

Proof. The set $\{1, z, z^2, \dots\}$ is an orthonormal basis for H^2. We have, for every n,

$$\mathcal{C}_\mu(z^n) = \sum_{k=n}^{\infty} \mu_k z^k,$$

and, hence,

$$\|\mathcal{C}_\mu(z^n)\|_{H^2}^2 = \sum_{k=n}^{\infty}|\mu_k|^2, \quad n \in \mathbb{N}.$$

Then

$$\sum_{n=0}^{\infty}\|\mathcal{C}_\mu(z^n)\|_{H^2}^2 = \sum_{n=0}^{\infty}\sum_{k=n}^{\infty}|\mu_k|^2 = \sum_{n=0}^{\infty} n|\mu_n|^2.$$

Since $\mathcal{D}(F_\mu) = \sum_{n=0}^{\infty} n|\mu_n|^2$ the equivalence (i) ⇔ (ii) follows. □

In order to state the analogue of Theorem 5 for the spaces A^2_α we have to introduce the weighted Dirichlet spaces. For $0 < p < \infty$ and $\beta > -1$, the weighted Dirichlet space $\mathcal{D}^p_\beta$ consists of those $f \in \mathrm{Hol}(\mathbb{D})$ such that $f' \in A^p_\beta$. The space $\mathcal{D}^2_0$ is the Dirichlet space $\mathcal{D}$. A simple computation shows that if $f \in \mathrm{Hol}(\mathbb{D})$, $f(z) = \sum_{n=0}^\infty a_n z^n$ $(z \in \mathbb{D})$, then

$$f \in \mathcal{D}^2_\beta \;\Leftrightarrow\; \sum_{n=1}^\infty n^{1-\beta}|a_n|^2 < \infty. \tag{3.5}$$

Theorem 6. *Let μ be a complex Borel measure on $\mathbb{D}$ and $\alpha > -1$. The following conditions are equivalent.*

(i) *The function F_μ belongs to the space $\mathcal{D}^2_{-1-\alpha}$.*
(ii) $\sum_{n=0}^\infty n^{2+\alpha}|\mu_n|^2 < \infty$.
(ii) *The operator $\mathcal{C}_\mu$ is Hilbert-Schmidt on A^2_α.*

Proof. The equivalence (i) $\Leftrightarrow$ (ii) follows from (3.5).

For $n = 0, 1, 2, \dots$, set

$$A_n(\alpha) = \sqrt{\frac{\Gamma(n+2+\alpha)}{n!\Gamma(2+\alpha)}}$$

and

$$e_n(z) = A_n(\alpha) z^n, \quad z \in \mathbb{D}.$$

Then (see [18, p. 4]) the sequence $\{e_n\}_{n=0}^\infty$ is an orthonormal basis of A^2_α.

For every n, we have

$$\mathcal{C}_\mu(e_n) = A_n(\alpha)\sum_{k=n}^\infty \mu_k z^k,$$

and, hence,

$$\|\mathcal{C}_\mu(e_n)\|^2_{A^2_\alpha} = A_n(\alpha)^2 \sum_{k=n}^\infty |\mu_k|^2.$$

Thus,

$$\sum_{n=0}^\infty \|\mathcal{C}_\mu(e_n)\|^2_{A^2_\alpha} = \sum_{n=0}^\infty |\mu_n|^2 \left(\sum_{k=0}^n A_k(\alpha)^2\right). \tag{3.6}$$

Since $A_k(\alpha)^2 \asymp k^{1+\alpha}$, (3.6) implies that

$$\sum_{n=0}^\infty \|\mathcal{C}_\mu(e_n)\|^2_{A^2_\alpha} \asymp \sum_{n=0}^\infty n^{2+\alpha}|\mu_n|^2$$

and then the equivalence (ii) $\Leftrightarrow$ (iii) follows. □

Acknowledgments

We wish to express our gratitude for the referee's comments.

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