Article

On Focal Borel Probability Measures

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Abstract: The novel concept of focality is introduced for Borel probability measures on compact Hausdorff topological spaces. We characterize focal Borel probability measures as those Borel probability measures that are strictly positive on every nonempty open subset. We also prove the existence of focal Borel probability measures on compact metric spaces. Lastly, we prove that the set of focal (regular) Borel probability measures is convex but not extremal in the set of all (regular) Borel probability measures.

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1. Introduction

The notions of depth and focality appear naturally in the optimal design of transcranial magnetic stimulation (TMS) coils. In such a context, these ideas refer to how deeply an electromagnetic field can be induced to a certain 3-dimensional body. In the excellent work of [1], the electric field penetration was quantified with the half-value depth, $d_{1/2}$, focality with the tangential spread, $S_{1/2}$, defined as the half-value volume ($V_{1/2}$) divided by the half-value depth:

$$S_{1/2} = \frac{V_{1/2}}{d_{1/2}}. \tag{1}$$

Formula (1) was implemented in [2] (Equation (4.1)) as part of a constraint in a single-optimization problem that pretends to minimize the stored energy in the coil:

$$\begin{cases} \min \psi^T L \psi \\ \text{subject to } S_{1/2} = S_{1/2}^0, \end{cases} \tag{2}$$

where $L \in \mathbb{R}^{N \times N}$ is the inductance matrix (symmetric and positive definite), $\psi \in \mathbb{R}^N$, $S_{1/2}$ is the focality, and $S_{1/2}^0$ is the corresponding focality of the coil 0.

The Euclidean metric and the Lebesgue measure are implicitly used in Formula (1). Those are the standard metric and measure employed in physics and engineering because, among other reasons, the Euclidean metric and the Lebesgue measure do not satisfy pathological properties such as vanishing on nonempty open subsets. However, in abstract topology and abstract measure theory, the existence of pathological metrics and measures is quite normal. Despite this, abstract measure theory has many applications not only in other areas of mathematics, but also in different disciplines such as physics or bioengineering. This manuscript takes the concept of focality as the motivating basis to add it to a more general and abstract scope. We introduce the novel concepts of focal continuous real-valued mappings and focal (regular) Borel probability measures, unveiling their geometric and

topological properties. The novelty of this approach consists in the relationship with focal continuous real-valued functions and in establishing connections to regular Borel measures with finite variation. Among other results, we prove the existence of focal Borel probability measures on compact metric spaces (Theorem 3). We also demonstrate that the set of focal (regular) Borel probability measures is a convex but not extremal subset of the set of (regular) Borel probability measures (Theorem 5). In this way, we give an example that provides interesting information about the geometry of the unit ball of the dual of the space of real-valued continuous functions on K, $\mathscr{C}(K)$, where K is a compact Hausdorff topological space.

2. Preliminaries

If X is a topological space, then $\mathcal{B}(X)$, or simply $\mathcal{B}$ if there is no confusion with X, stands for the Borel σ-algebra of X, that is, the smallest σ-algebra of X containing the closed subsets of X. The elements of $\mathcal{B}(X)$ are called the Borel subsets of X. A Borel measure μ on X is a σ-additive measure defined on $\mathcal{B}(X)$ with values in a Hausdorff topological left-module M over a Hausdorff topological ring R, that is, a mapping μ: $\mathcal{B}(X) \to {}_R M$ satisfying that for every pairwise-disjoint sequence $(A_n)_{n \in \mathbb{N}}$ of Borel subsets of X, $\mu(\bigcup_{n \in \mathbb{N}} A_n) = \sum_{n \in \mathbb{N}} \mu(A_n)$. We focus on regular Borel probability measures on compact Hausdorff topological spaces. Explicit examples are given in Appendix B.

If K is a compact Hausdorff topological space, then $\mathscr{C}(K)$ stands for the Banach space of real-valued continuous functions on K. If $f \in \mathscr{C}(K)$, then $\|f\|_\infty := \max |f|(K)$ (see [3] for further reading on the spaces of continuous functions). $\mathcal{P}(K)$ denotes the set of all Borel probability measures on K, that is, countably additive measures μ: $\mathcal{B}(K) \to [0,1]$ such that $\mu(K) = 1$. $\mathcal{P}(K)$ is trivially a convex subset of the vector space $\mathcal{M}(K)$ of real-valued Borel measures on K. According to [4,5] (see also [6] (Chapter 4)), $\mathscr{C}(K)^*$ can be isometrically identified with the Banach space

$$\mathrm{rca}(K) := \{\mu \in \mathcal{M}(K) : \mu \text{ is a regular Borel measure of finite variation}\}, \qquad (3)$$

endowed with the total variation via the action given by

$$\mu(f) := \int_K f \, d\mu. \qquad (4)$$

$\mathcal{P}(K) \cap \mathrm{rca}(K)$ is a convex subset of the unit sphere of $\mathrm{rca}(K)$ (see Appendix A for more details). Indeed, given $\mu_1, \mu_2 \in \mathcal{P}(K) \cap \mathrm{rca}(K)$ and $t \in [0,1]$, we have that $t\mu_1 + (1-t)\mu_2 \in \mathrm{rca}(K)$ because $\mathrm{rca}(K)$ is a linear space, and $t\mu_1 + (1-t)\mu_2 \in \mathcal{P}(K)$ because $t\mu_1(K) + (1-t)\mu_2(K) = 1$ and $t\mu_1(A) + (1-t)\mu_2(A) \in [0,1]$ for all Borel subset $A \in \mathcal{B}(K)$. Moreover, if $\mu \in \mathcal{P}(K) \cap \mathrm{rca}(K)$, then its total variation is $\mu(K) = 1$, so $\mathcal{P}(K) \cap \mathrm{rca}(K)$ is a convex subset of the unit sphere of $\mathrm{rca}(K)$. We refer the reader to [7] for a wider perspective on these concepts.

For a general metric space X, notations $\mathsf{B}_X(x, r)$ and $\mathsf{S}_X(x, r)$ stand for the closed ball of center $x \in X$ and radius $r > 0$ and the sphere of center $x \in X$ and radius $r > 0$, respectively. If X is a normed space, then B_X and S_X stand for the closed unit ball and the unit sphere, respectively.

3. Results

This section is divided into four subsections. In the first, a classical measure theory result on the measure of the union of increasing countable families of measurable subsets is extended to uncountable families. In the second, we define focality for real-valued continuous functions on a compact Hausdorff topological space. The third subsection focuses on the focality of (regular) Borel probability measures. Lastly, the fourth subsection shows that the set of focal (regular) Borel probability measures is a convex but not extremal subset of the set of (regular) Borel probability measures.

3.1. Increasing/Decreasing Families of Measurable Subsets

A classical measure theory result establishes that the measure of the union a countable increasing family of measurable subsets can be computed as the limit of the sequence of the measures of the subsets. This result was transported in [8] to the scope of measures defined on a effect algebra and valued on a topological module over a topological ring. Here, we extend [8] to uncountable families with countable cofinal subsets. However, we first recall [8] and prove it for the sake of completeness.

Theorem 1. *Let (Ω, Σ) be a measurable space. Let M be a Hausdorff topological module over a Hausdorff topological ring R. Let $\mu \colon \Sigma \to M$ be a countably additive measure. If $(A_n)_{n \in \mathbb{N}} \subseteq \Sigma$ is an increasing sequence of measurable subsets of Ω, then $(\mu(A_n))_{n \in \mathbb{N}}$ converges to $\mu(\bigcup_{n \in \mathbb{N}} A_n)$.*

Proof. For every $k \geq 2$, $\sum_{n=2}^{k}(\mu(A_n) - \mu(A_{n-1})) = \mu(A_k) - \mu(A_1)$. Therefore

$$
\begin{aligned}
\mu\left(\bigcup_{n \in \mathbb{N}} A_n\right) &= \mu\left(A_1 \cup \bigcup_{n \geq 2}(A_n \setminus A_{n-1})\right) = \mu(A_1) + \mu\left(\bigcup_{n \geq 2}(A_n \setminus A_{n-1})\right) \\
&= \mu(A_1) + \sum_{n=2}^{\infty}(\mu(A_n) - \mu(A_{n-1})) = \mu(A_1) + \lim_{n \to \infty}(\mu(A_n) - \mu(A_1)) \\
&= \lim_{n \to \infty} \mu(A_n).
\end{aligned}
$$

$\square$

Corollary 1. *Let (Ω, Σ) be a measurable space. Let M be a Hausdorff topological module over a Hausdorff topological ring R. Let $\mu \colon \Sigma \to M$ be a countably additive measure. If $(A_n)_{n \in \mathbb{N}} \subseteq \Sigma$ is a decreasing sequence of measurable subsets of Ω, then $(\mu(A_n))_{n \in \mathbb{N}}$ converges to $\mu(\bigcap_{n \in \mathbb{N}} A_n)$.*

Proof. Through Theorem 1, $(\mu(\Omega \setminus A_n))_{n \in \mathbb{N}}$ converges to

$$
\mu\left(\bigcup_{n \in \mathbb{N}}(\Omega \setminus A_n)\right) = \mu\left(\Omega \setminus \bigcap_{n \in \mathbb{N}} A_n\right) = \mu(\Omega) - \mu\left(\bigcap_{n \in \mathbb{N}} A_n\right).
$$

Lastly, it only suffices to observe that $\mu(\Omega \setminus A_n) = \mu(\Omega) - \mu(A_n)$ for all $n \in \mathbb{N}$. $\square$

If I is a directed set, and $J \subseteq I$ is cofinal (see, for example, [9] (p. 461)), then any decreasing family of sets indexed by I satisfies that $\bigcap_{i \in I} A_i = \bigcap_{j \in J} A_j$. Indeed, it is clear that $\bigcap_{i \in I} A_i \subseteq \bigcap_{j \in J} A_j$ and if $a \in \bigcap_{j \in J} A_j$, then for every $i \in I$ there exists $j \in J$ with $i \leq j$, so $A_j \subseteq A_i$, hence $a \in A_j \subseteq A_i$. Using the notion of cofinal set, we extend Corollary 1 to nets as follows.

Corollary 2. *Let (Ω, Σ) be a measurable space. Let M be a Hausdorff topological module over a Hausdorff topological ring R. Let $\mu \colon \Sigma \to M$ be a countably additive measure. Let I be a nonempty directed set that has a countable cofinal subset $J \subseteq I$. If $(A_i)_{i \in I}$ is a decreasing family of measurable subsets of Ω such that $\bigcap_{i \in I} A_i$ is measurable, then the net $(\mu(A_i))_{i \in I}$ converges to $\mu(\bigcap_{i \in I} A_i)$.*

Proof. Suppose, on the other hand, that $(\mu(A_i))_{i \in I}$ does not converge to $\mu(\bigcap_{i \in I} A_i)$. Then, we can find a neighborhood W of $\mu(\bigcap_{i \in I} A_i)$, such that, for all $i \in I$. there exists $k \in I$ with $k \geq i$, such that $\mu(A_k) \notin W$. Let us write $J = (j_n)_{n \in \mathbb{N}}$. We construct an increasing sequence $(k_n)_{n \in \mathbb{N}}$ on I using induction. For $n = 1$, we choose a $k_1 \in I$, such that $k_1 \geq j_1$ and $\mu(A_{k_1}) \notin W$. Assume that, for some $n \in \mathbb{N}$, we had already defined $k_1, \ldots, k_n$, and take $k_{n+1} \in I$, such that

$$
k_{n+1} \geq j_{n+1}, \quad k_{n+1} \geq k_n, \quad \mu\left(A_{k_{n+1}}\right) \notin W.
$$

Since $j_n \leq k_n$ for all $n \in \mathbb{N}$, and J is cofinal in I, then $(k_n)_{n \in \mathbb{N}}$ is cofinal in I. Therefore, $\bigcap_{i \in I} A_i = \bigcap_{n \in \mathbb{N}} A_{k_n}$ and $(A_{k_n})_{n \in \mathbb{N}}$ is decreasing. Via Corollary 1,

$$\mu\left(\bigcap_{n \in \mathbb{N}} A_{k_n}\right) = \lim_{n \to \infty} \mu(A_{k_n}).$$

However, the previous equality contradicts the fact that $\mu(A_{k_n}) \notin W$ for every $n \in \mathbb{N}$. □

The final corollary of this first subsection displays the version of the previous result for increasing uncountable families with a countable cofinal subset. We spare the reader the details of the proof.

Corollary 3. *Let (Ω, Σ) be a measurable space. Let M be a Hausdorff topological module over a Hausdorff topological ring R. Let $\mu \colon \Sigma \to M$ be a countably additive measure. Let I be a nonempty directed set that has a countable cofinal subset $J \subseteq I$. If $(A_i)_{i \in I}$ is an increasing family of measurable subsets of Ω such that $\bigcup_{i \in I} A_i$ is measurable, then net $(\mu(A_i))_{i \in I}$ converges to $\mu(\bigcup_{i \in I} A_i)$.*

3.2. Focality of Continuous Functions

We begin by defining the notion of focality for continuous real-valued functions with respect to a certain measure. However, we first need to introduce the regions of interest.

Definition 1 (α-Region). *Let K be a compact Hausdorff topological space. If $f \in \mathscr{C}(K)$ and $\alpha \in [0,1]$, then*

$$K_\alpha(f) := \{x \in K : |f(x)| \geq \alpha\|f\|_\infty\} \tag{5}$$

is usually called an α-region.

Obviously, the net of α-regions $(K_\alpha(f))_{\alpha \in [0,1]}$ decreases from $K_0(f) = K$ to $K_1(f) = \{x \in K : |f(x)| = \|f\|_\infty\}$. Clearly, every α-region is closed in K and hence compact. The open α-regions are defined as the topological interior of the α-regions.

Definition 2 (Open α-region). *Let K be a compact Hausdorff topological space. If $f \in \mathscr{C}(K)$ and $\alpha \in [0,1]$, then $O_\alpha(f) := \mathrm{int}(K_\alpha(f))$ is usually called an open α-region.*

Notice that

$$\{x \in K : |f(x)| > \alpha\|f\|_\infty\} \subseteq O_\alpha(f). \tag{6}$$

As a consequence, if $0 \leq \alpha < 1$, then every α-region has a nonempty interior because $\{x \in K : |f(x)| > \alpha\|f\|_\infty\}$ is a nonempty open subset of K contained in $K_\alpha(f)$.

The next result shows that, if μ is a Borel probability measure on K, then $\mu(K_1(f))$ can be obtained as the limit of the net $(\mu(K_\alpha(f)))_{\alpha \in [0,1)}$.

Proposition 1. *Let K be a compact Hausdorff topological space. Let $f \in \mathscr{C}(K)$. Let μ be a Borel probability measure on K. Then, net $(\mu(K_\alpha(f)))_{\alpha \in [0,1)}$ converges to $\mu(K_1(f))$.*

Proof. We apply Corollary 2. In the first place, the interval $[0,1)$ is totally ordered and has a countable cofinal subset $\left\{1 - \frac{1}{n} : n \in \mathbb{N}\right\}$. Next, $(K_\alpha(f))_{\alpha \in [0,1)}$ is a decreasing family of Borel subsets of K, in such a way that $\bigcap_{\alpha \in [0,1)} K_\alpha(f) = K_1(f)$ is a Borel subset of K. In accordance with Corollary 2,

$$\mu(K_1(f)) = \lim_{\alpha \in [0,1)} \mu(K_\alpha(f)).$$

□

In many physics problems [2], α-regions that are of interest are those with a positive measure. This motivates the following definition.

Definition 3 (Focal function). *Let K be a compact Hausdorff topological space. Let μ be a Borel probability measure on K. Function $f \in \mathscr{C}(K)$ is μ-focal if there exists $\alpha \in (0,1)$, such that $\mu(K_\alpha(f)) > 0$.*

Now, focal mappings allow for extending Formula (1) to abstract settings.

Definition 4 (Depth and focality). *Let K be a compact metric space. Let μ be a Borel probability measure on K. Let $f \in \mathscr{C}(K)$ be μ-focal, and take $\alpha \in (0,1)$ such that $\mu(K_\alpha(f)) > 0$; then, we can define the α-depth as*

$$\rho_\alpha := \max\{d(k, K_1(f)) : k \in K_\alpha(f)\} \tag{7}$$

and the α-focality as

$$\phi_\alpha := \frac{\rho_\alpha}{\mu(K_\alpha(f))}. \tag{8}$$

From [10] (Theorem IX.4.3, p. 185), we have the following remark (see also [11] for metrics in linear spaces).

Remark 1. *Let X be a metric space. Let $A \subseteq X$ a nonempty subset of X. Function*

$$\begin{aligned} d(\bullet, A) : \quad X &\rightarrow \quad [0, \infty) \\ x &\mapsto \quad d(x, A) \end{aligned} \tag{9}$$

is nonexpansive.

In general, it is clear that not all real-valued nonexpansive mappings on a metric space have the form described in (9). Nevertheless, distance functions combined with translations allow for us to obtain a wide variety of properties. For example, every nonexpansive real function on a metric space is bounded by a distance function and a constant:

Remark 2. *Let X be a metric space and $x_0 \in X$. Then, every Lipschitz function $f : X \rightarrow \mathbb{R}$ satisfies that $|f(x)| \leq |f(x_0)| + L(f)d(x, x_0)$ for all $x \in X$, where $L(f)$ is the Lipschitz constant of f.*

Furthermore, in connection with the α-regions, we have the following result. If K is a compact metric space, then K is bounded, that is, it has finite diameter $\mathrm{diam}(K) := \sup\{d(k, l) : k, l \in K\} < \infty$.

Proposition 2. *Let K be a nonsingleton compact metric space. Let $u \in K$ and $\alpha \in (0,1)$. Function*

$$\begin{aligned} \psi_u : \quad K &\rightarrow \quad [0, \mathrm{diam}(K)] \\ k &\mapsto \quad \psi_u(k) := \mathrm{diam}(K) - d(k, u) \end{aligned} \tag{10}$$

satisfies the following:

1. *ψ_u is positive and nonexpansive.*
2. *$\|\psi_u\|_\infty = \mathrm{diam}(K)$.*
3. *$K_1(\psi_u) = \{u\}$.*
4. *$K_\alpha(\psi_u) = \mathrm{B}_K(u, (1-\alpha)\mathrm{diam}(K)) = \{k \in K : d(k, u) \leq (1-\alpha)\mathrm{diam}(K)\}$.*

As a consequence, the collection $\{O_\alpha(f) : \alpha \in (0,1), f \in \mathscr{C}(K) \text{ nonexpansive}\}$ of all open α-regions forms a base of open subsets of K.

Proof. We only prove item 4. We spare the reader the details of the rest of the items. We have that

$$K_\alpha(\psi_u) = \{k \in K : \alpha \mathrm{diam}(K) \leq \mathrm{diam}(K) - d(k,u)\} = \mathsf{B}_K(u, (1-\alpha)\mathrm{diam}(K)).$$

Lastly, given $u \in K$ and $r > 0$, taking $\alpha = 1 - \min\{r, \mathrm{diam}(K)\}/(2\mathrm{diam}(K))$, we obtain

$$u \in \{k \in K : \alpha \|\psi_u\|_\infty < |\psi_u(k)|\} \subseteq O_\alpha(\psi_u) \subseteq K_\alpha(\psi_u) \subseteq \mathsf{B}_K\left(u, \tfrac{r}{2}\right) \subseteq \{k \in K : d(k,u) < r\}.$$

Then, every open subset of K is a union of sets $O_\alpha(\psi_u)$. $\square$

3.3. Focality of Measures

There exist Borel probability measures on compact Hausdorff topological spaces that vanish at certain nonempty open subsets. For instance, if K is a nonsingleton compact Hausdorff topological space and $k \in K$, we can consider the regular Borel probability measure

$$
\begin{aligned}
\delta_k : \quad \mathcal{B}(K) \quad &\to \quad \{0,1\} \\
A \quad &\mapsto \quad \delta_k(A) := \begin{cases} 1 & k \in A, \\ 0 & k \notin A. \end{cases}
\end{aligned}
\tag{11}
$$

Since K is Hausdorff and not a singleton, $U := K \setminus \{k\}$ is a nonempty open subset of K satisfying $\delta_k(U) = 0$.

Definition 5 (Focal measure). *Let K be a compact Hausdorff topological space. A Borel probability measure μ on K is focal if every $f \in \mathscr{C}(K)$ is μ-focal. The set of focal Borel probability measures on K are denoted by $\mathcal{P}_{\mathsf{f}}(K)$.*

We characterize focal Borel probability measures as those that do not vanish on nonempty open sets.

Theorem 2. *Let K be a compact Hausdorff topological space. A Borel probability measure μ on K is focal if and only if $\mu(U) > 0$ for every nonempty open subset $U \subseteq K$.*

Proof. If $\mu(U) > 0$ for every nonempty open subset $U \subseteq K$, then μ is clearly focal since every α-region $K_\alpha(f) = \{x \in K : |f(x)| \geq \alpha \|f\|_\infty\}$, for $\alpha \in (0,1)$ and $f \in \mathscr{C}(K)$, contains a nonempty open subset of K, $\{x \in K : |f(x)| > \alpha \|f\|_\infty\}$. Conversely, suppose that μ is focal. Fix an arbitrary nonempty open subset $U \subseteq K$. We show that $\mu(U) > 0$. Take any $u \in U$. Through Urysohn's Lemma, there exists a function $f \in \mathcal{C}(K)$, such that $f(x) = 0$ for all $x \in K \setminus U$ and $f(u) = 1$. Since f is μ-focal, there is $\alpha \in (0,1)$ with $\mu(K_\alpha(f)) > 0$. Clearly, $K_\alpha(f) \subseteq U$, so $\mu(U) \geq \mu(K_\alpha(f)) > 0$. $\square$

The following theorem assures the existence of focal Borel probability measures in compact metric spaces (see (3) to remember the notation $\mathrm{rca}(K)$ and [6] (Chapter 4) for more information). For this, we remind that compact metric spaces are separable (see, for example, [10] (Theorem VIII.7.3 and Theorem XI.4.1)).

Theorem 3. *If K is a compact metric space, then $\mathcal{P}_{\mathsf{f}}(K) \cap \mathrm{rca}(K) \neq \emptyset$.*

Proof. Let $(k_n)_{n \in \mathbb{N}} \subseteq K$ be a dense sequence in K and define $\mu := \sum_{n=1}^{\infty} \frac{\delta_{k_n}}{2^n}$. $\mu \in \mathscr{C}(K)^* = \mathrm{rca}(K)$ because $\mathscr{C}(K)^*$ is a Banach space and $\sum_{n=1}^{\infty} \frac{\delta_{k_n}}{2^n}$ is an absolutely convergent series in $\mathscr{C}(K)^*$ (keep in mind that $\|\delta_{k_n}\| = 1$ for all $n \in \mathbb{N}$). We show that $\mu \in \mathcal{P}_{\mathsf{f}}(K)$. Let U be a nonempty open subset of K. Since $(k_n)_{n \in \mathbb{N}}$ is dense in K, there exists $n_0 \in \mathbb{N}$ such that $k_{n_0} \in U$. Then

$$\mu(U) = \sum_{n=1}^{\infty} \frac{\delta_{k_n}(U)}{2^n} \geq \frac{\delta_{k_{n_0}}(U)}{2^{n_0}} = \frac{1}{2^{n_0}} > 0.$$

Lastly, Theorem 2 ensures that $\mu \in \mathcal{P}_{\mathtt{f}}(K)$. $\square$

In compact metric spaces, in order to check whether a measure is focal, it is only necessary to look at the nonexpansive mappings.

Definition 6 (Weakly focal measure). *Let K be a compact metric space. A Borel probability measure μ on K is weakly focal (w-focal) if every nonexpansive $f \in \mathscr{C}(K)$ is μ-focal. The set of weakly focal Borel probability measures on K are denoted by $\mathcal{P}_{\mathtt{wf}}(K)$.*

We show that w-focal Borel probability measures coincide with focal probability measures.

Theorem 4. *Let K be a compact metric space. A Borel probability measure μ on K is w-focal if and only if μ is focal.*

Proof. By definition, if μ is focal, then it is w-focal. Conversely, suppose that μ is weakly focal. We prove that $\mu(U) > 0$ for every nonempty open subset $U \subseteq K$ and then call on to Theorem 2. Indeed, fix an arbitrary nonempty open subset $U \subseteq K$. We may assume that $U \subsetneqq K$ since $\mu(K) = 1$. Take $f := d(\bullet, K \setminus U)$. Since U is not empty, for every $u \in U$, $f(u) = d(u, K \setminus U) > 0$ since $\{u\}$ is compact and $K \setminus U$ is closed. Therefore, $\|f\|_\infty > 0$. Since f is nonexpansive in view of Remark 1, there exists $\alpha \in (0,1)$ with $\mu(K_\alpha(f)) > 0$. Clearly, $K_\alpha(f) \subseteq U$, so $\mu(U) \geq \mu(K_\alpha(f)) > 0$. $\square$

3.4. Extremal Structure of the Set of Focal Borel Probability Measures

The following result on this manuscript shows that $\mathcal{P}_{\mathtt{f}}(K)$ is a convex subset of $\mathcal{P}(K)$, but it is not *extremal* in $\mathcal{P}(K)$. In the next definition we recall the notion of extremal subset.

Definition 7 (Extremal subset). *A subset E of a subset D of a real vector space Z is extremal in D if E satisfies the extremal condition with respect to D: if $d_1, d_2 \in D$ and there exists $\alpha \in (0,1)$ such that $\alpha d_1 + (1 - \alpha)d_2 \in E$, then $d_1, d_2 \in E$.*

We refer the reader to Appendix A for a further view on extremality theory and the geometry of normed spaces.

Theorem 5. *Let K be a nonsingleton compact Hausdorff topological space. If $\mathcal{P}_{\mathtt{f}}(K) \neq \varnothing$, then $\mathcal{P}_{\mathtt{f}}(K)$ is a convex subset of $\mathcal{P}(K)$ but it is not extremal in $\mathcal{P}(K)$.*

Proof. We show first that $\mathcal{P}_{\mathtt{f}}(K)$ is convex. Indeed, let $\mu_1, \mu_2 \in \mathcal{P}_{\mathtt{f}}(K)$ and $t \in [0,1]$. It is clear that $t\mu_1 + (1 - t)\mu_2$ is a Borel probability measure on K. Even more, if U is a nonempty open subset of K, then $(t\mu_1 + (1 - t)\mu_2)(U) = t\mu_1(U) + (1 - t)\mu_2(U) > 0$. As a consequence, $t\mu_1 + (1 - t)\mu_2 \in \mathcal{P}_{\mathtt{f}}(K)$ and hence $\mathcal{P}_{\mathtt{f}}(K)$ is convex. Let us prove now that $\mathcal{P}_{\mathtt{f}}(K)$ is not extremal in $\mathcal{P}(K)$. Fix any $\mu \in \mathcal{P}_{\mathtt{f}}(K)$. Since K is Hausdorff and has more than one points, there are two nonempty open subsets U, V in K such that $U \cap V = \varnothing$. Since $\mu \in \mathcal{P}_{\mathtt{f}}(K)$, we have that $\mu(U), \mu(V) > 0$, therefore $\mu(U), \mu(V) < 1$ and hence $\mu(K \setminus U), \mu(K \setminus V) > 0$. Consider the conditional probabilities of μ on U and $K \setminus U$, μ_U and $\mu_{K \setminus U}$, respectively, given by

$$
\begin{aligned}
\mu_U : \quad \mathcal{B}(K) \quad &\to \quad [0,1] \\
A \quad &\mapsto \quad \mu_U(A) := \frac{\mu(A \cap U)}{\mu(U)},
\end{aligned}
\tag{12}
$$

and

$$
\begin{aligned}
\mu_{K \setminus U} : \quad \mathcal{B}(K) \quad &\to \quad [0,1] \\
A \quad &\mapsto \quad \mu_{K \setminus U}(A) := \frac{\mu(A \cap (K \setminus U))}{\mu(K \setminus U)}.
\end{aligned}
\tag{13}
$$

Then, $\mu_U, \mu_{K\setminus U} \in \mathcal{P}(K) \setminus \mathcal{P}_{\mathbf{f}}(K)$ because $\mu_{K\setminus U}(U) = 0$ and $\mu_U(V) = 0$. We demonstrate that $\frac{1}{2}\mu_U + \frac{1}{2}\mu_{K\setminus U} \in \mathcal{P}_{\mathbf{f}}(K)$, reaching the conclusion that $\mathcal{P}_{\mathbf{f}}(K)$ is not extremal in $\mathcal{P}(K)$. Indeed, let W be any nonempty open subset of K. We have two options:

- $U \cap W \neq \emptyset$. Then

$$\left(\frac{1}{2}\mu_U + \frac{1}{2}\mu_{K\setminus U} \right)(W) = \frac{1}{2}\mu_U(W) + \frac{1}{2}\mu_{K\setminus U}(W) \geq \frac{1}{2}\mu_U(W) = \frac{1}{2}\frac{\mu(U \cap W)}{\mu(U)} > 0$$

  because $U \cap W$ is a nonempty open subset of K and $\mu \in \mathcal{P}_{\mathbf{f}}(K)$.
- $U \cap W = \emptyset$. In this case, $W \subseteq K \setminus U$, therefore

$$\left(\frac{1}{2}\mu_U + \frac{1}{2}\mu_{K\setminus U} \right)(W) \geq \frac{1}{2}\frac{\mu(W)}{\mu(K \setminus U)} > 0$$

  because W is a nonempty open subset of K and $\mu \in \mathcal{P}_{\mathbf{f}}(K)$.

As a consequence,

$$\frac{1}{2}\mu_U + \frac{1}{2}\mu_{K\setminus U} \in \mathcal{P}_{\mathbf{f}}(K).$$

□

In the upcoming results, we reproduce Theorem 5 for regular measures to adapt it to $\mathrm{rca}(K)$. Given a topological space X, a countably additive measure $\mu \colon \mathcal{B}(X) \to [0, \infty]$ is inner regular provided that every Borel subset B of X is inner regular: $\mu(B) = \sup\{\mu(F) : F \subseteq B, \ F \text{ compact}\}$. μ is also an outer regular if every Borel subset B of X is outer regular: $\mu(B) = \inf\{\mu(U) : U \supseteq B, \ U \text{ open}\}$. Lastly, μ is regular if it is inner and outer regular. If $B \in \mathcal{B}(X)$ and $\mu(B) = 0$, then B is trivially inner μ-regular, and if $\mu(B) = \mu(X)$, then B is trivially outer μ-regular. If X is Hausdorff, and μ is finite and inner regular, then μ is outer regular. Conversely, if X is compact, and μ is finite and outer regular, then μ is inner regular.

Lemma 1. *Let X be a topological space. Let $\mu \colon \mathcal{B}(X) \to [0, \infty]$ be a countably additive measure. Fix $A \in \mathcal{B}(X)$ with $0 < \mu(A) < \infty$. Consider*

$$\begin{aligned} \mu_A \colon \ \mathcal{B}(X) \ &\to \ [0, \infty] \\ B \ &\mapsto \ \mu_A(B) := \tfrac{\mu(A \cap B)}{\mu(A)}. \end{aligned} \tag{14}$$

Then:

1. *If μ is inner regular, then so is μ_A.*
2. *If μ is outer regular and A is closed, then μ_A is outer regular.*
3. *If μ is finite and outer regular, then μ_A is outer regular.*

Proof. Since μ is positive, it is clear that $\mu_A(B) \geq \sup\{\mu_A(F) : F \subseteq B, \ F \text{ compact}\}$ and $\mu_A(B) \leq \inf\{\mu_A(U) : U \supseteq B, \ U \text{ open}\}$ for each Borel subset $B \subseteq X$.

1. Fix an arbitrary $B \in \mathcal{B}(X)$. There exists a sequence $(F_n)_{n \in \mathbb{N}}$ of compact subsets of X, such that $F_n \subseteq B \cap A$ for every $n \in \mathbb{N}$ and $(\mu(F_n))_{n \in \mathbb{N}}$ converges to $\mu(B \cap A)$. Since $F_n = F_n \cap A$ for all $n \in \mathbb{N}$, we conclude that $(\mu_A(F_n))_{n \in \mathbb{N}}$ converges to $\mu_A(B)$. As a consequence, $\mu_A(B) = \sup\{\mu_A(F) : F \subseteq B, \ F \text{ compact}\}$.
2. Fix an arbitrary $B \in \mathcal{B}(X)$. There exists a sequence $(U_n)_{n \in \mathbb{N}}$ of open subsets of X such that $B \cap A \subseteq U_n$ for every $n \in \mathbb{N}$ and $(\mu(U_n))_{n \in \mathbb{N}}$ converges to $\mu(B \cap A)$. For every $n \in \mathbb{N}$, $V_n := U_n \cup (X \setminus A)$ is open and satisfies that $B \subseteq V_n$, $B \cap A \subseteq V_n \cap A = U_n \cap A \subseteq U_n$, and $\mu(B \cap A) \leq \mu(V_n \cap A) = \mu(U_n \cap A) \leq \mu(U_n)$. Therefore, $(\mu(V_n \cap A))_{n \in \mathbb{N}}$ converges to $\mu(B \cap A)$, meaning that $(\mu_A(V_n))_{n \in \mathbb{N}}$ converges to $\mu_A(B)$. As a consequence, $\mu_A(B) = \inf\{\mu_A(U) : U \supseteq B, \ U \text{ open}\}$.

3. Let $B \in \mathcal{B}(X)$ and denote $r = \inf\{\mu(W \cap A) : W \supseteq B, W \text{ open}\}$. We prove that $r \leq \mu(B \cap A)$. Since μ is outer regular, we have

$$\mu(B \cap A) = \inf\{\mu(U) : U \supseteq B \cap A, U \text{ open}\}.$$

Suppose that $\mu(B \cap A) < r$. Then, there exists an open subset U of X with $B \cap A \subseteq U$ such that $\mu(U) < r$. Given an open subset W of X with $B \subseteq W$, since μ is finite, it holds that

$$\mu(U) + \mu(B \cap (X \backslash A)) < r + \mu(B \cap (X \backslash A)) \leq \mu(W \cap A) + \mu(W \cap (X \backslash A)) = \mu(W).$$

Therefore,

$$\mu(U) + \mu(B \cap (X \backslash A)) < r + \mu(B \cap (X \backslash A)) \leq \inf\{\mu(W) : W \supseteq B, W \text{ open}\} = \mu(B).$$

However, we then arrive to the contradiction

$$\mu(B) = \mu(B \cap A) + \mu(B \cap (X \backslash A)) \leq \mu(U) + \mu(B \cap (X \backslash A)) < \mu(B).$$

Hence, $\mu(B \cap A) \geq r = \inf\{\mu(W \cap A) : W \supseteq B, W \text{ open}\}$, that is,

$$\mu_A(B) \geq \inf\{\mu_A(W) : W \supseteq B, W \text{ open}\}.$$

$\square$

The following example displays a pathological measure for which there exists an outer regular Borel subset that is not inner regular for a conditional measure.

Example 1. *Let X be a topological space such that there exists $a \in X$ with $\{a\}$ not closed and $X \setminus \{a\} \neq \varnothing$. Define a measure*

$$
\begin{aligned}
\mu : \quad \mathcal{B}(X) \quad &\to \quad [0, \infty] \\
C \quad &\mapsto \quad \mu(C) := \begin{cases} \infty & C \cap (X \setminus \{a\}) \neq \varnothing, \\ 1 & C = \{a\}, \\ 0 & C = \varnothing. \end{cases}
\end{aligned}
\tag{15}
$$

Let $A := \{a\}$ and $B := X \setminus \{a\}$. Notice that B is outer μ-regular since $\mu(B) = \infty = \mu(X)$. Next,

$$\mu_A(B) = \frac{\mu(A \cap B)}{\mu(A)} = \frac{\mu(\varnothing)}{\mu(\{a\})} = \frac{0}{1} = 0.$$

Finally, if $U \subseteq X$ is open and contains B, then $U = X$ since B is not open, thus

$$\mu_A(U) = \frac{\mu(A \cap U)}{\mu(A)} = \frac{\mu(\{a\})}{\mu(\{a\})} = 1.$$

This way

$$\mu_A(B) < \inf\{\mu_A(U) : B \subseteq U \text{ open}\}.$$

$\mathcal{P}_{\mathbf{f}}(K) \cap \mathrm{rca}(K)$ is a convex subset of $\mathcal{P}(K) \cap \mathrm{rca}(K)$, which is itself a convex subset of $\mathsf{S}_{\mathrm{rca}(K)}$, where $\mathsf{S}_{\mathrm{rca}(K)}$ denotes the unit sphere of $\mathrm{rca}(K)$. As usual, $\mathsf{B}_{\mathrm{rca}(K)}$ denotes the (closed) unit ball of $\mathrm{rca}(K)$.

Corollary 4. *Let K be a nonsingleton compact Hausdorff topological space. If $\mathcal{P}_{\mathbf{f}}(K) \cap \mathrm{rca}(K) \neq \varnothing$, then $\mathcal{P}_{\mathbf{f}}(K) \cap \mathrm{rca}(K)$ is not a face of $\mathsf{B}_{\mathrm{rca}(K)}$.*

Proof. Fix any $\mu \in \mathcal{P}_{\mathbf{f}}(K) \cap \mathrm{rca}(K)$. Since K is Hausdorff and has more than one points, there are two nonempty open subsets U, V in K such that $U \cap V = \varnothing$. Since $\mu \in \mathcal{P}_{\mathbf{f}}(K)$, we have that $\mu(U), \mu(V) > 0$; therefore, $\mu(U), \mu(V) < 1$; hence, $\mu(K \setminus U), \mu(K \setminus V) > 0$.

Consider the conditional probabilities of μ on $F := \mathrm{cl}(U)$ and $G := K \setminus U$, μ_F and μ_G. In view of Lemma 1, $\mu_F, \mu_G \in \mathcal{P}(K) \cap \mathrm{rca}(K)$. Thus, $\frac{1}{2}\mu_F + \frac{1}{2}\mu_G \in \mathcal{P}(K) \cap \mathrm{rca}(K)$. Since $\mu_F(V) = 0 = \mu_G(U)$, we conclude that $\mu_F, \mu_G \notin \mathcal{P}_{\mathtt{f}}(K)$. Let us show that $\frac{1}{2}\mu_F + \frac{1}{2}\mu_G \in \mathcal{P}_{\mathtt{f}}(K)$, which finalizes the proof. Indeed, let W be any nonempty open subset of K. We have two options:

- $U \cap W \neq \emptyset$. Then

$$\left(\frac{1}{2}\mu_F + \frac{1}{2}\mu_G\right)(W) \geq \frac{1}{2}\mu_F(W) = \frac{1}{2}\frac{\mu(F \cap W)}{\mu(F)} \geq \frac{1}{2}\frac{\mu(U \cap W)}{\mu(F)} > 0$$

  because $U \cap W$ is a nonempty open subset of K and $\mu \in \mathcal{P}_{\mathtt{f}}(K)$.
- $U \cap W = \emptyset$. In this case, $W \subseteq K \setminus U = G$, therefore

$$\left(\frac{1}{2}\mu_F + \frac{1}{2}\mu_G\right)(W) \geq \frac{1}{2}\frac{\mu(W)}{\mu(G)} > 0$$

  because W is a nonempty open subset of K and $\mu \in \mathcal{P}_{\mathtt{f}}(K)$.
  $\square$

Under the settings of Corollary 4, it is well known (see Appendix A and [12] (Theorem 3.7)) that $\mathcal{P}(K) \cap \mathrm{rca}(K)$ is, in fact, extremal in $\mathrm{B}_{\mathrm{rca}(K)}$.

4. Discussion and Conclusions

If K is a nonsingleton compact Hausdorff topological space, then $\mathcal{P}_{\mathtt{f}}(K)$ is a convex subset of $\mathcal{P}(K)$ but not a face of $\mathcal{P}(K)$. However, as recalled in Appendix A, $\mathcal{P}(K) \cap \mathrm{rca}(K)$ is a face of $\mathrm{B}_{\mathrm{rca}(K)}$, where $\mathrm{B}_{\mathrm{rca}(K)}$ denotes the unit ball of $\mathrm{rca}(K)$. So, we have the chain of inclusions

$$\mathcal{P}_{\mathtt{f}}(K) \cap \mathrm{rca}(K) \subseteq \mathcal{P}(K) \cap \mathrm{rca}(K) \subseteq \mathrm{B}_{\mathrm{rca}(K)},$$

where the first convex set is not a face of the second, whereas the second is a face of the third. This provides valuable information about the geometry of $\mathrm{B}_{\mathrm{rca}(K)} \equiv \mathrm{B}_{\mathscr{C}(K)^*}$.

It would be interesting to unveil other geometric or topological pathologies satisfied by the convex set of focal regular Borel probability measures.

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Appendix A. Geometry of Normed Spaces

For the sake of completeness and to provide a more general vision of extremal theory, we recall several basic concepts of the geometry of normed spaces. This also boosts the impact of Theorem 5.

Appendix A.1. Extremal Theory

Let X be a real vector space. Given two subsets $E \subseteq F \subseteq X$, E is extremal in F if E satisfies the extremal condition with respect to F, that is, if $f_1, f_2 \in F$ and exists $t \in (0,1)$ such that $tf_1 + (1-t)f_2 \in E$, then $f_1, f_2 \in E$. If F is convex, and E is an extremal convex subset of F, then E is called a face of F. If $e \in F$ and $\{e\}$ is extremal in F, then e is an extreme point of F.

Extreme points play a very important role in functional analysis. For instance, to study certain isometries, the extreme points of the unit ball of the dual of the corresponding normed space are often useful. Consult, for example, refs. [13,14] and their references to see some illustrations.

Now, consider a subset $F \subseteq X$ and a convex function $f : X \to \mathbb{R}$. The supporting set of f in F is defined as $H(f,F) := \{x \in F : f(x) = \sup f(F)\}$. If f is unbounded on F or bounded but the sup is not attained, then obviously $H(f,F) = \varnothing$. If $H(f,F) \neq \varnothing$, then it is not hard to show that $H(f,F)$ is extremal in F. Indeed, if $x, y \in F$ and exists $t \in (0,1)$ with $tx + (1-t)y \in H(f,F)$, then the chain of inequalities

$$\sup f(F) = f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) \leq \sup f(F)$$

forces that $f(x) = f(y) = \sup f(F)$, meaning that $x, y \in H(f,F)$. In case F is convex and f is linear, then $H(f,F)$ is convex as well and thus it becomes a face of F. This kind of face is called an exposed face.

Appendix A.2. $\mathcal{P}(K) \cap \mathrm{rca}(K)$ Is an Exposed Face of the Unit Ball of $\mathrm{rca}(K)$

Let K be a compact Hausdorff topological space. We showed in the Preliminaries Section that $\mathcal{P}(K) \cap \mathrm{rca}(K)$ is a convex subset of the unit sphere of $\mathrm{rca}(K)$. Here, we demonstrate that $\mathcal{P}(K) \cap \mathrm{rca}(K)$ is an exposed face of the unit ball of $\mathscr{C}(K)^*$. Indeed, let us denote by $\mathbf{1} \in \mathscr{C}(K)$ the constant function equal to 1, that is, the unity of the Banach algebra $\mathscr{C}(K)$. We can see $\mathbf{1}$ as an element of $\mathscr{C}(K)^{**}$ by relying on the canonical injection of a normed space into its bidual. Then, $\mathbf{1}$ acts on $\mathscr{C}(K)^*$ following Equation (4):

$$\begin{aligned} \mathbf{1}: \quad \mathscr{C}(K)^* &\to \mathbb{R} \\ \mu &\mapsto \mu(\mathbf{1}) = \int_K 1 d\mu = \mu(K). \end{aligned} \tag{A1}$$

Since $\|\mathbf{1}\|_\infty = 1$, we have that $\mathbf{1}$ has norm 1 in $\mathscr{C}(K)^{**}$. Notice that $\mathcal{P}(K) \cap \mathrm{rca}(K) = H\left(\mathbf{1}, \mathsf{B}_{\mathscr{C}(K)^*}\right)$, where $\mathsf{B}_{\mathscr{C}(K)^*}$ stands for the unit ball of $\mathscr{C}(K)^*$. Indeed, if $\mu \in \mathcal{P}(K) \cap \mathrm{rca}(K)$, then it is clear that $\|\mu\| = 1$ and $\mu(\mathbf{1}) = \int_K 1 d\mu = \mu(K) = 1$. As a consequence, $\mu \in H\left(\mathbf{1}, \mathsf{B}_{\mathscr{C}(K)^*}\right)$. Conversely, suppose that $\mu \in H\left(\mathbf{1}, \mathsf{B}_{\mathscr{C}(K)^*}\right)$. Then $\|\mu\| = 1 = \mu(K)$. In order to prove that $\mu \in \mathcal{P}(K)$ it only suffices to show that μ is positive. So, assume, on the other hand, that there exists $A \in \mathcal{B}(K)$, such that $\mu(A) < 0$. Then $1 = \mu(K) = \mu(A) + \mu(K \setminus A)$, hence $\mu(K \setminus A) > 1$, reaching the following contradiction with the total variation of μ:

$$1 = \|\mu\| \geq |\mu(A)| + |\mu(K \setminus A)| > 1.$$

As a consequence, we lastly conclude that $\mu \in \mathcal{P}(K) \cap \mathrm{rca}(K)$.

Appendix B. Nontrivial Examples of Focal (Regular) Borel Probability Measures

For the sake of completeness, we present several examples of focal (regular) Borel probability measures with values on Hausdorff topological modules over Hausdorff topological rings.

Appendix B.1. Counting Probability Measures

Let R be a Hausdorff topological ring. A series $\sum_{n=1}^{\infty} r_n$ in R is called subseries convergent if, for every $A \subseteq \mathbb{N}$, the series $\sum_{n \in A} r_n$ is convergent. Every subseries convergent series defines an interesting counting measure.

$$\mu: \begin{array}{ccc} \mathcal{P}(\mathbb{N}) & \to & R \\ A & \mapsto & \sum_{n \in A} r_n, \end{array} \tag{A2}$$

where $\mathcal{P}(\mathbb{N})$ stands for the power set of $\mathbb{N}$. Here, it is understood that $\mu(\varnothing) = 0$. Furthermore, if R is partially ordered and $\sum_{n=1}^{\infty} r_n$ is a convex series in the sense that $r_n \geq 0$ for all $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} r_n = 1$, then (A2) defines a generalized probability measure, since $\mu(A) \geq 0$ for all $A \subseteq \mathbb{N}$ and $\mu(\mathbb{N}) = 1$. This kind of counting probability measures are of special interest in quantum mechanics. Lastly, $\mathbb{N}$ is endowed with a discrete topology; therefore, μ is trivially regular, and it is also focal, provided that $r_n > 0$ for every $n \in \mathbb{N}$.

Appendix B.2. Counting Probability Measures on Quantum Systems

Let H be an infinite dimensional separable complex Hilbert space. According to the first postulate of quantum mechanics [15,16], H represents a quantum mechanical system. The C^*-algebra of bounded linear operators on H, $\mathcal{B}(H)$, is trivially a Hausdorff topological ring. Let $(e_n)_{n \in \mathbb{N}}$ be an orthonormal basis of H. Let $(t_n)_{n \in \mathbb{N}}$ be a real convex series. For every $n \in \mathbb{N}$, consider the bounded linear operator T_n on H given by $T_n(x) := t_n(x|e_n)e_n$. In accordance with [17] (Section 6), T_n is selfadjoint and positive. Notice that $\sum_{n=1}^{\infty} T_n$ is subseries convergent in $\mathcal{B}(H)$. Therefore,

$$\mu: \begin{array}{ccc} \mathcal{P}(\mathbb{N}) & \to & \mathcal{B}(H) \\ A & \mapsto & \sum_{n \in A} T_n, \end{array} \tag{A3}$$

defines a counting probability measure on the quantum system H, which is of special interest in Quantum Mechanics. If another quantum system K interacts with H by means of a bounded linear operator $S : K \to H$, then the counting probability measure (A3) can be redefined to take values on the Hausdorff topological $\mathcal{B}(H)$-module $\mathcal{B}(K, H)$ as follows:

$$\mu: \begin{array}{ccc} \mathcal{P}(\mathbb{N}) & \to & \mathcal{B}(K, H) \\ A & \mapsto & \sum_{n \in A} T_n \circ S. \end{array} \tag{A4}$$

Indeed, K is another infinite dimensional separable complex Hilbert space and the commutative additive group of $\mathcal{B}(K, H)$ is clearly a left $\mathcal{B}(H)$-module with left action given by left composition.

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