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Symmetry Analysis, Exact Solutions and Conservation Laws of a Benjamin–Bona–Mahony–Burgers Equation in 2+1-Dimensions

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† M.S. Bruzón sadly passed away prior to the revision of this paper. This is one of her last works.

Abstract: The Benjamin–Bona–Mahony equation describes the unidirectional propagation of small-amplitude long waves on the surface of water in a channel. In this paper, we consider a family of generalized Benjamin–Bona–Mahony–Burgers equations depending on three arbitrary constants and an arbitrary function $G(u)$. We study this family from the standpoint of the theory of symmetry reductions of partial differential equations. Firstly, we obtain the Lie point symmetries admitted by the considered family. Moreover, taking into account the admitted point symmetries, we perform symmetry reductions. In particular, for $G'(u) \neq 0$, we construct an optimal system of one-dimensional subalgebras for each maximal Lie algebra and deduce the corresponding (1+1)-dimensional nonlinear third-order partial differential equations. Then, we apply Kudryashov's method to look for exact solutions of the nonlinear differential equation. We also determine line soliton solutions of the family of equations in a particular case. Lastly, through the multipliers method, we have constructed low-order conservation laws admitted by the family of equations.

Keywords: conservation laws; exact solutions; Lie symmetries; symmetry reductions

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1. Introduction

The Benjamin–Bona–Mahony equation (BBM), or regularised long wave (RLW) equation,

$$u_t + u_x + uu_x - u_{xxt} = 0, \tag{1}$$

was first proposed by Benjamin et al. [1] as an alternative mathematical model to the Korteweg–de Vries for modelling long wave motions in nonlinear dispersive systems. The authors stressed that both models are applicable at the same level of approximation, however, from a computational mathematics point of view, the BBM equation has some advantages over the KdV equation. Generalized forms of the BBM equation have been widely studied. In this paper, we consider a generalized family of Benjamin–Bona–Mahony–Burgers (gBBMB) equations given by

$$u_t - \gamma(u_{txx} + u_{tyy}) - \alpha(u_{xx} + u_{yy}) + \beta(u_x + u_y) = (F_u u_x + F_u u_y) + f, \tag{2}$$

where $u = u(t, x, y)$ is an analytic function of the time coordinate t and the spatial coordinates x and y, $\gamma \neq 0$, α, β and f are arbitrary constants, whereas $F(u)$ is a nonlinear function. The gBBMB Equation (2) was considered in [2]. Here, the authors considered a new class of polynomial functions equipped with a parameter to approximate the solutions of the gBBMB Equation (2).

Regarding the physical interpretation, the original BBM equation was proposed as a model of long waves in channel flows where nonlinear dispersion is incorporated. Its solutions approximate solutions of the two-dimensional Euler equations. Hence, for the two-dimensional model considered here (2), the variables t, x, y keep the same physical interpretations as in Euler equations [3,4]. For two-dimensional water waves in a channel with a flat bottom, the independent variable t is interpreted as the elapsed time and x, y determine the position, the horizontal and vertical coordinates along the channel, respectively.

Currently, nonlinear equations involving parameters and arbitrary functions, which arise in diverse fields as mathematical biology and physics, nonlinear dynamics and plasma physics, have attracted the attention of numerous researchers. Nevertheless, the analysis of nonlinear equations involving arbitrary functions is often difficult and laborious. In particular, for dealing with the determination of exact solutions, several direct methods have been elaborated, among them Kudryashov method [5–8], tanh-sech method [9,10], Painlevé analysis [11,12], Adomian decomposition method [13–16] and other special methods [17,18]. However, these methods just work for a limited kind of equations.

Lie symmetries of a partial differential equation (PDE) are transformations acting on the space of independent and dependent variables which transforms the PDE solution space into itself. The analysis of Lie symmetries is one of the most effective algorithms to analyse PDE equations including the construction of invariant solutions, construction of mappings between equivalent equations of the same family, finding invertible mappings of nonlinear PDEs to linear PDEs or finding conservation laws. Among these applications, we highlight the construction of invariant solutions, i.e., interesting special classes of solutions which are invariant under a Lie group of point transformations. In the case of a PDE, the invariance under a Lie group of point transformations allows one to obtain, constructively, similarity solutions (invariant solutions), which are invariant under a subgroup of the full Lie symmetry group admitted by the PDE. Similarity solutions arise as solutions of a reduced system of differential equations (DEs) with fewer number of independent variables. For further information on Lie symmetries and their applications one can refer, as example, to [19–23] and references therein.

In many natural processes, such as physical or chemical ones, conservation laws emerge. Considering an isolated physical system, these laws characterise physical properties that do not change over time. Regarding PDEs, not all conservation laws have a physical interpretation. However, when a PDE has a large number of conservation laws it is usually an indicator of its integrability. Conservation laws are also studied for their applicability in numerical methods for PDEs since they are useful to investigate the existence, uniqueness and stability of solutions. Moreover, exact solutions can be constructed by using conservation laws.

With respect to the concept of conservation laws, different results can be found in the the recent literature. In [24], Ibragimov proved a theorem to find conservation laws which do not require the existence of a classical Lagrangian. This theorem is based on the concept of adjoint equation for nonlinear DEs [25–27]. Nontrivial conservation laws have been determined by using Ibragimov's conservation law theorem, see, for instance, Refs. [28–30]. Anco and Bluman presented a direct algorithmic method using adjoint-symmetries for finding all local conservation laws of a given DE system [31–34]. Moreover, in [35], this general method was further developed and reviewed in detail. In [36,37], symmetry properties of conservation laws of PDEs are analysed by using the multiplier method. In particular, in [36], it was proved that Ibragimov's formula [24] for determining conservation laws is equivalent to a standard formula for the action of an infinitesimal symmetry on a conservation law multiplier. The same author recently showed [38] that Ibragimov's method can lead to trivial conservation laws. Most importantly, the formula does not necessarily yield all non-trivial conservation laws unless the symmetry action on the set of these conservation laws is transitive, property which cannot be verified until all conservation laws have been determined. It is worth highlighting the symmetry multi-reduction method [39], which is a generalization of the double reduction method [40–42]. The method proposed in [39] gives an explicit algorithm for PDEs with $n \geq 2$ independent variables admitting a symmetry algebra whose dimension is at least $n-1$ that allows us to find all symmetry-invariant conservation laws, which will reduce to first integrals for the ODE that describe the symmetry-invariant solutions of the PDE.

In [43], the following generalized BBMB equations with power functions were studied

$$u_t + bu_x + a(u^m)_x + (u^n)_{txx} + c(u^k)_{xx} = 0. \tag{3}$$

Lie symmetries were used to reduce the equation into ordinary differential equations (ODEs) in order to obtain new solutions. Furthermore, conservation laws were obtained for special values of the parameters. Here we similarly analyse Equation (2). In fact, considering $n = k = 1$ and changing u^m to $F(u)$ in Equation (3), the one-dimensional version of Equation (2) arises.

The aim of this paper is to analyse Equation (2) from the viewpoint of Lie symmetries, conservation laws and analytical solutions. First, in Section 2, we determine a complete classification of the point symmetries admitted by PDE (2) depending on the arbitrary parameters γ, α, β and f and the arbitrary function $F(u)$. In Section 3, we use the one-dimensional point symmetry groups admitted by Equation (2) to reduce the PDE into ODEs. Then, in Section 4, exact solutions are obtained by using Kudryashov method. In Section 5, by using the multipliers method, a complete classification of the low-order conservation laws of PDE (6) has been achieved. In Section 6, we obtain line soliton solutions $u(t,x,y) = h(x + \mu y - \lambda t)$, where μ and λ represent the direction and the speed of propagation of the line soliton. Taking into account the line soliton formulation into PDE (2), a nonlinear third-order ODE for h is obtained. This equation is reduced to a second-order ODE through the corresponding differential invariants. Moreover, the second-order ODE can be easily integrated that leads to its complete quadrature. Undoing the change of variables we obtain a first-order separable ODE which can be integrated to determine the analytical expression of the line soliton solutions. Finally, in Section 7, we present the conclusions.

2. Lie Point Symmetries

To determine the Lie point symmetries of PDE (2), we consider a one-parameter group of infinitesimal transformations in (t, x, y, u) given by

$$\begin{array}{rcll} \tilde{t} &=& t &+ \varepsilon \tau(t,x,y,u) + \mathcal{O}(\varepsilon^2), \\ \tilde{x} &=& x &+ \varepsilon \xi(t,x,y,u) + \mathcal{O}(\varepsilon^2), \\ \tilde{y} &=& y &+ \varepsilon \phi(t,x,y,u) + \mathcal{O}(\varepsilon^2), \\ \tilde{u} &=& u &+ \varepsilon \eta(t,x,y,u) + \mathcal{O}(\varepsilon^2), \end{array} \tag{4}$$

where ε is the group parameter. An admitted infinitesimal generator is of the form

$$X = \tau(t,x,y,u)\partial_t + \xi(t,x,y,u)\partial_x + \phi(t,x,y,u)\partial_y + \eta(t,x,y,u)\partial_u. \tag{5}$$

We recall that a group of transformations (4) is a Lie point symmetry of PDE Equation (2) if and only if the solution space of PDE Equation (2) is invariant under the action of the one-parameter group of point transformations (4). Lie's fundamental theorems prove that Lie groups of transformations are entirely characterized by their infinitesimal generators. An infinitesimal formulation of Lie groups of transformations is crucial since it replaces nonlinear conditions of invariance of a given DE by equivalent, albeit far simpler, linear conditions which determine the infinitesimal generators of the group. Furthermore, for a given DE, the essential applications of Lie groups only need knowledge of the admitted infinitesimal generators. For further information one can refer, for example, to references [19–21,23]. For the sake of simplicity, it is helpful to introduce the function $G(u) = \beta - F_u$. Therefore, Equation (2) becomes

$$u_t - \gamma(u_{txx} + u_{tyy}) - \alpha(u_{xx} + u_{yy}) + G(u_x + u_y) - f = 0. \tag{6}$$

Point symmetries are determined by applying the symmetry invariance criterion [19–21,23], which requires that

$$X^{(3)}\left(u_t-\gamma(u_{txx}+u_{tyy})-\alpha(u_{xx}+u_{yy})+G(u_x+u_y)-f\right) \;=\; 0, \tag{7}$$

when Equation (6) holds. Here, $X^{(3)}$ represents the third prolongation of the infinitesimal generator (5), which is given by

$$X^{(3)}=X+\eta^{(1)}_{i_1}\frac{\partial}{\partial u_{i_1}}+\eta^{(2)}_{i_1i_2}\frac{\partial}{\partial u_{i_1i_2}}+\eta^{(3)}_{i_1i_2i_3}\frac{\partial}{\partial u_{i_1i_2i_3}},$$

with coefficients

$$\begin{aligned}
\eta^{(1)}_{i_1} &= D_{i_1}(\eta)-u_tD_{i_1}(\tau)-u_xD_{i_1}(\xi)-u_yD_{i_1}(\phi),\\
\eta^{(2)}_{i_1i_2} &= D_{i_2}(\eta^{(1)}_{i_1})-u_{i_1t}D_{i_2}(\tau)-u_{i_1x}D_{i_2}(\xi)-u_{i_1y}D_{i_2}(\phi),\\
\eta^{(3)}_{i_1i_2i_3} &= D_{i_3}(\eta^{(2)}_{i_1i_2})-u_{i_1i_2t}D_{i_3}(\tau)-u_{i_1i_2x}D_{i_3}(\xi)-u_{i_1i_2y}D_{i_3}(\phi),
\end{aligned}$$

with D the total derivative operator, $u_i=\frac{\partial u}{\partial x_i}$, $i=1,2,3$ with $x_1=t$, $x_2=x$ and $x_3=y$, and $i_j=1,2,3$ for $j=1,2,3$. The invariance condition (7) splits with respect to the differential consequences of u, yielding a set of 64 determining equations. By simplifying this system, we obtain $\tau=\tau(t)$, $\xi=\xi(x,y)$, $\phi=\phi(x,y)$, $\eta=\eta(t,x,y,u)$, and the parameters α, γ and f along with the arbitrary function $G(u)$ are related by the following conditions:

$$\begin{aligned}
\eta_{uu}=0,\quad \xi_y+\phi_x=0,\quad \xi_x-\phi_y=0,\\
\gamma\eta_{tu}+\alpha\tau_t=0,\quad 2\xi_x-\gamma\left(\eta_{xxu}+\eta_{yyu}\right)=0,\\
\xi_{xx}+\xi_{yy}-2\eta_{xu}=0,\quad \phi_{xx}+\phi_{yy}-2\eta_{yu}=0,\\
2\gamma\eta_{txu}-\eta\,G_u-G\left(\tau_t+\xi_x-\xi_y\right)=0,\\
2\gamma\eta_{tyu}-\eta\,G_u-G\left(\tau_t+\phi_y-\phi_x\right)=0,\\
\gamma\left(\eta_{txx}+\eta_{tyy}\right)+\alpha\left(\eta_{xx}+\eta_{yy}\right)-G\left(\eta_x+\eta_y\right)+f(\tau_t+2\xi_x-\eta_u)-\eta_t=0.
\end{aligned}\tag{8}$$

The determining system has been derived and solved with the aid of Maple commands "rifsimp" and "pdsolve". Moreover, it should be noted that the gBBMB Equation (6) are preserved under the equivalence transformation given by

$$\tilde{u}\longrightarrow u+k,$$

with k constant. So we obtain the result:

Theorem 1. *The classification of point symmetries admitted by the gBBMB Equation (6) depending on the arbitrary constants α, γ and f, and the arbitrary function $G(u)$ is given by the following cases:*

(i) *For arbitrary α, γ, f and $G(u)$ the admitted point symmetries are generated by:*

$X_1=\partial_t$,
time-translation.
$X_2=\partial_x$,
space-translation.
$X_3=\partial_y$,
space-translation.

(ii) *Extra point symmetries of the gBBMB Equation (6) are admitted in the following cases:*

(a) *If $\alpha=0$ and $G(u)=g_1\exp(mu)$*

- *For $f\neq 0$*

  $X_4=\exp(-fmt)(\partial_t+f\partial_u)$,
  dilation.

- *For $f=0$*

  $X_5=mt\partial_t-\partial_u$,
  dilation and shift.

(b) *If* $\alpha = 0$ *and* $G(u) = \dfrac{g_1}{u}$

$$X_6 = t\partial_t + u\partial_u,$$
scaling.

(c) *If* $\alpha = f = 0$ *and* $G(u) = g_1 u^n$

$$X_7 = nt\partial_t - u\partial_u,$$
scaling.

(d) *If* $G(u) = g_1$

$$X_\phi = \phi\,\partial_u,$$
$$X_\psi = (u - \psi)\partial_u,$$

where $\phi(t,x,y)$ *and* $\psi(t,x,y)$ *satisfies, respectively,*

$$\phi_t - \gamma(\phi_{txx} + \phi_{tyy}) - \alpha(\phi_{xx} + \phi_{yy}) + g_1(\phi_x + \phi_y) = 0,$$

and

$$\psi_t - \gamma(\psi_{txx} + \psi_{tyy}) - \alpha(\psi_{xx} + \psi_{yy}) + g_1(\phi_x + \phi_y) - f = 0.$$

In the above, $g_1 \neq 0$, $m \neq 0$, $n \neq 0$ *are arbitrary constants.*

3. Symmetry Reductions

In this section, we will restrict our attention to those cases where $G'(u) \neq 0$. By using the point symmetries admitted by PDE (6), we can reduce PDE (6) to an equation with fewer number of independent variables. Each point symmetry of PDE (6) leads to a characteristic system

$$\frac{dt}{\tau} = \frac{dx}{\xi} = \frac{dy}{\phi} = \frac{du}{\eta}. \tag{9}$$

Solving the characteristic equations, one obtains similarity variables z and r, and similarity solutions $w(z,r)$. The substitution of these variables into PDE (6) leads to third-order nonlinear PDEs for $w(z,r)$. In general, it is not always possible to determine all the possible group-invariant solutions, since the Lie symmetry group of a given PDE can contain an infinite number of Lie subgroups. The aim is to classify all the feasible group-invariant solutions into classes in a way that solutions belonging to the same class are equivalent, i.e., these solutions are related through an element of the Lie symmetry group; vice versa, solutions belonging to different classes are not equivalent and therefore there exists no element of the Lie symmetry group that maps one solution into the other. To address this problem, we will find an optimal system of one-dimensional subalgebras [23,44]. For that purpose, it is very useful to determine the most general symmetry Lie algebra that PDE (6) admits depending on the arbitrary function $G(u)$ and the arbitrary parameters α, γ and f. The following theorem shows a basis of generators for each maximal Lie algebra admitted by PDE (6).

Theorem 2. *The maximal Lie algebras for PDE (6), with* $G'(u) \neq 0$, *along with their non-zero commutator structure are given by:*

1. *For arbitrary* α, γ, f *and* $G(u)$

$$\mathcal{A}_1 = span(X_1, X_2, X_3).$$

2. *If* $\alpha = 0$, $f \neq 0$ *and* $G(u) = g_1 \exp(mu)$

$$\mathcal{A}_2 = span(X_1, X_2, X_3, X_4),$$
$$[X_1, X_4] = -fmX_4.$$

3. *If* $\alpha = f = 0$ *and* $G(u) = g_1 \exp(mu)$

$$\mathcal{A}_3 = span(X_1, X_2, X_3, X_5),$$
$$[X_1, X_5] = mX_1.$$

4. *If* $\alpha = 0$ *and* $G(u) = \dfrac{g_1}{u}$

$$\mathcal{A}_4 = span(X_1, X_2, X_3, X_6),$$
$$[X_1, X_6] = X_1.$$

5. *If* $\alpha = f = 0$ *and* $G(u) = g_1 u^n$

$$\mathcal{A}_5 = span(X_1, X_2, X_3, X_7),$$
$$[X_1, X_7] = nX_1.$$

Theorem 3. *For the gBBMB Equation (6), with* $G'(u) \neq 0$*, an optimal system of one-dimensional subalgebras for each maximal Lie algebra is given by:*

1. *For arbitrary* α, γ, f *and* $G(u)$

$$\langle X_1 + \nu X_2 + \mu X_3 \rangle, \quad \langle X_2 \rangle, \quad \langle X_3 \rangle.$$

2. *If* $\alpha = 0$, $f \neq 0$ *and* $G(u) = g_1 \exp(mu)$

$$\langle X_1 + \nu X_2 + \mu X_3 \rangle, \quad \langle X_4 + \lambda X_2 + \sigma X_3 \rangle, \quad \langle X_2 \rangle, \quad \langle X_3 \rangle.$$

3. *If* $\alpha = f = 0$ *and* $G(u) = g_1 \exp(mu)$

$$\langle X_1 + \nu X_2 + \mu X_3 \rangle, \quad \langle X_5 + \lambda X_2 + \sigma X_3 \rangle, \quad \langle X_2 \rangle, \quad \langle X_3 \rangle.$$

4. *If* $\alpha = 0$ *and* $G(u) = \dfrac{g_1}{u}$

$$\langle X_1 + \nu X_2 + \mu X_3 \rangle, \quad \langle X_6 + \lambda X_2 + \sigma X_3 \rangle, \quad \langle X_2 \rangle, \quad \langle X_3 \rangle.$$

5. *If* $\alpha = f = 0$ *and* $G(u) = g_1 u^n$

$$\langle X_1 + \nu X_2 + \mu X_3 \rangle, \quad \langle X_7 + \lambda X_2 + \sigma X_3 \rangle, \quad \langle X_2 \rangle, \quad \langle X_3 \rangle.$$

In the above, ν, μ, λ *and* σ *are arbitrary constants.*

Now, taking into account the optimal system of one-dimensional subalgebras for each maximal Lie algebra given in Theorem 3, we will consider some one-dimensional reductions which allow us to transform PDE (6) into PDEs in 1+1-dimensions. Moreover, since these PDEs admit symmetry groups, we can reduce the number of independent variables again to obtain third-order nonlinear ODEs.

3.1. Reduction under $X_1 + \nu X_2 + \mu X_3$

Let us start considering the symmetry generator $X_1 + \nu X_2 + \mu X_3$, where ν and μ are constants. Using this symmetry, we reduce (6) to a PDE with two independent variables. The symmetry gives the invariants

$$z = x - \nu t, \qquad r = y - \mu t \qquad u = w(z, r). \tag{10}$$

Using these invariants, Equation (6) reduces to

$$\begin{aligned} \gamma\nu(w_{zzz} + w_{rrz}) + \gamma\mu(w_{rrr} + w_{rzz}) - \alpha(w_{zz} + w_{rr}) \\ + G(w_z + w_r) - \nu w_z - \mu w_r - f = 0, \end{aligned} \tag{11}$$

which admits the symmetries

$$Z_1 = \partial_r, \quad Z_2 = \partial_z. \tag{12}$$

The symmetry $Z_1 + \delta Z_2$, with δ constant, provides the invariants

$$q = z - \delta r, \quad w = h(q), \tag{13}$$

and therefore PDE (11) transforms to the third-order nonlinear ODE

$$\gamma\left(\delta^2+1\right)(\nu - \delta\,\mu)h''' - \alpha\left(\delta^2+1\right)h'' + (G(1-\delta) + \delta\mu - \nu)h' - f = 0. \tag{14}$$

Taking $h'(q) = V(q)$, Equation (14) can be written as follows

$$V'' = \frac{1}{\gamma(\delta^2+1)(\nu - \delta\,\mu)}\left(\alpha\left(\delta^2+1\right)V' - (H(1-\delta) + \delta\mu - \nu)V + f\right), \tag{15}$$

where $H(\omega(V)) = G(\int V\,dq) = G(h)$.

3.2. Reduction under $X_4 + \lambda X_2 + \sigma X_3$

Now, we consider the symmetry $X_4 + \lambda X_2 + \sigma X_3$. This symmetry produces the invariants

$$z = x - \frac{\lambda}{fm}e^{fmt}, \qquad r = y - \frac{\sigma}{fm}e^{fmt}, \qquad u = ft + w(z,r), \tag{16}$$

where $w(z,r)$ verifies

$$\gamma\lambda(w_{zzz} + w_{rrz}) + \gamma\sigma(w_{rrr} + w_{rzz}) + g_1 e^{mw}(w_z + w_r) - \lambda w_z - \sigma w_r = 0. \tag{17}$$

PDE (17) admits symmetries (12). Taking into account invariants (13), PDE (17) is transformed into the third-order nonlinear ODE

$$\gamma\left(\delta^2+1\right)(\lambda - \delta\,\sigma)h''' + \left(g_1 e^{mh}(1-\delta) + \delta\sigma - \lambda\right)h' = 0. \tag{18}$$

3.3. Reduction under $X_5 + \lambda X_2 + \sigma X_3$

Consider the symmetry $X_5 + \lambda X_2 + \sigma X_3$. This symmetry yields the invariants

$$z = x - \frac{\lambda}{m}\log t, \qquad r = y - \frac{\sigma}{m}\log t, \qquad u = w(z,r) - \frac{1}{m}\log t, \tag{19}$$

where $w(z,r)$ satisfies

$$\gamma\lambda(w_{zzz} + w_{rrz}) + \gamma\sigma(w_{rrr} + w_{rzz}) + g_1 m e^{mw}(w_z + w_r) - \lambda w_z - \sigma w_r - 1 = 0. \tag{20}$$

This equation admits symmetries (12). From $Z_1 + \delta Z_2$ one obtains the invariants (13). Taking into account invariants (13), PDE (20) is transformed into the third-order nonlinear ODE

$$\gamma\left(\delta^2+1\right)(\lambda - \delta\,\sigma)h''' + \left(g_1 m e^{mh}(1-\delta) + \delta\sigma - \lambda\right)h' - 1 = 0. \tag{21}$$

3.4. Reduction under $X_6 + \lambda X_2 + \sigma X_3$

Consider the symmetry $X_6 + \lambda X_2 + \sigma X_3$. This symmetry produces the invariants

$$z = x - \lambda\log t, \qquad r = y - \sigma\log t, \qquad u = t\,w(z,r), \tag{22}$$

where $w(z,r)$ must satisfy

$$\begin{aligned}\gamma\lambda(w_{zzz} + w_{rrz}) + \gamma\sigma(w_{rrr} + w_{rzz}) - \gamma(w_{zz} + w_{rr})\\ + \frac{g_1}{w}(w_z + w_r) - \lambda w_z - \sigma w_r + w - f = 0.\end{aligned} \tag{23}$$

PDE (23) admits symmetries (12). By considering invariants (13), PDE (23) is transformed into the third-order nonlinear ODE

$$\gamma\left(\delta^2+1\right)(\lambda-\delta\,\sigma)h'''-\gamma\left(\delta^2+1\right)h''+\left(\frac{g_1}{h}(1-\delta)+\delta\sigma-\lambda\right)h'+h-f=0. \quad (24)$$

3.5. Reduction under $X_7+\lambda X_2+\sigma X_3$

Consider the symmetry $X_7+\lambda X_2+\sigma X_3$. This symmetry yields the invariants

$$z=x-\frac{\lambda}{n}\log t, \qquad r=y-\frac{\sigma}{n}\log t, \qquad u=t^{-1/n}w(z,r), \quad (25)$$

where $w(z,r)$ satisfies

$$\begin{aligned}\gamma\lambda(w_{zzz}+w_{rrz})+\gamma\sigma(w_{rrr}+w_{rzz})+\gamma(w_{zz}+w_{rr})\\ +g_1nw^n(w_z+w_r)-\lambda w_z-\sigma w_r-w=0.\end{aligned} \quad (26)$$

This equation admits symmetries (12). Taking into account invariants (13), PDE (26) is transformed into the third-order nonlinear ODE

$$\begin{aligned}\gamma(\delta^2+1)(\lambda-\delta\,\sigma)h'''+\gamma(\delta^2+1)h''\\ +(g_1nh^n(1-\delta)+\delta\sigma-\lambda)h'-h \quad =0.\end{aligned} \quad (27)$$

4. Exact Solutions via Kudryashov's Method

In this section, we determine the function G for which Equation (6) admits solutions which are obtained by employing the Kudryashov's method [8]. Kudryashov [6] called the simplest equation to any nonlinear ordinary differential equation of lesser order than the original equation with a known general solution.

- The Riccati equation was the first example of the simplest equation

$$V'+V^2-pV-q=0 \quad (28)$$

If $V(q)$ is a solution of Equation (28), then the equation

$$V''=2V^3-3pV^2+(p^2-2q)V+pq \quad (29)$$

has special solutions that are expressed via the general solution of Equation (28). It was proved by differentiating Equation (28) with respect to z and substituting V' from Equation (28) into expressions obtained.

Equation (15), with $H=\frac{\gamma(\delta^2+1)(\delta\,\mu-\nu)}{1-\delta}(2V^2-3pV)$ and $\alpha=0$, can be written in the same form that Equation (29) by considering

$$\begin{aligned}pq &= \frac{f}{\gamma(\delta^2+1)(\nu-\delta\,\mu)},\\ p^2-2q &= \frac{1}{\gamma(\delta^2+1)}.\end{aligned}$$

Thus, Equation (15) has special solutions that are expressed via the general solution of Riccati Equation (28).

In order to apply the Kudryashov's method to the nonlinear PDE in 1+1-dimensions (11), the first step is to reduce the nonlinear PDE into a nonlinear ODE, which we have already done using Lie symmetries in the previous section. Thus, we consider the ODE (15), which can be written in the form

$$V''=a_1HV+a_2V+a_3V'+a_4, \quad (30)$$

where $a_1=\frac{\delta-1}{\gamma(\delta^2+1)(\nu-\delta\,\mu)}$, $a_2=\frac{1}{\gamma(\delta^2+1)}$, $a_3=\frac{\alpha}{\gamma(\nu-\delta\,\mu)}$ and $a_4=\frac{f}{\gamma(\delta^2+1)(\nu-\delta\,\mu)}$. We suppose that the solution of ODE (30) can be expressed in terms of a polynomial of the form

$$V=\sum_{n=0}^{N}A_nY^n, \quad (31)$$

where $Y = Y(q)$ satisfies the first-order nonlinear ODE

$$Y'(q) = Y^2(q) - Y(q), \tag{32}$$

$A_n, n = 0, \dots, N$, are constants to be determined later, $A_N \neq 0$. We note that the solution of (32) is

$$\frac{1}{1+\exp(q)}. \tag{33}$$

Considering the homogeneous balance between V'' and $H(V)V$ in (30), we obtain:

- If $HV = V^2$ then $N = 2$.
- If $HV = V^3$ then $N = 1$.

We consider $HV = V^2$ and $N = 2$ then we can write (31) as

$$V = A_0 + A_1 Y(q) + A_2 Y^2(q), \tag{34}$$

with $A_2 \neq 0$. In the following we determine $A_n, n = 0, \dots, 2$. We substitute V, V^2, V' and V'' expressions in Equation (30). Equating each coefficient of Y^n, $n = 0, \dots, 2$, to zero, yields a set of simultaneous algebraic equations for A_n.

$$\begin{aligned} A_0^2 a_1 + A_0 a_2 + a_4 &= 0, \\ A_1(2A_0A_1 + A_2 + a_3 - 1) &= 0, \\ 2A_0A_1A_2 + A_1A_1^2 + A_1a_3 - 3A_1 + A_2A_2 + 2A_2, a_3 - 4A_2 &= 0, \\ 2(A_1A_1A_2 - A_1 + A_2a_3 - 5A_2) &= 0, \\ A_2(A_1A_2 - 6) &= 0. \end{aligned} \tag{35}$$

Solving system (35) for $a_1 = \frac{-36+a_2^2}{4a_4}$ and $a_3 = 5$, we obtain the set of solutions:

$$A_2 = \frac{24a_4}{-36+a_2^2}, \quad A_1 = -\frac{48a_4}{-36+a_2^2}, \quad A_0 = \frac{2(150-25a_2)a_4}{25(-36+a_2^2)}. \tag{36}$$

Consequently, the solution of Equation (30) with $a_1 = \frac{-36+a_2^2}{4a_4}$ and $a_3 = 5$ is

$$V(q) = -\frac{2a_4(a_2\cosh(q) + a_2 - 6\sinh(q) + 6)}{(a_2^2 - 36)(\cosh(q) + 1)}.$$

- The second example, presented by Kudryashov, was the Jacobi elliptic function equation

$$(V')^2 - V^4 - aV^3 - bV^2 - cV - d = 0. \tag{37}$$

If $V(q)$ is a solution of Equation (37) then the equation

$$V'' = 2V^3 + \frac{3}{2}aV^2 + bV + \frac{1}{2}c, \tag{38}$$

has special solutions that are expressed via the general solution of Equation (37). It was proved on similar lines given above.

Equation (15), with $H(V) = \frac{\gamma(\delta^2+1)(\delta\mu-\nu)}{1-\delta}(2V^2 + \frac{3}{2}aV)$ and $\alpha = 0$, can be written in the same form that of Equation (38) by considering

$$\begin{aligned} c &= \frac{2f}{\gamma(\delta^2+1)(\nu-\delta\mu)}, \\ b &= \frac{1}{\gamma(\delta^2+1)}. \end{aligned} \tag{39}$$

Thus, Equation (15) has special solutions that are expressed via the general solution of Jacobi Equation (37).

If V_1, V_2, V_3 and V_4 are the roots of

$$V^4 + aV^3 + bV^2 + cV + d = 0,$$

then Equation (37) with b and c given in (39) is

$$(V')^2 = (V - V_1)(V - V_2)(V - V_3)(V - V_4). \tag{40}$$

By a transformation [6], its solution could be written in terms of the Jacobi elliptic function sn(mq, k), where sn is the elliptic sine function.

- The Weierstrass elliptic function equation was the third example of the simplest equation

$$(V')^2 - 4V^3 - aV^2 - bV - c = 0. \tag{41}$$

If $V(q)$ is a solution of Equation (41), then the equation

$$V'' = 6V^2 + aV + \frac{1}{2}b, \tag{42}$$

has special solutions that are expressed via the general solution of Equation (41). It was proved on similar lines given above.

Equation (15), with $H(V) = \frac{6\gamma(\delta^2+1)(\delta\mu-\nu)}{1-\delta} V$ and $\alpha = 0$, can be written in the same form that Equation (42) by considering

$$\begin{aligned} b &= \frac{2f}{\gamma(\delta^2+1)(\nu - \delta\mu)}, \\ a &= \frac{1}{\gamma(\delta^2+1)}. \end{aligned}$$

Thus, Equation (15) has special solutions that are expressed via the general solution of Weierstrass Equation (41).

5. Conservation Laws

We construct conservation laws for the $(2+1)$-dimensional generalized BBMB Equation (6) by employing the multiplier method [23,34]. First, we determine low-order multipliers

$$Q = Q(t, x, y, u) \tag{43}$$

admitted by the Equation (6). Recall that multipliers $Q(t, x, y, u)$ for the equation under study (6) are obtained from the following equation known as the determining equation

$$\frac{\delta}{\delta u}\Big(\big(u_t - \gamma(u_{txx} + u_{tyy}) - \alpha(u_{xx} + u_{yy}) + G(u_x + u_y) - f\big)Q\Big) = 0, \tag{44}$$

which holds off of the set of solutions of Equation (6) and where $\frac{\delta}{\delta u}$ is the Euler operator with respect to the variable u [20].

Splitting Equation (44) with respect to u and its derivatives, we obtain a determining system depending on the arbitrary parameters α, γ, f and the arbitrary function $G(u)$ which Equation (6) involves. We solve the determining system by using "rifsimp" and "pdsolve" commands in Maple, and we determine the following classification for the low-order multipliers (43). We obtain the following result.

Theorem 4. *The low-order multipliers of differential order zero (43) admitted by the (2+1)-dimensional generalized BBMB Equation (6) depending on the arbitrary constants α, γ, f and an arbitrary function $G(u)$ are given in the cases:*

(i) *For arbitrary f and $G(u)$, and $\alpha \neq 0$, γ verifying $\frac{c}{c\gamma - \alpha} > 0$, with $c \neq 0$ any fixed real constant, the admitted multipliers are:*

$$Q_1 = \exp\left(ct + \frac{1}{\sqrt{2}}\sqrt{\frac{c}{c\gamma - \alpha}}(x-y)\right), \tag{45}$$

$$Q_2 = \exp\left(ct - \frac{1}{\sqrt{2}}\sqrt{\frac{c}{c\gamma - \alpha}}(x-y)\right). \tag{46}$$

(ii) *For arbitrary f and $G(u)$, and $\alpha \neq 0$, γ verifying $\frac{c}{c\gamma - \alpha} < 0$, with $c \neq 0$ any fixed real constant, the admitted multipliers are:*

$$Q_3 = e^{ct}\sin\left(\frac{1}{\sqrt{2}}\sqrt{\frac{-c}{c\gamma - \alpha}}(x-y)\right), \tag{47}$$

$$Q_4 = e^{ct}\cos\left(\frac{1}{\sqrt{2}}\sqrt{\frac{-c}{c\gamma - \alpha}}(x-y)\right). \tag{48}$$

(iii) *For $\alpha = 0$, $\gamma > 0$, arbitrary f and $G(u)$, the admitted multipliers are:*

$$Q_5 = -ft + u, \tag{49}$$

$$Q_6 = f_1(y-x), \tag{50}$$

$$Q_7 = f_2(t)\exp\left(\frac{x-y}{\sqrt{2\gamma}}\right), \tag{51}$$

$$Q_8 = f_3(t)\exp\left(-\frac{x-y}{\sqrt{2\gamma}}\right). \tag{52}$$

(iv) *For $\alpha = 0$, $\gamma < 0$, arbitrary f and $G(u)$, the admitted multipliers are Q_5, Q_6 and*

$$Q_9 = f_4(t)\sin\left(\frac{x-y}{\sqrt{-2\gamma}}\right), \tag{53}$$

$$Q_{10} = f_5(t)\cos\left(\frac{x-y}{\sqrt{-2\gamma}}\right). \tag{54}$$

In the above, $f_1(y-x), f_2(t), f_3(t), f_4(t)$ and $f_5(t)$ are arbitrary functions of their arguments.

A local conservation law for the (2+1)-dimensional generalized BBMB Equation (6) is a divergence expression of the form

$$D_tT(t,x,y,u,u_t,u_x,\dots) + D_xX(t,x,y,u,u_t,u_x,\dots) + D_yY(t,x,y,u,u_t,u_x,\dots) = 0, \tag{55}$$

that holds for the solutions $u(t,x,y)$ of Equation (6), and where D_t, D_x and D_y denote the total derivative operators with respect to t, x and y, respectively. T is a function called conserved density, and X, Y are functions called spatial fluxes. All these functions depend on t, x, y, u and derivatives of u.

Using the multipliers obtained previously (Theorem 4), we can calculate the conservation laws by integrating the following equation, called the characteristic equation,

$$D_tT + D_xX + D_yY = \Big(u_t - \gamma(u_{txx} + u_{tyy}) - \alpha(u_{xx} + u_{yy}) + G(u_x + u_y) - f\Big)Q, \tag{56}$$

or via an homotopy formula [33]. We obtain the following result.

Theorem 5. *Low-order conservation laws (55) admitted by the (2+1)-dimensional generalized BBMB Equation (6) depending on the arbitrary constants α, γ and f and an arbitrary function $G(u)$ are given by:*

(i) *For arbitrary f and $G(u)$, and $\alpha \neq 0$, γ verifying $\dfrac{c}{c\gamma-\alpha} > 0$, with $c \neq 0$ any fixed real constant, the admitted conservation laws are:*

$$
\begin{aligned}
T_1 = &\ \tfrac{\alpha u}{c\gamma-\alpha}\exp\Big(ct+\tfrac{1}{\sqrt{2}}\sqrt{\tfrac{c}{c\gamma-\alpha}}(x-y)\Big),\\
X_1 = &\ \Big(\tfrac{-2\gamma cf+\gamma cu_t+2\alpha f}{\sqrt{2}\sqrt{c(c\gamma-\alpha)}}-\alpha u_x-\gamma u_{tx}+\tfrac{\sqrt{2}}{2}\alpha\sqrt{\tfrac{c}{c\gamma-\alpha}}u\\
&+\int G(u)du\Big)\exp\Big(ct+\tfrac{1}{\sqrt{2}}\sqrt{\tfrac{c}{c\gamma-\alpha}}(x-y)\Big),\\
Y_1 = &\ -\Big(\sqrt{\tfrac{c}{2(c\gamma-\alpha)}}\gamma u_t+\gamma u_{ty}+\alpha u_y+\tfrac{\sqrt{2}}{2}\alpha\sqrt{\tfrac{c}{c\gamma-\alpha}}u\\
&-\int G(u)du\Big)\exp\Big(ct+\tfrac{1}{\sqrt{2}}\sqrt{\tfrac{c}{c\gamma-\alpha}}(x-y)\Big).
\end{aligned}
\tag{57}
$$

$$
\begin{aligned}
T_2 = &\ \tfrac{-\alpha u}{c\gamma-\alpha}\exp\Big(ct-\tfrac{1}{\sqrt{2}}\sqrt{\tfrac{c}{c\gamma-\alpha}}(x-y)\Big),\\
X_2 = &\ \Big(\tfrac{2\gamma cf-\gamma cu_t-2\alpha f}{\sqrt{2}\sqrt{c(c\gamma-\alpha)}}-\alpha u_x-\gamma u_{tx}-\tfrac{\sqrt{2}}{2}\alpha\sqrt{\tfrac{c}{c\gamma-\alpha}}u\\
&+\int G(u)du\Big)\exp\Big(ct-\tfrac{1}{\sqrt{2}}\sqrt{\tfrac{c}{c\gamma-\alpha}}(x-y)\Big),\\
Y_2 = &\ \Big(\sqrt{\tfrac{c}{2(c\gamma-\alpha)}}\gamma u_t-\gamma u_{ty}-\alpha u_y+\tfrac{\sqrt{2}}{2}\alpha\sqrt{\tfrac{c}{c\gamma-\alpha}}u\\
&+\int G(u)du\Big)\exp\Big(ct-\tfrac{1}{\sqrt{2}}\sqrt{\tfrac{c}{c\gamma-\alpha}}(x-y)\Big).
\end{aligned}
\tag{58}
$$

(ii) *For arbitrary f and $G(u)$, and $\alpha \neq 0$, γ verifying $\dfrac{c}{c\gamma-\alpha} < 0$, with $c \neq 0$ any fixed real constant, the admitted conservation laws are:*

$$
\begin{aligned}
T_3 = &\ \tfrac{\alpha ue^{ct}}{c\gamma-\alpha}\sin\Big(\tfrac{1}{\sqrt{2}}\sqrt{\tfrac{-c}{c\gamma-\alpha}}(x-y)\Big),\\
X_3 = &\ e^{ct}\big(\alpha u_x+\gamma u_{tx}-\int G(u)du\big)\sin\Big(\tfrac{1}{\sqrt{2}}\sqrt{\tfrac{-c}{c\gamma-\alpha}}(x-y)\Big)\\
&+\tfrac{e^{ct}}{\sqrt{c(c\gamma-\alpha)}}\Big(-\tfrac{\alpha cu}{2}-\alpha f+\Big(\gamma fc\sqrt{2}-\tfrac{\gamma c\sqrt{2}}{2}u_t\Big)\cos\Big(\tfrac{1}{\sqrt{2}}\sqrt{\tfrac{-c}{c\gamma-\alpha}}(x-y)\Big)\Big),\\
Y_3 = &\ e^{ct}\Big(\tfrac{1}{\sqrt{2}}\sqrt{\tfrac{c}{c\gamma-\alpha}}(\alpha u+\gamma u_t)\cos\Big(\tfrac{1}{\sqrt{2}}\sqrt{\tfrac{-c}{c\gamma-\alpha}}(x-y)\Big)\Big)\\
&+e^{ct}\Big(\big(\gamma u_{ty}+\alpha u_y-\int G(u)du\big)\sin\Big(\tfrac{1}{\sqrt{2}}\sqrt{\tfrac{-c}{c\gamma-\alpha}}(x-y)\Big)\Big).
\end{aligned}
\tag{59}
$$

$$
\begin{aligned}
T_4 = &\ \tfrac{\alpha ue^{ct}}{c\gamma-\alpha}\cos\Big(\tfrac{1}{\sqrt{2}}\sqrt{\tfrac{-c}{c\gamma-\alpha}}(x-y)\Big),\\
X_4 = &\ e^{ct}\big(\alpha u_x+\gamma u_{tx}-\int G(u)du\big)\cos\Big(\tfrac{1}{\sqrt{2}}\sqrt{\tfrac{-c}{c\gamma-\alpha}}(x-y)\Big)\\
&+\tfrac{e^{ct}}{\sqrt{c(c\gamma-\alpha)}}\Big(-\tfrac{\alpha cu}{2}-\alpha f+\Big(\gamma fc\sqrt{2}-\tfrac{\gamma c\sqrt{2}}{2}u_t\Big)\sin\Big(\tfrac{1}{\sqrt{2}}\sqrt{\tfrac{-c}{c\gamma-\alpha}}(x-y)\Big)\Big),\\
Y_4 = &\ e^{ct}\Big(\tfrac{1}{\sqrt{2}}\sqrt{\tfrac{c}{c\gamma-\alpha}}(\alpha u+\gamma u_t)\sin\Big(\tfrac{1}{\sqrt{2}}\sqrt{\tfrac{-c}{c\gamma-\alpha}}(x-y)\Big)\Big)\\
&+e^{ct}\Big(\big(\gamma u_{ty}+\alpha u_y-\int G(u)du\big)\cos\Big(\tfrac{1}{\sqrt{2}}\sqrt{\tfrac{-c}{c\gamma-\alpha}}(x-y)\Big)\Big).
\end{aligned}
\tag{60}
$$

(iii) *For $\alpha = 0$, $\gamma > 0$, arbitrary f and $G(u)$, the admitted conservation laws are:*

$$
\begin{aligned}
T_5 = &\ \tfrac{\gamma\left(u_x^2+u_y^2\right)}{2}+\tfrac{(ft-u)^2}{2},\\
X_5 = &\ \gamma(ft-u)u_{tx}-\int G(ft-u)du,\\
Y_5 = &\ \gamma(ft-u)u_{ty}+\int G(ft-u)du.
\end{aligned}
\tag{61}
$$

$$
\begin{aligned}
T_6 = &\ uf_1(y-x)-2\gamma u\tfrac{d^2}{d(y-x)^2}f_1(y-x),\\
X_6 = &\ -\gamma u_t\tfrac{d}{d(y-x)}f_1(y-x)-f\int f_1(y-x)dx-\gamma u_{tx}f_1(y-x)+f_1(y-x)\int G(u)du,\\
Y_6 = &\ \gamma u_t\tfrac{d}{d(y-x)}f_1(y-x)-\gamma u_{ty}f_1(y-x)+f_1(y-x)\int G(u)du.
\end{aligned}
\tag{62}
$$

$$
\begin{aligned}
T_7 = &\ -f\exp\Big(\tfrac{x-y}{\sqrt{2\gamma}}\Big)\int f_2(t)dt,\\
X_7 = &\ f_2(t)\Big(\tfrac{\sqrt{\gamma}}{\sqrt{2}}u_t-\gamma u_{tx}+\int G(u)du\Big)\exp\Big(\tfrac{x-y}{\sqrt{2\gamma}}\Big),\\
Y_7 = &\ f_2(t)\Big(\tfrac{-\sqrt{\gamma}}{\sqrt{2}}u_t-\gamma u_{ty}+\int G(u)du\Big)\exp\Big(\tfrac{x-y}{\sqrt{2\gamma}}\Big).
\end{aligned}
\tag{63}
$$

$$\begin{aligned}
T_8 = &\quad -f \exp\left(-\tfrac{x-y}{\sqrt{2\gamma}}\right) \int f_3(t)dt, \\
X_8 = &\quad f_3(t)\left(-\tfrac{\sqrt{\gamma}}{\sqrt{2}}u_t - \gamma u_{tx} + \int G(u)du\right) \exp\left(-\tfrac{x-y}{\sqrt{2\gamma}}\right), \\
Y_8 = &\quad f_3(t)\left(\tfrac{\sqrt{\gamma}}{\sqrt{2}}u_t - \gamma u_{ty} + \int G(u)du\right) \exp\left(-\tfrac{x-y}{\sqrt{2\gamma}}\right).
\end{aligned} \tag{64}$$

(iv) *For $\alpha = 0$, $\gamma < 0$, arbitrary f and $G(u)$, the admitted conservation laws are:*

$$\begin{aligned}
T_9 = &\quad ((u_{xx} + u_{yy})\gamma - u) \cos\left(\tfrac{x-y}{\sqrt{-2\gamma}}\right), \\
X_9 = &\quad \left(\gamma f_{4_t}(t) - \alpha f_4(t)\right) u_x \sin\left(\tfrac{x-y}{\sqrt{-2\gamma}}\right) + \left(\sqrt{\tfrac{\gamma}{2}} f_{4_t}(t) + \tfrac{\alpha}{\sqrt{2\gamma}}\right) u \cos\left(\tfrac{x-y}{\sqrt{-2\gamma}}\right) \\
&+ f_4 \int G\, du \sin\left(\tfrac{x-y}{\sqrt{-2\gamma}}\right) + f\sqrt{2\gamma} f_4(t) \cos\left(\tfrac{x-y}{\sqrt{-2\gamma}}\right), \\
Y_9 = &\quad \left(\gamma f_{4_t}(t) + \alpha f_4(t)\right) u_y \sin\left(\tfrac{x-y}{\sqrt{-2\gamma}}\right) - \left(\sqrt{\tfrac{\gamma}{2}} f_{4_t} + \tfrac{\alpha}{\sqrt{2\gamma}} f_4\right) u \cos\left(\tfrac{x-y}{\sqrt{-2\gamma}}\right) \\
&+ f_4 \int G\, du \sin\left(\tfrac{x-y}{\sqrt{-2\gamma}}\right).
\end{aligned} \tag{65}$$

$$\begin{aligned}
T_{10} = &\quad -((u_{xx} + u_{yy})\gamma - u) \sin\left(\tfrac{x-y}{\sqrt{-2\gamma}}\right), \\
X_{10} = &\quad \left(\gamma f_{3_t}(t) - \alpha f_3(t)\right) u_x \cos\left(\tfrac{x-y}{\sqrt{-2\gamma}}\right) - \left(\sqrt{\tfrac{\gamma}{2}} f_{3_t}(t) + \tfrac{\alpha}{\sqrt{2\gamma}}\right) u \sin\left(\tfrac{x-y}{\sqrt{-2\gamma}}\right) \\
&+ f_4 \int G\, du \cos\left(\tfrac{x-y}{\sqrt{-2\gamma}}\right) + f\sqrt{2\gamma} f_4(t) \sin\left(\tfrac{x-y}{\sqrt{-2\gamma}}\right), \\
Y_{10} = &\quad \left(\gamma f_{3_t}(t) + \alpha f_3(t)\right) u_y \cos\left(\tfrac{x-y}{\sqrt{-2\gamma}}\right) + \left(\sqrt{\tfrac{\gamma}{2}} f_{3_t} + \tfrac{\alpha}{\sqrt{2\gamma}} f_3\right) u \sin\left(\tfrac{x-y}{\sqrt{-2\gamma}}\right) \\
&+ f_3 \int G\, du \cos\left(\tfrac{x-y}{\sqrt{-2\gamma}}\right).
\end{aligned} \tag{66}$$

In the above, $f_1(y-x)$, $f_2(t)$, $f_3(t)$, $f_4(t)$ and $f_5(t)$ are arbitrary functions of their arguments.

6. Line Soliton Solution

Consider $G(u) = g_1 u^n + g_2$, $\alpha = f = 0$, where $n \neq -2, -1, 0$, $g_1 \neq 0$ and g_2 are arbitrary constants, and the two-dimensional subalgebra spanned by

$$X = X_1 + \lambda X_2 \qquad \text{and} \qquad Y = X_3 - \mu X_2, \tag{67}$$

which satisfies $[X, Y] = 0$. By applying the abelian subalgebra (67), the (2+1)-dimensional generalized BBMB Equation (6) can be reduced into a nonlinear third-order ODE through the use of two independent invariants z and h satisfying

$$X z = 0, \qquad X h = 0, \qquad Y z = 0, \qquad Y h = 0. \tag{68}$$

From (68), one obtains

$$z = x + \mu y - \lambda t, \qquad h = u. \tag{69}$$

The group invariant solution

$$u = h(x + \mu y - \lambda t), \tag{70}$$

is a line travelling wave. The amplitude h of a line travelling wave is invariant under translations in the perpendicular direction. The solution (70) depends on two parameters, $\mu = \tan\phi$ shows the direction of the wave propagation, i.e., the inclination of the line travelling wave in the (x, y)-plane, with ϕ the angle from the positive y-axis in counter-clockwise direction, and where $c = \frac{\lambda}{1+\mu^2}$ is the speed of the line wave. We are interested in line travelling waves whose amplitude exhibits exponential asymptotic decay for large $|z|$. These line travelling waves are known as line solitons. The group-invariant solution (70) transforms Equation (6) into the nonlinear ODE given by

$$\gamma\lambda(1 + \mu^2)h''' + \left((1 + \mu)(g_1 h^n + g_2) - \lambda\right)h' = 0. \tag{71}$$

For arbitrary n, λ and μ, ODE (71) only admits the obvious point symmetry (invariance under translations in z)

$$V = \partial_z. \tag{72}$$

The differential invariants corresponding to the infinitesimal generator V (72) are

$$\omega = h \quad \text{and} \quad \chi = h'. \tag{73}$$

After substituting invariants (73) into the third-order ODE (71) one finds that $\chi(\omega)$ verifies the second-order ODE

$$\gamma\lambda(1+\mu^2)\left(\chi\chi'' + \chi'^2\right) + (1+\mu)(g_1\omega^n + g_2) - \lambda = 0. \tag{74}$$

Fortunately, Equation (74) can be readily integrated to yield

$$\gamma\lambda(1+\mu^2)\chi\chi' + \left((1+\mu)\left(\frac{g_1}{n+1}\omega^n + g_2\right) - \lambda\right)\omega - k_1 = 0, \tag{75}$$

with k_1 arbitrary constant. Integrating again Equation (75) with respect to ω, we obtain the general solution of Equation (74), which is given by

$$\chi^2 = \frac{1}{\gamma\lambda(1+\mu^2)}\left(k_2 + 2k_1\omega + (\lambda - g_2(1+\mu))\omega^2 - \frac{2(1+\mu)g_1}{(n+1)(n+2)}\omega^{n+2}\right). \tag{76}$$

We recall that we look for a localized smooth solution such that h, h', $h'' \longrightarrow 0$ as $z \longrightarrow \pm\infty$. This implies $k_1 = k_2 = 0$. Undoing change of variables (73), one obtains

$$h' = \pm h\sqrt{\frac{1}{\gamma\lambda(1+\mu^2)}\left(\lambda - (1+\mu)g_2 - \frac{2(1+\mu)g_1}{(n+1)(n+2)}h^n\right)}. \tag{77}$$

Equation (77) is a separable first-order ODE. Suppose $\mu \neq -1$, after separating and integrating, one obtains

$$h(z) = \left(\frac{(n+1)(n+2)(\lambda-(1+\mu)g_2)}{2g_1(1+\mu)}\operatorname{sech}^2\left(\frac{n}{2}\sqrt{\frac{\lambda-(1+\mu)g_2}{\gamma\lambda(1+\mu^2)}}(z+z_0)\right)\right)^{1/n}, \tag{78}$$

where z_0 is an arbitrary constant. Note that $\operatorname{sech}^2(\phi)$ is smooth in ϕ and vanishes as $\phi \longrightarrow \pm\infty$.

Thus, for

$$\frac{\lambda - (1+\mu)g_2}{\gamma\lambda} > 0$$

we have Equation (78) is a smooth solution in z which asymptotically decays to 0 as $z \longrightarrow \pm\infty$. Finally, undoing change of variable (69) one obtains a line soliton solution for PDE (6) given by

$$u(t,x,y) = \left(\frac{(n+1)(n+2)(\lambda-(1+\mu)g_2)}{2g_1(1+\mu)}\operatorname{sech}^2\left(\frac{n}{2}\sqrt{\frac{\lambda-(1+\mu)g_2}{\gamma\lambda(1+\mu^2)}}(x+\mu y-\lambda t+z_0)\right)\right)^{1/n}. \tag{79}$$

In Figure 1, it is represented the line soliton solution (79) considering $\mu = 1$, $\gamma = 1$, $\lambda = 2$, $n = 1$, $g_1 = 3$, $g_2 = -1$ and $z_0 = 0$, for some fixed values of t. Here, it can be seen how the amplitude of the line soliton presents exponential asymptotic decay for large $|x + \mu y - \lambda t|$.

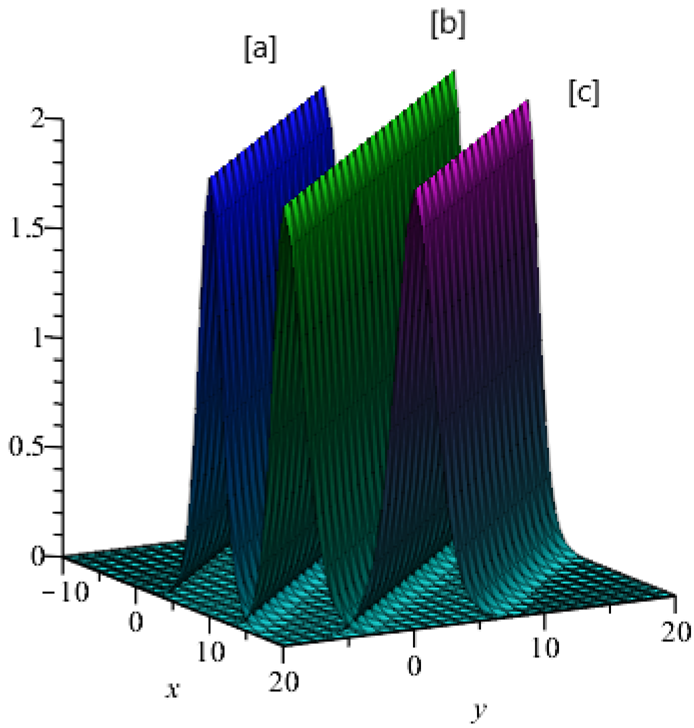

Figure 1. Line soliton solutions (79) for some fixed values of t, with $\mu = 1$, $\gamma = 1$, $\lambda = 2$, $n = 1$, $g_1 = 3$, $g_2 = -1$ and $z_0 = 0$: [a] $t = 0$ (blue), [b] $t = 5$ (green), [c] $t = 10$ (purple).

7. Conclusions

In this paper, we have classified the point symmetries admitted by the generalized family of BBMB equations (6). We have determined the most general symmetry Lie algebra along with its non-zero commutator structure that PDE (6) admits depending on three arbitrary parameters α, γ and f, and the arbitrary function $G(u)$. We have also constructed an optimal system of one-dimensional subalgebras for each maximal Lie algebra admitted by the considered equation. By using the subalgebras of this optimal system, we determine similarity variables and similarity solutions, which allow us to transform the generalized (2+1)-dimensional BBMB (6) into (1+1)-dimensional nonlinear third-order PDEs. Moreover, taking into account the symmetries of the reduced equations, we transform the considered family to nonlinear third-order ODEs. We have explored several exact solutions for this family. On the other hand, we have determined a classification of admitted low-order conservation laws of PDE (6) depending on the arbitrary parameters and the arbitrary function. Finally, line soliton solutions have been obtained in a particular case.

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