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Lie Symmetries and Conservation Laws for the Viscous Cahn-Hilliard Equation

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Abstract: In this paper, we study a viscous Cahn–Hilliard equation from the point of view of Lie symmetries in partial differential equations. The analysis of this equation is motivated by its applications since it serves as a model for many problems in physical chemistry, developmental biology, and population movement. Firstly, a classification of the Lie symmetries admitted by the equation is presented. In addition, the symmetry transformation groups are calculated. Afterwards, the partial differential equation is transformed into ordinary differential equations through symmetry reductions. Secondly, all low-order local conservation laws are obtained by using the multiplier method. Furthermore, we use these conservation laws to determine their associated potential systems and we use them to investigate nonlocal symmetries and nonlocal conservation laws. Finally, we apply the multi-reduction method to reduce the equation and find a soliton solution.

Keywords: Cahn–Hilliard equation; conservation laws; exact invariant solution; Lie symmetry method; multi-reduction method; symmetries; potential systems; viscous equation

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1. Introduction

Many nonlinear phenomena in fundamental sciences (physics, chemistry, biology, ...) are conveniently modelled by nonlinear partial differential equations (PDEs). Despite a powerful mathematical research activity carried out in recent years, the extremely diversified field of nonlinear PDEs still has a great number of difficult problems opened.

An important difference with respect to the linear PDEs lies in the fact that there is no general method to find explicit solutions for nonlinear PDEs. Consequently, nonlinear PDEs are strongly connected with the development of numerical methods. However, the application of the theory of Lie transformation groups plays an important role in nonlinear PDEs. Essentially, the basis of this technique is that, if a PDE is invariant under a Lie group of transformations, then a reduction in the number of independent variables exists. If the reduced equation is an ordinary differential equation (ODE) and a solution of this equation is found, then a solution of the original PDE can be recovered.

The Cahn–Hilliard equation, developed by John W. Cahn and John E. Hilliard, is an equation of mathematical physics describing the process of phase separation. This equation also appears in modelling other phenomena, such as the dynamics of two populations, the biomedical modelling of a bacterial film, and some thin film problems [1]. Nevertheless, many papers can be found in the literature studying physical models. For example, in [2] the authors construct a new type of solution for a $(2+1)$-dimensional nonlinear system of Schrödinger equations. There are also papers studying numerical solutions and comparing them with exact solutions, such as [3] for the Korteweg–de Vries (KdV) equation.

Specifically, several papers have been published studying the Cahn–Hilliard equation. For instance, in [4] the authors investigated different singular limits and convergence of an initial-boundary value problem of the viscous Cahn–Hilliard equation

$$\nu u_t = \Delta[\varphi(u) - \alpha\Delta u + \beta u_t]\,.$$

This equation is presented in a lower dimensional setting as

$$\nu u_t = (\varphi(u) - \alpha u_{xx} + \beta u_t)_{xx}, \tag{1}$$

where $u = u(x,t)$ is the solute concentration at point x and time t, $\varphi(u)$ is a "homogeneous free energy", $\alpha/2$ is the gradient energy coefficient describing the contribution of the diffuse interface to the decomposition, and ν, α, β are constants. Particularly, βu_{txx} is named the viscous term because of the fact that diffusion theory in physics has a second derivative with respect to x and a viscous constant, β.

If $\nu = 1$ and $\beta = 0$, then Equation (1) becomes the well-known Cahn–Hilliard equation, given by

$$u_t = (\varphi(u) - \alpha u_{xx})_{xx}.$$

This equation was introduced to study phase separation in binary alloy glasses and polymers. It is a good approach to spinodal decomposition [5].

An equation related to (1) was studied in [6,7], obtaining the Lie symmetries, solutions, and using the nonclassical method for solutions of the Cahn–Hilliard flux equation

$$u_t + k u_{xxxx} - (f(u)\, u_x)_x = 0,$$

describing diffusion for the decomposition of a one-dimensional binary solution. This equation is a specific case of the viscous Cahn–Hilliard Equation (1), without the viscous term.

Without loss of generality we set $\alpha = 1$ in Equation (1), yielding

$$\nu u_t = (\varphi(u) - u_{xx} + \beta u_t)_{xx}. \tag{2}$$

The present paper is devoted to studying Equation (2), focusing on symmetries, conservation laws, potential systems, and exact invariant solutions.

In the nineteenth century, Sophus Lie started to investigate the continuous groups of transformations leaving invariant the differential equations. By using the invariance condition, he developed what is now named the symmetry analysis of differential equations. The original idea was creating a general theory for the integration of ODEs, as E. Galois and N. Abel did for algebraic Equations [8]. This theory allows to obtain solutions of differential equations in an algorithmic way. Since 1960, researchers started to apply the methods of symmetry analysis of differential equations to find solutions of any problem of mathematical physics.

Lie's theory of symmetry analysis of differential equations is based on the invariance of a differential equation under a transformation of independent and dependent variables. This transformation generates a local group of point transformations with a diffeomorphism on the space of independent and dependent variables, that enables to map solutions of the equation to other solutions. Specifically, by using the Lie symmetry method, symmetries can be used to find exact invariant solutions.

A very important result of symmetry in physics and in mathematics is the existence of conservation laws. In 1918, Emmy Noether proved her well-known theorem connecting continuous symmetries and conservation laws. Furthermore, in [9] Anco and Bluman gave a general algorithmic method to find all local conservations laws for PDEs with any number of independent and dependent variables, called the multiplier method. The main advantage of this method is that does not require the use or existence of a variational principle and reduces the computation of conservation laws to solving a system of determining equations, similar to that for obtaining symmetries.

Conservation laws describe the time evolution of certain conserved quantities in physical models, such as mass, momentum, energy, and electric charge. Among its applications, it is important to point out the detection of the integrability and the existence, uniqueness, and stability of solutions of differential equations. Some papers have been published over the past few decades computing conservation laws by using the multiplier method. For instance, in [10–12] the authors apply the multiplier method to different types of wave equations in order to find conservation laws. In [13] a dispersive equation based on the well-known KdV equation is studied obtaining as well conservation laws by the application of this method. In the same way, conservation laws are determined in [14] for a general nonlinear diffusion-reaction equation.

In relation with the symmetry reductions, for any ODE arising from a symmetry reduction in an equation, which possesses a Lagrangian, the corresponding conservation law of the equation will similarly reduce to a first integral of the ODE. This happens as a consequence of a general result for symmetry reduction in Euler–Lagrange PDEs [15–17]. In addition, by applying the multi-reduction method [18], all first integrals arising from symmetry invariant conservation laws can be found directly, by using the symmetry.

In this work, we solve a group classification problem for Equation (2), by studying different expressions of function $\varphi(u)$ for which Lie symmetries are admitted. The symmetry groups are found through consistent applications of the Lie group formalism. Furthermore, we determine all low-order local conservation laws of Equation (2) by using the multiplier method. Moreover, we study their corresponding potential systems to obtain nonlocal symmetries and nonlocal conservation laws. Additionally, the original PDE is reduced to ODEs through symmetry reductions and by using the multi-reduction method, an exact invariant solution is found.

This article is structured as follows. Firstly, in Section 2, we classify the Lie symmetries admitted by Equation (2) and calculate the symmetry transformation groups, depending on $\varphi(u)$. In Section 3, we apply the multiplier method to Equation (2) in order to find all low-order conservation laws and we also study the potential systems to find nonlocal symmetries and nonlocal conservation laws. In addition, in Section 4 the multi-reduction method is applied to reduce the equation and obtain an exact invariant solution. Finally, in Section 5, we give some final concluding remarks on the results.

2. Lie Symmetries

First of all, we define an infinitesimal point symmetry as a set of vector fields corresponding to the associated Lie algebra of infinitesimal symmetries, defined by

$$X = \xi(x,t,u)\partial_x + \tau(x,t,u)\partial_t + \eta(x,t,u)\partial_u, \tag{3}$$

where $\xi(x,t,u)$, $\tau(x,t,u)$, $\eta(x,t,u)$ are the infinitesimals. We recall that a point transformation group is a point symmetry of Equation (2) if, and only if, the action of the group leaves invariant the solution space. Therefore, point symmetries are obtained by applying the symmetry invariance condition, given by

$$\mathrm{pr}^{(4)}X(\nu u_t - (\varphi(u) + u_{xx} - \beta u_t)_{xx})|_{\mathcal{E}} = 0,$$

where $\mathcal{E}$ is the solution space of Equation (2).

Each infinitesimal generator (3) is associated with a transformation, obtained by solving the system of ODEs

$$\frac{\partial \hat{x}}{\partial \epsilon} = \xi(\hat{x},\hat{t},\hat{u}), \qquad \frac{\partial \hat{t}}{\partial \epsilon} = \tau(\hat{x},\hat{t},\hat{u}), \qquad \frac{\partial \hat{u}}{\partial \epsilon} = \eta(\hat{x},\hat{t},\hat{u}), \tag{4}$$

verifying the initial conditions

$$\hat{x}|_{\epsilon=0} = x, \qquad \hat{t}|_{\epsilon=0} = t, \qquad \hat{u}|_{\epsilon=0} = u, \tag{5}$$

where ϵ is the group parameter.

Fourth-order prolongations involve a great number of calculations that can be avoided by using a geometrical way [19,20]. Every infinitesimal point symmetry can also be expressed by its characteristic form

$$\hat{X} = \hat{\eta}\,\partial_u, \quad \hat{\eta} = \eta(x,t,u) - \tau(x,t,u)u_t - \xi(x,t,u)u_x,$$

where $\hat{\eta}$ is called the characteristic.

Then, by applying the invariance condition

$$\mathrm{pr}^{(4)}\hat{X}(\nu u_t - (\varphi(u) + u_{xx} - \beta u_t)_{xx})|_{\mathcal{E}} = 0, \tag{6}$$

the infinitesimal point symmetries can be determined. Here, $pr^{(4)}$ represents the fourth-order prolongation of the generator $\hat{X}$ defined by

$$\begin{aligned}\mathrm{pr}^{(4)}\hat{X} \quad = \quad & \hat{X} + (D_x\hat{\eta})\,\partial_{u_x} + (D_t\hat{\eta})\,\partial_{u_t} + (D_x^2\hat{\eta})\,\partial_{u_{xx}} + (D_t^2\hat{\eta})\,\partial_{u_{tt}} \\ & + (D_x^3\hat{\eta})\,\partial_{u_{xxx}} + (D_t^3\hat{\eta})\,\partial_{u_{ttt}} + (D_x^4\hat{\eta})\,\partial_{u_{xxxx}} + (D_t^4\hat{\eta})\,\partial_{u_{tttt}} + \cdots .\end{aligned}$$

The symmetry determining Equation (6) splits with respect to the x-derivatives and t-derivatives of u yielding an overdetermined linear system of equations for the infinitesimals. Here the software Maple is used to compute the determining equations and afterwards, functions "rifsimp", "dsolve", and "pdsolve" are applied to solve the system. Specifically, "rifsimp" returns a tree with all the solution cases and then, for each solution case, function "dsolve" is applied to find function $\varphi(u)$ and "pdsolve" to find the infinitesimals $\xi(x,t,u)$, $\tau(x,t,u)$, and $\eta(x,t,u)$.

The results achieved on point symmetries are classified in the following cases:

1. For $\varphi(u)$ an arbitrary function,

$$X_1 = \partial_x, \quad X_2 = \partial_t. \tag{7}$$

2. For $\beta = 0$ and $\varphi(u) = a(u+c)^p + k$, where a, c, p, k are constants, with $p \neq 0, 1$, besides X_1 and X_2, Equation (2) admits an extra symmetry,

$$X_3^1 = x\,\partial_x + 4t\,\partial_t - \left(\frac{2u}{p-1} + \frac{2c}{p-1}\right)\partial_u. \tag{8}$$

3. For $\beta = 0$ and $\varphi(u) = ae^{pu} + k$, where a, p, k are constants, with $p \neq 0$, besides X_1 and X_2, Equation (2) admits an additional symmetry,

$$X_3^2 = x\,\partial_x + 4t\,\partial_t - \frac{2}{p}\partial_u. \tag{9}$$

In the above classification, $\nu \neq 0$ and $\varphi(u)$ is considered a nonlinear function. Next, by solving the system of ODEs (4), with the initial conditions (5), we obtain the following symmetry transformation groups:

1. For $\varphi(u)$ an arbitrary function, a space-translation and a time-translation,

$$\begin{aligned}(x,t,u) \quad & \rightarrow \quad (x+\epsilon, t, u), \\ (x,t,u) \quad & \rightarrow \quad (x, t+\epsilon, u).\end{aligned}$$

2. For $\beta = 0$ and $\varphi(u) = a(u+c)^p + k$, where a, c, p, k are constants, with $p \neq 0, 1$, a scaling and shift in u,

$$(x,t,u) \rightarrow (xe^{\epsilon}, te^{4\epsilon}, e^{-\frac{2\epsilon}{p-1}}(u+c) - c).$$

3. For $\beta = 0$ and $\varphi(u) = ae^{pu} + k$, where a, p, k are constants, with $p \neq 0$, a dilation,

$$(x,t,u) \rightarrow (te^{4\epsilon}, xe^{\epsilon}, -\frac{2\epsilon}{p} + u).$$

In the above classification, we also consider $\nu \neq 0$ and $\varphi(u)$ a nonlinear function.

The notion of Lie symmetries includes point symmetries and contact symmetries for PDEs with one dependent variable u [19–22].

A contact symmetry comes from considering an extension of the Lie symmetry method. For this type of symmetries, the symmetry transformation depends on x, t, u, and also on the first-order derivatives of u. The characteristic form of the infinitesimal generator is

$$\hat{X} = P(x,t,u,u_x,u_t)\partial_u$$

and the corresponding infinitesimal transformation is given by

$$P = \tau\,\partial_t + \xi\,\partial_x + \eta\,\partial_u + \eta^t\,\partial_{u_t} + \eta^x\,\partial_{u_x}.$$

Finally, we can state that there are not any contact symmetries admitted by the viscous Cahn–Hilliard Equation (2), with $\nu \neq 0$ and $\varphi(u)$ a nonlinear function.

Symmetry Reductions

Now, we use the symmetries obtained previously to compute the similarity variables and similarity solutions, in order to reduce the viscous Cahn–Hilliard Equation (2) to ODEs.

The similarity variable and the similarity solution are found by using the characteristic equations

$$\frac{dx}{\xi} = \frac{dt}{\tau} = \frac{du}{\eta}.$$

Hence, a reduction is obtained and substituting the variables into Equation (2), it can be reduced to an ODE.

From the first case with the space-translation and time-translation (7), we determine a travelling wave variable and solution for the subalgebra $\lambda X_1 + X_2$,

$$z = x - \lambda t, \quad u = h(z), \tag{10}$$

where $h(z)$ satisfies the travelling wave reduction, given by the fourth-order nonlinear ODE

$$h'''' + \lambda\beta h''' - \varphi'\,h'' - \varphi''(h')^2 - \lambda\nu h' = 0. \tag{11}$$

From the second case with the scaling and shift transformation in u (8), we obtain the similarity variable and solution for the subalgebra X_3^1,

$$z = \tfrac{x}{t^{1/4}}, \qquad u = h(z)\,t^{\frac{1}{-2p+2}} - c,$$

where $h(z)$ verifies the fourth-order nonlinear ODE

$$(p-1)h'''' - ap(p-1)h^{p-1}h'' - ap(p-1)^2h^{p-2}(h')^2 + \frac{\nu(p-1)}{4}z\,h' - \frac{\nu}{2}h = 0.$$

From the third case with the dilation (9), we find the similarity variable and solution for the subalgebra X_3^2,

$$z = \tfrac{x}{t^{1/4}}, \qquad u = h(z) - \tfrac{\ln(t)}{2p},$$

where $h(z)$ verifies the fourth-order nonlinear ODE

$$4ph'''' - 4ap^2e^{ph}(h'' + p(h')^2) - \nu pz\,h' - 2\nu = 0.$$

3. Low-Order Conservation Laws

Local conservation laws are continuity equations that yield basic conserved quantities for all solutions [19,23].

A local conservation law of the viscous Cahn–Hilliard Equation (2) is a space-time divergence expression

$$D_tT + D_xX|_{\mathcal{E}} = 0, \tag{12}$$

holding on the solution space $\mathcal{E}$ of Equation (2), where the conserved density T and the flux X are functions of x, t, u, and its derivatives. Here D_t and D_x denote total derivatives with respect to t and x.

Each local conservation law (12) has an equivalent characteristic form in terms of Q that holds as an identity off of the solution space $\mathcal{E}$ of Equation (2). By using this identity, a general method using multipliers was constructed to find all local conservation laws. Specifically, this method, called the multiplier method, is reduced to solving a linear system of determining equations, as in the Lie symmetry method [24,25].

The characteristic form of a local conservation law admitted by Equation (2), is given by

$$D_t T + D_x(X - \Psi) = (\nu u_t - (\varphi(u) + u_{xx} - \beta u_t)_{xx})Q, \tag{13}$$

where $\Psi|_{\mathcal{E}} = 0$.

If (13) is evaluated on the solution space $\mathcal{E}$, then it reduces to the local conservation law, since the flux term $\Psi|_{\mathcal{E}} = 0$ is trivial, and

$$Q = E_u(T) \tag{14}$$

is a function of x, t, u, and its derivatives, called the multiplier of the conservation law, where

$$E_u = \partial_u - D_t\partial_{u_t} - D_x\partial_{u_x} + D_t^2\partial_{u_{tt}} + D_tD_x\partial_{u_{tx}} + D_x^2\partial_{u_{xx}} - \cdots$$

is the Euler operator with respect to u. This operator has the important property that annihilates total derivatives.

The relation (14) between Q and T proves that all locally equivalent conservation laws have the same multiplier. Hence, there is a one-to-one correspondence between multipliers and conserved densities through the characteristic equation.

A function Q is a multiplier if, and only if,

$$(\nu u_t - (\varphi(u) + u_{xx} - \beta u_t)_{xx})Q \tag{15}$$

is a total divergence.

All multipliers can be determined by applying the Euler operator to the total divergence (15), due to its property of converting to zero any total derivative. This yields the determining equation

$$E_u((\nu u_t - (\varphi(u) + u_{xx} - \beta u_t)_{xx})Q) = 0.$$

The multiplier determining equation splits with respect to u_{tt}, u_{ttx}, u_{txx}, u_{ttxx}, u_{txxx}, u_{xxxx}, u_{txxxx}, u_{xxxxx}, u_{txxxxx}, u_{xxxxxx} yielding an overdetermined linear system of equations to the multiplier $Q(x,t,u,u_x,u_t,u_{xx},u_{tx},u_{xxx})$ among with $\varphi(u)$.

The solutions

$$Q(x,t,u,u_x,u_t,u_{xx},u_{tx},u_{xxx})$$

of the determining system are all the conservation law multipliers for the viscous Cahn–Hilliard Equation (2).

The local conservation laws of physical interest typically arise from low-order multipliers. The general expression of a low-order multiplier Q, in terms of u and its derivatives, is given by those variables that can be differentiated to determine a leading derivative of the given PDE. The leading derivatives of Equation (2) are u_{txx} and u_{xxxx}.

Clearly, u_{txx} can be obtained by differentiation of u, u_t, u_{tx}, and u_{xxxx} by differentiation of u, u_x, u_{xx}, and u_{xxx}. Consequently, the general form for a low-order multiplier for the viscous Cahn–Hilliard Equation (2) is

$$Q(x,t,u,u_x,u_t,u_{xx},u_{tx},u_{xxx}).$$

Therefore, we obtain as result that the low-order multipliers admitted by the viscous Cahn–Hilliard Equation (2), with $\nu \neq 0$ and $\varphi(u)$ a nonlinear function, are given by

$$Q_1 = 1, \quad Q_2 = x.$$

Each low-order multiplier Q corresponds to a conserved density T and flux X through the characteristic Equation (13). All low-order local conservation laws have the general form

$$T(x,t,u), \quad X(x,t,u,u_x,u_t,u_{xx},u_{tx},u_{xxx}).$$

Given a multiplier Q, we can obtain the conserved density T by using a standard method

$$T = \int_0^1 u\, Q(x,t,Qu,Qu_x,Qu_{xx},\dots)dQ$$

and the flux X by using the expression

$$\begin{aligned} X &= -D_x^{-1}(Q(\nu u_t - (\varphi(u) + u_{xx} - \beta u_t)_{xx}) - \frac{\partial T}{\partial u_x}(\nu u_t - (\varphi(u) + u_{xx} - \beta u_t)_{xx}) \\ &\quad + (\nu u_t - (\varphi(u) + u_{xx} - \beta u_t)_{xx}) D_x\left(\frac{\partial T}{\partial u_{xx}}\right) + \cdots . \end{aligned}$$

For the following conservation laws classification we apply the same Maple commands used for the Lie symmetries classification. We have applied "rifsimp" to find each solution case of the system formed by the multiplier determining equations. Afterwards, "dsolve" and "pdsolve" have been applied to solve each solution case, finding the corresponding multipliers. Finally, for each multiplier, we yield its conserved density T and flux X by using the expressions defined above.

As result, we find that the low-order local conservation laws admitted by the viscous Cahn–Hilliard Equation (2), with $\nu \neq 0$ and $\varphi(u)$ a nonlinear function, are the following:

1. For the low-order multiplier $Q_1 = 1$, the low-order local conservation law is

$$\begin{aligned} T_1 &= \nu u, \\ X_1 &= -\varphi'(u)\, u_x - \beta u_{tx} + u_{xxx}. \end{aligned} \tag{16}$$

2. For the low-order multiplier $Q_2 = x$, the low-order local conservation law is

$$\begin{aligned} T_2 &= \nu x\, u, \\ X_2 &= -x\,\varphi'(u)\, u_x + \varphi(u) - (\beta u_{tx} - u_{xxx})x - u_{xx} + \beta u_t. \end{aligned} \tag{17}$$

Potential Systems and Symmetries

From conservation laws, by using the corresponding conserved (potential) systems, we can obtain potential symmetries of Equation (2).

Firstly, for the low-order conservation laws (16) and (17) admitted by the viscous Cahn–Hilliard Equation (2), we determine their corresponding potential systems [23].

From the first low-order conservation law (16), with multiplier $Q_1 = 1$, the potential system is

$$\begin{aligned} v_x - \nu u &= 0, \\ v_t - \varphi'(u)\, u_x - \beta u_{tx} + u_{xxx} &= 0. \end{aligned} \tag{18}$$

From the second low-order conservation law (17), with multiplier $Q_2 = x$, the potential system is

$$\begin{aligned} v_x - \nu x\, u &= 0, \\ v_t - x\,\varphi'(u)\, u_x + \varphi(u) - (\beta u_{tx} - u_{xxx})x - u_{xx} + \beta u_t &= 0. \end{aligned} \tag{19}$$

These potential systems yield a further potential system by substituting u for an expression in terms of v_x,

$$v_t - \frac{\varphi'\left(\frac{v_x}{\nu}\right) v_{xx}}{\nu^2} - \frac{\beta v_{txx}}{\nu} + \frac{v_{xxxx}}{\nu} = 0 \tag{20}$$

and

$$v_t - \frac{\varphi'\left(\frac{v_x}{\nu x}\right) v_{xx}}{\nu^2 x} + \varphi\left(\frac{v_x}{\nu x}\right) - \left(\frac{\beta v_{txx}}{\nu x} - \frac{v_{xxxx}}{\nu x}\right)x - \frac{v_{xxx}}{\nu x} + \frac{\beta v_{tx}}{\nu x} = 0. \tag{21}$$

Afterwards, we classify all the potential symmetries of these potential systems.

A point symmetry of the potential systems (18) and (19), depending on u and v, is a one-parameter Lie group of transformations on (x,t,u,v) generated by a vector field, given by

$$Y = \xi(x,t,u,v)\partial_x + \tau(x,t,u,v)\partial_t + \eta(x,t,u,v)\partial_u + \phi(x,t,u,v)\partial_v, \tag{22}$$

which is required to leave invariant the solution space of the corresponding potential system.

The projection of a point symmetry, defined by (22), to the associated potential Equations (20) and (21) is

$$Y = \xi(x,t,v_x,v)\partial_x + \tau(x,t,v_x,v)\partial_t + \eta(x,t,v_x,v)\partial_{v_x} + \phi(x,t,v_x,v)\partial_v,$$

and generates a one-parameter Lie group of contact transformations on (x,t,v_x,v).

A projected symmetry is a prolonged point symmetry if, and only if, ξ, τ, ϕ do not depend on v_x.

Each point symmetry (22) projects to a symmetry of Equation (2), defined by

$$X = \xi(x,t,u,v)\partial_x + \tau(x,t,u,v)\partial_t + \eta(x,t,u,v)\partial_u, \tag{23}$$

where v is a nonlocal function of u arising from integrating the first equation of each potential system. Therefore, if the projected symmetry (23) presents dependence on v, then it yields a nonlocal symmetry of Equation (2). Otherwise, the projected symmetry is a point symmetry.

As the projected symmetry does not presents dependence on v, we state that all of the point symmetries admitted by the potential systems (18) and (19) project to point symmetries of the viscous Cahn–Hilliard Equation (2).

In addition, all the local conservation laws of the potential systems (18) and (19) can be determined by applying the multiplier method.

Thus, a new result is found with new nonlocal low-order conservation laws. We can state that all the low-order conservation laws admitted by the viscous Cahn–Hilliard potential systems (18) and (19), with $\nu \neq 0$, are the followings:

1. From the potential system (18), the low-order multiplier $Q_1 = 1$, with $\varphi\left(\frac{v_x}{\nu}\right)$ a nonlinear function, yields the low-order conservation law

$$\begin{aligned} T_1 &= v, \\ X_1 &= -\varphi\left(\frac{v_x}{\nu}\right) - \beta\frac{v_{tx}}{\nu} + \frac{v_{xxx}}{\nu}. \end{aligned}$$

2. From the potential system (19), the low-order multiplier $Q_2 = \frac{1}{x^2}$, with $\varphi\left(\frac{v_x}{\nu x}\right)$ a nonlinear function, yields the low-order conservation law

$$\begin{aligned} T_2 &= \frac{v}{x^2}, \\ X_2 &= \frac{v_{xxx}}{\nu x^2} - \frac{2v_{xx}}{\nu x^3} + \frac{2v_x}{\nu x^4} - \beta\frac{v_{xt}}{\nu x^2} - \frac{\varphi\left(\frac{v_x}{\nu x}\right)}{x}. \end{aligned}$$

These conservation laws project to nonlocal conservation laws of Equation (2), where $v = \int u\, dx$.

4. Multi-Reduction Method and Exact Invariant Solution

One of the most interesting applications of symmetries is finding group-invariant solutions of nonlinear PDEs. These solutions satisfy a reduced differential equation in fewer variables, which are the invariants of the chosen symmetry group. In order to obtain these group-invariant solutions in an explicit form, we need to solve the reduced differential equation, which requires finding sufficiently many first integrals to reduce its order so that a quadrature is obtained.

If the reduction in the PDE under this symmetry is an ODE, then the corresponding reduction in the conserved current, or the underlying local conservation law, yields a first integral of the ODE.

It happens that in most of the applications the considered conservation laws are invariant under translations. In a recent paper [18], a new method has been proposed, which main advantage is that it starts with a symmetry that reduces a PDE and then, it finds all conservation laws invariant under the symmetry. Each one will be inherited by the reduced differential equation. This extension is more interesting when a PDE in two independent variables, such as Equation (2), is being reduced to an ODE, as then a set of first integrals can be obtained allowing further reduction in the ODE.

To sum up, the idea is to apply the multi-reduction method proposed in [18] to find all symmetry-invariant conservation laws admitted by PDE (2), which allow us to reduce the given PDE to first integrals for the ODE, that describes the symmetry-invariant solutions of the PDE.

In general, a travelling wave has the form

$$u(x,t) = h(x - \lambda t), \tag{24}$$

where λ is the travelling wave speed.

Invariance of a given PDE under the translation symmetry

$$X = \partial_t + \lambda\partial_x, \tag{25}$$

gives rise to travelling wave solutions, with $z = x - \lambda t$ and $u = h(z)$ being the invariants.

Now, we focus on the conservation law of Equation (2) invariant under the translation symmetry (25).

Specifically, the conservation law admitted by the viscous Cahn–Hilliard Equation (2), which is invariant under the translation symmetry (25), is given by the multiplier $Q = 1$. Consequently, the only conservation law admitted by the viscous Cahn–Hilliard Equation (2), which is invariant under the translation symmetry (25), is given by (16).

Substitution of the travelling wave expression (24) into Equation (2) yields the nonlinear fourth-order ODE (11).

By using the conservation law invariant under translations, we obtain the first integral

$$\Psi_1 = h''' + \lambda\beta h'' - \varphi' h' - \lambda\nu h - C = 0,$$

where C is the integration constant.

Thus, we obtain a double reduction to a third-order autonomous equation

$$h''' + \lambda\beta h'' - \varphi' h' - \lambda\nu h - C = 0. \quad (26)$$

A way to find solutions of ODE (26) is considering a form for $h(z)$ and using the ODE to obtain the corresponding form for $\varphi'(h)$.

Any function $h(z)$ determines some corresponding function $\varphi'(h)$. For $\varphi'(h)$ to look reasonably simple, $h(z)$ is restricted to have a form such that the terms h, h', h'', h''' in the ODE evaluate to a similar form. This method succeeds in yielding different types of travelling wave solutions with speed λ.

It is readily apparent that this works when

$$\varphi'(h(z)) = k_1 tanh(z)^2 + k_2 tanh(z) + k_3 + k_4 \frac{1}{tanh(z)} + k_5 \frac{1}{tanh(z)^2},$$

where k_1, k_2, k_3, k_4, k_5 are constants to be found.

Hence, by using Equation (26), we find $h(z) = c_1 + c_2 tanh(z)^2$, with

$$k_1 = 12, \quad k_2 = -3\beta\lambda, \quad k_3 = -8, \quad k_4 = \lambda\left(\beta + \frac{\nu}{2}\right), \quad k_5 = 0, \quad c_1 = -c_2.$$

Consequently, for

$$\varphi'(h(z)) = 12 tanh(z)^2 - 3\beta\lambda\, tanh(z) - 8 + \lambda\left(\beta + \frac{\nu}{2}\right)\frac{1}{tanh(z)}$$

we find a soliton solution,

$$h(z) = sech(z)^2,$$

that plays a major role in theoretical physics and propagation of waves. They are defined as nonlinear solitary wave packets with finite amplitude and their physical interest is because of their nature, retaining their shapes and speed when interacting with each other.

By undoing the change of variables (10), we determine a new group-invariant and closed-form general solution

$$u(x,t) = sech(x - \lambda t)^2,$$

for

$$\varphi'(u) = 12 tanh(x - \lambda t)^2 - 3\beta\lambda\, tanh(x - \lambda t) - 8 + \lambda\left(\beta + \frac{\nu}{2}\right)\frac{1}{tanh(x - \lambda t)},$$

depending on constants λ, ν, β.

5. Conclusions

The present paper has obtained new results in aspects, such as symmetries, conservation laws, potential systems, and exact invariant solutions for the viscous Cahn–Hilliard Equation (2). Firstly, all Lie symmetries have been classified depending on function $\varphi(u)$, including Lie point symmetries, with the symmetry transformation groups, and contact symmetries. Secondly, all low-order conservation laws have been derived obtaining conserved quantities. Furthermore, their associated potential systems have been determined finding nonlocal symmetries and nonlocal conservation laws. Afterwards, as an application of the Lie point symmetries classification, symmetry reduction has been used to transform the original fourth-order partial differential equation into ordinary differential equations. Then, we have considered a travelling wave reduction and we have applied the multi-reduction method yielding a third-order equation by double reduction. Finally, a soliton solution of physical interest have been found in terms of a hyperbolic secant function.

Some similar PDEs contained in the family of Cahn–Hilliard equations have been previously studied. However, this complete analysis of the viscous Cahn–Hilliard equation from the point of view of Lie symmetries and conservation laws have not been performed before. Here, we study globally the Lie symmetries, including Lie point symmetries and contact symmetries, and the conservation laws, with also new results from the associated potential systems. In addition, the application of the multi-reduction method, recently proposed, is a novel result itself yielding a new soliton solution in this work.

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Abbreviations

The following abbreviations are used in this manuscript:

PDE	Partial Differential Equation
ODE	Ordinary Differential Equation

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