Results in Mathematics

Further Contributions on the Outer Multiset Dimension of Graphs

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Abstract. The outer multiset dimension $\dim_{\mathrm{ms}}(G)$ of a graph G is the cardinality of a smallest set of vertices that uniquely recognize all the vertices outside this set by using multisets of distances to the set. It is proved that $\dim_{\mathrm{ms}}(G) = n(G) - 1$ if and only if G is a regular graph with diameter at most 2. Graphs G with $\dim_{\mathrm{ms}}(G) = 2$ are described and recognized in polynomial time. A lower bound on the lexicographic product of G and H is proved when H is complete or edgeless, and the extremal graphs are determined. It is proved that $\dim_{\mathrm{ms}}(P_s \,\square\, P_t) = 3$ for $s \geq t \geq 2$.

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1. Introduction

Computing the metric dimension of graphs is one of the classical topics in metric graph theory. The dimension is applicable in several location related problems arising in different areas of investigation like for instance computer science, chemistry, biology, or social sciences. For a better understanding on definitions, terminology, contributions, and open questions on this issue we suggest the fairly complete and recently presented survey [16].

The theory of metric dimension has also been much developed by studying several variations of the classical concept with the purpose of a better understanding of the latter. The number of such variants has been significantly increased in the last recent years. The reader can now find a very rich area of

investigation concerning such variants. For a comprehensive background on a large number of such variants, it is suggested the other recent survey [10].

   Given a connected graph G, it is said that a vertex $v \in V(G)$ *resolves* (or *identifies*, or *determines*) two vertices $x, y \in V(G)$ if $d_G(x, v) \neq d_G(y, v)$; equivalently, x, y are *resolved* by v. Here and later $d_G(u, w)$ stands for the distance between u, w in G. It is also said that a set of vertices S *resolves* the set $V(G)$ if every two vertices of G are resolved by a vertex of S, and such set is called a *resolving set*. The *metric dimension* of G is defined as the cardinality of a smallest resolving set for G, and denoted by $\dim(G)$. A resolving set of cardinality $\dim(G)$ is called a *metric basis*. The concepts above were first (and independently) presented in [7,15].

   It can be easily noted that a resolving set S of a graph G has the property of uniquely identifying all the vertices of G by means of distances to the vertices in S. That is, consider $S = \{v_1, \ldots, v_k\}$ as an ordered set of vertices of a connected graph G, and for any vertex $v \in V(G)$, consider the vector

$$r(v|S) = (d_G(v, v_1), \ldots, d_G(v, v_k)).$$

It is readily seen that S is a resolving set for G if and only if all the vectors $r(v|S)$ with $v \in V(G)$ are pairwise different. The vector $r(v|S)$ is called the *metric representation* of v with respect to S.

   The core of the locating property of a resolving set S of a graph is based on the uniqueness of the metric representations with respect to S of the vertices of the graph. These metric representations are given by vectors of distances. A modified version of this was presented in [14]. That is, authors of [14] suggested the use of "multisets" instead of vectors in the definition of metric representations of a vertex with respect to a given set. For a given vertex $v \in V(G)$ and a set $S = \{w_1, \ldots, w_t\}$, the *multiset representation of u with respect to S* is

$$\mathrm{m}_G(u|S) = \{\!\{d_G(u, w_1), \ldots, d_G(u, w_t)\}\!\},$$

where $\{\!\{\cdot\}\!\}$ limits a multiset. Hence, the set S is a *multiset resolving set* for G if all the multisets $\mathrm{m}_G(u|S)$ with $u \in V(G)$ are pairwise different. The *multiset dimension* of G is then defined as the cardinality of a smallest multiset resolving set. A detail that one can immediately notice is that there could be vertices in a graph that have the same multiset representation with respect to every set of vertices of G (for instance twin vertices). Thus, the multiset dimension of such graphs is not properly defined. In such situations, the authors of [14] adopted the agreement that such graphs has infinite multiset dimension. The problem from [14] to characterize the graphs with infinite multiset dimension remains open. Some partial contributions on this direction were already described in [4].

   On the other hand, in order to avoid the problem of the possible infiniteness of the multiset version of the metric dimension, it was introduced an "outer" version of multiset resolving sets in [6] as follows. A set $S \subseteq V(G)$ is a *outer multiset resolving set* for G, if the multiset representations of vertices

$u \notin S$ with respect to S are pairwise different. A multiset resolving set of the smallest possible cardinality is called an *outer multiset basis*, and the cardinality of an outer multiset basis is the *outer multiset dimension* of G, denoted by $\dim_{ms}(G)$. This structure clearly avoids the problem of infiniteness of multiset dimension, since vertices that must be distinguished by a set S are only vertices outside S. In this work, we are aimed to continue developing this research line. To this end, we next give some basic definitions and terminologies that are necessary along our exposition.

For a positive integer k we will use the notation $[k] = \{1, \ldots, k\}$. A vertex u of a graph G is *diametral* if there exists a vertex $v \in V(G)$ such that $d_G(u, v) = \operatorname{diam}(G)$, where $\operatorname{diam}(G)$ denotes the *diameter* of G, that is, the largest distance between vertices of G. We also say that v is a vertex diametral to u. A subgraph H of a graph G is *isometric* if $d_H(u, v) = d_G(u, v)$ holds for all $u, v \in V(H)$. The open and the closed neighborhood of a vertex u of G will be denoted by $N_G(u)$ and by $N_G[u]$, respectively. The *degree* $\deg_G(u)$ of u is $|N_G(u)|$. Vertices x and y of G are *true twins* if $N_G[u] = N_G[v]$ and are *false twins* if $N_G(u) = N_G(v)$. Vertices u and v are just *twins*, if they are either true twins or false twins. The order of G will be denoted by $n(G)$. If in a multiset (of distances) an element d appears k times, then we may abbreviate it to d^k. For instance, $\{\{1, 1, 1, 2, 5, 5\}\} = \{\{1^3, 2, 5^2\}\}$. Finally, all graphs considered in this paper are connected and of order at least 2.

2. Graphs with Outer Multiset Dimension Order Minus One

For the classical metric dimension parameter, it is well known that $\dim(G) = n(G) - 1$ if and only if G is a complete graphs. In [6], some examples of non complete graphs G of order n such that $\dim_{ms}(G) = n-1$ were given. However, a complete characterization of the class of graphs achieving this equality was not given in [6]. We next settle this issue.

Theorem 2.1. *A graph G satisfies* $\dim_{ms}(G) = n(G) - 1$ *if and only if G is a regular graph with* $\operatorname{diam}(G) \le 2$.

Proof. It was observed in [6] that $\dim_{ms}(K_n) = n - 1$. Moreover, from [6, Example 3.3, Proposition 3.5] we also know that $\dim_{ms}(C_4) = 3$, $\dim_{ms}(C_5) = 4$, and $\dim_{ms}(C_n) = 3$ for $n \ge 6$, as well as that $\dim_{ms}(P_n) = 1$. It follows that the theorem holds for complete graphs and graphs G with $\Delta(G) \le 2$. In the rest of the proof we may thus assume that $\operatorname{diam}(G) \ge 2$ and $\Delta(G) \ge 3$.

Suppose first that G is a graph which satisfies $\dim_{ms}(G) = n(G) - 1$. Assume that G is not regular and select vertices u and v of G such that $\deg_G(u) \ne \deg_G(v)$. We claim that $V(G) \setminus \{u, v\}$ is an outer multiset resolving set. For this sake we only need to verify that $\operatorname{m}_G(u|S) \ne \operatorname{m}_G(v|S)$. Let $\operatorname{m}_G(u|S) = \{\{1^{s_u}, \ldots\}\}$ and $\operatorname{m}(v|S) = \{\{1^{s_v}, \ldots\}\}$. Then, no matter whether u and v are adjacent or not, $\deg_G(u) \ne \deg_G(v)$ implies that $s_u \ne s_v$ and hence $\operatorname{m}_G(u|S) \ne \operatorname{m}_G(v|S)$. This shows that G must be regular.

Suppose now that G is an r-regular graph, $r \geq 3$, which satisfies $\dim_{\mathrm{ms}}(G) = n(G) - 1$. We first claim that every vertex of G is diametral. Assume on the contrary that there exists a non-diametral vertex u, and let v be an arbitrary diametral vertex of G. Let $S = V(G) \setminus \{u, v\}$ and note that $\mathrm{diam}(G) \notin \mathrm{m}(u|S)$ while $\mathrm{diam}(G) \in \mathrm{m}(v|S)$. This means that S is an outer multiset resolving set which in turn implies that $\dim_{\mathrm{ms}}(G) < n(G) - 1$. This contradiction proves the claim that each vertex of G is diametral.

Assume that $\mathrm{diam}(G) \geq 3$. Let u be an arbitrary vertex of G and consider a diametral path starting in u and ending in u'. Let w be the neighbor of u on the diametral path. Then $w \neq u'$ because $\mathrm{diam}(G) \geq 3$. Let $S = V(G) \setminus \{u, w\}$. Since $\dim_{\mathrm{ms}}(G) = n(G) - 1$, the set S is not an outer multiset resolving set. In particular, if $\mathrm{m}_G(u|S) = \{\!\{\ldots, \mathrm{diam}(G)^{s_u}\}\!\}$ and $\mathrm{m}_G(v|S) = \{\!\{\ldots, \mathrm{diam}(G)^{s_w}\}\!\}$, then $s_u = s_w > 0$. Consider now the set $T = V(G) \setminus \{u, u', w\}$. Note first that (having in mind that $\mathrm{diam}(G) \geq 3$ and that G is r-regular) each of the vertices u and w has $r - 1$ neighbors in T while u' has r neighbors in T. Therefore, the pairs u, u' and w, u' are resolved by T. Moreover, since $u' \notin T$, there are $s_u - 1$ vertices in T at distance $\mathrm{diam}(G)$ from u, while there are s_w vertices in T that are at distance $\mathrm{diam}(G)$ from w. Since $s_u = s_w$ we have proved that T also resolves the pair u, w. It follows that T is an outer multiset resolving set, a contradiction to the assumption $\dim_{\mathrm{ms}}(G) = n(G) - 1$. We came to this contradiction because we assumed that $\mathrm{diam}(G) \geq 3$. We conclude that $\mathrm{diam}(G) = 2$.

Conversely, suppose that G is an r-regular graph, $r \geq 3$, with $\mathrm{diam}(G) = 2$. Let S be an outer multiset basis of G. Assume that $|S| \leq n(G) - 2$. Let $\overline{S} = V(G) \setminus S$. Since $\mathrm{diam}(G) = 2$, for each $x \in \overline{S}$ we have $\mathrm{m}_G(x|S) = \{\!\{1^{s_x}, 2^{s'_x}\}\!\}$, where $s_x + s'_x = |S|$. As S is an outer multiset resolving set, $s_x \neq s_y$ for each pair of vertices $x, y \in \overline{S}$, for otherwise $s_x = s_y$ would mean that $s'_x = s'_y$ and hence x and y have the same multiset representation. Select a vertex $u \in \overline{S}$ such that s_u is smallest possible. Then, because $s_x \neq s_y$ for each pair of vertices $x, y \in \overline{S}$, and since $|\overline{S}| \geq 2$, we have $s_u < r$. Let $r = s_u + t$. Then u has t neighbors in $\overline{S}$. Because the vertices x from $\overline{S}$ have pairwise different values s_x, and since s_u is the smallest among them, $\{s_x : x \in \overline{S}\} = \{s_u, s_u + 1, \ldots, s_u + t\} = \{s_u, s_u + 1, \ldots, r\}$. Hence, there exits a vertex $w \in \overline{S}$ with $s_w = r$. This means that w has r neighbors in S. But w is adjacent also to $u \in \overline{S}$, hence $\deg_G(w) \geq r + 1$. As this is not possible we conclude that $|S| = n(G) - 1$, that is, $\dim_{\mathrm{ms}}(G) = n(G) - 1$. $\square$

Examples of graphs from Theorem 2.1 are the Petersen graph, Hamming graphs (alias Cartesian products of complete graphs), and direct products of complete graphs. In addition, from Theorem 2.1 we immediately derive [6, Proposition 3.4] which asserts that if $k \geq 2$, then for the complete k-partite graph $K_{r,\ldots,r}$ we have $\dim_{\mathrm{ms}}(K_{r,\ldots,r}) = kr - 1$. On the other hand, we can easily get that if $k \geq 2$ and $2 \leq r_1 < r_2 < \cdots < r_k$, them $\dim_{\mathrm{ms}}(K_{r_1,\ldots,r_k}) = r_1 + \cdots + r_k - k$.

3. Graphs with Outer Multiset Dimension 2

The problem of characterizing the graphs having its classical metric dimension equal to 2 is one of the open problems in the area. See [3] for an example with partial contributions in this direction. In this section we center our attention into those graphs G with $\dim_{\mathrm{ms}}(G) = 2$. We propose a polynomial algorithm for their recognition and describe their structure.

Lemma 3.1. *If G is a graph with $\dim_{\mathrm{ms}}(G) = 2$ and $S = \{u, v\}$ is an outer multiset basis, then $d_G(u, v) \leq 2$.*

Proof. Suppose that $d_G(u, v) > 2$. Let P be a u, v-geodesic and let $u' \in N(u) \cap V(P)$ and $v' \in N(v) \cap V(P)$. Then u' and v' have the same multiset representation with respect to S, which is not possible. □

Lemma 3.1 leads to Algorithm 1 which decides in polynomial time whether $\dim_{\mathrm{ms}}(G) = 2$ holds for a given graph G.

Algorithm 1 Deciding whether a graph G satisfies $\dim_{\mathrm{ms}}(G) = 2$

1: **procedure** OUTER-MULTISET-DIMENSION-EQUAL-TWO(G)
2:     **if** G is a path **then**
3:         **return** $\dim_{\mathrm{ms}}(G) = 1$
4:     **else**
5:         compute the distance matrix of G
6:     **for all** $u \in V(G)$ **do**
7:         **for all** $v \in V(G) : d_G(u, v) \leq 2$ **do**
8:             **if** multisets in $\{\{\!\{d_G(u, x), d_G(v, x)\}\!\} : x \neq u, v\}$ are different **then**
9:                 $\dim_{\mathrm{ms}}(G) = 2$
10:             **else return** $\dim_{\mathrm{ms}}(G) > 2$
11: **end procedure**

Theorem 3.2. *Deciding whether a graph G of order n satisfies $\dim_{\mathrm{ms}}(G) = 2$ can be done in $\mathcal{O}(n^3)$ time.*

Proof. By Lemma 3.1, we only need to verify each pair of vertices at distance at most 2 whether it forms an outer multiset basis. This checking is implemented in Algorithm 1 whose correctness it thus guaranteed by Lemma 3.1.

The distance matrix of the graph G of order n and size m can be computed in time $\mathcal{O}(nm)$. Moreover, since we have shown above that the maximum degree of G is bounded by a (small) constant, $\mathcal{O}(nm) = \mathcal{O}(n^2)$. (We can pre-process G by checking the degrees of its vertices.) The main loop (Step 6) is performed n times, while for each vertex u, the inner loop (Step 7) in performed a constant number of times because the maximum degree of G and hence also the square of the maximum degree are constant. Checking whether

multisets in Step 8 are different can be done in $\mathcal{O}(n^2)$ time, hence the total complexity is $\mathcal{O}(n^3)$.   □

Despite the fact that deciding whether the outer multiset dimension of a graph equals 2 is polynomial, it is of interest to have more insight into the structure of such graphs. To do so, for a graph G, $X \subseteq V(G)$, and $k \geq 0$, we define

$$L_k(X) = \left\{ u \in V(G) : \min_{x \in X} d_G(u, x) = k \right\}.$$

Note that $L_0(X) = X$, and that the sets $L_k(X)$, $k \geq 0$, partition $V(G)$.

Lemma 3.3. *Let G be a graph with $\dim_{ms}(G) = 2$ and let S be an outer multiset basis of G. Then for every $k \geq 1$ we have $|L_k(S)| \leq 3$. Moreover, if the vertices of S are adjacent, then $|L_k(S)| \leq 2$ and $|L_{k+1}(S)| \leq |L_k(S)|$.*

Proof. Let $S = \{u, v\}$. Let $k \geq 1$ and let $x \in L_k(S)$. By definition of $L_k(S)$, we may without loss of generality assume that $d_G(x, u) = k$. Since $d_G(u, v) \leq 2$ by Lemma 3.1, we have $d_G(v, x) \in \{k, k+1, k+2\}$. It follows that $m(x|S)$ is one of $\{\!\{k, k\}\!\}$, $\{\!\{k, k+1\}\!\}$, and $\{\!\{k, k+2\}\!\}$. As S is an outer multiset basis this in turn implies that $|L_k(S)| \leq 3$.

Assume in the rest that $d_G(u, v) = 1$. Then $d_G(v, x) \in \{k, k+1\}$ and we can conclude similarly as above that $|L_k(S)| \leq 2$. Suppose now that $|L_k(S)| = 1$ and $|L_{k+1}(S)| = 2$ for some $k \geq 1$. Let $L_{k+1}(S) = \{x, y\}$ and $L_k(S) = \{z\}$. Then, by the definition of the sets $L_i(S)$, we infer that z is adjacent to both x and y. But this means that x and y have the same mutiset representation with respect to S. This contradiction proves that $|L_{k+1}(S)| \leq |L_k(S)|$.   □

Based on Lemma 3.3, we next characterize the graphs G with $\dim_{ms}(G) = 2$ having an outer multiset basis formed by two adjacent vertices. To this end, let $S = \{u, v\}$ be an outer multiset basis of G with $uv \in E(G)$. By Lemma 3.3, $|L_k(S)| \leq 2$ for each k. Suppose now that for some k we have $L_k(S) = \{x_k, y_k\}$ and $L_{k+1}(S) = \{x_{k+1}, y_{k+1}\}$. Assume without loss of generality that $m(x_k|S) = \{\!\{k, k\}\!\}$, $m(y_k|S) = \{\!\{k, k+1\}\!\}$, $m(x_{k+1}|S) = \{\!\{k+1, k+1\}\!\}$, and $m(y_{k+1}|S) = \{\!\{k+1, k+2\}\!\}$. Then x_k is not adjacent to y_{k+1}, but must be adjacent to x_{k+1}. In addition, y_k must be adjacent to y_{k+1}, and may be adjacent to x_{k+1}. Finally each of the edges $x_k y_k$ and $x_{k+1} y_{k+1}$ may be present.

The above description together with the fact of Lemma 3.3 that for each $k \geq 1$ we have $|L_{k+1}(S)| \leq |L_k(S)|$, lead to the family $\mathcal{F}$ of graphs G constructed in the following way. The vertex set of $G \in \mathcal{F}$ is $V(G) = \{u_0, \ldots, u_r\} \cup \{v_0, \ldots, v_s\}$ for some $r \geq 0$ and $s \geq 1$, and the edges of G are given as follows.

- $u_0 v_0, u_0 v_1 \in E(G)$.
- For every $i \in [r]$ and every $j \in [s]$, $u_{i-1} u_i \in E(G)$ and $v_{j-1} v_j \in E(G)$.
- For every $i \in [\min\{r, s\}]$, the edge $u_i v_i$ might exist or not in G.
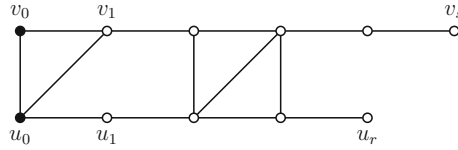- For every $j \in [\min\{r, s-1\}]$, the edge $u_i v_{i+1}$ might exist or not in G.

FIGURE 1. A fairly representative example of a graph from the family $\mathcal{F}$

Note that for instance, if $r = 0$ and $s = 1$ in the construction, then G is precisely the complete graph K_3. If $r = 0$ and $s = 2$, then G is the *paw*, that is, the graph obtained from K_3 by attaching a pendant edge to one of its vertices; and if $r = s = 1$, then G is either K_4 minus an edge, or the paw. For a representative example of a graph from the family $\mathcal{F}$ see Fig. 1. The outer multiset basis is indicated bold.

The above discussion gives us the following result.

Theorem 3.4. *A graph G has an outer multiset basis formed by two adjacent vertices if and only if $G \in \mathcal{F}$.*

In view of Lemma 3.1 and Theorem 3.4, in order to complete the characterization of the graphs G with $\dim_{\mathrm{ms}}(G) = 2$, it remains to describe how such graphs look like for the case in which all their outer multiset bases are formed by two non adjacent vertices. This can be done in a similar way as we did it when there exists an outer multiset basis consisting of two adjacent vertices, however a formal description is more technical and hence not given in detail. The reason for this is the fact that in this subcase it is not satisfied the property $|L_{k+1}(S)| \leq |L_k(S)|$ which holds when there exists an outer mutiset basis with two adjacent vertices. Nevertheless, we still have that $|L_k(S)| \leq 3$. Thus, instead of presenting a lengthy description, we only show a typical example in Fig. 2 with its outer multiset basis again in bold.

4. Lexicographic Products and Multiset Distance Irregular Graphs

In this section we consider the outer multiset dimension of lexicographic products of graphs. Recall that the *lexicographic product* $G \circ H$ of graphs G and H has the vertex set $V(G) \times V(H)$ and edges $(g,h)(g',h')$, where either $g = g'$ and $hh' \in E(H)$, or $gg' \in E(G)$. If $g \in V(G)$, then the set of vertices $\{(g,h) : h \in V(H)\}$ induces a subgraph of $G \circ H$ isomorphic to H which is called an *H-layer* and denoted by ${}^g H$.

The metric dimension of lexicographic products has been independently investigated in papers [8,13]. The main tool used in these articles is that one of transforming the metric dimension in the lexicographic product $G \circ H$ to
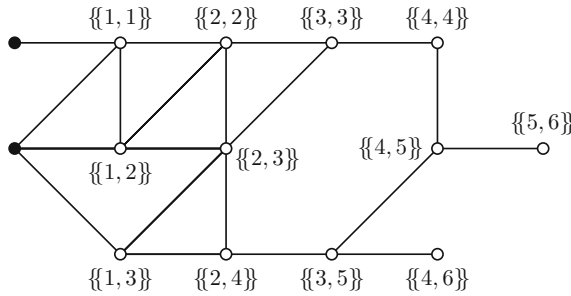
FIGURE 2. A fairly representative example of a graph G with $\dim_{\mathrm{ms}}(G) = 2$ and an outer multiset basis formed by two non adjacent vertices

the so-called adjacency dimension of H. Indeed, the adjacency dimension was explicitly introduced for the first time in [8] (and implicitly in [13]), for more information on it see [10].

We say that a graph G is *multiset distance irregular* if for every two vertices $u, v \in V(G)$, the multisets $\mathrm{m}_G(u|V(G))$ and $\mathrm{m}_G(v|V(G))$ are different.

Theorem 4.1. *If G is a graph with $n(G) \geq 2$ and $H \in \{K_k, \overline{K_k}\}$, $k \geq 2$, then*

$$\dim_{\mathrm{ms}}(G \circ H) \geq n(G)(k-1).$$

Moreover, equality holds if and only if G is multiset distance irregular.

Proof. Let $V(G) = \{g_1, \ldots, g_{n(G)}\}$. For g_i, $i \in [n(G)]$, let $d_i = \mathrm{ecc}_G(g_i)$ be the *eccentricity* of g_i which is the largest distance between g_i and any other vertex of G. Then write

$$\mathrm{m}_G(g_i|V(G)) = \{\!\{0, 1^{i_1}, \ldots, d_i^{i_d}\}\!\}. \tag{1}$$

Let (g_i, h) and (g_i, h') be different vertices from the ${}^{g_i}H$-layer. Since $H \in \{K_k, \overline{K_k}\}$, the vertices (g_i, h) and (g_i, h') are twins. By [6, Proposition 3.7] we know that every outer multiset resolving set of $G \circ H$ contains at least one of (g_i, h) and (g_i, h'). Inductively, every outer multiset resolving set of $G \circ H$ contains at least $k-1$ vertices from ${}^{g_i}H$. As this H-layer was arbitrary, the inequality follows.

Suppose now that the equality holds and let S be an outer multiset basis of $G \circ H$. By the above argument, each layer ${}^{g_i}H$ contains exactly one vertex which does not belong to S. We may without loss of generality assume that this vertex is (g_i, h), where h is some fixed vertex of H. In view of (1) we then have

$$\mathrm{m}_{G \circ H}((g_i, h)|S) = \{\!\{1^{i_1(k-1)}, \ldots, d_i^{i_d(k-1)}, 1^{(k-1)}\}\!\} \tag{2}$$

when $H = K_k$ and

$$\mathrm{m}_{G \circ H}((g_i, h)|S) = \{\!\{1^{i_1(k-1)}, \ldots, d_i^{i_d(k-1)}, 2^{(k-1)}\}\!\} \tag{3}$$
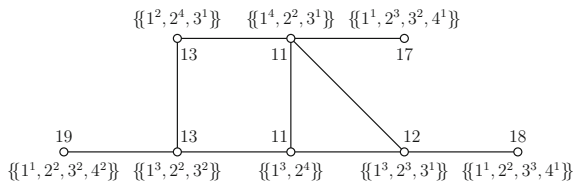
FIGURE 3. A graph which is multiset distance irregular but not transmission irregular

when $H = \overline{K_k}$. The above respective terms $1^{(k-1)}$ and $2^{(k-1)}$ reflect the distances between (g_i, h) and the other vertices from ^{g_i}H (which are all in S). Since S is an outer multiset basis of $G \circ H$, we have $\mathrm{m}_{G \circ H}((g_i, h)|S) \neq \mathrm{m}_{G \circ H}((g_j, h)|S)$ for each $i, j \in [n(G)]$, $i \neq j$. But then no matter whether we consider (2) or (3), we deduce that $\mathrm{m}_G(g_i|V(G)) \neq \mathrm{m}_G(g_j|V(G))$ holds for each $i, j \in [n(G)]$. Hence G is a multiset distance irregular.

Conversely, let G be a multiset distance irregular graph. Then, by definition, $\mathrm{m}_G(g_i|V(G)) \neq \mathrm{m}_G(g_j|V(G))$ holds for each $i, j \in [n(G)]$. Let $S \subseteq V(G \circ H)$ be a set that contains exactly $k - 1$ vertices of each H-layer. But then in view of (2) or (3), $\mathrm{m}_{G \circ H}((g_i, h)|S) \neq \mathrm{m}_{G \circ H}((g_j, h)|S)$ for each $i, j \in [n(G)]$, $i \neq j$. This means that S is a outer multiset resolving set. Since $|S| = n(G)(k - 1)$ we are done. $\qquad\square$

We believe that multiset distance irregular graphs are of independent interest. First of all, they are closely related to transmission irregular graphs which are defined as follows. The *transmission* $\mathrm{Tr}_G(v)$ of a vertex v of a graph G is the sum of distances between v and all the other vertices of the graph. G is *transmission irregular* if the vertices of G have pairwise different transmissions. (Transmission irregular graphs are also known as status injective graphs, cf. [12].) This concept was first studied in [1] with respect to the Wiener dimension because transmission irregular graphs are the graphs with a largest possible Wiener dimension. For a selection of different appealing constructions of transmission irregular graphs and results on these graphs see [2,5,17].

Clearly, if $\mathrm{m}_G(u|V(G)) = \mathrm{m}_G(v|V(G))$, then $\mathrm{Tr}_G(u) = \mathrm{Tr}_G(v)$. Hence transmission irregular graphs form a subset of multiset distance irregular graphs. The inclusion is strict as demonstrated by the graph X from Fig. 3 which is multiset distance irregular but not transmission irregular. Next to each vertex u the multiset $\mathrm{m}_G(u|(V(X) \setminus \{u\})$ is written as well as $\mathrm{Tr}_X(u)$. The multisets $\mathrm{m}_G(u|V(X))$ are indeed pairwise different, but there are two pairs of vertices with the same transmission.

5. Grid Graphs

Let $V(P_n) = \{0, 1, \ldots, n-1\}$ where i is adjacent to j if and only if $|i-j| = 1$. The grid graph $P_s \,\square\, P_t$ is the Cartesian product of two paths P_s and P_t, that is $V(P_s \,\square\, P_t) = \{(i, j) \,:\, 0 \le i \le s-1 \text{ and } 0 \le j \le t-1\}$ and $(i,j)(k,\ell) \in E(P_s \,\square\, P_t)$ when $|i-j| + |k-\ell| = 1$.

The metric dimension of grids was reported in several papers, among the earliest ones are [9,11]. Here we add the following result for the outer metric dimension of grids.

Theorem 5.1. *For all $s \ge t \ge 2$,*

$$\dim_{\mathrm{ms}}(P_s \square P_t) = 3.$$

Proof. In order to simplify the notation, let $G = P_s \square P_t$. It clearly cannot be $\dim_{\mathrm{ms}}(G) = 1$ since G is not a path. Also, notice that if $s = t = 2$, then by Theorem 2.1 we have that $\dim_{\mathrm{ms}}(G) = 3$. Accordingly, from now on we may w.l.g. assume that $s \ge 3$. If $s = 3$, then $t \in \{2, 3\}$. In such a case it can be easily checked that the set $\{(0,0), (2,0), (2,1)\}$ is an outer multiset basis. Hence, from now we assume that $s \ge 4$.

Now, if $\dim_{\mathrm{ms}}(G) = 2$ and $S = \{(i,j), (i',j')\}$ is an outer multiset basis, then by Lemma 3.1, we have that $d_G((i,j),(i',j')) \le 2$. If $i = i'$, we readily observe that the two vertices $(i+1, j), (i+1, j')$ (or the two vertices $(i-1, j), (i-1, j')$ if $i = s$) have the same multiset representation with respect to S, which is not possible. By symmetry, a similar conclusion is deduced if $j = j'$. It remains the case $d_G((i,j),(i',j')) = 2$, $i \ne i'$ and $j \ne j'$. Thus, it must happen that (w.l.g.) $S = \{(i,j), (i+1, j+1)\}$. Hence, $(i+1, j)$ and $(i, j+1)$ have the same multiset representation with respect to S, and this is also not possible. As a consequence of the two previous contradictions, we obtain that $\dim_{\mathrm{ms}}(G) \ge 3$.

In order to show that $\dim_{\mathrm{ms}}(G) \le 3$, we claim that $R = \{(0,0), (1,0), (s-1, 0)\}$ is an outer multiset resolving set for G. Let $R' = \{(0,0), (s-1, 0)\}$ and notice that R' is a metric basis for G, cf. [9]. Let $(i,j), (i',j')$ be any two vertices in $V(G) \setminus R$. We observe that the multiset representations of (i,j) and (i',j') with respect to R' are:

$$m_G((i,j)|R') = \{\!\{i+j, s-1-i+j\}\!\}$$
$$m_G((i',j')|R') = \{\!\{i'+j', s-1-i'+j'\}\!\}.$$

Since R' is a metric basis, we have one of the following three situations.

Case 1: $d_G((i,j),(0,0)) = d_G((i',j'),(0,0)) = \alpha$.
In such situation, it must happen $\beta = d_G((i,j),(s-1,0)) \ne d_G((i',j'),(s-1,0)) = \beta'$ since R' is a metric basis. On the other hand, since G is bipartite,

$$d_G((i,j),(1,0)) \in \{d_G((i,j),(0,0)) - 1, d_G((i,j),(0,0)) + 1\},$$

and

$$d_G((i',j'),(1,0)) \in \{d_G((i',j'),(0,0)) - 1, d_G((i',j'),(0,0)) + 1\}.$$

If $d_G((i,j),(1,0)) = d_G((i,j),(0,0)) + 1 = \alpha + 1$ and $d_G((i',j'),(1,0)) = d_G((i',j'),(0,0)) + 1 = \alpha + 1$, then we note that

$$m_G((i,j)|R) = (\alpha, \beta, \alpha + 1) \neq (\alpha, \beta', \alpha + 1) = m_G((i',j')|R)$$

since $\beta \neq \beta'$. Similarly, if $d_G((i,j),(1,0)) = d_G((i,j),(0,0)) - 1$ and $d_G((i',j'),(1,0)) = d_G((i',j'),(0,0)) - 1$, then we note that $m_G((i,j)|R) \neq m_G((i',j')|R)$.

We now consider the subcase when $d_G((i,j),(1,0)) = \alpha + 1$ and $d_G((i',j'),(1,0)) = \alpha - 1$. From the first equality, we deduce that $i = 0$ must occur. Thus, from the case assumption we obtain that $j = i + j = d_G((i,j),(0,0)) = d_G((i',j'),(0,0)) = i' + j'$. Notice that the multiset representations of (i,j) and (i',j') with respect to R are:

$$m_G((i,j)|R) = \{\!\{i+j, s-1-i+j, i+j+1\}\!\} = \{\!\{j, s-1+j, j+1\}\!\},$$

and

$$m_G((i',j')|R) = \{\!\{i'+j', s-1-i'+j', i'+j'-1\}\!\}.$$

Recall that $j = i' + j'$ and that $s - 1 + j \neq s - 1 - i' + j'$. If $m_G((i,j)|R) = m_G((i',j')|R)$, then it must happen that $s - 1 + j = i' + j' - 1$ and that $j + 1 = s - 1 - i' + j'$. However, since $j = i' + j'$, from $s - 1 + j = i' + j' - 1$ we deduce that $s = 0$, which is not possible. Thus, $m_G((i,j)|R) \neq m_G((i',j')|R)$.

To complete this case, we may assume $d_G((i,j),(1,0)) = \alpha - 1$ and $d_G((i',j'),(1,0)) = \alpha + 1$. We again obtain a similar conclusion, by using a procedure analogous to the one above, but taking into account that now it must first occur $i' = 0$ instead of $i = 0$.

Case 2: $d_G((i,j),(s-1,0)) = d_G((i',j'),(s-1,0))$.
In such situation, it must happen $d_G((i,j),(0,0)) \neq d_G((i',j'),(0,0))$ since R' is a metric basis. If $m_G((i,j)|R) = m_G((i',j')|R)$, then by using some similar arguments as in Case 1, it must happen that the multisets of (i,j) and (i',j') with respect to R are either

$$m_G((i,j)|R) = \{\!\{i+j, s-1-i+j, i+j+1\}\!\},$$

and

$$m_G((i',j')|R) = \{\!\{i'+j', s-1-i'+j', i'+j'-1\}\!\};$$

or

$$m_G((i,j)|R) = \{\!\{i+j, s-1-i+j, i+j-1\}\!\},$$

and

$$m_G((i',j')|R) = \{\!\{i'+j', s-1-i'+j', i'+j'+1\}\!\}.$$

Moreover, $i + j = i' + j' - 1$ and $i' + j' = i + j + 1$ must happen in the first situation, as well as $i + j = i' + j' + 1$ and $i' + j' = i + j - 1$ in the second one.

In addition, in the first possibility we have that $i = 0$ must happen, while in the second one $i' = 0$ must happen instead. Since $s - 1 - i + j = d_G((i,j),(s-1,0)) = d_G((i',j'),(s-1,0)) = s - 1 - i' + j'$, we obtain that

either $j = j' - i'$ (when $i = 0$), or $j' = j - i$ (when $i' = 0$). By using these equalities in $i + j = i' + j' - 1$ and $i + j = i' + j' + 1$, we deduce that $2i' = -1$ and that $2i' = 1$, respectively, which are both contradictions. Consequently, we again obtain $m_G((i, j)|R) \neq m_G((i', j')|R)$.

Case 3: $d_G((i, j), (0, 0)) \neq d_G((i', j'), (0, 0))$ and $d_G((i, j), (s - 1, 0)) \neq d_G((i', j'), (s - 1, 0))$.

Clearly, if $d_G((i, j), (0, 0)) \neq d_G((i', j'), (s - 1, 0))$ and $d_G((i, j), (s - 1, 0)) \neq d_G((i', j'), (0, 0))$, then we have that $m_G((i, j)|R) \neq m_G((i', j')|R)$, independently on which the distances $d_G((i, j), (1, 0))$ and $d_G((i', j'), (1, 0))$ are.

Now, if

$$i + j = d_G((i, j), (0, 0)) = d_G((i', j'), (s - 1, 0)) = s - 1 - i' + j'$$

and

$$s - 1 - i + j = d_G((i, j), (s - 1, 0)) = d_G((i', j'), (0, 0)) = i' + j',$$

then we deduce that $j = j'$. Thus, since $s \geq 4$ it must happen that $d_G((i, j), (1, 0)) \neq d_G((i', j'), (1, 0))$, for otherwise we get either $i = i'$ or (w.l.g.) $i = 0$ and $i' = 2$ (which are contradictions). This leads to $m_G((i, j)|R) \neq m_G((i', j')|R)$.

To conclude the proof, we need to consider the case when

$$i + j = d_G((i, j), (0, 0)) = d_G((i', j'), (s - 1, 0)) = s - 1 - i' + j'$$

and

$$s - 1 - i + j = d_G((i, j), (s - 1, 0)) \neq d_G((i', j'), (0, 0)) = i' + j'.$$

If $m_G((i, j)|R) = m_G((i', j')|R)$, then it must happen $d_G((i, j), (s - 1, 0)) = d_G((i', j'), (1, 0))$ and that $d_G((i, j), (1, 0)) = d_G((i', j'), (0, 0))$. Moreover, since

$$\begin{aligned} d_G((i, j), (1, 0)) &= d_G((i, j), (0, 0)) + 1 \text{ or } d_G((i, j), (1, 0)) \\ &= d_G((i, j), (0, 0)) - 1, \end{aligned}$$

and

$$\begin{aligned} d_G((i', j'), (1, 0)) &= d_G((i', j'), (0, 0)) + 1 \text{ or } d_G((i', j'), (1, 0)) \\ &= d_G((i', j'), (0, 0)) - 1, \end{aligned}$$

it must be satisfied that $|d_G((i, j), (s-1, 0)) - d_G((i', j'), (0, 0))| = 1$. However, by using the fact that $i + j = d_G((i, j), (0, 0)) = d_G((i', j'), (s - 1, 0)) = s - 1 - i' + j'$ in the difference $d_G((i, j), (s-1, 0)) - d_G((i', j'), (0, 0)) = (s - 1 - i + j) - (i' + j')$, we obtain that $d_G((i, j), (s-1, 0)) - d_G((i', j'), (0, 0)) = 2(j - j')$, which is an even number, a contradiction. Therefore, $m_G((i, j)|R) = m_G((i', j')|R)$. A similar conclusion is obtained if $i + j = d_G((i, j), (0, 0)) \neq d_G((i', j'), (s - 1, 0)) = s - 1 - i' + j'$ and $s - 1 - i + j = d_G((i, j), (s - 1, 0)) = d_G((i', j'), (0, 0)) = i' + j'$.

As a consequence of the arguments above, we conclude that R is an outer multiset resolving set, which completes the proof. □

6. Concluding Remarks

The discussion at the end of Sect. 2 suggests the following challenge.

Problem 6.1. *Investigate the outer multiset dimension of non-regular graphs of diameter* 2.

In Sect. 4, a lower bound is proved for the lexicographic products in which the second factor is complete or edgeless.

Problem 6.2. *Investigate the outer multiset dimension of general lexicographic products.*

Theorem 5.1 offers the following challenge.

Problem 6.3. *Extend Theorem 5.1 to multidimensional grids, that is, to Cartesian products of several paths.*

In this direction, hypercubes (that is, Cartesian products of paths of order 2) deserve a special attention.

An additional class for which it would be interesting to determine the outer multiset dimension is the following.

Problem 6.4. *Determine* $\dim_{\mathrm{ms}}(C_s \,\square\, C_t)$ *for* $s, t \geq 3$.

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Declarations

Conflict of interest The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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