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Lie symmetries and exact solutions for a fourth-order nonlinear diffusion equation

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In this paper, we consider a fourth-order nonlinear diffusion partial differential equation, depending on two arbitrary functions. First, we perform an analysis of the symmetry reductions for this parabolic partial differential equation by applying the Lie symmetry method. The invariance property of a partial differential equation under a Lie group of transformations yields the infinitesimal generators. By using this invariance condition, we present a complete classification of the Lie point symmetries for the different forms of the functions that the partial differential equation involves. Afterwards, the optimal systems of one-dimensional subalgebras for each maximal Lie algebra are determined, by computing previously the commutation relations, with the Lie bracket operator, and the adjoint representation. Next, the reductions to ordinary differential equations are derived from the optimal systems of one-dimensional subalgebras. Furthermore, we study travelling wave reductions depending on the form of the two arbitrary functions of the original equation. Some travelling wave solutions are obtained, such as solitons, kinks and periodic waves.

KEYWORDS

diffusion equations, exact solutions, Lie group analysis, symmetry reductions

MSC CLASSIFICATION

35C09, 35Q70, 35K55

1 | INTRODUCTION

It is well known that many phenomena in nature can be described by partial differential equations (PDEs). These phenomena appear in many scientific fields, such as mathematical biology, fluid mechanics, plasma physics, chemistry, and thermodynamics. In recent years, many researchers of applied mathematics have been focused on studying mathematical models to understand these physical phenomena. Specifically, several papers are published analysing nonlinear PDEs, due to their applications in several fields of science.

A diffusion equation[1] is a parabolic PDE describing density fluctuations in a material undergoing diffusion. It is mostly applied in physics and mathematics describing, for instance, the macroscopic behaviour of many micro-particles in Brownian motion, resulting from the random movements and collisions of the particles, and it is related to Markov processes, materials science, information theory, and biophysics. In particular, this equation is a special case of the convection-diffusion equation, when bulk velocity is zero. There are some variations of interest of the diffusion equation; let us present a few of them.

Keita et al[2] considered a class of fourth-order parabolic PDEs,

$$u_t = -\gamma \nabla \cdot (f(u)\nabla \Delta u) + \nabla \cdot (f(u)\nabla \varphi'(u)), \tag{1}$$

where $f(u)$ and $\varphi(u)$ are smooth functions and γ is a constant. Depending on the application, $u = u(x, t)$ can describe the height of a liquid phase spreading on a solid surface,[3,4] the concentration, the volume fraction, or the density of a phase in a mixture.[5,6] Equation (1) was written by Hocking[4] as a system of second-order equations, by introducing an auxiliary variable. Moreover, they designed a second-order fully discrete mixed finite element method to approximate these equations. They also presented numerical experiments to confirm the second-order accuracy in time, for both constant and non-constant mobility functions.

Equation (1), with $g(u) = 0$, leads to a lubrication equation that models surface tension, generated by the motion of thin viscous films and spreading droplets. This equation, with $f(u) = |u|$, also models a thin neck of fluid in the Hele-Shaw cell. In such problems, $u = u(x, t)$ is the local thickness of the film or neck.

Furthermore, a similar equation was analysed in Bruzón et al[7] and Gandarias and Medina[8] obtaining a classification of the point symmetries admitted by the generalized equation

$$u_t = -(f(u)u_{xxx})_x.$$

For a modified version, given by

$$u_t = -f(u)u_{xxxx},$$

the authors proved, by using symmetry reductions, that for some particular functional forms of $f(u)$, the one-dimensional lubrication model admits some solutions of physical interest, such as similarity solutions.

In this work, we study the fourth-order nonlinear diffusion equation

$$u_t + f(u)u_{xxxx} + g(u)u_{xx} = 0, \tag{2}$$

where $f(u)$ and $g(u)$ are smooth functions, satisfying $f(u) \neq 0$ to preserve the order of the PDE.

Lie symmetries were introduced in 1881 by Sophus Lie. They are a really powerful tool to find exact solutions to nonlinear PDEs, helping us to understand mathematical models. The notion of a symmetry for a PDE is a transformation of the solution manifold into itself. In particular, the Lie point symmetries are written in terms of the infinitesimal generators depending on the independent and dependent variables. There are many applications of Lie symmetries, such as reducing PDEs, determining invariant solutions, mapping solutions to other solutions, and detecting linearising transformations. Among these applications, it is important to point out the determination of invariant solutions under a Lie group of transformations, also known as similarity solutions. Moreover, a wide literature, including textbooks, has been published about the application of Lie transformation group theory to construct solutions of nonlinear PDEs, such as Olver,[9] Bluman and Anco,[10] Bluman et al,[11] and Bluman and Kumei[12] and references therein.

Motivated by the reasons above, there are several papers essentially analysing the Lie symmetries of PDEs.[13–16] Specifically, by using the Lie symmetry method, symmetries can be used to find exact invariant solutions.

The main goal of this paper is to provide a complete classification of the Lie point symmetries of Equation (2), depending on the form of $f(u)$ and $g(u)$, with $f(u) \neq 0$. Then, we determine the optimal systems of one-dimensional subalgebras for each maximal Lie algebra and afterwards, the similarity reductions, by using the generators of the optimal systems. Thus, we find the similarity variables and the similarity solutions to reduce the number of independent variables. Hence, we obtain ODEs that allow us to construct exact solutions of Equation (2), after undoing the change of variables.

The paper is structured as follows. First, we analyse Equation (2) from the point of view of the theory of symmetry reductions in PDEs. Therefore, in Section 2, we give a complete classification of the Lie point symmetries admitted by Equation (2), for different forms of $f(u)$ and $g(u)$. In Section 3, we determine the optimal systems of one-dimensional subalgebras for each maximal Lie algebra, by calculating the commutators and the adjoint actions. Next, in Section 4, we use the generators of the optimal systems to reduce the original equation to ODEs. Consequently, in Section 5, we construct travelling wave solutions, obtaining analytical solutions. We also show the graphical representations of the solutions for a better understanding of their nature. Finally, in Section 6, we give some concluding remarks on the results.

2 | LIE POINT SYMMETRIES

Let us consider a one-parameter Lie group of infinitesimal transformations on the space of independent and dependent variables:

$$\begin{aligned}\hat{x}(x,t,u;\epsilon) &= x + \epsilon\,\xi(x,t,u) + \mathcal{O}(\epsilon^2),\\ \hat{t}(x,t,u;\epsilon) &= t + \epsilon\,\tau(x,t,u) + \mathcal{O}(\epsilon^2),\\ \hat{u}(x,t,u;\epsilon) &= u + \epsilon\,\eta(x,t,u) + \mathcal{O}(\epsilon^2),\end{aligned} \tag{3}$$

where ϵ represents the group parameter. Each infinitesimal point symmetry (3) constitutes a generator

$$X = \xi(x,t,u)\,\partial_x + \tau(x,t,u)\,\partial_t + \eta(x,t,u)\,\partial_u \tag{4}$$

that leaves invariant Equation (2). By applying the symmetry invariance condition

$$\mathrm{pr}^{(4)}X\,(u_t + f(u)\,u_{xxxx} + g(u)\,u_{xx}) = 0, \text{ when } u_t + f(u)\,u_{xxxx} + g(u)\,u_{xx} = 0,$$

point symmetries can be determined, where $\mathrm{pr}^{(4)}X$ represents the fourth prolongation of the generator X.

There is a geometrical way to avoid the tedious computations that the fourth-order prolongation formula involves.[9,10] The action of the vector field (4) on the solution space of Equation (2) is similar to the action of the generator

$$\hat{X} = \hat{\eta}\,\partial_u,\ \ \hat{\eta} = \eta - \xi\,u_x - \tau\,u_t,$$

called the characteristic form of the point symmetry. Then, the solutions of Equation (2) are preserved under the transformation (3) if

$$\mathrm{pr}^{(4)}\hat{X}\,(u_t + f(u)\,u_{xxxx} + g(u)\,u_{xx}) = 0, \text{ when } u_t + f(u)\,u_{xxxx} + g(u)\,u_{xx} = 0.$$

Here,

$$\mathrm{pr}^{(4)}\hat{X} = \hat{X} + (D_x\hat{\eta})\,\partial_{u_x} + (D_t\hat{\eta})\,\partial_{u_t} + (D_x^2\hat{\eta})\,\partial_{u_{xx}} + (D_x^3\hat{\eta})\,\partial_{u_{xxx}} + (D_x^4\hat{\eta})\,\partial_{u_{xxxx}} \tag{5}$$

represents the fourth prolongation of the generator $\hat{X}$, where D_x and D_t are the total derivatives respect to x and t, respectively.

The symmetry determining equation (5) splits with respect to the x derivatives and t derivatives of u, leading to a overdetermined linear system of determining equations for the infinitesimals $\xi(x,t,u)$, $\tau(x,t,u)$, $\eta(x,t,u)$.

In general, applying Lie groups theory to differential equations yields many tedious calculations. Therefore, we use the powerful mathematical software Maple for the computations. The determining equations are obtained in Maple and with the help of Maple commands 'rifsimp', 'dsolve' and 'pdsolve', we solve the system. Specifically, 'rifsimp' returns a tree with all the solution cases, and then, for each solution case, we apply 'dsolve' and 'pdsolve' to determine the solutions.

A system with 39 determining equations is generated. From this system, we get that

$$\begin{aligned}\xi &= \xi(x,t),\\ \tau &= \tau(t),\\ \eta &= \alpha(x,t)\,u + \beta(x,t)\end{aligned}$$

are related through the following equations,

$$
\begin{aligned}
2\alpha_x + 3\xi_{xx} &= 0,\\
g(\alpha_{xx}u + \beta_{xx}) + f(\alpha_{xxxx}u + \beta_{xxxx}) + \alpha_t u + \beta_t &= 0,\\
4f\xi_x - f_u(\alpha u + \beta) - f\tau_t &= 0,\\
4f\alpha_{xxx} + 2g\alpha_x - f\xi_{xxxx} - g\xi_{xx} - \xi_t &= 0,\\
-4f^2\xi_{xxx} + 6f^2\alpha_{xx} + 2fg\xi_x - (f_u g - g_u f)(\alpha u + \beta) &= 0.
\end{aligned}
$$

After solving these equations, we define Theorem 1, which states a classification of the point symmetries.

Theorem 1. *Point symmetries are admitted by the fourth-order nonlinear diffusion equation (2), with $f(u) \neq 0$, in the following cases:*

1. *For $f(u)$ and $g(u)$ arbitrary functions,*

$$X_1 = \partial_t,$$
$$X_2 = \partial_x.$$

If $g(u) = 0$, in addition to X_1 and X_2, Equation (2) admits an extra point symmetry,

$$X_3 = 4t\,\partial_t + x\,\partial_x.$$

2. *For particular functions of $f(u)$ and $g(u)$, besides X_1 and X_2, additional point symmetries are admitted by Equation (2).*

   (a) *For $f(u) = f_0\,u^p$ and $g(u) = g_0\,u^q$, with p and q not simultaneously zero,*

$$X_4 = 2(p - 2q)\,t\,\partial_t + (p - q)x\,\partial_x + 2u\,\partial_u.$$

   i. *If $f(u) = f_0\,u^{8/3}$ and $g(u) = 0$, besides X_3 and $X_4|_{p=8/3,q=0}$, Equation (2) admits*

$$X_5 = x^2\,\partial_x + 3x\,u\,\partial_u.$$

   ii. *If $f(u) = f_0$ and $g(u) = g_0$, besides $X_4|_{p=q=0} \equiv u\,\partial_u$, Equation (2) admits an additional infinite-dimensional symmetry,*

$$X_\alpha = \alpha\,\partial_u,$$

   where $\alpha(x,t)$ satisfies $\alpha_t + f_0\,\alpha_{xxxx} + g_0\,\alpha_{xx} = 0$ corresponding to the linear case. Moreover, if $g(u) = 0$, X_3 is also obtained.

   (b) *For $f(u) = f_0\,e^{p\,u}$ and $g(u) = g_0\,e^{q\,u}$, with p and q not simultaneously zero,*

$$X_6 = 2(p - 2q)\,t\,\partial_t + (p - q)x\,\partial_x + 2\partial_u.$$

   Moreover, if $g(u) = 0$, X_3 is also obtained.

In the above point symmetries, $f_0 \neq 0$, $g_0 \neq 0$, and p, q are arbitrary constants. When $g_0 = 0$, we can set $q = 0$ without losing generality.

For PDEs with a single-dependent variable u, Lie symmetries include point symmetries and contact symmetries.[10]

Contact symmetries arise when considering an extension of the symmetry method. In this case, symmetry transformations will be allowed to depend on u_x and u_t in the definition of invariance.

The infinitesimal generator of a contact symmetry is given by

$$X = \xi\,\partial_x + \tau\,\partial_t + \eta\,\partial_u + \eta^x\,\partial_{u_x} + \eta^t\,\partial_{u_t}.$$

Theorem 2. *There are not any contact symmetries admitted by the fourth-order nonlinear diffusion equation (2), with $f(u) \neq 0$.*

3 | OPTIMAL SYSTEMS

To determine nonequivalent solutions of Equation (2) by the action of the group, we calculate the one-dimensional optimal systems of subalgebras.[9] Hence, we obtain a classification of subalgebras, where each class contains equivalent solutions related through an element of the Lie symmetry group. For this section, we focus on the cases with $f'(u) \neq 0$.

To begin, we present the maximal Lie algebras in Theorem 3, defined by $\mathcal{A}$, a k dimensional Lie algebra with basis formed by generators X_k, $k = 1, \ldots, 6$, representing the most general symmetry Lie algebras, depending on functions $f(u)$ and $g(u)$. Next, we compute the commutators and the adjoint representation, which shows the separate adjoint actions of each element in X_k, $k = 1, \ldots, 6$. This construction is done by applying the Lie bracket operator and summing the Lie series. Therefore, we state in Theorem 4 the subalgebras of the optimal systems, by using the commutators and the adjoint actions.

Theorem 3. *The maximal Lie algebras admitted by the fourth-order nonlinear diffusion equation (2), with $f'(u) \neq 0$, along with the non-zero commutation relations, are given in the following cases:*

1. *For $f(u)$ and $g(u)$ arbitrary functions,*
$$\mathcal{A}_1 = span(X_1, X_2).$$
2. *For $f(u)$ arbitrary function and $g(u) = 0$,*
$$\begin{aligned}&\mathcal{A}_2 = span(X_1, X_2, X_3),\\ &[X_1, X_3] = 4X_1,\ [X_2, X_3] = X_2.\end{aligned}$$
3. *For $f(u) = f_0\,u^p$ and $g(u) = g_0\,u^q$, with p and q not simultaneously zero,*
$$\begin{aligned}&\mathcal{A}_3 = span(X_1, X_2, X_4),\\ &[X_1, X_4] = 2(p-2q)X_1,\ [X_2, X_4] = (p-q)X_2.\end{aligned}$$
4. *For $f(u) = f_0\,u^p$ and $g(u) = 0$, with $p \neq 0$,*
$$\begin{aligned}&\mathcal{A}_4 = span(X_1, X_2, X_3, X_4),\\ &[X_1, X_3] = 4X_1,\ [X_1, X_4] = 2pX_1,\\ &[X_2, X_3] = X_2,\ [X_2, X_4] = pX_2.\end{aligned}$$
5. *For $f(u) = f_0\,u^{8/3}$ and $g(u) = 0$,*
$$\begin{aligned}&\mathcal{A}_5 = span(X_1, X_2, X_3, X_4, X_5),\\ &[X_1, X_3] = 4X_1,\ [X_1, X_4] = \frac{16}{3}X_1,\\ &[X_2, X_3] = X_2,\ [X_2, X_4] = \frac{8}{3}X_2,\ [X_2, X_5] = \frac{3}{2}X_4 - 2X_3,\\ &[X_3, X_5] = X_5,\ [X_4, X_5] = \frac{8}{3}X_5.\end{aligned}$$
6. *For $f(u) = f_0\,e^{p\,u}$ and $g(u) = g_0\,e^{q\,u}$, with p and q not simultaneously zero,*
$$\begin{aligned}&\mathcal{A}_6 = span(X_1, X_2, X_6),\\ &[X_1, X_6] = 2(p-2q)X_1,\ [X_2, X_6] = (p-q)X_2.\end{aligned}$$

7. *For $f(u)=f_0 e^{pu}$ and $g(u)=0$, with $p\neq 0$,*

$$\begin{aligned}&\mathcal{A}_7 = span(X_1,X_2,X_3,X_6),\\&[X_1,X_3]=4X_1,\ [X_1,X_6]=2pX_1,\\&[X_2,X_3]=X_2,\ [X_2,X_6]=pX_2.\end{aligned}$$

In the above Lie algebras, $f_0\neq 0$, $g_0\neq 0$, and p, q are arbitrary constants.

The optimal systems of Lie algebras up to dimension four were computed in Patera and Winternitz.[17] In the following theorem, we show the optimal systems of one-dimensional subalgebras for each maximal Lie algebra admitted by the fourth-order nonlinear diffusion equation (2), and we identify each Lie algebra of dimension up to four with the corresponding algebra in the work of Patera and Winternitz.[17]

Theorem 4. *The optimal systems of one-dimensional subalgebras for each maximal Lie algebra admitted by the fourth-order nonlinear diffusion equation (2), with $f'(u)\neq 0$, are given by as follows:*

1. *For $f(u)$ and $g(u)$ arbitrary functions,*

$$\{X_1+\lambda X_2, X_2\},$$

which corresponds, following the notation used in Patera and Winternitz,[17] to the two-dimensional algebra $2A_1$.

2. *For $f(u)$ arbitrary function and $g(u)=0$,*

$$\{X_1+\lambda X_2, X_2, X_3\},$$

which corresponds, following the notation used in Patera and Winternitz,[17] to the three-dimensional algebra $A_{3,3}$.

3. *For $f(u)=f_0u^p$ and $g(u)=g_0u^q$, with p and q not simultaneously zero,*

   (a) *if $p\neq q, 2q$,*

$$\{X_1+\lambda X_2, X_2, X_4\},$$

   which corresponds, following the notation used in Patera and Winternitz,[17] to the three-dimensional algebra $A_{3,3}$;

   (b) *if $p=q$,*

$$\{X_1+\lambda X_2, X_2, \mu X_2+X_4\},$$

   which corresponds, following the notation used in Patera and Winternitz,[17] to the three-dimensional algebra $A_1\oplus A_2$;

   (c) *if $p=2q$,*

$$\{X_1+\lambda X_2, X_2, \mu X_1+X_4\},$$

   which corresponds, following the notation used in Patera and Winternitz,[17] to the three-dimensional algebra $A_1\oplus A_2$.

4. *For $f(u)=f_0u^p$ and $g(u)=0$,*

   (a) *if $a_3\neq -p, -\frac{p}{2}$,*

$$\{X_1+\lambda X_2, X_2, X_3, a_3X_3+X_4\};$$

   (b) *if $a_3=-\frac{p}{2}$,*

$$\left\{X_1+\lambda X_2, X_2, X_3, X_4, X_1-\frac{p}{2}X_3+X_4, -\frac{p}{2}X_3+X_4\right\};$$

   (c) *if $a_3=-p$,*

$$\{X_1+\lambda X_2, X_2, X_3, X_4, X_2-pX_3+X_4, -pX_3+X_4\}.$$

It can be easily proved that considering the Lie algebra defined by $\left\{X_2, X_3-\frac{2}{p}X_4, X_1, X_3-\frac{1}{p}X_4\right\}$, the algebra $\mathcal{A}_4$ decomposes into the direct sum of two two-dimensional Lie algebras with identical commutator structure, namely, $[Y_1,Y_2]=c_{12}Y_2$, with c_{12} the structure constant. Therefore, this case corresponds, following the notation used in Patera and Winternitz,[17] to $2A_2$.

5. *For $f(u) = f_0 u^{8/3}$ and $g(u) = 0$,*
   (a) *if $a_4 \neq 0$,*
      i. *if $a_2 \neq 0$,*
         A. *if $\tilde{a}_3{}^2 - 4a_2a_5 > 0$, the classification of the optimal system for this case corresponds to the previous Case 4, with $p = \frac{8}{3}$;*
         B. *if $\tilde{a}_3{}^2 - 4a_2a_5 = 0$,*
            - *if $a_5 \neq 0$, the classification of the optimal system is the same as in the previous case, where $\tilde{a}_3{}^2 - 4a_2a_5 > 0$.*
            - *if $a_5 = 0$, the reduced optimal system also corresponds to the previous case, where $\tilde{a}_3{}^2 - 4a_2a_5 > 0$, but with $a_3 = -p = -\frac{8}{3}$;*
         C. *if $\tilde{a}_3{}^2 - 4a_2a_5 < 0$,*
            - *if $\tilde{a}_3 = \frac{4}{3}$, the reduced optimal system is*

$$\{X_1 + \mu X_2 + X_5, X_2 + X_5, -X_2 + X_5, X_1, X_2, X_5\}\,;$$

            - *if $\tilde{a}_3 \neq \frac{4}{3}$, the reduced optimal system is*

$$\{X_2 + X_4 + \lambda X_5, -X_2 + X_4 + \lambda X_5, X_4 + \lambda X_5, X_2, X_5\}\,;$$

      ii. *if $a_2 = 0$,*
         A. *if $a_3 \neq -p = -\frac{8}{3}$,*
            - *if $a_3 \neq -\frac{p}{2} = -\frac{4}{3}$, the classification of the optimal system for this case corresponds to the previous Case 4, with $a_3 \neq \frac{-p}{2} = -\frac{4}{3}$;*
            - *if $a_3 = -\frac{p}{2} = -\frac{4}{3}$, the reduced optimal system also corresponds to the previous Case 4, but with $a_3 = -\frac{p}{2} = -\frac{4}{3}$;*
         B. *if $a_3 = -p = -\frac{8}{3}$, the reduced optimal system is*

$$\left\{-\frac{8}{3}X_3 + X_4 + X_5, -\frac{8}{3}X_3 + X_4, X_3, X_4, X_5\right\}\,;$$

   (b) *if $a_4 = 0$, the reduced optimal system is*

$$\{X_2 + X_3 + \lambda X_5, -X_2 + X_3 + \lambda X_5, X_3 + \lambda X_5, X_1 + \mu X_2 + X_5, X_2 + X_5, -X_2 + X_5, X_1 + \gamma X_2, X_2, X_5\}\,.$$

6. *For $f(u) = f_0 e^{pu}$ and $g(u) = g_0 e^{qu}$, with p and q not simultaneously zero,*
   (a) *if $p \neq q, 2q$,*

$$\{X_1 + \lambda X_2, X_2, X_6\}\,,$$

   which corresponds, following the notation used in Patera and Winternitz,[17] to the three-dimensional algebra $A_{3,3}$;
   (b) *if $p = q$,*

$$\{X_1 + \lambda X_2, X_2, \mu X_2 + X_6\}\,,$$

   which corresponds, following the notation used in Patera and Winternitz,[17] to the three-dimensional algebra $A_1 \oplus A_2$;
   (c) *if $p = 2q$,*

$$\{X_1 + \lambda X_2, X_2, \mu X_1 + X_6\}\,,$$

   which corresponds, following the notation used in Patera and Winternitz,[17] to the three-dimensional algebra $A_1 \oplus A_2$.

7. *For* $f(u) = f_0 e^{pu}$ *and* $g(u) = 0$,

   (a) *if* $a_3 \neq -p, -\frac{p}{2}$,

$$\{X_1 + \lambda X_2, X_2, X_3, a_3 X_3 + X_6\};$$

   (b) *if* $a_3 = -\frac{p}{2}$,

$$\left\{X_1 + \lambda X_2, X_2, X_3, X_6, X_1 - \frac{p}{2}X_3 + X_6, -\frac{p}{2}X_3 + X_6\right\};$$

   (c) *if* $a_3 = -p$,

$$\{X_1 + \lambda X_2, X_2, X_3, X_6, X_2 - pX_3 + X_6, -pX_3 + X_6\}.$$

It can be easily proved that considering the Lie algebra defined by $\left\{X_2, X_3 - \frac{2}{p}X_6, X_1, X_3 - \frac{1}{p}X_6\right\}$, *the algebra* $\mathcal{A}_7$ *decomposes into the direct sum of two two-dimensional Lie algebras with identical commutator structure, namely,* $[Y_1, Y_2] = c_{12}Y_2$, *with* c_{12} *the structure constant. Therefore, this case corresponds, following the notation used in Patera and Winternitz,*[17] *to* $2A_2$.

In the above optimal systems, $\tilde{a_3} = a_3 + \frac{8}{3}$, where a_1, a_2, a_3, a_4, a_5, a_6 are coefficients of a particular generator of $X = a_1X_1 + a_2X_2 + a_3X_3 + a_4X_4 + a_5X_5 + a_6X_6$, depending on the case study. In addition, $f_0 \neq 0$, $g_0 \neq 0$, p and q not simultaneously zero, and λ, μ, γ are arbitrary constants.

4 | SYMMETRY REDUCTIONS

The number of independent variables of Equation (2) can be reduced by using its admitted point symmetries appearing in Theorem 1. Consequently, the original PDE (2) is transformed into ODEs.

The invariant surface condition has the form

$$\eta(x,t,u) - \xi(x,t,u)u_x - \tau(x,t,u)u_t = 0,$$

and it is solved by introducing the corresponding characteristic system for each point symmetry,

$$\frac{dx}{\xi} = \frac{dt}{\tau} = \frac{du}{\eta}.$$

For each one-dimensional subalgebra of the optimal systems, the similarity variable and the similarity solution are derived from the characteristic system. Then, by substituting the transformations into Equation (2), we determine the corresponding ODEs.

- For $X_1 + \lambda X_2$, with $f(u)$ and $g(u)$ arbitrary functions, we obtain the similarity variable and the similarity solution:

$$z = x - \lambda t, \qquad u = h(z). \tag{6}$$

  Substituting invariants (6) into Equation (2), we get

$$\Delta_1 \equiv f(h)h^{(4)} + g(h)h'' - \lambda h' = 0.$$

- For X_2, with $f(u)$ and $g(u)$ arbitrary functions, we obtain the invariants

$$z = t, \qquad u = h(z),$$

  where $h(z)$ satisfies the equation

$$\Delta_2 \equiv h' = 0.$$

- For X_3, with $f(u)$ arbitrary function and $g(u) = 0$, we obtain the invariants

$$z = \frac{x}{t^{1/4}}, \qquad u = h(z),$$

where $h(z)$ satisfies the equation

$$\Delta_3 \equiv 4f(h)h^{(4)} - zh' = 0.$$

- For X_4, with $f(u) = f_0 u^p$ and $g(u) = g_0 u^q$, where p and q are not simultaneously zero, we obtain the invariants

$$z = x t^{-\frac{p-q}{2p-4q}}, \qquad u = t^{\frac{1}{p-2q}} h(z),$$

where $h(z)$ satisfies the equation

$$\Delta_4 \equiv 2f_0(p-2q)h^p h^{(4)} + 2g_0(p-2q)h^q h'' - z(p-q)h' + 2h = 0.$$

- For X_5, with $f(u) = f_0 u^{8/3}$ and $g(u) = 0$, we obtain the invariants

$$z = t, \qquad u = x^3 h(z),$$

where $h(z)$ satisfies the equation

$$\Delta_5 \equiv h' = 0.$$

- For X_6, with $f(u) = f_0 e^{pu}$ and $g(u) = g_0 e^{qu}$, where p and q are not simultaneously zero, we obtain the invariants

$$z = x t^{-\frac{p-q}{2p-4q}}, \qquad u = \frac{\ln(t)}{p-2q} + h(z),$$

where $h(z)$ satisfies the equation

$$\Delta_6 \equiv 2f_0(p-2q)e^{ph} h^{(4)} + 2g_0(p-2q)e^{qh} h'' - z(p-q)h' + 2 = 0.$$

- For $\mu X_2 + X_4$, with $f(u) = f_0 u^p$ and $g(u) = g_0 u^q$, where $q = p$, not simultaneously zero, we obtain the invariants

$$z = \frac{2px + \mu \ln(t)}{2p}, \qquad u = t^{-1/p} h(z),$$

where $h(z)$ satisfies the equation

$$\Delta_7 \equiv p h^p (f_0 h^{(4)} + g_0 h'') + \frac{\mu}{2} h' - h = 0.$$

- For $\mu X_1 + X_4$, with $f(u) = f_0 u^p$ and $g(u) = g_0 u^q$, where $p = 2q$, not simultaneously zero, we obtain the invariants

$$z = x e^{-qt/\mu}, \qquad u = e^{2t/\mu} h(z),$$

where $h(z)$ satisfies the equation

$$\Delta_8 \equiv -\mu f_0 h^{2q} h^{(4)} - \mu g_0 h^q h'' + qzh' - 2h = 0.$$

- For $a_3X_3 + X_4$, with $f(u) = f_0\,u^p$ and $g(u) = 0$, we obtain the invariants

$$z = x\,t^{-\frac{a_3+p}{2p+4a_3}}, \qquad u = t^{\frac{1}{p+2a_3}}\,h(z),$$

where $h(z)$ satisfies the equation

$$\Delta_9 \equiv -2(2a_3+p)\,f_0\,h^p\,h^{(4)} + (a_3+p)zh' - 2h = 0.$$

- For $X_2 + X_5$, with $f(u) = f_0\,u^{8/3}$ and $g(u) = 0$, we obtain the invariants

$$z = t, \qquad u = (x^2+1)^{3/2}\,h(z),$$

where $h(z)$ satisfies the equation

$$\Delta_{10} \equiv h' + 9f_0\,h^{11/3} = 0.$$

- For $X_3 + \lambda X_5$, with $f(u) = f_0\,u^{8/3}$ and $g(u) = 0$, we obtain the invariants

$$z = \frac{t^{1/4}(\lambda x+1)}{x}, \qquad u = \frac{x^3\,h(z)}{t^{3/4}},$$

where $h(z)$ satisfies the equation

$$\Delta_{11} \equiv 4f_0\,h^{8/3}\,h^{(4)} + zh' - 3h = 0.$$

- For $\mu X_2 + X_6$, with $f(u) = f_0\,e^{p\,u}$ and $g(u) = g_0\,e^{q\,u}$, where $q = p$, not simultaneously zero, we obtain the invariants

$$z = \frac{2px + \mu\ln(t)}{2p}, \qquad u = \frac{p\,h(z) - \ln(t)}{p},$$

where $h(z)$ satisfies the equation

$$\Delta_{12} \equiv 2p\,e^{p\,h}(f_0h^{(4)} + g_0h'') + \mu\,h' - 2 = 0.$$

- For $\mu X_1 + X_6$, with $f(u) = f_0\,e^{p\,u}$ and $g(u) = g_0\,e^{q\,u}$, where $p = 2q$, not simultaneously zero, we obtain the invariants

$$z = x\,e^{-qt/\mu}, \qquad u = \frac{2t}{\mu} + h(z),$$

where $h(z)$ satisfies the equation

$$\Delta_{13} \equiv \mu\,f_0\,e^{2q\,h}\,h^{(4)} + \mu\,g_0\,e^{q\,h}\,h'' - qzh' + 2 = 0.$$

- For $a_3X_3 + X_6$, with $f(u) = f_0\,e^{p\,u}$ and $g(u) = 0$, we obtain the invariants

$$z = x\,t^{-\frac{a_3+p}{4a_3+2p}}, \qquad u = \frac{\ln(t)}{2a_3+p} + h(z),$$

where $h(z)$ satisfies the equation

$$\Delta_{14} \equiv 2(2a_3+p)\,f_0\,e^{p\,h}h^{(4)} - (a_3+p)zh' + 2 = 0.$$

- For $X_1 - \frac{p}{2}X_3 + X_4$, with $f(u) = f_0\,u^p$ and $g(u) = 0$, we obtain the invariants

$$z = x\,e^{-pt/2}, \qquad u = e^{2t}h(z),$$

where $h(z)$ satisfies the equation

$$\Delta_{15} \equiv -2f_0 h^p h^{(4)} + pzh' - 4h = 0.$$

- For $X_2 - pX_3 + X_4$, with $f(u) = f_0 u^p$ and $g(u) = 0$, we obtain the invariants

$$z = \frac{2px + \ln(t)}{2p}, \qquad u = t^{-1/p} h(z),$$

where $h(z)$ satisfies the equation

$$\Delta_{16} \equiv 2f_0 p h^p h^{(4)} + h' - 2h = 0.$$

- For $X_1 + \mu X_2 + X_5$, with $f(u) = f_0 u^{8/3}$ and $g(u) = 0$, we obtain the invariants

$$z = \frac{-t + \sqrt{\mu} + \arctan\left(\frac{x}{\sqrt{\mu}}\right)}{\sqrt{\mu}}, \qquad u = \left(1 + \frac{x^2}{\mu}\right)^{3/2} h(z),$$

where $h(z)$ satisfies the equation

$$\Delta_{17} \equiv f_0 h^{8/3}\left(\frac{h^{(4)}}{10} + h''\right) - \frac{\mu^4 h'}{10} + \frac{9\mu^2 f_0 h^{11/3}}{10} = 0.$$

- For $X_1 - \frac{p}{2}X_3 + X_6$, with $f(u) = f_0 e^{pu}$ and $g(u) = 0$, we obtain the invariants

$$z = xe^{-pt/2}, \qquad u = 2t + h(z),$$

where $h(z)$ satisfies the equation

$$\Delta_{18} \equiv f_0 e^{ph} h^{(4)} - \frac{pzh'}{2} + 2 = 0.$$

- For $X_2 - pX_3 + X_6$, with $f(u) = f_0 e^{pu}$ and $g(u) = 0$, we obtain the invariants

$$z = \frac{2px + \ln(t)}{2p}, \qquad u = \frac{-\ln(t) + ph(z)}{p},$$

where $h(z)$ satisfies the equation

$$\Delta_{19} \equiv 2f_0 p e^{ph} h^{(4)} + h' - 2 = 0.$$

In the above symmetry reductions, $a_1, a_2, a_3, a_4, a_5, a_6$ are coefficients of a particular generator of $X = a_1X_1 + a_2X_2 + a_3X_3 + a_4X_4 + a_5X_5 + a_6X_6$, depending on the case study. Besides, $f_0 \neq 0$, $g_0 \neq 0$, p and q not simultaneously zero, and λ, μ, γ are arbitrary constants.

5 | EXACT SOLUTIONS

Particularly, for the generator $X_1 + \lambda X_2$, considering f and g arbitrary functions, the similarity variable and the similarity solution are given by $z = x - \lambda t$, $u = h(z)$, so $u(x,t) = h(x - \lambda t)$. Consequently, the corresponding solution is a travelling wave solution.

In general, the derivative of trigonometric functions can be expressed based on themselves. Hence, functions f and g can be written as an algebraic function of h, so equation Δ_1 admits the trigonometric functions $\sin^n(h)$, $\cos^n(h)$, $\tan^n(h)$ as solutions. In the same way, we can use the hyperbolic functions $\sinh^n(h)$, $\cosh^n(h)$, $\tanh^n(h)$.

Next, we present some exact solutions.

- Let us consider $\lambda = 1$ and $f(h)$, $g(h)$ satisfying

$$\pm\sqrt{1-h} + f(h)(60h^4 - 60h^2 + 8) + g(h)(2 - 3h) = 0.$$

Thus, we have that $h(z) = sech^2(z)$ is a solution of Δ_1. Consequently, the fourth-order parabolic equation (2) admits

$$u(x,t) = sech^2(x - t) \tag{7}$$

as a solution.

Figure 1 shows a soliton moving along a line with constant velocity.

- Let us consider $\lambda = -\frac{1}{2}$. If $h(z) = \frac{1}{4}\tanh(z)$ is a solution of equation Δ_1, then $f(h)$ and $g(h)$ verify

$$f(h)(6144h^5 - 640h^3 + 16h) + g(h)(32h^3 - 2h) - 2h^2 + \frac{1}{8} = 0.$$

Hence, a solution of Equation (2) is

$$u(x,t) = \frac{1}{4}\tanh\left(x + \frac{1}{2}t\right), \tag{8}$$

which is a kink solution. The graphic of this solution is given in Figure 2.

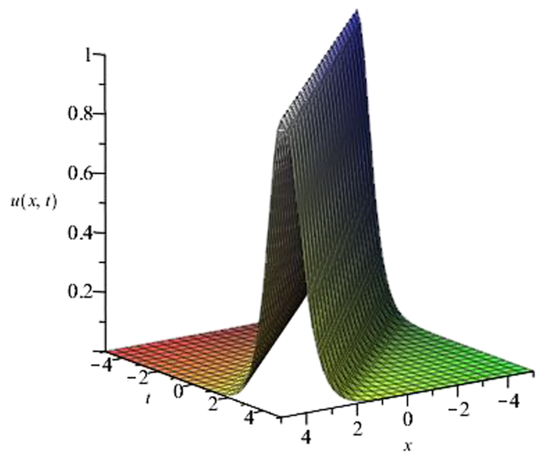

FIGURE 1 Solution (7) [Colour figure can be viewed at wileyonlinelibrary.com]

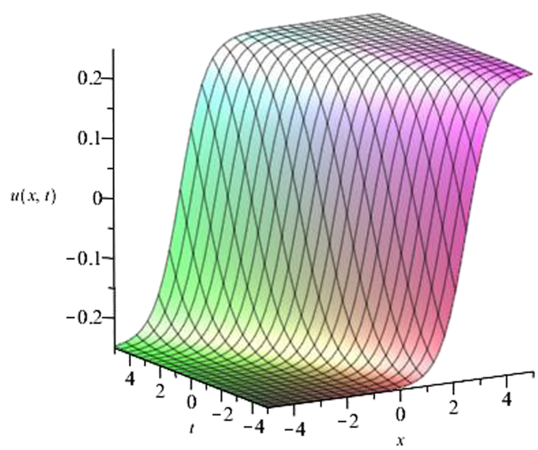

FIGURE 2 Solution (8) [Colour figure can be viewed at wileyonlinelibrary.com]

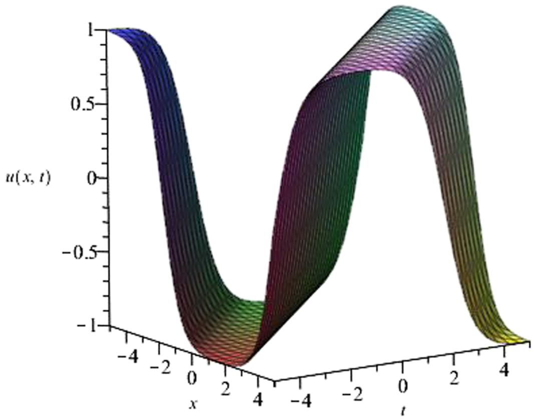

FIGURE 3 Solution $u(x,t) = sn(x+t,m)$, with $m = 0,99$ [Colour figure can be viewed at wileyonlinelibrary.com]

- Let us consider $\lambda = -1$. If $f(h)$ and $g(h)$ satisfy

$$f(h)(5h^5m^2 + 18h^4m^2 - 5h^3m^2 - 14h^2m^2 + hm^2 - 18h^3m - 2h^2m + 14hm + h) + g(h)(h^3m + h^2m - hm - h) + \sqrt{1-h^2}\sqrt{1-hm} = 0,$$

then we have $h(z) = sn(z,m)$, where sn is a Jacobi elliptic function, as a solution of ODE Δ_1. It is well known that $sn(z,1) = \tanh(z)$ and $sn(z,0) = \sin(z)$. Thus, the graphical representation of a Jacobi elliptic sine function has interesting shapes. For instance, in Figure 3, we can observe a periodic wave.

6 | CONCLUSIONS

In this paper, we have classified the Lie point symmetries of a fourth-order nonlinear diffusion equation, depending on functions $f(u)$ and $g(u)$, by applying the Lie symmetry method. Furthermore, we have presented the maximal Lie algebras with the non-zero commutators. The Lie point symmetries have been used to construct the optimal systems of one-dimensional subalgebras for each maximal Lie algebra, by using the commutation relations and the adjoint actions. Afterwards, these systems have been used to obtain symmetry reductions and to transform the original equation to ODEs, providing new group-invariant solutions of Equation (2). In addition, we have found some travelling wave solutions for Equation (2). Moreover, the exact solutions constructed are of physical interest, such as solitons, kinks and periodic waves.

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CONFLICT OF INTEREST

The authors declare no potential conflict of interest.

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