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A new formulation and branch-and-cut method for single-allocation hub location problems

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ABSTRACT

A new compact formulation for uncapacitated single-allocation hub location problems with fewer variables than the previous Integer Linear Programming formulations in the literature is introduced. Our formulation works even with costs not based on distances and not satisfying triangle inequality. Moreover, costs can be given in aggregated or disaggregated way. Different families of valid inequalities that strengthen the formulation are developed and a branch-and-cut algorithm based on a relaxed version of the formulation is designed, whose restrictions are inserted in a cut generation procedure together with two sets of valid inequalities. The performance of the proposed methodology is tested on well-known hub location data sets and compared to the most recent and efficient exact algorithms for single-allocation hub location problems. Extensive computational results prove the efficiency of our methodology, that solves large-scale instances in very competitive times.

1. Introduction

Hub location problems arise in various application settings, e.g., telecommunication and transportation systems where several origin/destination sites send and receive some product. Instead of serving each origin–destination pair directly (because this sort of linkage is too expensive to be carried out), transshipment points (hubs) collect the product from the origin and distribute it to the destination. These hubs centralize the product shipment, resulting in lower transportation costs and potential savings in the overall design and operational costs of the system. Therefore, the hubs systems are designed to exploit the scale economies attainable through the shared use of high capacity links between hubs. Alumur et al. (2021) include discussions of modeling economies of scale and real-world examples of hub systems, as passenger and freight airlines, less-than-truckload and truckload transportation, postal operations, express shipment and cargo delivery, liner shipping, public transit, and computer and telecommunication networks. Moreover, new applications are appearing, as the green hubs or hub systems for medical applications including in drone delivery networks. This wide range of applications indicates the power of the hubs location problems and the need for more and better models.

Many reviews about hub location problems, see Alumur and Kara (2008), Campbell and O'Kelly (2012), Farahani et al. (2013), Contreras and O'Kelly (2019) and Alumur et al. (2021), show the very wide range of activity in this field and the applications of these problems.

The hub location problem aims to locate hub facilities and to allocate origin–destination sites to hubs in order to establish a route between each origin and destination that goes only through located hubs. Different variants of hub location problems have been investigated according to different allocation strategies. Single allocation, where the product to be sent to and received from a given site must be routed through a single hub (O'Kelly, 1987; Ernst and Krishnamoorthy, 1996, 1998; Kara and Tansel, 2000; Correia et al., 2010; Contreras et al., 2010; Rostami et al., 2016, among others) or multiple allocation, when these sites can send and receive the amount of product to and from more than one hub (Ernst and Krishnamoorthy, 1998; Ebery et al., 2000; Hamacher et al., 2004; Marín, 2005; Contreras et al., 2011; García et al., 2012, among others). A strategy generalizing both single and multiple allocation models is the r-allocation, where each site can be connected to at most r hubs (Yaman, 2011; Corberán et al., 2019). When the number of hubs is set in advance, this problem is known as the p-hub median problem. The problem is said capacitated if hubs have finite capacities (Campbell, 1994; Ernst and Krishnamoorthy, 1999; Ebery et al., 2000; Correia et al., 2010; Contreras et al., 2012; Meier and Clausen, 2017). If no capacity is considered for each hub, the problems

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Table 1
Number of variables and constraints for different formulations of the USApHMP.

Formulation	Binary variables	Continuous variables	Constraints
Campbell (1994)	n^2+n	n^4	n^4+n^2+n+1
Skorin-Kapov et al. (1996)	n^2	n^4	$2n^3+n^2+n+1$
Ernst and Krishnamoorthy (1996)	n^2	n^3	$2n^2+n+1$
Ebery (2001)	n^2	n^2	$2n^2+n+1$
Meier and Clausen (2017)	n^2	n^2	n^4+n^2+n+1
EMMR$_r$ in Section 3	n^2	n	$2n^2+n+1$

are referred to as uncapacitated hub location problems. Several works that propose different cost structures can be found in the literature, e.g., Bryan (1998), O'Kelly and Bryan (1998), O'Kelly et al. (2015), and Lüer-Villagra et al. (2019). Reviews, synthesis and classification on different variants of the hub location problem can be found in Alumur and Kara (2008), Campbell and O'Kelly (2012), Contreras and O'Kelly (2019), Alumur et al. (2021), among others.

This paper deals with uncapacitated single-allocation hub location problems. In the uncapacitated single-allocation p-hub median problem (USApHMP), the aim is to choose p hubs and assign every site to them minimizing the overall transportation costs between origins and destinations through the hubs. In the uncapacitated single-allocation hub location problem (USAHLP) a cost for setting a hub is given and the number of hubs is a decision variable. The aim is to locate the hubs and to assign the remaining sites to the hubs minimizing the overall installation and transportation costs. Both problems are NP-hard. Moreover, even if the locations of the hubs are fixed, the allocation part of the problem remains NP-hard (Kara, 1999).

We concentrate on the USApHMP. O'Kelly (1987) presented the first mathematical formulation for this problem. Since then, different linearization strategies have been used in the literature to handle the quadratic term in the objective function of this model. Campbell (1994) proposed the first linear integer programming formulation for the USApHMP. After that, Skorin-Kapov et al. (1996) proposed a new mixed integer formulation (which they referred to as the path-based formulation) with fewer binary variables and constraints than the previous one. This formulation has been widely used by decomposition methods capable of handling its number of variables and constraints. An alternative to handle the large number of variables is to project them out from the path-based formulation (Labbé and Yaman, 2004; Labbé et al., 2005; De Camargo et al., 2011; De Camargo and De Miranda, 2012). Ernst and Krishnamoorthy (1996) proposed a different linear integer programming formulation (flow-based formulation) that reduces the size of the formulation of Skorin-Kapov et al. (1996), both in terms of variables and constraints. This formulation has been used to model many extensions of single-allocation hub location problems such as capacitated and balanced problems (Correia et al., 2011).

Ebery (2001) introduced a formulation that uses fewer variables than those of Ernst and Krishnamoorthy (1996) and Skorin-Kapov et al. (1996). However, the formulation of Skorin-Kapov et al. (1996) provides a better linear relaxation and the one introduced in Ernst and Krishnamoorthy (1996) is the most effective in terms of computation time requirement.

Alternative methods to linearize the binary quadratic terms in the formulation by O'Kelly (1987) are given in Meier et al. (2016), Ghaffarinasab and Kara (2019) and Rostami et al. (2022). The method described in Meier et al. (2016) uses a row generation procedure and applies whenever Euclidean distances are used. Ghaffarinasab and Kara (2019) proposed exact algorithms based on Benders decomposition for solving large-scale instances. They assume that the transportation costs between hubs are proportional to the distance between them. Rostami et al. (2022) provided a convex reformulation and a branch-and-cut algorithm based on outer approximation cuts.

Table 1 gives the number of variables and constraints of the aforementioned formulations, where n is the number of sites that represent the origins and destinations. As shown in the table, our paper constitutes a contribution to the existing literature of the USApHMP by providing a new formulation that uses fewer variables than the aforementioned ones. Our formulation is valid even for costs not based on distances and not satisfying triangle inequality. Moreover, costs can be given in aggregated or disaggregated way. This allows us to model more realistic cases in transportation systems where, for instance, fares are not proportional to travel distances or longer trips may have lower ticket prices than shorter trips. Moreover, some of the existing formulations for the USApHMP need to have the overall transportation cost from origin to destination disaggregated in the three components: origin–hub, hub–hub, hub–destination. We develop formulations for both cases, with aggregated/disaggregated transportation costs. Different families of valid inequalities are obtained considering extended formulations and later projecting out some of their variables by applying the Farkas' lemma. Moreover, separation procedures for these inequalities are developed. A comparison of the performance of the most recent and efficient solution methods existing in the literature (Meier et al., 2016; Ghaffarinasab and Kara, 2019; Rostami et al., 2022) shows the efficiency of our methodology, solving large-scale instances in competitive times. Although we focus on USApHMP, the formulation can be adapted to USAHLP. Furthermore, capacitated versions could also benefit from our results, as could other variants of the problem.

The remainder of this paper is organized as follows. In the next section, we describe the USApHMP. Section 3 introduces a new formulation to solve these problems which is reinforced with families of valid inequalities. Section 4 shows the procedure carried out to solve the USApHMP using our formulation. Computational experiments as well as the corresponding results are presented in Section 5. Finally, some conclusions and outlooks for future research are presented.

2. Description of the problem

Let $N := \{1, \dots, n\}$ be the set of sites representing the origins and destinations of the product to be transported and also the set of potential hub locations. Let $\omega_{ij} \geq 0$ be the amount of product to be sent from the origin i to the destination j, for all $i, j \in N$. There is a fixed number p of sites of N, $2 \leq p \leq n-1$ that must be chosen to be hubs. We require the product to be sent from the origin i to destination j through one or, at most, two hubs. Hence, there are three possibilities: (i) if neither the origin nor the destination are hubs, the product is sent through one or two hubs; (ii) if either the origin or the destination (but not both of them) is a hub, then the product may be sent directly or it may go through one intermediate hub; (iii) if both the origin and the destination are hubs, the product is sent directly.

Moreover, every site i which is not a hub must be allocated to a single hub, so that the amount of product sent from/to i to/from any other site must pass through this hub.

Define C_{ijkm} as the cost of transporting one unit of product from i to j through hubs k and m in this order. These costs can sometimes be disaggregated in three components (transportation costs between origin–hub, hub–hub, hub–destination) that consider some discount factor between hubs. We do not require costs based on distances neither satisfaction of the triangle inequality.

The USApHMP consists of determining a subset of p sites to become hubs and assign every site to them minimizing the overall transportation costs between origins and destinations through the hubs.

In the rest of this paper, i and j are used to index origins and destinations respectively, and k and m are used to index hub locations.

For the sake of simplicity, instead of considering costs C_{ijkm}, $\forall i,j,k,m \in N$, let define $\hat{C}_{ijkm} := \omega_{ij}C_{ijkm} + \omega_{ji}C_{jimk}$ for all $i,j(i<j) \in N$ and $k,m \in N$, as the cost of transporting the total amount of product between sites i and j (from i to j and from j to i) using k as the hub assigned to i and m as the one assigned to j.

3. New formulation for solving the USApHMP

The USApHMP was first formulated by O'Kelly (1987) using the following variables for the location of hubs as well as the allocations to hubs:

$$x_{ik} = \begin{cases} 1 & \text{if site } i \text{ is assigned to hub } k, \\ 0 & \text{otherwise,} \end{cases} \qquad \forall i,k(i \neq k) = 1,\dots,n.$$

When $i = k$, variable x_{kk} represents whether or not a hub is located at site k. The x variables are usually referred to as location/allocation variables.

The quadratic integer program for the USApHMP with aggregated costs proposed by O'Kelly (1987) is given by

$$\min \sum_{i\in N}\sum_{\substack{j\in N:\\ j>i}}\sum_{k\in N}\sum_{m\in N} \hat{C}_{ijkm}x_{ik}x_{jm}, \tag{1}$$

$$\text{s.t.} \sum_{k\in N} x_{kk} = p, \tag{2}$$

$$x_{ik} \leq x_{kk}, \qquad \forall i,k(i\neq k) \in N, \tag{3}$$

$$\sum_{k\in N} x_{ik} = 1, \qquad \forall i \in N, \tag{4}$$

$$x_{ik} \in \{0,1\}, \qquad \forall i,k \in N. \tag{5}$$

The objective function (1) is quadratic due to the product of the x-variables. The number of hubs to be located is set by (2). Constraints (3) ensure that a site i is not assigned to a site k unless a hub is opened at k. Constraints (4) ensure that each site is assigned to exactly one other site and (5) give the binary condition of x-variables.

Different linearization strategies have been proposed to handle the quadratic term in (1). In this paper, we present a new linear integer programming formulation with fewer variables than the formulations found in the literature for solving the USApHMP. Unlike other works such as (Meier et al., 2016), that requires Euclidean distances, or (Ghaffarinasab and Kara, 2019), that also assume that the costs are proportional to the distances, this formulation holds for costs that do not require any of the following assumptions: (i) costs are given by distances among sites, (ii) costs need to be symmetrical, (iii) costs satisfy the triangle inequality and (iv) costs are necessarily given in a disaggregated way (cost from origin to the first hub, cost between hubs and cost from the last hub to the destination).

For $i \in N$, we define

S_i = overall transportation cost with origin/destination at i to/from any site $j \in N$ with $j > i$.

That is, $S_i = \sum_{\substack{j\in N\\ j>i}}\sum_{m\in N}\sum_{k\in N} \hat{C}_{ijkm}x_{ik}x_{jm}$.

We propose a formulation based on the variables S_i as follows:

$$\text{(EMMR)} \quad \min \sum_{i\in N} S_i \tag{6}$$

$$\text{s.t. } (2)\text{–}(5),$$

$$S_i \geq \sum_{\substack{j\in N:\\ j>i}}\sum_{m\in N} \hat{C}_{ijkm}(x_{ik}+x_{jm}-1), \qquad \forall i,k \in N, \tag{7}$$

$$S_i \geq 0, \qquad \forall i \in N.$$

The meaning of constraints (7) is the following. If $x_{ik} = 1$ for some $i,k \in N$, then $S_i \geq \sum_{\substack{j\in N:\\ j>i}}\sum_{m\in N}\hat{C}_{ijkm}x_{jm}$. That is, if site i is allocated to hub k, then S_i is at least the cost of transporting the total amount of product from/to site i using hub k to/from any site $j \in N$ with $j > i$. Since (EMMR) is a minimization problem and considering constraints (4), S_i is exactly that overall transportation cost. Otherwise, if $x_{ik} = 0$ then $S_i \geq \sum_{\substack{j\in N:\\ j>i}}\sum_{m\in N}\hat{C}_{ijkm}(x_{jm}-1)$, which is a redundant constraint since the right hand side of this inequality is non-positive. Therefore, in (6) we are minimizing the sum of the transportation costs between origins and destinations through the hubs. Thus, this formulation solves the USApHMP.

To the best of our knowledge, this is the first formulation using only n continuous variables to solve the USApHMP (see Table 1). This formulation has n^2+n (n^2 binary and n continuous) variables and $2n^2+n+1$ linear constraints.

Clearly a small variation of formulation (EMMR) will yield a formulation for USAHLP by removing constraint (2) and including a term in the objective function which provides the costs of locating the hubs.

Let (EMMR$_r$) be the formulation obtained by replacing (7) in (EMMR) with

$$S_i \geq \sum_{\substack{j\in N\\ j>i}}\left(\sum_{m\in N}\hat{C}_{ijkm}x_{jm} + (x_{ik}-1)\max_{m\in N}\{\hat{C}_{ijkm}\}\right), \qquad \forall i,k \in N. \tag{8}$$

Proposition 3.1 (*EMMR$_r$*). *is a formulation for the USApHMP and reinforces (EMMR).*

Proof. First we prove that (EMMR$_r$) is a formulation for the USApHMP. For any $i,k \in N$ such that $x_{ik} = 0$, the right hand side of (8) is non-positive. Moreover, by (4), for any $i \in N$ there exists $k \in N$ such that $x_{ik} = 1$. Then for these i and k the right hand side of the corresponding constraint (8) will represent the overall transportation cost of the product from/to i to/from j with $j > i$. Since we are minimizing in the S-variables and constraints (4) are fulfilled, we obtain that the sum of S-variables represent the overall transportation cost between origins and destinations. Now, we prove that (EMMR$_r$) reinforces (EMMR). To do that, we need to prove that the right hand side of (8) is greater than or equal to the one of (7). But it is straightforward because $\sum_{m\in N}\hat{C}_{ijkm} \geq \max_{m\in N}\hat{C}_{ijkm}$ and $x_{ik} \leq 1$. □

3.1. Valid inequalities for (EMMR)

We have provided two formulations for USApHMP. However, a preliminary computational experience shows that (EMMR$_r$) still has a weak linear relaxation. For this reason, we focus on developing new valid inequalities to strengthen this formulation. Our approach consists of considering extended formulations of (EMMR$_r$) and obtaining valid inequalities from the projection of some variables of these formulations using Farkas' Lemma (Mangasarian, 1969). The version of the lemma used in this paper is the following:

Lemma 3.1. ***Farkas' Lemma.*** *Let A be an $n\times r$ dimensional matrix. A vector $b \in \mathbb{R}^n$ verifies $Ax \leq b$ for $x \in \mathbb{R}^n$ with $x \geq \mathbf{0}$ if and only if for any $y \geq \mathbf{0}$ verifying $A^T y \geq \mathbf{0}$, it is satisfied that $b^T y \geq 0$.*

3.1.1. First family of inequalities

The first family of valid inequalities is obtained by the linearization of the product of x-variables. To do that, for all i, j $(i<j)$, k, $m \in N$, let us define $X_{ijkm} = x_{ik}x_{jm}$ (these variables were also used in the formulation by Skorin-Kapov et al., 1996) and consider the following inequalities:

$$\sum_{m\in N} X_{ijkm} = x_{ik}, \qquad \forall i,j(>i),k \in N, \tag{9}$$

$$X_{ijkm} \geq x_{ik}+x_{jm}-1, \qquad \forall i,j(>i),k,m \in N, \tag{10}$$

$$S_i \geq \sum_{\substack{j\in N\\ j>i}}\sum_{m\in N}\hat{C}_{ijkm}X_{ijkm}, \qquad \forall i,k \in N, \tag{11}$$

$$X_{ijkm} \geq 0, \qquad \forall i,j(>i),k,m \in N. \tag{12}$$

It is straightforward that the above inequalities are valid for the formulation (EMMR$_r$). Hence, the following result provides valid inequalities to (EMMR$_r$) from the projection of the X-variables applying Farkas' Lemma to the resulting extended formulation obtained adding (9)–(12) to (EMMR$_r$).

Proposition 3.2. *Given $(\boldsymbol{x}, S)$, there exists X that satisfies (9)–(12) if and only if for any $i, k \in N$,*

$$S_i \geq \sum_{j \in N} \sum_{m \in N} \beta_{jm}(x_{ik} + x_{jm} - 1) + \sum_{\substack{j \in N \\ j > i}} \gamma_j x_{ik}, \tag{13}$$

for all $(\boldsymbol{\beta}, \boldsymbol{\gamma})$ such that

$$\hat{C}_{ijkm} - \beta_{jm} - \gamma_j \geq 0, \quad \forall j(> i), m \in N, \tag{14}$$

$$\beta_{jm} \geq 0, \quad \forall j(> i), m \in N. \tag{15}$$

Any inequality of the form (13) defined by $(\boldsymbol{\beta}, \boldsymbol{\gamma})$ which satisfies inequalities (14) and (15) is called a *projection inequality*. Observe that constraints (7) are particular cases of (13) by considering $\beta_{jm} = \hat{C}_{ijkm}$ and $\gamma_j = 0$, for all $i, k, j(> i), m \in N$. Our goal is to find those values of parameters $(\boldsymbol{\beta}, \boldsymbol{\gamma})$ satisfying (14)–(15) and providing the most violated valid inequality of the type (13). The following result provides these valid inequalities.

Proposition 3.3. *Given $(\bar{x}, \bar{S})$ a fractional solution of (EMMR$_r$), the following set of inequalities are the most violated within the family (13).*

$$S_i \geq \sum_{\substack{j \in N \\ j > i}} H_{ijk} x_{ik} + \sum_{\substack{j \in N \\ j > i}} \sum_{\substack{m \in N \\ \bar{x}_{ik} + \bar{x}_{jm} > 1}} (\hat{C}_{ijkm} - H_{ijk})(x_{ik} + x_{jm} - 1), \quad \forall i, k \in N \tag{16}$$

where $H_{ijk} := \min_{\substack{m \in N \\ \bar{x}_{ik} + \bar{x}_{jm} > 1}} \hat{C}_{ijkm}$. (In case m does not exist, such that $\bar{x}_{ik} + \bar{x}_{jm} > 1$ for some $i, k \in N$, H_{ijk} will be defined as 0).

Proof. Let $(\bar{x}, \bar{S})$ be a fractional solution of (EMMR$_r$). By maximizing the right hand side of (13) for this solution, we will obtain the most violated version of these inequalities. Hence, for each $i, k \in N$, we solve the following linear optimization problem:

$$\begin{aligned} \max_{\boldsymbol{\beta}, \boldsymbol{\gamma}} \quad & \sum_{\substack{j \in N \\ j > i}} \sum_{m \in N} \beta_{jm}(\bar{x}_{ik} + \bar{x}_{jm} - 1) + \sum_{\substack{j \in N \\ j > i}} \gamma_j \bar{x}_{ik}, \\ \text{s.t.} \quad & \beta_{jm} + \gamma_j \leq \hat{C}_{ijkm}, && \forall j(> i), m \in N, \\ & \beta_{jm} \geq 0, && \forall j(> i), m \in N. \end{aligned}$$

The problem above can be separated for any $i, j(> i), k \in N$ by considering for each $j(> i) \in N$, the variables $\hat{\beta}_m := \beta_{jm}$ and $\hat{\gamma} := \gamma_j$, as follows:

$$\begin{aligned} \max_{\hat{\beta}, \hat{\gamma}} \quad & \sum_{m \in N} \hat{\beta}_m(\bar{x}_{ik} + \bar{x}_{jm} - 1) + \hat{\gamma}\bar{x}_{ik}, \\ \text{s.t.} \quad & \hat{\beta}_m + \hat{\gamma} \leq \hat{C}_{ijkm}, && \forall m \in N, \\ & \hat{\beta}_m \geq 0, && \forall m \in N. \end{aligned}$$

Hence, since $\hat{\gamma} \leq \hat{C}_{ijkm} - \hat{\beta}_m$, then $\hat{\gamma} = \min_{m \in N}\{\hat{C}_{ijkm} - \hat{\beta}_m\}$. Moreover, taking into account the objective function of the problem above we have that:

- If $\bar{x}_{ik} + \bar{x}_{jm_0} - 1 \leq 0$ for some $m_0 \in N$ then $\hat{\beta}_{m_0} = 0$ and $\hat{\gamma} \leq \min_{m \in N} \hat{C}_{ijkm}$.
- If $\bar{x}_{ik} + \bar{x}_{jm_0} - 1 > 0$ and

$$\sum_{\substack{m \in N \\ \bar{x}_{ik} + \bar{x}_{jm} > 1}} (\bar{x}_{ik} + \bar{x}_{jm} - 1) \leq \bar{x}_{ik}, \tag{17}$$

then we have that $\hat{\gamma} \leq \min_{m \in N} \hat{C}_{ijkm}$ and $\hat{\beta}_{m_0} = \hat{C}_{ijkm_0} - \min_{m \in N} \hat{C}_{ijkm}$.

Therefore, since we are maximizing and $\bar{x}_{ik} \geq 0$ then $\hat{\gamma} = \min_{m \in N} \hat{C}_{ijkm}$.

Now, we prove that (17) always holds, i.e., it is not a condition. Let us define $\Omega_{ij} := \{m \in N : \bar{x}_{ik} + \bar{x}_{jm} > 1\}$. Hence,

$$\sum_{\substack{m \in N \\ \bar{x}_{ik} + \bar{x}_{jm} > 1}} (\bar{x}_{ik} + \bar{x}_{jm} - 1) = |\Omega_{ij}|(\bar{x}_{ik} - 1) + \sum_{\substack{m \in N \\ \bar{x}_{ik} + \bar{x}_{jm} > 1}} \bar{x}_{jm}.$$

Therefore, we have that (17) is fulfilled if and only if

$$\sum_{\substack{m \in N \\ \bar{x}_{ik} + \bar{x}_{jm} > 1}} \bar{x}_{jm} \leq \bar{x}_{ik} + (1 - \bar{x}_{ik})|\Omega_{ij}|.$$

The right term of the above inequality is a convex combination of 1 and $|\Omega_{ij}|$, then it will take values in the interval $[1, |\Omega_{ij}|]$. Hence, this inequality is fulfilled since the left term is less than or equal to 1. □

A reinforcement of valid inequalities (16) is provided in the following proposition.

Proposition 3.4. *Given $(\bar{x}, \bar{S})$ a fractional solution of (EMMR$_r$), the following inequalities*

$$S_i \geq \sum_{\substack{j \in N \\ j > i}} H_{ijk} + \sum_{\substack{j \in N \\ j > i}} \left(\sum_{\substack{m \in N \\ \bar{x}_{ik} + \bar{x}_{jm} > 1}} (\hat{C}_{ijkm} - H_{ijk}) x_{jm} + \max_{\substack{m \in N \\ \bar{x}_{ik} + \bar{x}_{jm} > 1}} \{\hat{C}_{ijkm}\}(x_{ik} - 1) \right), \forall i, k \in N \tag{18}$$

are valid for (EMMR$_r$) and reinforce constraints (16).

Proof. First, we prove that inequalities (18) are valid. (Observe that $\bar{x}$ fractional values are only used to define the summation indices conditions, but the validity of these inequalities should be shown for binary x-variables). We distinguish two cases.

- If $\sum_{\substack{m \in N \\ \bar{x}_{ik} + \bar{x}_{jm} > 1}} x_{jm} = 0$, then (18) can be rewritten as

$$S_i \geq \sum_{\substack{j \in N \\ j > i}} \left(H_{ijk} + \max_{\substack{m \in N \\ \bar{x}_{ik} + \bar{x}_{jm} > 1}} \{\hat{C}_{ijkm}\}(x_{ik} - 1) \right), \quad \forall i, k \in N. \tag{19}$$

  – Case $x_{ik} = 0$; the right hand side of these constraints is non-positive and the corresponding constraints are meaningless.
  – Case $x_{ik} = 1$; (19) is rewritten as

$$S_i \geq \sum_{\substack{j \in N \\ j > i}} H_{ijk}, \quad \forall i, k \in N.$$

  These constraints are valid since S_i is at least the sum of the minimum costs of the flow from/to site i to/from another site j via a hub with $j > i$.

- If $\sum_{\substack{m \in N \\ \bar{x}_{ik} + \bar{x}_{jm} > 1}} x_{jm} = 1$,

  – Case $x_{ik} = 1$; then S_i is at least the transportation cost from/to i of the overall flow to/from j with $i < j$ and since we are minimizing, S_i represents that transportation cost for each $i \in N$.
  – Case $x_{ik} = 0$; again the right hand side of (18) becomes a non-positive amount and then (18) is meaningless.

Now, we prove that constraints (18) reinforce (16):

$$\sum_{\substack{j \in N \\ j > i}} \left(H_{ijk} x_{ik} + \sum_{\substack{m \in N \\ \bar{x}_{ik} + \bar{x}_{jm} > 1}} (\hat{C}_{ijkm} - H_{ijk})(x_{ik} + x_{jm} - 1) \right) =$$

$$\sum_{\substack{j \in N \\ j > i}} \left(H_{ijk} x_{ik} + \sum_{\substack{m \in N \\ \bar{x}_{ik} + \bar{x}_{jm} > 1}} (\hat{C}_{ijkm} - H_{ijk}) x_{jm} + \sum_{\substack{m \in N \\ \bar{x}_{ik} + \bar{x}_{jm} > 1}} (\hat{C}_{ijkm} - H_{ijk})(x_{ik} - 1) \right) \leq$$

$$\sum_{\substack{j\in N\\ j>i}}\Big(H_{ijk}x_{ik}+\sum_{\substack{m\in N\\ \bar{x}_{ik}+\bar{x}_{jm}>1}}(\hat{C}_{ijkm}-H_{ijk})x_{jm}+(x_{ik}-1)(\max_{\substack{m\in N\\ \bar{x}_{ik}+\bar{x}_{jm}>1}}\{\hat{C}_{ijkm}\}-H_{ijk})\Big)=$$
$$\sum_{\substack{j\in N\\ j>i}}\Big(H_{ijk}+\sum_{\substack{m\in N\\ \bar{x}_{ik}+\bar{x}_{jm}>1}}(\hat{C}_{ijkm}-H_{ijk})x_{jm}+\max_{\substack{m\in N\\ \bar{x}_{ik}+\bar{x}_{jm}>1}}\{\hat{C}_{ijkm}\}(x_{ik}-1)\Big).$$

Hence, the result follows. □

3.1.2. Second family of inequalities

A second family of valid inequalities is obtained by considering the families of inequalities (9), (11), (12) and

$$\sum_{k\in N} X_{ijkm}=x_{jm}, \qquad \forall i,j(>i),m\in N. \tag{20}$$

Observe that it is straightforward that (20) are valid inequalities for the formulation (EMMR$_r$). Hence, the following result provides valid inequalities to (EMMR$_r$) by projecting out the X-variables using Farkas' Lemma to the resulting extended formulation obtained when (9), (11), (12) and (20) are added to (EMMR$_r$).

Proposition 3.5. *Given (x,S), there exists X that satisfies (9), (11), (12), (20) if and only if for any $i\in N$*

$$S_i\ge \sum_{\substack{j\in N\\ j>i}}\sum_{k\in N}\gamma_{jk}x_{ik}+\sum_{\substack{j\in N\\ j>i}}\sum_{m\in N}\mu_{jm}x_{jm}, \tag{21}$$

for all (γ,μ) such that

$$\hat{C}_{ijkm}-\gamma_{jk}-\mu_{jm}\ge 0, \quad \forall j(>i),k,m\in N. \tag{22}$$

Now, our goal is to consider values of parameters γ_{jk} and μ_{jm} for all $j(>i),m,k\in N$ sastisfying (22) and providing the most violated valid inequalities of the type (21). The following result provides these valid inequalities.

Proposition 3.6. *Given $(\bar{x},\bar{S})$ a fractional solution of (EMMR$_r$), the following set of inequalities is the most violated within the family (21):*

$$S_i\ge \sum_{\substack{j\in N\\ j>i}}\sum_{k\in N}\gamma^*_{jk}x_{ik}+\sum_{\substack{j\in N\\ j>i}}\sum_{m\in N}\mu^*_{jm}x_{jm}, \quad \forall i\in N, \tag{23}$$

where (γ^*,μ^*) are the optimal values of dual variables of a transportation problem with n origins and destinations where the supply of origin k is $\bar{x}_{ik}$, for any $k\in N$, and the demand of destination m is $\bar{x}_{jm}$, for any $m\in N$. The transportation cost from origin k to destination m is $\hat{C}_{ijkm}$.

Proof. Let $(\bar{x},\bar{S})$ be a fractional solution of (EMMR$_r$). The maximal violation of (21) is given by solving the following linear subproblem for each $i\in N$:

$$\begin{aligned}\max_{\gamma,\mu}\quad & \sum_{\substack{j\in N\\ j>i}}\sum_{k\in N}\gamma_{jk}\bar{x}_{ik}+\sum_{\substack{j\in N\\ j>i}}\sum_{m\in N}\mu_{jm}\bar{x}_{jm},\\ s.t.\quad & \gamma_{jk}+\mu_{jm}\le \hat{C}_{ijkm}, && \forall j(>i),k,m\in N.\end{aligned}$$

An alternative is solving a linear subproblem for each $i,j(>i)\in N$,

$$\begin{aligned}\max_{\gamma,\mu}\quad & \sum_{k\in N}\gamma_{jk}\bar{x}_{ik}+\sum_{m\in N}\mu_{jm}\bar{x}_{jm},\\ s.t.\quad & \gamma_{jk}+\mu_{jm}\le \hat{C}_{ijkm}, && \forall k,m\in N,\\ & \gamma_{jk},\mu_{jm}\ \text{unrestricted}, && \forall k,m\in N.\end{aligned}$$

The problem above is the dual formulation of a transportation problem with the input data described in the statement of the proposition. □

Observe that the transportation problems considered in the above result usually have a small size, since only few values $\bar{x}_{ik}$ and $\bar{x}_{jm}$ are different to zero.

In Section 3.1.2 a family of valid inequalities has been obtained by projecting out the X-variables using Farkas' Lemma to the resulting extended formulation. It is worth mentioning that other alternatives to project them out have been studied in the literature. In particular, using Benders decomposition (Benders, 1962) for hub location problems (Contreras et al., 2011; De Camargo et al., 2011; Ghaffarinasab and Kara, 2019) provides valid inequalities with similar shape. However, these are included in a Benders scheme solution where the values of $\bar{x}$ are integer, unlike our procedure where these valid inequalities are used as cuts within a branch and bound and cut procedure and these variables could be fractional.

4. Procedure for solving the USApHMP using (EMMR) with disaggregated costs

In the next section, we will compare the formulations and reinforcements proposed in the previous section with four of the most effective ways of solving USApHMP in the literature, the ones proposed by Ghaffarinasab and Kara (2019), Meier et al. (2016), Meier and Clausen (2017) and Rostami et al. (2022). Those models are only valid for the case of disaggregated costs. For this reason, we adapt the formulation (EMMR) and the valid inequalities introduced in the previous section to use disaggregated costs. For all $i,j\in N$, let $C_{ijkm}=\chi c_{ik}+\alpha c_{km}+\delta c_{mj}$, where c_{ij} denotes the transportation cost (per unit) between i and j, and χ, α and δ are collection, transfer and distribution factors, respectively. To simulate economies of scale, it is assumed that $\alpha<\chi$ and $\alpha<\delta$. Furthermore, let define $O_i:=\sum_{j\in N}\omega_{ij}$ as the units of product which must be sent from site i, $D_i:=\sum_{j\in N}\omega_{ji}$ as the units of product which must be sent to site i and $\hat{c}^{ij}_{km}=\omega_{ij}c_{km}+\omega_{ji}c_{mk}$ for all $i,j(i<j),k,m\in N$. Now, for all $i\in N$, we define the variables s_i as follows

s_i = overall cost of transferring between hubs
(without the transfer factor) the product
sent/received from origin/destination at i to/from any site
$j\in N$ with $j>i$.

The formulation (EMMR) with disaggregated costs can be written as follows:

$$\begin{aligned}\text{(EMMR\_D)}\quad \min\quad & \sum_{i\in N}\alpha s_i+\sum_{i\in N}\sum_{k\in N}(\chi O_i+\delta D_i)c_{ik}x_{ik}, && (24)\\ \text{s.t.}\quad & (2)\text{–}(5),\\ & s_i\ge \sum_{\substack{j\in N\\ j>i}}\sum_{m\in N}\hat{c}^{ij}_{km}(x_{ik}+x_{jm}-1), && \forall i,k\in N, \quad (25)\\ & s_i\ge 0, && \forall i\in N.\end{aligned}$$

For the sake of a better understanding, the factor α is considered in the objective function instead of being included in the constraints (25). The formulation is similar to (EMMR), but now the objective function given in (24) has been decomposed in the cost between hubs and non-hubs. Since s_i represents the cost between hubs (without the transfer factor α), constraints (7) have been adapted to consider only these costs. Thus, constraints (25) mean that for $i,k\in N$, if $x_{ik}=1$, then $s_i\ge\sum_{\substack{j\in N\\ j>i}}\sum_{m\in N}\hat{c}^{ij}_{km}x_{jm}$. That is, if site i is allocated to hub k, then s_i is at least the cost of transporting the total amount of product between hubs k and m (without the transfer factor α) with origin at site i and destination any $j\in N$ with $j>i$. Therefore, since in the objective function s_i appears with non-negative coefficient, this is a formulation for the USApHMP with disaggregated costs.

Analogously to Proposition 3.1, we obtain that constraints (25) are reinforced by:

$$s_i\ge\sum_{\substack{j\in N\\ j>i}}\Big(\sum_{m\in N}\hat{c}^{ij}_{km}x_{jm}+(x_{ik}-1)\max_{m\in N}\hat{c}^{ij}_{km}x_{jm}\Big), \qquad \forall i,k\in N. \tag{26}$$

Let (EMMR_D$_r$) be the formulation obtained by replacing (25) by (26) in (EMMR_D).

All the valid inequalities provided in Section 3.1 can be adapted for the case of disaggregated costs by replacing $\hat{C}_{ijkm}$ with $\hat{c}^{ij}_{km}$, $\forall i, j(i < j), k, m \in N$.

In spite that (EMMR_D$_r$) reinforces (EMMR_D), a preliminary computational experience shows that it leads to a weak linear relaxation. We develop a solution procedure that considers: (i) a restriction-relaxed formulation, (ii) the inclusion of cuts through a separation method and (iii) the computation of upper bounds.

4.1. Feasibility cut procedure

Formulation (EMMR_D$_r$) has been relaxed by replacing constraints (26) with aggregated constraints as follows:

$$
\begin{aligned}
(\text{EMMR\_D}^*_r)\quad \min\ & \sum_{i\in N} \alpha s_i + \sum_{i\in N}\sum_{k\in N} (\chi O_i + \delta D_i) c_{ik} x_{ik},\\
\text{s.t.}\ & (2)\text{–}(5),\\
& s_i \ge \sum_{k\in N}\sum_{\substack{j\in N\\ j>i}} \left(\sum_{m\in N} \hat{c}^{ij}_{km} x_{jm} + (x_{ik}-1)\max_{m\in N} \hat{c}^{ij}_{km} \right), \quad \forall i \in N, \qquad (27)\\
& s_i \ge 0, \quad \forall i \in N.
\end{aligned}
$$

Observe that (EMMR_D$^*_r$) is not a formulation for USApHMP. Indeed, constraints (27) do not provide the transportation cost between hubs of the product with origin/destination at site i. This is due to the fact that the right hand side of (27) could be negative since (27) aggregates (26) by summing over k. Therefore, to solve USApHMP using (EMMR_D$^*_r$), we will add constraints (26) in a branch-and-cut procedure to obtain feasible solutions. In the following, this method will be called Feasibility Cut Procedure (FCP) that consists of solving (EMMR_D$^*_r$) with a subset $K \subset N$ of constraints (27). For each new incumbent integer solution found in the branch-and-bound tree, we add the violated feasibility cuts of type (26), and continue with the solution procedure. Thus, we guarantee that the solutions found are feasible for (EMMR_D$_r$).

This algorithm is reinforced by adding some valid inequalities.

4.2. Adding valid inequalities

The valid inequalities given in Section 3.1 can be adapted to consider disaggregated costs and added to (EMMR_D$^*_r$) in a branch and cut procedure.

Given $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{s}})$ a fractional solution of (EMMR_D$^*_r$), the cuts given by (16) can be adapted to the disaggregated costs following similar arguments to obtain these cuts:

$$
s_i \ge \sum_{\substack{j\in N\\ j>i}} \sum_{\substack{m\in N\\ \bar{x}_{ik}+\bar{x}_{jm}>1}} \hat{c}^{ij}_{km}(x_{ik} + x_{jm} - 1), \quad \forall i, k \in N. \qquad (28)
$$

Moreover, the cuts given by (18) can also be adapted to the disaggregated costs to obtain the following cuts:

$$
s_i \ge \sum_{\substack{j\in N\\ j>i}} \left(\sum_{\substack{m\in N\\ \bar{x}_{ik}+\bar{x}_{jm}>1}} \hat{c}^{ij}_{km} x_{jm} + \max_{\substack{m\in N\\ \bar{x}_{ik}+\bar{x}_{jm}>1}} \{\hat{c}^{ij}_{km}\}(x_{ik}-1) \right), \quad \forall i, k \in N. \qquad (29)
$$

Both families of cuts have been analyzed in a preliminary computational study. Although (29) provided deeper cuts, (28) had better performance in terms of computing times (this preliminary study reported an improvement of more than 20% on average). Therefore, the computational analyzes of this paper have been carried out using (28).

Analogously, the cuts given by (23) can be adapted to this case, as follows:

$$
s_i \ge \sum_{\substack{j\in N\\ j>i}} \sum_{k\in N} \gamma^*_{jk} x_{ik} + \sum_{\substack{j\in N\\ j>i}} \sum_{m\in N} \mu^*_{jm} x_{jm}, \quad \forall i \in N, \qquad (30)
$$

where $(\boldsymbol{\gamma}^*, \boldsymbol{\mu}^*)$ are obtained by solving, for all $i, j(>i) \in N$, the following subproblems

$$
\begin{aligned}
(TP^d_{ij})\quad \max_{\gamma,\mu}\ & \sum_{k\in N} \gamma_{jk}\bar{x}_{ik} + \sum_{m\in N} \mu_{jm}\bar{x}_{jm},\\
\text{s.t}\ & \gamma_{jk} + \mu_{jm} \le \hat{c}^{ij}_{km}, && \forall k, m \in N,\\
& \gamma_{jk}, \mu_{jm} \text{ unrestricted}, && \forall k, m \in N.
\end{aligned}
$$

Note that (TP$^d_{ij}$) is the dual of a transportation problem with n origins and destinations where the supply of origin k is $\bar{x}_{ik}$, for any $k \in N$, and the demand of destination m is $\bar{x}_{jm}$, for any $m \in N$. The transportation cost from origin k to destination m is $\hat{c}^{ij}_{km}$. Therefore, for adding (30), we first solve the dual of different transportation problems to obtain the parameters $(\boldsymbol{\gamma}^*, \boldsymbol{\mu}^*)$ and then add to (EMMR_D$^*_r$) each constraint of type (30) that is violated.

The algorithm for solving the transportation problem is based on the primal–dual algorithm with preprocessing described in Haddadi and Slimani (2012).

Algorithm 1: Solution procedure for solving USApHMP

Input: dataset, depth_max, pass_max, pass2_max, tolerance.
pass:=1, pass2:=1;
repeat at each node of the branching tree
  Solve the linear relaxation of (EMMR_D$^*_r$);
  Let $(\bar{x}, \bar{s})$ be the optimal solution of the LP-relaxation;
  if *$\bar{x}$ is integer* **then**
    for $i, k \in N$ **do**
      if $\bar{s}_i < \sum_{\substack{j\in N\\ j>i}} \left(\sum_{m\in N} \hat{c}^{ij}_{km}\bar{x}_{jm} + (\bar{x}_{ik}-1)\max_m \hat{c}^{ij}_{km} \right)$ **then**
        Add constraint $s_i \ge \sum_{\substack{j\in N\\ j>i}} \left(\sum_{m\in N} \hat{c}^{ij}_{km} x_{jm} + (x_{ik}-1)\max_m \hat{c}^{ij}_{km} \right)$
  get depth;
  if *depth<depth_max and pass <pass_max* **then**
    for $i \in V$ **do**
      for $j(>i) \in N$ **do**
        Solve (TP$^d_{ij}$);
        Let $(\boldsymbol{\gamma}^*_j, \boldsymbol{\mu}^*_j)$ the optimal solution;
        if $\bar{s}_i < \sum_{\substack{j\in N\\ j>i}} \sum_{k\in N} \gamma^*_{jk}\bar{x}_{ik} + \sum_{\substack{j\in N\\ j>i}} \sum_{m\in N} \mu^*_{jm}\bar{x}_{jm}$ **then**
          Add constraint $s_i \ge \sum_{\substack{j\in N\\ j>i}} \sum_{k\in N} \gamma^*_{jk} x_{ik} + \sum_{\substack{j\in N\\ j>i}} \sum_{m\in N} \mu^*_{jm} x_{jm}$
      for $k \in N$ **do**
        if $\bar{s}_i < \sum_{\substack{j\in N\\ j>i}} \sum_{\substack{m\in N\\ \bar{x}_{ik}+\bar{x}_{jm}>1}} \hat{c}^{ij}_{km}(\bar{x}_{ik} + \bar{x}_{jm} - 1)$ **then**
          Add constraint $s_i \ge \sum_{\substack{j\in N\\ j>i}} \sum_{\substack{m\in N\\ \bar{x}_{ik}+\bar{x}_{jm}>1}} \hat{c}^{ij}_{km}(x_{ik} + x_{jm} - 1)$
    pass:=pass+1
  if *pass2<pass2_max* **then**
    Obtain $\hat{P} := \{k \in N : \bar{x}_{kk} > 0\}$;
    Solve the USApHMP restricted to $\hat{P}$;
    Let UB be the optimal objective value;
    Provide UB as upper bound of the USApHMP;
    pass2:=pass2+1;
until *optimal solution is found*;

4.3. Upper bound computation

After adding valid inequalities, another procedure to obtain upper bounds and reduce the size of the branching tree is developed. This procedure consists of two steps.

1. We solve the relaxation of (EMMR_D$_r^*$). Let $(\bar{x}, \bar{s})$ be a fractional solution and $\hat{P}$ the set of $k \in N$ such that $\bar{x}_{kk} > 0$.
2. We solve the USApHMP by using the Ernst and Krishnamoorthy formulation (one of the most effective formulations in the literature) with the location of hubs and allocation to hubs restricted to the set $\hat{P}$. This allows us to determine a primal solution that gives an upper bound for the optimal solution of the USApHMP.

The complete solution procedure described above is summarized in the Algorithm 1.

5. Computational study

The procedure described in Section 4 has been tested in an extensive computational experiment, for solving the USApHMP and USAHLP.

Three commonly used data sets from the hub location literature are used for the test: the Civil Aeronautics Board (CAB), Australian Post (AP) and TR data set. CAB data set, introduced by O'Kelly (1987), is based on airline passenger interactions between 25 US cities. The second data set we have used is the AP data set (Ernst and Krishnamoorthy, 1996), which is based on a postal delivery in Sydney and consists of 200 postal districts. The TR data set (Tan and Kara, 2007) is based on the cargo flows between 81 cities of Turkey.

For our tests we used an Intel Xeon W-2245 CPU of 3.90 GHz with the Microsoft Windows 10 operating system. All formulations and solution methods have been implemented in Xpress v8.10 with all automatic cuts and preprocessing disabled and the number of threads was limited to four (this is due to the fact that the computer used in Ghaffarinasab and Kara (2019) used a machine with four threads, see Appendix for more details).

We set the following parameters when generating our cuts of type (28) and (30). We limited the total number of iterations for generating cuts *pass_max* := 40, the maximum depth of the branching tree *depth_max* := 2, and the cut tolerance (minimum violation for generating a cut) *tolerance* := 0.1. There is no limit for the number of cuts added in each iteration.

The first observation that comes after running the experiment is that solving instances from data set CAB with $n = 25$ and different values of p provides computational times below two seconds (see Appendix, Tables S.1 and S.2). Therefore, we concentrate on instances from AP and TR data sets.

5.1. Performance analysis of our formulations and Algorithm 1

We first provide a computational analysis to compare (EMMR_D$_r$) and FCP to different compact formulations proposed in the literature (Ebery, 2001; Ernst and Krishnamoorthy, 1996; Skorin-Kapov et al., 1996). Table 2 reports the results obtained by solving the USApHMP for AP data set with $n = 75$ and the time limit set to 7200 s. The first column shows the number of hubs. The second column reports the formulation used to solve the USApHMP. The following three columns provide the objective value of the best solution, the best bound and the gap between both values, respectively. The next column shows the number of nodes explored in the branching tree and the last column reports the computational time. From this table we can conclude that the formulation proposed by Ernst and Krishnamoorthy (1996) provides the best computational times. As expected given their reduced sizes, (EMMR_D$_r$) and FCP take long times. In what follows we will show the crucial impact of considering feasibility cuts (26), cuts (28) and (30), as well as an upper bound to build a much more competitive solution method.

Secondly, we will analyze the performance of Algorithm 1. We only show the results for solving the USApHMP with AP data set. Similar conclusions were obtained for TR data sets. Tables 3 and 4 show the impact in the performance of each part of Algorithm 1 for solving USApHMP, using AP data set with $n = 100$ (Table 3) and $n = 200$ (Table 4). The first and second columns show the size of the instances and the number of hubs, respectively. The third column reports the procedures used to solve the USApHMP: (EMMR_D$_r$) or the relaxed version where constraints (26) are inserted in a cut generation procedure (FCP), with or without the valid inequalities (28) and (30), and also the upper bound (UB) described in Section 4.3 or steps 22–27 of Algorithm 1. The following three columns provide the objective value of the best solution, the best bound and the gap between both values, respectively. The next column shows the number of nodes explored in the branching tree. The number of constraints added as cuts of types (26), (28) and (30) are shown in columns labeled as #(26), #(28) and #(30), respectively. Note that cuts of type (26) are only added in (FCP) because (26) are constraints defining formulation (EMMR_D$_r$). In this table, the time limit was set to 7200 s.

We can observe the importance of using the (FCP) with (28) and (30), since the optimal solutions are obtained in lower computing times than (EMMR_D$_r$)+(28)+(30), for $n = 100$ (and consequently, than the ones given using formulation of Ernst and Krishnamoorthy (1996), since we have obtained times for $n = 100$ similar to the ones obtained by them for $n = 75$). Moreover, instances with $n = 200$ are not solved whenever (FCP) is not used due to lack of memory. The upper bound (UB) performs better for large values of n and p, e.g. $n = 200$ with $p = 5$ and $p = 10$, where the instances are only solved with (EMMR_D$_r$)+UB+(28)+(30).

5.2. Comparison with alternative solution procedures

Our goal is to present a comparison between our branch-and-cut algorithm and the most recent and efficient exact algorithms existing in the literature for the USApHMP and UHLPSA: Meier et al. (2016), Ghaffarinasab and Kara (2019) and Rostami et al. (2022). To the best of our knowledge, these constitute the state-of-the-art of exact algorithms for solving theses problems. We use the same three data sets in our computational experiments as (Ghaffarinasab and Kara, 2019). Meier et al. (2016) provided a method that explicitly required the assumption of Euclidean distances, and for this reason they can only use CAB and AP data sets (they choose use only AP data set because CAB data set consists of instances with up to 25 nodes). The TR data set is based on the cargo flows between 81 cities of Turkey and uses travel distances. Hence, the methodology of Meier et al. (2016) cannot be applied to TR data set. Recently, Rostami et al. (2022) proposed a convex reformulation and a branch-and-cut algorithm based on outer approximation cuts for a large class of binary quadratic programs, which include the USApHMP and UHLPSA as particular cases. They only show results for the USApHMP with AP data.

Meier et al. (2016) use $C\#$ to call Gurobi 5.6 running on a processor at 3.4 GHz and 16 GB of RAM (more details are not provided by the authors) while Ghaffarinasab and Kara (2019) use Java to call CPLEX 12.6 on a machine running Windows 7 Intel (R) Core(TM) i3-3220 CPU of 3.30 GHz and 16 GB of RAM. Finally Rostami et al. (2022) use C^{++} to call Gurobi 6.5 on a machine running Linux Intel Xeon(R) CPU E3-1270 (2 quad-core CPUs with 3.60 GHz) with 64 Gb of RAM. Observe the different machines, RAM, Operating Systems, programming languages and solvers with their versions. It is neither easy nor straightforward to give a percentage that reflects the difference between the computational experiments performed using the different methodologies aforementioned, since there are many factors that influence the performance.

With the aim of providing more insights about the good performance of Algorithm 1 in comparison with the results obtained by

Table 2
Results with AP data set for USApHMP with $n = 75$ using different compact formulations.

p	Formulation	Best solution	Best bound	Gap bb	Nodes	Time
2	Ebery (2001)	180118.91	180118.91	0	11481	816.1
	Ernst and Krishnamoorthy (1996)	180118.91	180118.91	0	56	77.7
	Skorin-Kapov et al. (1996)	180118.91	180118.91	0	1	1637.3
	(EMMR_D$_r$)	180118.91	180118.91	0	229	79.7
	FCP	180118.91	180118.91	0	291	7.2
3	Ebery (2001)	161322.52	157453.98	2.40	179833	7200
	Ernst and Krishnamoorthy (1996)	161056.74	161056.74	0	97	179.3
	Skorin-Kapov et al. (1996)	161056.74	161056.74	0	1	1729.2
	(EMMR_D$_r$)	161056.74	161056.74	0	7245	307.6
	FCP	161056.74	161056.74	0	9493	176.7
4	Ebery (2001)	145967.42	136980.27	6.16	176179	7200
	Ernst and Krishnamoorthy (1996)	145734.20	145734.20	0	279	218.4
	Skorin-Kapov et al. (1996)	145734.20	145734.20	0	1	1928.4
	(EMMR_D$_r$)	145734.20	145734.20	0	109535	2852.4
	FCP	145734.20	145734.20	0	132895	2330.0
5	Ebery (2001)	136238.53	123775.37	9.15	147532	7200
	Ernst and Krishnamoorthy (1996)	136011.35	136011.35	0	1135	396.8
	Skorin-Kapov et al. (1996)	136011.35	136011.35	0	3	3043.0
	(EMMR_D$_r$)	136220.08	128825.12	5.43	207949	7200
	FCP	136166.96	128471.55	5.65	256864	7200
10	Ebery (2001)	116206.27	86356.97	25.69	240113	7200
	Ernst and Krishnamoorthy (1996)	106364.90	106364.90	0	781	394.1
	Skorin-Kapov et al. (1996)	106364.90	106364.90	0	1	2080.4
	(EMMR_D$_r$)	109345.71	87130.33	20.32	300689	7200
	FCP	111892.72	85614.15	23.49	170678	7200

Table 3
Impact of each part in the performance of the Algorithm 1 with AP data set for USApHMP with $n = 100$.

n	p	Formulation	Best solution	Best bound	Gap bb	Nodes	#(26)	#(28)	#(30)	Time
100	2	(EMMR_D$_r$)	180223.77	180223.77	0	1781	–	–	–	613.6
		(FCP)	180223.77	180223.77	0	1663	1346	–	–	61.2
		(EMMR_D$_r$)+(28)	180223.77	180223.77	0	1575	–	99	–	528.4
		(FCP)+(28)	180223.77	180223.77	0	1557	1890	99	–	65.8
		(EMMR_D$_r$)+(28)+(30)	180223.77	180223.77	0	6	–	152	256	397.2
		(FCP)+(28)+(30)	180223.77	180223.77	0	8	85	151	256	53.1
		(EMMR_D$_r$)+UB+(28)+(30)	180223.77	180223.77	0	6	–	152	256	386.0
		(FCP)+UB+(28)+(30)	180223.77	180223.77	0	10	85	152	256	51.8
	3	(EMMR_D$_r$)	160847.00	160847.00	0	44261	–	–	–	4424.5
		(FCP)	160847.00	160847.00	0	42009	34085	–	–	1589.6
		(EMMR_D$_r$)+(28)	160847.00	160847.00	0	27981	–	135	–	3001.5
		(FCP)+(28)	160847.00	160847.00	0	48863	40931	135	–	1943.6
		(EMMR_D$_r$)+(28)+(30)	160847.00	160847.00	0	8	–	269	412	394.8
		(FCP)+(28)+(30)	160847.00	160847.00	0	5	156	269	412	42.9
		(EMMR_D$_r$)+UB+(28)+(30)	160847.00	160847.00	0	6	–	269	412	397.0
		(FCP)+UB+(28)+(30)	160847.00	160847.00	0	6	155	269	412	43.0
	4	(EMMR_D$_r$)	146007.00	132084.54	9.54	19053	–	–	–	7200
		(FCP)	146377.34	141545.13	3.30	92181	242789	–	–	7200
		(EMMR_D$_r$)+(28)	146168.62	139126.64	4.82	44639	–	217	–	7200
		(FCP)+(28)	146529.10	141196.24	3.63	87979	230866	217	–	7200
		(EMMR_D$_r$)+(28)+(30)	145896.58	145896.58	0	1	–	286	462	391.4
		(FCP)+(28)+(30)	145896.58	145896.58	0	1	190	286	462	47.6
		(EMMR_D$_r$)+UB+(28)+(30)	145896.58	145896.58	0	1	–	286	462	389.3
		(FCP)+UB+(28)+(30)	145896.58	145896.58	0	1	190	286	462	47.2
	5	(EMMR_D$_r$)	138759.71	120259.65	13.33	41942	–	–	–	7200
		(FCP)	139715.91	121246.54	13.22	72227	226708	–	–	7200
		(EMMR_D$_r$)+(28)	138037.16	120414.84	12.76	43242	–	215	–	7200
		(FCP)+(28)	138988.86	122325.28	11.99	72240	226403	215	–	7200
		(EMMR_D$_r$)+(28)+(30)	136929.44	136929.44	0	68	–	251	675	533.8
		(FCP)+(28)+(30)	136929.44	136929.44	0	78	290	251	675	96.1
		(EMMR_D$_r$)+UB+(28)+(30)	136929.44	136929.44	0	165	–	251	675	603.8
		(FCP)+UB+(28)+(30)	136929.44	136929.44	0	97	189	251	675	123.8
	10	(EMMR_D$_r$)	114445.35	81505.41	28.78	70396	–	–	–	7200
		(FCP)	118096.90	80954.53	31.45	60645	174315	–	–	7200
		(EMMR_D$_r$)+(28)	113482.51	82734.38	27.09	64804	–	205	–	7200
		(FCP)+(28)	114765.33	81697.19	28.81	61074	169485	205	–	7200
		(EMMR_D$_r$)+(28)+(30)	106469.57	106469.57	0	1	–	309	510	380.4
		(FCP)+(28)+(30)	106469.57	106469.57	0	1	154	309	510	47.3
		(EMMR_D$_r$)+UB+(28)+(30)	106469.57	106469.57	0	1	–	309	510	380.4
		(FCP)+UB+(28)+(30)	106469.57	106469.57	0	1	154	309	510	47.1

Table 4
Impact of each part in the performance of the Algorithm 1 with AP data set for USApHMP with $n = 200$.

n	p	Formulation	Best solution	Best bound	Gap bb	Nodes	#(26)	#(28)	#(30)	Time
200	2	(EMMR_D$_r$)	–	–	–	–	–	–	–	–
		(FCP)	182459.79	181840.02	0.34	22101	13633	–	–	7200
		(EMMR_D$_r$)+(28)	–	–	–	–	–	–	–	–
		(FCP)+(28)	182459.88	182094.02	0.20	23484	15706	298	–	7200
		(EMMR_D$_r$)+(28)+(30)	–	–	–	–	–	–	–	–
		(FCP)+(28)+(30)	182459.25	182459.25	0	6	154	474	400	633.7
		(EMMR_D$_r$)+UB+(28)+(30)	–	–	–	–	–	–	–	–
		(FCP)+UB+(28)+(30)	182459.25	182459.25	0	6	154	474	400	630.2
	3	(EMMR_D$_r$)	–	–	–	–	–	–	–	–
		(FCP)	163233.36	141310.38	13.43	6173	25318	–	–	7200
		(EMMR_D$_r$)+(28)	–	–	–	–	–	–	–	–
		(FCP)+(28)	163389.10	143762.50	12.01	5073	21092	282	–	7200
		(EMMR_D$_r$)+(28)+(30)	–	–	–	–	–	–	–	–
		(FCP)+(28)+(30)	162887.03	162887.03	0	1	283	561	1322	1222.1
		(EMMR_D$_r$)+UB+(28)+(30)	–	–	–	–	–	–	–	–
		(FCP)+UB+(28)+(30)	162887.03	162887.03	0	1	283	561	1322	1220.5
	4	(EMMR_D$_r$)	–	–	–	–	–	–	–	–
		(FCP)	150350.65	125512.11	16.52	5308	19452	–	–	7200
		(EMMR_D$_r$)+(28)	–	–	–	–	–	–	–	–
		(FCP)+(28)	151755.20	125872.13	17.06	4574	19595	351	–	7200
		(EMMR_D$_r$)+(28)+(30)	–	–	–	–	–	–	–	–
		(FCP)+(28)+(30)	147767.30	147767.30	0	17	470	489	1578	1372.5
		(EMMR_D$_r$)+UB+(28)+(30)	–	–	–	–	–	–	–	–
		(FCP)+UB+(28)+(30)	147767.30	147767.30	0	17	470	489	1578	1413.5
	5	(EMMR_D$_r$)	–	–	–	–	–	–	–	–
		(FCP)	145572.43	111128.85	23.66	5262	19955	–	–	7200
		(EMMR_D$_r$)+(28)	–	–	–	–	–	–	–	–
		(FCP)+(28)	144045.70	110855.15	23.04	4550	32012	285	–	7200
		(EMMR_D$_r$)+(28)+(30)	–	–	–	–	–	–	–	–
		(FCP)+(28)+(30)	140165.43	140019.49	0.10	1418	857	286	1686	7200
		(EMMR_D$_r$)+UB+(28)+(30)	–	–	–	–	–	–	–	–
		(FCP)+UB+(28)+(30)	140062.65	140062.65	0	429	380	286	1686	3934.7
	10	(EMMR_D$_r$)	–	–	–	–	–	–	–	–
		(FCP)	125808.52	74912.65	40.46	5869	20491	–	–	7200
		(EMMR_D$_r$)+(28)	–	–	–	–	–	–	–	–
		(FCP)+(28)	128609.71	75023.11	41.66	4535	25308	432	–	7200
		(EMMR_D$_r$)+(28)+(30)	–	–	–	–	–	–	–	–
		(FCP)+(28)+(30)	110243.95	110146.36	0.09	1020	879	602	1598	7200
		(EMMR_D$_r$)+UB+(28)+(30)	–	–	–	–	–	–	–	–
		(FCP)+UB+(28)+(30)	110147.65	110147.65	0	280	318	602	1598	1780.5

Ghaffarinasab and Kara (2019), we have carried out a new computational analysis in an Intel Core i3 computer with 16 GB of RAM (a similar computer to the one used in Ghaffarinasab and Kara, 2019). Moreover, two different versions of solver have been used: Xpress 8.10 available since 2020 and Xpress 8.04 available since 2016 (see Appendix B, Table S.3). We can observe that our methodology still improve the computational experiments given in Ghaffarinasab and Kara (2019) with the oldest version of the solver and the oldest processor. Observe that this percentage of improvement is more than 80% for a size of 200 with the oldest processor and version.

Since it is not always possible to perform computational experiments on computers with similar characteristics and our experiments confirm that our algorithm performs well, improving the results obtained by Ghaffarinasab and Kara (2019), a more extensive and in-depth computational study is presented in the following.

5.2.1. Results for the USApHMP

In this section, we compare the performance of our solution procedure with the ones of Ghaffarinasab and Kara (2019), Meier et al. (2016) and Rostami et al. (2022) (denoted in our tables as GK(2019), MCRB(2016) and REL(2022), respectively) for solving the USApHMP on AP and TR data sets using the same values of n and p considered in these works.

In Appendix, Tables S.1 and S.2 show a comparison of our computational times and those obtained in Ghaffarinasab and Kara (2019) for the CAB data set. Since they are very similar and below two seconds, we concentrate on instances from AP and TR data sets. The collection and distribution factors χ and δ are set to 3 and 2, respectively for AP data set and set to 1 for CAB and TR data sets (like in the aforementioned papers). The transfer factor α is set to 0.75 for AP data set and it takes different values from 0.2 to 1 for CAB and TR data sets.

Tables 5–7 show the numerical results obtained by solving the USApHMP with AP data set. Tables 5 and 6 have the following structure: the first and second columns give the values of n and p, respectively. The next column reports the optimal objective values. The number of nodes explored in the branching tree is given in column labeled *Nodes*. The following three columns give the number of inequalities added in the process as cuts, depending on the type, #(26), #(28) and #(30). The total time in seconds to obtain the optimal solution with Algorithm 1 is reported in *Time*.

In Table 5, the results for values of n ranging from 50 up to 200, $p \in \{2, 3, 4, 5\}$ and $\alpha = 0.75$ are presented. This table includes the solution times for solving the problem by the method of Rostami et al. (2022). Results not reported in Rostami et al. (2022) are denoted by '-'. Regarding the times showed in Rostami et al. (2022), our procedure is always faster than their method. Table 6 shows the results with $p \in \{5, 10, 15, 20\}$. In this case, we compare the computational times with the ones of Ghaffarinasab and Kara (2019). It can be observed that Algorithm 1 gives the optimal solutions in much less computational time than the results provided by Ghaffarinasab and Kara (2019) for all the instances.

Table 5
Results with AP data set for USApHMP.

n	p	Algorithm 1						REL(2022)
		Objective	Nodes	#(26)	#(28)	#(30)	Time	Time
50	2	178484.29	1	35	71	56	1.3	7.6
	3	158569.93	1	49	106	129	2.2	8.1
	4	143378.05	1	82	128	150	2.3	9.0
	5	132366.95	1	63	148	166	2.3	8.0
60	2	179920.21	1	37	113	90	3.5	15.0
	3	160338.58	1	54	126	234	6.2	36.4
	4	144719.69	1	104	145	252	5.5	23.4
	5	132850.29	1	71	156	279	6.4	35.6
70	2	180093.19	1	52	125	66	4.8	20.5
	3	160933.23	1	81	186	242	8.3	32.1
	4	145619.65	1	111	175	304	10.1	39.3
	5	135835.20	2	69	168	422	15.4	86.8
75	2	180118.91	1	55	132	71	6.2	26.6
	3	161056.74	1	74	169	239	10.8	44.7
	4	145734.20	1	117	189	424	17.5	56.5
	5	136011.35	2	71	216	423	22.0	167.8
90	2	179821.61	4	73	176	229	30.3	76.6
	3	160437.43	21	174	251	389	33.4	143.5
	4	145133.69	1	173	240	457	31.0	215.6
	5	135808.25	1	155	219	632	46.3	268.4
100	2	180223.77	10	85	151	256	51.6	103.6
	3	160847.00	6	155	269	412	42.8	283.9
	4	145896.58	1	190	286	462	47.1	191.4
	5	136929.44	97	189	251	675	123.7	386.7
125	2	180372.18	1	73	326	356	123.3	220.6
	3	161117.17	8	225	377	718	154.1	460.8
	4	146173.22	2	223	335	730	149.0	431.2
	5	137175.68	127	222	363	972	234.2	933.1
150	2	180898.84	1	82	378	514	320.7	487.5
	3	161490.48	1	226	496	793	288.6	1578.4
	4	146521.33	1	261	409	882	279.6	1191.8
	5	137425.91	175	229	331	1042	704.7	3345.5
175	2	182120.64	1	121	364	325	361.8	–
	3	162553.71	1	266	525	964	583.0	–
	4	147316.45	32	368	399	1392	758.0	–
	5	139354.51	205	425	94	1624	1284.2	–
200	2	182459.25	6	154	474	400	628.3	–
	3	162887.03	1	283	561	1322	1219.8	–
	4	147767.30	17	470	489	1578	1371.3	–
	5	140062.65	429	380	286	1686	3954.1	–

The last experiment carried out with the AP data set is shown in Table 7, where we compare Algorithm 1 to the methodology proposed in Meier et al. (2016) for solving the USApHMP. As mentioned, their method is based on the assumption that the transportation costs are proportional to the Euclidean distances, whereas our algorithm applies to general cost structure that are not necessarily based in distances. Table 7 includes, for each value of n, the mean of instances with $p \in \{5, 10, 15, 20\}$. Although the algorithms by Meier et al. (2016) look better for large values of p, we can observe that our procedure provides competitive computational times on average. Moreover, the overall mean provides a general idea of the time improvement of our procedure, since it reports an average of 1406.33 s whereas the results in Meier et al. (2016) reported 2560.39 s on average.

Table 8 reports the computational results for TR data set, where H represents the candidate sites for locating hubs and column α gives the different values of α tested. We also include the times obtained by Ghaffarinasab and Kara (2019). We can observe that our methodology improve the computational experiments given in Ghaffarinasab and Kara (2019).

5.2.2. Results for the USAHLP

Algorithm 1 can be adapted for solving the USAHLP. Analogously to the USApHMP models, we also include the performance of solution procedures by Ghaffarinasab and Kara (2019) for solving the USAHLP on AP and TR data sets. The results are reported in Tables 9 and 10.

There are two types of fixed hub establishment cost values and two types of hub capacity values in the AP data set, tight and loose. These instances are denoted as LT, TT, LL and TL. The first letter indicates whether loose or tight fixed costs apply (loose fixed costs are low, being dominated by transportation costs, while it is the other way around for tight fixed costs). The second letter indicates if the capacities are loose or tight. Since we do not consider the hub capacities in our work, we use LT and TT instances, as in Ghaffarinasab and Kara (2019). Table 9 shows the results with different values of n ranging from 10 to 200. The solution times for solving the USAHLP with the method proposed in Ghaffarinasab and Kara (2019) are included. It is observed that our solution method provides better computational times than those of Ghaffarinasab and Kara (2019) for all the instances.

The solution times for the TR data set are given in Table 10. Column *FCS* corresponds to the fixed cost scaling factor considered in Ghaffarinasab and Kara (2019). Results reported in this table indicate that our procedure reduces the times respect to the method provided in Ghaffarinasab and Kara (2019).

Table 6
Results with AP data set for USApHMP.

n	p	Algorithm 1						GK(2019)
		Objective	Nodes	#(26)	#(28)	#(30)	Time	Time
100	5	136929.44	97	189	251	675	123.7	313.80
	10	106469.57	1	154	309	510	47.0	109.18
	15	90533.52	7	108	304	617	93.3	144.40
	20	80270.96	1	131	382	655	53.3	61.47
125	5	137175.68	127	222	363	972	234.2	1286.48
	10	107092.09	38	255	330	456	162.0	414.08
	15	91494.56	171	298	417	1091	769.4	1271.86
	20	81471.65	3	172	486	1060	165.7	213.76
150	5	137425.91	175	229	331	1042	704.7	2989.83
	10	107478.12	70	277	439	768	340.5	1148.01
	15	92050.58	21	182	474	1063	419.6	1695.15
	20	82229.39	4	201	535	934	302.6	531.99
175	5	139354.51	205	425	94	1624	1284.2	31347.15
	10	109744.35	253	386	513	1085	2331.3	10551.64
	15	94123.66	4851	2349	614	1961	11234.3	19602.93
	20	83843.54	225	435	730	1328	1601.1	1778.11
200	5	140062.65	429	380	286	1686	3954.1	127546.79
	10	110147.65	280	318	602	1598	1775.3	46706.90
	15	94459.20	43	274	592	1366	1038.4	26640.56
	20	84955.37	900	406	734	1485	1491.9	27224.48

Table 7
Results with AP data set for USApHMP.

n	p	Algorithm 1		MCRB(2016)
		Objective	Time	Time
100	5	136929.44	123.7	356.39
	10	106469.57	47.0	32.22
	15	90533.52	93.3	85.95
	20	80270.96	53.3	33.54
		Mean	79.3	127.02
125	5	137175.68	234.2	1104.31
	10	107092.09	162.0	184.62
	15	91494.56	769.4	465.08
	20	81471.65	165.7	111.03
		Mean	332.8	466.26
150	5	137425.91	704.7	1474.76
	10	107478.12	340.5	412.53
	15	92050.58	419.6	405.57
	20	82229.39	302.6	185.34
		Mean	441.9	619.55
175	5	139354.51	1284.2	10699.58
	10	109744.35	2331.3	3023.02
	15	94123.66	11234.3	8143.73
	20	83843.54	1601.1	271.02
		Mean	4112.7	5534.33
200	5	140062.65	3954.1	17628.38
	10	110147.65	1775.3	4957.31
	15	94459.20	1038.4	1107.48
	20	84955.37	1491.9	526.08
		Mean	2064.9	6054.81
		Total mean	1406.3	2560.39

6. Conclusions

We have presented a new formulation for uncapacitated single-allocation hub location problems with fewer variables (n^2 assignment binary variables and n continuous variables) than the previous Integer Linear Programming formulations known in the literature. The formulation holds for general cost structures that are not based on distances and do not necessarily satisfy triangle inequality, and costs could also be aggregated along the whole origin-hub-hub-destination path, instead of being decomposed by arc. Different families of valid inequalities are developed to strengthen the original formulation by proposing extended formulations and projecting out some of their variables using Farkas' Lemma. Moreover, we have proposed a branch-and-cut algorithm to solve these models based on a relaxed version of the new formulation whose restrictions are inserted in a cut generation procedure together with two sets of valid inequalities.

The performance of the proposed algorithm has been tested on three well-known hub location data sets, namely the CAB, TR and AP data sets. The numerical results clearly show the importance of using the relaxed version of the new formulation instead of the original formulation, together with (28) and (30), since the optimal solutions are obtained in lower computing times.

The experiments were also compared to the most recent and efficient exact algorithms known for single-allocation hub location problems. The reported results show that our algorithm outperforms the previous ones in the standard benchmarks for hub location problems, providing competitive computing times for solving large-scale instances.

An interesting avenue for future research is to adapt the new formulation and modify the proposed algorithm to solve other variants of the single allocation hubs location problems, which include hub capacities or uncertainty by generating different scenarios of flows among sites.

Data availability

Data will be made available on request.

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Table 8
Results with TR data set for USApHMP.

α	p	$\|H\|$	Algorithm 1						GK(2019)
			Objective	Nodes	#(26)	#(28)	#(30)	Time	Time
0.2	2	22	784.84	1	131	26	137	2.0	9.57
		81	781.32	13	164	72	244	26.9	33.81
	4	22	580.44	46	211	28	322	10.5	23.05
		81	575.17	57	218	43	386	51.5	66.36
	6	22	454.09	40	144	27	102	4.4	10.76
		81	448.60	51	128	56	127	14.3	27.83
	8	22	399.61	50	170	32	131	5.0	13.60
		81	396.48	131	149	37	237	26.3	33.36
	10	22	359.72	177	158	32	130	6.7	16.12
		81	357.70	143	147	36	225	27.0	23.97
0.4	2	22	860.76	4	157	21	238	4.2	12.15
		81	850.17	12	186	85	389	30.3	41.19
	4	22	690.28	149	258	28	388	16.0	33.33
		81	683.24	407	337	22	607	115.0	293.15
	6	22	586.30	89	169	25	224	9.1	21.85
		81	579.38	223	218	40	420	48.8	57.97
	8	22	531.86	87	174	21	258	9.7	21.45
		81	530.39	607	278	34	418	77.4	323.08
	10	22	494.42	1543	260	36	258	18.6	20.18
		81	493.30	1445	410	33	501	105.0	320.94
0.6	2	22	916.90	8	203	30	390	7.5	15.80
		81	916.69	27	301	98	532	60.8	142.99
	4	22	790.69	106	341	39	667	33.9	78.08
		81	777.03	265	251	18	690	127.5	363.33
	6	22	699.64	217	205	30	366	13.2	36.83
		81	691.35	125	186	33	589	62.8	106.54
	8	22	655.24	331	242	20	439	18.5	34.73
		81	651.35	231	195	38	755	143.4	686.88
	10	22	622.19	1861	248	42	523	30.6	50.77
		81	619.08	359	230	38	736	132.1	797.82
0.8	2	22	961.83	61	275	45	472	15.7	31.75
		81	961.83	137	175	118	665	103.2	180.57
	4	22	871.59	456	348	42	838	44.0	112.19
		81	861.99	229	282	46	724	142.0	1084.03
	6	22	805.51	93	234	29	539	16.2	66.53
		81	792.28	265	216	36	632	79.0	280.25
	8	22	770.82	524	290	18	583	25.5	66.84
		81	762.10	647	322	36	763	156.1	800.64
	10	22	742.51	609	239	45	651	26.2	42.25
		81	737.71	1641	447	28	809	190.0	2165.9
1	2	22	992.72	35	142	1	652	19.0	73.84
		81	992.72	81	265	158	753	122.0	272.75
	4	22	946.94	2225	543	56	994	76.5	365.88
		81	932.56	433	317	48	855	230.5	508.60
	6	22	904.83	2423	507	57	855	64.3	175.61
		81	883.85	207	241	32	806	209.4	400.02
	8	22	876.57	1477	351	33	985	61.6	226.93
		81	862.10	811	286	38	973	229.6	1845.28
	10	22	857.28	5055	798	63	974	80.3	150.02
		81	843.38	673	270	44	1000	190.1	1476.57

Table 9
Results with AP data set for USAHLP.

Instance	n	Algorithm 1						GK(2019)
		Objective	Nodes	#(26)	#(28)	#(30)	Time	Time
LT	10	224250.05	1	16	18	9	0	0.69
	20	234690.96	1	27	41	18	0.1	0.36
	25	236650.63	1	28	55	32	0.2	0.48
	40	240986.23	2	50	68	154	1.5	2.10
	50	237421.99	1	75	95	94	1.9	4.42
	60	228855.08	3	36	138	99	5.8	10.52
	70	226188.20	1	93	164	100	5.2	21.76
	75	235847.50	1	103	140	202	10.6	41.55
	90	225475.48	2	130	259	244	27.1	98.92
	100	238016.28	1	58	116	182	26.3	82.42
	125	227949.00	1	167	418	447	115.9	411.56
	150	225450.09	1	198	474	436	207.5	1259.22
	175	227655.38	2	210	398	393	330.9	2044.77
	200	233802.98	15	149	570	428	1816.7	5494.77
TT	10	263399.94	1	13	10	17	0	0.22
	20	271128.18	1	16	20	53	0.1	0.32
	25	295667.84	1	28	35	24	0.1	0.68
	40	293164.84	1	32	32	38	0.4	0.67
	50	300420.99	1	48	76	58	1.3	3.92
	60	264742.11	1	81	141	81	4.3	10.62
	70	261294.99	1	0	0	0	1.4	8.41
	75	288778.29	1	48	77	66	4.2	15.38
	90	257415.86	1	72	181	136	17.2	50.40
	100	305097.95	1	58	57	33	12.7	60.23
	125	258839.68	1	76	229	180	49.1	188.05
	150	234778.74	12	120	273	139	132.5	478.38
	175	247876.80	1	149	526	607	554.5	1639.21
	200	272188.11	118	727	499	1210	16489.1	20292.35

Table 10
Results with TR data set for USAHLP.

FCS	α	$\|H\|$	Algorithm 1						GK(2019)
			Objective	Nodes	#(26)	#(28)	#(30)	Time	Time
0.05	0.2	22	549.96	144	196	96	135	6.1	14.10
		81	547.57	83	207	51	110	18.2	28.76
	0.4	22	682.84	337	208	26	199	9.8	13.04
		81	678.60	172	196	51	189	31.7	42.59
	0.6	22	808.93	303	223	31	409	14.3	43.17
		81	803.24	319	216	55	386	55.5	97.51
	0.8	22	925.87	875	356	38	755	33.4	85.90
		81	918.64	129	252	66	603	83.1	247.77
	1	22	1015.94	747	296	83	855	40.2	126.49
		81	1015.94	286	253	77	867	166.8	975.20
0.1	0.2	22	683.10	43	175	115	149	4.7	11.29
		81	681.66	71	170	64	221	23.6	31.20
	0.4	22	806.02	154	177	33	182	8.3	16.43
		81	806.02	257	261	62	323	39.6	48.93
	0.6	22	920.28	145	180	41	345	12.7	29.88
		81	920.28	138	231	82	459	46.2	144.50
	0.8	22	1007.68	103	204	45	391	11.8	56.41
		81	1007.68	106	222	102	435	49.5	159.45
	1	22	1056.26	1	178	70	176	1.8	47.02
		81	1056.26	1	187	127	376	24.9	54.94
0.15	0.2	22	772.66	61	213	29	174	6.4	17.09
		81	765.28	28	110	65	99	11.5	32.26
	0.4	22	884.34	49	171	40	199	6.5	20.09
		81	884.34	231	211	88	236	34.5	55.45
	0.6	22	983.63	30	203	60	284	7.0	18.03
		81	983.63	69	232	96	276	28.6	42.32
	0.8	22	1067.22	58	183	69	395	10.2	21.16
		81	1067.22	85	194	108	429	57.3	156.06
	1	22	1071.79	1	107	35	72	0.9	3.77
		81	1071.79	1	99	67	72	6.3	14.56

Table S.1
Results using CAB data set for USApHMP with $n = 25$.

p	α	Algorithm 1						GK(2019)
		Objective	Nodes	#(26)	#(28)	#(30)	Time	Time
2	0.2	1000.91	1	26	26	23	0.08	0.52
3	0.2	767.35	8	44	37	43	0.22	0.45
4	0.2	629.63	3	31	42	35	0.13	0.21
5	0.2	538.37	1	30	41	30	0.11	0.18
2	0.4	1101.63	1	26	26	23	0.08	0.11
3	0.4	901.70	6	45	41	75	0.34	0.33
4	0.4	787.52	2	36	42	54	0.24	0.28
5	0.4	707.69	29	33	31	39	0.28	0.20
2	0.6	1201.21	1	50	57	108	0.23	0.13
3	0.6	1033.56	19	47	36	130	0.78	0.40
4	0.6	939.21	2	40	41	92	0.28	0.47
5	0.6	876.59	30	28	29	85	0.67	0.34
2	0.8	1294.08	1	50	52	150	0.36	0.30
3	0.8	1158.83	1	59	46	158	0.44	0.45
4	0.8	1087.66	6	42	43	170	0.50	0.69
5	0.8	1034.10	2	34	38	166	0.52	0.49
2	1	1359.19	18	51	65	194	0.77	0.72
3	1	1256.63	1	60	50	156	0.53	0.58
4	1	1211.23	4	42	45	261	1.05	1.00
5	1	1173.24	51	50	31	263	1.47	1.04

Table S.2
Results using CAB data set for USAHLP with $n = 25$.

FC	α	Algorithm 1						GK(2019)
		Objective	Nodes	#(26)	#(28)	#(30)	Time	Time
100	0.2	1029.63	1	32	49	43	0.12	0.59
150	0.2	1217.35	13	33	51	74	0.28	0.46
200	0.2	1367.35	11	44	37	43	0.20	0.33
250	0.2	1500.91	2	57	39	27	0.22	0.28
100	0.4	1187.52	1	37	45	69	0.19	0.38
150	0.4	1351.70	2	42	43	96	0.23	0.39
200	0.4	1501.63	2	39	41	71	0.31	0.47
250	0.4	1601.63	2	55	41	34	0.14	0.15
100	0.6	1333.56	1	33	37	131	0.38	0.48
150	0.6	1483.56	1	36	40	115	0.33	0.49
200	0.6	1601.21	1	62	50	62	0.16	0.23
250	0.6	1701.21	2	54	54	62	0.20	0.19
100	0.8	1458.83	1	33	40	145	0.41	0.80
150	0.8	1594.08	1	57	51	127	0.31	0.35
200	0.8	1690.58	1	62	49	84	0.19	0.37
250	0.8	1740.58	1	39	36	24	0.08	0.19
100	1	1556.63	4	37	40	181	0.70	0.54
150	1	1640.58	1	49	53	91	0.23	0.30
200	1	1690.58	1	58	50	47	0.13	0.13
250	1	1740.58	1	39	36	24	0.08	0.12

We thank S. Haddadi for providing us the code of the primal–dual algorithm used for solving the transportation problem.

Appendix A. Computational studies for the CAB data set

This section is devoted to show the performance of our algorithm for solving instances from data set CAB with $n = 25$ and different values of p. Tables S.1 and S.2 report the results for solving the USApHMP and USAHLP, respectively. Both tables include the solution times for solving the problem by the method of Ghaffarinasab and Kara (2019).

The numerical tables have the following structure: the first column gives the value of p (for the USApHMP) or the value of fixed cost for installing hubs (for the USAHLP). Second column gives the value of discount factor α. The next column reports the optimal objective values. The number of nodes explored in the branching tree is given in column labeled *Nodes*. The following three columns give the number of inequalities added in the process as cuts, depending on the type, #(26), #(28) and #(30). The total time to obtain the optimal solution with Algorithm 1 is reported in *Time*. The times obtained with the method of Ghaffarinasab and Kara (2019) are given in the last column. It is observed that both method (our method and the ones of Ghaffarinasab and Kara, 2019) provide computational times below two seconds.

Appendix B. Comparing experiments in different computers

This section is devoted to provide more insights about the good performance of Algorithm 1 in comparison with the results obtained by Ghaffarinasab and Kara (2019). We have carried out a new computational analysis in an Intel Core i3-6100 3.7 GHz computer with 16 GB of RAM (a similar computer to the one used in Ghaffarinasab and Kara, 2019). Moreover, two different versions of the solver have been used: Xpress 8.10 available since 2020 and Xpress 8.04 available since 2016. We have even fixed the number of threads of our machine to four, as in Ghaffarinasab and Kara (2019).

Table S.3 reports the computational results provided by Ghaffarinasab and Kara (2019) in first column and the Algorithm 1 implemented in an Intel Xeon and Xpress 8.10 (second column), Intel Xeon and Xpress 8.0.4 (third column), Intel Core i3 and Xpress 8.10 (fourth column) and Intel Core i3 and Xpress 8.0.4 (fifth column). These results correspond to the AP instances for the USApHMP with $n \in \{100, 125, 150, 175, 200\}$ and $p \in \{5, 10, 15, 20\}$. Moreover, the percentage of improvement respect to Ghaffarinasab and Kara (2019) obtained with the two computers and the two versions of the solvers is given.

Table S.3
Comparative computational results considering different computers and different versions of the solver.

n	*p*	Ghaffarinasab and Kara (2019)	Intel Xeon Xpress 8.10	Intel Xeon Xpress 8.0.4	Intel Core i3 Xpress 8.10	Intel Core i3 Xpress 8.0.4
100	5	313.8	123.7	118.4	197.6	213.6
	10	109.1	47.0	49.4	79.1	74.6
	15	144.4	93.9	87.7	148.3	136.4
	20	61.4	53.3	55.2	91.7	103.4
Mean		157.2	79.5(**49.5%**)	77.7(**50.6%**)	129.2(**17.8%**)	132.0(**16.0%**)
125	5	1286.5	234.2	462.9	403.7	615.9
	10	414.0	162.0	135.2	251.9	231.2
	15	1271.8	769.4	374.1	1042.2	658.0
	20	213.7	165.7	178.2	345.9	372.2
Mean		796.5	332.8(**58.2%**)	287.6(**63.9%**)	510.9(**35.9%**)	469.3(**41.1%**)
150	5	2989.8	704.7	1208.7	1066.1	1447.7
	10	1148.0	340.5	551.1	649.3	807.1
	15	1695.1	419.6	502.5	842.4	967.4
	20	531.9	302.6	323.0	617.2	663.0
Mean		1591.2	441.9(**72.2%**)	646.3(**59.4%**)	793.8(**50.1%**)	971.3(**39.0%**)
175	5	31347.1	1284.2	11106.5	2395.4	20183.5
	10	10551.6	2331.3	1855.1	2091.2	2205.1
	15	19602.9	11234.3	19442.4	19046.4	25652.2
	20	1778.1	1601.1	887.8	2371.3	2003.0
Mean		15819.9	4112.7(**74.0%**)	8322.9(**47.4%**)	6476.1(**59.1%**)	12510.9(**20.9%**)
200	5	127546.7	3954.1	4542.9	6046.2	8400.4
	10	46706.9	1775.3	8813.9	3385.1	22817.1
	15	26640.5	1038.4	2253.5	2039.2	2156.7
	20	27224.4	1491.4	3332.6	2960.9	4937.3
Mean		57029.7	2064.8(96.4%)	4735.7(**91.7%**)	3607.8(**93.7%**)	9577.9(**83.2%**)

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