Interval and Fuzzy Optimization. Applications to Data Envelopment Analysis

Thesis report submitted in partial fulfillment

of the requirements for the degree of

Doctor of Philosophy

by

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Under the supervision of

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Ph. Doctorate in Mathematics
Research themes of Statistics and Operational Research
University of Cádiz
Jerez de la Frontera, Cádiz, 2022

Abstract

Enhancing concern in the efficiency assessment of a set of peer entities termed Decision Making Units (DMUs) in many fields from industry to healthcare has led to the development of efficiency assessment models and tools. Data Envelopment Analysis (DEA) is one of the most important methodologies to measure efficiency assessment through the comparison of a group of DMUs. It permits the use of multiple inputs/outputs without any functional form. It is vastly applied to production theory in Economics and benchmarking in Operations Research.

In conventional DEA models, the observed inputs and outputs possess precise and real-valued data. However, in the real world, some problems consider imprecise and integer data. For example, the number of defect-free lamps, the fleet size, the number of hospital beds or the number of staff can be represented in some cases as imprecise and integer data.

This thesis considers several novel approaches for measuring the efficiency assessment of DMUs where the inputs and outputs are interval and fuzzy data. First, an axiomatic derivation of the fuzzy production possibility set is presented and a fuzzy enhanced Russell graph measure is formulated using a fuzzy arithmetic approach. The proposed approach uses polygonal fuzzy sets and LU-fuzzy partial orders and provides crisp efficiency measures (and associated efficiency ranking) as well as fuzzy efficient targets. The second approach is a new integer interval DEA, with the extension of the corresponding arithmetic and LU-partial orders to integer intervals. Also, a new fuzzy integer DEA approach for efficiency assessment is presented. The proposed approach considers a hybrid scenario involving trapezoidal fuzzy integer numbers and trapezoidal fuzzy numbers. Fuzzy integer arithmetic and partial orders are introduced. Then, using appropriate axioms, a fuzzy integer DEA technology can be derived. Finally, an inverse (DEA) based on the non-radial slacks-based model in the presence of uncertainty, employing both integer and continuous interval data is presented.

Data Envelopment Analysis (DEA); Efficiency assessment; Decision Making Units (DMUs); Inverse DEA; Integer intervals; Fuzzy integer

Resumen

El aumento de la preocupación por la evaluación de la eficiencia de la conjunto de entidades pares denominadas Unidades de Toma de decisiones en muchos campos, desde la industria hasta la atención médica, ha llevado al desarrollo de modelos y herramientas de evaluación de la eficiencia. El Análisis Envolvente de Datos es una de las metodologías más importantes para medir la evaluación de la eficiencia a través de la comparación de un grupo de Unidades de Toma de decisiones. Permite el uso de múltiples entradas/salidas sin ninguna forma funcional. Se aplica ampliamente a la teoría de la producción en Economía y al evaluación comparativa en Investigación Operativa.

En los modelos de Análisis Envolvente de Datos convencionales, las entradas y salidas observadas poseen datos precisos y de valor real. Sin embargo, en el mundo real, algunos problemas consideran datos enteros e imprecisos. Por ejemplo, el número de lámparas sin defectos, el tamaño de la flota, el número de camas de hospital o el número de empleados pueden representarse en algunos casos como datos imprecisos y enteros.

La tesis considera varios enfoques novedosos para medir la evaluación de la eficiencia de Unidades de Toma de decisiones donde las entradas y salidas son datos de intervalos y difusos. En primer lugar, se presenta una derivación axiomática del conjunto de posibilidades de producción difuso y se formula una medida del gráfico de Russell mejorado difuso utilizando un enfoque aritmético difuso. El enfoque propuesto utiliza conjuntos difusos poligonales y órdenes parciales difusos LU y proporciona medidas de eficiencia nítidas (y clasificación de eficiencia asociada), así como objetivos de eficiencia difusos. El segundo enfoque es un nuevo Análisis Envolvente de Datos con intervalos de enteros, con la extensión de los correspondientes órdenes parciales LU y aritméticas a intervalos enteros. Además, se presenta un nuevo enfoque de Análisis Envolvente de Datos con enteros difusos para la evaluación de la eficiencia. El enfoque propuesto considera un escenario híbrido que involucra números enteros difusos trapezoidales y números difusos trapezoidales. Se introducen la aritmética de enteros difusos y los órdenes parciales. Luego, usando los axiomas apropiados, se puede derivar una tecnología para el Análisis Envolvente de Datos de enteros difusos. Finalmente, se presenta un Análisis Envolvente de Datos inverso basado en el modelo de holguras no radiales en presencia de incertidumbre, empleando datos con intervalos enteros y continuos.

Análisis Envolvente de Datos; Evaluación de la eficiencia; Unidades de Toma de Decisiones; Análisis Envolvente de Datos inverso; Intervalos enteros; Entero difuso

Acknowledgments

First and foremost, I express my deepest gratitude towards my supervisor Prof. Manuel Arana-Jiménez for the constant help and encouragement from the starting of the thesis work. I would like to extend my thanks to Prof. M. Carmen Sánchez-Gil and Prof. Sebastián Lozano for their help and suggestion during my research.

It gives me great pleasure in acknowledging the support Statistics and Operation Research Department for extending help and facilities.

Also, I would like to express my thanks to my father, my mother, my husband, my brother, and especially my daughter for support and care during my Ph.D. program.

Finally, I thank my God for all thing without whom this would have been impossible.

Jerez de la Frontera
March 16, 2022

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Chapter 1

Introduction

One of the widespread and main problems in all kinds of organizations, including public bodies, firms, and schools, is assessing and improving efficiency. Reduced costs, increased revenue, reduced natural resources depletion, reduced pollution, and increased sustainability are some benefits of evaluating efficiency. Therefore, it is necessary to develop efficiency assessment models and tools. Data envelopment analysis (DEA) is a non-parametric methodology to assess the efficiency of a set of Decision Making Units (DMUs) that consume multiple inputs to produce multiple outputs (Charnes et al. [31], Banker et al. [28]). A Production Possibility Set (PPS) can be derived from inputs and outputs and some certain axioms. The non-dominated subset of the PPS is defined as the efficient frontier. A DMU that is belongs to the efficient frontier is efficient. Otherwise, it is inefficient and the process of projecting it onto the efficient frontier determines an efficient target operating point. There are different ways to define the PPS (e.g. Constant Returns to Scale (CRS) and Variable Returns to Scale (VRS)) as well as different ways of projecting inefficient DMU onto the efficient frontier and evaluating the corresponding efficiency scores, e.g. using efficiency potential (Lozano and Calzada-Infante [107], Soltani and Lozano [129]), multidirectional approaches (Lozano and Soltani [108]) or lexicographic approaches (Lozano and Soltani [109, 110]), among others.

DEA is very well organized in the case of crisp data, but there are situations in which data is uncertain and imprecise. Dyson and Shale [49] study uncertainty in DEA and review different approaches for dealing with uncertainty. A classic approach for dealing with uncertainty is stochastic DEA methods, such as Chance Constrained DEA (Cooper et al. [38], [39]) or Monte Carlo simulation (e.g. Kao and Liu [85, 86]). In these approaches, it is assumed that the input and output data are random variables whose joint distribution function is known (generally a multivariate normal) or can be fitted from historical data (e.g. a beta distribution). The resulting efficient frontier is random. Olesen and Petersen [116] discuss a review of stochastic DEA approaches, including Stochastic Frontier Analysis and Chance-constrained DEA. Also Salahi et al. [123] and Izadikhah [70] work on Robust optimization and Chance-constrained DEA approaches, respectively.

Interval and fuzzy data is an alternative way to random data to study data uncertainty. Thus, often the data variability is not due to randomness. For example, the typical input variables such as the number of employees of a certain company or business unit are not random but can fluctuate from one month to another (or even from one week to another) throughout the year and it may not be reached a crisp number to represent variable. Another example happens with typical output variables like the number of customers attended may not exist and the corresponding numbers are imprecise. Interval and Fuzzy data have proved very helpful to model these types of uncertainties.

Hence, considering DEA to fuzzy data has been many hot topics in the literature (see Hatami-Marbini et al. [63] and Emrouznejad et al. [51] for a survey and a taxonomy), although many of the existing approaches have drawbacks (Soleimani-damaneh et al. [128]). A number of Fuzzy DEA (FDEA) approaches apply the α-level set (i.e. α-cut) concept (e.g. Kao and Liu [84], Saati et al. [121]). Another large group of FDEA studies considers a fuzzy ranking approach (e.g. Guo and Tanaka [59], León et al. [99], Ghasemi et al. [55]). Other researchers define a possibility (e.g. Lertworasirikul et al. [101], or a fuzzy arithmetic approach (e.g. Wang et al. [139]) or consider fuzzy random/type-2 fuzzy sets (e.g. Tavana et al. [136]). Another category of FDEA approaches, which has been thoroughly reviewed in Peikani et al. [118], is fuzzy chance-constrained DEA. Therefore FDEA is a growing area of efficiency analysis under uncertainty, with many new approaches and applications being reported (e.g. Kachouei et al. [83], Ebrahimnejad and Amani [50]). Also, FDEA can be applied to solve multiobjective fuzzy optimization problems (e.g. [25]-[27]).

In terms of interval DEA, there have been also many research, most of them use radial multiplier formulations (e.g. Despotis and Smirlis [42], Zhu [155]), although there are also additive imprecise DEA approaches (e.g. Lee et al. [100]), FDH interval DEA models (e.g. Jahanshaloo et al. [78]), non-radial, non-oriented imprecise DEA approaches (e.g. Azizi et al. [22]), ideal point approaches (e.g. Jahanshahloo et al. [73]), inverted DEA approaches (e.g. Inuiguchi and Mizoshita [69]), interval DEA with negative data (e.g. Hatami-Marbini et al. [62]), flexible measure interval DEA approaches (e.g. Kordrostami and Jahani Sayyad Noveiri [96]) and common weights imprecise DEA approaches (e.g. Hatami-Marbini et al. [61]). Applications involve in manufacturing industry (e.g. Wang et al. [139]), banks and bank branches (e.g. Jahanshaloo et al [74], Inuiguchi and Mizoshita [69], Hatami-Marbini et al. [62]), power plants (e.g. Khalili-Damghani et al. [91]), etc.

Another approach in DEA is integer DEA, it is first considered by Lozano and Villa [111] and consequently study in Kuosmanen and Kazemi Matin [98] and Kazemi Matin and Kuosmanen [90]. Advanced integer DEA models utilize Directional Distance Function (DDF) (e.g. Tan et al. [135]), super-efficiency (e.g. Du et al. [46], Chen et al. [36]), flexible measures (e.g. Kordrostami et al. [95]), two-stage systems (e.g. Ajirlo et al. [2]) or congestion (e.g. Khoveyni et al. [93]). A related problem is variables that can only take certain discrete values (e.g. Amirteimoori and Kordrostami Amir2014). Integer DEA has been used, for example, to hotel performance (Wu et al.[146]), sports (e.g. Wu et al. [147], Chen et al. [35]) and transportation (e.g. Lozano et al. [112], Yu and Hsu [150]).

As far as we know, The first fuzzy integer DEA approach is introduced by Kordrostami et al. [94] that develop the fuzzy integer DEA model of Jie et al. [81] applies the fuzzy number ranking method and the graded mean integration representation method to assess the efficiency score of DMUs where all the data are fuzzy integer data (triangular fuzzy numbers). They also study the hybrid scenario, in which some of the fuzzy inputs and outputs are integer and the rest are real-valued.

Also, another outstanding research area is inverse DEA. The concept of the inverse DEA model is firstly introduced by Zhang and Cui [152]. They study the input increases of a DMU are evaluated for its given output increases under the CCR efficiency fixed constraints, although inverse DEA is formally studied by Wei et al. [143]. They considered the first question in inverse DEA (output-estimation)."If the inputs of DMU_o increase, how much should the outputs of DMU_o increase to preserve the efficiency score of DMU_o?" Wei et al. [143] and Yan et al. [148] propose a linear programming problem when DMU_o is weakly efficient and a multiple-objective linear programming (MOLP) problem when DMU_o is inefficient to answer this question. The second question in inverse DEA (input-estimation) is studied by Hadi-Vancheh et al. ([65], [64]). "If the outputs of DMU_o increase, how much should the outputs of DMU_o increase to preserve the efficiency score of DMU_o?" They develope the models, which have introduced by Wei et al. [143]. Input-estimation and output-estimation have been studied by Jahanshahloo et al. ([76], [77]), provided that DMU_o maintains or improves the efficiency score. Also, both questions have been investigated under inter-temporal dependence by Jahanshahloo et al. [80]. The third question in inverse DEA (input-output estimation) is introduced by Jahanshahloo et al. [75]. " If the inputs and outputs of DMU_o increase, how much should the inputs and outputs of DMU_o increase to preserve the efficiency score of DMU_o?" This question is answered only for the efficient DMU_o. They applied MOLP for input-output estimation. The third question in inverse DEA was extended by Ghobadi [57], which improves the efficiency score of DMU_o. Most of the literature has studied on radial inverse DEA such as CCR [31], BBC [28], ST [124],and FG [52]. However, When slacks are of importance, radial inverse DEA can not answer questions in inverse DEA. Thus, some researchers consider inverse DEA based on non-radial models. As far as we know, Jahanshahloo et al. [75] apply a non-radial inverse DEA based on the Enhanced Russel model. They assume that the efficiency scores of each dimension remain unchanged. Then Zhang and Cui in [153] introduce a non-radial inverse DEA model, supposing that the overall efficiency score remains unchanged, covering all radial and non-radial measures that are monotonous.

The following chapters are devoted to study results and fuzzy optimization techniques to solve DEA and inverse DEA, discussing the future work of this thesis. It is organized as follows. In chapter 2, basic notations and results related to intervals and fuzzy numbers to define new integer intervals and fuzzy integer numbers are introduced. Then optimization techniques, DEA and inverse DEA are presented. Chapter 3 is based on our published paper titled "Efficiency assessment using fuzzy production possibility set and enhanced Russel Graph measure" (see [17]). We will use the enhanced Russell Graph measure (ERM), a non-radial and non-oriented approach, studied by Pastor et al. [117] under uncertain data that inputs and outputs are given by polygonal fuzzy numbers. The corresponding fuzzy PPS form inputs and outputs are derived and a fuzzy ERM model is modeled. To solve the proposed non-linear FERM model, a crisp optimization model is modeled to linearized. We finally have a simple and effective FDEA approach for evaluating the efficiency and projecting the production units. It is measured with a crisp efficiency score instead of a fuzzy efficiency. Chapter 4 is based on our published paper named "Integer interval DEA: an axiomatic derivation of the technology and an additive, slacks-based model" (see [18]). A new integer interval DEA technology and a new slacks-based DEA approach including two phases are considered. Chapter 5 is related to our submitted paper termed "Fuzzy integer Data Envelopment Analysis DEA" (see [19]). It defines a new fuzzy integer DEA technology and a new slacks-based fuzzy integer DEA approach that develop our previous work in Arana-Jimenez et al. [18] from interval integer DEA to fuzzy integer DEA. We consider that inputs and outputs can be either trapezoidal fuzzy integer numbers (TF_Z) or trapezoidal (real-valued) fuzzy numbers (tF_C). Chapter 6 is based on our submitted paper named "Using slacks-based model to solve inverse DEA with integer intervals for input estimation" (see [149]). The concepts about integer intervals are used to extend inverse DEA. We use non-radial slacks-based measure, which has more properties of radial models, on integer interval framework. We consider the following question: "If the output of DMU_o increases such that its inefficiency score is not less than t-percent, how much should the input of DMU_o increase?"

Chapter 2

Preliminaries on intervals, fuzzy numbers, and DEA

2.1 Introduction

In this chapter, the basic notations and results pertaining to intervals and fuzzy numbers are introduced to define new integer intervals and fuzzy integer numbers. Fuzzy logic was defined by Zadeh [151] to study the situations which are imprecise. More researchers have studied this subject and its applications, but there is a few literature in integer cases. The innovation of our work is to developed integer interval and fuzzy integer numbers. Then a brief introduction of optimization techniques, DEA, and inverse DEA is discussed.

2.2 Intervals

In this section, we introduce those arithmetic operations between intervals that are used in the next the sections and chapters.

Given $\mathbb{R}$ be the real number set. We denote by $\mathcal{K}_C = \left\{\left[\underline{a}, \overline{a}\right] \mid \underline{a}, \overline{a} \in \mathbb{R} \text{ and } \underline{a} \leq \overline{a}\right\}$ the family of all bounded closed intervals in $\mathbb{R}$.

Definition 2.2.1. *Given* $A = [\underline{a}, \overline{a}] \in \mathcal{K}_C$*,* $B = [\underline{b}, \overline{b}] \in \mathcal{K}_C$

- *Addition:* $A + B := \{a + b \mid a \in A, b \in B\} = [\underline{a} + \underline{b}, \overline{a} + \overline{b}]$,
- *Opposite value:* $-A = \{-a : a \in A\} = [-\overline{a}, -\underline{a}]$,
- *Multiplication:* $A \cdot B := \{a \cdot b \mid a \in A, b \in B\} = [min(A \cdot B), max(A \cdot B)]$,
  where $A \cdot B = \{\underline{a} \cdot \underline{b}, \underline{a} \cdot \overline{b}, \overline{a} \cdot \underline{b}, \overline{a} \cdot \overline{b}\}$.
- *Multiplication by scalar: for any* λ,

$$\lambda \cdot A := \begin{cases} [\lambda \cdot \underline{a}, \lambda \cdot \overline{a}] & \lambda \geq 0 \\ [\lambda \cdot \overline{a}, \lambda \cdot \underline{a}] & \lambda < 0 \end{cases}$$

- *difference:* $A - B = A + (-B) := \{a - b \mid a \in A, b \in B\} = [\underline{a} - \overline{b}, \overline{a} - \underline{b}]$,

Example 2.2.1. *Consider the following examples of the defined operations for continuous intervals. Note that, when applied to continuous intervals, all these operations produce continuous interval domains. For addition,* $[-5,2]+[-4,-1]=[-9,1]$*; for opposite value,* $-[2,7]=[-7,-2]$*; for multiplication,* $[2,4]\cdot[4,6]=[8,24]$*; for multiplication by scalar,* $3\cdot[2,4]=[6,12]$*,* $-3\cdot[2,4]=[-12,-6]$*; for difference,* $[-6,7]-[2,3]=[-6,7]+[-3,-2]=[-9,5]$*.*

For more detail on interval arithmetic, we refer to Moore [114, 115], Alefeld and Herberger [3] ,and Stefanini and Arana-Jimenez [133].

Also, It is necessary also to define a partial order relationship for intervals, which are well known in the literature, (see, e.g., [144, 133, 30] and the references therein).

Definition 2.2.2. *Given two intervals* $A=[\underline{a},\overline{a}], B=[\underline{b},\overline{b}]\in\mathcal{K}_C$*, we say that:*

(i) $[\underline{a},\overline{a}]\leqq[\underline{b},\overline{b}]$ *if and only if* $\underline{a}\leq\underline{b}$ *and* $\overline{a}\leq\overline{b}$*.*

(ii) $[\underline{a},\overline{a}]\leq[\underline{b},\overline{b}] if and only if [\underline{a},\overline{a}]\leqq[\underline{b},\overline{b}]$ *and* $[\underline{a},\overline{a}]\neq[\underline{b},\overline{b}]$*.*

(iii) $[\underline{a},\overline{a}]<[\underline{b},\overline{b}]$ *if and only if* $\underline{a}<\underline{b}$ *and* $\overline{a}<\overline{b}$*.*

2.2.1 Integer intervals

Apt and Zoeteweij [7] defined some arithmetic operations on integer intervals. Recently, Arana-Jiménez et al. [18] have extended them and established a new notation, as following.

Let A and B be integer intervals, we have following arithmetic operations:

- Addition: $A+B:=\{a+b \mid a\in A, b\in B\}$,
- Subtraction: $A-B:=\{a-b \mid a\in A, b\in B\}$,
- Multiplication: $A*B:=\{a*b \mid a\in A, b\in B\}$,
- Multiplication by scalar: for any integer λ,

$$\lambda * A := \begin{cases}\lambda * a & \lambda\geq 0\\ -\lambda * a & \lambda<0\end{cases}$$

Example 2.2.2. *To illustrate the previous arithmetic operations between integer intervals, consider the following examples. For the case of sum and subtraction,* $\{3,4,5\}+\{2,3,4\}=\{5,6,7,8,9\}$*,* $\{3,4,5,6\}-\{2,3,4,5,6,7\}=\{-4,,-3,-2,-1,0,1,2,3,4\}$*; and for the case of multiplication,* $\{2,3,4\}*\{4,5,6\}=\{8,10,12,12,15,16,18,20,24\}$*,* $3*\{2,3,4\}=\{6,9,12\}$*. Note, from the last example, that* $\{2,3,4\}*\{4,5,6\}$ *does not contains all integer numbers from* 8 *to* 24*, and also* $3*\{2,3,4\}$ *does not contains all integer numbers from* 6 *to* 12*.*

Therefore, for A, B integer intervals and a λ an integer the following holds:

- $A+B, A-B$ are integer intervals.
- $A*B$ does not correspond to an integer interval, in general. And the same for $\lambda * A$.

To deal with this problem, it is necessary to introduce a new multiplication operation for the multiplication between two integer interval to be an integer interval also. Let $\mathbb{Z}$ be the integer set. We denote by $\mathcal{K}_{\mathbb{Z}} = \left\{ \left[\underline{a}, \overline{a}\right]_{\mathbb{Z}} \mid \underline{a}, \overline{a} \in \mathbb{Z} \text{ and } \underline{a} \leq \overline{a} \right\}$ a closed integer interval in $\mathbb{Z}$.

Definition 2.2.3. *Let $A = [\underline{a}, \overline{a}] \in \mathcal{K}_{\mathbb{Z}}$, $B = [\underline{b}, \overline{b}] \in \mathcal{K}_{\mathbb{Z}}$*

- *Addition: $[\underline{a}, \overline{a}]_{\mathbb{Z}} + [\underline{b}, \overline{b}]_{\mathbb{Z}} = [\underline{a} + \underline{b}, \overline{a} + \overline{b}]_{\mathbb{Z}}$*
- *Subtraction: $[\underline{a}, \overline{a}]_{\mathbb{Z}} - [\underline{b}, \overline{b}]_{\mathbb{Z}} = [\underline{a} - \overline{b}, \overline{a} - \underline{b}]_{\mathbb{Z}}$*
- *Multiplication: $[\underline{a}, \overline{a}]_{\mathbb{Z}} \cdot [\underline{b}, \overline{b}]_{\mathbb{Z}} = [min(A \cdot B), max(A \cdot B)]_{\mathbb{Z}}$,*
  where $A \cdot B = \{\underline{a} \cdot \underline{b}, \underline{a} \cdot \overline{b}, \overline{a} \cdot \underline{b}, \overline{a} \cdot \overline{b}\}$.
- *Multiplication by scalar: for any integer λ,*

$$\lambda \cdot A := \begin{cases} [\lambda \cdot \underline{a}, \lambda \cdot \overline{a}]_{\mathbb{Z}} & \lambda \geq 0 \\ [\lambda \cdot \overline{a}, \lambda \cdot \underline{a}]_{\mathbb{Z}} & \lambda < 0 \end{cases}$$

Example 2.2.3. *Consider the following examples of the above operations for integer intervals.* $[4,5]_{\mathbb{Z}} + [-1,2]_{\mathbb{Z}} = [3,7]_{\mathbb{Z}}$, $[-4,5]_{\mathbb{Z}} - [-1,2]_{\mathbb{Z}} = [-6,4]_{\mathbb{Z}}$, $[2,4]_{\mathbb{Z}} \cdot [4,6]_{\mathbb{Z}} = [8,24]_{\mathbb{Z}}$, $3 \cdot [2,4]_{\mathbb{Z}} = [6,12]_{\mathbb{Z}}$. *It can be seen that the arithmetic operations for integer intervals defined above always produce integer intervals.*

Moreover, to extend the previous multiplication by an integer scalar to the case of a real scalar, we define the smaller integer bigger than and the largest integer less than a real number, i.e. given $x \in \mathbb{R}$, we define

$$i(x) = min\{z \in \mathbb{Z} \colon z \geq x\}, \tag{2.1}$$

$$I(x) = max\{z \in \mathbb{Z} \colon z \leq x\}. \tag{2.2}$$

Therefore, given $A = [\underline{a}, \overline{a}]_{\mathbb{Z}} \in \mathcal{K}_{\mathbb{Z}}$ and $\lambda \in \mathbb{R}$:

$$\lambda \cdot A := \begin{cases} [\lambda \cdot \underline{a}, \lambda \cdot \overline{a}] \cap \mathbb{Z} & \lambda \geq 0 \\ [\lambda \cdot \overline{a}, \lambda \cdot \underline{a}] \cap \mathbb{Z} & \lambda < 0 \end{cases}$$

$$= \begin{cases} [i(\lambda \cdot \underline{a}), I(\lambda \cdot \overline{a})]_{\mathbb{Z}} & \lambda \geq 0 \\ [i(\lambda \cdot \overline{a}), I(\lambda \cdot \underline{a})]_{\mathbb{Z}} & \lambda < 0 \end{cases}$$

$$= \left[\min\{i(\lambda \cdot \underline{a}), i(\lambda \cdot \overline{a})\}, \max\{I(\lambda \cdot \underline{a}), I(\lambda \cdot \overline{a})\}\right]_{\mathbb{Z}}.$$

Let us also define the **continuous extension** of an integer interval $[\underline{a}, \overline{a}]_{\mathbb{Z}} \in \mathcal{K}_{\mathbb{Z}}$ as

$$C([\underline{a}, \overline{a}]_{\mathbb{Z}}) := [\underline{a}, \overline{a}] \in \mathcal{K}_C \tag{2.3}$$

Conversely, we define the **integer projection** of $[\underline{a}, \overline{a}] \in \mathcal{K}_C$ as

$$\mathbb{Z}([\underline{a}, \overline{a}]) := [i(\underline{a}), I(\overline{a})]_{\mathbb{Z}} \in \mathcal{K}_{\mathbb{Z}} \tag{2.4}$$

If $\underline{a} \leq \overline{a}$ and $\underline{a}, \overline{a} \in \mathbb{Z}$ then $\mathbb{Z}([\underline{a}, \overline{a}]) = [\underline{a}, \overline{a}]_{\mathbb{Z}} \in \mathcal{K}_{\mathbb{Z}}$. In this case, $[\underline{a}, \overline{a}] \in \boldsymbol{\mathcal{K}}_{C \to \mathbb{Z}}$, which is the set of intervals whose endpoints are integer. Note also that $\mathbb{Z}(C([\underline{a}, \overline{a}]_{\mathbb{Z}})) = [\underline{a}, \overline{a}]_{\mathbb{Z}}$ and in general, given $[\underline{a}, \overline{a}] \in \mathcal{K}_C$, then $C(\mathbb{Z}([\underline{a}, \overline{a}])) \subseteq [\underline{a}, \overline{a}]$.

Now, an adaptation of LU-fuzzy partial orders on intervals to integer intervals will be used.

Definition 2.2.4. *Given two integer intervals $A = [\underline{a}, \overline{a}]_{\mathbb{Z}}, B = [\underline{b}, \overline{b}]_{\mathbb{Z}} \in \mathcal{K}_{\mathbb{Z}}$, we say that:*

(i) $[\underline{a}, \overline{a}]_{\mathbb{Z}} \leqq [\underline{b}, \overline{b}]_{\mathbb{Z}}$ *if and only if* $\underline{a} \leq \underline{b}$ *and* $\overline{a} \leq \overline{b}$.

(ii) $[\underline{a}, \overline{a}]_{\mathbb{Z}} \prec [\underline{b}, \overline{b}]_{\mathbb{Z}}$ *if and only if* $\underline{a} < \underline{b}$ *and* $\overline{a} < \overline{b}$.

In a similar manner, we define the relationships $A \geqq B$ and $A \succ B$ for intervals and integer intervals, which really means $B \leqq A$ and $B \prec A$, respectively. Note that, for the sake of simplicity, we use the same symbols of partial orders to compare intervals in $\mathcal{K}_C$ as to compare integer intervals in $\mathcal{K}_{\mathbb{Z}}$.

2.3 Fuzzy numbers

We denote by $\mathcal{K}_C = \left\{ \left[\underline{a}, \overline{a}\right] \mid \underline{a}, \overline{a} \in \mathbb{R} \text{ and } \underline{a} \leq \overline{a} \right\}$ the family of all bounded closed intervals in $\mathbb{R}$. A fuzzy set on $\mathbb{R}^n$ is a mapping $u : \mathbb{R}^n \to [0, 1]$. For each fuzzy set u, we denote its α-level set as $[u]^\alpha = \{x \in \mathbb{R}^n \mid u(x) \geq \alpha\}$ for any $\alpha \in (0, 1]$, and its support as $supp(u) = \{x \in \mathbb{R}^n \mid u(x) > 0\}$. The closure of $supp(u)$ defines the 0-level of u, i.e. $[u]^0 = cl(supp(u))$ where $cl(M)$ means the closure of the subset $M \subset \mathbb{R}^n$. Following Dubois & Prade [47, 48], a fuzzy set u on $\mathbb{R}$ is said to be a fuzzy number if (i) u is normal, this is there exists $x_0 \in \mathbb{R}$ such that $u(x_0) = 1$, (ii) upper semi-continuous function, (iii) convex, and (iv) $[u]^0$ is compact. $\mathcal{F}_C$ denotes the family of all fuzzy numbers. The α-levels of a fuzzy number can be represented as $[u]^\alpha = \left[\underline{u}_\alpha, \overline{u}_\alpha\right] \in \mathcal{K}_C$, $\underline{u}_\alpha, \overline{u}_\alpha \in \mathbb{R}$. Although there are many parametrical families of fuzzy numbers (see [24] and [60] for a complete description of these families), two of the most used families of fuzzy numbers are triangular and trapezoidal fuzzy numbers, because of their easy modeling and interpretation (see, for instance, [47, 48, 88, 92, 105, 131]). As an extension of these two families of fuzzy numbers, and inspired in other definitions of parametric fuzzy numbers (see, for instance, [131, 132, 60, 24]), below we review the concept of polygonal fuzzy numbers, introduced by Báez et al. [24] as a particular case of polygonal fuzzy sets.

Definition 2.3.1. *Given a partition of the interval* $[0, 1]$, $\mathcal{P}_k = \{\alpha_i : i = 0, 1 \ldots, k\}$, *with* $0 = \alpha_0 < \alpha_1 <, \ldots, < \alpha_k = 1$, *a fuzzy number* $\tilde{a}$ *is said to be a k-polygonal fuzzy number with respect to* $\mathcal{P}_k$ *if its* α*-levels satisfy* $[\tilde{a}]^\alpha = (1 - \lambda)[\tilde{a}]^{\alpha_{i-1}} + \lambda[\tilde{a}]^{\alpha_i}$, *where* $0 \leq \alpha_{i-1} < \alpha \leq \alpha_i \leq 1$ *for some* $i = 1, \ldots, k-1$ *and* $\lambda = \lambda(\alpha) = (\alpha - \alpha_{i-1})/(\alpha_i - \alpha_{i-1})$.

In terms of their membership function instead of their α-levels, a polygonal fuzzy number can alternatively be defined as follows.

Proposition 2.3.1. *Given a partition of the interval* $[0, 1]$, $\mathcal{P}_k = \{\alpha_i : i = 0, 1 \ldots, k\}$, *with* $0 = \alpha_0 < \alpha_1 <, \ldots, < \alpha_k = 1$, *for* $k \in N$, *a fuzzy number* $\tilde{a}$ *is a k-polygonal fuzzy number with respect to* $\mathcal{P}_k$ *if and only if there exist* $a_0^-, a_1^-, \ldots, a_k^-, a_k^+, \ldots, a_1^+, a_0^+ \in \mathbb{R}$ *with* $a_0^- \leq a_1^- \leq \cdots \leq a_k^- \leq a_k^+ \leq \cdots \leq a_1^+ \leq a_0^+$, *such that its membership function is*

$$\tilde{a}(x) = \begin{cases} \dfrac{x - a_{i-1}^-}{a_i^- - a_{i-1}^-}(\alpha_i - \alpha_{i-1}) + \alpha_{i-1}, & \text{if } i \in \{1, \ldots, k\} \text{ and } a_{i-1}^- \leq x < a_i^-, \\ 1, & \text{if } a_k^- \leq x \leq a_k^+, \\ \dfrac{a_{i-1}^+ - x}{a_{i-1}^+ - a_i^+}(\alpha_i - \alpha_{i-1}) + \alpha_{i-1}, & \text{if } i \in \{1, \ldots, k\} \text{ and } a_i^+ < x \leq a_{i-1}^+, \\ 0, & \text{otherwise.} \end{cases} \tag{2.5}$$

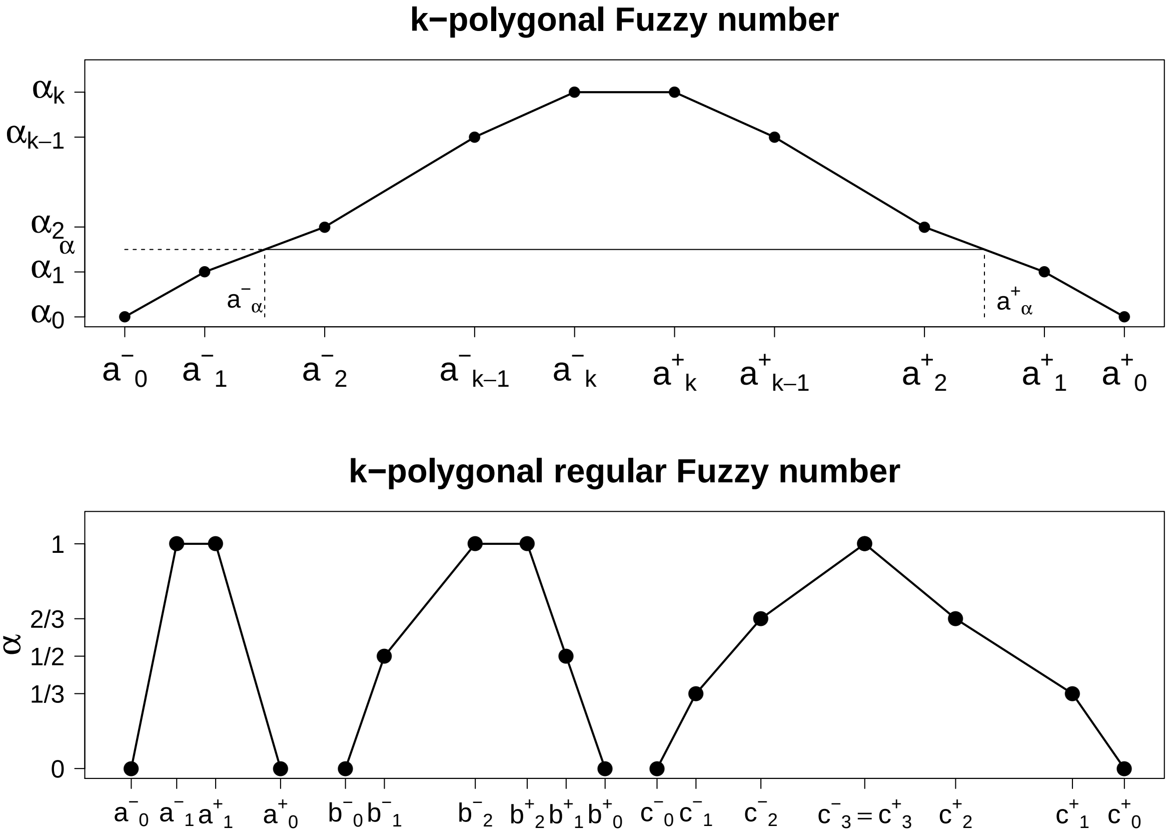

Figure 2.1: Representation for a general $k-$ polygonal Fuzzy number $\tilde{a} = a_0^-, a_1^-, \ldots, a_k^-, a_k^+, \ldots, a_1^+, a_0^+$, with respect a partition $\alpha_0 = 0 < \alpha_1 \leq \ldots \leq \alpha_k = 1$ (top) and some examples of $k = 1$, $k = 2$ and $k = 3$ regular polygonal fuzzy numbers (bottom).

As a result of Proposition 2.3.1, we can denote a k-polygonal fuzzy number with respect to $\mathcal{P}_k = \{\alpha_i : i = 0, 1 \ldots, k\}$ as $\tilde{a} = a_0^-, a_1^-, \ldots, a_k^-, a_k^+, \ldots, a_1^+, a_0^+$. For the sake of simplicity, in the sequel, we will assume that $\alpha_i = \frac{i}{k}$, and then $\tilde{a}$ is said to be a regular k-polygonal fuzzy number. We denote the set of all regular k-polygonal fuzzy numbers as $RPFN_k$. Thus, $\tilde{a} = (a_0^-, a_1^-, a_1^+, a_0^+)$ corresponds to a trapezoidal fuzzy number. And if $a_1^- = a_1^+$ then $\tilde{a}$ is a triangular fuzzy number, and it can be noted as $\tilde{a} = (a^-, a, a^+)$. Finally, we denote $\tilde{0}$ and $\tilde{1}$ as the regular polygonal fuzzy numbers whose components are all 0 and 1, respectively.

For better illustration, we introduce the definition of trapezoidal fuzzy numbers that have been extensively studied in the literature (for example [47], [48], [88], [92], [105], [131]). Also we present examples of both, a general $k-$ polygonal fuzzy number, on the top, and regular 1-polygonal and 2-polygonal fuzzy numbers, on the bottom in Figure 2.1 .

Definition 2.3.2. *Given a_1, a_2, a_3 and $a_4 \in \mathbb{R}$ with $a_1 \leq a_2 \leq a_3 \leq a_4$. If the fuzzy set $u : \mathbb{R} \to [0, 1]$ is defined as*

$$u(x) = \begin{cases} \frac{x - a_1}{a_2 - a_1} & if\ x \in [a_1, a_2] \\ 1 & if\ x \in [a_2, a_3] \\ \frac{a_4 - x}{a_4 - a_3} & if\ x \in [a_3, a_4] \\ 0 & c.c. \end{cases} \tag{2.6}$$

*then u is said to be a **trapezoidal fuzzy number**, $u \in \boldsymbol{TF_C}$, and it is also denoted as $u \equiv \tilde{a} = (a_1, a_2, a_3, a_4)$. If $a_2 = a_3$, then we have a triangular fuzzy ($\boldsymbol{tF_C}$).*

As an extension for fuzzy numbers (see [29], [56], [103]), the membership function $u * v$, with $* \in \{+, \cdot\}$, is defined as

$$(u * v)(z) := Sup\ min\{u(x), u(y)\}.$$

And given $\lambda \in \mathbb{R}$,

$$(\lambda \cdot u)(x) := \begin{cases} u(x/\lambda) & \lambda \neq 0 \\ 0 & \lambda = 0 \end{cases}$$

Therefore, the addition, multiplication, and multiplication by scalar of fuzzy numbers is also a fuzzy number, and it can be considered by means of α-levels (see, for instance, Theorem 2.6, [56]). In this regard, given $u, v \in \mathcal{F}_C$ represented by means of α-levels as $[u]^\alpha = [\underline{u}_\alpha, \overline{u}_\alpha] \in \mathcal{K}_C$, and $[v]^\alpha = [\underline{v}_\alpha, \overline{v}_\alpha] \in \mathcal{K}_C$, respectively, and $\lambda \in \mathbb{R}$, then for any $\alpha \in [0, 1]$:

$$[u + v]^\alpha := [u]^\alpha + [v]^\alpha = [\underline{u}_\alpha + \underline{v}_\alpha, \overline{u}_\alpha + \overline{v}_\alpha], \tag{2.7}$$

$$\begin{aligned}[u \cdot v]^\alpha := [u]^\alpha \cdot [v]^\alpha = \Big[& min\{\underline{u}_\alpha \cdot \underline{v}_\alpha, \underline{u}_\alpha \cdot \overline{v}_\alpha, \overline{u}_\alpha \cdot \underline{v}_\alpha, \overline{u}_\alpha \cdot \overline{v}_\alpha\}, \\ & max\{\underline{u}_\alpha \cdot \underline{v}_\alpha, \underline{u}_\alpha \cdot \overline{v}_\alpha, \overline{u}_\alpha \cdot \underline{v}_\alpha, \overline{u}_\alpha \cdot \overline{v}_\alpha\}\Big],\end{aligned} \tag{2.8}$$

$$\begin{aligned}[\lambda \cdot u]^\alpha := \lambda \cdot [u]^\alpha &= \begin{cases} [\lambda \cdot \underline{u}_\alpha, \lambda \cdot \overline{u}_\alpha] & \lambda \geq 0 \\ [\lambda \cdot \overline{u}_\alpha, \lambda \cdot \underline{u}_\alpha] & \lambda < 0 \end{cases} \\ &= \Big[min\{\lambda \cdot \underline{u}_\alpha, \lambda \cdot \overline{u}_\alpha\}, max\{\lambda \cdot \underline{u}_\alpha, \lambda \cdot \overline{u}_\alpha\}\Big]\end{aligned} \tag{2.9}$$

As mentioned above, if $u, v \in \mathcal{F}_C$, and $\lambda \in \mathbb{R}$, then it holds that $u + v \in \mathcal{F}_C$, $u \cdot v \in \mathcal{F}_C$, and $\lambda \cdot u \in \mathcal{F}_C$.

Regarding the previous arithmetic operations (2.7), (2.8), and (2.9), we present the following definitions for two arithmetic operations, namely sum, multiplication, and multiplication by scalar.

Definition 2.3.3. *Given two regular k-polygonal fuzzy numbers $\tilde{a} = a_0^-, a_1^-, \ldots, a_k^-, a_k^+, \ldots, a_1^+, a_0^+$ and $\tilde{b} = b_0^-, b_1^-, \ldots, b_k^-, b_k^+, \ldots, b_1^+, b_0^+$, the following basic arithmetical operations can be defined:*

(i) Addition

$$\tilde{a} + \tilde{b} = (a_0^- + b_0^-, a_1^- + b_1^-, \ldots, a_k^- + b_k^-, a_k^+ + b_k^+, \ldots, a_1^+ + b_1^+, a_0^+ + b_0^+) \tag{2.10}$$

(ii) The multiplication of two k-fuzzy polygonal numbers, $\tilde{a}\tilde{b} = \tilde{c} = (c_0^-, c_1^-, \ldots, c_k^-, c_k^+, \ldots, c_1^+, c_0^+)$, where

$$\begin{cases} c_i^- = \min\{a_i^- b_i^-, a_i^- b_i^+, a_i^+ b_i^-, a_i^+ b_i^+\} \\ c_i^+ = \max\{a_i^- b_i^-, a_i^- b_i^+, a_i^+ b_i^-, a_i^+ b_i^+\} \end{cases} \quad i = 0, 1, \ldots, k \tag{2.11}$$

(ii) Multiplication by a scalar $\lambda \in \mathbb{R}$,

$$\lambda\tilde{a} = \begin{cases} (\lambda a_0^-, \lambda a_1^-, \ldots, \lambda a_k^-, a_k^+, \ldots, \lambda a_1^+, \lambda a_0^+) & if\ \lambda \geq 0; \\ (\lambda a_0^+, \ldots, \lambda a_{k-1}^+, \lambda a_k^+, \lambda a_k^-, \lambda a_{k-1}^-, \ldots, \lambda a_0^-) & if\ \lambda < 0. \end{cases} \tag{2.12}$$

Note that the multiplication (2.11) is not the same as defined previously in (2.8). But it is indeed an approximation of the latter which ensures an internal operation.

For illustration, we present the following arithmetic rules on the trapezoidal fuzzy numbers (TF_C) (see, for example, Kumar et al. [97], Kaufmann, Gupta [88], khan et al. [92]). Given two trapezoidal fuzzy numbers $\tilde{a} = (a_1, a_2, a_3, a_4) \in TF_C$, $\tilde{b} = (b_1, b_2, b_3, b_4) \in TF_C$, and $\lambda \in \mathbb{R}$,

$$\begin{aligned}
\tilde{a} + \tilde{b} &:= (a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4) &(2.13)\\
\tilde{a} \cdot \tilde{b} &:= \Big(\min\{a_1b_1, a_1b_4, a_4b_1, a_4b_4\}, \min\{a_2b_2, a_2b_3, a_3b_2, a_3b_3\}, \\
&\quad \max\{a_2b_2, a_2b_3, a_3b_2, a_3b_3\}, \max\{a_1b_1, a_1b_4, a_4b_1, a_4b_4\}\Big) &(2.14)\\
\lambda \cdot \tilde{a} &:= \begin{cases} (\lambda \cdot a_1, \lambda \cdot a_2, \lambda \cdot a_3, \lambda \cdot a_4) & \lambda \geq 0 \\ (\lambda \cdot a_1, \lambda \cdot a_2, \lambda \cdot a_3, \lambda \cdot a_4) & \lambda < 0 \end{cases} &(2.15)
\end{aligned}$$

Note that the multiplication (2.14) is not the same as defined previously in (2.8). But it is indeed an approximation of the latter which ensures an internal operation, i.e. $\tilde{a} \cdot \tilde{b} \in TF_C$. For further information on triangular and trapezoidal fuzzy numbers, the reader is referred to, for example, [24], [131], [132].

Let us recall th LU-fuzzy partial orders, which are well known in the literature (see, e.g., [144, 133] and the references therein).

Definition 2.3.4. *Given two fuzzy numbers $u, v \in \mathcal{F}_C$ represented by means of α-levels as $[u]^\alpha = [\underline{u}_\alpha, \overline{u}_\alpha] \in \mathcal{K}_C$, $[v]^\alpha = [\underline{v}_\alpha, \overline{v}_\alpha] \in \mathcal{K}_C$, respectively, then:*

(i) $u \leqq v$ if and only if $[\underline{u}_\alpha, \overline{u}_\alpha] \leqq [\underline{v}_\alpha, \overline{v}_\alpha]$, for all $\alpha \in [0, 1]$.

(ii) $u \prec v$ if and only if $[\underline{u}_\alpha, \overline{u}_\alpha] \prec [\underline{v}_\alpha, \overline{v}_\alpha]$, for all $\alpha \in [0, 1]$.

The relationships $\mu \geqq \nu$ and $\mu \succ \nu$ means $\nu \leqq \mu$ and $\nu \prec \mu$, respectively. In Arana [8] a reformulation of the previous definition for triangular fuzzy numbers by means of the relationship between their parameters is presented. The following extends this result to regular k-polygonal fuzzy numbers.

Proposition 2.3.2. *Given two regular k-polygonal fuzzy numbers $\tilde{a} = a_0^-, a_1^-, \ldots, a_k^-, a_k^+, \ldots, a_1^+, a_0^+$ and $\tilde{b} = b_0^-, b_1^-, \ldots, b_k^-, b_k^+, \ldots, b_1^+, b_0^+$ with respect to $\{\alpha_i : i = 0, 1 \ldots, k\}$, it follows that*

(i) $\tilde{a} \leqq \tilde{b}$ if and only if $a_i^- \leq b_i^-$, and $a_i^+ \leq b_i^+$, for all $i = 0, 1, \ldots, k$.

(ii) $\tilde{a} \prec \tilde{b}$ if and only if $a_i^- < b_i^-$, and $a_i^+ < b_i^+$, for all $i = 0, 1, \ldots, k$.

We can also consider a natural extension of Definition 2.3.4 to vectors of fuzzy numbers. Thus, given two vectors of fuzzy numbers $\mu = (\mu_1, \ldots, \mu_H)$ and $\nu = (\nu_1, \ldots, \nu_H)$, we say that $\mu \leqq (\prec)\nu$ if $\mu_h \leqq (\prec)\nu_h$ for all h.

2.3.1 Fuzzy integer numbers

To extend fuzzy numbers and applications, Wang et al. in [141] have presented a definition for fuzzy integer numbers. This definition, under our notation, is as follows.

Definition 2.3.5. *A fuzzy set $u : \mathbb{R} \to [0, 1]$ is called a **fuzzy integer number** if its support is a closed integer interval, denoted $[\underline{u}(0), \overline{u}(0)]_{\mathbb{Z}}$, and satisfies the following:*

1) u is normal; i.e., there exists $x' \in [\underline{u}(0), \overline{u}(0)]_{\mathbb{Z}}$ such that $u(x') = 1$, *2) $u(x_i) \leq u(x_j)$ for any $x_i, x_j \in [\underline{u}(0), x']_{\mathbb{Z}}$ with $x_i \leq x_j$,*

3) $u(x_i) \geq u(x_j)$ for any $x_i, x_j \in [x', \overline{u}(0)]_{\mathbb{Z}}$ with $x_i \leq x_j$,

Let $\mathcal{F}_{\mathbb{Z}}$ be the family of all fuzzy integer numbers and note that $\mathcal{F}_{\mathbb{Z}} \subseteq \mathcal{F}_C$. A fuzzy integer number can be represented by means of its α-levels as $[u]^{\alpha}_{\mathbb{Z}} = \left[\underline{u}_{\alpha}, \overline{u}_{\alpha}\right]_{\mathbb{Z}} \in \mathcal{K}_Z$, where $\underline{u}_{\alpha}, \overline{u}_{\alpha} \in \mathbb{Z}$.

The arithmetic operations of integer intervals has been presented, let us do the same for fuzzy integer numbers. Once more, the multiplication between two fuzzy integer numbers and the multiplication by a scalar introduced in [141] do not preserve the closeness of these operations, i.e. the α-levels of the multiplication and multiplication by scalar operations are not necessarily a closed integer interval. In order to overcome this issue, let us consider that the product of these α-levels is equal to the product of the corresponding α-levels of the fuzzy integer numbers involved. Mathematically, given $u, v \in \mathcal{F}_{\mathbb{Z}}$, represented by means of their α-levels, for any $\alpha \in [0,1]$, as $[u]^{\alpha}_{\mathbb{Z}} = [\underline{u}_{\alpha}, \overline{u}_{\alpha}]_{\mathbb{Z}}$, and $[v]^{\alpha}_{\mathbb{Z}} = [\underline{v}_{\alpha}, \overline{v}_{\alpha}]_{\mathbb{Z}}$, respectively, and given $\lambda \in \mathbb{R}$, let

$$[u+v]^{\alpha}_{\mathbb{Z}} = [u]^{\alpha}_{\mathbb{Z}} + [v]^{\alpha}_{\mathbb{Z}} = [\underline{u}_{\alpha} + \underline{v}_{\alpha}, \overline{u}_{\alpha} + \overline{v}_{\alpha}]_{\mathbb{Z}}, \tag{2.16}$$

$$\begin{aligned}[u \cdot v]^{\alpha}_{\mathbb{Z}} = [u]^{\alpha}_{\mathbb{Z}} \cdot [v]^{\alpha}_{\mathbb{Z}} = \Big[& min\{\underline{u}_{\alpha} \cdot \underline{v}_{\alpha}, \underline{u}_{\alpha} \cdot \overline{v}_{\alpha}, \overline{u}_{\alpha} \cdot \underline{v}_{\alpha}, \overline{u}_{\alpha} \cdot \overline{v}_{\alpha}\}, \\ & max\{\underline{u}_{\alpha} \cdot \underline{v}_{\alpha}, \underline{u}_{\alpha} \cdot \overline{v}_{\alpha}, \overline{u}_{\alpha} \cdot \underline{v}_{\alpha}, \overline{u}_{\alpha} \cdot \overline{v}_{\alpha}\}\Big]_{\mathbb{Z}},\end{aligned} \tag{2.17}$$

$$\begin{aligned}[\lambda \cdot u]^{\alpha}_{\mathbb{Z}} = \lambda \cdot [u]^{\alpha}_{\mathbb{Z}} &= \begin{cases} [\lambda \cdot \underline{u}_{\alpha}, \lambda \cdot \overline{u}_{\alpha}] \cap \mathbb{Z} = [i(\lambda \cdot \underline{u}_{\alpha}), I(\lambda \cdot \overline{u}_{\alpha})]_{\mathbb{Z}} & \lambda \geq 0 \\ [\lambda \cdot \overline{u}_{\alpha}, \lambda \cdot \underline{u}_{\alpha}] \cap \mathbb{Z} = [i(\lambda \cdot \overline{u}_{\alpha}), I(\lambda \cdot \underline{u}_{\alpha})]_{\mathbb{Z}} & \lambda < 0 \end{cases} \\ &= \Big[min\{i(\lambda \cdot \underline{u}_{\alpha}), i(\lambda \cdot \overline{u}_{\alpha})\}, max\{I(\lambda \cdot \underline{u}_{\alpha}), I(\lambda \cdot \overline{u}_{\alpha})\}\Big]_{\mathbb{Z}}.\end{aligned} \tag{2.18}$$

In this manner, given $u, v \in \mathcal{F}_{\mathbb{Z}}$ and $\lambda \in \mathbb{R}$, it holds that $u + v \in \mathcal{F}_{\mathbb{Z}}$, $u \cdot v \in \mathcal{F}_{\mathbb{Z}}$, and $\lambda \cdot u \in \mathcal{F}_{\mathbb{Z}}$.

Given Equations (2.3) and (2.4), the continuous extension of a fuzzy integer number $u \in \mathcal{F}_{\mathbb{Z}}$ can be defined by means of α-levels as $C([\underline{u}_{\alpha}, \overline{u}_{\alpha}]_{\mathbb{Z}}) = [\underline{u}_{\alpha}, \overline{u}_{\alpha}] \in \mathcal{K}_C$. Conversely, the integer projection of a given fuzzy number $u \in \mathcal{F}_C$ by means of α-levels $[u]^{\alpha} = [\underline{u}_{\alpha}, \overline{u}_{\alpha}] \in \mathcal{K}_C$ is $\mathbb{Z}([\underline{u}_{\alpha}, \overline{u}_{\alpha}]) = [i(\underline{u}_{\alpha}), I(\overline{u}_{\alpha})]_{\mathbb{Z}} \in \mathcal{K}_{\mathbb{Z}}$. $\mathbb{Z}([\underline{u}_{\alpha}, \overline{u}_{\alpha}]) = [I(\underline{u}_{\alpha}), i(\overline{u}_{\alpha})]_{\mathbb{Z}} \in \mathcal{K}_{\mathbb{Z}}$. If $\underline{u}_{\alpha} \leq \overline{u}_{\alpha}$ and $\underline{u}_{\alpha}, \overline{u}_{\alpha} \in \mathbb{Z}$ then $\mathbb{Z}([\underline{u}_{\alpha}, \overline{u}_{\alpha}]) = [\underline{u}_{\alpha}, \overline{u}_{\alpha}]_{\mathbb{Z}} \in \mathcal{K}_{\mathbb{Z}}$. In this case, we say that $[\underline{u}_{\alpha}, \overline{u}_{\alpha}] \in \mathcal{K}_{C \to \mathbb{Z}}$, which is the set of intervals whose endpoints are integer. Note also that, in this case, $\mathbb{Z}(C([\underline{u}_{\alpha}, \overline{u}_{\alpha}]_{\mathbb{Z}})) = [\underline{u}_{\alpha}, \overline{u}_{\alpha}]_{\mathbb{Z}}$, but in general, given $[\underline{u}_{\alpha}, \overline{u}_{\alpha}] \in \mathcal{K}_C$, then $C(\mathbb{Z}([\underline{u}_{\alpha}, \overline{u}_{\alpha}])) \subseteq [\underline{u}_{\alpha}, \overline{u}_{\alpha}]$.

Wang et al. [141] introduce the definition of trapezoidal fuzzy integer numbers ($TF_{\mathbb{Z}}$). We rewrite it with our notation as follows.

Definition 2.3.6. *Given a_1, a_2, a_3 and $a_4 \in \mathbb{Z}$ with $a_1 \leq a_2 \leq a_3 \leq a_4$. If the fuzzy set $u : \mathbb{R} \to [0,1]$ is defined as*

$$u(x) = \begin{cases} \frac{x - a_1}{a_2 - a_1} & if\ x \in [a_1, a_2]_{\mathbb{Z}} \\ 1 & if\ x \in [a_2, a_3]_{\mathbb{Z}} \\ \frac{a_4 - x}{a_4 - a_3} & if\ x \in [a_3, a_4]_{\mathbb{Z}} \\ 0 & otherwise \end{cases} \tag{2.19}$$

*then u is said to be an **trapezoidal fuzzy integer number ($TF_{\mathbb{Z}}$)**, and it is denoted as $u \equiv \tilde{a} = (a_1, a_2, a_3, a_4)_{\mathbb{Z}}$. If $a_2 = a_3$, then we have a triangular fuzzy integer number (**$tF_{\mathbb{Z}}$**).*

For any $\alpha \in [0, 1]$, the α-level of a trapezoidal fuzzy integer number is a closed integer interval and can be represented as

$$\begin{aligned} [u]_{\mathbb{Z}}^{\alpha} &= [\underline{u}_{\alpha}, \overline{u}_{\alpha}]_{\mathbb{Z}} = [a_1 + (a_2 - a_1)\alpha, a_4 - (a_4 - a_3)\alpha] \cap \mathbb{Z} = \\ &= [i(a_1 + (a_2 - a_1)\alpha), I(a_4 - (a_4 - a_3)\alpha)]_{\mathbb{Z}} \end{aligned} \tag{2.20}$$

The same discussion given above, in Equations (2.16), (2.17), and (2.18), for the basic arithmetic by means of α-levels for general fuzzy integer numbers $\mathcal{F}_{\mathbb{Z}}$ applies also to the trapezoidal fuzzy integer numbers $TF_{\mathbb{Z}}$. However, for trapezoidal (triangular) fuzzy integer numbers, usually represented by their components $\tilde{a} = (a_1, a_2, a_3, a_4)_{\mathbb{Z}}$, we must be aware that the intuition may fail when they are seen as the projection of their continuous extension. The following example illustrates more clearly this issue, as well as the basic arithmetic with $TF_{\mathbb{Z}}$.

Example 2.3.1. *In this example, we show that the extension of arithmetic rules of trapezoidal fuzzy numbers (TF_C) to trapezoidal fuzzy integer numbers ($TF_{\mathbb{Z}}$) does not work. Let be $\tilde{a} = (1, 2, 2, 3)_{\mathbb{Z}}$ and $\tilde{b} = (4, 6, 8, 9)_{\mathbb{Z}}$ two trapezoidal fuzzy integer numbers. For the case of the addition, defined by α-levels as per Equation (2.16), we have that $\tilde{a} + \tilde{b} \in \mathcal{F}_{\mathbb{Z}}$, but $\tilde{a} + \tilde{b} \neq (1+4, 2+6, 2+8, 3+9)_{\mathbb{Z}} = (5, 8, 10, 12)_{\mathbb{Z}}$. In other words, the sum rule given by Equation (2.13) for the continuous case (TF_C) does not apply. This can be observed in the middle panel of Figure 2.2. For each fuzzy integer number, the α–level, for each of the four values of α considered ($\alpha \in \{0, 0.25, 0.5, 1\}$), are shown. Note that, in the case of $\tilde{a}$, for any $0 < \alpha < 1$ we have that $[\tilde{a}]_{\mathbb{Z}}^{\alpha} = [\tilde{a}]_{\mathbb{Z}}^{1}$. Those intermediate levels are plotted with grey colour just to illustrate this feature. Hence, we can say that $\tilde{a}$ has two characteristic α–levels, those of $\alpha = 0$ and $\alpha = 1$, which are plotted in black colour. Something similar occurs in the case of $\tilde{b}$, i.e. it has three characteristic α–levels, $\alpha = 0$, $\alpha = 0.5$ and $\alpha = 1$. For any $0 < \alpha < 0.5$, $[\tilde{b}]_{\mathbb{Z}}^{\alpha} = [\tilde{b}]_{\mathbb{Z}}^{0.5}$, and for any $0.5 < \alpha < 1$ we have that $[\tilde{b}]_{\mathbb{Z}}^{\alpha} = [\tilde{b}]_{\mathbb{Z}}^{1}$.*

The middle panel shows $\tilde{a} + \tilde{b}$ calculated as per Equation (2.16). This sum has the same three characteristic α–levels as $\tilde{b}$ since the sum operation is defined using alpha–levels. We have represented with a white square symbol those integers that belong to $(5, 8, 10, 12)_{\mathbb{Z}}$, but not to $\tilde{a} + \tilde{b}$. Thus, for example, for $\alpha = 0.5$, by (2.16) we have that

$$[\tilde{a} + \tilde{b}]_{\mathbb{Z}}^{0.5} = [\tilde{a}]_{\mathbb{Z}}^{0.5} + [\tilde{b}]_{\mathbb{Z}}^{0.5} = [2, 2]_{\mathbb{Z}} + [5, 8]_{\mathbb{Z}} = [7, 10]_{\mathbb{Z}},$$

whereas $[(5, 8, 10, 12)_{\mathbb{Z}}]_{\mathbb{Z}}^{0.5} = [7, 11]_{\mathbb{Z}}$. The latter can be seen as the integer projection of the continuous extension, this is $\mathbb{Z}\left([\tilde{a}]^{0.5} + [\tilde{b}]^{0.5} = [\tilde{a} + \tilde{b}]^{0.5}\right) = \mathbb{Z}\big([1.5, 2.5] + [5, 8.5]\big) = \mathbb{Z}\big([6.5, 11]\big) = [7, 11]_{\mathbb{Z}}$.

In this manner, we highlight the fact that: i) $\tilde{a} + \tilde{b} \in \mathcal{F}_{\mathbb{Z}}$, but $\tilde{a} + \tilde{b} \notin TF_{\mathbb{Z}}$, as defined in (2.19); and ii) the natural or intuitive idea or extension of the sum as the integer projection of the continuous extension, which in this example corresponds to $(5, 8, 10, 12)_{\mathbb{Z}}$, is not the proposed definition of the sum of trapezoidal fuzzy integer numbers, given by Equation (2.16). As it has been mentioned, the reason for this is that arithmetic operations for fuzzy numbers are defined using α–levels.

Similarly, for the multiplication case, plotted at the bottom panel of Figure 4.1, this difference is even more significant. Defined by α-levels in Equation (2.17), we have that $\tilde{a} \cdot \tilde{b} \in \mathcal{F}_{\mathbb{Z}}$, but $\tilde{a} \cdot \tilde{b} \neq (1 \cdot 4, 2 \cdot 6, 2 \cdot 8, 3 \cdot 9)_{\mathbb{Z}} = (4, 12, 16, 27)_{\mathbb{Z}}$, i.e. the product rule given by Equation (2.14) for the continuous case (TF_C) does not apply here. Thus, for $\alpha = 0.5$, following Equation (2.17), we have that

$$[\tilde{a} \cdot \tilde{b}]_{\mathbb{Z}}^{0.5} = [\tilde{a}]_{\mathbb{Z}}^{0.5} \cdot [\tilde{b}]_{\mathbb{Z}}^{0.5} = [2, 2]_{\mathbb{Z}} \cdot [5, 8]_{\mathbb{Z}} = [10, 16]_{\mathbb{Z}},$$

whereas

$$\mathbb{Z}\left([\tilde{a}]^{0.5} \cdot [\tilde{b}]^{0.5} = [\tilde{a} \cdot \tilde{b}]^{0.5}\right) = \mathbb{Z}\big([1.5, 2.5] \cdot [5, 8.5]\big) = \mathbb{Z}\big([7.5, 21.25]\big) = [8, 21]_{\mathbb{Z}} = [(4, 12, 16, 27)_{\mathbb{Z}}]_{\mathbb{Z}}^{0.5}$$

.

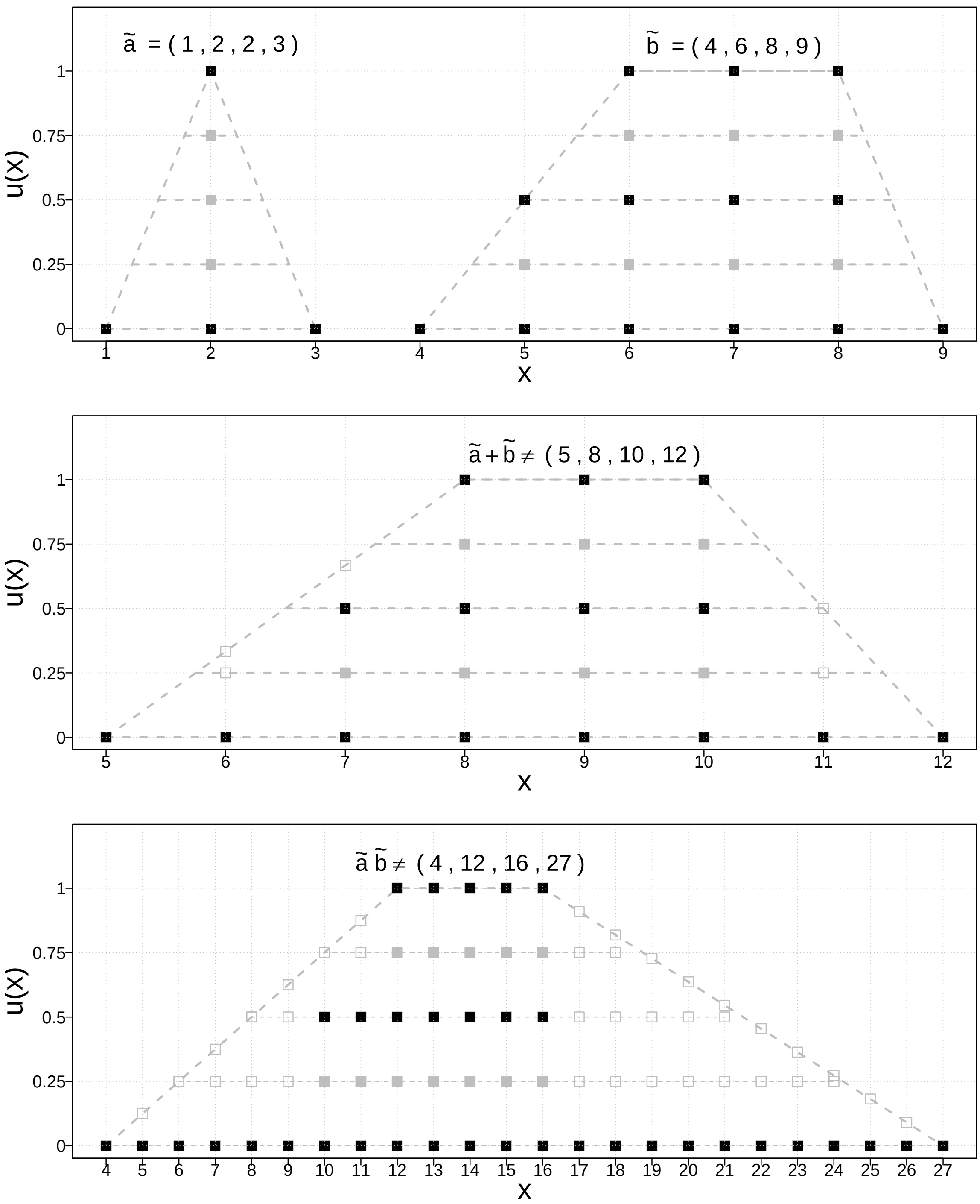

Figure 2.2: The two trapezoidal fuzzy integer numbers $\tilde{a}$, $\tilde{b}$ of Example 2, represented for different α-levels, and the corresponding α-levels of the fuzzy integer numbers resulting from their addition and multiplication.

Let us now introduce the continuous extension of a trapezoidal fuzzy integer number ($TF_{\mathbb{Z}}$) to a trapezoidal fuzzy number (TF_C) as $C((a_1, a_2, a_3, a_4)_{\mathbb{Z}}) = (a_1, a_2, a_3, a_4) \in TF_C$. Conversely, if $a_1 \leq a_2 \leq a_3 \leq a_4$ and $a_1, a_2, a_3, a_4 \in \mathbb{Z}$, we introduce the integer projection of a trapezoidal fuzzy number (TF_C) to a trapezoidal fuzzy integer number ($TF_{\mathbb{Z}}$) as $\mathbb{Z}(a_1, a_2, a_3, a_4) = (a_1, a_2, a_3, a_4)_{\mathbb{Z}} \in TF_{\mathbb{Z}}$. Note that, in the case $a_1, a_2, a_3, a_4 \in \mathbb{Z}$, both operations are inverse, i.e. $\mathbb{Z}(C((a_1, a_2, a_3, a_4)_{\mathbb{Z}})) = (a_1, a_2, a_3, a_4)_{\mathbb{Z}} \in TF_{\mathbb{Z}}$. In this case, we say that $(a_1, a_2, a_3, a_4) \in TF_{C \to \mathbb{Z}}$, where $TF_{C\to\mathbb{Z}}$ is the set of trapezoidal fuzzy numbers whose four parameters are integer.

Definition 2.3.7. *Given two fuzzy integer numbers $u, v \in \mathcal{F}_{\mathbb{Z}}$ represented by means of α-levels as $[u]^{\alpha}_{\mathbb{Z}} = [\underline{u}_{\alpha}, \overline{u}_{\alpha}]_{\mathbb{Z}} \in \mathcal{K}_{\mathbb{Z}}$, $[v]^{\alpha}_{\mathbb{Z}} = [\underline{v}_{\alpha}, \overline{v}_{\alpha}]_{\mathbb{Z}} \in \mathcal{K}_{\mathbb{Z}}$, respectively, then:*

(i) $u \leqq v$ if and only if $[\underline{u}_{\alpha}, \overline{u}_{\alpha}]_{\mathbb{Z}} \leqq [\underline{v}_{\alpha}, \overline{v}_{\alpha}]_{\mathbb{Z}}$, for all $\alpha \in [0, 1]$.

(ii) $u < v$ if and only if $[\underline{u}_{\alpha}, \overline{u}_{\alpha}]_{\mathbb{Z}} < [\underline{v}_{\alpha}, \overline{v}_{\alpha}]_{\mathbb{Z}}$, for all $\alpha \in [0, 1]$.

We can also introduce the relations $u \geqq v$ and $u > v$ for fuzzy numbers and fuzzy integer numbers as equivalent to $v \leqq u$ and $v < u$, respectively. In a similar way as for intervals, for the sake of simplicity, we use the same symbols for partial orders when comparing fuzzy numbers as when comparing fuzzy integer numbers. This unified notation will make easier, in the next section, the formulation of a hybrid DEA model with inputs and outputs that can be either fuzzy numbers or fuzzy integer numbers. Furthermore, as an application of the criterion given in [8, 13], we can determine the partial order between trapezoidal fuzzy numbers by means of the order between their corresponding parameters. Thus, given $(a_1, a_2, a_3, a_4), (b_1, b_2, b_3, b_4) \in TF_C$, it is clear that $(a_1, a_2, a_3, a_4) \leqq (<)(b_1, b_2, b_3, b_4)$ if and only if $a_i \leqq (<)b_i$, for $i \in \{1, 2, 3, 4\}$. Similarly, given $(a_1, a_2, a_3, a_4)_{\mathbb{Z}}, (b_1, b_2, b_3, b_4)_{\mathbb{Z}} \in TF_{\mathbb{Z}}$, then $(a_1, a_2, a_3, a_4)_{\mathbb{Z}} \leqq (<)(b_1, b_2, b_3, b_4)_{\mathbb{Z}}$ if and only if $C((a_1, a_2, a_3, a_4)_{\mathbb{Z}}) = (a_1, a_2, a_3, a_4) \leqq (<)(b_1, b_2, b_3, b_4) = C((b_1, b_2, b_3, b_4)_{\mathbb{Z}})$, that is, $a_i \leqq (<)b_i$, for $i \in \{1, 2, 3, 4\}$. Therefore, the order relation between two trapezoidal fuzzy integer numbers is the the same as the order relation between their corresponding trapezoidal fuzzy continuous extensions.

2.4 Optimization

In real-world situations, optimization has extensive applications, for instance, optimization can be used to improve efficiency, effectiveness, profit and also to decrease costs, energy consumption. We next introduce specific optimization programming problems that have been applied in this thesis.

2.4.1 Interval and Fuzzy Programming

Generally, the optimization problem is to find the minimum (or maximum) of a specific objective function under some constraints. In the conventional optimization problem, the objective function and constraints are vector functions, but in real-world problems, the objective function, as well as the constraints are under a different form of uncertainty. In this thesis, we will consider optimization problems under interval and fuzzy uncertainty. The interval and fuzzy optimization problems are represented as below:

$$
\begin{aligned}
\text{Min} \quad & f(x) \\
\text{s.t.} \quad & g_j(x) \leqq 0 \qquad j = 1, ..., m, \\
& h_w(x) = 0 \qquad w = 1, ..., p, \\
& x \; is \; a \; n - dimensional \; interval \; or \; fuzzy \; number
\end{aligned}
\tag{2.21}
$$

where f is the objective function to be minimized over the n-dimensional interval or fuzzy number, $g_j(x) \leqq 0$ are inequality constraints, $h_w(x) = 0$ are equality constraints, and $\leqq$ is inequality sign for intervals and fuzzy numbers.

To solve interval and fuzzy optimization problem, we take into account the arithmetic operations and partial order relations which is introduced in this chapter. Therefore, interval and fuzzy optimization problem can be formulated as a crisp optimization problem as below:

$$
\begin{aligned}
\text{Min} \quad & f(x) \\
\text{s.t.} \quad & g_j(x) \le 0 \qquad j = 1, ..., m, \\
& h_w(x) = 0 \qquad w = 1, ..., p, \\
& x \in \mathbb{R}^n
\end{aligned}
\tag{2.22}
$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is the objective function to be minimized over the n-dimensional vector x and $g_j(x) \le 0$ are inequality constraints, and $h_w(x) = 0$ are equality constraints.

Generally, the optimization problem is classified into four main sub-fields, Linear Programming (LP), Non-Linear Programming (NLP), Integer Linear Programming (ILP), Integer Non-Linear Programming (INLP). The problems that allow both integer and continuous variables are termed Mixed Integer Linear Programming (MILP) or Mixed Integer Non Linear Programming (MINLP). In LP the objective function and constraints are linear and variables are continuous, but in NLP one or more constraints or objective function are non-linear and the variables are also continuous. In IP the objective function and constraints are linear and variables are integer, while in INLP the one or more constraints or objective function are non-linear and the variables are also an integer. LP belongs to complexity class P. Finding the solution for this kind of optimization problem is easy. The simplex or interior point method can be applied. On contrary, NLP and INLP are NP-hard, but when NLP and INLP are convex, they belong to P because of the applicability of the interior point method. If solution region of a problem is convex and the objective function is convex in a min problem or concave in a max problem, the problem is said convex. In general, solving a convex problem is easier. ILP is NP-complete. In particular, the special case of 0-1 ILP in which variables are binary.

2.4.2 Multiple Objective Programming (MOP)

Multiple objective optimization is an outstanding topic of a mathematical optimization problem that has been applied in many scientific areas such as economics, engineering, industry. MOP involves more than one objective function to be optimized. In this thesis, we will study MOP, since we will use related MOP with crisp data to solve intervals and fuzzy optimization problems. MOP are represented as below:

$$
\begin{aligned}
\text{Min} \quad & (f_1(x), f_2(x), ..., f_k(x)) \\
\text{s.t.} \quad & g_j(x) \le 0 \qquad j = 1, ...m, \\
& h_w(x) = 0 \qquad w = 1, ..., p, \\
& x \in \mathbb{R}^n
\end{aligned}
\tag{2.23}
$$

where $f(x) = (f_1(x), f_2(x), ..., f_k(x))$ is the multiple objective function to be minimized over the n-dimensional vector x, $g_j(x) \le 0$ are inequality constraints, and $h_w(x) = 0$ are equality constraints.

The set $X = \{x \in \mathbb{R}^n : g_j(x) \leq 0, j = 1, 2, ..., m, h_w(x) = 0, w = 1, ..., p\}$ is feasible solution of the model (2.23). In MOP, there is not usually a feasible solution that minimizes all objective function. Therefore, It is necessary to introduce Pareto solution and weakly Pareto solution, defined by Pareto dominated relation. The set of Pareto optimal is termed Pareto frontier.

Definition 2.4.1. *(see [153]). Let $x^* \in X$ be a feasible solution to the problem* (2.23). *$x^* \in X$ is said to be a Pareto (efficient) solution to the problem* (2.23) *if there isn't feasible solution $x \in X$ of* (2.23) *such that $f_i(x) \leq f_i(x^*)$ for all $i = 1, 2, ..., k$ and $f_i(x) < f_i(x^*)$ for at least one i.*

Definition 2.4.2. *(see [153]). Let $x^* \in X$ be a feasible solution to the problem* (2.23). *$x^* \in X$ is said to be a weakly Pareto (weakly efficient) solution to the problem* (2.23) *if there isn't feasible solution $x \in X$ of* (2.23) *such that $f_i(x) < f_i(x^*)$ for all $i = 1, 2, ..., k$.*

There are different methods to generate weakly Pareto (weakly efficient) solutions of MOP. One of the most usual methods is weighted sum problems (see [9] and [10]). Given MOP (2.23) and $w \in \{w = (w_1, w_2, \cdots, w_k) \in \mathbb{R}^k | \sum_{i=1}^{k} w_i = 1, w_i > 0\}$, We define the related sum problem as follows.

$$
\begin{aligned}
\text{Min} \quad & \sum_{i=1}^{k} w_i f_i(x) && && (2.24)\\
\text{s.t.} \quad & g_j(x) \leq 0 && j = 1, ..., m, &&\\
& h_w(x) = 0 && w = 1, ..., p, &&\\
& x \in \mathbb{R}^n &&&&
\end{aligned}
$$

Theorem 2.4.1. *$w \in \{w = (w_1, w_2, \cdots, w_k) \in \mathbb{R}^k | \sum_{i=1}^{k} w_i = 1, w_i > 0\}$ such that x is optimal solution of* (2.24)*, then $x \in X$ is a Pareto solution of* (2.23).

Theorem 2.4.2. *$w \in \{w = (w_1, w_2, \cdots, w_k) \in \mathbb{R}^k | \sum_{i=1}^{k} w_i = 1, w_i > 0\}$ such that x is optimal solution of* (2.24)*, then $x \in X$ is a weakly Pareto solution of* (2.23).

Another approach to solve MOLP is $\epsilon-$ constraints Method, introduced by Chankong and Haimes [33]. Decision makers opt for one objective out of k to be minimized and the rest of the objectives to be restricted within user-specific values.

$$
\begin{aligned}
\text{Min} \quad & f_\mu(x) && && (2.25)\\
\text{s.t.} \quad & f_i(x) \leq \epsilon_i && i = 1, ...k \quad i \neq \mu, &&\\
& g_j(x) \leq 0 && j = 1, ..., m, &&\\
& h_w(x) = 0 && w = 1, ..., p, &&\\
& x \in \mathbb{R}^n &&&&
\end{aligned}
$$

Theorem 2.4.3. *[113] If x^* is a optimal solution of the problem* (2.25) *then x^* is a weakly Pareto solution of the problem* (2.23).

Let us recall, as in most of real-life situation some or all inputs/outputs can be taken as integer data, for instance, the number of patient in any hospital, the problems that some or all inputs/outputs are considered as integer, are of importance. To this matter, mix integer programming plays an important role (see [11], [12], and the bibliography there in).

2.5 Data Envelopment Analysis (DEA)

Next, in this section is presented the brief introduction of basic concepts and models in Data Envelopment Analysis (DEA).
Data envelopment analysis (DEA) is a non-parametric methodology for evaluating the efficiency of a set of homogeneous units commonly named Decision Making Units (DMUs), consuming multiple inputs to produce multiple outputs. DEA is introduced by Charnes et el. [31], refereed as the CCR model and Banker et el. [28] developed a modification of the CCR model as the BCC model. Then DEA turn to one of the predominant fields in Operation Research and Management.

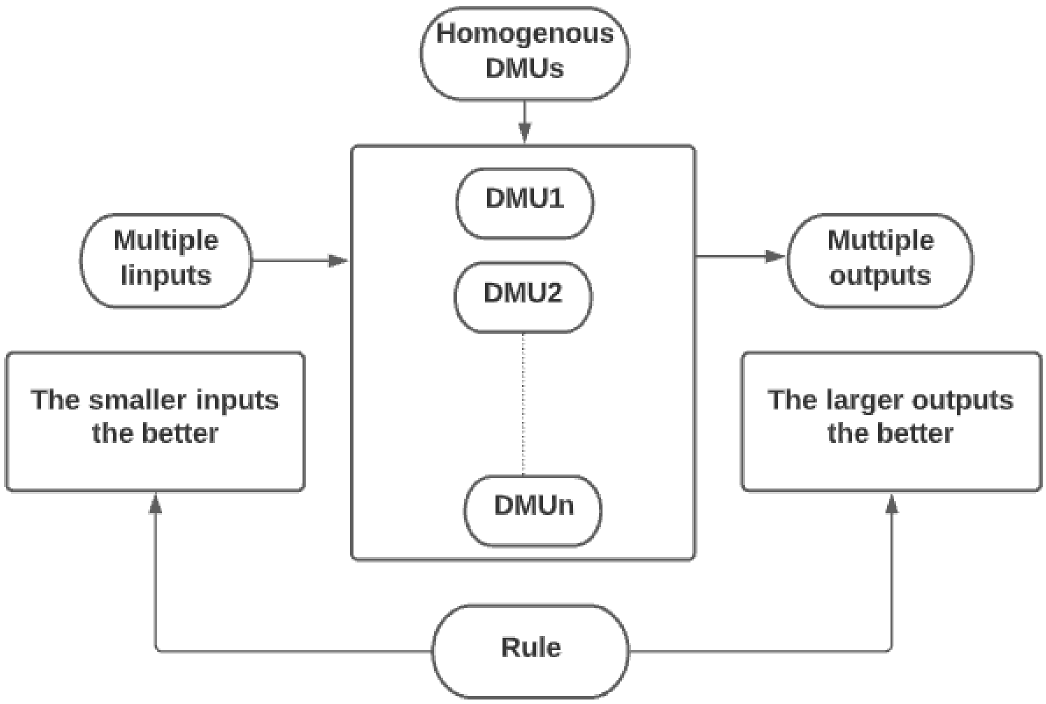

Figure 2.3: The structure of DMUs

Figure 2.3 represents DMUs as a production entity with some performance metrics consuming multiple inputs to produce multiple outputs. In a set of homogeneous DMUs, the inputs and outputs of the DMUs are the same for all DMUs, can be different organizations from schools to hospitals. The inputs are a resource that enters to DMUs such as labor, fixed cost and the outputs are the quantity produced by DMUs such as products yields, profit. The performance metrics are classified as the smaller inputs are the better and the larger outputs are the better.
Efficiency is a comprehensive score for each DMU, obtained by comparing the production of each DMU with the others. Overall, to measure efficiency, there are two main parametric

and non-parametric methods. DEA is one of the most common non-parametric methods to assist efficiency. The main alternative to DEA is Stochastic Frontier Analysis (SFA) which is a non-parametric method. In contrast to the parametric method, non-parametric methods don't require any information on production function to determine efficiency. A production function is an equation that explains the input-output relationship. The most common production function used in SFA is the regression function and Cobb-Doaglas function. Therefore, DEA utilizes multiple outputs, but SFA considers only one output at a time. Conversely, SFA is a better predictor of future performance.

2.5.1 Production Possibility Set (PPS)

In this section, a Production Possibility Set (PPS), DEA technology, is inferred from the input and output data of a set of DMUs, and applying some basic axioms (like Envelopment, free disposability, and convexity). The PPS contains all the operating points that are deemed feasible.

Let us consider a set of n DMUs. For $j \in J = \{1, \ldots, n\}$, each DMU_j has m inputs $X_j = (x_{1j}, \ldots, x_{mj}) \in \mathbb{R}^m$, produces s outputs $Y_j = (y_{1j}, \ldots, y_{sj}) \in \mathbb{R}^s$. The production possibility set (PPS) or technology, defined as below:

$$T = \{(X, Y)|X \; can \; produce \; Y\}$$

introduce by Charnes et el. [31], PPS satisfies in the following axioms:

(A1) Envelopment: $(X_j, Y_j) \in T$, for all $j \in J$.

(A2) Free disposability: $(X, Y) \in T$, $(X', Y') \in \mathbb{R}^{m+s}$, $X' \geqq X$, $Y' \leqq Y \Rightarrow (X', Y') \in T$.

(A3) Convexity: $(X, Y), (X', Y') \in T$, then $\lambda(X, Y) + (1 - \lambda)(X', Y') \in T$, for all $\lambda \in [0, 1]$.

(A4) Scalability (constant-returns to scale (CRS)): $(X, Y) \in T \Rightarrow (\lambda X, \lambda Y) \in T$, for all $\lambda \in \mathbb{R}_+$.

According to the minimum extrapolation principle (see [28]), the PPS, which contains all the feasible input-output bundles, is the intersection of all the sets that satisfy axioms (A1)-(A4) and PPS under constant-returns to scale (CRS) can be expressed as

$$T_{DEA} = \left\{(x, y) \in \mathbb{R}_+^{m+s} : x \geq \sum_{j=1}^{n} \lambda_j X_j, y \leq \sum_{j=1}^{n} \lambda_j Y_j, \lambda_j \geq 0\right\}. \tag{2.26}$$

where λ_j represents the intensity variable attached to DMU_j.
By adding the constraint $\sum_{j=1}^{n} \lambda_j = 1$, $\sum_{j=1}^{n} \lambda_j \geq 1$, and $\sum_{j=1}^{n} \lambda_j \leq 1$ to (2.26), we can obtain the PPS under the assumption variable-returns to scale (VRS), increasing-returns to scale (IRS), and decreasing-returns to scale (DRS), respectively. The Figure 2.4 represent PPS under different returns to scale with a simple numerical example with three single-input single-output DMUs represented by A, B, and C .

2.5.2 Technical efficiency and classical DEA models

First, let us present the definition of efficiency, introduced by Charnes et al. [31], as below:

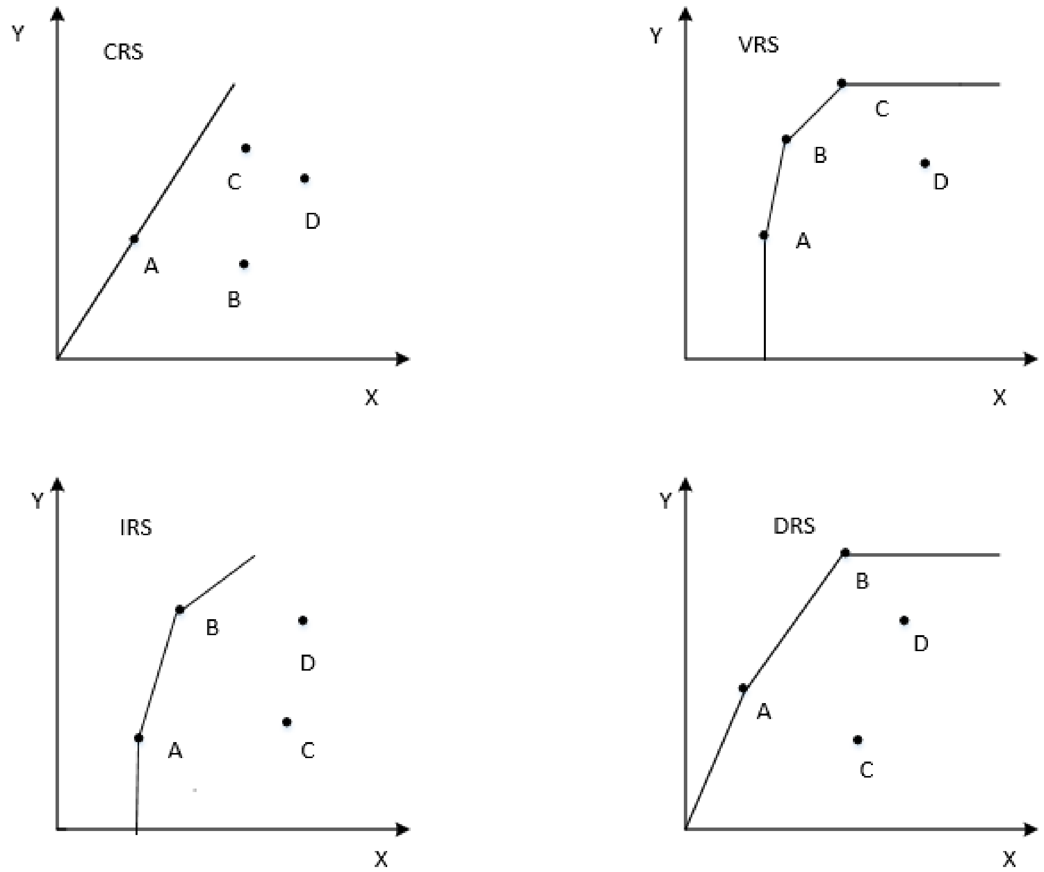

Figure 2.4: The PPS under different return to scale

Definition 2.5.1. *(Pareto-Koopmans efficiency) DMU o is said to be efficient if and only if for any* $(X, Y) \in T_{DEA}$ *such that* $X \leqq X_o$ *and* $Y \geqq Y_o$*, then* $(X, Y) = (X_o, Y_o)$*.*

As already discussed, the non-dominated subset of the PPS is said the efficient frontier. DMUs that belong to the efficient frontier are labeled efficient while the DMUs that do not belong to the efficient frontier are labeled inefficient and can be projected onto the efficient frontier. The projection of a DMU onto efficient frontier is called its target and the distance from the DMU to the target, which is a measure of the potential improvements that the DMU can achieve, is used to compute a quantitative efficiency score.

In DEA, there are two models of efficiency with different characteristics; radial and non-radial models. In radial DEA models, the efficiency of a DMU_o is evaluated in two ways; input-oriented models and output-oriented models. The input-oriented models decrease all the inputs of DMU_o equi-proportionally without decreasing the outputs. Therefore, the problem is formulated as

$$min\{\theta_o \in \mathbb{R}_+ | (\theta_o X_o, Y_o) \in T\}$$

In other words, it is to project DMU_o on the PPS to measure how much its inputs can be decreased at maximum. Figure 2.5 represents a simple example with eight two-inputs one-output DMUs under PPS, introduced by Charnes et al. [31], CCR model. The outputs of DMUs are the same. We can see that the efficiency of DMU_o is $\theta = \frac{OH'}{OH}$ by projection it to H' on the frontier.

The first DEA model with PPS under constant-returns to scale (CRS), CCR model, intro-

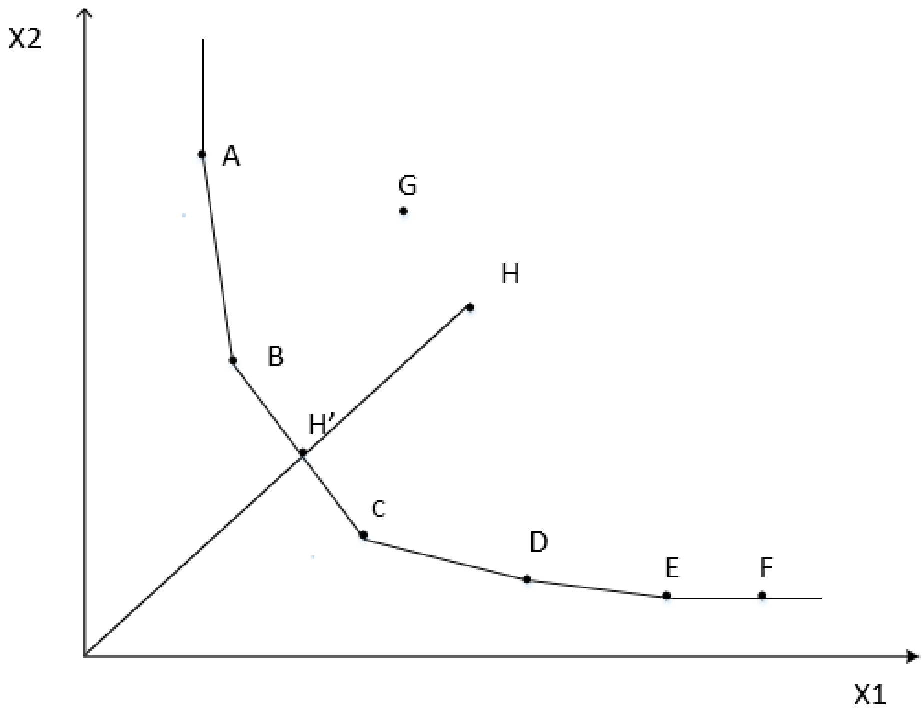

Figure 2.5: PPS, frontier, and efficiency measurement under CRS assumption

duced by Charnes et al. [31] can be represented as below:

$$
\begin{aligned}
\text{(CCR)} \quad \text{Min} \quad & \theta & \quad (2.27)\\
\text{s.t.} \quad & \sum_{j=1}^{n} \lambda_j x_{ij} \leq \theta x_{io}, \quad i = 1, \ldots, m, \\
& \sum_{j=1}^{n} \lambda_j y_{rj} \geq y_{ro}, \quad r = 1, \ldots, s, \\
& \lambda_j \geq 0, \quad j = 1, \ldots, n.
\end{aligned}
$$

where (θ^*, λ^*) are optimal solution and $0 \leq \theta^* \leq 1$.

Note that, according to model (2.27), Banker et al. [28] introduce BCC model with PPS under variable-returns to scale (VRS).

Theorem 2.5.1. *If DMU_o is efficient, then $\theta = 1$*

Although Theorem 2.5.1 is a necessary condition, it is not sufficient to guarantee the efficiency of DMU_o. For explanation, in Figure 2.5, eficient frontier is represented in black line. It is clear that DMUs F and E have $\theta < 1$ and thus according to Definition 2.5.1, they are inefficient. DMUs A, B, C, D, and S have $\theta = 1$, but it does not imply that DMUs are efficient. According to Definition 2.5.1, A, B, C, and D are efficient, but S is not efficient because $X_{1D} < x_{1S}$ and

$X_{2D} = X_{2S}$. Therefore, to get more information about the efficiency of each DMU, we need to formulate the model (2.27) as below:

$$
\begin{aligned}
\text{(CCR)} \quad \text{Min} \quad & \theta \\
\text{s.t.} \quad & \sum_{j=1}^{n} \lambda_j x_{ij} = \theta x_{io} - s_i^-, \quad i = 1, \ldots, m, \\
& \sum_{j=1}^{n} \lambda_j y_{rj} = y_{ro} + s_r^+, \quad r = 1, \ldots, s, \\
& \lambda_j \geq 0, \quad j = 1, \ldots, n. \\
& s_i^- \geq 0, \quad \forall i, \\
& s_r^+ \geq 0, \quad \forall r.
\end{aligned} \tag{2.28}
$$

where s_i^- and s_r^+ are slacks for inputs and outputs respectively.

Definition 2.5.2. *For each DMU_o, $o \in \{1, \ldots, n\}$, we say that the DMU o is*

(i) Pareto efficient if $\theta^* = 1$, $s_i^- = 0, \forall i$ and $s_r^+ = 0, \forall r$,

(ii) weakly efficient if $\theta^* = 1$ and $s_i^- \neq 0$ and $s_r^+ \neq 0$ for some i, r,

(iii) inefficient if $\theta > 0$.

Let $(\theta^*, s_i^{-*}, s_r^{+*}, \boldsymbol{\lambda}^*)$ be the optimal solution for (2.28) for a given DMU_o, we can compute its input and output targets X_o^{target} and Y_o^{target} as

$$
x_{io}^{target} = \theta^* x_{io} - s_i^{-*}, \qquad i = 1, \ldots, M, \tag{2.29}
$$

$$
y_{io}^{target} = y_{ro} + s_r^{+*}, \qquad r = 1, \ldots, S. \tag{2.30}
$$

Output-oriented models increase all the outputs equi-proportionally without increasing the inputs, i.e.

$$
max\{\phi_o \in \mathbb{R}_+ | (X_o, \phi_o Y_o) \in T\}
$$

.

$$
\begin{aligned}
\text{(CCR)} \quad \text{Min} \quad & \phi \\
\text{s.t.} \quad & \sum_{j=1}^{n} \lambda_j x_{ij} \leq x_{io}, \quad i = 1, \ldots, m, \\
& \sum_{j=1}^{n} \lambda_j y_{rj} \geq \phi y_{ro}, \quad r = 1, \ldots, s, \\
& \lambda_j \geq 0, \quad j = 1, \ldots, n.
\end{aligned} \tag{2.31}
$$

where (ϕ^*, λ^*) are optimal solution and $\phi^* \geq 1$ and if $\phi^* = 1$ then DMU_o is at least weakly efficient.

The non-radial models, for instance, the Additive Model, the Slack-based Model (SBM), and the Enhanced Russell Graph Efficiency Measure (ERM), are other DEA approaches which the reductions of inputs and outputs are not equi-proportional.

One of the important models is the Slack-based non-radial model (SBM), introduced by Tone [137].

$$
\begin{aligned}
(\text{SBM}) \;\; \rho_0 = \quad & \text{Min} \;\; \frac{1 - \dfrac{1}{m}\displaystyle\sum_{i=1}^{m} \frac{s_i^x}{x_{io}}}{1 + \dfrac{1}{s}\displaystyle\sum_{r=1}^{s} \frac{s_r^y}{y_{ro}}} \\
& \text{s.t.} \quad \sum_{j=1}^{N} \lambda_j x_{ij} \leq x_{ip} - s_i^x, \quad i = 1, \ldots, M, \\
& \qquad \sum_{j=1}^{N} \lambda_j y_{rj} \geq y_{rp} + s_r^y, \quad r = 1, \ldots, S, \\
& \qquad \lambda_j \geq 0, \quad j = 1, \ldots, N, \\
& \qquad s_i^x, s_r^y \geq 0, \quad i = 1, \ldots, M, \;\; r = 1, \ldots, S.
\end{aligned}
\tag{2.32}
$$

where λ_j, $j = 1, \ldots, n$, are the intensity variables used for defining the corresponding efficient target of DMU_o. The inefficiency measure $I(X_o, Y_o)$ is units invariant and non-negative.

Theorem 2.5.2. *DMU_o is efficient if and only if $\rho = 1$.*

The targets $(\tilde{X}_o^{target}, \tilde{Y}_o^{target})$ of the DMU_o is presented as below:

$$\tilde{X}_o^{target} = \sum_{j=1}^{n} \lambda_j^* \tilde{X}_j \tag{2.33}$$

$$\tilde{Y}_o^{target} = \sum_{j=1}^{n} \lambda_j^* \tilde{Y}_j \tag{2.34}$$

Theorem 2.5.3. $\rho(\tilde{X}_o^{target}, \tilde{Y}_o^{target}) = 1$.

Also, Pastor et al. [117] proposed another non-radoal model, Enhanced Russell Graph

Efficiency Measure (ERM):

$$(\text{ERM}) \;\; R = \quad \text{Min} \quad \frac{\frac{1}{m}\sum_{i=1}^{m}\theta_i}{\frac{1}{s}\sum_{r=1}^{s}\gamma_r} \tag{2.35}$$
$$\begin{aligned} \text{s.t.} \quad & \sum_{j=1}^{n}\lambda_j x_{ij} \leq \theta_i x_{io}, \quad i = 1,\ldots,m, \\ & \sum_{j=1}^{n}\lambda_j y_{rj} \geq \gamma_r y_{ro}, \quad r = 1,\ldots,s, \\ & \theta_i \leq 1, \gamma_r \geq 1, \quad \forall i,r, \\ & \lambda_j \geq 0, \quad j = 1,\ldots,n. \end{aligned}$$

where λ_j, $j = 1,\ldots,n$, are the intensity variable of each DMU_j for defining the corresponding efficient target of DMU_p. R is interpreted as the the ratio between the average input reduction and the average output increase, and $0 < R \leq 1$. and .

Although the objective function of (2.35) is non linear, it is possible to reformulate the problem and get an linear programming problem.

Theorem 2.5.4. *DMU_o is efficient if and only if $R = 1$.*

Besides, this model provides the targets $(\tilde{X}_o^{target}, \tilde{Y}_o^{target})$ associated to a DMU_o, given as

$$\tilde{X}_o^{target} = \sum_{j=1}^{n}\lambda_j^*\tilde{X}_j \tag{2.36}$$

$$\tilde{Y}_o^{target} = \sum_{j=1}^{n}\lambda_j^*\tilde{Y}_j \tag{2.37}$$

Theorem 2.5.5. $R(\tilde{X}_o^{target}, \tilde{Y}_o^{target}) = 1$.

2.6 Inverse DEA

Two main questions have been considered in the most investigation related to inverse DEA. As discussed before, the first question in inverse DEA (output-estimation) is considered by Wei et al. [143]. "If the inputs of DMU_o increase, how much should the outputs of DMU_o increase to preserve the efficiency score of DMU_o?" The second question in inverse DEA (input-estimation) is considered by Hadi-Vancheh et al. [65, 64]. "If the outputs of DMU_o increase, how much should the outputs of DMU_o increase to preserve the efficiency score of DMU_o?" They developed the models, which were introduced by Wei et al. [143].

2.6.1 Output-estimation in inverse DEA

To answer the first question, we need to calculate the minimum increase of output (β_o^*) if the input of DMU_o increase from X_o to $\alpha_o = X_o + \triangle X_o$, where $\triangle X_o \gneqq 0$ such that the efficiency score of DMU_o remains constant. In fact,

$$\beta_o^* = (\beta_{1o}^*, \beta_{2o}^*, ..., \beta_{so}^*)^t = Y_o + \triangle Y_o, \quad \triangle Y_o \gneqq 0.$$

To solve the above question, Wei et al. [143] introduced the following MOLP:

$$\begin{aligned}
\text{Max} \quad & (\beta_{1o}, ..., \beta_{so}) \\
\text{s.t.} \quad & \sum_{j=1}^{N} \lambda_j x_{ij} \leq \alpha_{io}, \quad i = 1, \ldots, m, \\
& \sum_{j=1}^{N} \lambda_j y_{rj} \geq \phi_o^* \beta_{ro}, \quad r = 1, \ldots, s, \\
& \beta_{ro} \geq y_{ro}, \quad i = 1, \ldots, s, \\
& \lambda_j \geq 0, \quad j = 1, \ldots, N.
\end{aligned} \tag{2.38}$$

where ϕ^* is the optimal value of the model (2.31).

Definition 2.6.1. *(see [153]). Let (λ^*, β_0^*) be a feasible solution to the model* (2.38). *(λ^*, β_0^*) is said to be a Pareto (efficient) solution to the model* (2.38) *if there isn't feasible solution (λ, β_0) of the model* (2.38) *such that $\beta_{ro} \leq \beta_{ro}^*$ for all $r = 1, 2, ..., s$ and $\beta_{ro} < \beta_{ro}^*$ for at least one r.*

Definition 2.6.2. *(see [153]). Let (λ^*, β_0^*) be a feasible solution to the model* (2.38). *(λ^*, β_0^*) is said to be a weakly Pareto (weakly efficient) solution to the model* (2.38) *if there isn't feasible solution (λ, β_0) of the model* (2.38) *such that $\beta_{ro} \leq \beta_{ro}^*$ for all $r = 1, 2, ..., s$.*

Furthermore, assume that the DMU_{n+1} represents DMU_o. After modification of inputs and outputs, the following model is introduced to estimate the efficiency of the DMU_{n+1}:

$$\begin{aligned}
\phi_o^{+*} \quad \text{Max} \quad & \phi \\
\text{s.t.} \quad & \sum_{j=1}^{n} \lambda_j x_{ij} + \lambda_{n+1} \alpha_{io} \leq \alpha_{io}, \quad i = 1, \ldots, m, \\
& \sum_{j=1}^{n} \lambda_j y_{rj} + \lambda_{n+1} \beta_{ro}^* \geq \phi \beta_{ro}^*, \quad r = 1, \ldots, s, \\
& \lambda_j \geq 0, \quad j = 1, \ldots, n.
\end{aligned} \tag{2.39}$$

where (λ^*, ϕ^*) are optimal solution.

Definition 2.6.3. *If the optimal values of the model* (2.31) *and* (2.39) *are equal, it is said to be the efficiency score remains unchanged; that is, $\phi(\alpha_o, \beta_o^*) = \phi(X_0, Y_0)$.*

To answer the question, Wei et al. [143] established the following theorems.

Theorem 2.6.1. *Assume that $\phi_o > 1$ and the inputs of DMU_o are increased from X_0 to $\alpha_0 = X_0 + \triangle X_o \quad (\triangle X_o \gneqq 0)$.*

(1) Let (λ^*, β_o^*) be a Weak Pareto solution to the model (2.38) *and if outputs of DMU_o are increase to β^* then efficiency score of the DMU_o under new inputs and outputs remains unchanged $\Big(\phi(\alpha_o, \beta_o^*) = \phi(X_o, Y_o)\Big)$.*
(2) Conversely, let (λ^*, β_o^*) be a feasible solution to the model (2.38). *If the efficiency score of the DMU_o under new inputs and outputs remains unchanged $\Big(\phi(\alpha_o, \beta_o^*) = \phi(X_o, Y_o)\Big)$, then (λ^*, β_o^*) is a Weak Pareto solution to the model* (2.38).

For at least weakly efficient DMUs ($\phi = 1$), Wei et al. [143] suggested the following Linear Programming (LP):

$$
\begin{aligned}
\text{(CCR)} \quad \text{Min} \quad & \phi \\
\text{s.t.} \quad & \sum_{j=1}^{n} \lambda_j x_{ij} \leq \alpha_{io}, \quad i = 1, \ldots, m, \\
& \sum_{j=1}^{n} \lambda_j y_{rj} \geq \phi y_{ro}, \quad r = 1, \ldots, s, \\
& \theta_i \leq 1, \gamma_r \geq 1, \quad \forall i, r, \\
& \lambda_j \geq 0, \quad j = 1, \ldots, n.
\end{aligned} \tag{2.40}
$$

Theorem 2.6.2. *Assume that $\phi_o = 1$ and the inputs of DMU_o are increased from X_0 to $\alpha_0 = X_0 + \triangle X_o \quad (\triangle X_o \gneqq 0)$. If ϕ^* is the optimal value of the model* (2.40) *then efficiency score of the DMU_o under new inputs and outputs remains unchanged $\Big(\phi(\alpha_o, \phi^* Y_o) = \phi(X_o, Y_o)\Big)$.*

2.6.2 Input-estimation in inverse DEA

To answer the second question, we need to calculate the minimum increase of input (α_o^*) if the output of DMU_o increase from Y_o to $\beta_o = Y_o + \triangle Y_o$, where $\triangle Y_o \gneqq 0$ such that the efficiency score of DMU_o remains constant. In fact,

$$\alpha_o^* = (\alpha_{1o}^*, \alpha_{2o}^*, \ldots,_{so}^*)^t = X_o + \triangle X_o, \quad \triangle X_o \gneqq 0.$$

To solve the above question, Hadi-Vencheh et al. [65] introduced the following MOLP:

$$
\begin{aligned}
\text{Min} \quad & (\alpha_{1o}, \ldots, \alpha_{mo}) \\
\text{s.t.} \quad & \sum_{j=1}^{N} \lambda_j x_{ij} \leq \theta_o^* \alpha_{io}, \quad i = 1, \ldots, m, \\
& \sum_{j=1}^{N} \lambda_j y_{rj} \geq \beta_{ro}, \quad r = 1, \ldots, s, \\
& \alpha_{mo} \geq x_{mo}, \quad i = 1, \ldots, m, \\
& \lambda_j \geq 0, \quad j = 1, \ldots, N.
\end{aligned} \tag{2.41}
$$

where θ^* is the optimal value of the model (2.27).

Definition 2.6.4. *(see [153]). Let* (λ^*, α_0^*) *be a feasible solution to the model* (2.41). (λ^*, α_0^*) *is said to be a Pareto (efficient) solution to the model* (2.41) *if there isn't feasible solution* (λ, α_0) *of the model* (2.41) *such that* $\alpha_{io} \leq \alpha_{io}^*$ *for all* $i = 1, 2, ..., m$ *and* $\alpha_{io} < \alpha_{io}^*$ *for at least one* i.

Definition 2.6.5. *(see [153]). Let* (λ^*, α_0^*) *be a feasible solution to the model* (2.41). (λ^*, α_0^*) *is said to be a weakly Pareto (weakly efficient) solution to the model* (2.41) *if there isn't feasible solution* (λ, α_0) *of the model* (2.41) *such that* $\alpha_{io} \leq \alpha_{io}^*$ *for all* $i = 1, 2, ..., m$.

Furthermore, assume that the DMU_{n+1} represents DMU_o. After modification of inputs and outputs, the following model is introduced to estimate the efficiency of the DMU_{n+1}:

$$
\begin{aligned}
\theta_o^{+*} \quad \text{Min} \quad & \theta \\
\text{s.t.} \quad & \sum_{j=1}^{n} \lambda_j x_{ij} + \lambda_{n+1}\alpha_{io}^* \leq \theta\alpha_{io}^*, \quad i = 1, \ldots, m, \\
& \sum_{j=1}^{n} \lambda_j y_{rj} + \lambda_{n+1}\beta_{ro} \geq \beta_{ro}, \quad r = 1, \ldots, s, \\
& \lambda_j \geq 0, \quad j = 1, \ldots, n.
\end{aligned}
\tag{2.42}
$$

where (λ^*, θ^*) are optimal solution.

Definition 2.6.6. *If the optimal values of the model* (2.31) *and* (2.42) *are equal, it is said to be the efficiency score remains unchanged; that is,* $\theta(\alpha_o, \beta_o^*) = \theta(X_0, Y_0)$.

To answer the question, Hadi-Vencheh et al. [65] established the following theorems.

Theorem 2.6.3. *[65] Assume that* (λ^*, θ_o^*) *is Pareto solution of the model* (2.27) *and the outputs of* DMU_o *are increased from* Y_0 *to* $\beta_0 = Y_0 + \triangle Y_o \quad (\triangle Y_o \gneqq 0)$.

Let (λ^*, α_o^*) *be a Pareto solution to the model* (2.41) *and if inputs of* DMU_o *are increase to* α^* *then efficiency score of the* DMU_o *under new inputs and outputs remains unchanged* $\theta(\phi(\alpha_o, \beta_o^*) = \theta(X_o, Y_o)\big)$.

Theorem 2.6.4. *[64]Let* (λ^*, α_o^*) *be a weak Pareto solution to the model* (2.41) *such that* $\alpha_o^* > X_o$ *and if inputs of* DMU_o *are increase to* α^* *then efficiency score of the* DMU_o *under new inputs and outputs remains unchanged* $\theta(\phi(\alpha_o, \beta_o^*) = \theta(X_o, Y_o)\big)$.

Chapter 3

Fuzzy DEA

3.1 Introduction

In this chapter, we will use the Enhanced Russell Graph Measure (ERM) proposed in Pastor et al. [117], which, in the case of crisp data, is equivalent to the slacks-based measure of efficiency (SBM) of Tone [137]. Although still a very active research field, the DEA methodology is very well developed in the case of crisp data. There are, however, situations in which there is uncertainty in the data. In this chapter, we apply DEA to fuzzy data. Most existing FDEA approaches use a radial metric and an input or an output orientation. There are many situations, however, in which both inputs and outputs should be improved and not necessarily in the same proportions. In those cases, a non-radial and non-oriented approach, like the proposed fuzzy ERM (FERM) model, is adequate. In this chapter an axiomatic approach to derive a Fuzzy PPS (FPPS) is presented and a FERM model is proposed. Only a few researchers have attempted to explicitly build a FPPS from the observations. Thus, Allahviranloo et al. [4] use the Extension principle to define a FPPS while Raei Nojehdehi et al. [120] use a geometrical approach based on t-norms.

As regards, FERM approaches, note that a fuzzy ranking approach is used in Jahanshahloo et al. [78], Ahmady et al. [1], and Izadihah et al. [71]. Hsiao et al. [68] and Puri and Yadav [119] use the α-level set approach of Kao and Liu [84] while Saati and Memariani [122] use the α-level set approach of Saati et al. [121]. Chen et al. [36] also use α-cuts and the Extension Principle. Both Hsiao et al. [68] and Chen et al. [36] also formulate a Fuzzy Super SBM model. Wu et al. [145] proposed the α-level set FERM approach with undesirable outputs. Izadikhah and Khoshroo [72] also considered undesirable outputs and proposed a possibility approach to solve a super-efficiency FERM model. Finally, Azadi et al. [23]proposed a possibility approach based on a multiplier ERM formulation.

The FERM approach proposed in this chapter uses polygonal fuzzy numbers, a LU-fuzzy partial order, and an axiomatic derivation of the fuzzy PPS. With respect to polygonal fuzzy numbers, Stefanini et al. [131] proposed using a uniform subdivision of the interval [0, 1] to get a finite number of α-cuts. Báez et al. [24] study the polygonal fuzzy numbers as a particular case of the parametric representation of fuzzy numbers with linear interpolation. As an application of polygonal fuzzy numbers, Chen and Adam [34] has recently proposed a new transformation-based weighted fuzzy interpolative reasoning method.

In this chapter, we consider polygonal fuzzy number to model or approximate inputs and outputs. No ranking functions or expected values on the polygonal fuzzy numbers are needed to formulate the corresponding crisp model. Another difference between the existing and the proposed FERM approach is that we compute a crisp efficiency score instead of a fuzzy efficiency score. Fuzzy efficiency scores are more consistent with the fuzzy nature of the data but the crisp efficiency scores are simpler to understand and apply for practitioners. In this chapter, we have opted for crisp efficiency scores. Computing fuzzy efficiency scores using a fully fuzzy approach and LU-fuzzy partial orders is also possible (e.g. Arana-Jiménez et al. [14]) but the process is necessarily much more involved. In this paper, we have searched for a compromise between fuzziness/information loss and simplicity.

Therefore, the contributions of the proposed approach are several. One of them is the use of polygonal fuzzy numbers and a LU-fuzzy partial order. Another is the axiomatic derivation of the fuzzy production possibility set that contains all the fuzzy operating points that are deemed feasible. Using this fuzzy DEA technology, a simple fuzzy optimization model allows computing a crisp efficiency score and a fuzzy target for each production unit. To solve the proposed FERM model a crisp optimization model is formulated that, although in principle is non-linear, can be appropriately linearized. In the end we have a simple and effective FDEA approach for assessing the efficiency and projecting the production units.

3.2 Crisp production possibility set and ERM model

Let us consider a set of n DMUs. For $j \in J = \{1, \dots, n\}$, each DMU_j has m inputs $X_j = (x_{1j}, \dots, x_{mj}) \in \mathbb{R}^m$, produces s outputs $Y_j = (y_{1j}, \dots, y_{sj}) \in \mathbb{R}^s$. In the classic Charnes et al. [31] DEA model, the production possibility set (PPS) or technology, denoted by T, satisfies axioms envelopment, free disposability, convexity, and scalability introduced in Subsection 2.5.1. According to the minimum extrapolation principle (see [28]), The DEA PPS, which contains all the feasible input-output bundles, can be introduced. In this chapter, we express the DEA PPS under constant-return to scale (CRS) introduced in Subsection 2.5.1.

As mentioned in Subsection 2.5.2, radial and non-radial DEA models are two models of efficiency in DEA. In radial DEA models, there are the input-oriented models in which decrease all the inputs of DMU_o equi-proportionally without decreasing the outputs and the output-oriented models in which increase all the outputs equi-proportionally without increasing the inputs. In non-radial models, the reductions of inputs and outputs are not equi-proportional. In this regard, let us recall the Russel Graph Measure (ERM) (2.35) as a non-radial DEA model which combines the input and output Russel measures in an additive way.

3.3 Proposed fuzzy PPS and fuzzy ERM model

Let us consider a set of n DMUs, $j \in J = \{1, \dots, n\}$, each DMU_j has m inputs $\tilde{X}_j = (\tilde{x}_{1j}, \dots, \tilde{x}_{mj}) \in RPFN_k \times \cdots \times RPFN_k = (RPFN_k)^m_+$, produces s outputs $\tilde{Y}_j = (\tilde{y}_{1j}, \dots, \tilde{y}_{sj}) \in (RPFN_k)^s_+$. Let us consider the following axioms, which are analogous to axioms introduced in Subsection 2.5.1, but considering fuzzy inputs and outputs and using the corresponding partial order introduced in Definition 2.3.4:

(B1) *Envelopment:* $(\tilde{X}_j, \tilde{Y}_j) \in T$, *for all* $j \in J$.

(B2) *Free disposability:* $(\tilde{X}, \tilde{Y}) \in T$, $\tilde{X}' \geqq \tilde{x}$, $\tilde{Y}' \leqq \tilde{Y}$, $(\tilde{X}', \tilde{Y}') \in (RPFN_k)^{m+s}_+ \Rightarrow (\tilde{X}', \tilde{Y}') \in T$.

(B3) *Convexity:* $(\tilde{X}, \tilde{Y}), (\tilde{X}', \tilde{Y}') \in T$, *then* $\alpha(\tilde{X}, \tilde{Y}) + (1 - \alpha)(\tilde{X}', \tilde{Y}') \in T$, *for all* $\alpha \in [0, 1]$.

(B4) *Scalability:* $(\tilde{X}, \tilde{Y}) \in T \Rightarrow (\alpha\tilde{X}, \alpha\tilde{Y}) \in T$, *for all* $\alpha \in \mathbb{R}_+$.

Following the minimum extrapolation principle, the fuzzy production possibility set is the intersection of all sets that satisfy axioms (B1)-(B4).

$$T_{FDEA} = \left\{ (\tilde{X}, \tilde{Y}) \in (RPFN_k)_+^{m+s} : \tilde{X} \geqq \sum_{j=1}^{n} \lambda_j \tilde{X}_j, \tilde{Y} \leqq \sum_{j=1}^{n} \lambda_j \tilde{Y}_j, \lambda_j \geq 0, \forall j \right\},$$

Theorem 3.3.1. *Under axioms (B1), (B2), (B3) and (B4),* T_{FDEA} *is the fuzzy production possibility set that results from the minimum extrapolation principle.*

Proof. Denote by T_{true} the result of the minimum extrapolation principle axioms (B1), (B2), (B3) and (B4). To prove the theorem it is necessary to show that $T_{FDEA} = T_{true}$. To this end, let us divide the proof into two parts.
(i) $T_{true} \subseteq T_{FDEA}$.
It is sufficient to prove that T_{FDEA} satisfies (B1), (B2), (B3) and (B4), since this implies that T_{FDEA} contains the intersection of all sets that satisfies the previous axioms, and consequently contains T_{true}. Therefore, let us check the axioms (B1), (B2), (B3) and (B4) by T_{FDEA}.

- Check (B1). It is clear since, given $j \in J$, then $(\tilde{X}_j, \tilde{Y}_j)$, with $\lambda_j = 1$ and $\lambda'_i = 0$, for all $i' \neq j$, satisfies conditions in T_{FDEA}.

- Check (B2). Given $(\tilde{X}, \tilde{Y}) \in T_{FDEA}$, $\tilde{X}' \geqq \tilde{X}$, $\tilde{Y}' \leqq \tilde{Y}$, $(\tilde{X}', \tilde{Y}') \in (RPFN_k)_+^{m+s}$, we have to prove that $(\tilde{X}', \tilde{Y}') \in T_{FDEA}$. By hypothesis, there exists $\lambda \geqq 0$ such that

$$\tilde{X} \geqq \sum_{j=1}^{n} \lambda_j \tilde{X}_j, \quad \tilde{Y} \leqq \sum_{j=1}^{n} \lambda_j \tilde{Y}_j. \tag{3.1}$$

  Combining (3.1) with $\tilde{X}' \geqq \tilde{X}$, $\tilde{Y}' \leqq \tilde{Y}$, it follows that

$$\tilde{X}' \geqq \tilde{X} \geqq \sum_{j=1}^{n} \lambda_j \tilde{X}_j, \quad \tilde{Y}' \leqq \tilde{Y} \leqq \sum_{j=1}^{n} \lambda_j \tilde{Y}_j. \tag{3.2}$$

  Therefore, $(\tilde{X}', \tilde{Y}') \in T_{FDEA}$.

- Check (B4). Given $(\tilde{X}, \tilde{Y}) \in T_{FDEA}$, there exists $\lambda \geqq 0$ such that (3.1) holds. Given $\alpha \in \mathbb{R}_+$, define $\bar{\lambda} = \alpha\lambda = (\alpha\lambda_1, \ldots, \alpha\lambda_n) \geqq 0$. It follows that $(\alpha\tilde{X}, \alpha\tilde{Y}) \in (RPFN_k)_+^{m+s}$ and

$$\alpha\tilde{X} \geqq \sum_{j=1}^{n} \alpha\lambda_j \tilde{X}_j = \sum_{j=1}^{n} \bar{\lambda}_j \tilde{X}_j, \quad \alpha\tilde{Y} \leqq \sum_{j=1}^{n} \alpha\lambda_j \tilde{Y}_j = \sum_{j=1}^{n} \bar{\lambda}_j \tilde{Y}_j.$$

  Therefore, $(\alpha\tilde{X}, \alpha\tilde{Y}) \in T_{FDEA}$.

- Check (B3). Let us consider $(\tilde{X}, \tilde{Y}), (\tilde{X}', \tilde{Y}') \in T_{FDEA}$, and $\alpha \in [0, 1]$. By hypothesis, there exist $\lambda, \lambda' \geqq 0$ such that

$$\tilde{X} \geqq \sum_{j=1}^{n} \lambda_j \tilde{X}_j, \quad \tilde{X}' \geqq \sum_{j=1}^{n} \lambda'_j \tilde{X}_j, \tag{3.3}$$

$$\tilde{Y} \leqq \sum_{j=1}^{n} \lambda_j \tilde{Y}_j, \quad \tilde{Y}' \leqq \sum_{j=1}^{n} \lambda'_j \tilde{Y}_j. \tag{3.4}$$

Multiplying by α each side in the first fuzzy inequality in (3.3), by $(1-\alpha)$ each side in the second fuzzy inequality in (3.3), and then combining the fuzzy inequalities, we get

$$\alpha\tilde{X} + (1-\alpha)\tilde{X}' \geqq \sum_{j=1}^{n}(\alpha\lambda_j + (1-\alpha)\lambda'_j)\tilde{X}_j, \tag{3.5}$$

Proceeding in a similar way with $\tilde{y}$ and $\tilde{y}'$ and inequalities (3.4), we have

$$\alpha\tilde{Y} + (1-\alpha)\tilde{Y}' \leqq \sum_{j=1}^{n}(\alpha\lambda_j + (1-\alpha)\lambda'_j)\tilde{Y}_j, \tag{3.6}$$

Define $\lambda'' = (\lambda''_1, \dots, \lambda''_n)$, with $\lambda''_j = \alpha\lambda_j + (1-\alpha)\lambda'_j \geq 0$, for all $j = 1, \dots, n$, and substitute them in expressions (3.5) and (3.6). It follows that $(\alpha\tilde{X} + (1-\alpha)\tilde{X}', \alpha\tilde{X} + (1-\alpha)\tilde{Y}') = \alpha(\tilde{X}, \tilde{Y}) + (1-\alpha)(\tilde{X}', \tilde{Y}') \in T_{FDEA}$.

(ii) $T_{FDEA} \subseteq T_{true}$.
We need to prove that every element of T_{FDEA} belongs to T_{true}. To this purpose, consider $(\tilde{x}, \tilde{y}) \in T_{FDEA}$, which means that there exists $\lambda \geqq 0$, $\lambda \in \mathbb{R}^n$, such that

$$\tilde{X} \geqq \sum_{j=1}^{n}\lambda_j\tilde{X}_j, \quad \tilde{Y} \leqq \sum_{j=1}^{n}\lambda_j\tilde{Y}_j. \tag{3.7}$$

We have that $(\tilde{X}_j, \tilde{Y}_j) \in T_{true}$ by (B1), for all $j \in J$. Then, by (B4), it follows that $(\lambda_j\tilde{X}_j, \lambda_j\tilde{Y}_j) \in T_{true}$, for all $j \in J$. Reasoning by induction, let us prove that

$$(\sum_{j=1}^{s}\lambda_j\tilde{X}_j, \sum_{j=1}^{s}\lambda_j\tilde{Y}_j) \in T_{true}, \quad s = 1, \dots, n. \tag{3.8}$$

- Check that in the case $s = 1$ it holds. This case is immediate since $(\tilde{X}_1, \tilde{Y}_1) \in T_{true}$, and then by (B3), $(\lambda_1\tilde{X}_1, \lambda_1\tilde{Y}_1) \in T_{true}$.
- Check that if cases $s \leq t$ are true, this implies that the case $s = t+1$ is also true. We can write $(\sum_{j=1}^{t+1}\lambda_j\tilde{X}_j, \sum_{j=1}^{t+1}\lambda_j\tilde{Y}_j)$ as the convex sum of two elements of T_{true}, multiplied by a scalar. Define $\alpha = 0.5$ and $\alpha' = 2$, then:

$$\begin{aligned}(\sum_{j=1}^{t+1}\lambda_j\tilde{X}_j, \sum_{j=1}^{t+1}\lambda_j\tilde{Y}_j) &= (\sum_{j=1}^{t}\lambda_j\tilde{X}_j, \sum_{j=1}^{t}\lambda_j\tilde{Y}_j) + (\lambda_{t+1}\tilde{X}_{t+1}, \lambda_{t+1}\tilde{Y}_{t+1}) \\ &= \alpha'(\alpha(\sum_{j=1}^{t}\lambda_j\tilde{X}_j, \sum_{j=1}^{t}\lambda_j\tilde{Y}_j) + (1-\alpha)(\lambda_{t+1}\tilde{X}_{t+1}, \lambda_{t+1}\tilde{Y}_{t+1})).\end{aligned}$$

  Then, by (B3) and (B4) it follows that $(\sum_{j=1}^{t+1}\lambda_j\tilde{X}_j, \sum_{j=1}^{t+1}\lambda_j\tilde{Y}_j) \in T_{true}$, and therefore (3.8) holds.

As a consequence of (3.8), we have that $(\sum_{j=1}^{n}\lambda_j\tilde{X}_j, \sum_{j=1}^{n}\lambda_j\tilde{Y}_j) \in T_{true}$. Since $(\tilde{X}, \tilde{Y})$ verifies (3.7), then by (B2) we conclude that $(\tilde{X}, \tilde{Y}) \in T_{true}$. Therefore, $T_{FDEA} \subseteq T_{true}$ and the proof is complete. □

Given the above fuzzy production possibility set, we can present the following definition of efficiency.

Definition 3.3.1. $(\tilde{X}, \tilde{Y}) \in T_{FDEA}$ *is said to be efficient if* $\tilde{X}' \leqq \tilde{X}$, $\tilde{Y}' \geqq \tilde{Y} \implies (\tilde{X}', \tilde{Y}') = (\tilde{X}, \tilde{Y})$, *for all* $(\tilde{X}', \tilde{Y}') \in T_{FDEA}$.

After the characterization result for the fuzzy PPS, given in Theorem 3.3.1, and in order provide a measure for the efficiency of each DMU, we can formulate the following fuzzy ERM (FERM) model, as an extension of the correponding crisp ERM model.

$$
\begin{aligned}
\text{(FERM)}\ \ R^F(\tilde{X}_o, \tilde{Y}_o) = \quad \text{Min} \quad & \frac{\frac{1}{m}\sum_{i=1}^{m} \theta_i}{\frac{1}{s}\sum_{r=1}^{s} \gamma_r} \\
\text{s.t.} \quad & \sum_{j=1}^{n} \lambda_j \tilde{x}_{ij} \leqq \theta_i \tilde{x}_{io}, \quad i = 1, \ldots, m, \\
& \sum_{j=1}^{n} \lambda_j \tilde{y}_{rj} \geqq \gamma_r \tilde{y}_{ro}, \quad r = 1, \ldots, s, \\
& \theta_i \leq 1, \gamma_r \geq 1, \quad \forall i, r, \\
& \lambda_j \geq 0, \quad j = 1, \ldots, n.
\end{aligned}
\tag{3.9}
$$

where the different inputs $\tilde{x}_{ij}$ and outputs $\tilde{y}_{rj}$ belong to $(RPFN_k)_+$, i.e.

$$
\begin{aligned}
\tilde{x}_{ij} &= (x^-_{ij0}, x^-_{ij1}, \ldots, x^-_{ijk}, x^+_{ijk}, \ldots, x^+_{ij1}, x^+_{ij0}), \quad i = 1, \ldots, M, \ \ j = 1, \ldots, N \\
\tilde{y}_{rj} &= (y^-_{rj0}, y^-_{rj1}, \ldots, y^-_{rjk}, y^+_{rjk}, \ldots, y^+_{rj1}, y^+_{rj0}), \quad r = 1, \ldots, S, \ \ j = 1, \ldots, N
\end{aligned}
$$

Note that one of the advantages of using k-polygonal fuzzy numbers is that you can have all the flexibility you need for modeling the fuzzy input and output data and yet be able to describe the whole membership function using a finite set of alpha values. Since the linear combination of the observed inputs and outputs are also k-polygonal fuzzy numbers the above feature also applies to the constraints of the proposed model (3.9). That is why, according to Proposition 2.3.2, the following crisp model, which only considers a finite number of alpha values, is equivalent to model which considers all alpha values.

The above model can be written in parameterized form as follows:

$$R^F(\tilde{X}_o, \tilde{Y}_o) = \quad \text{Min} \quad \frac{\frac{1}{m}\sum_{i=1}^{m}\theta_i}{\frac{1}{s}\sum_{r=1}^{s}\gamma_r} \tag{3.10}$$

$$\begin{aligned}
\text{s.t.} \quad & \sum_{j=1}^{n}\lambda_j x^+_{ijl} \leq \theta_i x^+_{iol}, \quad i = 1,\ldots,m, \;\; l = 0,\ldots,k, \\
& \sum_{j=1}^{n}\lambda_j x^-_{ijl} \leq \theta_i x^-_{iol}, \quad i = 1,\ldots,m, \;\; l = 0,\ldots,k, \\
& \sum_{j=1}^{n}\lambda_j y^+_{rjl} \geq \gamma_r y^+_{rol}, \quad r = 1,\ldots,s, \;\; l = 0,\ldots,k, \\
& \sum_{j=1}^{n}\lambda_j y^-_{rjl} \geq \gamma_r y^-_{rol}, \quad r = 1,\ldots,s, \;\; l = 0,\ldots,k, \\
& \theta_i \leq 1, \gamma_r \geq 1, \quad \forall i, r, \\
& \lambda_j \geq 0, \quad j = 1,\ldots,n.
\end{aligned}$$

Note that FERM inherits some ERM properties such as that $0 < R^F \leq 1$, and that R^F is units invariant. Furthermore, it is not difficult to show that if DMU_o is efficient then $R^F(\tilde{X}_o, \tilde{Y}_o) = 1$. However, as shown in Example 3.3.1, unlike in the crisp case, $R^F(\tilde{X}_o, \tilde{Y}_o) = 1$ is not a sufficient condition for efficiency. Hence, model (3.10) needs to be modified to correct this, as argued below.

Example 3.3.1. *This small and simple example aims to illustrate that $R^F(\tilde{X}_o, \tilde{Y}_o) = 1$ does not imply the efficiency of DMU_o in the fuzzy case, as it occurs for the crisp model* (2.35). *Let us assume that there are only two DMUs that consume a single input and produce a single output and that the two same output value but different inputs. Specifically, let $\tilde{X}_1 = (\tilde{x}_1)$, where $\tilde{x}_1 = (1, 1.75, 2.5, 3.5, 4.75, 6)$, $\tilde{X}_2 = (\tilde{x}_2)$, where $\tilde{x}_2 = (2, 3, 3, 3.5, 4.75, 6)$ and $\tilde{Y}_1 = \tilde{Y}_2 = (\tilde{y})$. DMU_1 is efficient, but DMU_2 is not efficient since it is clear that $\tilde{X}_1 \leqq \tilde{X}_2$, $\tilde{Y}_2 = \tilde{Y}_1$, as it is directly derived from Figure 3.1, see left plot. However, according to model* (3.10), *not only θ and γ are equal to one for both DMUs, but $R^F(\tilde{X}_1, \tilde{Y}_1) = 1$, and $R^F(\tilde{X}_2, \tilde{Y}_2) = 1$. In fact, all convex combinations of the two DMUs have $R^F(\tilde{X}_o, \tilde{Y}_o) = 1$, for $o = 1, 2$. Therefore, an optimal value of unity for model* (3.10) *does not imply the efficiency of the DMU. Moreover, among all the alternative optimal targets for DMU_2, $\tilde{X}_2^{target} = \lambda_1\tilde{X}_1 + \lambda_2\tilde{X}_2$, only one is truly efficient, namely $\tilde{X}_2^{target} = \tilde{X}_1$, which corresponds to $\lambda_1 = 1$ and $\lambda_2 = 0$. Hence, model* (3.10) *is not an adequate extension of the crisp case.*

An initial attempt to solve this discrepancy between a ratio value $R^F(\tilde{X}_o, \tilde{Y}_o) = 1$ and the DMU's efficiency characterization, as discussed in Example 3.3.1, might be to consider a different variable θ_{il} and γ_{rl}, for each level $l = 0, \ldots, k$, i.e.

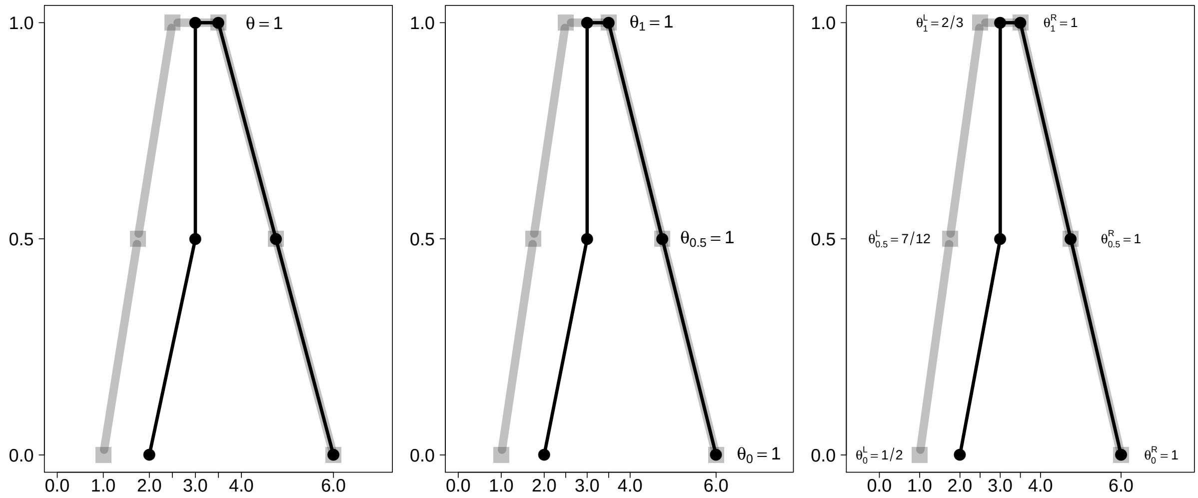

Figure 3.1: Consider two DMUs that consume a single input and produce a single and unit output. The input of DMU_1 and DMU_2 are $\tilde{X}_1 = (1, 1.75, 2.5, 3.5, 4.75, 6)$, and $\tilde{X}_2 = (2, 3, 3, 3.5, 4.75, 6)$, respectively. *Left:* DMU_2 is clearly inefficient, $\tilde{X}_1 \leqq \tilde{X}_2$, $\tilde{X}_1 \neq \tilde{X}_2$ and they have the same output but when (3.10) is solved we get that $\theta = 1$ and hence $R^F(\tilde{X}_2, \tilde{Y}_2) = 1$. This is in contrast with the crisp case, where only efficient DMUs have an ERM efficiency score equal to unity. *Center:* If the variables θ_l and γ_l are let free for each l–level, the issue remains. Since the right limits of both $\tilde{X}_1$ and $\tilde{X}_2$ coincide, the solution of model (3.11) would be $\theta_0 = \theta_{0.5} = \theta_1 = 1$ and $R^F_m(\tilde{X}_2, \tilde{Y}_2) = 1$. *Right:* Using separate left and right variables θ^L_l and θ^R_l, the modified (MFERM) model (3.12) computes $R^F_M(\tilde{X}_2, \tilde{Y}_2) < 1$ and thus indicates that DMU_2 is inefficient.

$$
\begin{aligned}
R^F_m(\tilde{X}_o, \tilde{Y}_o) = \quad \text{Min} \quad & \frac{\dfrac{1}{m(k+1)} \displaystyle\sum_{i=1}^{m} \sum_{l=0}^{k} \theta_{il}}{\dfrac{1}{s(k+1)} \displaystyle\sum_{r=1}^{s} \sum_{l=0}^{k} \gamma_{rl}} \qquad\qquad (3.11) \\
\text{s.t.} \quad & \sum_{j=1}^{n} \lambda_j x^+_{ijl} \leq \theta_{il} x^+_{iol}, \quad i = 1, \ldots, m, \;\; l = 0, \ldots, k, \\
& \sum_{j=1}^{n} \lambda_j x^-_{ijl} \leq \theta_{il} x^-_{iol}, \quad i = 1, \ldots, m, \;\; l = 0, \ldots, k, \\
& \sum_{j=1}^{n} \lambda_j y^+_{rjl} \geq \gamma_{rl} y^+_{rol}, \quad r = 1, \ldots, s, \;\; l = 0, \ldots, k, \\
& \sum_{j=1}^{n} \lambda_j y^-_{rjl} \geq \gamma_{rl} y^-_{rol}, \quad r = 1, \ldots, s, \;\; l = 0, \ldots, k, \\
& \theta_{il} \leq 1, \quad \gamma_{rl} \geq 1, \quad \forall i, r, l, \\
& \lambda_j \geq 0, \quad j = 1, \ldots, n.
\end{aligned}
$$

However, coming back to Example 3.3.1, the solution of model (3.11) for DMU_2 input would be $\theta_0 = \theta_{0.5} = \theta_1 = 1$, i.e., since the right limits for both $\tilde{X}_1$ and $\tilde{X}_2$ coincide, this leads to all three θ_l (and γ_l) variables being equal to unity, and therefore to a modified (FERM) value $R^F_m(\tilde{X}_2, \tilde{Y}_2) = 1$, which would be misleading as DMU_2 is not efficient. See central plot of Figure 3.1.

We have checked that in a fuzzy framework, both efficient and inefficient DMUs could have the same objective value of 1 for the straightforward extension of ERM model (3.10), or the modified (3.11). For example if the upper/lower limits of the $l-$levels coincide, may lead to this case. This motivates the inclusion of left and right variables $\{\theta^L_l; \theta^R_l; \gamma^L_l; \gamma^R_l\}$, for each $l-$level.

$$
\begin{aligned}
\text{(MFERM)}\ \ R^F_M(\tilde{X}_o, \tilde{Y}_o) = \ \ & \text{Min}\ \ \frac{\dfrac{1}{2m(k+1)}\displaystyle\sum_{i=1}^{m}\sum_{l=0}^{k}(\theta^L_{il} + \theta^R_{il})}{\dfrac{1}{2s(k+1)}\displaystyle\sum_{r=1}^{s}\sum_{l=0}^{k}(\gamma^L_{rl} + \gamma^R_{rl})} \qquad (3.12)\\
& \text{s.t.}\ \ \sum_{j=1}^{n}\lambda_j x^-_{ijl} \le \theta^L_{il} x^-_{iol}, \quad i = 1,\dots,m,\ \ l = 0,\dots,k,\\
& \qquad \sum_{j=1}^{n}\lambda_j x^+_{ijl} \le \theta^R_{il} x^+_{iol}, \quad i = 1,\dots,m,\ \ l = 0,\dots,k,\\
& \qquad \sum_{j=1}^{n}\lambda_j y^-_{rjl} \ge \gamma^L_{rl} y^-_{rol}, \quad r = 1,\dots,s,\ \ l = 0,\dots,k,\\
& \qquad \sum_{j=1}^{n}\lambda_j y^+_{rjl} \ge \gamma^R_{rl} y^+_{rol}, \quad r = 1,\dots,s,\ \ l = 1,\dots,k,\\
& \qquad \theta^L_{il}, \theta^R_{il} \le 1, \quad i = 1,\dots,m,\ l = 0,\dots,k,\\
& \qquad \gamma^L_{rl}, \gamma^R_{rl} \ge 1, \quad r = 1,\dots,s,\ l = 0,\dots,k,\\
& \qquad \lambda_j \ge 0, \quad j = 1,\dots,n.
\end{aligned}
$$

The advantage of model (3.12) over (3.9) is that, as shown below in Theorem 3.3.2, it has the efficiency indication property, i.e. $R^F_M(\tilde{X}_o, \tilde{Y}_o)$=1 if and only if the DMU_o is efficient.

Note that, by using separate left and right input reduction factors for each input and left and right output expansion factors for each output, all feasible input reductions and output increases are exhausted. Back to Example 3.3.1, not all the above variables θ^L_l and θ^R_l are equal to one for the inefficient DMU2, and $R^F_M(\tilde{X}_2, \tilde{Y}_2) < 1$. This can be seen in the right panel of Figure 3.1, which shows the optimal left and right values of the input reduction variables for the different alpha values.

Despite the objective function of (3.12) is non linear, it is possible, following a similar procedure to that proposed by Pastor et al. [117], to reformulate the problem and get an equivalent problem with a linear objective function as follows:

$$
\begin{gathered}
\frac{1}{\beta} = \frac{1}{2s(k+1)}\sum_{r=1}^{s}\sum_{l=0}^{k}(\gamma^L_{rl} + \gamma^R_{rl}), \quad \lambda'_j = \beta\lambda_j,\\
\theta'^L_{il} = \beta\theta^L_{il}, \quad \theta'^R_{il} = \beta\theta^R_{il}, \quad \gamma'^L_{rl} = \beta\gamma^L_{rl}, \quad \gamma'^R_{rl} = \beta\gamma^R_{rl}, \quad \forall j, i, r, l.
\end{gathered} \qquad (3.13)
$$

Reformulating (3.12) using (3.13), we have:

$$
\begin{aligned}
\text{(LMFERM)}\ \ R_M^F(\tilde{X}_o,\tilde{Y}_o) = \ \ \text{Min}\ \ & \frac{1}{2m(k+1)}\sum_{i=1}^{m}\sum_{l=0}^{k}(\theta_{il}^{\prime L}+\theta_{il}^{\prime R}) \\
\text{s.t.}\ \ & \frac{1}{2s(k+1)}\sum_{r=1}^{s}\sum_{l=0}^{k}(\gamma_{rl}^{\prime L}+\gamma_{rl}^{\prime R}) = 1, \\
& \sum_{j=1}^{n}\lambda_j^{\prime}x_{ijl}^{-} \leq \theta_{il}^{\prime L}x_{iol}^{-}, \quad i=1,\ldots,m,\ \ l=0,\ldots,k, \\
& \sum_{j=1}^{n}\lambda_j^{\prime}x_{ijl}^{+} \leq \theta_{il}^{\prime R}x_{iol}^{+}, \quad i=1,\ldots,m,\ \ l=0,\ldots,k, \\
& \sum_{j=1}^{n}\lambda_j^{\prime}y_{rjl}^{-} \geq \gamma_{rl}^{\prime L}y_{rol}^{-}, \quad r=1,\ldots,s,\ \ l=0,\ldots,k, \\
& \sum_{j=1}^{n}\lambda_j^{\prime}y_{rjl}^{+} \geq \gamma_{rl}^{\prime R}y_{rol}^{+}, \quad r=1,\ldots,s,\ \ l=1,\ldots,k, \\
& \theta_{il}^{\prime L},\ \theta_{il}^{\prime R} \leq \beta, \quad i=1,\ldots,m,\ l=0,\ldots,k, \\
& \gamma_{rl}^{\prime L},\ \gamma_{rl}^{\prime R} \geq \beta, \quad r=1,\ldots,s,\ l=0,\ldots,k, \\
& \beta > 0, \quad \lambda_j^{\prime} \geq 0, \quad j=1,\ldots,n.
\end{aligned} \tag{3.14}
$$

The above optimization problem (LMFERM) is now a linear program whose feasibility region and objective function are equivalent to those given in (3.12), with the change of variables given in (3.13).

Theorem 3.3.2. *$(\tilde{X},\tilde{Y}) \in T_{FDEA}$ is efficient if and only if $R_M^F(\tilde{X},\tilde{Y})$=1.*

Proof. (i) Let us begin with the first part of the proof, that is, suppose that $(\tilde{X},\tilde{Y}) \in T_{FDEA}$ is efficient and we have to prove that $R_M^F(\tilde{X},\tilde{Y})$=1. By contradiction, suppose that the statement is not true, i.e. $R_M^F(\tilde{X},\tilde{Y}) < 1$. Let $(\theta^{\prime},\gamma^{\prime},\beta,\lambda^{\prime})$ the optimal solution for (LMFERM), then let (θ,γ,λ) the corresponding optimal solution of model (MFERM). There must exist $l_0 \in \{0,\ldots,k\}$, $i_0 \in \{1,\ldots,m\}$ and $r_0 \in \{1,\ldots,s\}$ such that $\theta_{i_0l_0}^{L} < 1$ or $\theta_{i_0l_0}^{R} < 1$ or $\gamma_{r_0l_0}^{L} > 1$ or $\gamma_{r_0l_0}^{R} > 1$. For the sake of simplicity, we continue the proof for the case $\theta_{i_0l_0}^{R} < 1$; the proof for the other cases is similar. From the constrains of (MFERM), it follows that $\sum_{j=1}^{n}\lambda_j x_{ijl}^{-} \leq \theta_{il}^{L}x_{il}^{-} \leq x_{il}^{-}$, $\sum_{j=1}^{n}\lambda_j x_{ijl}^{+} \leq \theta_{il}^{R}x_{il}^{+} \leq x_{il}^{+}$, $\sum_{j=1}^{n}\lambda_j y_{rjl}^{-} \geq \gamma_{rl}^{L}y_{rl}^{-} \geq y_{rl}^{-}$, $\sum_{j=1}^{n}\lambda_j y_{rjl}^{+} \geq \gamma_{rl}^{R}y_{rl}^{+} \geq y_{rl}^{+}$ for all i,l. In particular, $\sum_{j=1}^{n}\lambda_j x_{i_0jl_0}^{-} < \theta_{i_0l_0}^{L}x_{i_0l_0}^{-} \leq x_{i_0l_0}^{-}$. Therefore, $\sum_{j=1}^{n}\lambda_j\tilde{X}_j \leqq \tilde{X}$, $\sum_{j=1}^{n}\lambda_j\tilde{Y}_j \geqq \tilde{Y}$, $(\sum_{j=1}^{n}\lambda_j\tilde{X}_j,\sum_{j=1}^{n}\lambda_j\tilde{Y}_j) \neq (\tilde{X},\tilde{Y})$, with $(\sum_{j=1}^{n}\lambda_j\tilde{X}_j,\sum_{j=1}^{n}\lambda_j\tilde{Y}_j) \in T_{FDEA}$, contradicting the assumption that $(\tilde{X},\tilde{Y})$ is efficient.

(ii) For the second part of the proof, let us consider that $R_M^F(\tilde{X},\tilde{Y})$=1 and we have to prove that $(\tilde{X},\tilde{Y}) \in T_{FDEA}$ is efficient. To this matter, suppose the contrary, that is, that $(\tilde{X},\tilde{Y})$ is not efficient. This means that there exists $(\tilde{X}', \tilde{Y}') \in T_{FDEA}$ such that $\tilde{X}' \leqq \tilde{X}$, $\tilde{Y}' \geqq \tilde{Y}$ and $(\tilde{X}', \tilde{Y}') \neq (\tilde{X}, \tilde{Y})$. Then there exist $i_0 \in \{1, \ldots, m\}$ or $r_0 \in \{1, \ldots, s\}$ such that $\tilde{x}'_{i_0} \leqq \tilde{x}_{i_0}$ and $\tilde{x}'_{i_0} \neq \tilde{x}_{i_0}$, or $\tilde{y}'_{r_0} \geqq \tilde{y}_{r_0}$ and $\tilde{y}'_{r_0} \neq \tilde{y}_{r_0}$. For the sake of simplicity, we continue the proof for the case $\tilde{x}'_{i_0} \leqq \tilde{x}_{i_0}$ and $\tilde{x}'_{i_0} \neq \tilde{x}_{i_0}$; the proof for the other case is similar. It follows that there exists $l_0 \in \{0, \ldots, k\}$ such that $[\tilde{x}'^{-}_{i_0l_0}, \tilde{x}'^{+}_{i_0l_0}] \leqq [\tilde{x}^{-}_{i_0l_0}, \tilde{x}^{+}_{i_0l_0}]$, $[\tilde{x}'^{-}_{i_0l_0}, \tilde{x}'^{+}_{i_0l_0}] \neq [\tilde{x}^{-}_{i_0l_0}, \tilde{x}^{+}_{i_0l_0}]$, and then $\tilde{x}'^{-}_{i_0l_0} < \tilde{x}^{-}_{i_0l_0}$ or $\tilde{x}'^{+}_{i_0l_0} < \tilde{x}^{+}_{i_0l_0}$. Suppose that $\tilde{x}'^{-}_{i_0l_0} < \tilde{x}^{-}_{i_0l_0}$. Then there exists $\delta < 1$, $\delta \geq 0$, such that $\tilde{x}'^{-}_{i_0l_0} \leq \delta\tilde{x}^{-}_{i_0l_0}$. Since $(\tilde{X}', \tilde{Y}') \in T_{FDEA}$, then there exists $\lambda \in \mathbb{R}^n_+$, with $\tilde{X} \geqq \tilde{X}' \geqq \sum_{j=1}^n \lambda_j \tilde{X}_j$, $\tilde{Y} \leqq \tilde{Y}' \leqq \sum_{j=1}^n \lambda_j \tilde{Y}_j$, that is, $\sum_{j=1}^n \lambda_j x^{-}_{i_0jl_0} \leq \tilde{x}'^{-}_{i_0l_0} \leq \delta\tilde{x}^{-}_{i_0l_0}$, $\quad i = 1, \ldots, m, \quad l = 0, \ldots, k$. Define $\theta'^{L}_{i_0l_0} = \delta < 1$, $\lambda' = \lambda$, and the remaining variables equal to one in (LMFERM). Then, such $(\theta', \gamma', \beta, \lambda')$ is feasible for (LMFERM), with $\frac{1}{2m(k+1)} \sum_{i=1}^m \sum_{l=0}^k (\theta'^{L}_{il} + \theta'^{R}_{il}) < 1$, contradicting the assumption that $R^F(\tilde{X}, \tilde{Y}) = 1$.

□

Besides, this model provides the targets $(\tilde{X}_o^{target}, \tilde{Y}_o^{target})$ associated to a DMU_o, given as

$$\tilde{X}_o^{target} = \sum_{j=1}^n \lambda_j^* \tilde{X}_j \tag{3.15}$$

$$\tilde{Y}_o^{target} = \sum_{j=1}^n \lambda_j^* \tilde{Y}_j \tag{3.16}$$

Note that above (MFERM) model, and equivalently model (LMFERM), computes the efficiency score of a DMU_o and this is indicated in the arguments of $R^F(\tilde{X}_o, \tilde{Y}_o)$. It can, however, be used to project any feasible operating point $(\tilde{X}, \tilde{Y})$ to compute its efficiency $R^F_M(\tilde{X}, \tilde{Y})$. This will be useful for proving the following result.

Theorem 3.3.3. $R^F_M(\tilde{X}_o^{target}, \tilde{Y}_o^{target}) = 1$.

Proof. To prove the result, suppose the contrary, i.e. $R^F_M(\tilde{X}_p^{target}, \tilde{Y}_p^{target}) < 1$. Due to the way $(\tilde{X}_p^{target}, \tilde{Y}_p^{target})$ is computed there must exist an optimal solution $(\theta^{*'}, \gamma^{*'}, \beta^{*'}, \lambda^{*'})$ for (LMFERM), which derives an optimal solution $(\theta^*, \gamma^*, \lambda^*)$ of model (MFERM) and an optimal target of DMU_p given by $\tilde{X}_p^{target} = \sum\limits_{j=1}^n \lambda_j^* \tilde{X}_j$ and $\tilde{Y}_p^{target} = \sum\limits_{j=1}^n \lambda_j^* \tilde{Y}_j$. Analogously, let $(\theta^{**}, \gamma^{**}, \lambda^{**})$ be the optimal solution of the (MFERM) model that projects $(\tilde{X}_p^{target}, \tilde{Y}_p^{target})$. It follows that

$$R^F_M(\tilde{X}_p^{target}, \tilde{Y}_p^{target}) = \frac{\dfrac{1}{2m(k+1)} \displaystyle\sum_{i=1}^m \sum_{l=0}^k (\theta^{**L}_{il} + \theta^{**R}_{il})}{\dfrac{1}{2s(k+1)} \displaystyle\sum_{r=1}^s \sum_{l=0}^k (\gamma^{**L}_{rl} + \gamma^{**R}_{rl})}. \tag{3.17}$$

Since $\theta^{**L}_{il}, \theta^{**R}_{il} \leq 1$, and $\gamma^{**L}_{rl}, \gamma^{**R}_{rl} \geq 1$ for all i, r, l, it is clear that $R^F_M(\tilde{X}_p^{target}, \tilde{Y}_p^{target}) < 1$ implies that there must exist $l_0 \in \{0, \ldots, k\}$, $i_0 \in \{1, \ldots, m\}$ and $r_0 \in \{1, \ldots, s\}$ such that $\theta^{**L}_{i_0l_0} < 1$ or $\theta^{**R}_{i_0l_0} < 1$ or $\gamma^{**L}_{r_0l_0} > 1$ or $\gamma^{**R}_{r_0l_0} > 1$. For the sake of simplicity of this proof, we only consider the first case $\theta^{**L}_{i_0l_0} < 1$; the proof for the other cases is similar. Define $\overline{\theta}^L_{il} = \theta^{*L}_{il}\theta^{**L}_{il}$, $\overline{\theta}^R_i = \theta^{*R}_{il}\theta^{**R}_{il}$, $\overline{\gamma}^L_{rl} = \gamma^{*L}_{rl}\gamma^{**L}_{rl}$ and $\overline{\gamma}^R_{rl} = \gamma^{*R}_{rl}\gamma^{**R}_{rl}$ for all $i \in \{1, \ldots, m\}$, $r \in \{1, \ldots, s\}$ and $l \in \{1, \ldots, k\}$. It is clear that

$$\overline{\theta}_{il}^{L} \leq \theta_{il}^{*L} \leq 1,\ \overline{\theta}_{il}^{R} \leq \theta_{il}^{*R} \leq 1,\ \forall i, l, \quad \overline{\theta}_{i_0}^{L} < \theta_{i_0}^{*L} \leq 1, \tag{3.18}$$

$$\overline{\gamma}_{rl}^{L} \geq \gamma_{rl}^{*L} \geq 1, \quad \overline{\gamma}_{rl}^{R} \geq \gamma_{rl}^{*R} \geq 1\ \forall r, l. \tag{3.19}$$

Therefore, from definition of the target (3.15) and (3.16), and since $(\theta^{**}, \gamma^{**}, \lambda^{**})$ is an optimal solution of (MFERM) for $(\tilde{X}_o^{target}, \tilde{Y}_o^{target})$, and for all i, r, l

$$\sum_{j=1}^{n} \lambda_j^{**} x_{ijl}^{-} \leq \theta_{il}^{**L} x_{iol}^{target-} = \theta_{il}^{**L} \sum_{j=1}^{n} \lambda_j^{*} x_{ijl}^{-} \leq \theta_{il}^{**L} \theta_{il}^{*L} x_{iol}^{-} = \overline{\theta}_{il}^{L} x_{iol}^{-}, \tag{3.20}$$

$$\sum_{j=1}^{n} \lambda_j^{**} x_{ijl}^{-} \leq \theta_{il}^{**R} x_{iol}^{target-} = \theta_{il}^{**R} \sum_{j=1}^{n} \lambda_j^{*} x_{ijl}^{-} \leq \theta_{il}^{**R} \theta_{il}^{*R} x_{iol}^{-} = \overline{\theta}_{il}^{R} x_{iol}^{-}, \tag{3.21}$$

$$\sum_{j=1}^{n} \lambda_j^{**} y_{rjl}^{-} \geq \gamma_{rl}^{**L} y_{rol}^{target-} = \gamma_{rl}^{**L} \sum_{j=1}^{n} \lambda_j^{*} y_{rjl}^{-} \geq \gamma_{rl}^{**L} \gamma_{rl}^{*L} y_{rol}^{-} = \overline{\gamma}_{rl}^{L} y_{rol}^{-}, \tag{3.22}$$

$$\sum_{j=1}^{n} \lambda_j^{**} y_{rjl}^{-} \geq \gamma_{rl}^{**R} y_{rol}^{target-} = \gamma_{rl}^{**R} \sum_{j=1}^{n} \lambda_j^{*} y_{rjl}^{-} \geq \gamma_{rl}^{**R} \gamma_{rl}^{*R} y_{rol}^{-} = \overline{\gamma}_{rl}^{R} y_{rol}^{-}, \tag{3.23}$$

This implies that $(\overline{\theta}, \overline{\gamma}, \lambda^{**})$ is a feasible solution of model (MFERM) for DMU_o, which combined with (3.18) and (3.19) implies that

$$\frac{\frac{1}{2m(k+1)} \sum_{i=1}^{m} \sum_{l=0}^{k} (\overline{\theta}_{il}^{L} + \overline{\theta}_{il}^{R})}{\frac{1}{2s(k+1)} \sum_{r=1}^{s} \sum_{l=0}^{k} (\overline{\gamma}_{rl}^{L} + \overline{\gamma}_{rl}^{R})} < \frac{\frac{1}{2m(k+1)} \sum_{i=1}^{m} \sum_{l=0}^{k} (\theta_{il}^{*L} + \theta_{il}^{*R})}{\frac{1}{2s(k+1)} \sum_{r=1}^{s} \sum_{l=0}^{k} (\gamma_{rl}^{*L} + \gamma_{rl}^{*R})} = R_M^F(\tilde{X}_o, \tilde{Y}_o), \tag{3.24}$$

contradicting the assumption that $(\theta^{*}, \gamma^{*}, \lambda^{*})$ is an optimal solution of (MFERM). This completes the proof.

□

In summary, extending the conventional ERM efficiency, the proposed MFERM approach uses a linearizable, non-oriented, non-radial optimization model that exhausts all feasible input reductions and output expansions at all l-levels, providing crisp efficiency scores (and corresponding efficiency ranking) as well as fuzzy efficient targets.

3.4 Numerical examples

3.4.1 Triangular fuzzy numbers dataset

In order to illustrate the proposed fuzzy ERM model consider the dataset from Arya & Yadav [20] shown in Table 3.1 and which correspond to 12 Community Health Centers (CHCs) in Meerut district of Uttar Pradesh, India. The table shows the fuzzy input and output of each

Table 3.1: Data for Example **??** (from Arya & Yadav [20]), together FERM targets as well as the parameters $\{\theta^L_{il}; \theta^R_{il}; \gamma^L_{rl}; \gamma^R_{rl}\}$ for $l = 0, 1$. Note that, as we have triangular fuzzy numbers, $\theta^L_{i1} = \theta^R_{i1} = \theta_{i1}$. Idem for $\gamma^L_{r1} = \gamma^R_{r1} = \gamma_{r1}$.

DMU	FERM		Inputs			Outputs		
j	R^F_M	i, r	$\widetilde{X}_{ij}$	$\widetilde{X}^{target}_{ij}$	$\{\theta^L_{i0}; \theta_{i1}; \theta^R_{i0}\}$	$\widetilde{Y}_{rj}$	$\widetilde{Y}^{target}_{rj}$	$\{\gamma^L_{r0}; \gamma_{r1}; \gamma^R_{r0}\}$
1	1	1	(10 , 13 , 15)	(10 , 13 , 15)	{ 1 ; 1 ; 1 }	(3640 , 3650 , 3655)	(3640 , 3650 , 3655)	{ 1 ; 1 ; 1 }
		2	(3 , 5 , 8)	(3 , 5 , 8)	{ 1 ; 1 ; 1 }	(134130 , 134137 , 134145)	(134130 , 134137 , 134145)	{ 1 ; 1 ; 1 }
2	1	1	(10 , 12 , 14)	(10 , 12 , 14)	{ 1 ; 1 ; 1 }	(4150 , 4160 , 4170)	(4150 , 4160 , 4170)	{ 1 ; 1 ; 1 }
		2	(3 , 5 , 7)	(3 , 5 , 7)	{ 1 ; 1 ; 1 }	(116055 , 116062 , 116068)	(116055 , 116062 , 116068)	{ 1 ; 1 ; 1 }
3	1	1	(9 , 12 , 14)	(9 , 12 , 14)	{ 1 ; 1 ; 1 }	(4360 , 4370 , 4380)	(4360 , 4370 , 4380)	{ 1 ; 1 ; 1 }
		2	(2 , 4 , 5)	(2 , 4 , 5)	{ 1 ; 1 ; 1 }	(94060 , 94066 , 94072)	(94060 , 94066 , 94072)	{ 1 ; 1 ; 1 }
4	0.33	1	(6 , 8 , 11)	(2.188 , 2.901 , 3.377)	{ 0.365; 0.363 ; 0.307 }	(485 , 492 , 500)	(1000.675 , 1003.052 , 1005.182)	{ 2.063 ; 2.039 ; 2.01 }
		2	(1 , 1 , 3)	(0.525 , 1 , 1.336)	{ 0.525 ; 1 ; 0.445 }	(24320 , 24329 , 24335)	(24332 , 24333.475 , 24335)	{ 1 ; 1 ; 1 }
5	0.834	1	(8 , 10 , 13)	(8 , 10 , 11.602)	{ 1 ; 1 ; 0.892 }	(2460 , 2464 , 2470)	(3128.539 , 3136.549 , 3142.62)	{ 1.272 ; 1.273 ; 1.272 }
		2	(3 , 4 , 6)	(2.393 , 3.995 , 5.975)	{ 0.798 ; 0.999 ; 0.996 }	(99740 , 99748 , 99760)	(99748.821 , 99754.418 , 99760)	{ 1 ; 1 ; 1 }
6	0.441	1	(10 , 11 , 12)	(6.75 , 9 , 10.5)	{ 0.675 ; 0.818 ; 0.875 }	(1360 , 1368 , 1375)	(3270 , 3277.5 , 3285)	{ 2.404 ; 2.396 ; 2.389 }
		2	(2 , 3 , 4)	(1.5 , 3 , 3.75)	{ 0.75 ; 1 ; 0.937 }	(49395 , 49401 , 49410)	(70545 , 70549.5 , 70554)	{1.428 ; 1.428 ; 1.428 }
7	0.414	1	(9 , 10 , 12)	(4.5 , 6 , 7)	{ 0.5 ; 0.6 ; 0.583 }	(1055 , 1062 , 1070)	(2180 , 2185 , 2190)	{ 2.066 ; 2.057 ; 2.047 }
		2	(1 , 2 , 6)	(1 , 2 , 2.5)	{ 1 ; 1 ; 0.417 }	(37765 , 37772 , 37780)	(47030 , 47033 , 47036)	{ 1.245 ; 1.245 ; 1.245 }
8	1	1	(9 , 11 , 15)	(9 , 11 , 15)	{ 1 ; 1 ; 1 }	(1295 , 1302 , 1310)	(1295 , 1302 , 1310)	{ 1 ; 1 ; 1 }
		2	(1 , 4 , 7)	(1 , 4 , 7)	{ 1 ; 1 ; 1 }	(82835 , 82841 , 82850)	(82835 , 82841 , 82850)	{ 1 ; 1 ; 1 }
9	0.602	1	(10 , 12 , 15)	(9.226 , 12 , 14.453)	{ 0.923 ; 1 ; 0.964 }	(1660 , 1671 , 1680)	(3502.927 , 3512.248 , 3520.663)	{ 2.11 ; 2.102 ; 2.096 }
		2	(2 , 5 , 7)	(2 , 4.226 , 6.132)	{ 1 ; 0.845 ; 0.876 }	(100590 , 100596 , 100605)	(100591.641 , 100597.868 , 100605)	{ 1 ; 1 ; 1 }
10	0.3	1	(10 , 16 , 20)	(9 , 12 , 14)	{ 0.9 ; 0.75 ; 0.7 }	(1010 , 1018 , 1025)	(4360 , 4370 , 4380)	{ 4.317 ; 4.293 ; 4.273 }
		2	(2 , 4 , 6)	(2 , 4 , 5)	{ 1 ; 1 ; 0.833 }	(64345 , 64351 , 64360)	(94060 , 94066 , 94072)	{ 1.462 ; 1.462 ; 1.462 }
11	0.423	1	(9 , 11 , 14)	(8.25 , 11 , 12.833)	{ 0.917 ; 1 ; 0.917 }	(1500 , 1504 , 1510)	(3996.667 , 4005.833 , 4015)	{ 2.664 ; 2.663 ; 2.659 }
		2	(3 , 5 , 8)	(1.833 , 3.667 , 4.583)	{ 0.611 ; 0.733 ; 0.573 }	(80050 , 80056 , 80061)	(86221.667 , 86227.167 , 86232.667)	{ 1.077 ; 1.077 ; 1.077 }
12	1	1	(5 , 8 , 10)	(5 , 8 , 10)	{ 1 ; 1 ; 1 }	(1960 , 1965 , 1972)	(1960 , 1965 , 1972)	{ 1 ; 1 ; 1 }
		2	(1 , 4 , 6)	(1 , 4 , 6)	{ 1 ; 1 ; 1 }	(58160 , 58167 , 58175)	(58160 , 58167 , 58175)	{ 1 ; 1 ; 1 }

Table 3.2: FERM efficiency scores compared with some existing fuzzy SBM approaches.

DMU	Our approach (3.14)		Arya & Yadav [20]	Hsiao et al.[68]		Izadikhah et al. (2017) [71]	
	$R_M^F(\tilde{X}_o, \tilde{Y}_o)$	rank	$SBME_o$	$[SBM_o^L, SBM_o^U]_{\alpha=0}$	$[SBM_o^L, SBM_o^U]_{\alpha=1}$	Interval efficiency	rank
H1	1	1	1	$[1,1]$	1	$[0.84, 1.00]$	3 ($\rho_1 \overset{0.628}{>} \rho_5$)
H2	1	1	1	$[1,1]$	1	$[0.88, 1.00]$	2 ($\rho_2 \overset{0.564}{>} \rho_1$)
H3	1	1	1	$[1,1]$	1	$[0.89, 1.00]$	1 ($\rho_3 \overset{0.523}{>} \rho_2$)
H4	0.33	11	1	$[0.12, 1]$	1	$[0.30, 1.00]$	7 ($\rho_4 \overset{0.64}{>} \rho_9$)
H5	0.834	6	1	$[0.39, 1]$	1	$[0.73, 1.00]$	4 ($\rho_5 \overset{0.695}{>} \rho_8$)
H6	0.441	8	1	$[0.24, 1]$	0.53	$[0.42, 0.52]$	9 ($\rho_6 \overset{0.572}{>} \rho_{11}$)
H7	0.414	10	1	$[0.16, 1]$	0.66	$[0.36, 0.55]$	11 ($\rho_7 \overset{0.536}{>} \rho_{12}$)
H8	1	1	1	$[0.19, 1]$	0.51	$[0.39, 1.00]$	5 ($\rho_8 \overset{0.572}{>} \rho_{12}$)
H9	0.602	7	1	$[0.26, 1]$	0.52	$[0.44, 0.62]$	8 ($\rho_9 \overset{0.719}{>} \rho_6$)
H10	0.3	12	1	$[0.14, 1]$	0.30	$[0.26, 0.35]$	12
H11	0.423	9	0.66	$[0.22, 1]$	0.50	$[0.40, 0.51]$	10 ($\rho_{11} \overset{0.506}{>} \rho_7$)
H12	1	1	1	$[0.32, 1]$	1	$[0.57, 0.76]$	6 ($\rho_{12} \overset{0.514}{>} \rho_4$)

DMU. The first input is the total sum of doctors and staff nurses while the second one is the number of pharmacists. The two outputs correspond to the number of inpatients and outpatients, respectively. All the inputs and outputs are given as triangular fuzzy numbers, which correspond to $k = 1$ regular polygonal fuzzy numbers.

The efficiency $R_M^F(\tilde{X}_o, \tilde{Y}_o)$ of each DMU_o, $o = 1, \ldots, 12$, has been computed using (3.14), and is shown in the second column of Tables 3.1 and 3.2. It can be seen that DMUs H1, H2, H3, H8 and H12 are labelled efficient. In Table 4.2 we also add a ranking, based on this efficiency measurement, so that this ranking can be compared with those of other methods. Efficient DMUs are all ranked equal.

Table 3.1 shows the fuzzy targets computed for each DMU using (3.15) and (3.16). Due to horizontal space constraints, the structure of the table is unusual in the sense that for each DMU there are two rows, each showing the data and the results for each of the two inputs and two outputs. Note that the target coincides with the observed data in the case of the efficient DMUs and dominate it, in the sense of the partial order introduced in Definition 2.3.4, in the case of inefficient DMUs.

For comparison purposes, Table 3.2 also shows the efficiency scores computed by other approaches from the literature. The fourth column shows the $SBME_o$ efficiency score computed by the fuzzy SBM model of Arya & Yadav [20]. Note that their method labelled all DMUs as efficient except one, namely H11. Thus, it seems that the proposed FERM approach has, at least for this dataset, more discriminant power than this method. The fifth and sixth columns of Table 3.2 correspond to the results of the Fuzzy SBM model of Chen et al. [36] and Hsiao et al. [68]. Since they use fuzzy slack variables, their efficiency scores are fuzzy. The corresponding α-cuts for $\alpha = 0, 1$ levels are shown. Note that the FERM efficiency scores are within the corresponding $\alpha = 0$ cut for all DMUs.

Finally, the two last columns of Table 3.2 correspond to the Fuzzy ERM approach of Izadikhah et al.[71], where a pair of models are used to generate upper and lower limits of interval efficiency score, based on the enhanced Russell model applied to interval data. The whole dataset is converted into intervals for applying this methodology, computing the nearest weighted interval approximation of fuzzy numbers, with the weighting function $(4a^3, 4a^3)$. We use a preference degree measurement ρ_j, see [71] for more details, to establish some partial order between intervals and ranking the DMUs. In this case, although at first sight it appears that this method has more discriminant power from [71], we also remark the different efficiency interpretations. Izadikhah et al.[71] approach computes an efficiency score interval (converting or approximating all data to intervals). Whereas the proposed method is based on technical efficiency within a fuzzy technology framework. According to the latter, the efficient DMUs are characterised by $R_M^F(\tilde{X}_o, \tilde{Y}_o) = 1$ (see Theorem 3.3.2). In this regard, we found in some cases that efficient DMUs are ranked after inefficient ones by Izadikhah et al. [71] approach. However, despite these differences, we see agreement between the results of both methods. Thus, except for DMU 12, $R_M^F(\tilde{X}_o, \tilde{Y}_o)$ always lies within the corresponding efficiency score interval. The level of agreement between the results of our approach and [71] has been tested using a Wilcoxon signed-rank test, which is a non-parametric paired difference test for matched pairs or dependent samples. With a $p-$value= 0.2146, we cannot reject the null hypothesis of homogeneity, i.e. that the median difference is zero.

3.4.2 Modified dataset with 2−polygonal fuzzy numbers

In this subsection we illustrate the proposed approach with fuzzy data that are not triangular fuzzy numbers. Actually, polygonal fuzzy numbers allow a great deal of flexibility in modelling the uncertainty in the input and output data. To keep things simple, however, we will just consider 2−polygonal fuzzy numbers, instead of triangular (1−polygonal). We have modified the inputs and outputs of the DMUs for getting $RPFN_2$ data. The modifications have been randomly generated keeping the same closure, see Table 3.3.

As an example, Figure 3.2 shows the input/output data for DMU H4, as well as the corresponding input and output targets. The black circles and solid lines show the observed 2−polygonal fuzzy inputs and outputs. The MFERM targets are represented with dashed lines and black squares. For example, for input 2, the corresponding variables are $\{\theta^L_{2,0}; \theta^L_{2,1}; \theta^L_{2,2}; \theta^R_{2,2}; \theta^R_{2,1}; \theta^R_{2,0}\} = \{0.525; 0.849; 1.000; 1.000; 0.683; 0.445\}$. In the case of output 1, the target support is significantly to the right of the observed data, implying a large inefficiency as regards that output. For the other inefficient DMUs this separation between the targets and observed data values are even greater and happens mainly for output 2.

In addition to the fuzzy input and output for the example in Subsection 3.4.2, Table 3.3 shows the corresponding MFERM efficiency scores. In general, they show an improvement with respect to those of Example **??**. The efficient or inefficient character of the DMUs remains the same, but the inefficient DMUs have improved their efficiency. This is not guaranteed to happen necessarily although it is not strange. That is because the use of a higher $k-$polygonal fuzzy numbers leads to an increased number of constraints (and hence a smaller feasibility region) in (PLFERM) model, which is a minimization problem.

The lower part of Table 3.3 also shows the computed MFERM targets. The efficient DMUs Figure 3.1 are projected onto themselves. For the inefficient DMUs, as graphically shown in Figure for DMU H4, the fuzzy targets always dominate the observed inputs and outputs and sometimes by a large amount.

The interpretation of the fuzzy targets is similar to that of the fuzzy data from which they derive. Thus, for each α-level, the corresponding α-cuts of the observed input and the corresponding target can be compared, and the same occurs with the observed output and the target output. The proposed approach guarantees that, for each α-level, the lower limit of the target input is less than the lower limit of the observed input and that the upper limit of the target input is less than the upper limit of the observed input. Moreover, the relative difference between the two lower limits and between the upper limits must be at least the value of the corresponding θ^L_{il}, θ^R_{il}, γ^L_{rl} or γ^R_{rl} variable computed by the model (LMFERM) (3.14), and equations (3.13), which represents the maximum margin for improvement for that input or output.

Table 3.3: Modified dataset using 2− polygonal fuzzy inputs and outputs and corresponding FERM targets

DMU	R_M^F	Fuzzy input data		Fuzzy output data	
		$\widetilde{X}_{1j}$	$\widetilde{X}_{2j}$	$\widetilde{Y}_{1j}$	$\widetilde{Y}_{2j}$
H1	1	(10 , 10.81 , 13 , 13.04 , 15)	(3 , 3.12 , 5 , 5.48 , 8)	(3 640 , 3 640.21 , 3 650 , 3 650.43 , 3 655)	(134 130 , 134 135.68 , 134 137 , 134 142.45 , 134 145)
H2	1	(10 , 11.7 , 12 , 13.09 , 14)	(3 , 4.67 , 5 , 6.12 , 7)	(4 150 , 4 151.64 , 4 160 , 4 165.69 , 4170)	(116 055 , 116 061.78 , 116 062 , 116 062.42 , 116 068)
H3	1	(9 , 10.09 , 12 , 13.68 , 14)	(2 , 3.69 , 4 , 4.48 , 5)	(4 360 , 4 365.12 , 4 370 , 4 378.88 , 4 380)	(94 060 , 94 060.14 , 94 066 , 94 066.43 , 94 072)
H4	0.343	(6 , 6.69 , 8 , 10.45 , 11)	(1 , 1 , 1 , 1.63 , 3)	(485 , 486.62 , 492 , 496.94 , 500)	(24 320 , 24 320.38 , 24 329 , 24 330.7 , 24 335)
H5	0.839	(8 , 8.5 , 10 , 10.13 , 13)	(3 , 3.82 , 4 , 4.18 , 6)	(2 460 , 2 463.42 , 2 464 , 2 468.91 , 2 470)	(99 740 , 99 743.96 , 99 748 , 99 754.3 , 99 760)
H6	0.458	(10 , 10.05 , 11 , 11.62 , 12)	(2 , 2.38 , 3 , 3.71 , 4)	(1 360 , 1 364.87 , 1 368 , 1 370.22 , 1 375)	(49 395 , 49 396.58 , 49 401 , 49 407.19 , 49 410)
H7	0.444	(9 , 10 , 10 , 10 , 12)	(1 , 2 , 2 , 2 , 6)	(1055 , 1062 , 1062 , 1062 , 1070)	(37 765 , 37 772 , 37 772 , 37 772 , 37 780)
H8	1	(9 , 11 , 11 , 11 , 15)	(1 , 4 , 4 , 4 , 7)	(1 295 , 1 302 , 1 302 , 1 302 , 1 310)	(82 835 , 82 841 , 82 841 , 82 841 , 82 850)
H9	0.625	(10 , 12 , 12 , 12 , 15)	(2 , 5 , 5 , 5 , 7)	(1 660 , 1 671 , 1 671 , 1 671 , 1 680)	(100 590 , 100 596 , 100 596 , 100 596 , 100 605)
H10	0.303	(10 , 16 , 16 , 16 , 20)	(2 , 4 , 4 , 4 , 6)	(1 010 , 1 018 , 1 018 , 1 018 , 1 025)	(64 345 , 64 351 , 64 351 , 64 351 , 64 360)
H11	0.447	(9 , 11 , 11 , 11 , 14)	(3 , 5 , 5 , 5 , 8)	(1 500 , 1 504 , 1 504 , 1 504 , 1 510)	(80 050 , 80 056 , 80 056 , 80 056 , 80 061)
H12	1	(5 , 8 , 8 , 8 , 10)	(1 , 4 , 4 , 4 , 6)	(1 960 , 1 965 , 1 965 , 1 965 , 1 972)	(58 160 , 58 167 , 58 167 , 58 167 , 58 175)

DMU	Fuzzy input targets		Fuzzy output targets	
	$\widetilde{X}_{1p}^{target}$	$\widetilde{X}_{2p}^{target}$	$\widetilde{Y}_{1p}^{target}$	$\widetilde{Y}_{2p}^{target}$
H1	(10 , 10.81 , 13 , 13.04 , 15)	(3 , 3.12 , 5 , 5.48 , 8)	(3640 , 3640.21 , 3650 , 3650.43 , 3655)	(134130 , 134135.68 , 134137 , 134142.45 , 134145)
H2	(10 , 11.7 , 12 , 13.09 , 14)	(3 , 4.67 , 5 , 6.12 , 7)	(4150 , 4151.64 , 4160 , 4165.69 , 4170)	(116055 , 116061.78 , 116062 , 116062.42 , 116068)
H3	(9 , 10.09 , 12 , 13.68 , 14)	(2 , 3.69 , 4 , 4.48 , 5)	(4360 , 4365.12 , 4370 , 4378.88 , 4380)	(94060 , 94060.14 , 94066 , 94066.43 , 94072)
H4	(2.19 , 2.43 , 2.9 , 3.22 , 3.38)	(0.52 , 0.85 , 1 , 1.11 , 1.34)	(1000.68 , 1001.65 , 1003.05 , 1004.75 , 1005.18)	(24332 , 24332.31 , 24333.48 , 24333.83 , 24335)
H5	(7.63 , 8.28 , 9.94 , 10.13 , 11.49)	(2.22 , 2.46 , 3.77 , 4.14 , 5.89)	(2879.32 , 2879.95 , 2887.04 , 2888.18 , 2891.38)	(99748.7 , 99752.56 , 99754.01 , 99757.74 , 99760)
H6	(5.80 , 6.51 , 7.74 , 8.82 , 9.03)	(1.29 , 2.38 , 2.58 , 2.89 , 3.22)	(2812.14 , 2815.44 , 2818.59 , 2824.32 , 2825.04)	(60667.43 , 60667.52 , 60671.30 , 60671.57 , 60675.17)
H7	(4.02 , 4.5 , 5.36 , 6.11 , 6.25)	(0.89 , 1.65 , 1.79 , 2 , 2.23)	(1946.43 , 1948.71 , 1950.89 , 1954.86 , 1955.36)	(41991.07 , 41991.13 , 41993.75 , 41993.94 , 41996.43)
H8	(9 , 11 , 11 , 11 , 15)	(1 , 4 , 4 , 4 , 7)	(1295 , 1302 , 1302 , 1302 , 1310)	(82835 , 82841 , 82841 , 82841 , 82850)
H9	(8.86 , 10.02 , 11.44 , 12 , 13.83)	(2 , 3.35 , 4.14 , 4.47 , 6.32)	(3075.44 , 3078.96 , 3084.12 , 3087.16 , 3091.23)	(100592.4 , 100596 , 100598.4 , 100600.5 , 100605.5)
H10	(8.04 , 9.01 , 10.71 , 12.21 , 12.5)	(1.79 , 3.29 , 3.57 , 4 , 4.46)	(3892.86 , 3897.43 , 3901.79 , 3909.71 , 3910.71)	(83982.14 , 83982.27 , 83987.5 , 83987.88 , 83992.86)
H11	(7.38 , 8.23 , 9.8 , 11 , 11.42)	(1.72 , 2.93 , 3.33 , 3.72 , 4.34)	(3454.53 , 3458.19 , 3462.62 , 3468.96 , 3470.21)	(80055.34 , 80056 , 80060.29 , 80061.13 , 80065.34)
H12	(5 , 8 , 8 , 8 , 10)	(1 , 4 , 4 , 4 , 6)	(1960 , 1965 , 1965 , 1965 , 1972)	(58160 , 58167 , 58167 , 58167 , 58175)

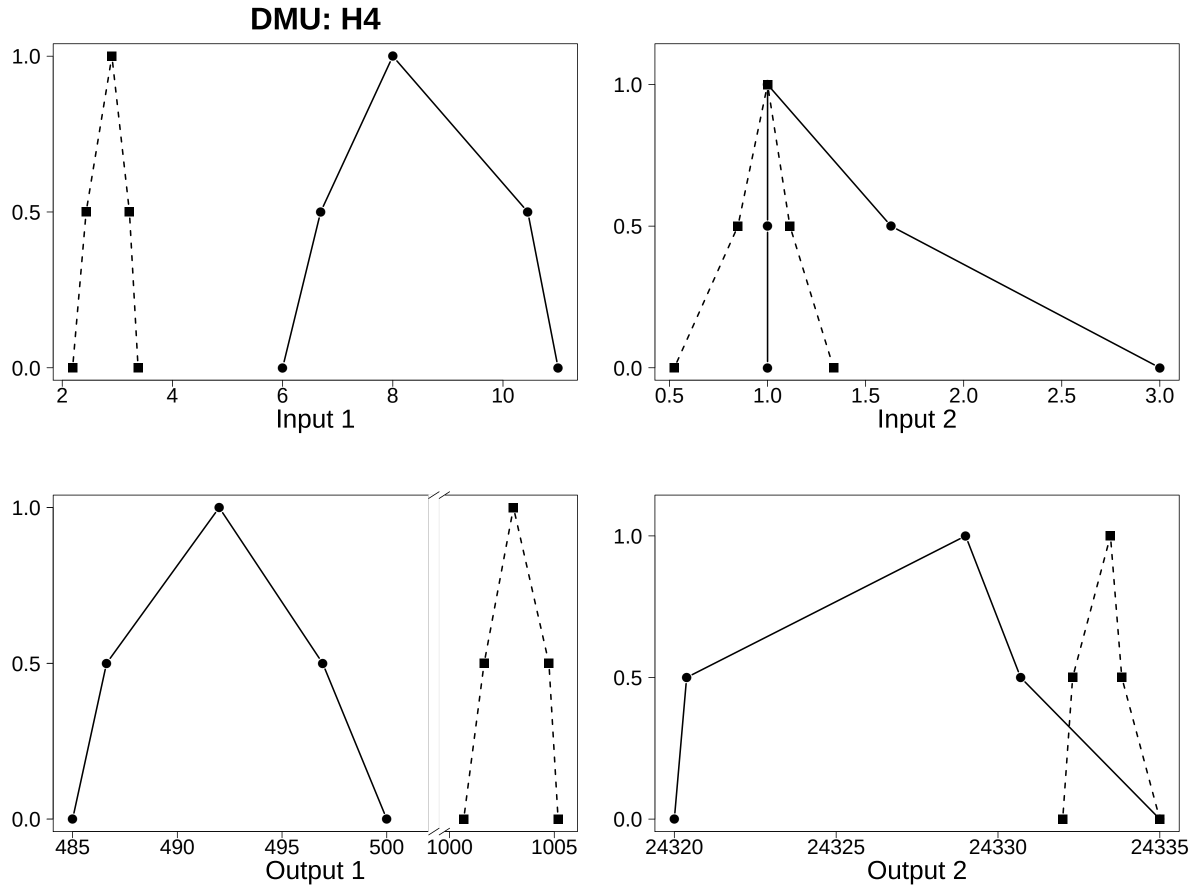

Figure 3.2: Observed and target 2−polygonal fuzzy inputs and outputs for DMU H4 for example in Section 3.4.2. The black circles and solid lines show the observed inputs, $\tilde{x}_{14}$ and $\tilde{x}_{24}$, and outputs, $\tilde{y}_{14}$ and $\tilde{y}_{24}$. The corresponding targets, $\tilde{x}_{i4}^{target}$ and $\tilde{y}_{r4}^{target}$, are represented with black squares and dashed lines. To compute the MFERM input and output targets, the observed data $\tilde{x}_{i4}$ and $\tilde{y}_{r4}$ are multiplied by the corresponding variables $\{\theta_{il}^L; \theta_{il}^R; \gamma_{rl}^L; \gamma_{rl}^R\}$, at levels $l = 0, 1, 2$, thus exhausting all possible input and output improvements. In the case of output 1, the support of the observed data and the MFERM target differ significantly, and hence the x-axis has been broken (the gap is marked with two vertical lines).

Thus, for example, as indicated in Table 3.3 and shown in Figure 3.2, the 0.0-level interval of input 1 of the observed DMU H4 is the interval [6, 11] while the 0.0-level interval for the corresponding target is [2.19, 3.38]. Assuming that the units of that input refer to full-time equivalents (FTE), the target indicates that, for that $\alpha-$level, the minimum value of the variable can be reduced 3.81 FTE and the maximum value can be reduced 7.62 FTE. The same reasons that make the observed input uncertain and fuzzy apply to the corresponding target. The proposed approach does neither ignore nor eliminate those reasons. What is clear is that, at the 0.0-level, DMU H4 can reduce input 1 by at least 3.81 FTE. Note also the feasible reduction for the 0.5-level (from [6.7, 10.45] to [2.43, 3.22]), and for the 1.0-level (from [8, 8] to [2.9, 2.9]). Overall, $\{\theta_{1,0}^L; \theta_{1,1}^L; \theta_{1,2}^L; \theta_{1,2}^R; \theta_{1,1}^R; \theta_{1,0}^R\} = \{0.365; 0.364; 0.363; 0.363; 0.308; 0.307\}$ for this input. As it can be seen in Figure 3.2, something similar happens in the case of input 2, leading in this case to $\{\theta_{1,0}^L; \theta_{1,1}^L; \theta_{1,2}^L; \theta_{1,2}^R; \theta_{1,1}^R; \theta_{1,0}^R\} = \{0.525; 0.849; 1.000; 1.000; 0.683; 0.445\}$. In the case of the outputs, the interpretation is similar. Thus, for output 1, number of inpatients, the 0.0-level, 0.5-level and 1.0 level sets for corresponding observed and the target fuzzy numbers are [485, 500] vs. [1000.68, 1005.18], [486.62, 496.94] vs. [1001.65, 1004.75] and [492, 492] vs. [1003.05, 1003.05], respectively.

Therefore, for DMU H4, the increase in the number of inpatients is apparent and significant, as the value of the corresponding variables

$$\{\gamma_{1,0}^L; \gamma_{1,1}^L; \gamma_{1,2}^L; \gamma_{1,2}^R; \gamma_{1,1}^R; \gamma_{1,0}^R\} = \{2.063; 2.058; 2.039; 2.039; 2.022; 2.010\}$$

computed by model (3.14) and equations (3.13), attests.

By contrast, for output 2, the number of outpatients,

$$\{\gamma_{2,0}^L; \gamma_{2,0.5}^L; \gamma_{2,1}^L; \gamma_{2,1}^R; \gamma_{2,0.5}^R; \gamma_{2,0}^R\} = \{1.0005; 1.0005; 1.0002; 1.0002; 1.0001; 1.000\}$$

because the upper limit of one of the 0.0-set cannot be increased, as it can be seen in Figure 3.2. This leads to the corresponding MFERM efficiency score shown in Table 5.1

$$\begin{aligned} R_M^F(\tilde{X}_{H4}, \tilde{Y}_{H4}) &= \frac{112\,(0.365 + 0.364 + 0.363 + 0.363 + 0.308 + 0.307+}{112\,(2.063 + 2.058 + 2.039 + 2.039 + 2.022 + 2.010+} \\ &\quad \frac{+0.525 + 0.849 + 1.0 + 1.0 + 0.683 + 0.445)}{+2.010 + 1.0005 + 1.0005 + 1.0002 + 1.0002 + 1.0001 + 1.0)} = \\ &= 0.343, \end{aligned}$$

In summary, what the example in Section 3.4.2 illustrates is that the use of $k-$polygonal fuzzy numbers provides all the flexibility that be needed to represent the uncertainty in the data and that the proposed FERM approach can handle all those situations providing appropriate efficiency scores and the corresponding fuzzy targets.

3.4.3 Case study: electric power distribution company

This numerical example is taken from Izadikhah et al. (2017) [71], where 17 Iranian suppliers of self-supporting cable for Markazi Province Electric Power Distribution Company (MPEPDC) in Iran are evaluated. There are 2 inputs, x_1 = the overall suppliers ranking (ordinal variable) and x_2 = unit price by considering volume discount (interval type). Furthermore, there are 6 outputs: y_1 = production capacity and y_2 = financial potential, both of interval type, and y_3 = environmental standards and regulations, y_4 = research and developments for eco-design product, y_5 = safety and health standards, and y_6 = customer satisfaction. These last four outputs are fuzzy triangular numbers (see cf. Tables 10 to 14 in Izadikak for more details). Note that, since interval data $[a_1, a_2]$ correspond with trapezoidal fuzzy numbers $x_0^-, x_1^-, x_1^+, x_0^+ = (a_1, a_1, a_2, a_2)$, we model such type of data as $(RPFN_1)_+$. Recall also that the triangular fuzzy outputs are particular cases of the trapezoidal ones, where $x_1^- = x_1^+$. We adopt the same approximation of the ordinal input x_1 as an interval as done in [71].

Table 3.4 shows the results of applying the proposed (PLFERM) model to this dataset, and the comparison with the interval efficiency scores obtained by Izadikhah et al. (2017) [71]. The proposed approach identifies nine DMUs as efficient DMUs. This relatively high number is not surprising for a fuzzy problem with a such number of DMUs and variables. As already discussed above in Section 3.4.2, obviating the differences between efficiency interpretations in both methods, we find sufficient agreement since most $R_M^F(\tilde{X}_p, \tilde{Y}_p)$ fall within the corresponding efficiency intervals. Moreover, if we apply a Wilcoxon signed-rank test of the results of our approach and [71], we get a $p-$value= 0.285. This means that we cannot reject the null hypothesis of homogeneity, i.e. that the median difference is equal to zero, which supports the existence of sufficient agreement between both efficiency measurements. As in Subsection 3.4.1, Izadikhah et al. (2017) [71] provides a ranking even between the efficient DMUs and, again, we

Table 3.4: FERM efficiency scores compared to interval efficiencies and ranking from Izadikhahet al. (2017) [71], corresponding to the Numerical example 3.4.3.

DMU	Our approach		Izadikhah et al. (2017) [71]	
	$R^F_M(\tilde{X}_o, \tilde{Y}_o)$	Rank	Interval efficiency	Rank
1	0.598	13	[0.5,0.72]	6
2	0.309	16	[0.24,0.51]	16
3	0.498	14	[0.32,0.80]	8
4	1	1	[0.84,0.99]	2
5	1	1	[0.29,0.59]	15
6	1	1	[0.45,0.69]	7
7	1	1	[0.57,1]	3
8	0.657	12	[0.32,0.76]	10
9	1	1	[0.28,0.82]	9
10	1	1	[0.24,0.75]	13
11	0.898	10	[0.33,0.74]	11
12	1	1	[0.84,1]	1
13	1	1	[0.34,0.61]	14
14	1	1	[0.46,1]	4
15	0.256	17	[0.20,0.42]	17
16	0.679	11	[0.47,0.99]	5
17	0.354	15	[0.27,0.74]	12

find that in some cases efficient DMUs are ranked after inefficient ones. The proposed approach is not able to discriminate between the efficient DMUs, but can easily rank the inefficient DMUs just sorting by their $R^F_M(\tilde{X}_p, \tilde{Y}_p)$ value.

3.5 Conclusions

This chapter presents a new approach for efficiency assessment and target setting when the input and output data are fuzzy. It is based on polygonal fuzzy numbers and LU-fuzzy partial orders. First, from the observed fuzzy data, and using simple axioms analogous to the ones considered in the crisp case, the fuzzy PPS containing all feasible operating points is inferred. Based on this PPS a fuzzy ERM DEA model is proposed to compute, for each DMU, a crisp efficiency score and a fuzzy target. The use of polygonal fuzzy numbers provides ample flexibility for modeling the uncertainty in the data. In addition, the non-radial approach used, which exhausts all possible input and output slacks, provides increased discriminant power.

The proposed approach has two main limitations. One is that it computes crisp efficiency scores. The second one is that it does not work (i.e. it leads to unbounded solutions) when the left limit of any input or output of a DMU is zero.

As regards topics for continuing this research, one is formulating a super-efficiency approach so that the efficient DMUs can be classified into extreme efficient and non-extreme efficient and the former can be ranked. Another topic is fully fuzzy approach for computing fuzzy efficiency scores. Other interesting extensions of the proposed approach include handling negative data and undesirable outputs given as fuzzy sets. Finally, a meaningful comparison between fuzzy DEA and stochastic DEA is also due (see, e.g., Wanke et al. [142]).

Chapter 4

Integer interval DEA

4.1 Introduction

As mentioned before, DEA was first introduced by Charnes et al. [31]. Then many researchers have investigated this area of science, but integer interval DEA, DEA under uncertainty, has not been worked until now. This chapter studies the situation when we have inputs and outputs that are both integer and interval-valued, as mathematical modeling of the uncertainty on integer data. To the best of our knowledge, the closest existing DEA approach is the fuzzy integer DEA model of Kordrostami et al. [94], which extends the integer DEA model of Jie et al. [81]. The approach proposed in this paper has numerous differences with respect to Kordrostami et al. [94]. Thus, while [94] considers fuzzy integer data in our case the uncertainty is modeled with interval integer data. While [94] uses a fuzzy ranking approach, which derives a defuzzification of the data instead of fully keeping the fuzzy information given by the original data, in the present approach we establish the order relation between the elements of the PPS using interval orders, together with interval arithmetic. Also, while [94] uses a radial-oriented approach, we use an additive, non-oriented approach. While [94] computes a crisp target, we compute an integer interval target. More important, while [94] uses the integer PPS of Kuosmanen and Kazemi Matin [98], we carry out an axiomatic derivation of a new integer interval PPS. This PPS can be used as a base to derive, in a continuation of this research, a fuzzy integer interval PPS, and a corresponding DEA model with fuzzy integer data using partial orders and arithmetic on fuzzy sets.

4.2 Crisp production possibility set and slack-based measure

Let us consider a set of n DMUs. For $j \in J = \{1, \dots, n\}$, each DMU_j has m inputs $X_j = (x_{1j}, \dots, x_{mj}) \in \mathbb{R}^m$, produces s outputs $Y_j = (y_{1j}, \dots, y_{sj}) \in \mathbb{R}^s$. In the classic Charnes et al. [31] DEA model, the production possibility set (PPS) or technology, denoted by T, satisfies the axioms Envelopment, Free disposability, Convexity, and Scability presented in Subsection 2.5.1.

Let us recall that following the minimum extrapolation principle (see [28]), the DEA PPS, which contains all the feasible input-output bundles, is the intersection of all the sets that satisfy axioms and can be expressed as

$$T_{DEA} = \left\{ (X,Y) \in \mathbb{R}_+^{m+s} : X \geq \sum_{j=1}^{n} \lambda_j X_j, Y \leq \sum_{j=1}^{n} \lambda_j Y_j, \lambda_j \geq 0 \right\}.$$

Also, a DMU o is said to be efficient if and only if for any $(X,Y) \in T_{DEA}$ such that $X \leqq X_o$ and $Y \geqq Y_o$, then $(X,Y) = (X_o, Y_o)$. This can be determined solving the following normalized slacks-based DEA model

$$\begin{aligned}
\text{(DEA)}\ \ I(X_o,Y_o) = \quad \text{Max} \quad & \sum_{i=1}^{M} \frac{s_i^x}{x_{io}} + \sum_{r=1}^{S} \frac{s_r^y}{y_{ro}} \\
\text{s.t.} \quad & \sum_{j=1}^{N} \lambda_j x_{ij} \leq x_{io} - s_i^x, \quad i = 1,\dots,M, \\
& \sum_{j=1}^{N} \lambda_j y_{rj} \geq y_{ro} + s_r^y, \quad r = 1,\dots,S, \\
& \lambda_j \geq 0, \quad j = 1,\dots,N, \\
& s_i^x, s_r^y \geq 0, \quad i = 1,\dots,M, \ \ r = 1,\dots,S.
\end{aligned} \tag{4.1}$$

where λ_j, $j = 1,\dots,n$, are the intensity variables used for defining the corresponding efficient target of DMU_o. The inefficiency measure $I(X_o,Y_o)$ is units invariant and non-negative. Moreover, a DMU_o is efficient if and only if $I(X_o,Y_o) = 0$.

4.3 Proposed integer interval PPS and slack-based measure of inefficiency

Let us consider a set of N DMUs. Each DMU_j, with $j \in J = \{1,\dots,N\}$, consumes M inputs given by $X_j = (x_{1j},\dots,x_{Mj}) \in (\mathcal{K}_{\mathbb{Z}+})^M$, with $x_{ij} = [\underline{x_{ij}}, \overline{x_{ij}}]_{\mathbb{Z}} \in \mathcal{K}_{\mathbb{Z}+}$ for $i \in \{1,\dots,M\}$. Each DMU_j also produces S outputs given by $Y_j = (y_{1j},\dots,y_{Sj}) \in (\mathcal{K}_{\mathbb{Z}+})^S$, with $y_{rj} = [\underline{y_{rj}}, \overline{y_{rj}}]_{\mathbb{Z}} \in \mathcal{K}_{\mathbb{Z}+}$ for $r \in \{1,\dots,S\}$. Their continuous extensions are $C(X_j) = \big(C(x_{1j}),\dots,C(x_{Mj})\big)$ and $C(Y_j) = \big(C(y_{1j}),\dots,C(y_{Sj})\big)$, with $C(x_{ij}) = [\underline{x_{ij}}, \overline{x_{ij}}] \in \mathcal{K}_C$, and $C(y_{rj}) = [\underline{y_{rj}}, \overline{y_{rj}}] \in \mathcal{K}_C$, respectively.

Let us consider the following axioms, which are analogous to axioms introduced in Subsection 2.5.1, but considering integer interval inputs and outputs and using the corresponding partial order introduced in Definitions 2.2.2 and 2.2.4:

(B1) *Envelopment:* $(X_j, Y_j) \in T$, *for all* $j \in J$.

(B2) *Free disposability:* $(X,Y) \in T, (X',Y') \in (\mathcal{K}_{\mathbb{Z}}+)^{M+S}$, *such that* $X' \geqq X, Y' \leqq Y \ \Rightarrow\ (X',Y') \in T$.

(B3) *Convexity:* $(X,Y),(X',Y') \in T$, $\alpha \in [0,1]$, *such that* $\alpha(C(X),C(Y)) + (1-\alpha)(C(X'),C(Y')) \in (\mathcal{K}_{C\to\mathbb{Z}})^{M+S} \Rightarrow (X'',Y'') = \mathbb{Z}\alpha(C(X),C(Y)) + (1-\alpha)(C(X'),C(Y')) \in T$.

(B4) *Scalability:* $(X,Y) \in T, \alpha \geq 0,$ *and* $\alpha(C(X),C(Y)) \in (\mathcal{K}_{C\to\mathbb{Z}})^{M+S} \Rightarrow (X'',Y'') = \mathbb{Z}(\alpha(C(X),C(Y))) \in T$.

Theorem 4.3.1. *Under axioms (B1), (B2), (B3) and (B4), the interval production possibility set that results from the minimum extrapolation principle is*

$$T_{IIDEA} = \left\{ (X,Y) \in (\mathcal{K}_{\mathbb{Z}}+)^{M+S} : C(X) \geqq \sum_{j=1}^{N} \lambda_j C(X_j), C(Y) \leqq \sum_{j=1}^{N} \lambda_j C(Y_j), \lambda_j \geq 0, \forall j \right\}$$

Proof. Denote by T_{true} the result of the minimum extrapolation principle axioms (B1), (B2), (B3) and (B4). To prove the theorem it is necessary to show that $T_{FDEA} = T_{true}$. To this end, let us divide the proof into two parts.
(i) $T_{true} \subseteq T_{IIDEA}$.
It is sufficient to prove that T_{IIDEA} satisfies (B1), (B2), (B3) and (B4), since this implies that T_{IIDEA} contains the intersection of all sets that satisfies the previous axioms, and consequently contains T_{true}. Therefore, let us check the axioms (B1), (B2), (B3) and (B4) by T_{IIDEA}.

- Check (B1). It is clear since, given $j \in J$, then (X_j, Y_j), with $\lambda_j = 1$ and $\lambda_{j'} = 0$, for all $j' \neq j$, satisfies conditions in T_{IIDEA}.
- Check (B2). Given $(X,Y) \in T_{IIDEA}$, $X' \geqq X$, $Y' \leqq Y$, $(X', Y') \in (\mathcal{K}_{\mathbb{Z}}+)^{m+s}$, we have to prove that $(X', Y') \in T_{IIDEA}$. By hypothesis, there exists $\lambda \geqq 0$ such that

$$C(X) \geqq \sum_{j=1}^{n} \lambda_j C(X_j), \quad C(Y) \leqq \sum_{j=1}^{n} \lambda_j C(Y_j). \tag{4.2}$$

  Combining (4.2) with $X' \geqq X$, $Y' \leqq Y$, it follows that

$$C(X') \geqq C(X) \geqq \sum_{j=1}^{n} \lambda_j C(X_j), \quad C(Y') \leqq C(Y) \leqq \sum_{j=1}^{n} \lambda_j C(Y_j). \tag{4.3}$$

  Therefore, $(X', Y') \in T_{IIDEA}$.

- Check (B3). Let us consider $(X,Y), (X',Y') \in T_{IIDEA}$, and $\alpha \geq 0$, what means that there exist $\lambda, \lambda' \geqq 0$ such that

$$C(X) \geqq \sum_{j=1}^{n} \lambda_j C(X_j), \quad C(X') \geqq \sum_{j=1}^{n} \lambda'_j C(X_j), \tag{4.4}$$

$$C(Y) \leqq \sum_{j=1}^{n} \lambda_j C(Y_j), \quad C(Y') \leqq \sum_{j=1}^{n} \lambda'_j C(Y_j). \tag{4.5}$$

  Multiplying by α each side in the first interval inequality in (4.4), by $(1-\alpha)$ each side in the second interval inequality in (4.4), and then combining the interval inequalities, we get

$$\alpha C(X) + (1-\alpha) C(X') \geqq \sum_{j=1}^{n} (\alpha \lambda_j + (1-\alpha)\lambda'_j) C(X_j), \tag{4.6}$$

  Proceeding in a similar way with y and y' and inequalities (4.5), we have

$$\alpha C(Y) + (1-\alpha) C(Y') \leqq \sum_{j=1}^{n} (\alpha \lambda_j + (1-\alpha)\lambda'_j) C(Y_j), \tag{4.7}$$

We can see that $(\alpha C(X)+(1-\alpha)C(X'), \alpha C(Y)+(1-\alpha)C(Y')) = \alpha(C(X),C(Y))+(1-\alpha)(C(X'),C(Y'))$. By hypothesis, $\alpha(C(X),C(Y))+(1-\alpha)(C(X'),C(Y')) \in (\mathcal{K}_{C\to\mathbb{Z}})^{m+s}$. Define $\lambda''=(\lambda''_1,\ldots,\lambda''_n)$, with $\lambda''_j=\alpha\lambda_j+(1-\alpha)\lambda'_j \geq 0$, for all $j=1,\ldots,n$, and substitute them in expressions (4.6) and (4.7). Then, it follows that $(X'',Y'')=\mathbb{Z}(\alpha(C(X),C(Y))+(1-\alpha)(C(X'),C(Y'))) \in T_{IIDEA}$.

- Check (B4). Given $(X,Y) \in T_{IIDEA}$, there exists $\lambda \geqq 0$ such that (4.2) holds. Given $\alpha \in \mathbb{R}_+$, and $(\alpha C(X), \alpha C(Y)) \in (\mathcal{K}_{C\to\mathbb{Z}})^{m+s}$, it follows that there exists $\mathbb{Z}((\alpha C(X),\alpha C(Y))) = (\alpha X, \alpha Y) \in (\mathcal{K}_{\mathbb{Z}}+)^{m+s}$. Define $\bar\lambda=\alpha\lambda=(\alpha\lambda_1,\ldots,\alpha\lambda_n)\geqq 0$. Then, multiplying by α each side in the inequalities in (4.2),

$$C(\alpha X) \geqq \sum_{j=1}^{n} \alpha\lambda_j C(X_j) = \sum_{j=1}^{n} \bar\lambda_j C(X_j), \quad C(\alpha Y) \leqq \sum_{j=1}^{n} \alpha\lambda_j C(Y_j) = \sum_{j=1}^{n} \bar\lambda_j C(Y_j).$$

  Therefore, $(\alpha X, \alpha Y) \in T_{IIDEA}$

(ii) $T_{IIDEA} \subseteq T_{true}$.
We need to prove that every element of T_{IIDEA} belongs to T_{true}. To this purpose, consider $(X,Y) \in T_{IIDEA}$, which means that there exists $\lambda \geqq 0$, $\lambda \in \mathbb{R}^n$, such that

$$C(X) \geqq \sum_{j=1}^{n} \lambda_j C(X_j), \quad C(Y) \leqq \sum_{j=1}^{n} \lambda_j C(Y_j), \tag{4.8}$$

what is equivalent to say

$$[\underline{x_i}, \overline{x_i}] \geqq \sum_{j=1}^{n} \lambda_j [\underline{x_{ij}}, \overline{x_{ij}}] = \Big[\sum_{j=1}^{n} \lambda_j \underline{x_{ij}}, \sum_{j=1}^{n} \lambda_j \overline{x_{ij}}\Big], \quad i=1,\ldots,m, \tag{4.9}$$

$$[\underline{y_r}, \overline{y_r}] \leqq \sum_{j=1}^{n} \lambda_j [\underline{y_{ij}}, \overline{y_{ij}}] = \Big[\sum_{j=1}^{n} \lambda_j \underline{y_{ij}}, \sum_{j=1}^{n} \lambda_j \overline{y_{ij}}\Big], \quad r=1,\ldots,s. \tag{4.10}$$

The relationships (4.9) and (4.10) imply

$$\underline{x_i} \geq \sum_{j=1}^{n} \lambda_j \underline{x_{ij}}, \quad \overline{x_i} \geq \sum_{j=1}^{n} \lambda_j \overline{x_{ij}}, \quad i=1,\ldots,m, \tag{4.11}$$

$$\underline{y_r} \leq \sum_{j=1}^{n} \lambda_j \underline{y_{rj}}, \quad \overline{y_r} \leq \sum_{j=1}^{n} \lambda_j \overline{y_{rj}}, \quad r=1,\ldots,s. \tag{4.12}$$

Taking into account the inequalities given by (4.11) and (4.12), we consider the following two cases:

- Suppose that there exists some index and some inequality, among those given by (4.11) and (4.12), such that the inequality becomes equality. For the sake of simplicity, suppose that the equality is verified for an inequality in the first group of (4.11), that is, there exists $i \in \{1,\ldots,m\}$ such that $\underline{x_i} = \sum_{j=1}^{n} \lambda_j \underline{x_{ij}} \in \mathbb{Z}_+$. The latter implies that $\lambda_j \in \mathbb{Q}_+$, for all j, with $\mathbb{Q} \subseteq \mathbb{R}$ the subset of rational numbers. Then, there exist $u_j, v_j \in \mathbb{Z}_+$, $v_j \neq 0$, with u_j a pair number, such that $\lambda_j = \frac{u_j}{v_j}$, for all j. Define $v=\prod_{j=1}^{n} v_j$, and $n_j = v\lambda_j \in \mathbb{Z}_+$. We point out that n_j is a pair number, that is, $0.5n_j \in \mathbb{Z}_+$, what is used in a next step in this proof. If we multiply each side of the interval inequalities given in (4.8), then if follows

$$vC(X) \geqq \sum_{j=1}^{n} n_j C(X_j), \quad vC(Y) \leqq \sum_{j=1}^{n} n_j C(Y_j). \tag{4.13}$$

We have that $(X_j, Y_j) \in T_{true}$ by (B1), for all $j \in J$. Since (n_jX_j, n_jY_j) and $(0.5n_jX_j, 0.5n_jY_j) \in (\mathcal{K}_{\mathbb{Z}_+})^{m+s}$, then, by (B4), it follows that (n_jX_j, n_jY_j) and $(0.5n_jX_j, 0.5n_jY_j) \in T_{true}$, for all $j \in J$, and the relationships (4.13) can be written as

$$vX \geqq \sum_{j=1}^{n} n_j X_j, \quad vY \leqq \sum_{j=1}^{n} n_j Y_j. \tag{4.14}$$

Following, and reasoning by induction, let us prove that

$$\Big(\sum_{j=1}^{k} n_j X_j, \sum_{j=1}^{k} n_j Y_j\Big) \in T_{true}, \quad k = 1, \dots, n. \tag{4.15}$$

To this matter, first we check that in the case $k = 1$ it holds, such as it has been proved before. Following, we check that if cases $k \leq t$ are true, this implies that the case $k = t+1$ is also true. We can write $(\sum_{j=1}^{t+1} n_jX_j, \sum_{j=1}^{t+1} n_jY_j)$ as the convex sum of two elements of T_{true}, multiplied by a scalar. Define $\alpha = 0.5$ and $\alpha' = 2$, then:

$$\begin{aligned}
\Big(\sum_{j=1}^{t+1} n_j X_j, \sum_{j=1}^{t+1} n_j Y_j\Big) &= \Big(\sum_{j=1}^{t} n_j X_j, \sum_{j=1}^{t} n_j Y_j\Big) + \Big(n_{t+1}X_{t+1}, n_{t+1}Y_{t+1}\Big) \\
&= \mathbb{Z}\Big(\alpha'\Big(\alpha\Big(\sum_{j=1}^{t} n_j C(X_j), \sum_{j=1}^{t} n_j C(Y_j)\Big) \ + \\
&\quad + \ (1-\alpha)\Big(n_{t+1}C(X_{t+1}), n_{t+1}C(Y_{t+1})\Big)\Big)\Big).
\end{aligned}$$

Since

$$\begin{aligned}
\alpha\Big(\sum_{j=1}^{t} n_j C(X_j), \sum_{j=1}^{t} n_j C(Y_j)\Big) + (1-\alpha)\Big(n_{t+1}C(X_{t+1}), n_{t+1}C(Y_{t+1})\Big) = \\
\Big(\sum_{j=1}^{t+1} 0.5 n_j C(X_j), \sum_{j=1}^{t+1} 0.5 n_j C(Y_j)\Big) \in (\mathcal{K}_{C\to\mathbb{Z}})^{m+s},
\end{aligned}$$

then, by (B3), it follows that

$$\mathbb{Z}\Big(\alpha\Big(\sum_{j=1}^{t} n_j C(X_j), \sum_{j=1}^{t} n_j C(Y_j)\Big) + (1-\alpha)\Big(n_{t+1}C(X_{t+1}), n_{t+1}C(Y_{t+1})\Big)\Big) \in T_{true}.$$

If it is multiplied by $\alpha' = 2$, then, by (B4), it follows that

$$\alpha'\Big(\mathbb{Z}\Big(\alpha\Big(\sum_{j=1}^{t} n_j C(X_j), \sum_{j=1}^{t} n_j C(Y_j)\Big) + (1-\alpha)\Big(n_{t+1}C(X_{t+1}), n_{t+1}C(Y_{t+1})\Big)\Big)\Big) \in T_{true}.$$

Thus, $\Big(\sum_{j=1}^{t+1} n_jX_j, \sum_{j=1}^{t+1} n_jY_j\Big) \in T_{true}$, and therefore (4.15) holds. As a consequence of (4.15), we have that $\Big(\sum_{j=1}^{n} n_jX_j, \sum_{j=1}^{n} n_jY_j\Big) \in T_{true}$. Since (vX, vY) satisfies (4.14), then it also satisfies (4.13). Then, by (B2), we have that $(vX, vY) \in T_{true}$. And since $\frac{1}{v}(vX, vY) = (X, Y) \in (\mathcal{K}_{\mathbb{Z}_+})^{m+s}$, then, by (B2), it follows that $(X, Y) \in T_{true}$.

- Suppose that there exists no index and inequality, among those given by (4.11) and (4.12), such that the inequality becomes equality. In such a case, all inequalities are sharp, and it is not difficult to see that there exists $\delta > 0$, small enough, such that

$$\underline{x_i} > \sum_{j=1}^{n} (\lambda_j + \delta)\underline{x_{ij}}, \quad \overline{x_i} > \sum_{j=1}^{n} (\lambda_j + \delta)\overline{x_{ij}}, \quad i = 1, \ldots, m, \tag{4.16}$$

$$\underline{y_r} < \sum_{j=1}^{n} (\lambda_j + \delta)\underline{y_{rj}}, \quad \overline{y_r} < \sum_{j=1}^{n} (\lambda_j + \delta)\overline{y_{rj}}, \quad r = 1, \ldots, s. \tag{4.17}$$

We choose $\lambda'_j \in (\lambda_j, \lambda_j + \delta) \cap \mathbb{Q}_+ \neq \emptyset$, for $j \in \{1, \ldots, n\}$. Then, from (4.11), (4.12), (4.16) and (4.17), it follows

$$\underline{x_i} > \sum_{j=1}^{n} \lambda'_j \underline{x_{ij}}, \quad \overline{x_i} > \sum_{j=1}^{n} \lambda'_j \overline{x_{ij}}, \quad i = 1, \ldots, m, \tag{4.18}$$

$$\underline{y_r} < \sum_{j=1}^{n} \lambda'_j \underline{y_{rj}}, \quad \overline{y_r} < \sum_{j=1}^{n} \lambda'_j \overline{y_{rj}}, \quad r = 1, \ldots, s. \tag{4.19}$$

In particular,

$$\underline{x_i} \geq \sum_{j=1}^{n} \lambda'_j \underline{x_{ij}}, \quad \overline{x_i} \geq \sum_{j=1}^{n} \lambda'_j \overline{x_{ij}}, \quad i = 1, \ldots, m, \tag{4.20}$$

$$\underline{y_r} \leq \sum_{j=1}^{n} \lambda'_j \underline{y_{rj}}, \quad \overline{y_r} \leq \sum_{j=1}^{n} \lambda'_j \overline{y_{rj}}, \quad r = 1, \ldots, s. \tag{4.21}$$

Reasoning as above, we conclude that $(X, Y) \in T_{true}$. Therefore, $T_{FDEA} \subseteq T_{true}$ and the proof is complete. □

After the characterization result for the T_{IIDEA} given in Theorem 4.3.1, we can formulate the following integer interval DEA (IIDEA) model, which is a slacks-based measure of inefficiency,

$$\begin{aligned}
\text{(IIDEA)} \;\; I(X_o, Y_o) = \quad \text{Max} \quad & \sum_{i=1}^{M} \frac{\underline{s_i^x} + \overline{s_i^x}}{\underline{x_{io}} + \overline{x_{io}}} + \sum_{r=1}^{S} \frac{\underline{s_r^y} + \overline{s_r^y}}{\underline{y_{ro}} + \overline{y_{ro}}} \\
\text{s.t.} \quad & \sum_{j=1}^{N} \lambda_j C(x_{ij}) \leqq C(x_{io}) - C(s_i^x), \quad i = 1, \ldots, M, \\
& \sum_{j=1}^{N} \lambda_j C(y_{rj}) \geqq C(y_{ro}) + C(s_r^y), \quad r = 1, \ldots, S, \\
& \lambda_j \geq 0, \quad j = 1, \ldots, N, \\
& s_i^x, s_r^y \in \mathcal{K}_{\mathbb{Z}+}, \quad i = 1, \ldots, M, \;\; r = 1, \ldots, S.
\end{aligned} \tag{4.22}$$

where it is assumed that all inputs $x_{ij} = [\underline{x_{ij}}, \overline{x_{ij}}]_{\mathbb{Z}}$, and outputs $y_{rj} = [\underline{y_{rj}}, \overline{y_{rj}}]_{\mathbb{Z}}$ are non-negative integer intervals and belong to $\mathcal{K}_{\mathbb{Z}+}$, $\forall i, j, r$.

Let us denote a feasible solution for (IIDEA) as $(\boldsymbol{s}^{x*}, \boldsymbol{s}^{y*}, \boldsymbol{\lambda}^*)$, where $\boldsymbol{s}^{x*} = (s_1^{x*}, \ldots, s_M^{x*}) \in (\mathcal{K}_{\mathbb{Z}+})^M$, $\boldsymbol{s}^{y*} = (s_1^{y*}, \ldots, s_S^{y*}) \in (\mathcal{K}_{\mathbb{Z}+})^S$, and $\boldsymbol{\lambda}^* = (\lambda_1^*, \ldots, \lambda_N^*) \in \mathbb{R}^N$. We will deal directly with (IIDEA) model, without any ranking function. Note that its objective function is a real number, i.e. $I(X_o, Y_o) \in \mathbb{R}$.

Definition 4.3.1. *A DMU_o is said to be efficient if and only if $(X, Y) \in T_{IIDEA}$, $x \leqq X_o$ and $Y \geqq Y_p$ implies $(X, Y) = (X_p, Y_p)$.*

Given the above integer-interval (IIDEA) model (4.22), efficient DMUs have a null inefficiency measure, i.e.

Theorem 4.3.2. *If DMU_o is efficient, then $I(X_o, Y_o) = 0$.*

Proof. Suppose that $I(X_o, Y_o) > 0$, with $(\boldsymbol{s}^{x*}, \boldsymbol{s}^{y*}, \boldsymbol{\lambda}^*)$ an optimal solution for (IIDEA). Let $x^* = (x_1^*, \ldots, x_M^*) \in (\mathcal{K}_{\mathbb{Z}_+})^M$, where $x_i^* = x_{io} - s_i^{x*} = [\underline{x_{io}} - \overline{s_i^{x*}}, \overline{x_{io}} - \underline{s_i^{x*}}]_{\mathbb{Z}}$ for each $i = 1, \ldots, M$. And let $y^* = (y_1^*, \ldots, y_S^*) \in (\mathcal{K}_{\mathbb{Z}}+)^S$, defined as $y_r^* = y_{ro} + s_r^{y*} = [\underline{y_{ro}} + \underline{s_r^{y*}}, \overline{y_{ro}} + \overline{s_r^{y*}}]_{\mathbb{Z}}$ for $r = 1, \ldots, S$. By the model constraints,

$$C(X^*) \geqq \sum_{j=1}^{N} \lambda_j^* C(X_j) \quad \text{and} \quad C(Y^*) \leqq \sum_{j=1}^{N} \lambda_j^* C(Y_j)$$

and hence, $(X^*, Y^*) \in T_{IIDEA}$. It is clear also that $X^* \leqq X_o$ and $Y^* \geqq Y_o$.

If $I(X_o, Y_o) > 0$, then $(\boldsymbol{s}^{x*}, \boldsymbol{s}^{y*}) \neq 0$, i.e., $\boldsymbol{s}^{x*} \geqq 0$, with $s_{i_0}^{x*} \neq 0$ for some i_0, or/and $\boldsymbol{s}^{y*} \geqq 0$, with $s_{r_0}^{y*} \neq 0$ for some r_0. In the first case, it must happen that $\overline{s_{i_0}^{x*}} > 0$ and therefore $X^* \leqq X_o$, with $X^* \neq X_o$. This means that $(X^*, Y^*) \in T_{IIDEA}$, $X^* \leqq X_o, X^* \neq X_o$, and $Y^* \geqq Y_o$, which implies that DMU_o is not efficient, reaching a contradiction. Analogously, we also reach a contradiction for the second case. □

To solve (IIDEA) model at its current stage (4.22), we take into account the arithmetic operations (Definition 2.2.3 and order relations (Definition 2.2.4) defined in the previous section. Therefore, the Integer Interval Data Envelopment Analysis problem (IIDEA) can be reformulated or parameterized as

$$
\begin{aligned}
\text{(PIIDEA)}\ I(X_o, Y_o) = \quad & \text{Max} \quad \sum_{i=1}^{M} \frac{\underline{s_i^x} + \overline{s_i^x}}{\underline{x_{io}} + \overline{x_{io}}} + \sum_{r=1}^{S} \frac{\underline{s_r^y} + \overline{s_r^y}}{\underline{y_{ro}} + \overline{y_{ro}}} \qquad (4.23)\\
& \text{s.t.} \quad \sum_{j=1}^{N} \lambda_j \underline{x_{ij}} \le \underline{x_{io}} - \overline{s_i^x}, \quad i = 1, \ldots, M,\\
& \quad\quad\ \ \sum_{j=1}^{N} \lambda_j \overline{x_{ij}} \le \overline{x_{io}} - \underline{s_i^x}, \quad i = 1, \ldots, M,\\
& \quad\quad\ \ \sum_{j=1}^{N} \lambda_j \underline{y_{rj}} \ge \underline{y_{ro}} + \underline{s_r^y}, \quad r = 1, \ldots, S,\\
& \quad\quad\ \ \sum_{j=1}^{N} \lambda_j \overline{y_{rj}} \ge \overline{y_{ro}} + \overline{s_r^y}, \quad r = 1, \ldots, S,\\
& \quad\quad\ \ \underline{s_i^x} \le \overline{s_i^x}, \quad i = 1, \ldots, M,\\
& \quad\quad\ \ \underline{s_r^y} \le \overline{s_r^y}, \quad r = 1, \ldots, S,\\
& \quad\quad\ \ \lambda_j \ge 0, \quad j = 1, \ldots, N,\\
& \quad\quad\ \ \underline{s_i^x}, \overline{s_i^x}, \underline{s_r^y}, \overline{s_r^y} \in \mathbb{Z}_+, \quad i = 1, \ldots, M, \ r = 1, \ldots, S.
\end{aligned}
$$

The first four sets of constraints are just the corresponding transformation of the inputs/outputs constraints from model (4.22), given the order relation for integer intervals, Definition 2.2.4. The fifth and sixth set of constraints ensure the slacks $s_i^x = [\underline{s_i^x}, \overline{s_i^x}]_{\mathbb{Z}_+}$ and $s_r^y = [\underline{s_r^y}, \overline{s_r^y}]_{\mathbb{Z}_+}$ are integer intervals $\mathcal{K}_{\mathbb{Z}_+}$.

The relationship between the (IIDEA) and (PIIDEA) solutions is demonstrated in the following proposition.

Proposition 4.3.1. *$(\boldsymbol{s}^{x*}, \boldsymbol{s}^{y*}, \boldsymbol{\lambda}^*)$ with $\boldsymbol{s}^{x*} \in (\mathcal{K}_{\mathbb{Z}_+})^M$, $\boldsymbol{s}^{y*} \in (\mathcal{K}_{\mathbb{Z}_+})^S$ and $\boldsymbol{\lambda}^* \in \mathbb{R}_+^N$ is an optimal solution of (IIDEA) if and only if its corresponding components or parameterization $(\underline{s_1^{x*}}, \overline{s_1^{x*}}, \ldots, \underline{s_M^{x*}}, \overline{s_M^{x*}}, \underline{s_1^{y*}}, \overline{s_1^{y*}}, \ldots, \underline{s_S^{y*}}, \overline{s_S^{y*}} \lambda_1^*, \ldots, \lambda_N^*)$, with $\lambda_j^* \in \mathbb{R}_+$, $j = 1, \ldots, N$, $\underline{s_i^{x*}}, \overline{s_i^{x*}} \in \mathbb{Z}_+$, $i = 1, \ldots, M$, and $\underline{s_r^{y*}}, \overline{s_r^{y*}} \in \mathbb{Z}_+$ for $r = 1, \ldots, S$, is an optimal solution of (PIIDEA).*

Proof. The constraint in (IIDEA) (4.22) are equivalent to the constraint conditions in (PIIDEA) (4.23), given Definitions 2.2.3 and 2.2.4. The rest of the proof is straightforward. □

Although Theorem 4.3.2 establishes it as a necessary condition, a null inefficiency measure, i.e. $I(X_o, Y_o) = 0$, is not sufficient to guarantee the efficiency of DMU_j in the integer intervals case, as it happens in the crisp model (4.1). This can be seen in the following example.

Example 4.3.1. *Consider six DMUs that consume two different inputs and produce a constant amount of output. Figure 4.1 shows the inputs of these DMUs, that produce a single and constant output. Therefore, by decreasing each input we move towards the efficiency frontier, represented with a thick grey line and delimited by DMUs 1, 2, and 6. As data are integer intervals, the inputs of each DMU are the set of integer points within such integer intervals,shown in the Figure with different shaped symbols*

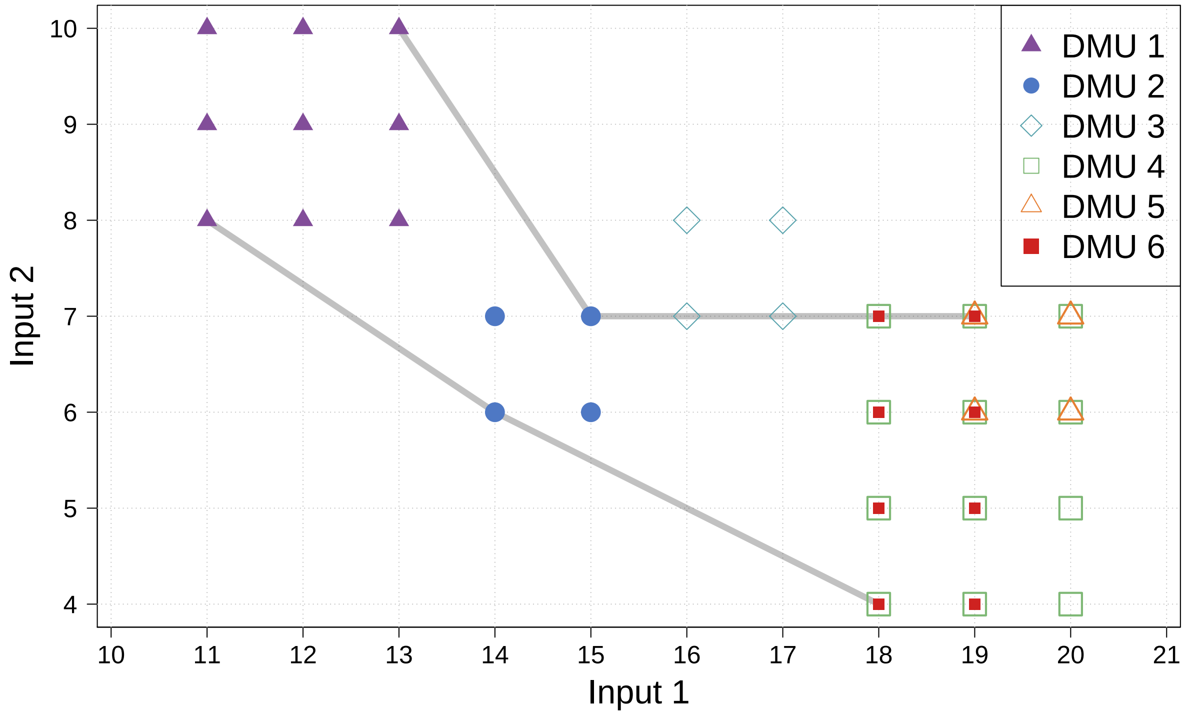

Figure 4.1: Consider these six *DMUs* that consume two inputs and produce a single constant output (see Example 4.3.1). The data are integer intervals and the set of integer points corresponding to each DMU is represented using different shaped symbols (see legend). The thick grey line represents the efficiency frontier. In this small example we can observe the different classes of DMUs in terms of their efficiency characterization. According to Definition 4.3.1, DMU_1, DMU_2 and DMU_6 (plotted with filled points) are efficient while the rest of DMUs are not efficient. While the inefficiency scores of the former are null, $I(X_3, Y_3) > 0$ and $I(X_5, Y_5) > 0$. Note that although DMU_4 is not efficient ($X_{1,6} \leqq X_{1,4}$, $X_{1,6} \neq X_{1,4}$) its inefficiency score $I(X_4, Y_4) = 0$. It is an example of weakly efficient DMU (see Definition 4.3.2), and this is why it is necessary a second phase for a correct efficiency characterization.

(filled points are used for the efficient DMUs). In this small example, we can observe the different classes of DMUs in terms of their efficiency characterization. Note that DMU_3 and DMU_5 have non-zero slacks for both inputs and thus $I(X_3, Y_3) > 0$ and $I(X_5, Y_5) > 0$. According to Theorem 4.3.2, they are inefficient. On the contrary, DMU_1, DMU_2, DMU_4 and DMU_6 have zero slacks for both inputs, and hence $I(X_1, Y_1) = 0$, $I(X_2, Y_2) = 0$, $I(X_4, Y_4) = 0$ and $I(X_6, Y_6) = 0$. But this does not imply that these DMUs are efficient in the integer intervals framework. According to Definition 4.3.1, it is clear that DMU_1, DMU_2 and DMU_6 are efficient. However, DMU_4 is not efficient, since $X_{1,6} \leqq X_{1,4}$, $X_{2,6} = X_{2,4}$ and $Y_6 = Y_4$, as it can be observed in Figure 4.1. Therefore, in order to exhaust all possible input and output slacks, a phase II is required to determine the efficiency character of the DMUs with null inefficiency measure $I(X_o, Y_o)$. This is performed by model (4.24) below, which uses additional integer-valued specific left and right slack variables, L^x, R^x, L^y and R^y. These variables allows us to detect if there still exists some remaining slack, for any input or output, that can be removed. The optimal solution of model (4.24) for DMU_4 has a non-zero objective function value $H(X_4, Y_4) > 0$, which tells us that DMU_4 is not efficient but weakly efficient (see Definition 4.3.2).

Therefore, given an optimal solution for (4.23) $(\boldsymbol{s}^{x*}, \boldsymbol{s}^{y*}, \boldsymbol{\lambda}^*)$, we can formulate the following Phase II model to exhaust all remaining input and output slacks.

$$
\begin{aligned}
(PIIDEA)_2 \;\; H(X_o, Y_o) = \quad \text{Max} \quad & \sum_{i=1}^{M} (L_i^x + R_i^x) + \sum_{r=1}^{S} (L_r^y + R_r^y) \\
\text{s.t.} \quad & \sum_{j=1}^{N} \lambda_j \underline{x_{ij}} \leq \underline{x_{io}} - \overline{s_i^{x*}} - R_i^x, \quad i = 1, \ldots, M, \\
& \sum_{j=1}^{N} \lambda_j \overline{x_{ij}} \leq \overline{x_{io}} - \underline{s_i^{x*}} - L_i^x, \quad i = 1, \ldots, M, \\
& \sum_{j=1}^{N} \lambda_j \underline{y_{rj}} \geq \underline{y_{ro}} + \underline{s_r^{y*}} + L_r^y, \quad r = 1, \ldots, S, \\
& \sum_{j=1}^{N} \lambda_j \overline{y_{rj}} \geq \overline{y_{ro}} + \overline{s_r^{y*}} + R_r^y, \quad r = 1, \ldots, S, \\
& \lambda_j \geq 0, \quad j = 1, \ldots, N, \\
& L_i^x, R_i^x, L_r^y, R_r^y \in \mathbb{Z}_+, \quad i = 1, \ldots, M, \;\; r = 1, \ldots, S.
\end{aligned}
\tag{4.24}
$$

Theorem 4.3.3. *Given a DMU_p with $I(X_o, Y_o) = 0$, then $H(X_o, Y_o) = 0$ if and only if DMU_o is efficient.*

Proof. If $I(X_o, Y_o) = 0$, for a maximizing problem with non-negative variables, it is clear that $\underline{s_i^x} = \overline{s_i^x} = \underline{s_r^y} = \overline{s_r^y} = 0$, $\forall i$ and $\forall r$. Moreover, if $H(X_o, Y_o) = 0$ as well, owing to similar reasoning, this implies that all the variables $L_i^x, R_i^x, L_r^y, R_r^y$ are equal to zero, for all $i = 1, \ldots, M$, and $r = 1, \ldots, S$. Now let us assume that DMU_o is not efficient. This means that there exist $(x^*, y^*) \in T_{IIDEA}$ such that $X^* \leqq X_o$ and $Y^* \geqq Y_o$, with $(X^*, Y^*) \neq (X_o, Y_o)$. I.e., $x_{i_0}^* \leqq x_{i_0p}, x_{i_0}^* \neq x_{i_0p}$ for some $i_0 \in \{1, \ldots, M\}$, or $y_{r_0}^* \geqq y_{r_0p}, y_{r_0}^* \neq y_{r_0p}$ for some $r_0 \in \{1, \ldots, S\}$. In the first case, by Definition 2.2.4, as either $\underline{x_{i_0}^*} < \underline{x_{i_0p}}$ or $\overline{x_{i_0}^*} < \overline{x_{i_0p}}$, we can compute a new feasible solution for $(PIIDEA)_2$, such that $R_{i_0}^{x^*} = \underline{x_{i_0p}} - \underline{x_{i_0}^*} > 0$ or $L_{i_0}^{x^*} = \overline{x_{i_0p}} - \overline{x_{i_0}^*} > 0$. Its feasibility holds since $(x^*, y^*) \in T_{IIDEA}$, i.e.

$$
\begin{aligned}
\sum_{j=1}^{N} \lambda_j \underline{x_{i_0j}} &\leq \underline{x_{i_0p}} - \overline{s_{i_0}^{x*}} - R_{i_0}^{x^*} = \underline{x_{i_0}^*} \\
\sum_{j=1}^{N} \lambda_j \overline{x_{ij}} &\leq \overline{x_{i_0p}} - \underline{s_{i_0}^{x*}} - L_{i_0}^{x^*} = \overline{x_{i_0}^*}.
\end{aligned}
$$

In this way we have reached a contradiction, since we have found a feasible solution with an objective function value larger than the supposed optimal value $H(X_o, Y_o) = 0$. For the second case, we also reach a contradiction with a similar reasoning, just defining a new solution with $L_{r_0}^{y*} = \underline{y_{r_0o}} - \underline{y_{r_0}^*} > 0$ or $R_{r_0}^{y*} = \overline{y_{r_0o}} - \overline{y_{r_0}^*} > 0$. Therefore, if $I(X_o, Y_o) = 0$, and $H(X_o, Y_o) = 0$ then DMU_o is efficient.

Finally, to proof that the efficiency of a DMU_o implies both $I(X_o, Y_o) = 0$ and $H(X_o, Y_o) = 0$, we only need to proof the latter since the necessary condition $I(X_o, Y_o) = 0$ was established in Theorem 4.3.2. Now let us suppose the opposite, $H(X_o, Y_o) > 0$. Then we can compute $(X^*, Y^*) \in T_{IIDEA}$ such that $X^* \leqq X_o$ and $Y^* \geqq Y_o$, with $(X^*, Y^*) \neq (X_o, Y_o)$, as follows. We have four possibilities, $L^{x}_{i_0} > 0$, or $R^{x}_{i_0} > 0$ for some $i_0 \in \{1, \ldots, M\}$, or, $L^{y}_{r_0} > 0$, or $R^{y*}_{r_0} > 0$ for some $r_0 \in \{1, \ldots, S\}$. For the two first cases, let $Y^* = Y_p$, and $x^*_i = x_{io}$ for all $i \in \{1, \ldots, M\}$, with $i \neq i_0$. And $\underline{x^*_{i_0}} = \underline{x_{i_0 p}} - \overline{s^{x*}_{i_0}} - R^{x}_{i_0}$, $\overline{x^*_{i_0}} = \overline{x_{io}} - \underline{s^{x*}_{i}} - L^{x}_{i_0}$. Then, $X^* \leqq X_o$ and $Y^* \geqq Y_o$, with $(X^*, Y^*) \neq (X_o, Y_o)$, which is a contradiction to the fact that DMU_p is efficient. Analogously, for the other two cases, let $X^* = X_o$, and $Y^*_r = y_{ro}$ for all $r \in \{1, \ldots, S\}$, with $r \neq r_0$. And $\underline{y^*_{r_0}} = \underline{y_{r_0 p}} - \overline{s^{y*}_{r_0}} - L^{y}_{r_0}$, $\overline{y^*_{r_0}} = \overline{y_{r_0 p}} - \underline{s^{y*}_{r_0}} - R^{y}_{r_0}$. Again, $X^* \leqq X_o$ and $Y^* \geqq Y_o$, with $(X^*, Y^*) \neq (X_o, Y_o)$, which is a contradiction to the fact that DMU_o is efficient. □

Let $(\boldsymbol{s}^{x*}, \boldsymbol{s}^{y*}, \boldsymbol{\lambda}^*)$ be the optimal solution for (4.23) and let $(\boldsymbol{L}^{x*}, \boldsymbol{R}^{x*}, \boldsymbol{L}^{y*}, \boldsymbol{R}^{y*}, \boldsymbol{\lambda}^{**})$ the optimal solution for (4.24) for a given DMU_o, we can compute its input and output targets X_o^{target} and Y_o^{target} as

$$\underline{x_{io}^{target}} = \underline{x_{io}} - \overline{s^{x*}_{i}} - R^{x*}_{i}, \qquad \overline{x_{io}^{target}} = \overline{x_{io}} - \underline{s^{x*}_{i}} - L^{x*}_{i}, \qquad i = 1, \ldots, M, \tag{4.25}$$

$$\underline{y_{ro}^{target}} = \underline{y_{ro}} + \underline{s^{y*}_{r}} + L^{y*}_{r}, \qquad \overline{y_{ro}^{target}} = \overline{y_{ro}} + \overline{s^{y*}_{r}} + R^{y*}_{r}, \qquad r = 1, \ldots, S. \tag{4.26}$$

Theorem 4.3.4. $(X_o^{target}, Y_o^{target})$ *is efficient.*

Proof. By the constraints of (4.24), it follows that $(X_o^{target}, Y_o^{target}) \in T_{IIDEA}$. Suppose that $(X_o^{target}, Y_o^{target})$ is not efficient. Then, there must exist $(X', Y') \in T_{IIDEA}$ such that $X' \leqq X_o^{target}$ and $Y' \geqq Y_o^{target}$, with $(X', Y') \neq (X_o^{target}, Y_o^{target})$. This implies that for some $\lambda' \geq 0$,

$$C(X') \geqq \sum_{j=1}^{N} \lambda'_j C(X_j), \quad C(Y') \leqq \sum_{j=1}^{N} \lambda'_j C(Y_j),$$

which is equivalent to

$$\underline{x'_i} \geq \sum_{j=1}^{N} \lambda'_j \underline{x_{ij}}, \qquad \overline{x'_i} \geq \sum_{j=1}^{N} \lambda'_j \overline{x_{ij}}, \qquad i = 1, \ldots, M,$$

$$\underline{y'_r} \leq \sum_{j=1}^{N} \lambda'_j \underline{y_{rj}}, \qquad \overline{y'_r} \leq \sum_{j=1}^{N} \lambda'_j \overline{y_{rj}}, \qquad r = 1, \ldots, S.$$

Besides,

$$\underline{x'_i} \leq \underline{x_{ip}^{target}} \qquad \overline{x'_i} \leq \overline{x_{ip}^{target}} \qquad i = 1, \ldots, M,$$

$$\underline{y'_r} \geq \underline{y_{rp}^{target}} \qquad \overline{y'_r} \geq \overline{y_{rp}^{target}} \qquad r = 1, \ldots, S.$$

where at least one of these inequalities is strict for some $i_0 \in \{1, \ldots, M\}$ or $r_0 \in \{1, \ldots, S\}$, since $(X', Y') \neq (X_o^{target}, Y_o^{target})$.

Combining the above constraints, it follows that

$$\sum_{j=1}^{N}\lambda'_j\underline{x_{ij}} \le \underline{x_{ip}} - \overline{s_i^{x*}} - R_i^{x*}, \qquad \sum_{j=1}^{N}\lambda'_j\overline{x_{ij}} \le \overline{x_{ip}} - \underline{s_i^{x*}} - L_i^{x*}, \quad i = 1,\dots,M,$$

$$\sum_{j=1}^{N}\lambda'_j\underline{y_{rj}} \ge \underline{y_{rp}} + \underline{s_r^{y*}} + L_r^{y*}, \qquad \sum_{j=1}^{N}\lambda'_j\overline{y_{rj}} \ge \overline{y_{rp}} + \overline{s_r^{y*}} + R_r^{y*}, \quad r = 1,\dots,S,$$

where at least one of these inequalities is strict for some $i_0 \in \{1,\dots,M\}$ or $r_0 \in \{1,\dots,S\}$. Therefore, there exists some $\delta_{i_0}^L, \delta_{i_0}^R, \epsilon_{r_0}^L, \epsilon_{r_0}^R \in \mathbb{Z}_+$, where at least one of them is non-zero, such that

$$\sum_{j=1}^{N}\lambda'_j\underline{x_{i_0j}} \le \underline{x_{i_0p}} - \overline{s_{i_0}^{x*}} - R_{i_0}^{x*} - \delta_{i_0}^R, \qquad \sum_{j=1}^{N}\lambda'_j\overline{x_{i_0j}} \le \overline{x_{i_0p}} - \underline{s_{i_0}^{x*}} - L_{i_0}^{x*} - \delta_{i_0}^L,$$

$$\sum_{j=1}^{N}\lambda'_j\underline{y_{r_0j}} \ge \underline{y_{r_0p}} + \underline{s_{r_0}^{y*}} + L_{r_0}^{y*} + \epsilon_{r_0}^L, \qquad \sum_{j=1}^{N}\lambda'_j\overline{y_{r_0j}} \ge \overline{y_{r_0p}} + \overline{s_{r_0}^{y*}} + R_{r_0}^{y*} + \epsilon_{r_0}^R,$$

If we define the new variables for the corresponding sharp constraints, as

$$\begin{array}{lllll} L_{i_0}^{x**} = L_{i_0}^{x*} + \delta_{i_0}^L, & R_{i_0}^{x**} = R_{i_0}^{x*} + \delta_{i_0}^R\ ; & L_i^{x**} = L_i^{x*}, & R_i^{x**} = R_i^{x*} & i = 1,\dots,M,\ i \ne i_0 \\ L_{r_0}^{y**} = L_{r_0}^{y*} + \epsilon_{r_0}^L, & R_{r_0}^{y**} = R_{r_0}^{y*} + \epsilon_{r_0}^R\ ; & L_r^{y**} = L_r^{y*}, & R_r^{y**} = R_r^{y*} & r = 1,\dots,S,\ r \ne r_0 \end{array}$$

then $(\boldsymbol{L}^{x**}, \boldsymbol{R}^{x**}, \boldsymbol{L}^{y**}, \boldsymbol{R}^{y**}, \boldsymbol{\lambda}')$ would be a feasible solution in (4.24) with a larger objective function value than the supposed optimum, which implies a contradiction.

□

Definition 4.3.2. *For each DMU o, $o \in \{1,\dots,N\}$, consider the inefficiency measurements $I(X_o, Y_o)$ computed in the (IIDEA), and $H(X_o, Y_o)$ obtained in Phase II, $(PIIDEA)_2$. We say that the DMU o is*

(i) efficient if $I(X_o, Y_o) = 0$ and $H(X_o, Y_o) = 0$,

(ii) weakly efficient if $I(X_o, Y_o) = 0$ and $H(X_o, Y_o) > 0$,

(iii) inefficient if $I(X_o, Y_o) > 0$.

4.4 Extension to the hybrid data scenario

Consider the hybrid scenario in which, in addition to integer interval data, there exist some inputs or outputs that are given as continuous intervals. Then, and following Lozano and Villa [111], we can partition each index set into two subsets; one for continuous variables, and another for integer variables. In this manner, for input and output variables, we have $O^X = O^{XI} \cup O^{XNI}$, $O^Y = O^{YI} \cup O^{YNI}$, respectively, where $O^X = \{1,\dots,M\}$ and $O^Y = \{1,\dots,S\}$. So, inputs $x_{ij} = [\underline{x_{ij}}, \overline{x_{ij}}]_{\mathbb{Z}} \in \mathcal{K}_{\mathbb{Z}_+}$, for all $i \in O^{XI}$, and $x_{ij} = [\underline{x_{ij}}, \overline{x_{ij}}] \in \mathcal{K}_{C_+}$, for all $i \in O^{XNI}$; and outputs $y_{rj} = [\underline{y_{rj}}, \overline{y_{rj}}]_{\mathbb{Z}} \in \mathcal{K}_{\mathbb{Z}_+}$ for all $r \in O^{YI}$, and $y_{rj} = [\underline{y_{rj}}, \overline{y_{rj}}] \in \mathcal{K}_{C_+}$ for all $r \in O^{YNI}$.

The model (IIDEA) becomes as follows, under the consideration of a hybrid interval DEA.

$$
\begin{aligned}
\text{(HIDEA)}\ \ I(X_o, Y_o) = \quad \text{Max} \quad & \sum_{i=1}^{M} \frac{\underline{s_i^x} + \overline{s_i^x}}{\underline{x_{io}} + \overline{x_{io}}} + \sum_{r=1}^{S} \frac{\underline{s_r^y} + \overline{s_r^y}}{\underline{y_{ro}} + \overline{y_{ro}}} \\
\text{s.t.} \quad & \sum_{j=1}^{N} \lambda_j C(x_{ij}) \leqq C(x_{io}) - C(s_i^x), \quad i \in O^{XI}, \\
& \sum_{j=1}^{N} \lambda_j x_{ij} \leqq x_{io} - s_i^x, \quad i \in O^{XNI}, \\
& \sum_{j=1}^{N} \lambda_j C(y_{rj}) \geqq C(y_{ro}) + C(s_r^y), \quad r \in O^{YI}, \\
& \sum_{j=1}^{N} \lambda_j y_{rj} \geqq y_{ro} + s_r^y, \quad r \in O^{YNI}, \\
& \lambda_j \geq 0, \quad j = 1, \ldots, N, \\
& s_i^x, s_r^y \in \mathcal{K}_{\mathbb{Z}+}, \quad i \in O^{XI}, \ r \in O^{YI}. \\
& s_i^x, s_r^y \in \mathcal{K}_{C+}, \quad i \in O^{XNI}, \ r \in O^{YNI}.
\end{aligned}
\tag{4.27}
$$

To solve (HIDEA) model (4.27), we consider its following parameterization, which can be considered as the Phase I of the solution method.

$$
\begin{aligned}
\text{(PIHIDEA)}\ \ I(X_o, Y_o) = \quad \text{Max} \quad & \sum_{i=1}^{M} \frac{\underline{s_i^x} + \overline{s_i^x}}{\underline{x_{io}} + \overline{x_{io}}} + \sum_{r=1}^{S} \frac{\underline{s_r^y} + \overline{s_r^y}}{\underline{y_{ro}} + \overline{y_{ro}}} \\
\text{s.t.} \quad & \sum_{j=1}^{N} \lambda_j \underline{x_{ij}} \leq \underline{x_{io}} - \overline{s_i^x}, \quad i \in O^X, \\
& \sum_{j=1}^{N} \lambda_j \overline{x_{ij}} \leq \overline{x_{io}} - \underline{s_i^x}, \quad i \in O^X, \\
& \sum_{j=1}^{N} \lambda_j \underline{y_{rj}} \geq \underline{y_{ro}} + \underline{s_r^y}, \quad r \in O^Y, \\
& \sum_{j=1}^{N} \lambda_j \overline{y_{rj}} \geq \overline{y_{ro}} + \overline{s_r^y}, \quad r \in O^Y, \\
& \underline{s_i^x} \leq \overline{s_i^x}, \quad i \in O^X, \\
& \underline{s_r^y} \leq \overline{s_r^y}, \quad r \in O^Y, \\
& \lambda_j \geq 0, \quad j = 1, \ldots, N, \\
& \underline{s_i^x}, \overline{s_i^x}, \underline{s_r^y}, \overline{s_r^y} \in \mathbb{Z}_+, \quad i \in O^{XI}, \ r \in O^{YI}, \\
& \underline{s_i^x}, \overline{s_i^x}, \underline{s_r^y}, \overline{s_r^y} \geq 0, \quad i \in O^{XNI}, \ r \in O^{YNI}.
\end{aligned}
\tag{4.28}
$$

As it can be seen, the only difference with respect the corresponding model (4.23) is the that only the slacks of the integer inputs and outputs are forced to be integer. The slacks of the other inputs and outputs are considered continuous variables.

Given $(\boldsymbol{s}^{x*}, \boldsymbol{s}^{y*}, \boldsymbol{\lambda}^{*})$, optimal solution for (4.28), we proceed with the phase II of the method.

$$
\begin{aligned}
(PHIDEA)_2 \;\; H(X_o, Y_o) = \quad \text{Max} \quad & \sum_{i=1}^{M} L_i^x + R_i^x + \sum_{r=1}^{S} L_r^y + R_r^y \\
\text{s.t.} \quad & \sum_{j=1}^{N} \lambda_j \underline{x_{ij}} \leq \underline{x_{io}} - \overline{s_i^{x*}} - R_i^x, \quad i \in O^X, \\
& \sum_{j=1}^{N} \lambda_j \overline{x_{ij}} \leq \overline{x_{io}} - \underline{s_i^{x*}} - L_i^x, \quad i \in O^X, \\
& \sum_{j=1}^{N} \lambda_j \underline{y_{rj}} \geq \underline{y_{ro}} + \underline{s_r^{y*}} + L_r^y, \quad r \in O^Y, \\
& \sum_{j=1}^{N} \lambda_j \overline{y_{rj}} \geq \overline{y_{ro}} + \overline{s_r^{y*}} + R_r^y, \quad r \in O^Y, \\
& \lambda_j \geq 0, \quad j = 1, \ldots, N, \\
& L_i^x, R_i^x, L_r^y, R_r^y \in \mathbb{Z}_{+}, \quad i \in O^{XI}, \;\; r \in O^{YI}, \\
& L_i^x, R_i^x, L_r^y, R_r^y \geq 0, \quad i \in O^{XNI}, \;\; r \in O^{YNI}.
\end{aligned} \tag{4.29}
$$

Given a DMU_o with $I(X_o, Y_o) = 0$, then $H(X_o, Y_o) = 0$ if and only if DMU_o is efficient. In other words, a DMU_o is efficient if and only if both $I(X_o, Y_o) = 0$ and $H(X_o, Y_o) = 0$.

Let $(\boldsymbol{s}^{x*}, \boldsymbol{s}^{y*}, \boldsymbol{\lambda}^{*})$ be the optimal solution of (4.28), and $\boldsymbol{L}^{x*}, \boldsymbol{R}^{x*}, \boldsymbol{L}^{y*}, \boldsymbol{R}^{y*}, \boldsymbol{\lambda}^{**}$ the optimal solution of (4.29) for a given DMU_o, then we can compute its input and output targets X_o^{target} and Y_o^{target} as

$$
\underline{x_{io}^{target}} = \underline{x_{io}} - \overline{s_i^{x*}} - R_i^{x*}, \qquad \overline{x_{io}^{target}} = \overline{x_{io}} - \underline{s_i^{x*}} - L_i^{x*}, \qquad i \in O^X, \tag{4.30}
$$

$$
\underline{y_{ro}^{target}} = \underline{y_{ro}} + \underline{s_r^{y*}} + L_r^{y*}, \qquad \overline{y_{ro}^{target}} = \overline{y_{ro}} + \overline{s_r^{y*}} + R_r^{y*}, \qquad r \in O^Y. \tag{4.31}
$$

4.5 Numerical experiments

4.5.1 Small illustrative case

Let us go back to the small dataset of Example 4.3.1 again to illustrate the proposed approach step by step, as well as the need for Phase II for the efficiency characterization and the computation of the targets. Recall that there are six DMUs, with two inputs and a single constant output (see Table 4.1). All the variables are assumed to be integer intervals.

Table 4.1: Data for Case 4.5.1

DMU (j)	1	2	3	4	5	6
x_{1j}	$(11,13)$	$(14,15)$	$(16,17)$	$(18,20)$	$(19,20)$	$(18,19)$
x_{2j}	$(8,10)$	$(6,7)$	$(7,8)$	$(4,7)$	$(6,7)$	$(4,7)$
y_{1j}	$(10,10)$	$(10,10)$	$(10,10)$	$(10,10)$	$(10,10)$	$(10,10)$

Table 4.2: Results for Phases I & II, and DMU efficiency status classification for Case 4.5.1.

	DMU	1	2	3	4	5	6
Phase I	$I(X_o, Y_o)$	0.00	0.00	0.25	0.00	0.26	0.00
	s_1^x	$(0,0)$	$(0,0)$	$(2,2)$	$(0,0)$	$(5,5)$	$(0,0)$
	s_2^x	$(0,0)$	$(0,0)$	$(1,1)$	$(0,0)$	$(0,0)$	$(0,0)$
	s_1^y	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$
Phase II	$H(X_o, Y_o)$	0	0	0	1	0	0
	L_1^x	0	0	0	1	0	0
	R_1^x	0	0	0	0	0	0
	L_2^x	0	0	0	0	0	0
	R_2^x	0	0	0	0	0	0
	L_1^y	0	0	0	0	0	0
	R_1^y	0	0	0	0	0	0
	X_1^{target}	$(11,13)$	$(14,15)$	$(14,15)$	$(18,19)$	$(14,15)$	$(18,19)$
	X_2^{target}	$(8,10)$	$(6,7)$	$(6,7)$	$(4,7)$	$(6,7)$	$(4,7)$
	Y_1^{target}	$(10,10)$	$(10,10)$	$(10,10)$	$(10,10)$	$(10,10)$	$(10,10)$
	Eff. Status	efficient	efficient	inefficient	weakly efficient	inefficient	efficient

Among these six DMUs, there are three classified as efficient, two inefficient, and one weakly efficient case, as established in Definition 4.3.2. Below we show, using DMU 1 as an example, the model solved and the results of the phases of the proposed approach.

<u>*Phase I*</u>: The corresponding (PIIDEA) (4.23) problem for DMU_1 is

$$
\begin{aligned}
I(X_1, Y_1) = \quad & Max \quad \frac{\underline{s}_1^x + \overline{s}_1^x}{11+13} + \frac{\underline{s}_2^x + \overline{s}_2^x}{8+10} + \frac{\underline{s}_1^y + \overline{s}_1^y}{10+10} \\
& \text{s.t.} \quad \left.\begin{array}{lll} 11\lambda_1 + 14\lambda_2 + 16\lambda_3 + 18\lambda_4 + 19\lambda_5 + 18\lambda_6 & \leq & 11 - \overline{s}_1^x \\ 13\lambda_1 + 15\lambda_2 + 17\lambda_3 + 20\lambda_4 + 20\lambda_5 + 19\lambda_6 & \leq & 13 - \underline{s}_1^x \end{array}\right\} \; i = 1 \\
& \left.\begin{array}{lll} 8\lambda_1 + 6\lambda_2 + 7\lambda_3 + 4\lambda_4 + 6\lambda_5 + 4\lambda_6 & \leq & 8 - \overline{s}_2^x \\ 10\lambda_1 + 7\lambda_2 + 8\lambda_3 + 7\lambda_4 + 7\lambda_5 + 7\lambda_6 & \leq & 10 - \underline{s}_2^x \end{array}\right\} \; i = 2 \\
& \left.\begin{array}{lll} 10\lambda_1 + 10\lambda_2 + 10\lambda_3 + 10\lambda_4 + 10\lambda_5 + 10\lambda_6 & \geq & 10 + \overline{s}_1^y \\ 10\lambda_1 + 10\lambda_2 + 10\lambda_3 + 10\lambda_4 + 10\lambda_5 + 10\lambda_6 & \geq & 10 + \underline{s}_1^y \end{array}\right\} \; r = 1 \\
& \underline{s}_i^x \leq \overline{s}_i^x \quad i = 1, 2, \\
& \underline{s}_1^y \leq \overline{s}_1^y \\
& \lambda_j \geq 0, \quad j = 1, \ldots, 6, \\
& \underline{s}_i^x, \overline{s}_i^x, \underline{s}_1^y, \overline{s}_1^y \in \mathbb{Z}_+ \quad i = 1, 2
\end{aligned}
$$

The optimal solution of the above Linear Program (LP) is $(\boldsymbol{s}^{x*}, \boldsymbol{s}^{y*}, \boldsymbol{\lambda}^*) = \underline{s}_1^{x*} = 0, \overline{s}_1^{x*} = 0, \underline{s}_2^{x*} = 0,$

$\overline{s_2^{x*}} = 0, \underline{s_1^{y*}} = 0, \overline{s_1^{y*}} = 0, \lambda_1^* = 1, \lambda_2^* = 0, \lambda_3^* = 0, \lambda_4^* = 0, \lambda_5^* = 0, \lambda_6^* = 0$. As $I(X_1, Y_1) = 0$, it is a candidate to be an efficient DMU, but we cannot be sure yet. To confirm its efficiency status we need to solve the Phase II model below.

Phase II: Given the solution obtained in Phase I for DMU_1, specifically the slacks $(\boldsymbol{s}^{x*}, \boldsymbol{s}^{y*})$, the corresponding $(PIIDEA)_2$ model (4.24) is formulated as

$$
\begin{array}{rll}
H(X_1, Y_1) = & \text{Max} & L_1^x + R_1^x + L_2^x + R_2^x + L_1^y + R_1^y \\
& \text{s.t.} & \left.\begin{array}{lcl} 11\lambda_1 + 14\lambda_2 + 16\lambda_3 + 18\lambda_4 + 19\lambda_5 + 18\lambda_6 & \leq & 11 - R_1^x \\ 13\lambda_1 + 15\lambda_2 + 17\lambda_3 + 20\lambda_4 + 20\lambda_5 + 19\lambda_6 & \leq & 13 - L_1^x \end{array}\right\} \; i = 1 \\
& & \left.\begin{array}{lcl} 8\lambda_1 + 6\lambda_2 + 7\lambda_3 + 4\lambda_4 + 6\lambda_5 + 4\lambda_6 & \leq & 8 - R_2^x \\ 10\lambda_1 + 7\lambda_2 + 8\lambda_3 + 7\lambda_4 + 7\lambda_5 + 7\lambda_6 & \leq & 10 - L_2^x \end{array}\right\} \; i = 2 \\
& & \left.\begin{array}{lcl} 10\lambda_1 + 10\lambda_2 + 10\lambda_3 + 10\lambda_4 + 10\lambda_5 + 10\lambda_6 & \geq & 10 + R_1^y \\ 10\lambda_1 + 10\lambda_2 + 10\lambda_3 + 10\lambda_4 + 10\lambda_5 + 10\lambda_6 & \geq & 10 + L_1^y \end{array}\right\} \; r = 1 \\
& & \lambda_j \geq 0, \quad j = 1, \ldots, 6, \\
& & L_i^x, R_i^x, L_r^y, R_r^y \in \mathbb{Z}_+, \quad i = 1, 2, \;\; r = 1.
\end{array}
$$

The optimal solution of the above LP problem is $(\boldsymbol{L}^{x*}, \boldsymbol{R}^{x*}, \boldsymbol{L}^{y*}, \boldsymbol{R}^{y*}, \boldsymbol{\lambda}^{**}) = (L_1^{x*} = 0, R_1^{x*} = 0, L_2^{x*} = 0, R_2^{x*} = 0, L_1^{y*} = 0, R_1^{y*} = 0, \lambda_1^{**} = 1, \lambda_2^{**} = 0, \lambda_3^{**} = 0, \lambda_4^{**} = 0, \lambda_5^{**} = 0, \lambda_6^{**} = 0)$. The left and right slack variables $L_i^x, R_i^x, L_r^y, R_r^y$ represent the potential improvements that may remain and correspond to moving, if possible, towards the efficiency frontier. Only for efficient DMUs these variables are all null, as it happens for DMU_1.

In this case, the corresponding input and output targets, as per (4.25) and (4.26), coincide with those of the observed DMU, i.e.

$$\underline{x_{11}^{target}} = \underline{x_{11}} - \overline{s_1^{x*}} - R_1^{x*} = 11 - 0 - 0 = 11, \qquad \overline{x_{11}^{target}} = \overline{x_{11}} - \underline{s_1^{x*}} - L_1^{x*} = 13 - 0 - 0 = 13,$$

$$\underline{x_{21}^{target}} = \underline{x_{21}} - \overline{s_2^{x*}} - R_2^{x*} = 8 - 0 - 0 = 8, \qquad \overline{x_{21}^{target}} = \overline{x_{21}} - \underline{s_2^{x*}} - L_2^{x*} = 10 - 0 - 0 = 10,$$

$$\underline{y_{11}^{target}} = \underline{y_{11}} + \underline{s_1^{y*}} + L_1^{y*} = 10 + 0 + 0 = 10, \qquad \overline{y_{11}^{target}} = \overline{y_{11}} + \overline{s_1^{y*}} + R_1^{y*} = 10 + 0 + 0 = 10.$$

As it can be seen in Table 4.2, in the case of DMU_4, the solution of the Phase I is $I(X_4, Y_4) = 0$, similar to what happens for DMU_1, DMU_2 and DMU_6. Unlike them, however, for DMU_4, the Phase II solution $L_1^x = 1$ and $H(X_4, Y_4) = 1$ indicates that the upper limit of the first input of DMU_4 can be feasibly reduced by one unit and hence DMU_4 is not efficient.

4.5.2 Larger real-world application

In this section, we take a real problem, which not only is bigger but also includes both integer and continuous variables. This case may be found more often than the pure integer one in the real world.

The dataset considered comes from Majid Azadi et al. [23]. The original data are given as triangular fuzzy numbers. To adapt them as intervals we have considered the corresponding zero $\alpha-$levels. The DMUs correspond to 26 suppliers of raw materials with four crisp inputs

Table 4.3: Phase I results for Case 4.5.2. This is a hybrid problem. The second input and both outputs are integer, whereas the other three inputs are continuous.

o	$I(X_o, Y_o)$	Input slacks intervals				Output slacks intervals	
		s_1^x	$\mathbf{s_2^x}$	s_3^x	s_4^x	$\mathbf{s_1^y}$	$\mathbf{s_2^y}$
1	0.00	(0.00 , 0.00)	(0 , 0)	(0.00 , 0.00)	(0.00 , 0.00)	(0 , 0)	(0 , 0)
2	2.29	(0.44 , 0.44)	(0 , 0)	(6.70 , 6.70)	(0.00 , 0.00)	(0 , 0)	(75 , 75)
3	1.21	(0.00 , 0.00)	(1 , 1)	(40.09 , 40.09)	(0.00 , 0.00)	(0 , 6)	(59 , 62)
4	1.37	(0.00 , 0.00)	(53 , 53)	(0.00 , 0.00)	(12.06 , 12.06)	(0 , 2)	(78 , 79)
5	0.56	(169.13 , 169.13)	(0 , 0)	(2.37 , 2.37)	(0.10 , 0.10)	(0 , 10)	(33 , 37)
6	0.00	(0.00 , 0.00)	(0 , 0)	(0.00 , 0.00)	(0.00 , 0.00)	(0 , 0)	(0 , 0)
7	0.00	(0.00 , 0.00)	(0 , 0)	(0.00 , 0.00)	(0.00 , 0.00)	(0 , 0)	(0 , 0)
8	0.00	(0.00 , 0.00)	(0 , 0)	(0.00 , 0.00)	(0.00 , 0.00)	(0 , 0)	(0 , 0)
9	0.00	(0.00 , 0.00)	(0 , 0)	(0.00 , 0.00)	(0.00 , 0.00)	(0 , 0)	(0 , 0)
10	1.91	(0.05 , 0.05)	(39 , 39)	(7.19 , 7.19)	(0.08 , 0.08)	(0 , 0)	(143, 143)
11	1.03	(18.42 , 18.42)	(0 , 0)	(21.92 , 21.92)	(26.45 , 26.45)	(0 , 0)	(0 , 0)
12	1.14	(0.00 , 0.00)	(21 , 21)	(0.19, 0.19)	(16.16, 16.16)	(0 , 7)	(67 , 70)
13	1.37	(142.78 , 142.78)	(60 , 60)	(0.00 , 0.00)	(15.82 , 15.82)	(0 , 4)	(58 , 59)
14	2.31	(0.00 , 0.00)	(24 , 24)	(5.98 , 5.98)	(52.49 , 52.49)	(0 , 0)	(152 , 152)
15	3.61	(0.00 , 0.00)	(110 , 110)	(10.36 , 10.36)	(36.90 , 36.90)	(0 , 0)	(158 , 158)
16	0.00	(0.00 , 0.00)	(0 , 0)	(0.00 , 0.00)	(0.00 , 0.00)	(0 , 0)	(0 , 0)
17	0.00	(0.00 , 0.00)	(0 , 0)	(0.00 , 0.00)	(0.00 , 0.00)	(0 , 0)	(0 , 0)
18	6.94	(0.00 , 0.00)	(141 , 141)	(13.97 , 13.97)	(27.17 , 27.17)	(0 , 13)	(311 , 315)
19	0.00	(0.00 , 0.00)	(0 , 0)	(0.00 , 0.00)	(0.00 , 0.00)	(0 , 0)	(0 , 0)
20	2.79	(0.02 , 0.02)	(92 , 92)	(36.33 , 36.33)	(03.16 , 3.16)	(20 , 24)	(200 , 202)
21	1.96	(0.00 , 0.00)	(56 , 56)	(28.37 , 28.37)	(38.25, 38.25)	(0 , 0)	(74 , 74)
22	0.00	(0.00 , 0.00)	(0 , 0)	(0.00 , 0.00)	(0.00 , 0.00)	(0 , 0)	(0 , 0)
23	2.14	(0.00 , 0.00)	(56 , 56)	(13.03 , 13.03)	(24.06, 24.06)	(0 , 16)	(210 , 216)
24	0.95	(0.04 , 0.04)	(62 , 62)	(19.74 , 19.74)	(0.09 , 0.09)	(0 , 0)	(38 , 38)
25	3.90	(29.81 , 29.81)	(91, 91)	(0.03 , 0.03)	(2.52 , 2.52)	(36 , 45)	(252 , 255)
26	1.35	(0.06,0.06)	(58,58)	(11.52,11.52)	(18.33,18.33)	(0,4)	(50,52)

and two integer interval outputs. The inputs are the economic criteria given by the total cost of shipments (TC), and the number of shipments per month (NS) and the social criteria given by the eco-design cost (ED) and the cost of work safety and labor health (CS). Except for the NS input, the rest of the inputs are continuous variables. The two outputs are the number of shipments to arrive on time (NOT) and the number of bills received from the supplier without errors (NB). Both outputs are integer interval variables.

The results from the Phase I, model (4.28) (see 4.4), are shown in Table 4.3. The results of the Phase II model (4.29), as well as the input and output targets, which are interval variables, and the corresponding efficiency status are given in Table 4.4. As we can see in the table, all DMUs are classified as either efficient or inefficient, i.e., there are no weakly efficient DMUs in this case.

Table 4.5 compares the results of the proposed approach with the inefficiency scores and the corresponding targets when the integrality of the integer variables is ignored. These results correspond to relaxing the integrality of the corresponding input and output slacks in models (4.28) and (4.29), which are the hybrid equivalent of models (4.23) and (4.24). Because they are relaxations of the original models, they can compute slightly higher inefficiency scores. However, we claim that those results are not valid because they correspond to targets that, as shown in Table 4.5, do not always respect the integer character of some of the variables (the second input, and the two outputs in the current instance). On the contrary, the proposed approach considers both the integer and the interval-valued character of those variables.

For the sake of comparison, Table 4.6 also includes the results when other existing approaches are applied, in particular Kordrostami et al. [94], which also considers a hybrid case of integer and continuous variables. As we discussed before, these authors consider fuzzy data, whereas we consider that the uncertainty is given in terms of interval data. To apply their models we consider interval data as a particular case of trapezoidal fuzzy data (a, b, c, d), when $a = b$ and $c = d$. We do not include the results from their alternative model (3.9), since they are the same in the case of interval data.

As before discussed, Among the main differences between our approach and Kordrostami et al. [94], we mentioned that Kordrostami et al. [94] use a fuzzy ranking approach and get crisp targets (see last columns of Table 4.6), while we use integer interval arithmetic and compute integer interval targets. In addition, they use a radial oriented approach (while we apply an additive, non-oriented approach) and they use the integer PPS of Kuosmanen & Matin [98] (while we use a specific integer interval PPS).

In spite of these differences, analysing the results of both approaches, we can check that they are in good agreement. In particular, the corresponding efficient characterisations coincide. Thus, the efficient DMUs identified by the proposed approach, with both inefficient null values $I(X_o, Y_o) = H(X_o, Y_o) = 0$, have also an efficiency score of 1 with the Kordrostami et al. approach. And those inefficient DMUS, with $I(X_o, Y_o) > 0$, have an efficiency score less than the unity. Also, the Spearman rank-order correlation coefficient between both approaches is $\rho = -0.91$. Finally, regarding the targets for the efficient DMUs for the second input variables, which are not interval since the original input data was not interval, they coincide with Kordrostami et al.'s targets. Moreover, the integer (non-interval) output targets from Kordrostami et al.'s model are contained within the integer interval targets computed from the proposed approach.

4.6 Conclusions

This chapter presents a new integer and interval-valued DEA approach and associated slacks-based measure of inefficiency. It requires solving two crisp linear optimization models that allow the computation of the corresponding input and output targets, as well as determining the efficiency status of each DMU. Computational experiments have been presented to validate the proposed approach.

It has been shown that a null value of the Phase I inefficiency score is a necessary but not sufficient condition for efficiency, i.e. the Phase I model cannot discriminate between efficient and weakly efficient DMUs. This is analogous to what happens with radial DEA models in crisp cases although it does not happen in the slacks-based case. This highlights the differences between crisp and interval data scenarios. Hence the need for the Phase II model, which also provides efficient input and output targets.

The proposed approach can handle data that are simultaneously uncertain and integer. Existing interval DEA approaches do not consider integer data and, conversely, integer DEA approaches assume crisp data. Although at the cost of requiring interval arithmetic and relational operators, with a higher number of constraints in its parameterization form, the proposed approach is able to address the joint integer interval scenario. It does so in a rigorous way, defining the corresponding integer interval PPS, its corresponding efficient subset, and finally, formulating the models that compute the inefficiency scores and the efficient targets.

As regards potential research directions, we envisage extending the proposed integer interval arithmetic and LU-partial order approach to the data case with fuzzy integer intervals. The approach should be non-oriented and guarantee efficient (i.e. non-dominated) fuzzy targets. As a first step, the fuzzy integer interval DEA technology needs to be axiomatically derived. Another interesting line of research, often neglected in the fuzzy DEA literature, is that of applying this type of approaches to real-world situations.

Table 4.4: Results for Phase II, including targets and DMU efficiency status classification for Case 4.5.2. Second input and both outputs are integer, whereas the other three inputs are continuous.

o	$I(X_o, Y_o)$	$H(X_o, Y_o)$	Phase II slack variables												TARGETS						Efficiency Status
			Input s.v.								Output s.v.				TC	**NS**	ED	CS	**NOT**	**NB**	
			L_1^x	R_1^x	$\mathbf{L_2^x}$	$\mathbf{R_2^x}$	L_3^x	R_3^x	L_4^x	R_4^x	$\mathbf{L_1^y}$	$\mathbf{R_1^y}$	$\mathbf{L_2^y}$	$\mathbf{R_2^y}$	X_1^{target}	$\mathbf{X_2^{target}}$	X_3^{target}	X_4^{target}	$\mathbf{Y_1^{target}}$	$\mathbf{Y_2^{target}}$	
1	0.00	0	0	0	0	0	0	0	0	0	0	0	0	0	(316.00 , 316.00)	(251 , 251)	(61.00 , 61.00)	(18.00 , 18.00)	(199 , 239)	(76 , 90)	Eff.
2	2.29	4	0	0	0	0	0	0	0	0	3	0	1	0	(280.56 , 280.56)	(164 , 164)	(38.30 , 38.30)	(21.00 , 21.00)	(156 , 193)	(105 , 117)	Ineff.
3	1.21	0	0	0	0	0	0	0	0	0	0	0	0	0	(309.00 , 309.00)	(197 , 197)	(42.91 , 42.91)	(40.00 , 40.00)	(203 , 249)	(137 , 154)	Ineff.
4	1.37	0	0	0	0	0	0	0	0	0	0	0	0	0	(291.00 , 291.00)	(165 , 165)	(37.00 , 37.00)	(32.94 , 32.94)	(167, 209)	(163 , 178)	Ineff.
5	0.56	0	0	0	0	0	0	0	0	0	0	0	0	0	(427.87 , 427.87)	(178 , 178)	(49.63 , 49.63)	(28.90 , 28.90)	(197 , 247)	(196 , 214)	Ineff.
6	0.00	0	0	0	0	0	0	0	0	0	0	0	0	0	(341.00 , 341.00)	(142 , 142)	(19.00 , 19.00)	(33.00 , 33.00)	(129 , 169)	(129 , 143)	Eff.
7	0.00	0	0	0	0	0	0	0	0	0	0	0	0	0	(475.00 , 475.00)	(149 , 149)	(74.00 , 74.00)	(18.00 , 18.00)	(193 , 233)	(111 , 125)	Eff.
8	0.00	0	0	0	0	0	0	0	0	0	0	0	0	0	(254.00 , 254.00)	(172 , 172)	(53.00 , 53.00)	(35.00 , 35.00)	(134 , 174)	(250 , 264)	Eff.
9	0.00	0	0	0	0	0	0	0	0	0	0	0	0	0	(328.00 ,328.00)	(135 , 135)	(83.00 , 83.00)	(47.00 , 47.00)	(184 , 224)	(58 , 72)	Eff.
10	1.91	1	0	0	0	0	0	0	0	0	1	0	0	0	(309.95 , 309.95)	(134 , 134)	(33.81 , 33.81)	(15.92 , 15.92)	(114 , 153)	(231 , 245)	Ineff.
11	1.03	6	0	0	0	0	0	0	0	0	5	0	1	0	(302.58 , 302.58)	(121 , 121)	(35.08 , 35.08)	(18.55 , 18.55)	(130 , 165)	(154 , 167)	Ineff.
12	1.14	0	0	0	0	0	0	0	0	0	0	0	0	0	(329.00 , 329.00)	(183 , 183)	(37.81 , 37.81)	(36.84 , 36.84)	(195 , 242)	(157 , 174)	Ineff.
13	1.37	0	0	0	0	0	0	0	0	0	0	0	0	0	(332.22 , 332.22)	(152 , 152)	(32.00 , 32.00)	(26.18 , 26.18)	(156 , 200)	(197 , 212)	Ineff.
14	2.31	0	0	0	0	0	0	0	0	0	0	0	0	0	(259.00 , 259.00)	(165 , 165)	(50.02 , 50.02)	(32.51 , 32.51)	(129 , 169)	(249 , 263)	Ineff.
15	3.61	8	0	0	0	0	0	0	0	0	6	0	2	0	(274.00 , 274.00)	(107 , 107)	(27.64 , 27.64)	(14.10 , 14.10)	(91 , 125)	(228 , 240)	Ineff.
16	0.00	0	0	0	0	0	0	0	0	0	0	0	0	0	(264.00 , 264.00)	(158 , 158)	(25.00 , 25.00)	(35.00 , 35.00)	(193 , 233)	(45 , 59)	Eff.
17	0.00	0	0	0	0	0	0	0	0	0	0	0	0	0	(327.00 , 327.00)	(124 , 124)	(32.00 , 32.00)	(16.00, 16.00)	(107 , 147)	(271 , 285)	Eff.
18	6.94	0	0	0	0	0	0	0	0	0	0	0	0	0	(429.00 , 429.00)	(166 , 166)	(43.03 , 43.03)	(21.83 , 21.83)	(142 , 195)	(357 , 375)	Ineff.
19	0.00	0	0	0	0	0	0	0	0	0	0	0	0	0	(262.00 , 262.00)	(138 , 138)	(25.00 , 25.00)	(31.00 , 31.00)	(122 , 162)	(173 , 187)	Eff.
20	2.79	0	0	0	0	0	0	0	0	0	0	0	0	0	(384.98 , 384.98)	(146 , 146)	(37.67 , 37.67)	(18.84 , 18.84)	(126 , 173)	(319 , 335)	Ineff.
21	1.96	1	0	0	0	0	0	0	0	0	1	0	0	0	(249.00 , 249.00)	(161 , 161)	(40.63 , 40.63)	(33.75 , 33.75)	(151 , 190)	(165 , 178)	Ineff.
22	0.00	0	0	0	0	0	0	0	0	0	0	0	0	0	(337.00 , 337.00)	(203 , 203)	(27.00 , 27.00)	(33.00 , 33.00)	(104 , 144)	(271 , 285)	Eff.
23	2.05	0	0	0	0	0	0	0	0	0	0	0	0	0	(365.00 , 365.00)	(236 , 236)	(71.97 , 71.97)	(49.94 , 49.94)	(185 , 241)	(353 , 373)	Ineff.
24	0.95	2	0	0	0	0	0	0	0	0	2	0	0	0	(295.96 , 295.96)	(123 , 123)	(29.26 , 29.26)	(17.91 , 17.91)	(114 , 152)	(216 , 229)	Ineff.
25	3.90	0	0	0	0	0	0	0	0	0	0	0	0	0	(398.19 , 398.19)	(151 , 151)	(38.97 , 38.97)	(19.48 , 19.48)	(130 , 179)	(330 ,347)	Ineff.
26	1.35	0	0	0	0	0	0	0	0	0	0	0	0	0	(326.94,326.94)	(160,160)	(31.48,31.48)	(29.67,29.67)	(173,217)	(163,179)	Ineff.

Table 4.5: Comparison of the results for Case 4.5.2, without integrality constraints.

	Hybrid case		Continuous case		TARGETS for continous case						
o	$I(X_o, Y_o)$	$H(X_o, Y_o)$	$I(X_o, Y_o)$	$H(X_o, Y_o)$	TC X_1^{target}	NS X_2^{target}	ED X_3^{target}	CS X_4^{target}	NOT Y_1^{target}	NB Y_2^{target}	Efficiency Status
1	0.00	0	0.00	0.00	(316.00, 316.00)	(251.00, 251.00)	(61.00, 61.00)	(18.00, 18.00)	(199.00, 239.00)	(76.00, 90.00)	Eff.
2	2.29	4	2.31	5.22	(281.00, 281.00)	(164.00, 164.00)	(38.31, 38.31)	(21.00, 21.00)	(156.86, 193.00)	(105.12, 117.76)	Ineff.
3	1.21	0	1.22	0.00	(309.00, 309.00)	(196.90, 196.90)	(42.91, 42.91)	(40.00, 40.00)	(203.00, 249.94)	(137.58, 154.01)	Ineff.
4	1.37	0	1.37	0.00	(291.00, 291.00)	(164.62, 164.62)	(37.00, 37.00)	(32.94, 32.94)	(167.00, 209.46)	(163.13, 177.99)	Ineff.
5	0.56	0	0.56	0.00	(426.42, 426.42)	(178.00, 178.00)	(49.32, 49.32)	(29.00, 29.00)	(197.00, 247.29)	(195.67, 213.27)	Ineff.
6	0.00	0	0.00	0.00	(341.00, 341.00)	(142.00, 142.00)	(19.00, 19.00)	(33.00, 33.00)	(129.00, 169.00)	(129.00, 143.00)	Eff.
7	0.00	0	0.00	0.00	(475.00, 475.00)	(149.00, 149.00)	(74.00, 74.00)	(18.00, 18.00)	(193.00, 233.00)	(111.00, 125.00)	Eff.
8	0.00	0	0.00	0.00	(254.00, 254.00)	(172.00, 172.00)	(53.00, 53.00)	(35.00, 35.00)	(134.00, 174.00)	(250.00, 264.00)	Eff.
9	0.00	0	0.00	0.00	(328.00, 328.00)	(135.00, 135.00)	(83.00, 83.00)	(47.00, 47.00)	(184.00, 224.00)	(58.00, 72.00)	Eff.
10	1.91	1	1.91	2.35	(310.00, 310.00)	(133.61, 133.61)	(33.67, 33.67)	(16.00, 16.00)	(114.74, 153.00)	(231.88, 245.27)	Ineff.
11	1.03	6	1.03	6.78	(302.58, 302.58)	(121.00, 121.00)	(35.08, 35.08)	(18.55, 18.55)	(130.02, 165.00)	(154.76, 167.00)	Ineff.
12	1.14	0	1.14	0.00	(329.00, 329.00)	(183.11, 183.11)	(38.00, 38.00)	(36.92, 36.92)	(195.00, 242.75)	(158.11, 174.83)	Ineff.
13	1.37	0	1.38	0.00	(330.82, 330.82)	(151.38, 151.38)	(32.00, 32.00)	(26.08, 26.08)	(156.00, 199.92)	(196.33, 211.70)	Ineff.
14	2.31	0	2.32	0.63	(259.00, 259.00)	(164.73, 164.73)	(50.08, 50.08)	(32.51, 32.51)	(129.47, 169.00)	(249.36, 263.19)	Ineff.
15	3.61	8	3.61	8.38	(274.00, 274.00)	(106.21, 106.21)	(27.67, 27.67)	(14.10, 14.10)	(91.21, 125.00)	(228.28, 240.11)	Ineff.
16	0.00	0	0.00	0.00	(264.00, 264.00)	(158.00, 158.00)	(25.00, 25.00)	(35.00, 35.00)	(193.00, 233.00)	(45.00, 59.00)	Eff.
17	0.00	0	0.00	0.00	(327.00, 327.00)	(124.00, 124.00)	(32.00, 32.00)	(16.00, 16.00)	(107.00, 147.00)	(271.00, 285.00)	Eff.
18	7.20	0	7.21	0.00	(429.00, 429.00)	(165.09, 165.09)	(42.88, 42.88)	(21.71, 21.71)	(142.00, 194.76)	(356.79, 375.26)	Ineff.
19	0.00	0	0.00	0.00	(262.00, 262.00)	(138.00, 138.00)	(25.00, 25.00)	(31.00, 31.00)	(122.00, 162.00)	(173.00, 187.00)	Eff.
20	2.79	0	2.79	0.00	(385.00, 385.00)	(145.99, 145.99)	(37.68, 37.68)	(18.84, 18.84)	(125.98, 173.07)	(319.07, 335.55)	Ineff.
21	1.96	1	1.97	1.85	(249.00, 249.00)	(160.88, 160.88)	(40.75, 40.75)	(33.80, 33.80)	(151.37, 190.00)	(165.05, 178.57)	Ineff.
22	0.00	0	0.00	0.00	(337.00, 337.00)	(203.00, 203.00)	(27.00, 27.00)	(33.00, 33.00)	(104.00, 144.00)	(271.00, 285.00)	Eff.
23	2.05	0	2.14	0.00	(365.00, 365.00)	(235.92, 235.92)	(71.98, 71.98)	(46.94, 46.94)	(185.00, 241.15)	(353.38, 373.04)	Ineff.
24	0.95	2	0.95	3.46	(296.00, 296.00)	(122.55, 122.55)	(29.10, 29.10)	(18.00, 18.00)	(114.56, 152.00)	(216.19, 229.29)	Ineff.
25	3.90	0	3.90	0.00	(398.53, 398.53)	(151.12, 151.12)	(39.00, 39.00)	(19.50, 19.50)	(130.41, 179.16)	(330.28, 347.34)	Ineff.
26	1.35	0	1.36	0.00	(327.00, 327.00)	(159.84, 159.84)	(31.48, 31.48)	(29.67, 29.67)	(173.00, 217.77)	(163.42, 179.09)	Ineff.

Table 4.6: Inefficiency measurements (Phases I and II) for Case 4.5.2, compared to the Efficiency score and integer input and output targets from Kordrostami et al. [94], see their model (3.8).

	proposed approach					Kordrostami et al. [94]			
o	$I(X_o, Y_o)$	$H(X_o, Y_o)$	X_2^{target}	Y_1^{target}	Y_2^{target}	Efficiency	X_2^{target}	Y_1^{target}	Y_2^{target}
1	0.00	0	(251, 251)	(199, 239)	(76, 90)	1.00	251	219	83
2	2.29	4	(164, 164)	(156, 193)	(105, 117)	0.95	155	173	64
3	1.21	0	(197, 197)	(203, 249)	(137, 154)	0.93	177	223	85
4	1.37	0	(165, 165)	(167, 209)	(163, 178)	0.84	151	187	92
5	0.56	0	(178, 178)	(197, 247)	(196, 214)	0.95	169	217	170
6	0.00	0	(142, 142)	(129, 169)	(129, 143)	1.00	142	149	136
7	0.00	0	(149, 149)	(193, 233)	(111, 125)	1.00	149	213	118
8	0.00	0	(172, 172)	(134, 174)	(250, 264)	1.00	172	154	257
9	0.00	0	(135, 135)	(184, 224)	(58, 72)	1.00	135	204	65
10	1.91	1	(134, 134)	(114, 153)	(231, 245)	0.83	143	133	95
11	1.03	6	(121, 121)	(130, 165)	(154, 167)	0.96	116	145	160
12	1.14	0	(183, 183)	(195, 242)	(157, 174)	0.85	172	215	97
13	1.37	0	(152, 152)	(156, 200)	(197, 212)	0.78	154	176	146
14	2.31	0	(165, 165)	(129, 169)	(249, 263)	0.79	129	149	104
15	3.61	8	(107, 107)	(91, 125)	(228, 240)	0.53	91	105	75
16	0.00	0	(158, 158)	(193, 233)	(45, 59)	1.00	158	213	52
17	0.00	0	(124, 124)	(107, 147)	(271, 285)	1.00	124	127	278
18	7.20	0	(166, 166)	(142, 195)	(357, 375)	0.49	137	162	53
19	0.00	0	(138, 138)	(122, 162)	(173, 187)	1.00	138	142	180
20	2.79	0	(146, 146)	(126, 173)	(319, 335)	0.60	140	128	126
21	1.96	1	(161, 161)	(151, 190)	(165, 178)	0.91	141	170	97
22	0.00	0	(203, 203)	(104, 144)	(271, 285)	1.00	203	124	278
23	2.05	0	(236, 236)	(185, 241)	(353, 373)	0.77	179	205	150
24	0.95	2	(123, 123)	(114, 152)	(216, 229)	0.89	141	132	184
25	3.90	0	(151, 151)	(130, 179)	(330, 347)	0.63	110	114	85
26	1.35	0	(160, 160)	(173, 217)	(163, 179)	0.80	163	193	120

Chapter 5

Fuzzy integer DEA

5.1 Introduction

In this chapter, fuzzy DEA approach that can handle continuous and integer data, i.e., variables that are constrained hybrid fuzzy data, is presented.

As mentioned before, Kordrostami et al. [94] is first introduced the integer DEA model of Jie et al. [81]. Then, we extend our previous work in Arana-Jimenez et al. [18] from interval integer DEA to fuzzy integer DEA. We consider that inputs and outputs can be either trapezoidal fuzzy integer numbers (TF_Z) or trapezoidal (real-valued) fuzzy numbers (tF_C). While [94] applies the crisp integer PPS of Kuosmanen and Kazemi Matin [98], we define a new hybrid fuzzy PPS. More importantly, different from [94], which uses a fuzzy ranking approach, in this paper, we introduce a new partial order relationship using hybrid fuzzy orders, along with hybrid fuzzy arithmetic. Also, while [94] uses a radial input-oriented approach, we use an additive, non-oriented approach and while [94] calculates a crisp target, the approach proposed in this paper aims at a fuzzy target.

5.2 Crisp production possibility set and slacks-based additive model

Let us assume a set of n DMUs, $j \in J = \{1, \dots, n\}$, in which each DMU_j consume m inputs $X_j = (x_{1j}, \dots, x_{mj}) \in \mathbb{R}^m$ to produces s outputs $Y_j = (y_{1j}, \dots, y_{sj}) \in \mathbb{R}^s$. In the conventional (i.e. crisp) DEA approach ([31]), the production possibility set (PPS) or DEA technology, defined by T, is derived using axioms Envelopment, Free disposability, Convexity, and Scalability introduced in Subsection 2.5.1.

Let us recall that the DEA PPS can be mathematically expressed as

$$T_{DEA} = \left\{ (X, Y) \in \mathbb{R}_+^{m+s} : X \geq \sum_{j=1}^{n} \lambda_j X_j, Y \leq \sum_{j=1}^{n} \lambda_j Y_j, \lambda_j \geq 0 \right\}.$$

Also, a certain DMU_0 is said to be efficient if and only if it is non-dominated, i.e. if for any $(X, Y) \in T_{DEA}$ such that $X \leqq X_o$ and $Y \geqq Y_o$, then $(X, Y) = (X_o, Y_o)$. This can be checked solving the normalized slacks-based additive model 4.1.

5.3 Proposed fuzzy integer PPS and slacks-based fuzzy integer DEA model

Arana-Jiménez et al. [18] studied the hybrid scenario for interval inputs and outputs, that is, with integer and continuous interval data. As an extension of integer intervals to fuzzy integer numbers, we can consider the hybrid scenario, containing both trapezoidal fuzzy integer numbers ($TF_{\mathbb{Z}}$) and trapezoidal fuzzy numbers (TF_C). Following Lozano and Villa [111], each index set is separated into two subsets, one of them for integer numbers and another for continuous numbers. O^{XI} and O^{XNI} are the index sets for fuzzy integer input variables and fuzzy input variables, respectively and O^{YI} and O^{YNI} are the index sets for fuzzy integer output variables and fuzzy output variables, respectively, with $|O^{XI}|+|O^{XNI}| = m$, $|O^{YI}|+|O^{YNI}| = s$, $O^X = O^{XI} \cup O^{XNI} = \{1, \ldots, m\}$, $O^Y = O^{YI} \cup O^{YNI} = \{1, \ldots, s\}$. Consider inputs $X = (X^I, X^{NI})$ and outputs $Y = (Y^I, Y^I)$, such that $(X, Y) = (X^I, X^{NI}, Y^I, Y^{NI}) \in (TF_{\mathbb{Z}})^{|O^{XI}|} \times (TF_C)^{|O^{XNI}|} \times (TF_{\mathbb{Z}})^{|O^{YI}|} \times (TF_C)^{|O^{YNI}|}$.

Let us suppose a set of n DMUs, $j \in J = \{1, \ldots, n\}$, in which every DMU_j consumes m inputs denoted by $X_j = (x_{1j}, \ldots, x_{mj}) \in (TF_{\mathbb{Z}})^{|O^{XI}|} \times (TF_C)^{|O^{XNI}|}$, with $x_{ij} = (x_{ij1}, x_{ij2}, x_{ij3}, x_{ij4})_{\mathbb{Z}} \in TF_{\mathbb{Z}}$ for $i \in O^{XI}$, $x_{ij} = (x_{ij1}, x_{ij2}, x_{ij3}, x_{ij4}) \in TF_C$ for $i \in O^{XNI}$ and produces s outputs denoted by $Y_j = (y_{1j}, \ldots, y_{sj}) \in (TF_{\mathbb{Z}})^{|O^{YI}|} \times (TF_C)^{|O^{YNI}|}$, with $y_{rj} = (y_{rj1}, y_{rj2}, y_{rj3}, y_{rj4}))_{\mathbb{Z}} \in TF_{\mathbb{Z}}$ for $r \in O^{YI}$, $y_{rj} = (y_{rj1}, y_{rj2}, y_{rj3}, y_{rj4})) \in TF_C$ for $r \in O^{YNI}$.

As an extension of the axioms given in [18] for integer intervals, let us introduce the following axioms, which correspond to axioms introduced in Subsection 2.5.1, but considering fuzzy integer inputs and outputs and utilizing the corresponding partial orders introduced in Definitions 2.3.4 and 2.3.7. Recall that T represents the PPS, i.e. the DEA technology (fuzzy integer DEA technology in this case), which contains all the feasible operating points.

(B1) *Envelopment:* $(X_j, Y_j) \in T$, *for all* $j \in J$.

(B2) *Free disposability:* $(X, Y), (\hat{X}, \hat{Y}) \in T$, *such that* $\hat{X} \geqq X$, $\hat{Y} \leqq Y \;\Rightarrow\; (\hat{X}, \hat{Y}) \in T$.

(B3) *Convexity:* $(X, Y), (\hat{X}, \hat{Y}) \in T$, $\epsilon \in [0, 1]$, *such that* $\epsilon(C(X^I), X^{NI}, C(Y^I), Y^{NI}) + (1-\epsilon)(C(\hat{X}^I), \hat{X}^{NI}, C(\hat{Y}^I), \hat{Y}^{NI}) \in (TF_{C\to\mathbb{Z}})^{|O^{XI}|} \times (TF_C)^{|O^{XNI}|} \times (TF_{C\to\mathbb{Z}})^{|O^{YI}|} \times (TF_C)^{|O^{YNI}|}$ $\Rightarrow (\hat{\hat{X}}, \hat{\hat{Y}}) = (\mathbb{Z}(\epsilon C(X^I) + (1-\epsilon)C(\hat{X}^I)), \epsilon X^{NI} + (1-\epsilon)\hat{X}^{NI}, \mathbb{Z}(\epsilon C(Y^I) + (1-\epsilon)C(\hat{Y}^I)), \epsilon Y^{NI} + (1-\epsilon)\hat{Y}^{NI}) \in T$.

(B4) *Scalability:* $(X, Y) \in T$, $\epsilon \geq 0$, *and* $\epsilon(C(X^I), X^{NI}, C(Y^I), Y^{NI}) \in (TF_{C\to\mathbb{Z}})^{|O^{XI}|} \times (TF_C)^{|O^{XNI}|} \times (TF_{C\to\mathbb{Z}})^{|O^{YI}|} \times (TF_C)^{|O^{YNI}|} \Rightarrow (\hat{\hat{X}}, \hat{\hat{Y}}) = (\mathbb{Z}(\epsilon C(X^I)), \epsilon X^{NI}, \mathbb{Z}(\epsilon C(Y^I)), \epsilon Y^{NI})) \in T$

In a similar manner, as an extension of the PPS given in [18] for integer intervals, let us introduce the following fuzzy integer production possibility set:

$$T_{FIDEA} = \Big\{(X, Y) = (X^I, X^{NI}, Y^I, Y^{NI}) \in (TF_{\mathbb{Z}})^{|O^{XI}|} \times (TF_C)^{|}O^{XNI}| \times (TF_{\mathbb{Z}})^{|O^{YI}|} \times (TF_C)^{|O^{YNI}|},$$
$$\text{s.t. } \Big(C(X^I), X^{NI}\Big) \geqq \sum_{j=1}^{N} \lambda_j \Big(C(X_j^I), X_j^{NI}\Big), \text{ and}$$
$$\Big(C(Y^I), Y^{NI}\Big) \leqq \sum_{j=1}^{N} \lambda_j \Big(C(Y_j^I), Y_j^{NI}\Big), \lambda_j \geq 0, \forall j \;\Big\}$$

Associated to T_{FIDEA}, the following fuzzy integer DEA (FIDEA) model, which is an extension of the slacks-based additive model (4.1), is proposed:

$$
\begin{aligned}
\text{(FIDEA)}\ \ I(X_o, Y_o) = \quad \text{Max} \quad & \sum_{i=1}^{m} \frac{s^x_{i1} + s^x_{i2} + s^x_{i3} + s^x_{i4}}{x_{io1} + x_{io2} + x_{io3} + x_{io4}} + \sum_{r=1}^{s} \frac{s^y_{r1} + s^y_{r2} + s^y_{r3} + s^y_{r4}}{y_{ro1} + y_{ro2} + y_{ro3} + y_{ro4}} \qquad (5.1)\\
\text{s.t.} \quad & \sum_{j=1}^{n} \lambda_j C(x_{ij}) \leqq C(x_{io}) - C(s^x_i), \quad i \in O^{XI},\\
& \sum_{j=1}^{n} \lambda_j x_{ij} \leqq x_{io} - s^x_i, \quad i \in O^{XNI},\\
& \sum_{j=1}^{n} \lambda_j C(y_{rj}) \geqq C(y_{ro}) + C(s^y_r), \quad r \in O^{YI},\\
& \sum_{j=1}^{n} \lambda_j y_{rj} \geqq y_{ro} + s^y_r, \quad r \in O^{YNI},\\
& \lambda_j \geq 0, \quad j = 1, \ldots, n,\\
& s^x_i, s^y_r \in TF_{\mathbb{Z}_+}, \quad i \in O^{XI},\ \ r \in O^{YI},\\
& s^x_i, s^y_r \in TF_{C_+}, \quad i \in O^{XNI},\ \ r \in O^{YNI}.
\end{aligned}
$$

A feasible solution for (FIDEA) is denoted by $(\boldsymbol{s}^{x*}, \boldsymbol{s}^{y*}, \boldsymbol{\lambda}^*)$, where $\boldsymbol{s}^{x*} = (s^{x*}_1, \ldots, s^{x*}_n) \in (TF_{\mathbb{Z}})^{|O^{XI}|} \times (TF_C)^{|O^{XNI}|}$, $\boldsymbol{s}^{y*} = (s^{y*}_1, \ldots, s^{y*}_s) \in (TF_{\mathbb{Z}})^{|O^{YI}|} \times (TF_C)^{|O^{YNI}|}$, and $\boldsymbol{\lambda}^* = (\lambda^*_1, \ldots, \lambda^*_n) \in \mathbb{R}^n$. Moreover, its objective function is a real number, i.e. $I(X_o, Y_o) \in \mathbb{R}$.

Definition 5.3.1. *A DMU_o is efficient if and only if $(X, Y) \in T_{FIDEA}$, $X \leqq X_o$ and $Y \geqq Y_o$ implies $(X, Y) = (X_o, Y_o)$.*

Efficient DMUs have a null inefficiency measure in the FIDEA model (5.1) , i.e.

Theorem 5.3.1. *If DMU_o is efficient, then $I(X_o, Y_o) = 0$.*

Proof. Assume that $I(X_o, Y_o) > 0$, with $(\boldsymbol{s}^{x*}, \boldsymbol{s}^{y*}, \boldsymbol{\lambda}^*)$ an optimal solution for (FIDEA). Let $X^* = (x^*_1, \ldots, x^*_m) \in (TF_{\mathbb{Z}})^{|O^{XI}|} \times (TF_C)^{|O^{XNI}|}$, where $x^*_i = x_{io} - s^{x*}_i = x_{io1} - s^{x*}_{i4}, x_{io2} - s^{x*}_{i3}, x_{io3} - s^{x*}_{i2}, x_{io4} - s^{x*}_{i1}$ for each $i = 1, \ldots, m$. And let $y^* = (y^*_1, \ldots, y^*_s) \in (TF_{\mathbb{Z}})^{|O^{YI}|} \times (TF_C)^{|O^{YNI}|}$, where $y^*_r = y_{ro} + s^{y*}_r = (y_{ro1} + s^{y*}_{r1}, y_{ro2} + s^{y*}_{r2}, y_{ro3} + s^{y*}_{r3}, y_{ro4} + s^{y*}_{r4})$ for each $r = 1, \ldots, s$.

By the model constraints,

$$
(C(X^{I^*}), X^{NI^*}) \geqq \sum_{j=1}^{n} \lambda^*_j (C(X^I_j), X^{NI}_j),
$$
$$
(C(Y^{I^*}), Y^{NI^*}) \leqq \sum_{j=1}^{n} \lambda^*_j (C(Y^I_j), Y^{NI}_j).
$$

and therefore, $(X^*, Y^*) \in T_{FIDEA}$. Also, $X^* \leqq X_o$ and $Y^* \geqq Y_o$.

If $I(X_o, Y_o) > 0$, then $(\boldsymbol{s}^{x*}, \boldsymbol{s}^{y*}) \neq 0$, i.e., $\boldsymbol{s}^{x*} \geqq 0$ and $\boldsymbol{s}^{y*} \geqq 0$ but $s^{x*}_{i_0} \neq 0$ for some i_0, or/and $s^{y*}_{r_0} \neq 0$ for some r_0. In the first case, it must happen that $s^{x*}_{i_0 4} > 0$ and hence $X^* \leqq X_o$, with

$X^* \neq X_o$. Therefore, as per Definition 5.3.1, DMU_o is not efficient, thus incurring in contradiction. Similarly, in the second case, it must happen that $s^{y*}_{r_01} > 0$ and hence $Y^* \geqq Y_o$, with $Y^* \neq Y_o$, implying, as per Definition 5.3.1, that DMU_o is not efficient, which is a contradiction. □

Let us rewrite the above model in parameterized form by utilizing the arithmetic operations and partial order relations defined in the previous section as follows:

$$
\begin{aligned}
\text{(PFIDEA)}\ I(X_0, Y_0) = \quad \text{Max} \quad & \sum_{i=1}^{m} \frac{s^x_{i1} + s^x_{i2} + s^x_{i3} + s^x_{i4}}{x_{io1} + x_{io2} + x_{io3} + x_{io4}} + \sum_{r=1}^{s} \frac{s^y_{r1} + s^y_{r2} + s^y_{r3} + s^y_{r4}}{y_{ro1} + y_{ro2} + y_{ro3} + y_{ro4}} \qquad (5.2) \\
\text{s.t.} \quad & \sum_{j=1}^{n} \lambda_j x_{ijk} \leq x_{iok} - s^x_{i(5-k)}, \quad i \in O^X,\ k = 1,2,3,4, \\
& \sum_{j=1}^{n} \lambda_j y_{rjk} \geq y_{rok} + s^y_{rk}, \quad r \in O^Y,\ k = 1,2,3,4, \\
& s^x_{ik} \leq s^x_{i(k+1)}, \quad i \in O^X,\ k = 1,2,3, \\
& s^y_{rk} \leq s^y_{r(k+1)}, \quad r \in O^Y,\ k = 1,2,3, \\
& s^x_{ik}, s^y_{rk} \in \mathbb{Z}_+, \quad i \in O^{XI},\ r \in O^{YI},\ \ k = 1,2,3,4, \\
& s^x_{ik}, s^y_{rk} \in \mathbb{R}_+, \quad i \in O^{XNI},\ r \in O^{YNI},\ \ k = 1,2,3,4, \\
& \lambda_j \geq 0, \quad j = 1, \ldots, n.
\end{aligned}
$$

The first two sets of constraints are just the transformation of the corresponding input and outputs constraints from the model (5.1), according to the partial order relations for fuzzy integer numbers, considering in Definition 2.2.4. The third and fourth of constraints certify that the slacks are trapezoidal fuzzy numbers (integer or continuous, depending on the index set).

The following proposition represents the relationship between the (FIDEA) and (PFIDEA) solutions.

Proposition 5.3.1. *$(\boldsymbol{s}^{x*}, \boldsymbol{s}^{y*}, \boldsymbol{\lambda}^*)$ with $\boldsymbol{s}^{x*} \in (TF_{\mathbb{Z}})^{|O^{XI}|} \times (TF_C)^{|O^{XNI}|}$, $\boldsymbol{s}^{y*} \in (TF_{\mathbb{Z}})^{|O^{YI}|} \times (TF_C)^{|O^{YNI}|}$ and $\boldsymbol{\lambda}^* \in \mathbb{R}^N_+$ is an optimal solution of (FIDEA) if and only if its corresponding parameterization $(s^{x*}_{11}, s^{x*}_{12}, s^{x*}_{13}, s^{x*}_{14} \cdots s^{x*}_{m1}, s^{x*}_{m2}, s^{x*}_{m3}, s^{x*}_{m4}, s^{y*}_{11}, s^{y*}_{12}, s^{y*}_{13}, s^{y*}_{14} \cdots s^{y*}_{s1}, s^{y*}_{s2},$
$s^{y*}_{s3}, s^{y*}_{s4},, \lambda^*_1, \ldots, \lambda^*_n)$ is an optimal solution of (PFIDEA).*

Proof. Definitions 2.3.4 and **??** imply that the constraint sets in (FIDEA) (5.1) are equivalent to the constraint sets in (PFIDEA) (5.2). The rest of the proof is straightforward. □

Although a null inefficiency score, i.e. $I(X_o, Y_o) = 0$, is a necessary and sufficient condition in the crisp model (4.1), it is not sufficient to guarantee the efficiency of DMU_o in the fuzzy integer case. Theorem 5.3.1 only establishes it as a necessary condition. Therefore, given an optimal solution for (5.2) $(\boldsymbol{s}^{x*}, \boldsymbol{s}^{y*}, \boldsymbol{\lambda}^*)$, we need to formulate the following Phase II model to exhaust all remaining input and output slacks.

$$
\begin{aligned}
(PFIDEA)_2 \;\; H(X_0,Y_0) = \quad & \text{Max} \;\; \sum_{i=1}^{m} \frac{v_{i1}^x + v_{i2}^x + v_{i3}^x + v_{i4}^x}{x_{io1}+x_{io2}+x_{io3}+x_{io4}} + \sum_{r=1}^{s} \frac{v_{r1}^y + v_{r2}^y + v_{r3}^y + v_{r4}^y}{y_{ro1}+y_{ro2}+y_{ro3}+y_{ro4}} \qquad (5.3)\\
& \text{s.t.} \;\; \sum_{j=1}^{n} \lambda_j x_{ijk} \leq x_{iok} - s_{i(5-k)}^{x*} - v_{i(5-k)}^{x}, \quad i \in O^X,\ k = 1,2,3,4,\\
& \sum_{j=1}^{n} \lambda_j y_{rjk} \geq y_{rok} + s_{rk}^{y*} + v_{ik}^{y}, \quad r \in O^Y,\ k = 1,2,3,4,\\
& v_{ik}^x, v_{rk}^y \in \mathbb{Z}_+, \quad i \in O^{XI},\ r \in O^{YI},\ k = 1,2,3,4,\\
& v_{ik}^x, v_{rk}^y \in \mathbb{R}_+, \quad i \in O^{XNI},\ r \in O^{YNI}\ k = 1,2,3,4,\\
& \lambda_j \geq 0, \quad j = 1,\ldots,n.
\end{aligned}
$$

Theorem 5.3.2. *Given a DMU_o with $I(X_o,Y_o) = 0$, then $H(X_o,Y_o) = 0$ if and only if DMU_o is efficient.*

Proof. For a maximizing problem with non-negative variables like (5.2), if $I(X_o,Y_o) = 0$ then $s_{i1}^{x*} = s_{i2}^{x*} = s_{i3}^{x*} = s_{i4}^{x*} = s_{r1}^{y*} = s_{r2}^{y*} = s_{r3}^{y*} = s_{r4}^{y*} = 0$, for all $i \in O^X$, and $r \in O^Y$. For the same reason, if $H(X_o,Y_o) = 0$ then $v_{i1}^{x*} = v_{i2}^{x*} = v_{i3}^{x*} = v_{i4}^{x*} = v_{r1}^{y*} = v_{r2}^{y*} = v_{r3}^{y*} = v_{r4}^{y*} = 0$, for all $i \in O^X$, and $r \in O^Y$.

Now let us assume that DMU_o is not efficient. That means that there exists $(X^*,Y^*) \in T_{FIDEA}$ such that $X^* \leqq X_o$ and $Y^* \geqq Y_o$ with $(X^*,Y^*) \neq (X_o,Y_o)$, implying that $x_{i_0}^* \leqq x_{i_0o}$ and $y_{r_0}^* \geqq y_{r_0o}$ and $x_{i_0}^* \neq x_{i_0o}$ for some $i_0 \in O^X$, or $y_{r_0}^* \neq y_{r_0o}$ for some $r_0 \in O^Y$. In any case, we can obtain a new feasible solution of $(PFIDEA)_2$ $(\hat{\boldsymbol{v}}^x, \hat{\boldsymbol{v}}^y, \boldsymbol{\lambda})$ such that

$$
\begin{aligned}
\hat{v}_{ik}^x &= x_{iok} - x_{ik}^* \geq 0 \quad i \in O^X,\ k = 1,2,3,4,\\
\hat{v}_{rk}^y &= y_{rk}^* - y_{rok} \geq 0 \quad r \in O^Y,\ k = 1,2,3,4.
\end{aligned}
$$

The feasibility of this solution of model $(PFIDEA)_2$ holds since $(X^*,Y^*) \in T_{FIDEA}$, and therefore, there exist $\lambda_j \geq 0$, $j = 1,\ldots,n$ such that

$$
\begin{aligned}
\sum_{j=1}^{n} \lambda_j x_{ijk} &\leq x_{iok} - s_{i(5-k)}^{x*} - \hat{v}_{i(5-k)}^{x} = x_{ik}^* \quad i \in O^X,\ k = 1,2,3,4,\\
\sum_{j=1}^{n} \lambda_j y_{rjk} &\geq y_{rok} + s_{rk}^{y*} + \hat{v}_{rk}^{y} = y_{rk}^* \quad r \in O^Y,\ k = 1,2,3,4.
\end{aligned}
$$

Moreover, the objective function of this feasible solution is

$$
\sum_{i=1}^{m} \frac{\hat{v}_{i1}^x + \hat{v}_{i2}^x + \hat{v}_{i3}^x + \hat{v}_{i4}^x}{x_{io1}+x_{io2}+x_{io3}+x_{io4}} + \sum_{r=1}^{s} \frac{\hat{v}_{r1}^y + \hat{v}_{r2}^y + \hat{v}_{r3}^y + \hat{v}_{r4}^y}{y_{ro1}+y_{ro2}+y_{ro3}+y_{ro4}} > 0
$$

Thus, we have found a feasible solution with an objective function value larger than the supposed optimal value $H(X_o,Y_o) = 0$, thus reaching a contradiction. □

Let $(\boldsymbol{s}^{x*}, \boldsymbol{s}^{y*}, \boldsymbol{\lambda}^*)$ be the optimal solution for (5.2) and let $(\boldsymbol{v}^{x*}, \boldsymbol{v}^{y*}, \boldsymbol{\lambda}^{**})$ be the optimal solution for (5.3) for a given DMU_o. we can calculate the corresponding fuzzy input and output targets X_o^{target} and Y_o^{target} as

$$x_{iok}^{target} = x_{iok} - s_{i(5-k)}^{x*} - v_{i(5-k)}^{x*}, \qquad i \in O^X, \qquad k = 1,2,3,4, \tag{5.4}$$

$$y_{rok}^{target} = y_{rok} + s_{rk}^{y*} + v_{rk}^{y*}, \qquad r \in O^Y, \qquad k = 1,2,3,4. \tag{5.5}$$

Theorem 5.3.3. $(X_o^{target}, Y_o^{target})$ *is efficient.*

Proof. $(X_o^{target}, Y_o^{target}) \in T_{FIDEA}$ by the constraints of (5.3). Assume that $(X_o^{target}, Y_o^{target})$ is not efficient. Then, there exists $(\hat{X}, \hat{Y}) \in T_{FIDEA}$ such that $\hat{X} \leqq X_o^{target}$ and $\hat{Y} \geqq Y_o^{target}$, with $(\hat{X}, \hat{Y}) \neq (X_o^{target}, Y_o^{target})$. This implies that there exist $\hat{\lambda}_j \geq 0$, $j = 1, \ldots, n$ such that

$$(C(\hat{X}^I), \hat{X}^{NI}) \geqq \sum_{j=1}^{n} \hat{\lambda}_j (C(X_j^I), X_j^{NI}),$$
$$(C(\hat{Y}^I), \hat{Y}^{NI}) \leqq \sum_{j=1}^{n} \hat{\lambda}_j (C(Y_j^I), Y_j^{NI}).$$

which is equivalent to

$$\hat{x}_{ik} \geq \sum_{j=1}^{n} \hat{\lambda}_j x_{ijk}, \quad i \in O^X, k = 1,2,3,4,$$
$$\hat{y}_{rk} \leq \sum_{j=1}^{n} \hat{\lambda}_j y_{rjk}, \quad r \in O^Y, k = 1,2,3,4.$$

Moreover,

$$\hat{x}_{ik} \leq x_{iok}, \quad i \in O^X, k = 1,2,3,4,$$
$$\hat{y}_{rk} \geq y_{rok}, \quad r \in O^Y, k = 1,2,3,4.$$

with at least one of the above inequalities holding strict. It follows that

$$\sum_{j=1}^{n} \hat{\lambda}_j x_{ijk} \leq x_{iok} - s_{i(5-k)}^{x*} - v_{i(5-k)}^{x*} \quad i \in O^X, k = 1,2,3,4,$$
$$\sum_{j=1}^{n} \hat{\lambda}_j y_{rjk} \geq y_{rok} + s_{rk}^{y*} + v_{rk}^{y*} \quad r \in O^Y, k = 1,2,3,4.$$

with, again, at least one of the inequalities holding strict. Defining

$$v_{i(5-k)}^{x**} = x_{iok} - s_{i(5-k)}^{x*} - \sum_{j=1}^{n} \hat{\lambda}_j x_{ijk} \geq v_{i(5-k)}^{x*} \quad i \in O^X, k = 1,2,3,4,$$
$$v_{rk}^{y**} = \sum_{j=1}^{n} \hat{\lambda}_j y_{rjk} - y_{rok} - s_{rk}^{y*} \geq v_{rk}^{y*} \quad r \in O^Y, k = 1,2,3,4.$$

Since at least one of the above inequalities holds strict, we would have a feasible solution of model (5.3), namely $(\boldsymbol{v}^{x**}, \boldsymbol{v}^{y**}, \hat{\lambda})$, with a larger objective function value than the supposed optimum, which is a contradiction.

$\square$

Definition 5.3.2. *Given the optimal objective function values $I(X_o, Y_o)$ of model (5.2) and $H(X_o, Y_o)$ of model (5.3), DMUo is*

(i) efficient if $I(X_o, Y_o) = 0$ and $H(X_o, Y_o) = 0$,

(ii) inefficient if $I(X_o, Y_o) > 0$ or $H(X_o, Y_o) > 0$,

(iii) weakly efficient if $I(X_o, Y_o) = 0$ and $H(X_o, Y_o) > 0$.

Observe from the previous definition that if a given a *DMUo* is not efficient, then $I(X_o, Y_o) + H(X_o, Y_o) > 0$, which is consistent with Theorem 5.3.2. Furthermore, this suggest ranking the DMUs based on their overall efficiency score $I(X_o, Y_o) + H(X_o, Y_o)$.

5.4 Numerical experiments

In order to illustrate the proposed Fuzzy integer DEA approach, let us consider the dataset from Majid Azadi et al. [23], which involves 26 DMUs (suppliers of raw materials). There are four crisp inputs and two fuzzy integer outputs. The crisp inputs are the total cost of shipments *TC*, the number of shipments per month *NS*, the eco-design cost *ED*, and the cost of work safety and labour health *CS*. The *NS* input is an integer variable, while the rest of the inputs are continuous variables. Of the two fuzzy outputs, one output is the number of shipments to arrive on time, *NOT*, and the other is the number of bills received from the supplier without errors, *NB*. Both outputs are given as triangular fuzzy integer numbers, which are a particular case of trapezoidal fuzzy integer numbers where $a_2 = a_3$. Similarly, any crisp data a can be regarded as $(a, a, a, a) \in TF_C$ or TF_Z. .

Note that the same dataset was used and adapted in our previous work for the Integer Interval DEA approach (see [18]), using the support of the fuzzy triangular numbers as the corresponding integer interval output data. As indicated in the Introduction section, the current work represents the extension from integer interval data to integer fuzzy numbers, which are more powerful for modelling the uncertainty in real-world problems.

Note also that we have changed a bit the original dataset. Specifically, for the first half of the DMUs, we picked randomly the points a_2 and a_3 between the original extremes a_1 and a_4 of the outputs, i.e. keeping the same closure. All the input data remain unchanged, as well as the outputs, for the remaining DMUs 15 to 26, for comparison purposes. The reason for these changes is that all the triangular fuzzy numbers the original data given by Majid Azadi et al.[23] were symmetric, i.e.in $a_2 = a_3 = (a_1 + a_4)/2$. In that case, as it was expected, the inefficient score $I(X_o, Y_o)$ obtained when we applied the proposed model for fuzzy integer numbers (FIDEA) (5.1), is essentially the same as the one computed by the Integer Interval DEA model (IIDEA) proposed in our previous work [18]. Of course, the Integer Interval DEA model (IIDEA) is not able to handle fuzzy integer data nor to compute fuzzy integer targets, as the proposed fuzzy integer DEA (FIDEA) approach. Therefore, the original data were modified to highlight the potential of this more general approach. Actually, trapezoidal fuzzy numbers allow more flexibility in modeling the uncertainty in the input and output data than just intervals. And we

Table 5.1: Input and output data of the numerical example based on [23]. This is a hybrid problem in which the second input is integer (and crisp) and both outputs are trapezoidal fuzzy integer numbers ($TF_{\mathbb{Z}}$). The label of those three variables are highlighted in bold. The other three inputs are continuous (and crisp). The crisp variables can be regarded as a particular case of trapezoidal fuzzy numbers where $a_1 = a_2 = a_3 = a_4$.

o	Input data				Output data	
	X_1	$\mathbf{X_2}$	X_3	X_4	$\mathbf{Y_1}$	$\mathbf{Y_2}$
1	316	251	61	18	(199 , 219 , 229 , 239)	(76 , 80 , 85 , 90)
2	281	164	45	21	(153 , 173 , 183 , 193)	(28 , 30 , 35 , 42)
3	309	198	83	40	(203 , 213 , 223 , 243)	(78 , 83 , 87 , 92)
4	291	218	37	45	(167 , 177 , 187 , 207)	(85 , 90 , 95 , 99)
5	597	178	52	29	(197 , 217 , 227 , 237)	(163 , 170 , 173 , 177)
6	341	142	19	33	(129 , 140 , 150 , 169)	(129 , 133 , 138 , 143)
7	475	149	74	18	(193 , 210 , 220 , 233)	(111 , 115 , 120 , 125)
8	254	172	53	35	(134 , 154 , 164 , 174)	(250 , 257 , 260 , 264)
9	328	135	83	47	(184 , 200 , 210 , 224)	(58 , 63 , 68 , 72)
10	310	173	41	16	(113 , 130 , 140 , 153)	(88 , 92 , 97 , 102)
11	321	121	57	45	(125 , 145 , 155 , 165)	(153 , 160 , 160 , 167)
12	329	204	38	53	(195 , 205 , 215 , 235)	(90 , 95 , 100 , 104)
13	475	212	32	42	(156 , 170 , 180 , 196)	(139 , 143 , 148 , 153)
14	259	189	56	85	(129 , 149 , 149 , 169)	(97 , 97 , 104 , 111)
15	274	217	38	51	(85 , 105 , 105 , 125)	(68 , 75 , 75 , 82)
16	264	158	25	35	(193 , 213 , 213 , 233)	(45 , 52 , 52 , 59)
17	327	124	32	16	(107 , 127 , 127 , 147)	(271 , 278 , 278 , 285)
18	429	307	57	49	(142 , 162 , 162 , 182)	(46 , 53 , 53 , 60)
19	262	138	25	31	(122 , 142 , 142 , 162)	(173 , 180 , 180 , 187)
20	385	238	74	22	(106 , 126 , 126 , 146)	(119 , 126 , 126 , 133)
21	249	217	69	72	(150 , 170 , 170 , 190)	(90 , 97 , 97 , 104)
22	337	203	27	33	(104 , 124 , 124 , 144)	(271 , 278 , 278 , 285)
23	365	292	85	71	(185 , 205 , 205 , 225)	(143 , 150 , 150 , 157)
24	296	185	49	18	(112 , 132 , 132 , 152)	(177 , 184 , 184 , 191)
25	428	242	39	22	(94 , 114 , 114 , 134)	(78 , 85 , 85 , 92)
26	327	218	43	48	(173 , 193 , 193 , 213)	(113 , 120 , 120 , 127)

Table 5.2: Efficiency scores and slacks ($TF_{\mathbb{Z}}$ or tF_C) from **Phase I**. The second input is an integer (crisp) variable, the other three inputs are continuous (crisp) and both outputs are trapezoidal fuzzy integer numbers $TF_{\mathbb{Z}}$. The crisp variables can be regarded as the particular case of trapezoidal fuzzy numbers where $a_1 = a_2 = a_3 = a_4$.

DMU_o	$I(X_o, Y_o)$	Input slacks				Output slacks	
		s_1^x	$\mathbf{s_2^x}$	s_3^x	s_4^x	$\mathbf{s_1^y}$	$\mathbf{s_2^y}$
1	0	0	0	0	0	(0 , 0 , 0 , 0)	(0 , 0 , 0 , 0)
2	1.71	0.36	0	4.81	0	(0 , 0 , 0 , 3)	(54 , 54 , 54 , 54)
3	1.22	0	1	40.09	0	(0 , 6 , 6 , 6)	(59 , 60 , 60 , 62)
4	1.35	0	53	0.12	12.11	(0 , 2 , 2 , 2)	(76 , 76 , 76 , 78)
5	0.46	162.62	0	0.28	0.38	(0 , 0 , 0 , 12)	(26 , 26 , 26 , 30)
6	0	0	0	0	0	(0 , 0 , 0 , 0)	(0 , 0 , 0 , 0)
7	0	0	0	0	0	(0 , 0 , 0 , 0)	(0 , 0 , 0 , 0)
8	0	0	0	0	0	(0 , 0 , 0 , 0)	(0 , 0 , 0 , 0)
9	0	0	0	0	0	(0 , 0 , 0 , 0)	(0 , 0 , 0 , 0)
10	1.74	0	32	5.34	0.03	(0 , 0 , 0 , 4)	(134 , 134 , 134 , 135)
11	0.7	0	0	10.3	23.14	(0 , 0 , 0 , 5)	(0 , 0 , 0 , 0)
12	1.12	0	21	0.19	16.16	(0 , 6 , 6 , 7)	(66 , 66 , 66 , 70)
13	1.33	142.38	59	0	15.42	(0 , 0 , 0 , 6)	(54 , 54 , 54 , 56)
14	2.34	0	24	5.98	52.49	(0 , 0 , 0 , 0)	(152 , 152 , 152 , 152)
15	3.61	0	110	10.36	36.9	(0 , 0 , 0 , 0)	(158 , 158 , 158 , 158)
16	0	0	0	0	0	(0 , 0 , 0 , 0)	(0 , 0 , 0 , 0)
17	0	0	0	0	0	(0 , 0 , 0 , 0)	(0 , 0 , 0 , 0)
18	7.2	0	141	13.97	27.17	(0 , 6 , 7 , 13)	(311 , 313 , 313 , 315)
19	0	0	0	0	0	(0 , 0 , 0 , 0)	(0 , 0 , 0 , 0)
20	2.8	0.02	92	36.33	3.16	(20 , 23 , 23 , 27)	(200 , 201 , 201 , 202)
21	1.96	0	56	28.37	38.25	(0 , 0 , 0 , 0)	(74 , 74 , 74 , 74)
22	0	0	0	0	0	(0 , 0 , 0 , 0)	(0 , 0 , 0 , 0)
23	2.07	0	58	16.47	24.04	(7 , 15 , 23 , 23)	(184 , 187 , 190 , 190)
24	0.95	0.04	62	19.74	0.09	(0 , 0 , 0 , 0)	(38 , 38 , 38 , 38)
25	3.89	29.81	91	0.03	2.52	(36 , 40 , 40 , 45)	(252 , 253 , 253 , 255)
26	1.35	0.06	58	11.52	18.33	(0 , 2 , 2 , 4)	(50 , 51 , 51 , 52)

check that not only the resulting efficiency scores are different, but also they tend to be lower when comparing the results of (FIDEA) with those of (IIDEA).

Tables 5.2 and 5.3 show the results from Phase I, model (5.2), and Phase II, model (5.3), together with the computed targets, (5.4) and (5.5), respectively. The corresponding efficiency status, given also in Table 5.3, coincide with those of the (IIDEA) model. All DMUs are classified as either efficient or inefficient, and there are no weakly efficient DMUs in this example. As regards the ranking of the DMUs provided by the proposed approach this is based on the overall efficiency score $I(X_o, Y_o) + H(X_o, Y_o)$.

For the sake of comparison, Table 5.4 shows the efficiency score and (crisp) targets of the Integer Fuzzy DEA approach of Kordrostami et al. [94] versus the efficiency scores and the (fuzzy) targets computed by the proposed approach. Note that we only show the integer input and output targets. The efficient DMUs identified by the two approaches coincide although the ranking of the inefficient DMUs differ. Recall that Kordrostami et al. [94] use a radial oriented approach based on the integer PPS of Kuosmanen & Matin [98]. As opposed to that, we apply an additive, non-oriented approach based on a specific fuzzy integer PPS. Despite these differences, analysing the results of both approaches, we can check that the results are in good agreement (the Spearman rank-order correlation coefficient is $\rho = 0.923$). As regards the targets, for the second (crisp) input variable, they are similar, although Kordrostami et al. [94] provide smaller values in general. More important, the fuzzy character of the targets of

Table 5.3: Results for **Phase II**, including targets and DMU efficiency status classification. Second input and both outputs are integer (labels highlighted in bold) whereas all othe rvariables are continuous. All the inputs slacks variables are zero $v_{ik}^{x} = 0$, for all $i = 1, \ldots, M$ and $k = 1, 2, 3, 4$, and hence they are not included in the table.

DMU_o	$I(X_o, Y_o)$	$H(X_o, Y_o)$	Output slacks variables								X_1^{target}	$\mathbf{X_2^{target}}$	X_3^{target}	X_4^{target}	$\mathbf{Y_1^{target}}$	$\mathbf{Y_2^{target}}$	Efficiency Status	Rank
			$\mathbf{v_{11}^{y}}$	$\mathbf{v_{12}^{y}}$	$\mathbf{v_{13}^{y}}$	$\mathbf{v_{14}^{y}}$	$\mathbf{v_{21}^{y}}$	$\mathbf{v_{22}^{y}}$	$\mathbf{v_{23}^{y}}$	$\mathbf{v_{24}^{y}}$								
1	0	0	0	0	0	0	0	0	0	0	316	251	61	18	(199 , 219 , 229 , 239)	(76 , 80 , 85 , 90)	Eff.	1
2	1.71	0.06	8	5	0	0	1	4	1	0	280.64	164	40.19	21	(161 , 178 , 183 , 196)	(83 , 88 , 90 , 96)	Ineff.	18
3	1.22	0.02	0	7	2	0	0	2	0	0	309	197	42.91	40	(203 , 226 , 231 , 249)	(137 , 145 , 147 , 154)	Ineff.	14
4	1.35	0.03	0	9	2	0	1	4	0	0	291	165	36.88	32.89	(167 , 188 , 191 , 209)	(162 , 170 , 171 , 177)	Ineff.	17
5	0.46	0.01	2	6	0	0	0	1	0	0	434.38	178	51.72	28.62	(199 , 223 , 227 , 249)	(189 , 197 , 199 , 207)	Ineff.	10
6	0	0	0	0	0	0	0	0	0	0	341	142	19	33	(129 , 140 , 150 , 169)	(129 , 133 , 138 , 143)	Eff.	1
7	0	0	0	0	0	0	0	0	0	0	475	149	74	18	(193 , 210 , 220 , 233)	(111 , 115 , 120 , 125)	Eff.	1
8	0	0	0	0	0	0	0	0	0	0	254	172	53	35	(134 , 154 , 164 , 174)	(250 , 257 , 260 , 264)	Eff.	1
9	0	0	0	0	0	0	0	0	0	0	328	135	83	47	(184 , 200 , 210 , 224)	(58 , 63 , 68 , 72)	Eff.	1
10	1.74	0.04	6	8	0	0	1	4	0	0	310	141	35.66	15.97	(119 , 138 , 140 , 157)	(223 , 230 , 231 , 237)	Ineff.	19
11	0.7	0.03	9	6	0	0	1	0	1	0	321	121	46.7	21.86	(134 , 151 , 155 , 170)	(154 , 160 , 161 , 167)	Ineff.	11
12	1.12	0.02	0	7	0	0	1	4	0	0	329	183	37.81	36.84	(195 , 218 , 221 , 242)	(157 , 165 , 166 , 174)	Ineff.	13
13	1.33	0.02	1	9	0	0	1	4	0	0	332.62	153	32	26.58	(157 , 179 , 180 , 202)	(194 , 201 , 202 , 209)	Ineff.	15
14	2.34	0.04	0	0	9	0	0	7	2	0	259	165	50.02	32.51	(129 , 149 , 158 , 169)	(249 , 256 , 258 , 263)	Ineff.	22
15	3.61	0.04	6	3	3	0	2	1	1	0	274	107	27.64	14.1	(91 , 108 , 108 , 125)	(228 , 234 , 234 , 240)	Ineff.	24
16	0	0	0	0	0	0	0	0	0	0	264	158	25	35	(193 , 213 , 213 , 233)	(45 , 52 , 52 , 59)	Eff.	1
17	0	0	0	0	0	0	0	0	0	0	327	124	32	16	(107 , 127 , 127 , 147)	(271 , 278 , 278 , 285)	Eff.	1
18	7.2	0	0	0	0	0	0	0	0	0	429	166	43.03	21.83	(142 , 168 , 169 , 195)	(357 , 366 , 366 , 375)	Ineff.	26
19	0	0	0	0	0	0	0	0	0	0	262	138	25	31	(122 , 142 , 142 , 162)	(173 , 180 , 180 , 187)	Eff.	1
20	2.8	0	0	0	0	0	0	0	0	0	384.98	146	37.67	18.84	(126 , 149 , 149 , 173)	(319 , 327 , 327 , 335)	Ineff.	23
21	1.96	0.02	1	0	6	0	0	0	2	0	249	161	40.63	33.75	(151 , 170 , 176 , 190)	(164 , 171 , 173 , 178)	Ineff.	20
22	0	0	0	0	0	0	0	0	0	0	337	203	27	33	(104 , 124 , 124 , 144)	(271 , 278 , 278 , 285)	Eff.	1
23	2.07	0	0	0	0	0	0	0	0	0	365	234	68.53	46.96	(192 , 220 , 228 , 248)	(327 , 337 , 340 , 347)	Ineff.	21
24	0.95	0.01	2	1	1	0	0	0	0	0	295.96	123	29.26	17.91	(114 , 133 , 133 , 152)	(215 , 222 , 222 , 229)	Ineff.	12
25	3.89	0	0	0	0	0	0	0	0	0	398.19	151	38.97	19.48	(130 , 154 , 154 , 179)	(330 , 338 , 338 , 347)	Ineff.	25
26	1.35	0	0	0	0	0	0	0	0	0	326.94	160	31.48	29.67	(173 , 195 , 195 , 217)	(163 , 171 , 171 , 179)	Ineff.	16

Table 5.4: Efficiency scores (from Phases I and II) of proposed approach versus Efficiency score and integer input and output targets of Kordrostami et al. [94]

	Proposed approach					Kordrostami et al. [94]				
DMU_o	$I(X_o, Y_o) + H(X_o, Y_o)$	Rank	X_2^{target}	Y_1^{target}	Y_2^{target}	Efficiency	Rank	X_2^{target}	Y_1^{target}	Y_2^{target}
1	0	1	251	(199 , 219 , 229 , 239)	(76 , 80 , 85 , 90)	1	1	252	223	83
2	1.78	18	164	(161 , 178 , 183 , 196)	(83 , 88 , 90 , 96)	0.96	12	157	177	63
3	1.23	14	197	(203 , 226 , 231 , 249)	(137 , 145 , 147 , 154)	0.91	13	175	220	85
4	1.38	17	165	(167 , 188 , 191 , 209)	(162 , 170 , 171 , 177)	0.83	18	149	184	93
5	0.47	10	178	(199 , 223 , 227 , 249)	(189 , 197 , 199 , 207)	0.97	11	172	221	171
6	0	1	142	(129 , 140 , 150 , 169)	(129 , 133 , 138 , 143)	1	1	143	147	136
7	0	1	149	(193 , 210 , 220 , 233)	(111 , 115 , 120 , 125)	1	1	150	215	120
8	0	1	172	(134 , 154 , 164 , 174)	(250 , 257 , 260 , 264)	1	1	173	158	259
9	0	1	135	(184 , 200 , 210 , 224)	(58 , 63 , 68 , 72)	1	1	136	205	66
10	1.78	19	141	(119 , 138 , 140 , 157)	(223 , 230 , 231 , 237)	0.83	17	143	135	95
11	0.73	11	121	(134 , 151 , 155 , 170)	(154 , 160 , 161 , 167)	0.98	10	118	149	160
12	1.15	13	183	(195 , 218 , 221 , 242)	(157 , 165 , 166 , 174)	0.84	16	169	212	98
13	1.35	15	153	(157 , 179 , 180 , 202)	(194 , 201 , 202 , 209)	0.78	20	155	176	146
14	2.37	22	165	(129 , 149 , 158 , 169)	(249 , 256 , 258 , 263)	0.78	21	128	149	102
15	3.65	24	107	(91 , 108 , 108 , 125)	(228 , 234 , 234 , 240)	0.52	25	91	105	75
16	0	1	158	(193 , 213 , 213 , 233)	(45 , 52 , 52 , 59)	1	1	158	213	52
17	0	1	124	(107 , 127 , 127 , 147)	(271 , 278 , 278 , 285)	1	1	124	127	278
18	7.2	26	166	(142 , 168 , 169 , 195)	(357 , 366 , 366 , 375)	0.49	26	137	162	53
19	0	1	138	(122 , 142 , 142 , 162)	(173 , 180 , 180 , 187)	1	1	138	142	180
20	2.8	23	146	(126 , 149 , 149 , 173)	(319 , 327 , 327 , 335)	0.59	24	136	126	126
21	1.98	20	161	(151 , 170 , 176 , 190)	(164 , 171 , 173 , 178)	0.91	14	141	170	97
22	0	1	203	(104 , 124 , 124 , 144)	(271 , 278 , 278 , 285)	1	1	203	124	278
23	2.07	21	234	(192 , 220 , 228 , 248)	(327 , 337 , 340 , 347)	0.77	22	178	205	150
24	0.96	12	123	(114 , 133 , 133 , 152)	(215 , 222 , 222 , 229)	0.89	15	139	132	184
25	3.89	25	151	(130 , 154 , 154 , 179)	(330 , 338 , 338 , 347)	0.63	23	110	114	85
26	1.35	16	160	(173 , 195 , 195 , 217)	(163 , 171 , 171 , 179)	0.79	19	162	193	120

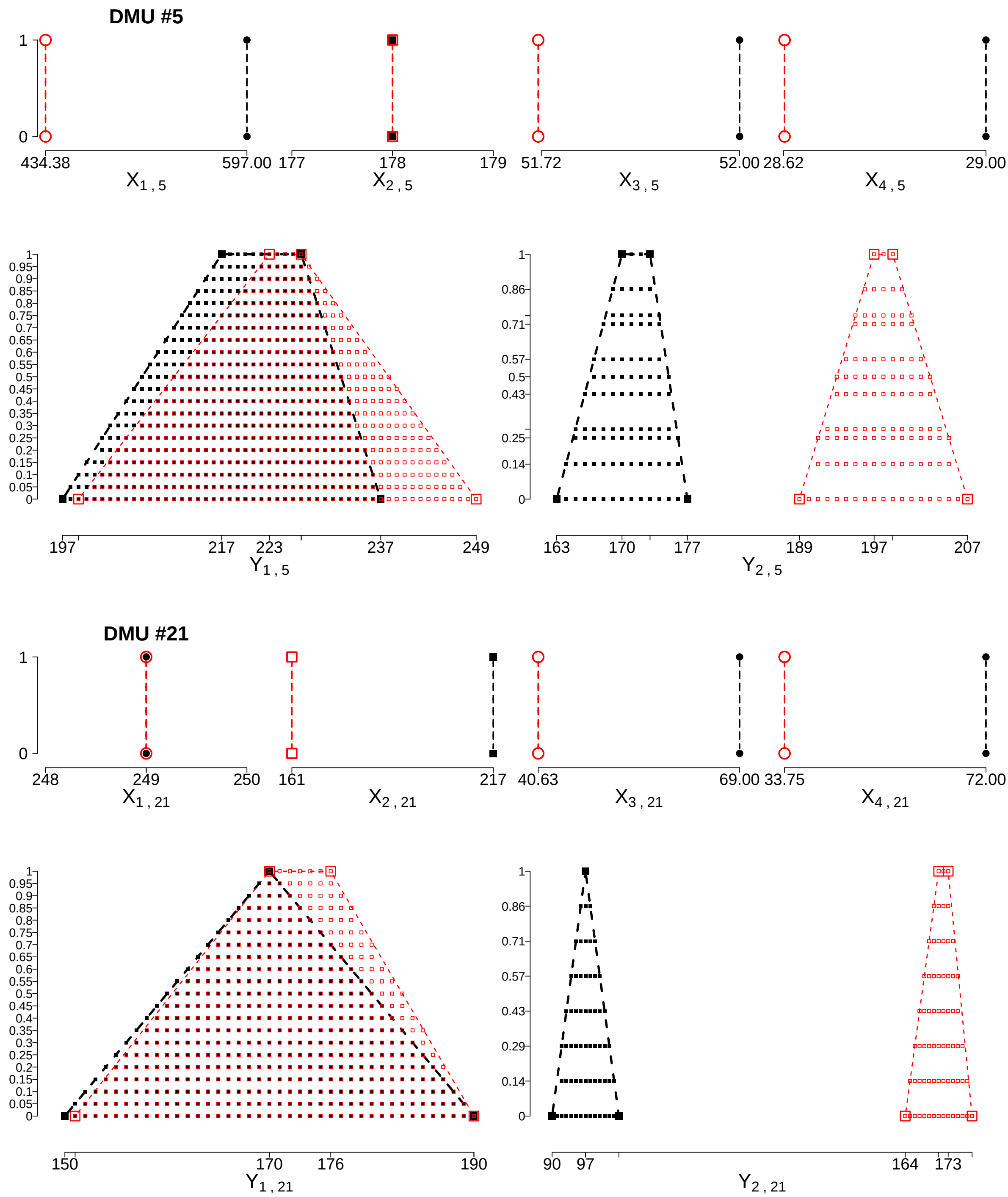

Figure 5.1: Observed input and output data versus their corresponding targets for two inefficient DMUs. Observed data are plotted in black colour while targets are shown in red colour.

the proposed approach maintain the uncertainty originally present in the fuzzy data and, in addition, they are usually higher than those Kordrostami et al. [94].

Figure 5.1 compares the observed input and output data versus the corresponding targets, as per Equations (5.4) and (5.5), for two inefficient DMUs. The data and the targets are plotted in black and red, respectively. Note how for the input variables the target inputs are lower than the observed data while the opposite occurs for the output variables.

Figure 5.2 graphically shows the target computation process using the results of Phases I

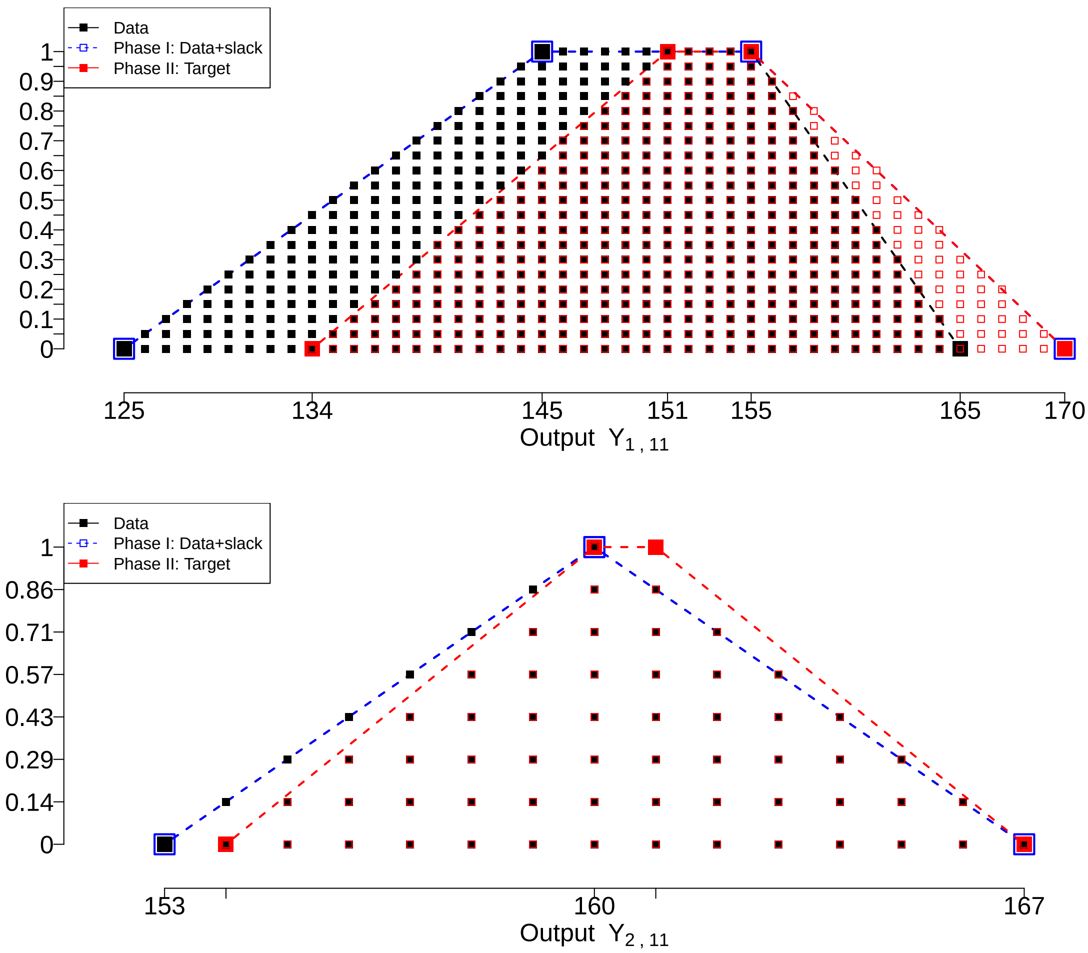

Figure 5.2: Target computation process, using the solutions of Phase I and Phase II, for DMU 11. $s_1^{y*} = (0, 0, 0, 5)$, $v_{1,1}^{y*} = 9$, $v_{1,2}^{y*} = 6$, $v_{1,3}^{y*} = 0$, $v_{1,4}^{y*} = 0$, $s_2^{y*} = (0, 0, 0, 0)$, $v_{2,1}^{y*} = 1$, $v_{2,2}^{y*} = 0$, $v_{2,3}^{y*} = 1$, $v_{2,4}^{y*} = 0$.

and II. The first approximation is obtained by the computation of the fuzzy slacks in (5.2). The result of adding these Phase I output slacks $s_r^{y*} \in TF_{\mathbb{Z}_+}$ to the observed output y_{rj} is plotted in blue color. The final output targets are plotted in red color. These output targets are computed using Equation (5.5), i.e. adding to the observed output not only the Phase I output slacks s_r^{y*} but also the $v_{rk}^{y*} \in \mathbb{Z}_+$ variables from the optimal solution of the Phase II model (5.3). As it can be seen, Phase II checks if there still any remaining reduction (for inputs) or increase (for outputs). Those improvements are computed by the corresponding input and output slack variables v_{ik}^x and v_{rk}^y, $k = 1, 2, 3, 4$.

Finally, for completeness and illustration purposes, in order to fully understand how the proposed approach works let us consider an additional weakly efficient DMU', plotted in Figure 5.3. This is constructed based on the efficient DMU_1 in the following way. DMU' has the same inputs as DMU_1, i.e. $X' = X_1$, but the outputs have been slightly reduced so that with $Y' \leqq Y_1$, and $Y_1 \neq Y'$. More specifically, if $y_r' = (y_{r,1}', y_{r,2}', y_{r,3}', y_{r,4}')$ and $y_{1,r} = (y_{1,r,1}, y_{1,r,2}, y_{1,r,3}, y_{1,r,4})$, with $r = 1, 2$, then $y_{r,k}' < y_{1,r,k}$ for $k = 1, 2$, and $y_{r,k}' = y_{1,r,k}$ for $k = 3, 4$. Thus, $I(X', Y') = 0$ because the Phase I model (5.2) cannot find any non-null fuzzy slacks that would improve DMU' within the fuzzy integer PPS. The Phase II model (5.3), however, checks that there are some possible increases in the v_{rk}^y outputs slacks for $k = 1, 2$, leading to $H(X', Y') > 0$. Hence, DMU' is labeled

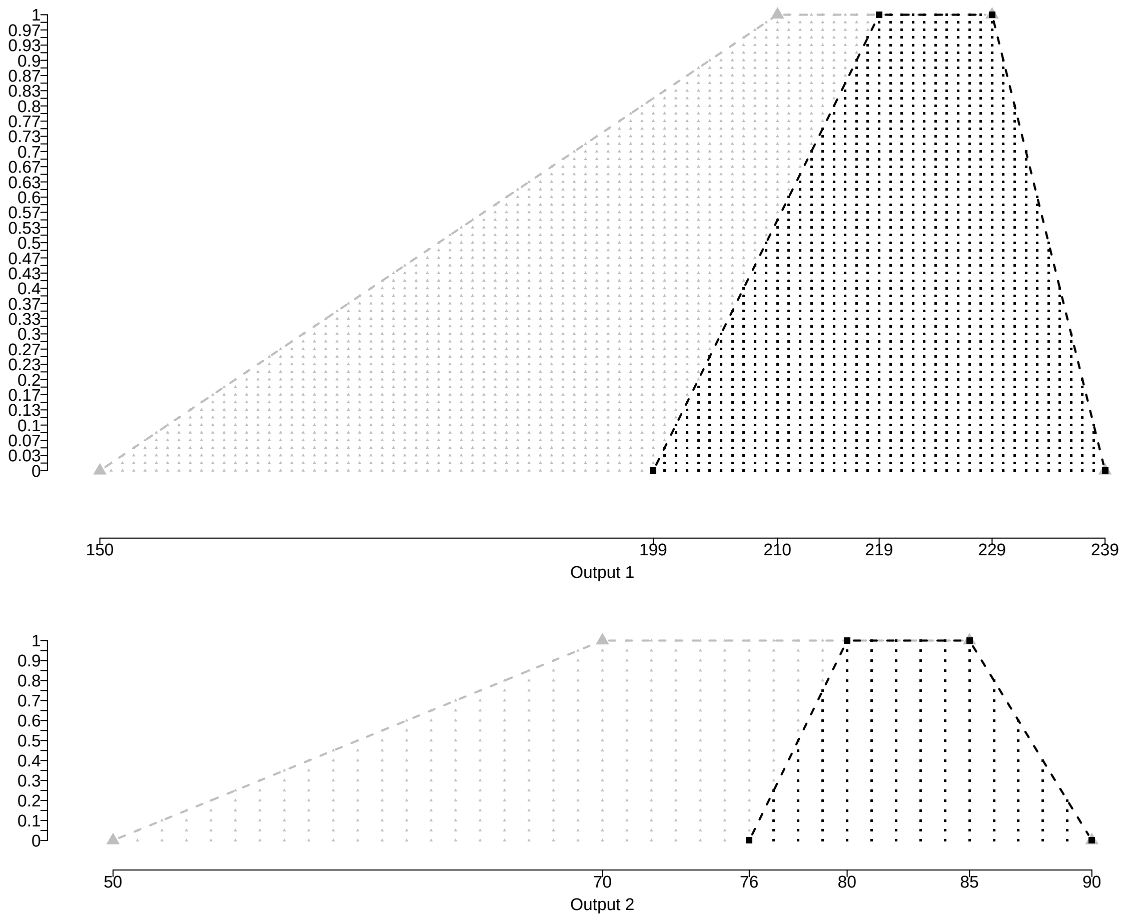

Figure 5.3: The weakly Efficient DMU′ (grey colour) versus the efficient DMU_1 that dominates it (black colour). The inputs of both are the same, $X' = X_1$, but $Y' \leqq Y_1$, $Y_1 \neq Y'$. $I(X', Y') = 0$ but $H(X', Y') = v^{y*}_{1,1} + v^{y*}_{1,2} + v^{y*}_{2,1} + v^{y*}_{2,2} = (199-150)+(219-210)+(76-50)+(80-70) = 49+9+26+10 = 94$. Therefore, it is not possible to add non null output slacks $s^{y*}_r \in TF_{\mathbb{Z}_+}$, but it is still possible to shift to the right some of the fuzzy output parameters so as to improve the efficiency of DMU′. The target is indeed DMU_1. As in Figure 4.1, we only represent the integer points within each α-level.

as weakly efficient as per Definition 5.3.2.

5.5 Conclusions

In this chapter, a general fuzzy integer DEA approach is presented to assess and rank the efficiency of the DMUs. Conventional integer DEA approaches do not consider uncertainty while, with the exception of Kordrostami et al. [94], existing fuzzy DEA approaches do not consider fuzzy integer variables. The proposed approach considers a hybrid scenario that may involve trapezoidal fuzzy integer numbers ($TF_{\mathbb{Z}}$) and trapezoidal fuzzy numbers (TF_C). As an extension of the integer PPS given in [18], a fuzzy integer PPS is derived using fuzzy integer arithmetic and fuzzy integer partial orders. Once the corresponding FIDEA technology is established, a non-oriented slacks-based fuzzy integer DEA model is proposed. This allows not only computing efficiency scores but also efficient fuzzy targets.

The proposed approach involves two phases because, unlike in the crisp case, in the fuzzy DEA scenario a null inefficiency score of the Phase I, i.e., $I(X_0, Y_0) = 0$, does not imply efficiency, being a necessary but not sufficient condition. In other words, the phase I model can not distinguish between efficient and weakly efficient DMUs. The efficient fuzzy integer targets are actually computed by the Phase II model.

We have applied the proposed approach to a numerical dataset with both crisp and fuzzy integer variables. The efficiency scores and DMU ranking of the proposed approach are compared with those of the radial, input-oriented approach of Kordrostami et al. [94]. Unlike Kordrostami et al. [94], however, the proposed approach is able to compute efficient input and output targets.

Regarding future research, a first step would extend the existing fuzzy integer arithmetic and partial orders to polygonal fuzzy integer numbers, which are more general than trapezoidal fuzzy integer numbers and thus allow more flexibility for modeling the uncertainty in the input and output data. Also, other types of DEA models, for example involving undesirable outputs, non discretionary variables, or multiple processes (so-called network DEA) ought to be developed.

Chapter 6

Inverse DEA

6.1 Introduction

This chapter is to consider inverse DEA with non-radial slacks-based measure, which has more properties of radial models, on integer interval framework. We consider the following question: "If the output of DMU_o increases such that its inefficiency score is not less than t-percent, how much should the input of DMU_o increase?" As explained before, Zhang and Cui in [153] propose a non-radial inverse DEA model with crisp data, supposing that the overall efficiency score remains unchanged, covering all radial and non-radial measures that are monotonous. In other words, they introduced a basic form of all inverse DEA models because monotonicity is one of the main properties of DEA measures.

6.2 Inverse DEA models with crisp data

Let us assume a set of N DMUs in which each DMU_j, $j \in J = \{1, \ldots, N\}$, consume M inputs $X_j = (x_{1j}, \ldots, x_{Mj}) \in \mathbb{R}^M$ to produces S outputs $Y_j = (y_{1j}, \ldots, y_{Sj}) \in \mathbb{R}^S$. In the classic Charnes et al. [31] DEA model, the production possibility set (PPS) or technology, defined by T, satisfies in the axioms Envelopment, Free disposability, Convexity, Scalability introduced in Subsection 2.5.1. According to the minimum extrapolation principle in [28] and these axioms, the DEA PPS is defined. The DEA PPS under constant-return to scale (CRS) is presented in this paper.

Let us recall, a DMU_o is said to be efficient if and only if for any $(X, Y) \in T_{DEA}$ such that $X \leqq X_o$ and $Y \geqq Y_o$, then $(X, Y) = (X_o, Y_o)$. Let us recall that this can be got solving the following normalized slacks-based DEA model .

$$
\begin{aligned}
\text{(DEA)}\ \ I^{*}(X_o, Y_o) = \quad \text{Max} \quad & \sum_{i=1}^{M} \frac{s_i^x}{x_{io}} + \sum_{r=1}^{S} \frac{s_r^y}{y_{ro}} \\
\text{s.t.} \quad & \sum_{j=1}^{N} \lambda_j x_{ij} \leq x_{io} - s_i^x, \quad i = 1, \dots, M, \\
& \sum_{j=1}^{N} \lambda_j y_{rj} \geq y_{ro} + s_r^y, \quad r = 1, \dots, S, \\
& \lambda_j \geq 0, \quad j = 1, \dots, N, \\
& s_i^x, s_r^y \geq 0, \quad i = 1, \dots, M, \ \ r = 1, \dots, S.
\end{aligned} \tag{6.1}
$$

Where λ_j, $j = 1, \dots, N$, are the intensity variables used for defining the corresponding efficient target of DMU_o. A DMU_o is efficient if and only if $I^{*}(X_o, Y_o) = 0$.

Now, the following question is considered based on investigations carried out in previous literature. If the outputs of DMU_o increase, how much should the inputs of the DMU_o increase to decrease the inefficiency score of DMU_o to the amount of t-percent. The aim of the question is to calculate the minimum increase of input (α_o^*) if the output of DMU_o increase from Y_o to $\beta_o = Y_o + \triangle Y_o$, where $\triangle Y_o \gneqq 0$ provided that the inefficiency score of DMU_o decrease to the amount of t-percent. In fact,

$$
\alpha_o^* = (\alpha_{1o}^*, \alpha_{2o}^*, ..., \alpha_{Mo}^*)^t = X_o + \triangle X_o, \qquad \triangle X_o \gneqq 0.
$$

Furthermore, we consider that the new DMU belongs to the technology. For the sake of simplicity, assume that the new DMU represents DMU_o. After modification of inputs and outputs, the following model is presented to estimate the inefficiency of the new DMU:

$$
\begin{aligned}
\text{(DEA)}\ \ I^{*}(\alpha_o^*, \beta_o) = \quad \text{Max} \quad & \sum_{i=1}^{M} \frac{s_i^x}{\alpha_{io}^*} + \sum_{r=1}^{S} \frac{s_r^y}{\beta_{ro}} \\
\text{s.t.} \quad & \sum_{j=1}^{N} \lambda_j x_{ij} \leq \alpha_{io}^* - s_i^x, \quad i = 1, \dots, M, \\
& \sum_{j=1}^{N} \lambda_j y_{rj} \geq \beta_{ro} + s_r^y, \quad r = 1, \dots, S, \\
& \lambda_j \geq 0, \quad j = 1, \dots, N, \\
& s_i^x, s_r^y \geq 0, \quad i = 1, \dots, M, \ \ r = 1, \dots, S.
\end{aligned} \tag{6.2}
$$

Definition 6.2.1. *(1) If the optimal values of the model* (6.1) *and* (6.2) *are equal, it is said to be the inefficiency score remains unchanged; that is,* $I^{*}(\alpha_o^*, \beta_0) = I^{*}(X_o, Y_o)$.
(2) If the optimal values of the model (6.1) *are less than model* (6.2), *it is said to be the inefficiency score decrease to the amount of t-percent; that is,* $I^{*}(\alpha_o^*, \beta_0) = (1 - t)I^{*}(X_o, Y_o)$.

To solve the above question, the following Multiple Objective Non-Linear Programming (MONLP) is considered:

$$
\begin{aligned}
\text{(MONLP)} \quad \text{Min} \quad & (\alpha_{1o}, ..., \alpha_{Mo}) && (6.3)\\
\text{s.t.} \quad & \sum_{j=1}^{N} \lambda_j x_{ij} \le \alpha_{io} - s_i^x, \quad i = 1, \ldots, M,\\
& \sum_{j=1}^{N} \lambda_j y_{rj} \ge \beta_{ro} + s_r^y, \quad r = 1, \ldots, S,\\
& \sum_{i=1}^{M} \frac{s_i^x}{\alpha_{io}} + \sum_{r=1}^{S} \frac{s_r^y}{\beta_{ro}} = (1-t)I^*\\
& \alpha_{io} \ge x_{io}, \quad i = 1, \ldots, M,\\
& \lambda_j \ge 0, \quad j = 1, \ldots, N,\\
& s_i^x, s_r^y \ge 0, \quad i = 1, \ldots, M, \ r = 1, \ldots, S.
\end{aligned}
$$

Where I^* is the optimal value of the model (6.1) and $0 \le t \le 1$, note that when $t = 1$, $I^*(\alpha_o^*, \beta_o) = 0$, which means the new *DMU* is efficient and when $t = 0$, $I^*(\alpha_o^*, \beta_o) = I^*(X_o, Y_o)$. Therefore, when t increases, the inefficiency score decreases.

Definition 6.2.2. *(see [153]). Let* $(\lambda^*, \alpha_0^*, s^{x*}, s^{y*})$ *be a feasible solution to the model* (6.3). $(\lambda^*, \alpha_0^*, s^{x*}, s^{y*})$ *is said to be a Pareto (efficient) solution to the model* (6.3) *if there isn't feasible solution* $(\lambda, \alpha_0, s^x, s^y)$ *of* (6.3) *such that* $\alpha_{io} \le \alpha_{io}^*$ *for all* $i = 1, 2, ..., M$ *and* $\alpha_{io} < \alpha_{io}^*$ *for at least one* i.

Definition 6.2.3. *(see [153]). Let* $(\lambda^*, \alpha_0^*, s^{x*}, s^{y*})$ *be a feasible solution to the model* (6.3). $(\lambda^*, \alpha_0^*, s^{x*}, s^{y*})$ *is said to be a weakly Pareto (weakly efficient) solution to the model* (6.3) *if there isn't feasible solution* $(\lambda, \alpha_0, s^x, s^y)$ *of* (6.3) *such that* $\alpha_{io} \le \alpha_{io}^*$ *for all* $i = 1, 2, ..., M$.

There are different methods to generate weakly Pareto (weakly efficient) solutions of *MOLP* and *MONLP*. One of the most usual methods is weighted sum problems (see [9] and [10]). Following formulation is this type of optimization problem. Given MONLP (6.3) and $w = (w_1, w_2, \cdots, w_M) \in \mathbb{R}^M$, $w_i > 0$, $\sum_{i=1}^{M} w_i = 1$, We define the related sum problem as follows.

$$
\begin{aligned}
\text{(MONLP)w} \quad \text{Min} \quad & \sum_{i=1}^{M} w_i \alpha_{io} && (6.4)\\
\text{s.t.} \quad & \sum_{j=1}^{N} \lambda_j x_{ij} \le \alpha_{io} - s_i^x, \quad i = 1, \ldots, M,\\
& \sum_{j=1}^{N} \lambda_j y_{rj} \ge \beta_{ro} + s_r^y, \quad r = 1, \ldots, S,\\
& \sum_{i=1}^{M} \frac{s_i^x}{\alpha_{io}} + \sum_{r=1}^{S} \frac{s_r^y}{\beta_{ro}} = (1-t)I^*,\\
& \alpha_{io} \ge x_{io}, \quad i = 1, \ldots, M,\\
& \lambda_j \ge 0, \quad j = 1, \ldots, N,\\
& s_i^x, s_r^y \ge 0, \quad i = 1, \ldots, M, \ r = 1, \ldots, S.
\end{aligned}
$$

Theorem 6.2.1. *Assume that $I^*(X_o, Y_o)$ be the inefficiency score of DMU_o under the monotonous measure in the model* (6.1) *and the outputs of DMU_o are increased from Y_0 to $\beta_0 = Y_0 + \triangle Y_o \quad (\triangle Y_o \leqq 0)$.*

(1) Let $(\lambda^*, \alpha_o^*, s^{x*}, s^{y*})$ be a Pareto solution to the model (6.3) *then inefficiency score of the DMU_o under new inputs and outputs decrease to the amount of t-percent.*
(2) Conversely, let $(\lambda^*, \alpha_o^*, s^{x*}, s^{y*})$ be a feasible solution to the problem (6.3). *If the inefficiency score of the new DMU decreases to the amount of t-percent, then $(\lambda^*, \alpha_o^*, s^{x*}, s^{y*})$ must be a Pareto solution to the model* (6.3).

Note that there is a similar resulting. If the input of DMU_o increases, how much should the output of DMU_o increase to decrease to the amount of t-percent the inefficiency score of DMU_o. In other words, we calculate $I^*(\alpha_o, \beta_o^*)$.

6.3 Inverse DEA models with integer and continuous interval data

In this section, the non-radial slacks-based model is extended to an integer interval framework, which is considered by Arana-Jimenez et al. [18]. In other words, we provide the question, which is mentioned in previous sections, in the presence of integer interval data using a non-radial slacks-based model.

Let us assume a set of N DMUs, $j \in J = \{1, \dots, N\}$, in which each DMU_j consumes M inputs denoted by $X_j = (x_{1j}, \dots, x_{Mj}) \in (\mathcal{K}_{\mathbb{Z}^+})^M$, with $x_{ij} = [\underline{x_{ij}}, \overline{x_{ij}}]_{\mathbb{Z}} \in \mathcal{K}_{\mathbb{Z}+}$ for $i \in \{1, \dots, M\}$ to produces S outputs denoted by $Y_j = (y_{1j}, \dots, y_{Sj}) \in (\mathcal{K}_{\mathbb{Z}^+})^S$, with $y_{rj} = [\underline{y_{rj}}, \overline{y_{rj}}]_{\mathbb{Z}} \in \mathcal{K}_{\mathbb{Z}+}$ for $r \in \{1, \dots, S\}$. Their continuous extensions are $C(X_j) = \Big(C(x_{1j}), \dots, C(x_{Mj})\Big) \in (\mathcal{K}_{C^+})^M$ and $C(Y_j) = \Big(C(y_{1j}), \dots, C(y_{Sj})\Big) \in (\mathcal{K}_{C^+})^S$.

Let us consider the following axioms, which are corresponding to axioms introduced in Subsection 2.5.1, but considering integer fuzzy inputs and outputs and utilizing the corresponding partial order introduced in Definitions 2.2.2 and 2.2.4:

(B1) *Envelopment: $(X_j, Y_j) \in T$, for all $j \in J$.*

(B2) *Free disposability: $(X, Y) \in T$, $(X', Y') \in (\mathcal{K}_{\mathbb{Z}^+})^{M+S}$, such that $X' \geqq X$, $Y' \leqq Y \;\Rightarrow\; (X', Y') \in T$.*

(B3) *Convexity: $(X, Y), (X', Y') \in T$, $\alpha \in [0, 1]$, such that $\alpha(C(X), C(Y)) + (1 - \alpha)(C(X'), C(Y')) \in (\mathcal{K}_{C \to Z})^{M+S} \Rightarrow (X'', Y'') = \mathbb{Z}\alpha(C(X), C(Y)) + (1 - \alpha)(C(X'), C(Y')) \in T$.*

(B4) *Scalability: $(X, Y) \in T$, $\alpha \geq 0$, and $\alpha(C(X), C(Y)) \in (\mathcal{K}_{c \to z})^{M+S} \Rightarrow (X'', Y'') = \mathbb{Z}(\alpha(C(X), C(Y))) \in T$.*

Theorem 6.3.1. *Under axioms (B1), (B2), (B3) and (B4), the interval production possibility set that results from the minimum extrapolation principle is*

$$T_{IFDEA} = \left\{ (X, Y) \in (\mathcal{K}_{\mathbb{Z}^+})^{M+S} : C(X) \geqq \sum_{j=1}^{N} \lambda_j C(X_j), C(Y) \leqq \sum_{j=1}^{N} \lambda_j C(Y_j), \lambda_j \geq 0, \forall j \right\}.$$

After the characterization result for the T_{IIDEA} given in Theorem 6.3.1, the following integer interval DEA (IIDEA) model, which is a slacks-based measure of inefficiency, can be extended from the non-radial slacks-based model.

$$
\begin{aligned}
\text{(IIDEA)}\ \ II^*(X_o, Y_o) = \quad & \text{Max} \ \ \sum_{i=1}^{M} \frac{\underline{s_i^x} + \overline{s_i^x}}{\underline{x_{io}} + \overline{x_{io}}} + \sum_{r=1}^{S} \frac{\underline{s_r^y} + \overline{s_r^y}}{\underline{y_{ro}} + \overline{y_{ro}}} \\
& \text{s.t.} \quad \sum_{j=1}^{N} \lambda_j C(x_{ij}) \leqq C(x_{io}) - C(s_i^x), \quad i = 1, \ldots, M, \\
& \qquad \sum_{j=1}^{N} \lambda_j C(y_{rj}) \geqq C(y_{ro}) + C(s_r^y), \quad r = 1, \ldots, S, \\
& \qquad \lambda_j \geq 0, \quad j = 1, \ldots, N, \\
& \qquad s_i^x, s_r^y \in \mathcal{K}_{\mathbb{Z}+}, \quad i = 1, \ldots, M, \ \ r = 1, \ldots, S,
\end{aligned}
\tag{6.5}
$$

where inputs x_{ij} and outputs y_{rj} belong to $\mathcal{K}_{\mathbb{Z}}$, i.e.,

$$
\begin{aligned}
x_{ij} &= [\underline{x_{ij}}, \overline{x_{ij}}]_{\mathbb{Z}}, \quad i = 1, \ldots, M, \ \ j = 1, \ldots, N, \\
y_{rj} &= [\underline{y_{rj}}, \overline{y_{rj}}]_{\mathbb{Z}}, \quad r = 1, \ldots, S, \ \ j = 1, \ldots, N.
\end{aligned}
$$

A feasible solution for (IIDEA) is denoted by $(\boldsymbol{s}^{x*}, \boldsymbol{s}^{y*}, \boldsymbol{\lambda}^*)$, where $\boldsymbol{s}^{x*} = (s_1^{x*}, \ldots, s_M^{x*}) \in (\mathcal{K}_z)^M$, $\boldsymbol{s}^{y*} = (s_1^{y*}, \ldots, s_S^{y*}) \in (\mathcal{K}_z)^S$, and $\boldsymbol{\lambda}^* = (\lambda_1^*, \ldots, \lambda_N^*) \in \mathbb{R}^N$. Moreover, (IIDEA) model will deal directly without any ranking function. Also, its objective function is a real number, i.e. $II(X_o, Y_o) \in \mathbb{R}$.

Definition 6.3.1. *A DMU_o is considered to be efficient if and only if $(x, y) \in T_{IFDEA}$, $x \leqq X_o$ and $y \geqq Y_o$ implies $(x, y) = (X_o, Y_o)$.*

Theorem 6.3.2. *If DMU_o is efficient, then $II(X_o, Y_o) = 0$.*

Arana-Jimenez et al. [18] extended the previous axioms, interval production possibility set, and result to the hybrid data scenario, that is, with integer and continuous integer data. The extended and corresponding non-radial slacks-based model is the following:

$$
\begin{aligned}
\text{(HIDEA)}\ \ II^{*}(X_o, Y_o) = \quad \text{Max} \quad & \sum_{i=1}^{M} \frac{\underline{s_i^x} + \overline{s_i^x}}{\underline{x_{io}} + \overline{x_{io}}} + \sum_{r=1}^{S} \frac{\underline{s_r^y} + \overline{s_r^y}}{\underline{y_{ro}} + \overline{y_{ro}}} \\
\text{s.t.} \quad & \sum_{j=1}^{N} \lambda_j C(x_{ij}) \leqq C(x_{io}) - C(s_i^x), \quad i \in O^{XI}, \\
& \sum_{j=1}^{N} \lambda_j x_{ij} \leqq x_{io} - s_i^x, \quad i \in O^{XNI}, \\
& \sum_{j=1}^{N} \lambda_j C(y_{rj}) \geqq C(y_{ro}) + C(s_r^y), \quad r \in O^{YI}, \\
& \sum_{j=1}^{N} \lambda_j y_{rj} \geqq y_{ro} + s_r^y, \quad r \in O^{YNI}, \\
& \lambda_j \geq 0, \quad j = 1, \ldots, N, \\
& s_i^x, s_r^y \in \mathcal{K}_{\mathbb{Z}+}, \quad i \in O^{XI},\ \ r \in O^{YI}, \\
& s_i^x, s_r^y \in \mathcal{K}_{C+}, \quad i \in O^{XNI},\ \ r \in O^{YNI},
\end{aligned}
\tag{6.6}
$$

with O^{XI} and O^{XNI} the index sets for integer input variables and continuous input variables, respectively, O^{YI} and O^{YNI} the index sets for integer output variables and continuous output variables, respectively, with $XI + XNI = M$, $YI + YNI = S$, $O^X = O^{XI} \cup O^{XNI} = \{1, \ldots, M\}$, $O^Y = O^{YI} \cup O^{YNI} = \{1, \ldots, S\}$. Let us write the above model in parameterized form as follows:

$$\text{(PHIDEA)} \quad II^*(X_o, Y_o) = \quad \text{Max} \quad \sum_{i=1}^{M} \frac{\underline{s_i^x} + \overline{s_i^x}}{\underline{x_{io}} + \overline{x_{io}}} + \sum_{r=1}^{S} \frac{\underline{s_r^y} + \overline{s_r^y}}{\underline{y_{ro}} + \overline{y_{ro}}} \tag{6.7}$$

$$\begin{aligned}
\text{s.t.} \quad & \sum_{j=1}^{N} \lambda_j \underline{x_{ij}} \leq \underline{x_{io}} - \overline{s_i^x}, \quad i \in O^X, \\
& \sum_{j=1}^{N} \lambda_j \overline{x_{ij}} \leq \overline{x_{io}} - \underline{s_i^x}, \quad i \in O^X, \\
& \sum_{j=1}^{N} \lambda_j \underline{y_{rj}} \geq \underline{y_{ro}} + \underline{s_r^y}, \quad r \in O^Y, \\
& \sum_{j=1}^{N} \lambda_j \overline{y_{rj}} \geq \overline{y_{ro}} + \overline{s_r^y}, \quad r \in O^Y, \\
& \underline{s_i^x} \leq \overline{s_i^x}, \quad i \in O^X, \\
& \underline{s_r^y} \leq \overline{s_r^y}, \quad r \in O^Y, \\
& \lambda_j \geq 0, \quad j = 1, \ldots, N, \\
& \underline{s_i^x}, \overline{s_i^x}, \underline{s_r^y}, \overline{s_r^y} \in \mathbb{Z}_+, \quad i \in O^{XI}, \ r \in O^{YI}, \\
& \underline{s_i^x}, \overline{s_i^x}, \underline{s_r^y}, \overline{s_r^y} \geq 0, \quad i \in O^{XNI}, \ r \in O^{YNI}.
\end{aligned}$$

The first four sets of constraints are just the corresponding transformation of the inputs/outputs constraints from the model (6.6), with regard to the partial order relation for integer interval numbers, considering in Definition 2.2.4. The two last constraints certify the integer and continuous slacks. Therefore, it is not difficult to derive the following proposition, which establishes the relationship between the (HIDEA) and (PHIDEA) solutions.

Proposition 6.3.1. *$(\boldsymbol{s}^{x*}, \boldsymbol{s}^{y*}, \boldsymbol{\lambda}^*)$ with $\boldsymbol{s}^{x*} \in (\mathcal{K}_{\mathbb{Z}_+})^{XI} * (\mathcal{K}_C)^{XNI}, XI + XNI = M$, $\boldsymbol{s}^{y*} \in (\mathcal{K}_{\mathbb{Z}_+})^{YI} * (\mathcal{K}_C)^{YNI}, YI + YIN = S$ and $\boldsymbol{\lambda}^* \in \mathbb{R}_+^N$ is an optimal solution of (HIDEA) if and only if its corresponding components or parameterization $(\underline{s_1^{x*}}, \overline{s_1^{x*}}, \ldots, \underline{s_M^{x*}}, \overline{s_M^{x*}}, \underline{s_1^{y*}}, \overline{s_1^{y*}}, \ldots, \underline{s_S^{y*}}, \overline{s_S^{y*}} \lambda_1^*, \ldots, \lambda_N^*)$, with $\lambda_j^* \in \mathbb{R}_+$, $j = 1, \ldots, N$, $\underline{s_i^{x*}}, \overline{s_i^{x*}}, \underline{s_r^{y*}}, \overline{s_r^{y*}} \in \mathbb{Z}_+$ for $i \in O^{XI}, r \in O^{YI}$ and $\underline{s_i^{x*}}, \overline{s_i^{x*}}, \underline{s_r^{y*}}, \overline{s_r^{y*}} \in \mathbb{R}_+$ for $i \in O^{XNI}, r \in O^{YNI}$, is an optimal solution of (PHIDEA).*

In this new framework with integer and continuous interval data, we reconsider the inverse DEA concept from the classic concept under continuous crisp data. It is known that, in general, given a real number, it is not guaranteed that one can attain such a real number utilizing an arithmetic combination of a finite collection of integer numbers. The latter makes that, in general, given β_0 an increase of a Y_0, there exists no α_0 an increase of X_0 such that inefficiency $II^*(X_0, Y_0)$ or a given t-percent of it is attained, i.e., $II^*(\alpha_0, \beta_0) = (1 - t)II^*(X_0, Y_0)$. Furthermore, transformations of a formulation of DEA problems via change of variables are, in general, not consistent with the integer condition of the original variables; that is, the result of a transformed integer variable is not necessarily an integer. In this regard, if one follows the procedure proposed by Zhang and Ciu [153] applied to our hybrid DEA model using a variable, with the division between variables, then an integer variable becomes a non necessarily integer variable. These remarks make us approach the question of inverse DEA as follows. The aim of the question is to estimate the minimum increase of input, (α_o^*), if the output of DMU_o increases from Y_o to β_o, such the new DMU is given by (α_o^*, β) belongs to the technology, and its inefficiency score of is not less than t-percent. Here, $\alpha_o^* = (\alpha_{1o}^*, \alpha_{2o}^*, ..., \alpha_{Mo}^*) \in (\mathcal{K}_{\mathcal{Z}+})^{XI} * (\mathcal{K}_C)^{XNI}$, $\alpha_o^* \geqq X_0$, $\beta_o^* = (\alpha_{1o}^*, \beta_{2o}^*, ..., \beta_{So}^*) \in (\mathcal{K}_{\mathcal{Z}+})^{YI} * (\mathcal{K}_C)^{YNI}$, $\beta_0 \geqq Y_0$. After these previous considerations, the following slacks-based model estimate the inefficiency of the new DMU:

$$
\begin{aligned}
\text{(PHIDEA)} \;\; II^*(\alpha_o^*, \beta_o) = \quad & \text{Max} \;\; \sum_{i=1}^{M} \frac{\underline{s_i^x} + \overline{s_i^x}}{\underline{\alpha_{io}^*} + \overline{\alpha_{io}^*}} + \sum_{r=1}^{S} \frac{\underline{s_r^y} + \overline{s_r^y}}{\underline{\beta_{ro}} + \overline{\beta_{ro}}} \\
& \text{s.t.} \;\; \sum_{j=1}^{N} \lambda_j \underline{x_{ij}} \leq \underline{\alpha_{io}^*} - \overline{s_i^x}, \quad i \in O^X, \\
& \sum_{j=1}^{N} \lambda_j \overline{x_{ij}} \leq \overline{\alpha_{io}^*} - \underline{s_i^x}, \quad i \in O^X, \\
& \sum_{j=1}^{N} \lambda_j \underline{y_{rj}} \geq \underline{\beta_{ro}} + \underline{s_r^y}, \quad r \in O^Y, \\
& \sum_{j=1}^{N} \lambda_j \overline{y_{rj}} \geq \overline{\beta_{ro}} + \overline{s_r^y}, \quad r \in O^Y, \\
& \underline{s_i^x} \leq \overline{s_i^x}, \quad i \in O^X, \\
& \underline{s_r^y} \leq \overline{s_r^y}, \quad r \in O^Y, \\
& \lambda_j \geq 0, \quad j = 1, \ldots, N, \\
& \underline{s_i^x}, \overline{s_i^x}, \underline{s_r^y}, \overline{s_r^y} \in \mathbb{Z}_+, \quad i \in O^{XI}, \;\; r \in O^{YI}, \\
& \underline{s_i^x}, \overline{s_i^x}, \underline{s_r^y}, \overline{s_r^y} \geq 0, \quad i \in O^{XNI}, \;\; r \in O^{YNI}.
\end{aligned}
\tag{6.8}
$$

To solve integer interval problem, the following (IP) problem is established:

$$
\begin{aligned}
\text{(IP)} \quad \text{Min} \quad & (\alpha_{1o}, , ..., \alpha_{Mo}) \\
\text{s.t.} \quad & (\alpha_o, \beta_o) \in T, \\
& \alpha_o \geqq X_o, \\
& II^*(\alpha_o, \beta_o) \geq (1-t) II^*.
\end{aligned}
\tag{6.9}
$$

Definition 6.3.2. *Let $\alpha_o^* \in (\mathcal{K}_{\mathcal{Z}+})^{XI} * (\mathcal{K}_C)^{XNI}$ be a feasible solution to the model* (6.9)*. It is said to be an interval Pareto solution to the model* (6.9) *if there isn't feasible solution α_o of* (6.9) *such that $\alpha_o \leqq \alpha_o^*, \alpha_o \neq \alpha_o^*$.*

Definition 6.3.3. *Let $\alpha_o^* \in (\mathcal{K}_{\mathcal{Z}+})^{XI} * (\mathcal{K}_C)^{XNI}$ be a feasible solution to the model* (6.9)*. It is said to be an interval weakly Pareto solution to the model* (6.9) *if there isn't feasible solution α_o of* (6.9) *such that $\alpha_o \leqq \alpha_o^*$.*

After parametrization of (IP), the following (MONLP) is established:

$$
\begin{aligned}
\text{(MONLP)}\quad \text{Min}\quad & (\underline{\alpha_{1o}}, \overline{\alpha_{1o}}, \ldots, \underline{\alpha_{Mo}}, \overline{\alpha_{Mo}}) \qquad\qquad (6.10)\\
\text{s.t.}\quad & \sum_{j=1}^{N} \lambda_j \underline{x_{ij}} \le \underline{\alpha_{io}} - \overline{s_i^x}, \quad i \in O^X,\\
& \sum_{j=1}^{N} \lambda_j \overline{x_{ij}} \le \overline{\alpha_{io}} - \underline{s_i^x}, \quad i \in O^X,\\
& \sum_{j=1}^{N} \lambda_j \underline{y_{rj}} \ge \underline{\beta_{ro}} + \underline{s_r^y}, \quad r \in O^Y,\\
& \sum_{j=1}^{N} \lambda_j \overline{y_{rj}} \ge \overline{\beta_{ro}} + \overline{s_r^y}, \quad r \in O^Y,\\
& \sum_{i=1}^{M} \frac{\underline{s_i^x} + \overline{s_i^x}}{\underline{\alpha_{io}} + \overline{\alpha_{io}}} + \sum_{r=1}^{S} \frac{\underline{s_r^y} + \overline{s_r^y}}{\underline{\beta_{ro}} + \overline{\beta_{ro}}} \ge (1-t)II^*,\\
& \underline{\alpha_{io}} \ge \underline{x_{io}}, \quad i \in O^X,\\
& \overline{\alpha_{io}} \ge \overline{x_{io}}, \quad i \in O^X,\\
& \underline{s_i^x} \le \overline{s_i^x}, \quad i \in O^X,\\
& \underline{s_r^y} \le \overline{s_r^y}, \quad r \in O^Y,\\
& \underline{\alpha_{io}} \le \overline{\alpha_{io}}, \quad i \in O^X,\\
& \lambda_j \ge 0, \quad j = 1, \ldots, N,\\
& \underline{s_i^x}, \overline{s_i^x}, \underline{s_r^y}, \overline{s_r^y} \in \mathbb{Z}_+, \quad i \in O^{XI},\ r \in O^{YI},\\
& \underline{s_i^x}, \overline{s_i^x}, \underline{s_r^y}, \overline{s_r^y} \ge 0, \quad i \in O^{XNI},\ r \in O^{YNI},\\
& \underline{\alpha_{io}}, \overline{\alpha_{io}} \in \mathbb{Z}_+, \quad i \in O^{XI},\\
& \underline{\alpha_{io}}, \overline{\alpha_{io}} \ge 0, \quad i \in O^{XNI},
\end{aligned}
$$

where II^* is the optimal value of model (6.7) and $0 \le t \le 1$.

Given MONLP (6.10) and $w = (w_1, w_2, \cdots, w_{2M}) \in \mathbb{R}^{2M}$, $w_i > 0$, $\sum_{i=1}^{2M} w_i = 1$, we introduce the following related sum problem.

$$
\begin{aligned}
\text{(MONLP)w} \quad \text{Min} \quad & \sum_{i=1}^{2M} w_i \alpha_{io} && (6.11)\\
\text{s.t.} \quad & \sum_{j=1}^{N} \lambda_j \underline{x_{ij}} \leq \underline{\alpha_{io}} - \overline{s_i^x}, \quad i \in O^X, \\
& \sum_{j=1}^{N} \lambda_j \overline{x_{ij}} \leq \overline{\alpha_{io}} - \underline{s_i^x}, \quad i \in O^X, \\
& \sum_{j=1}^{N} \lambda_j \underline{y_{rj}} \geq \underline{\beta_{ro}} + \underline{s_r^y}, \quad r \in O^Y, \\
& \sum_{j=1}^{N} \lambda_j \overline{y_{rj}} \geq \overline{\beta_{ro}} + \overline{s_r^y}, \quad r \in O^Y, \\
& \sum_{i=1}^{M} \frac{\underline{s_i^x} + \overline{s_i^x}}{\underline{\alpha_{io}} + \overline{\alpha_{io}}} + \sum_{r=1}^{S} \frac{\underline{s_r^y} + \overline{s_r^y}}{\underline{\beta_{ro}} + \overline{\beta_{ro}}} \geq (1-t)II^*, \\
& \underline{\alpha_{io}} \geq \underline{x_{io}}, \quad i \in O^X, \\
& \overline{\alpha_{io}} \geq \overline{x_{io}}, \quad i \in O^X, \\
& \underline{s_i^x} \leq \overline{s_i^x}, \quad i \in O^X, \\
& \underline{s_r^y} \leq \overline{s_r^y}, \quad r \in O^Y, \\
& \underline{\alpha_{io}} \leq \overline{\alpha_{io}}, \quad i \in O^X, \\
& \lambda_j \geq 0, \quad j = 1, \ldots, N, \\
& \underline{s_i^x}, \overline{s_i^x}, \underline{s_r^y}, \overline{s_r^y} \in \mathbb{Z}_+, \quad i \in O^{XI}, \; r \in O^{YI}, \\
& \underline{s_i^x}, \overline{s_i^x}, \underline{s_r^y}, \overline{s_r^y} \geq 0, \quad i \in O^{XNI}, \; r \in O^{YNI}, \\
& \underline{\alpha_{io}}, \overline{\alpha_{io}} \in \mathbb{Z}_+, \quad i \in O^{XI}, \\
& \underline{\alpha_{io}}, \overline{\alpha_{io}} \geq 0, \quad i \in O^{XNI}.
\end{aligned}
$$

Let us pointed out that the previous problem is a mixed- integer nonlinear optimization problem, which is NP-hard, in general. To deal with it and compute examples (following), on the one hand, we include penalties on integer variables in the objective function, following a proposal used in [15] and [102], among others. Then, we apply the R-package called "nloptr", which used methods based on gradients to provide a solution. From now on, and for the sake of simplicity, we use a similar notation to refer to vector interval solutions of (IP) and their parameterizations as real vector solutions of (MONLP). In this regard, for instance, $\boldsymbol{\alpha}_o = (\underline{\alpha_{1o}}, \overline{\alpha_{1o}}, ..., \underline{\alpha_{Mo}}, \overline{\alpha_{Mo}})$ can be interpreted as a vector of intervals or as a vector of real numbers, depending on the problem at hand. The inequality relationships are used according to the previous interpretation, being $\leqq$ for intervals, and $\leqq$ for vectors of real numbers, for instance.

The following theorem represents the relationship between the (IP) and (MONLP) solutions.

Theorem 6.3.3. *$\boldsymbol{\alpha}_o^* \in (\mathcal{K}_{\mathbb{Z}_+})^{XI} * (\mathcal{K}_C)^{XNI}$, $XI + XNI = M$ is an interval Pareto solution of (IP) if and only if there exist $\boldsymbol{\lambda}^* \in \mathbb{R}_+^N$, $\boldsymbol{s}^{x*} \in (\mathcal{K}_{\mathbb{Z}_+})^{XI} * (\mathcal{K}_C)^{XNI}$, $XI + XNI = M$ and $\boldsymbol{s}^{y*} \in (\mathcal{K}_{\mathbb{Z}_+})^{YI} * (\mathcal{K}_C)^{YNI}$, $YI + YNI =$*

S such that the corresponding parameterization of $(\boldsymbol{\lambda}^*, \boldsymbol{\alpha}_o^*, \boldsymbol{s}^{x*}, \boldsymbol{s}^{y*})$, $(\lambda_1^*, \dots, \lambda_N^*, \underline{\alpha_{1o}^*}, \overline{\alpha_{1o}^*}, \dots, \underline{\alpha_{Mo}^*}, \overline{\alpha_{Mo}^*}, \underline{s_1^{x*}}, \overline{s_1^{x*}}, \dots, \underline{s_M^{x*}}, \overline{s_M^{x*}}, \underline{s_1^{y*}}, \overline{s_1^{y*}}, \dots, \underline{s_S^{y*}}, \overline{s_S^{y*}})$*, is a Pareto solution of (MONLP).*

Proof. (i) Suppose that $\boldsymbol{\alpha}_o^*$ is an interval Pareto solution of (IP). It implies that, if one considers the related optimization problem to calculate $II^*(\alpha_o^*, \beta_o)$, then there exist $\boldsymbol{\lambda}^* \in \mathbb{R}_+^N$, $\boldsymbol{s}^{x*} \in (\mathcal{K}_{\mathbb{Z}_+})^{XI} * (\mathcal{K}_C)^{XNI}$, $XI + XNI = M$ and $\boldsymbol{s}^{y*} \in (\mathcal{K}_{\mathbb{Z}_+})^{YI} * (\mathcal{K}_C)^{YNI}$, $YI + YNI = S$ such that

$$II^*(\alpha_o^*, \beta_o) = \sum_{i=1}^{M} \frac{\underline{s_i^{x*}} + \overline{s_i^{x*}}}{\underline{\alpha_{io}^*} + \overline{\alpha_{io}^*}} + \sum_{r=1}^{S} \frac{\underline{s_r^{y*}} + \overline{s_r^{y*}}}{\underline{\beta_{ro}} + \overline{\beta_{ro}}} \geq (1-t)II^*.$$

The latter means that $(\boldsymbol{\lambda}^*, \boldsymbol{\alpha}_o^*, \boldsymbol{s}^{x*}, \boldsymbol{s}^{y*})$, in its parameterization form, is a feasible solution of (MONLP). Now, reasoning by contradiction, suppose that$(\boldsymbol{\lambda}^*, \boldsymbol{\alpha}_o^*, \boldsymbol{s}^{x*}, \boldsymbol{s}^{y*})$ is not a Pareto solution of (MONLP), which implies that there exists $(\boldsymbol{\lambda}^{**}, \boldsymbol{\alpha}_o^{**}, \boldsymbol{s}^{x**}, \boldsymbol{s}^{y**})$ a feasible solution of (MONLP) such that $\alpha_o^{**} \leqq \alpha_o^*, \alpha_o^{**} \neq \alpha_o^*$. Therefore, $(\alpha_o^{**}, \beta_o) \in T, \alpha_o^{**} \geqq X_o$ and

$$II^*(\alpha_o^{**}, \beta_o) = \sum_{i=1}^{M} \frac{\underline{s_i^{x**}} + \overline{s_i^{x**}}}{\underline{\alpha_{io}^{**}} + \overline{\alpha_{io}^{**}}} + \sum_{r=1}^{S} \frac{\underline{s_r^{y**}} + \overline{s_r^{y**}}}{\underline{\beta_{ro}} + \overline{\beta_{ro}}} \geq (1-t)II^*.$$

In consequence, we have that α_o^{**} is a feasible solution of (IP), with $\alpha_o^{**} \leqq \alpha_o^*, \alpha_o^{**} \neq \alpha_o^*$, what is a contradiction with α_o^* is an interval Pareto solution of (IP).

(ii) Suppose that $(\lambda_1^*, \dots, \lambda_N^*, \underline{\alpha_{1o}^*}, \overline{\alpha_{1o}^*}, \dots, \underline{\alpha_{Mo}^*}, \overline{\alpha_{Mo}^*}, \underline{s_1^{x*}}, \overline{s_1^{x*}}, \dots, \underline{s_M^{x*}}, \overline{s_M^{x*}}, \underline{s_1^{y*}}, \overline{s_1^{y*}}, \dots, \underline{s_S^{y*}}, \overline{s_S^{y*}})$ is Pareto solution of (MONLP). From the problem (6.10), we derive

$$II^*(\alpha_o^*, \beta_o) = \sum_{i=1}^{M} \frac{\underline{s_i^{x*}} + \overline{s_i^{x*}}}{\underline{\alpha_{io}^*} + \overline{\alpha_{io}^*}} + \sum_{r=1}^{S} \frac{\underline{s_r^{y*}} + \overline{s_r^{y*}}}{\underline{\beta_{ro}} + \overline{\beta_{ro}}} \geq (1-t)II^*.$$

Then $(\alpha_o^*, \beta_o) \in T$, $\alpha_o^* \geqq X_o$, that is, α_o^* is a feasible solution of (IP). Proceeding by contradiction, suppose α_o^* is not an interval Pareto solution of (IP), i.e. there exists α_o^{**} feasible for (IP), with $\alpha_o^{**} \leqq \alpha_o^*$, $\alpha_o^{**} \neq \alpha_o^*$. Since α_o^{**} is feasible solution of (IP), it implies that there exists $(\lambda_1^{**}, \dots, \lambda_N^{**}, \underline{\alpha_{1o}^{**}}, \overline{\alpha_{1o}^{**}}, \dots, \underline{\alpha_{Mo}^{**}}, \overline{\alpha_{Mo}^{**}}, \underline{s_1^{x**}}, \overline{s_1^{x**}}, \dots, \underline{s_M^{x**}},$
$\overline{s_M^{x**}}, \underline{s_1^{y**}}, \overline{s_1^{y**}}, \dots, \underline{s_S^{y**}}, \overline{s_S^{y**}})$ feasible solution of (MONLP), with $\alpha_o^{**} \leqq \alpha_o^*$, $\alpha_o^{**} \neq \alpha_o^*$, what is a contradiction with α_o^* Pareto solution of (MONLP).

□

As a consequence of the previous theorem, we have the following one that shows that the above integer interval (MONLP) can be used for input level estimation.

Theorem 6.3.4. *Assume that* II^* *is the inefficiency score of* DMU_o *in the model* (6.7) *and the output of* DMU_o *are increased from* Y_0 *to* $\beta_0 = (\underline{\beta_{1o}}, \overline{\beta_{1o}}, \underline{\beta_{2o}}, \overline{\beta_{2o}}, ..., \underline{\beta_{So}}, \overline{\beta_{So}}) = Y_0 + \triangle Y_o$, $\triangle Y_o \gneqq 0$.

(1) Let $(\lambda_1^*, \dots, \lambda_N^*, \underline{\alpha_{1o}^*}, \overline{\alpha_{1o}^*}, \dots, \underline{\alpha_{Mo}^*}, \overline{\alpha_{Mo}^*}, \underline{s_1^{x*}}, \overline{s_1^{x*}}, \dots, \underline{s_M^{x*}}, \overline{s_M^{x*}}, \underline{s_1^{y*}}, \overline{s_1^{y*}}, \dots, \underline{s_S^{y*}}, \overline{s_S^{y*}})$ *be a Pareto solution to the model* (6.10)*, then the inefficiency score of* DMU_o *under new inputs and outputs is not less than t-percent.*

(2) Conversely, if the new DMU_o *belongs to the technology, and the inefficiency score of the new* DMU_o *is not less than t-percent, then there exist* $\boldsymbol{\lambda}^*, \boldsymbol{s}^{x*}, \boldsymbol{s}^{y*}$ *such that* $(\lambda_1^*, \dots, \lambda_N^*, \underline{\alpha_{1o}^*}, \overline{\alpha_{1o}^*}, \dots, \underline{\alpha_{Mo}^*}, \overline{\alpha_{Mo}^*},$

$\underline{s_1^{x*}}, \overline{s_1^{x*}}, \ldots, \underline{s_M^{x*}}, \overline{s_M^{x*}}, \underline{s_1^{y*}}, \overline{s_1^{y*}}, \ldots, \underline{s_S^{y*}}, \overline{s_S^{y*}}$) is a feasible solution for (MONLP). Furthermore, if any decrease in the input α_o^* of the new DMU_o in the Pareto sense makes not fulfill the previous conditions, then it follows that α_o^* is a Pareto solution of (MONLP).

Proof. If $(\lambda_1^*, \ldots, \lambda_N^*, \underline{\alpha_{1o}^*}, \overline{\alpha_{1o}^*}, \ldots, \underline{\alpha_{Mo}^*}, \overline{\alpha_{Mo}^*}, \underline{s_1^{x*}}, \overline{s_1^{x*}}, \ldots, \underline{s_M^{x*}}, \overline{s_M^{x*}}, \underline{s_1^{y*}}, \overline{s_1^{y*}}, \ldots, \underline{s_S^{y*}}, \overline{s_S^{y*}})$ is a Pareto solution of the problem (MONLP), then by Theorem 6.3.3 it follows that $(\boldsymbol{\lambda}^*, \boldsymbol{\alpha}_o^*, \boldsymbol{s}^{x*}, \boldsymbol{s}^{y*})$ is interval Pareto solution of (IP), and then $II^*(\alpha_o^*, \beta_o) \geq (1-t)II^*$. Therefore, (1) is proof. Conversely, if the inefficiency score of DMU_o is not less than t-percent, $II^*(\alpha_o^*, \beta_o) \geq (1-t)II^*$, it means that (α_o^*, β_o) is feasible for (IP), and there exist $\boldsymbol{\lambda}^*, \boldsymbol{s}^{x*}, \boldsymbol{s}^{y*}$ such that $(\lambda_1^*, \ldots, \lambda_N^*, \underline{\alpha_{1o}^*}, \overline{\alpha_{1o}^*}, \ldots, \underline{\alpha_{Mo}^*}, \overline{\alpha_{Mo}^*}, \underline{s_1^{x*}}, \overline{s_1^{x*}}, \ldots, \underline{s_M^{x*}}, \overline{s_M^{x*}}, \underline{s_1^{y*}}, \overline{s_1^{y*}}, \ldots, \underline{s_S^{y*}}, \overline{s_S^{y*}})$ is a feasible solution of (MONLP). Furthermore, since (α_o^*, β_o) is feasible for (IP) and there aren't $\alpha_o^{**} \leqq \alpha_o^*$, $\alpha_o^{**} \neq \alpha_o^*$ then α_o^* is an interval Pareto solution of (IP), and then, by Theorem 6.3.3, is a Pareto solution of (MONLP).

$\square$

6.4 Numerical experiments

In this section, we introduce a problem that contains both integer and continuous variables. The data set coming from Zhang and Cui [153] are shown in Table 6.1. There are 12 DMUs. Every DMU consume three inputs and produce two outputs. The first input and the second output are continuous, and the other data are integer. Firstly, we calculate the inefficiency score of the model (6.7). It is indicated in Table 6.2. Then due to the dependency between DEA and MONLP, we can relate inverse DEA mode into single objective programming by means of weighted problems. To illustrate the example, the result is shown for DMU_1 and DMU_2 in Table 6.3 and Table 6.4 for three values, respectively. In Table 6.3, we increase the output of DMU_1 from $Y_1 = ([67, 67], [751, 751])$ to $\beta_1 = ([80, 85], [780, 850])$ and put $t = 0.3$. After solving the model (6.10) by using weighted sum problem, $w = (0.2, 0.3, 0.1, 0.2, 0.1, 0.1)$, we can get $\alpha_1^* = ([350.00, 350.11], [47, 47], [13, 13])$. According to the the model (6.8), $II(^*_1, \beta_1) = 1.01$ which is not less than $(1-t)II^*(X_1, Y_1) = 0.994$. Also, if we increase from $Y_1 = ([67, 67], [751, 751])$ to $\beta_1 = ([70, 73], [760, 770])$ and put $t = 0.3$, a Pareto solution for MONLP will be $\alpha_1^* = ([350.00, 350.00], [47, 47], [13, 13])$, which means the inefficiency score is not less than $(1-t)II^*(X_1, Y_1) = 0.994$. In addition, again we increase from $Y_1 = ([67, 67], [751, 751])$ to $\beta_1 = ([67, 70], [760, 765])$ and put $t = 0.3$ and get $\alpha_1^* = ([350.00, 350.00], [47, 47], [13, 13])$ that is not less than $(1-t)II^*(X_1, Y_1) = 0.994$. Also, in table 4, we consider the problem for the outputs of DMU_2 and get new inputs. For example, when we increase $Y_2 = ([70, 76], [608, 620])$ to $\beta_2 = ([80, 85], [620, 630])$, we calculate $\alpha_2^* = ([304.73, 304.75], [35, 38], [14, 14])$ that the inefficiency score of new DMU is not less than $(1-t)II^*(X_2, Y_2) = 0.259$. Also, After changing $Y_2 = ([70, 76], [608, 620])$ to $\beta_2 = ([72, 78], [619, 625])$, we get $\alpha_2^* = ([298.00, 299.11], [35, 38], [14, 14])$ which $II^*(\alpha_2^*, \beta_2)$ is not less than $(1-t)II^*(X_2, Y_2) = 0.259$. Finally, we increase $Y_2 = ([70, 76], [608, 620])$ to $\beta_2 = ([75, 78], [610, 622])$, and a pareto solution will be $\alpha_2^* = ([298.00, 299.11], [35, 38], [14, 14])$ which the inefficiency score of new DMU is not less than $(1-t)II^*(X_2, Y_2) = 0.259$.

Table 6.1: Data of 12 DMUs.

DMU (j)	1	2	3	4	5	6	7	8	9	10	11	12
x_{1j}	[350, 350]	[298, 298]	[420, 424]	[281, 281]	[301, 301]	[360, 360]	[540, 540]	[276, 276]	[300, 350]	[444, 444]	[323, 323]	[444, 444]
x_{2j}	[39, 39]	[25, 28]	[31, 31]	[16, 16]	[16, 16]	[29, 29]	[15, 21]	[33, 33]	[25, 25]	[61, 67]	[25, 25]	[64, 64]
x_{3j}	[9, 9]	[8, 8]	[7, 7]	[9, 9]	[4, 7]	[17, 17]	[10, 10]	[5, 5]	[5, 5]	[6, 6]	[3, 7]	[3, 9]
y_{1j}	[67, 67]	[70, 76]	[75, 75]	[70, 70]	[75, 75]	[83, 83]	[70, 81]	[78, 78]	[75, 75]	[74, 74]	[25, 25]	[104, 104]
y_{2j}	[751, 751]	[608, 620]	[584, 584]	[665, 665]	[442, 449]	[1070, 1070]	[457, 457]	[583, 595]	[1074, 1074]	[1072, 1072]	[350, 350]	[1199, 1199]

Table 6.2: Results of slacks-based model

DMU (j)	1	2	3	4	5	6	7	8	9	10	11	12
$I^*(X_p, Y_p)$	1.42	0.37	1.66	0.0	0.0	0.0	0.0	0.0	0.0	1.02	2.68	0.0
s_1^x	[0.0, 50.00]	[0.0, 9.51]	[4.00, 60.00]	[0.0, 0.0]	[0.0, 0.0]	[0.0, 0.0]	[0.0, 0.0]	[0.0, 0.0]	[0.0, 0.0]	[24.00, 84.00]	[65.24, 89.12]	[0.0, 0.0]
s_2^x	[14, 14]	[0, 0]	[1, 1]	[0, 0]	[0, 0]	[0, 0]	[0, 0]	[0, 0]	[0, 0]	[31, 31]	[0, 0]	[0, 0]
s_3^x	[4, 4]	[2, 3]	[1, 1]	[0, 0]	[0, 0]	[0, 0]	[0, 0]	[0, 0]	[0, 0]	[0, 0]	[0, 0]	[0, 0]
s_1^y	[8, 8]	[0, 0]	[15, 15]	[0, 0]	[10, 10]	[10, 10]	[0, 0]	[0, 0]	[0, 0]	[16, 16]	[32, 32]	[0, 0]
s_2^y	[323.00, 323.00]	[28.37, 28.37]	[704.80, 704.80]	[0.0, 0.0]	[0.0, 0.0]	[0.0, 0.0]	[0.0, 0.0]	[0.0, 0.0]	[0.0, 0.0]	[216.80, 216.80]	[407.58, 407.58]	[0.0, 0.0]

Table 6.3: Results of inverse slacks-based model for DMU_1, $w = (0.2, 0.3, 0.1, 0.2, 0.1, 0.1)$ and $t = 0.3$

optimal value, slacks and inputs	$\beta_1 = ([80, 85], [780, 850])$	$\beta_1 = ([70, 73], [760, 770])$	$\beta_1 = (67, 70, 760, 765)$
α_1	$([350.00, 350.11], [47, 47], [13, 13])$	$([350.00, 350.00], [47, 47], [13, 13])$	$([350.00, 350.00], [47, 47], [13, 13])$
$I^*(\alpha^*, \beta)$	1.01	1.58	1.63
optimal value	191.76	191.70	191.74
s_1^x	$[0.00, 6.00]$	$[31.15, 34.74]$	$[30.80, 32.53]$
s_2^x	$[1, 1]$	$[1, 1]$	$[1, 1]$
s_3^x	$[3, 6]$	$[3, 6]$	$[3, 6]$
s_1^y	$[1, 2]$	$[1, 2]$	$[1, 2]$
s_2^y	$[1.00, 2.00]$	$[17.12, 17.30]$	$[20.55, 21.55]$

Table 6.4: Results of inverse slacks-based model for DMU_2, $w = (0.2, 0.3, 0.1, 0.2, 0.1, 0.1)$ and $t = 0.3$

optimal value, slacks and inputs	$\beta_2 = ([80, 85], [620, 630])$	$\beta_2 = ([72, 78], [619, 625])$	$\beta_2 = ([75, 78], [610, 622])$
α_2	$([304.73, 304.75], [35, 38], [14, 14])$	$([298.00, 299.11], [35, 38], [14, 14])$	$([298.00, 298.97], [35, 38], [14, 14])$
$I^*(\alpha^*, \beta)$	0.65	0.89	0.91
optimal value	176.05	165.19	165.06
s_1^x	$[0.00, 0.00]$	$[0.00, 2.00]$	$[0.00, 2.00]$
s_2^x	$[1, 1]$	$[1, 1]$	$[1, 1]$
s_3^x	$[2, 3]$	$[2, 3]$	$[2, 3]$
s_1^y	$[1, 2]$	$[1, 2]$	$[1, 2]$
s_2^y	$[1.00, 2.00]$	$[1.00, 2.00]$	$[1.00, 2.00]$

6.5 Conclusions

In this chapter, we present a new inverse DEA problem on the non-radial slacks-based model with an integer and continuous data set. The main question on inverse DEA on the input estimation has been discussed. in this regard, we use Pareto solutions of the MONLP to determine sufficient and necessary conditions of input estimation. It is shown that in this new framework, with integer and continuous interval data, it is not guaranteed when Y_o increase to β_o, there is an increase of X_o such that $I^*(\alpha_o, \beta_o) = (1 - t)I^*(X_o, Y_o)$, what happens with crisp data. This is of difference between crisp and interval data. Therefore, the method can be applied to increase inputs for a slacks-based model such that the inefficiency score of DMU_o is not less than t-percent. Necessary and sufficient conditions are established for each DMU with integer and interval variables. The present work establishes the first response to inverse DEA under integer interval-type uncertainty on data, which is an important step to address a future study under fuzzy data, which will lead our future research.

Chapter 7

Conclusions

This study has presented several new approaches for efficiency assessment when inputs and outputs are interval and fuzzy. We consider these approaches as follows.

The first approach presents a new study for efficiency assessment and target setting when the input and output data are fuzzy. It is based on polygonal fuzzy numbers and LU-fuzzy partial orders. From the observed fuzzy data, and using simple axioms analogous to the ones considered in the crisp case, the fuzzy PPS containing all feasible operating points is inferred. Based on this PPS a fuzzy ERM DEA model is proposed to compute, for each DMU, a crisp efficiency score and a fuzzy target. The second approach can handle data that are simultaneously uncertain and integer. Existing interval DEA approaches do not consider integer data and, conversely, integer DEA approaches assume crisp data. Although at the cost of requiring interval arithmetic and relational operators, with a higher number of constraints in its parameterization form, the proposed approach is able to address the joint integer interval scenario. It does so in a rigorous way, defining the corresponding integer interval PPS, its corresponding efficient subset, and finally, formulating the models that compute the inefficiency scores and the efficient targets. Another proposed approach considers a hybrid scenario that may involve trapezoidal fuzzy integer numbers ($TF_{\mathbb{Z}}$) and trapezoidal fuzzy numbers (TF_C). As an extension of the integer PPS given in [18], a fuzzy integer PPS is derived using fuzzy integer arithmetic and fuzzy integer partial orders. Once the corresponding FIDEA technology is established, a non-oriented slacks-based fuzzy integer DEA model is proposed. This allows not only computing efficiency scores but also efficient fuzzy targets. Finally, a new inverse DEA problem on the non-radial slacks-based model with integer and continuous data set is presented. The main question on inverse DEA on the input estimation has been discussed.

All in all, the contribution of this work is vast. Addressing the gap in the literature of DEA and inverse DEA with integer interval and fuzzy integer data. Using polygonal fuzzy numbers provides sample flexibility for modeling the uncertainty in the data and the non-radial approach, which exhausts all possible input and output slacks, provides increased discriminant power.

As regards potential research directions, this work establishes the first response to inverse DEA under integer interval-type uncertainty on data, which is an important step to address a future study under fuzzy data, which will lead our future research. Also, a first step would extend the triangular fuzzy integer arithmetic and partial orders to polygonal fuzzy integer numbers, which are more general than trapezoidal fuzzy integer numbers and thus allow more flexibility for modeling the uncertainty in the input and output data. Also, other types of DEA models, for example involving undesirable outputs, non discretionary variables or multiple processes (so-called network DEA) ought to be developed. Another interesting line of research, often neglected in the interval and fuzzy DEA literature, is that of applying this type of approache to real-world situations, e.g. manufacturing, healthcare, or transportation, in which there may be uncertainty in the input and output data.

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