Received: 11 November 2021 | Revised: 25 May 2022 | Accepted: 29 June 2022

DOI: 10.1002/mma.8597

RESEARCH ARTICLE

WILEY

Value reducts and bireducts: A comparative study

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Communicated by: J. Banasiak

Funding information
Department of Economy, Knowledge, Business and University of the Regional Government of Andalusia, Grant/Award Number: FEDER-UCA18-108612; European Cooperation in Science and Technology, Grant/Award Number: CA17124; European Regional Development Fund, Grant/Award Number: PID2019-108991GB-I00; State Research Agency (AEI)

In Rough Set Theory, the notion of bireduct allows to simultaneously reduce the sets of objects and attributes contained in a dataset. In addition, value reducts are used to remove some unnecessary values of certain attributes for a specific object. Therefore, the combination of both notions provides a higher reduction of unnecessary data. This paper is focused on the study of bireducts and value reducts of information and decision tables. We present theoretical results capturing different aspects about the relationship between bireducts and reducts, offering new insights at a conceptual level. We also analyze the relationship between bireducts and value reducts. The studied connections among these notions provide important profits for the efficient information analysis, as well as for the detection of unnecessary or redundant information.

KEYWORDS
bireducts, reducts, rough set theory, value reducts

MSC CLASSIFICATION
06A06, 06A15, 18A32, 03B60

1 | INTRODUCTION

Introduced by Pawlak in the 1980s, Rough Set Theory (RST) is a formal tool whose purpose is modeling and processing incomplete information contained in relational databases.[1] Due to the huge amount of data collected in databases, it is increasingly common to find problems for the management and treatment of the stored information.[2–6] This fact makes necessary to carry out a preprocessing of data in order to remove redundant information.[7–11]

Pawlak[12] provided a three-step method for analyzing information tables. The three-step method of rough set analysis addresses three different types of reductions that have to be carried out for the simplification of an information table.[13] The first step deals with the search of a minimal subset of attributes capable of containing the information of the complete set of attributes. The second step studies the dependencies of the attribute values to simplify decision rules. Finally, the third step analyzes the dependencies between the decision rules in order to eliminate redundancies. All these steps show the relevance of the notion of reduct for the reduction mechanisms of information and decision tables in RST. Indeed, a generic definition of reduct was provided in Yao and Fu.[13] This generic definition collects the sufficiency and non-redundancy conditions that must be satisfied by a reduct and is capable of accommodating the different interpretations of reduct, provided by each of the steps included in the Pawlak's method.

The notion of reduct, as a minimal set of attributes keeping the original information (associated with the first step mentioned above), has been widely studied within this mathematical theory.[14–18] In addition, Pawlak[12] also defined value reducts in RST in order to keep reducing the size of relational databases (associated with the second step). The idea of this notion is to consider the indispensable attributes for each object instead of indispensable attributes for all the objects. Hence, it is possible to disregard also certain values of attributes for each object, while the information of the original table is preserved. However, the notion of value reduct has not been so intensively studied so far. On the other hand, the notion of bireduct goes further and pursues the simultaneous removal of both, objects and attributes, by avoiding incompatibilities, redundancies and removing the existing noise.[19–25]

This paper aims to know better all the previously mentioned notions at a conceptual level, that is, the notion of reduct, bireduct and value reduct, in order to take advantages the potentiality of these notions for the information analysis. We will see that some bireducts are related to reducts, showing that these bireducts offer the reduction of the first and the third steps commented above. Moreover, we will prove that other bireducts include the value reduct, and so they also provide the second step of the usual reduction of redundant information on a given dataset. Therefore, the computation and analysis of bireducts and value reducts provide relevant knowledge of information tables. We will also analyze bireducts whose subset of objects contains all objects, bireducts whose subset of attributes contains all attributes and bireducts whose subsets of objects or attributes are the empty set. Moreover, we have shown that, given a subset of attributes B of a bireduct, each object in the universe belongs at least to an information bireduct associated with B. The connection that has been established between the notions of reduct and bireduct provides important profits. For example, only from the computation of reducts, we can build sub-tables (bireducts) for the efficient information analysis, removing redundancies in the attributes, as well as in the objects. Furthermore, the study carried out about decision tables also provides important results. One of these results lets us simplify the procedure to obtain decision bireducts, since it allows us to dispense with all the objects belonging to the positive region during the computation process of bireducts, which significantly reduces the number of objects to be considered.

In addition, we will take special attention to the comparison of bireducts and value reducts in order to discover the existing relationships between these notions. This second part will also be very interesting because, although originally both notions were defined independently, we will show remarkable links between them. In particular, we will prove that every value reduct B for an object x is the subset of attributes of a bireduct (X, B) in which x belongs to the subset of objects of the bireduct, $x \in X$. All the notions and results will appear accompanied with examples to make easier their understanding.

Finally, it is convenient to mention that some of the notions considered in this paper are those introduced by Pawlak in his book.[12] However, this paper takes into account a slightly different nomenclature. In particular, the names of "information table" and "decision table" are considered instead of "knowledge representation system," the name "reduct" refers in this paper to "reduct of knowledge" in Pawlak,[12] and "value reduct" and "d-value reduct" are used instead of "reduction of categories" and "relative reduct of categories," respectively.

The paper is organized as follows: preliminary notions of RST, together with an example, are recalled in Section 2. Afterwards, Section 3 presents the notion of bireduct together with different technical properties, as well as its relationship with reducts. Section 4 introduces the relationship between bireducts and value reducts through some results and examples. Finally, conclusions and future works are presented in Section 5.

2 | ROUGH SET THEORY

This section presents the basic definitions of RST which will play a crucial role in our study. First of all, it is convenient to recall that databases are represented as information tables in this framework. Formally, information tables are defined below.

Definition 1. Let U and $\mathcal{A}$ be non-empty sets of objects and attributes, respectively. An information table is a tuple $(U, \mathcal{A}, \mathcal{V}_{\mathcal{A}}, \overline{\mathcal{A}})$ such that $\mathcal{V}_{\mathcal{A}} = \{V_a | a \in \mathcal{A}\}$, where V_a is the set of values associated with the attribute a over U, and $\overline{\mathcal{A}} = \{\bar{a} : U \to V_a | a \in \mathcal{A}\}$.

Next, an equivalence relation is defined on the set of objects of an information table. This equivalence relation will be useful to compare objects according to a given subset of attributes.

Definition 2. Let $(U, \mathcal{A}, V_{\mathcal{A}}, \overline{\mathcal{A}})$ be an information table. The indiscernibility mapping $I : \mathcal{P}(\mathcal{A}) \to \mathcal{P}(U \times U)$ is defined, for each $B \subseteq \mathcal{A}$, as the equivalence relation

$$I(B) = \{(x, y) \in U \times U | \bar{a}(x) = \bar{a}(y), \quad \text{for all} \quad a \in B\}$$

which is called B-indiscernibility relation. Each class of $I(B)$ can be written as $[x]_{I(B)} = \{y \in U | (x, y) \in I(B)\}$. The partition determined by $I(B)$ on the set of objects U is denoted as $U/I(B) = \{[x]_{I(B)} | x \in U\}$.

Notice that in the particular case of $B = \emptyset$, we have that $I(\emptyset) = U \times U$. Based on an indiscernibility relation, the notions of indiscernible and discernible objects are introduced as follows.

Definition 3. Given an information table $(U, \mathcal{A}, V_{\mathcal{A}}, \overline{\mathcal{A}})$ and a subset of attributes $B \subseteq \mathcal{A}$, we say that $x, y \in U$ are *B-indiscernible* if $y \in [x]_{I(B)}$. Otherwise, we say x, y are B-discernible.

Now, we define a fundamental notion in the analysis and extraction of information from databases, that is, the notion of reduct. Reducts are introduced to cover the need to reduce information tables, removing their redundant and/or superfluous variables without losing information.

Definition 4. Given an information table $(U, \mathcal{A}, V_{\mathcal{A}}, \overline{\mathcal{A}})$ and a subset of attributes $B \subseteq \mathcal{A}$, we say that:

- $a \in B$ is *dispensable* in B if $I(B) = I(B \backslash \{a\})$. Otherwise, a is *indispensable* in B.
- B is *independent* if all its attributes are indispensable in B.
- $B' \subseteq B$ is a *reduct* of B if B' is independent and $I(B) = I(B')$.

Notice that, a reduct of $\mathcal{A}$ is a subset of attributes that preserves the partition and therefore, it classifies objects in the same way as if we consider the whole set of attributes. Consequently, attributes not belonging to a reduct are unnecessary.

The following notion allows to remove some values of certain attributes, when it is not possible to delete such attributes without altering the information contained in the considered database.

Definition 5. Let $(U, \mathcal{A}, V_{\mathcal{A}}, \overline{\mathcal{A}})$ be an information table, $B \subseteq \mathcal{A}$ and $x \in U$. We say that

- The value of an attribute $a \in B$ is *dispensable* in B for x, if the equality $[x]_{I(B)} = [x]_{I(B \backslash \{a\})}$ holds. Otherwise, the value of a is *indispensable* in B for x.
- B is *independent* for x, if the value of a is indispensable in B for x, for each attribute $a \in B$.
- $B' \subseteq B$ is a value reduct of B for x, if B' is independent for x and $[x]_{I(B)} = [x]_{I(B')}$.

A helpful particular case of information table arises when an attribute is highlighted, which is defined below.

Definition 6. Let U and $\mathcal{A}$ be non-empty sets of objects and attributes, respectively. A decision table is a tuple $(U, \mathcal{A}_d, \mathcal{V}_{\mathcal{A}_d}, \overline{\mathcal{A}_d})$ such that $\mathcal{A}_d = \mathcal{A} \cup \{d\}$ with $d \notin \mathcal{A}$, $\mathcal{V}_{\mathcal{A}_d} = \{V_a | a \in \mathcal{A}_d\}$, where V_a is the set of values associated with attribute a over U, and $\overline{\mathcal{A}_d} = \{\bar{a} : U \to V_a | a \in \mathcal{A}_d\}$. In this case, the attributes of $\mathcal{A}$ are called condition attributes and d is called decision attribute.

In what follows, the notions of dispensable, independent and reduct are translated to the decision table framework. These concepts are based on the well-known notion of positive region which is given below.

Definition 7. Let $(U, \mathcal{A}_d, \mathcal{V}_{\mathcal{A}_d}, \overline{\mathcal{A}_d})$ be a decision table and $B \subseteq \mathcal{A}$. The positive region of the partition $U/I(\{d\})$ with respect to B is defined as

$$POS_B(\{d\}) = \bigcup_{X \in U/I(\{d\})} B_*(X)$$

where $B_*(X) = \{x \in U | [x]_{I(B)} \subseteq X\}$.

Definition 8. Given a decision table $(U, \mathcal{A}_d, \mathcal{V}_{\mathcal{A}_d}, \overline{\mathcal{A}_d})$ and a subset of attributes $B \subseteq \mathcal{A}$, we say that

- $a \in B$ is d-dispensable in B, if $POS_B(\{d\}) = POS_{(B \backslash \{a\})}(\{d\})$. Otherwise, attribute a is *d-indispensable* in B.
- B is *d-independent* if all its attributes are d-indispensable in B.
- $B' \subseteq B$ is a *d-reduct* of B if B' is d-independent and the equality $POS_B(\{d\}) = POS_{B'}(\{d\})$ holds.

The dispensable nature of the values of the attributes included in a decision table is studied by using the following definition.

Definition 9. Given a decision table $(U, \mathcal{A}_d, \mathcal{V}_{\mathcal{A}_d}, \overline{\mathcal{A}_d})$, a subset of attributes $B \subseteq \mathcal{A}$ and $x \in U$, we say that

- The value of an attribute $a \in B$ is d-dispensable in B for x, if

$$[x]_{I(B)} \subseteq [x]_{I(\{d\})} \text{ and } [x]_{I(B \backslash \{a\})} \subseteq [x]_{I(\{d\})}$$

Otherwise, the value of attribute a is *d-indispensable* in B for x.
- B is *d-independent* for x, if the value of a is d-indispensable in B for x, for each attribute $a \in B$.
- $B' \subseteq B$ is a *d-value reduct* of B for x, if B' is d-independent for x and the inclusion $[x]_{I(B)} \subseteq [x]_{I(\{d\})}$ implies that $[x]_{I(B')} \subseteq [x]_{I(\{d\})}$.

This section finishes with an illustrative example in order to clarify the definitions presented previously. Specifically, the following example was introduced in Stawicki et al[25] for determining the suitability of a particular day for playing sport according to meteorological conditions.

Example 10. Consider an information table $(U, \mathcal{A}, \mathcal{V}_{\mathcal{A}}, \overline{\mathcal{A}})$ where the set of objects $U = \{1, 2, \dots, 14\}$ represents days of a certain month and the set of attributes $\mathcal{A} = \{Outlook(O), Temperature(T), Humidity(H), Wind(W)\}$ indicates meteorological conditions. This information table is represented in Table 1.

To begin with, we will see that $\mathcal{A}$ is actually the only reduct of the information table. It is easy to verify that there are no objects with the same values for the attributes of $\mathcal{A}$ and therefore, $[x]_{I(\mathcal{A})} = \{x\}$ for all $x \in U$. However, if we dispense with an attribute, for example we consider day 1 and the subset of attributes $\{O, T, H\}$, we obtain

$$\begin{aligned}
[1]_{I(\{O,T,H\})} =& \{y \in U \mid 1, y \text{ are } \{O, T, H\}\text{-indiscernible}\} \\
=& \{y \in U \mid \overline{O}(y) = \overline{O}(1) = sunny, \\
& \overline{T}(y) = \overline{T}(1) = \text{hot}, \\
& \overline{H}(y) = \overline{H}(1) = \text{high}\} \\
=& \{1, 2\}
\end{aligned}$$

Since $[1]_{I(\mathcal{A})} = \{1\} \neq \{1, 2\} = [1]_{I(\{O,T,H\})}$, we can conclude that $I(\mathcal{A}) \neq I(\{O, T, H\})$. Hence, by Definition 4, attribute W is indispensable in A. Following an analogous reasoning, we obtain $[1]_{I(\{O,H,W\})} = \{1, 8\}$ and therefore, attribute T is also indispensable in A. In addition, O is indispensable in A because $[1]_{I(\{T,H,W\})} = \{1, 3\}$. Notice that, from object 1, we cannot deduce that attribute H be indispensable since $[1]_{I(\mathcal{A})} = \{1\} = [1]_{I(\{O,T,W\})}$. However, from object 3, we have that $[3]_{I(\mathcal{A})} = \{3\} \neq \{3{,}13\} = [3]_{I(\{O,T,W\})}$. Then, by Definition 4, attribute H is indispensable in A.

Consequently, each attribute in $\mathcal{A}$ is indispensable for the information processing and we can ensure that $\mathcal{A}$ is the only reduct of the information table. Therefore, we need all the attributes to manage information presented in Table 1. In other words, if we omit any attribute, we lose information from the RST point of view.

Now, we will compute the value reducts corresponding to $\mathcal{A}$ by using Definition 5. For that, we need to eliminate the values of attributes which are dispensable in $\mathcal{A}$ for each object.

For instance, we calculate a value reduct of $\mathcal{A}$ for the object 1. We have that the equality $[1]_{I(\mathcal{A})} = [1]_{I(\{O,T,W\})}$ holds. In addition, $[1]_{I(\{O,T,W\})} \neq [1]_{I(\{O,T,W\} \setminus \{a\})}$ for all $a \in \{O, T, W\}$, since $[1]_{I(\{O,T\})} = \{1, 2\}$, $[1]_{I(\{O,W\})} = \{1, 8, 9\}$ and $[1]_{I(\{T,W\})} = \{1, 3, 13\}$. By Definition 5, the value of the attributes O, T, W is indispensable for the object 1 and $\{O, T, W\}$ is independent for the object 1. Then, $\{O, T, W\}$ is a value reduct of $\mathcal{A}$ for 1. In fact, it is the only value reduct of $\mathcal{A}$ for 1.

TABLE 1 Table associated with the information table $(U, \mathcal{A}, \mathcal{V}_{\mathcal{A}}, \overline{\mathcal{A}})$ given in Example 10

Day	Outlook	Temperature	Humidity	Wind
1	Sunny	Hot	High	Weak
2	Sunny	Hot	High	Strong
3	Overcast	Hot	High	Weak
4	Rain	Mild	High	Weak
5	Rain	Cool	Normal	Weak
6	Rain	Cool	Normal	Strong
7	Overcast	Cool	Normal	Strong
8	Sunny	Mild	High	Weak
9	Sunny	Cool	Normal	Weak
10	Rain	Mild	Normal	Weak
11	Sunny	Mild	Normal	Strong
12	Overcast	Mild	High	Strong
13	Overcast	Hot	Normal	Weak
14	Rain	Mild	High	Strong

Following an analogous procedure with the rest of objects, we obtain Table 2. The information collected in Table 2 indicates the value reducts corresponding to $\mathcal{A}$ for the considered information table $(U, \mathcal{A}, \mathcal{V}_{\mathcal{A}}, \overline{\mathcal{A}})$. From Table 2, we can conclude that O and T are the most important attributes of the information table because its values are indispensable for the most of objects.

Now, we will include a new column in Table 1 in order to obtain a decision table $(U, \mathcal{A}_d, \mathcal{V}_{\mathcal{A}_d}, \overline{\mathcal{A}_d})$ whose decision attribute is $\{d\} = \{Play\}$. The decision table is represented in Table 3 and it is studied below.

As it was shown previously, each pair of objects are $\mathcal{A}$-discernible and so, we deduce the equality $POS_{\mathcal{A}}(\{d\}) = U$. Now, we will prove that $\{O, H, W\}$ is a d-reduct of $\mathcal{A}$. First of all, we compute $U/I(\{O, H, W\})$, obtaining $[1]_{I(O,H,W)} = \{1, 8\}, [5]_{I(O,H,W)} = \{5,10\}$ and the rest of equivalence classes are the trivial classes. On the other hand, the set $U/I(\{d\})$ is composed of the classes $[1]_{I(\{d\})} = \{1, 2, 6, 8, 14\}$ and $[3]_{I(\{d\})} = \{3, 4, 5, 7, 9,10, 11,12, 13\}$, that is,

$$U/I(\{d\}) = \{[1]_{I(\{d\})}, [3]_{I(\{d\})}\}$$

As a consequence, we obtain

$$\begin{aligned}
POS_{\{O,H,W\}}(\{d\}) &= \{x \in U | [x]_{I(\{O,H,W\})} \subseteq [x]_{I(\{d\})}\} = U \\
POS_{\{O,H\}}(\{d\}) &= [1]_{I(\{O,H\})} \cup [3]_{I(\{O,H\})} \cup [7]_{I(\{O,H\})} \cup [9]_{I(\{O,H\})} \\
&= \{1, 2, 3, 7, 8, 9, 11,12, 13\} \\
POS_{\{O,W\}}(\{d\}) &= [3]_{I(\{O,W\})} \cup [4]_{I(\{O,W\})} \cup [6]_{I(\{O,W\})} \cup [7]_{I(\{O,W\})} \\
&= \{3, 4, 5, 6, 7,10, 12,13, 14\} \\
POS_{\{H,W\}}(\{d\}) &= [5]_{I(\{H,W\})} \\
&= \{5, 9, 10,13\}
\end{aligned}$$

Day	Outlook	Temperature	Humidity	Wind
1	Sunny	Hot	-	Weak
2	-	Hot	-	Strong
3	Overcast	Hot	High	-
4	Rain	-	High	Weak
5	Rain	Cool	-	Weak
6	Rain	Cool	-	Strong
7	Overcast	Cool	-	-
8	Sunny	Mild	High	-
9	Sunny	Cool	-	-
10	Rain	Mild	Normal	-
11	Sunny	Mild	Normal	-
12	Overcast	Mild	-	-
13	-	Hot	Normal	-
14	Rain	Mild	-	Strong

TABLE 2 Value reducts corresponding to $\mathcal{A}$ of the information table $(U, \mathcal{A}, \mathcal{V}_{\mathcal{A}}, \overline{\mathcal{A}})$ given in Example 10

Day	Outlook	Temperature	Humidity	Wind	Play
1	Sunny	Hot	High	Weak	No
2	Sunny	Hot	High	Strong	No
3	Overcast	Hot	High	Weak	Yes
4	Rain	Mild	High	Weak	Yes
5	Rain	Cool	Normal	Weak	Yes
6	Rain	Cool	Normal	Strong	No
7	Overcast	Cool	Normal	Strong	Yes
8	Sunny	Mild	High	Weak	No
9	Sunny	Cool	Normal	Weak	Yes
10	Rain	Mild	Normal	Weak	Yes
11	Sunny	Mild	Normal	Strong	Yes
12	Overcast	Mild	High	Strong	Yes
13	Overcast	Hot	Normal	Weak	Yes
14	Rain	Mild	High	Strong	No

TABLE 3 Table associated with the decision table $(U, \mathcal{A}_d, \mathcal{V}_{\mathcal{A}_d}, \overline{\mathcal{A}_d})$ given in Example 10

TABLE 4 d-value reducts corresponding to $\{O, H, W\}$ of the decision table $(U, \mathcal{A}_d, \mathcal{V}_{\mathcal{A}_d}, \overline{\mathcal{A}_d})$ given in Example 10

Day	Outlook	Temperature	Humidity	Wind	Play
1	Sunny	-	High	-	No
2	Sunny	-	High	-	No
3	Overcast	-	-	-	Yes
4	Rain	-	-	Weak	Yes
5	Rain	-	-	Weak	Yes
6	Rain	-	-	Strong	No
7	Overcast	-	-	-	Yes
8	Sunny	-	High	-	No
9	Sunny	-	Normal	-	Yes
10	Rain	-	-	Weak	Yes
11	Sunny	-	Normal	-	Yes
12	Overcast	-	-	-	Yes
13	Overcast	-	-	-	Yes
14	Rain	-	-	Strong	No

By Definition 8, each attribute in $\{O, H, W\}$ is d-indispensable and $\{O, H, W\}$ is d-independent. Consequently, $\{O, H, W\}$ is a d-reduct of $\mathcal{A}$. This fact implies that we can eliminate all values of *Temperature* for each object without losing information. Hence, we can decide if a day is suitable to play only by using *Outlook*, *Humidity* and *Wind*.

Next step is studying d-value reducts of $\{O, H, W\}$, since it is a d-reduct of $\mathcal{A}$, for all $x \in U$ by using Definition 9, which clearly also are d-value reducts of $\mathcal{A}$. For instance, we will show $\{O, H\}$ is a d-value reduct of $\{O, H, W\}$ for the object 1. According to the information collected in Table 3, we obtain

- $[1]_{I(\{O,H,W\})} = \{1, 8\} \subseteq [1]_{I(\{d\})} = \{1, 2, 6, 8, 14\}$.
- $[1]_{I(\{O,H\})} = \{1, 2, 8\} \subseteq [1]_{I(\{d\})}$.
- $[1]_{I(\{O\})} = \{1, 2, 8, 9, 11\} \nsubseteq [1]_{I(\{d\})}$.
- $[1]_{I(\{H\})} = \{1, 2, 3, 4, 8, 12, 14\} \nsubseteq [1]_{I(\{d\})}$.

Applying Definition 9, we have that the value of attributes O, H is d-indispensable and $\{O, H\}$ is d-independent. Therefore, $\{O, H\}$ is a d-value reduct of $\{O, H, W\}$ for the object 1. Hence, we can decide if a day is suitable to play by means of the values of *Outlook* and *Humidity* for the object 1. Following an analogous procedure with the rest of objects, we obtain Table 4, which indicates some d-value reducts corresponding to $\{O, H, W\}$ for the considered decision table $(U, \mathcal{A}_d, \mathcal{V}_{\mathcal{A}_d}, \overline{\mathcal{A}_d})$. In addition, it is convenient to emphasize that there may exist more than one d-value reduct for an object. For instance, as Table 4 shows, $\{O, H\}$ is a d-value reduct of $\{O, H, W\}$ for the object 9, and it is easy to see that $\{H, W\}$ is a d-value reduct of $\{O, H, W\}$ for this object too.

Therefore, value reducts offer a different point of view of the relationships among the attributes, which is very useful to obtain an extra level of reduction and only consider the strictly necessary attributes and values in the computation, such as in the determination of decision rules.

3 | BIREDUCTS. PROPERTIES AND THE RELATIONSHIP WITH REDUCTS

Bireducts arise as an extension of the classical notions of reducts in RST providing an extra flexibility for operating with subsets of attributes and subsets of objects from which those attributes can be portrayed efficiently.[19,22–25] Specifically, bireducts are used to reduce the number of attributes by avoiding the occurrence of incompatibilities and removing the existing noise in tabular databases.

This section introduces diverse properties of the different notions of bireduct: information bireduct, decision bireduct and U-decision bireduct, and the relationship with reducts, which will be illustrated with examples. These results complement the ones given in the initial study presented in Stawicki et al.[25]

3.1 | Information bireducts

Information bireducts generalize information reducts by operating with subsets of attributes and subsets of objects.[19,23,25] Specifically, information bireducts allow to distinguish objects by the subset of considered attributes, so they are good subrepresentations of the information table. In this section, an information table $(U, \mathcal{A}, \mathcal{V}_{\mathcal{A}}, \overline{\mathcal{A}})$ will be fixed. The notion of information bireduct is formally defined as follows.

Definition 11. Given $X \subseteq U$ and $B \subseteq \mathcal{A}$, we say that

- B is *X-irreducible* if there is no $B' \subset B$ such that all pairs $x, y \in X$ are B'-discernible.
- X is *B-inextensible* if there is no $X' \subset U$ with $X \subset X'$ such that all pairs $x, y \in X'$ are B-discernible.
- The pair (X, B) is an information bireduct if all pairs $x, y \in X$ are B-discernible, B is X-irreducible and X is B-inextensible.

Notice that information bireducts generate information subtables with only different rows (all objects are B-discernible). The following technical property guarantees that, given $B \subseteq \mathcal{A}$, if there exist information bireducts in the information table with B as subset of attributes, then each object of the universe belongs to some information bireduct.

Proposition 12. *Given a subset of attributes $B \subseteq \mathcal{A}$ and the subsets of objects $X_1, \dots, X_n \subseteq U$ such that the pairs $(X_1, B), \dots, (X_n, B)$ are all the information bireducts, then $U = \bigcup_{i=1}^{n} X_i$.*

Proof. It is clear that $\bigcup_{i=1}^{n} X_i \subseteq U$. We will prove the other inclusion by reductio ad absurdum. Given $y \in U$, we will suppose that $y \notin \bigcup_{i=1}^{n} X_i$. Therefore, $y \notin X_i$ for all $i \in \{1, \dots, n\}$. Consequently, without loss of generality, we can suppose that there exists $x_1 \in X_1$ such that x_1, y are B-indiscernible. Hence, $(X_1 \backslash \{x_1\} \cup \{y\}, B)$ is an information bireduct which leads us to a contradiction, since by hypothesis $(X_1, B), \dots, (X_n, B)$ are all information bireducts with B as subset of attributes and $X_i \neq X_1 \backslash \{x_1\} \cup \{y\}$ for all $i \in \{1, \dots, n\}$ because $y \notin \bigcup_{i=1}^{n} X_i$. As a consequence, $U \subseteq \bigcup_{i=1}^{n} X_i$. In conclusion, $U = \bigcup_{i=1}^{n} X_i$. □

The notion of information bireduct as well as the last property will be applied to a particular case next.

Example 13. Consider the information table given in Example 10 and the subset of attributes $B = \{O, T, W\} \subseteq \mathcal{A}$ in order to compute all information bireducts. First of all, we will calculate what objects are B-indiscernible. Taking into account the information collected in Table 1 and applying Definition 2, we obtain

$$\begin{aligned} U/I(B) = \{ & [1]_{I(B)}, [2]_{I(B)}, [3]_{I(B)}, [4]_{I(B)}, [5]_{I(B)}, \\ & [6]_{I(B)}, [7]_{I(B)}, [8]_{I(B)}, [9]_{I(B)}, [11]_{I(B)}, \\ & [12]_{I(B)}, [14]_{I(B)} \} \end{aligned}$$

where $[3]_{I(B)} = \{3{,}13\}$, $[4]_{I(B)} = \{4{,}10\}$ and the other classes are the trivial classes.

Therefore, according to Definition 3, the pairs of objects 3, 13 and 4, 10 are B-indiscernible and the other pairs of objects are B-discernible. Notice that, removing from U an object of each pair 3, 13 and 4, 10, we obtain the following sets $U \backslash \{10{,}13\}$, $U \backslash \{3{,}10\}$, $U \backslash \{4{,}13\}$ and $U \backslash \{3, 4\}$, which are all B-inextensible subsets of objects by construction. Consequently, by Definition 11, the candidates to information bireducts are the pairs $(U \backslash \{10{,}13\}, B)$, $(U \backslash \{3{,}10\}, B)$, $(U \backslash \{4{,}13\}, B)$ and $(U \backslash \{3, 4\}, B)$.

On the other hand, B must be X-irreducible, with $X \in \{U \backslash \{10{,}13\}, U \backslash \{3{,}10\}, U \backslash \{4{,}13\}, U \backslash \{3, 4\}\}$, in order to satisfy the conditions required in the notion of information bireduct. It can be seen immediately that the objects 1,2 are $\{O, T\}$-indiscernible, the objects 1,8 are $\{O, W\}$-indiscernible and the objects 5,9 are $\{T, W\}$-indiscernible. Hence, for every of the four possibilities for X, there is no $B' \subset \mathcal{A}$ with $B' \subset B$ such that all pairs $x, y \in X$ are B'-discernible, that is, B is X-irreducible. Therefore, we can ensure that $(U \backslash \{10{,}13\}, B)$, $(U \backslash \{3{,}10\}, B)$, $(U \backslash \{4{,}13\}, B)$ and $(U \backslash \{3, 4\}, B)$ are all the information bireducts.

Finally, according to Proposition 12, it is straightforwardly obtained that each object belongs to some information bireduct, that is,

$$U = U \backslash \{10{,}13\} \bigcup U \backslash \{3{,}10\} \bigcup U \backslash \{4{,}13\} \bigcup U \backslash \{3, 4\}$$

Therefore, the set of information bireducts with a fixed subset of attributes B allows the division of information tables into subtables, which are non-redundant and each object is represented in at least one of these subtables, so no information is lost. In this way, it is possible to rebuild the original table by using the generated subtables. Moreover, an important

TABLE 5 Table associated with $(U, \mathcal{A}, \mathcal{V}_{\mathcal{A}}, \overline{\mathcal{A}})$ given in Example 15

Person	Height	Age	Gender
1	Medium	Young	Woman
2	Medium	Young	Woman
3	Medium	Young	Man
4	Short	Old	Woman

particular case is when the subset of attributes B is a reduct of $\mathcal{A}$. In this case, each information subtable contains the same information that the original table, providing a great representation of the database.

Next, two particular cases of information bireducts are introduced. The first one was already proved in Benítez-Caballero et al,[26] and it relates the notion of reduct to the notion of information bireduct.

Proposition 14 (Benítez-Caballero et al.[26]). *Given $B \subseteq \mathcal{A}$, if the pair (U, B) is an information bireduct, then B is a reduct of $\mathcal{A}$.*

Since (U, B) is an information bireduct, we can conclude that B distinguishes all the objects of U and for each subset $B' \subseteq B$ there exists $x_{B'}, y_{B'} \in U$ such that they are B'-indiscernible. As a consequence, B is a reduct of $\mathcal{A}$. However, if B is a reduct, there may have objects that B is not able to distinguish because $\mathcal{A}$ do not distinguish them. Therefore, the reciprocal of the last property is not true in general, as it is illustrated in the following example.

Example 15. Let $(U, \mathcal{A}, \mathcal{V}_{\mathcal{A}}, \overline{\mathcal{A}})$ be the information table represented in Table 5, where the set of objects $U = \{1, 2, 3, 4\}$ represents patients, the set of attributes is given by $\mathcal{A} = \{\text{Height}(h), \text{Age}(a), \text{Gender}(g)\}$.

From Table 5 it is easy to see that the equality $I(\{h, g\}) = I(\mathcal{A})$ holds, since $U/I(\mathcal{A}) = U/I(\{h, g\}) = \{[1]_{I(\mathcal{A})}, [3]_{I(\mathcal{A})}, [4]_{I(\mathcal{A})}\}$ where $[1]_{I(\mathcal{A})} = \{1, 2\}$ and the other classes are the trivial classes. Notice that $[1]_{I(\{h\})} = \{1, 2, 3\} \neq [1]_{I(\{h,g\})}$ and $[1]_{I(\{g\})} = \{1, 2, 4\} \neq [1]_{I(\{h,g\})}$, then attributes h and g are indispensable. Therefore, by using Definition 4, we conclude that $\{h, g\}$ is a reduct of $\mathcal{A}$. Now, we will show that $(U, \{h, g\})$ is not an information bireduct, that is, the counterpart of Proposition 14 is not satisfied.

The pair of objects 1,2 are $\{h, g\}$-indiscernible whereas the other pairs of objects are $\{h, g\}$-discernible. As a consequence, all information bireducts with $\{h, g\}$ as subset of attributes are $(\{1, 3, 4\}, \{h, g\})$ and $(\{2, 3, 4\}, \{h, g\})$. Therefore, $(U, \{h, g\})$ is not an information bireduct.

As it is showed in the previous example, reducts play an important role in the calculus of information bireducts. Now, we present the result that summarizes this property.

Proposition 16. *Given $B \subseteq \mathcal{A}$, if B is a reduct of $\mathcal{A}$ then there exists $X \subseteq U$ such that (X, B) is an information bireduct.*

Proof. Suppose that B is a reduct of $\mathcal{A}$. Consider a subset of objects $X \subseteq U$ such that each pair of objects $x, y \in X$ are B-discernible and X is B-inextensible. On the other hand, since B is a reduct of $\mathcal{A}$, for each $B' \subset B$ there exist $x_{B'}, y_{B'} \in X$ B-discernible and B'-indiscernible. As a consequence, B is X-irreducible and (X, B) is an information bireduct.

Thanks to this result, it is possible to relate the notion of information bireduct to the notion of reduct. Therefore, if a reduct B of an information table is known, a non-redundant subtable can be built by using this subset of attributes and which contains the same information as the original table. Thus, the reduction given by a bireduct associated with a reduct is equivalent to the first and the third steps of Pawlak three-step rough set analysis.

In the next result, we study information bireducts whose subset of attributes is the total set $\mathcal{A}$, obtaining a double equivalence between information bireducts and reducts of $\mathcal{A}$. □

Corollary 17. *There exists $X \subseteq U$ such that $(X, \mathcal{A})$ is an information bireduct if and only if $\mathcal{A}$ is a reduct of $\mathcal{A}$.*

Proof. Suppose that $(X, \mathcal{A})$ is an information bireduct. Hence, each pair $x, y \in X$ are $\mathcal{A}$-discernible. We will prove that $\mathcal{A}$ is a reduct of $\mathcal{A}$ by reductio ad absurdum. Suppose that there exists $B \subset \mathcal{A}$ being B a reduct of $\mathcal{A}$. Then, $I(B) = I(\mathcal{A})$. Therefore, each pair $x, y \in X$ are B-discernible, obtaining a contradiction because $\mathcal{A}$ is X-irreducible.

The counterpart is obtained from Proposition 16. □

From Corollary 17, we can conclude that the reciprocal of Proposition 16 is only satisfied when $B = \mathcal{A}$.

Next result studies information bireducts whose subset of attributes is the empty set, that is, there are no attribute to distinguish the elements in the subset of objects.

Proposition 18. *Let (X, B) be an information bireduct, with $X \subseteq U$ and $B \subseteq \mathcal{A}$. Then $B = \emptyset$ if and only if $X = \{x\}$ for all $x \in U$.*

Proof. Suppose $B = \emptyset$ and there exist $x, y \in X$ such that $x \neq y$. By hypothesis $(X, \emptyset)$ is an information bireduct, then we obtain that x, y are $\emptyset$-discernible. This fact lead us to a contradiction, since each pair of objects in U are $\emptyset$-indiscernible. Consequently, $X = \{x\}$ for all $x \in U$.

Now, we prove the counterpart supposing that $X = \{x\}$ for all $x \in U$. By hypothesis (X, B) is an information bireduct, then X is B-inextensible and we obtain that x, y are B-indiscernible for all $y \in U$. Therefore, for every $B' \subseteq B$, we have that x, y are B'-indiscernible for all $y \in U$. Since B is X-irreducible, it implies that $B = \emptyset$. □

Consequently, when there exists a subset of attributes $B \subseteq \mathcal{A}$ that cannot discern any pair of objects, we cannot find any information bireducts with any subset $B' \subseteq B$ as subset of attributes, with the exception of the empty set. If the subset of attributes considered is the empty set, which cannot discern objects, then each object must be taken individually.

The study of particular cases of information bireducts is concluded remarking that there are no information bireducts whose subset of objects is the empty set. This occurs because we can always consider at least one object, as it is shown in Proposition 18.

This section finishes highlighting a feature of Definition 11 which could be strange in some practical examples. Notice that, if there exist $x, y \in U$ such that $[x]_{I(\mathcal{A})} = [y]_{I(\mathcal{A})}$ then x, y are B-indiscernible, for all $B \subseteq \mathcal{A}$, as a consequence x, y cannot belong to the same information bireduct (X, B). However, the choice of B has not caused x, y to be B-indiscernible due to they are $\mathcal{A}$-indiscernible. Since this fact is independent of B, it has no sense considering these objects to compute the information bireducts. This fact would lead us to obtain two information bireducts whose only difference is that one of them contains the object x and the other one the object y, being in essence exactly the same information provided by both bireducts. In order to avoid this redundancy, another type of bireducts defined in information tables could be considered taking into account this fact. For that, next definition proposes to include all $\mathcal{A}$-indiscernible objects in equivalence classes so they can belong to a same information bireduct.

Definition 19. Let $(U, \mathcal{A}, \mathcal{V}_{\mathcal{A}}, \overline{\mathcal{A}})$ be an information table. An information table reduced by classes is a tuple $(U^*, \mathcal{A}_s, \mathcal{V}_{\mathcal{A}_s}, \overline{\mathcal{A}_s})$ such that $U^* = \{[x]_{I(\mathcal{A})} | x \in U\}$, $\mathcal{A}_s = \mathcal{A} \cup \{s\}$ with $s \notin \mathcal{A}$, $\mathcal{V}_{\mathcal{A}_s} = \{V_a | a \in \mathcal{A}_s\}$ being V_a the set of values associated with the attribute a over U^*, and $\overline{\mathcal{A}_s}$ is the set which collects the mappings associated with each $a \in \mathcal{A}$ defined as

$$\begin{array}{rrcl} \overline{a^*}: & U^* & \to & V_a \\ & [x]_{I(\mathcal{A})} & \mapsto & \bar{a}(x) \end{array}$$

and the mapping associated with the attribute s defined as

$$\begin{array}{rrcl} \bar{s}: & U^* & \to & V_s \\ & [x]_{I(\mathcal{A})} & \mapsto & card([x]_{I(\mathcal{A})}) \end{array}$$

Notice that the attribute s provides the number of objects $\mathcal{A}$-indiscernible with x and $V_s \subset \mathbb{N}$.

Following the same philosophy that the last two results, we present the following properties of information tables reduced by classes.

Proposition 20. *Let $(U^*, \mathcal{A}_s, \mathcal{V}_{\mathcal{A}_s}, \overline{\mathcal{A}_s})$ be an information table reduced by classes and $B \subseteq \mathcal{A}$. The pair (U^*, B) is an information bireduct if and only if B is a reduct of $\mathcal{A}$.*

Proof. Suppose that B is a reduct of $\mathcal{A}$. Given $x^*, y^* \in U^*$, by the definition of U^*, we have that x^*, y^* are $\mathcal{A}$-discernible. Since $I(\mathcal{A}) = I(B)$ because B is a reduct, then we deduce that x^*, y^* are B-discernible. Therefore, the pair (U^*, B) is an information bireduct.

The counterpart is obtained from Proposition 14 taking into account that the elements of U^* are the classes $[x]_{I(\mathcal{A})}$, for all $x \in U$. □

The last property shows that the information tables considered in Benítez-Caballero et al[26] are actually information tables reduced by classes.

Proposition 21. *Let $(U^*, \mathcal{A}_s, \mathcal{V}_{\mathcal{A}_s}, \overline{\mathcal{A}_s})$ be an information table reduced by classes and (X, B) be an information bireduct, with $X \subseteq U^*$, $B \subseteq \mathcal{A}$. Then $B = \emptyset$ if and only if $X = [x]_{I(\mathcal{A})}$ for all $x \in U$.*

Proof. The proof is obtained directly by Proposition 18, taking into account that the elements of U^* are the classes $[x]_{I(\mathcal{A})}$, for all $x \in U$. □

The previous properties and examples highlight the need of studying information bireducts for a useful management of information in data analysis. Now, we carry out the study of decision bireducts following a similar structure to this section.

3.2 | Decision bireducts

Decision bireducts are useful to analyze if a group of different subsets of attributes classifies suitably a subset of objects in an easier way.[19,24,25] Now, a decision table $(U, \mathcal{A}_d, \mathcal{V}_{\mathcal{A}_d}, \overline{\mathcal{A}_d})$ will be fixed. The formal notion of decision bireduct together with an illustrative example are given below.

Definition 22. Given $X \subseteq U$ and $B \subseteq \mathcal{A}$, we say that the pair (X, B) is a decision bireduct if every pair $x, y \in X$ is B-discernible when they are d-discernible, B is X-irreducible and X is B-inextensible with respect to this property.

It is important to emphasize that decision bireducts generate decision subtables where there are no contradictions. Therefore, this notion is very useful for decision-making. Next, we illustrate this notion in the following example.

Example 23. Consider the decision table given in Example 10 and the subset of attributes $B = \{O, T, H\} \subseteq \mathcal{A}$ in order to compute all decision bireducts with subset of attributes B. Taking into account the information contained in Table 3 and applying Definition 2, we obtain

$$U/I(B) = \{[1]_{I(B)}, [3]_{I(B)}, [4]_{I(B)}, [5]_{I(B)}, [7]_{I(B)}, [8]_{I(B)}, \ [9]_{I(B)}, [10]_{I(B)}, [11]_{I(B)}, [12]_{I(B)}, [13]_{I(B)}\}$$

where $[1]_{I(B)} = \{1, 2\}$, $[4]_{I(B)} = \{4{,}14\}$, $[5]_{I(B)} = \{5, 6\}$ and the other classes are the trivial classes. Recall that

$$U/I(\{d\}) = \{[1]_{I(\{d\})}, [3]_{I(\{d\})}\}$$

where $[1]_{I(\{d\})} = \{1, 2, 6, 8, 14\}$, $[3]_{I(\{d\})} = \{3, 4, 5, 7, 9{,}10, 11{,}12, 13\}$. Therefore, it is easy to see that the pairs of objects 4,14 and 5,6 are the only d-discernible and B-indiscernible objects simultaneously. Hence, these pairs of objects cannot appear in the same decision bireduct.

If we withdraw an object of each pair 4,14 and 5,6 from U, we obtain the sets $U\backslash\{6{,}14\}$, $U\backslash\{5{,}14\}$, $U\backslash\{4, 6\}$ and $U\backslash\{4, 5\}$ which are B-inextensible by construction. Moreover, all objects included in these sets are B-discernible when they are d-discernible. As a result, by Definition 22, the candidates to decision bireducts are $(U\backslash\{6{,}14\}, B)$, $(U\backslash\{5{,}14\}, B)$, $(U\backslash\{4, 6\}, B)$ and $(U\backslash\{4, 5\}, B)$. In order to guarantee that they are actually decision bireducts, we need to see that B is X-irreducible, being $X \in \{U\backslash\{6{,}14\}, U\backslash\{5{,}14\}, U\backslash\{4, 6\}, U\backslash\{4, 5\}\}$.

From Table 3 and the classes of the quotient sets $U/I(\{d\})$ and $U/I(B)$, we can ensure that

- The pairs of objects 8,11 and 10,14 are B-discernible, d-discernible and $\{O, T\}$-indiscernible.
- The pairs of objects 1,3; 2,3; 4,8; 8,12; 6,7; 6,9 and 12,14 are B-discernible, d-discernible and $\{T, H\}$-indiscernible.
- The pair of objects 6,10 are B-discernible, d-discernible and $\{O, H\}$-indiscernible.

With respect to the two first items, we have that at least one pair appears in all candidates to decision bireduct. Therefore, attributes H and O cannot be removed from any candidate. In addition, the pair of objects 6,10 only appear in candidates $(U\backslash\{5{,}14\}, B)$ and $(U\backslash\{4, 5\}, B)$, so that it is not possible to eliminate the attribute T in $(U\backslash\{5{,}14\}, B)$ and $(U\backslash\{4, 5\}, B)$.

Consequently, B is X-irreducible only for the pairs $(U\backslash\{5{,}14\}, B)$ and $(U\backslash\{4, 5\}, B)$ and then, they are the only decision bireducts whose subset of attributes is B. We also have that $(U\backslash\{6{,}14\}, \{O, H\})$, and $(U\backslash\{4, 6\}, \{O, H\})$ are decision bireducts.

It is important to mention that Proposition 12 is not satisfied by decision bireducts. In fact, the previous example is a counterexample. Specifically, we have that object 5 does not belong to any decision bireduct with B as subset of attributes.

On the other hand, next result makes easier the computation of decision bireducts thanks to the notion of positive region.

Proposition 24. *Given $X \subseteq U$ and $B \subseteq \mathcal{A}$, if the pair (X, B) is a decision bireduct then $POS_B(\{d\}) \subseteq X$.*

Proof. Let (X, B) be a decision bireduct. We will prove that $POS_B(\{d\}) \subseteq X$ by reductio ad absurdum. Let $x \in POS_B(\{d\})$ such that $x \notin X$. As (X, B) is a decision bireduct, the subset X is B-inextensible. Then, there exists $y \in X$ such that x, y are d-discernible and B-indiscernible. As a consequence, $[x]_{I(B)} \nsubseteq [x]_{I(\{d\})}$. Therefore, $x \notin POS_B(\{d\})$, obtaining a contradiction. □

This result allows to simplify the procedure to obtain decision bireducts, since the positive region is very useful to the construction of the subset of objects X. On the other hand, it is important to emphasize that the inclusion is usually strict, that is $POS_B(\{d\}) \neq X$. For instance, in Example 23, $(U \backslash \{5,14\}, B)$ and $(U \backslash \{4,5\}, B)$ are all decision bireducts with $B = \{O, T, H\}$ as subset of attributes. In addition, from the computation of $U/I(\{d\})$ and $U/I(B)$ it is easy to see that $POS_B(\{d\}) = U \backslash \{4, 5, 6, 14\}$. Therefore, in both cases, $POS_B(\{d\}) \neq X$.

Following the same scheme of the previous section, we introduce a result which relates the definitions of decision bireduct and d-reduct.

Proposition 25. *Let $B \subseteq \mathcal{A}$ be a subset of attributes. If the pair (U, B) is a decision bireduct then $POS_B(\{d\}) = U$. Moreover, B is a d-reduct of $\mathcal{A}$.*

Proof. First of all, we will prove that $POS_B(\{d\}) = U$. Clearly, $POS_B(\{d\}) \subseteq U$. Now, given $x \in U$ and $y \in [x]_{I(B)} \subseteq U$, we have that x, y are B-indiscernible. Since (U, B) is a decision bireduct, then x, y are also d-indiscernible. Therefore, $y \in [x]_{I(\{d\})}$. Consequently, $[x]_{I(B)} \subseteq [x]_{I(\{d\})}$ which implies that $x \in POS_B(\{d\})$. Hence, the equality $POS_B(\{d\}) = U$ holds.

Furthermore, we must prove that B is a d-reduct of $\mathcal{A}$. First of all, notice that $POS_{\mathcal{A}}(\{d\}) = U$ since $POS_B(\{d\}) \subseteq POS_{\mathcal{A}}(\{d\})$. Now, we will prove the irreducibility condition by reductio ad absurdum. We suppose that there exists $B' \subset B$ such that $POS_{B'}(\{d\}) = POS_B(\{d\}) = U$. Since (U, B) is a decision bireduct, we obtain that for each $B' \subset B$ there exist $x_{B'}, y_{B'} \in U$ such that they are d-discernible, B-discernible and B'-indiscernible. As a consequence, $x_{B'}, y_{B'} \in POS_B(\{d\})$ but $x_{B'}, y_{B'} \notin POS_{B'}(\{d\})$. Hence, $POS_B(\{d\}) \neq POS_{B'}(\{d\})$, obtaining a contradiction. As a result, B is d-independent and therefore B is a d-reduct of $\mathcal{A}$. □

The following example shows that B be a d-reduct of $\mathcal{A}$ is not enough to ensure that (U, B) is a decision bireduct.

Example 26. We consider the information table of Example 15, and we add a decision attribute $\{d\} = \{\text{Test}(t)\}$ which represents a medical test. This decision table is represented in Table 6.

Consider the subset of attributes $B = \{h, g\}$. It is easy to see that

$$\begin{aligned} U/I(\{d\}) &= \{[1]_{I(\{d\})}, [2]_{I(\{d\})}\} \\ U/I(B) &= \{[1]_{I(B)}, [3]_{I(B)}, [4]_{I(B)}\} \end{aligned}$$

where $[1]_{I(\{d\})} = \{1, 3\}, [2]_{I(\{d\})} = \{2, 4\}, [1]_{I(B)} = \{1, 2\}$ and the other classes are the trivial classes. In addition, in Example 15, we showed that $U/I(\mathcal{A}) = U/I(B)$. Hence, it is obtained that $POS_{\mathcal{A}}(\{d\}) = POS_B(\{d\}) = \{3, 4\}$. On the other hand, it is easy to see that $POS_{\{h\}}(\{d\}) = \{4\}$ and $POS_{\{g\}}(\{d\}) = \{3\}$, so both attributes are d-indispensable in B, and therefore, B is a d-reduct of $\mathcal{A}$. However, since $POS_B(\{d\}) = \{3, 4\} \neq U$, by Proposition 25, we obtain that (U, B) is not a decision bireduct. Indeed, in this case, there is no subset of objects $X \subseteq U$ such that (X, B) is a decision bireduct, as we show next.

We have that the pair of objects 1,2 are B-indiscernible and d-discernible, whereas the other pairs of objects are B-discernible if they are d-discernible. As a consequence, the only candidates to decision bireduct with B as subset of attributes are $(\{1, 3, 4\}, B)$ and $(\{2, 3, 4\}, B)$. However, it is easy to see that B is not X-irreducible with $X \in \{\{1, 3, 4\}, \{2, 3, 4\}\}$ because $(\{1, 3, 4\}, \{h\})$ and $(\{2, 3, 4\}, \{g\})$ are decision bireducts. Therefore, since B is a d-reduct of $\mathcal{A}$ and does not exist $X \subseteq U$ such that (X, B) is a decision bireduct, we have that Proposition 16 is not true for decision tables.

	Height	Age	Gender	Test
1	Medium	Young	Woman	+
2	Medium	Young	Woman	–
3	Medium	Young	Man	+
4	Short	Old	Woman	–

TABLE 6 Table associated with the decision table $(U, \mathcal{A}_d, \mathcal{V}_{\mathcal{A}_d}, \overline{\mathcal{A}_d})$ given in Example 26

TABLE 7 Table associated with the decision table $(U, \mathcal{A}_d, \mathcal{V}_{\mathcal{A}_d}, \overline{\mathcal{A}_d})$ given in Example 26 modified

	Height	Age	Gender	Test
1	Medium	Young	Woman	+
2	Medium	Young	Woman	–
3	Medium	Young	Man	+
4	Short	Old	Woman	–
5	Medium	Old	Woman	+
6	Medium	Old	Woman	–

In addition, we will show that Corollary 17 is not true in decision tables. With this purpose, we add two objects to the original table. This decision table is represented in Table 7.

We will show that $(\{2,3,4,5\}, \mathcal{A})$ is a decision bireduct. From Table 7, it is easy to see that all elements in $\{2,3,4,5\}$ are $\mathcal{A}$-discernible if they are d-discernible. In addition, $\{2,3,4,5\}$ is $\mathcal{A}$-inextensible because the pairs of objects $1,2$ and 5,6 are d-discernible and $\mathcal{A}$-indiscernible.

Now, we will prove that $\mathcal{A}$ is $\{2,3,4,5\}$-irreducible. On the one hand, the pair of objects $2,3$ are $\{h,a\}$-indiscernible and d-discernible, so that the attribute g is necessary to distinguish these objects. On the other hand, objects $4,5$ are $\{a,g\}$-indiscernible and d-discernible, so that the attribute h is also necessary. Finally, the objects $2,5$ are $\{h,g\}$-indiscernible and d-discernible, so the attribute a is also necessary. As a consequence, $\mathcal{A}$ is $\{2,3,4,5\}$-irreducible and therefore $(\{2,3,4,5\}, \mathcal{A})$ is a decision bireduct.

To conclude, we will show that $\mathcal{A}$ is not a d-reduct of $\mathcal{A}$. As we have mentioned, the pairs of objects 1,2 and 5,6 are the only pairs of objects d-discernible and $\mathcal{A}$-indiscernible. Therefore, $POS_{\mathcal{A}}(\{d\}) = \{3,4\}$. On the other hand, as $[3]_{I(\{h,g\})} = \{3\}$ and $[4]_{I(\{h,g\})} = \{4\}$ and $POS_{\{h,g\}}(\{d\}) \subseteq POS_{\mathcal{A}}(\{d\})$ it is obtained that $POS_{\{h,g\}}(\{d\}) = POS_{\mathcal{A}}(\{d\}) = \{3,4\}$. Finally, we obtain that $3 \notin POS_{\{h\}}(\{d\})$ due to $[3]_{I(\{h\})} = U \backslash \{4\}$. As a consequence, the attribute g is indispensable in $\{h,g\}$. For the same reason, $4 \notin POS_{\{g\}}(\{d\})$ and the attribute h is indispensable in $\{h,g\}$. Therefore, $\{h,g\}$ is a d-reduct of $\mathcal{A}$. Therefore, there exists $X \subseteq U$ such that $(X, \mathcal{A})$ is a decision bireduct and $\mathcal{A}$ is not d-reduct of $\mathcal{A}$, and so Corollary 17 is not true in decision tables.

The next result recalls that, if a decision table satisfies that $POS_{\mathcal{A}}(\{d\}) = U$ then the reciprocal of Proposition 25 is true, as it is showed in Stawicki et al.[25]

Proposition 27 (Stawicki et al.[25]). *Let $(U, \mathcal{A}_d, \mathcal{V}_{\mathcal{A}_d}, \overline{\mathcal{A}_d})$ be a decision table such that $POS_{\mathcal{A}}(\{d\}) = U$ and $B \subseteq \mathcal{A}$. Then, the pair (U, B) is a decision bireduct if and only if B is a d-reduct of $\mathcal{A}$.*

Notice that, the decision table given in Example 10 satisfies the equality $POS_{\mathcal{A}}(\{d\}) = U$. Taking into account the set of attributes $B = \{O, H, W\}$, we obtained that $\{O, H, W\}$ is a d-reduct of $\mathcal{A}$ in the aforementioned example. Then, applying Proposition 27, we can ensure that $(U, \{O, H, W\})$ is a decision bireduct.

The following property studies the connection between the indiscernibility classes of the objects in a decision table and the bireducts with no attribute.

Proposition 28. *Let (X, B) be a decision bireduct with $X \subseteq U$ and $B \subseteq \mathcal{A}$. Then, $B = \emptyset$ if and only if $X = [x]_{I(\{d\})}$ for all $x \in U$.*

Proof. Suppose that $B = \emptyset$. Hence, each pair of objects $x, y \in X$ are B-indiscernible. Therefore, since (X, B) is a decision bireduct, each pair of objects $x, y \in X$ must be d-indiscernible. As a consequence, $X = [x]_{I(\{d\})}$ for all $x \in U$.

We suppose now $X = [x]_{I(\{d\})}$ for all $x \in U$. By hypothesis, every $x, y \in X$ are d-indiscernible. Therefore, since B does not need to discern any pair of objects, by the irreducibility condition of B, this set must be the empty set. □

Notice that, unlike to information bireducts, a table reduced by classes is not needed.

The correspondence between the set of objects X of a decision bireduct $(X, \emptyset)$ and the indiscernibility classes $[x]_{I(\{d\})}$, for all $x \in U$, is illustrated below.

Example 29. Considering the subsets of objects $X_1 = \{1,2,6,8,14\}$ and $X_2 = \{3,4,5,7,9,10,11,12,13\}$ of the decision table given in Example 10, from the information displayed in Table 3, it is easy to see that $(X_1, \emptyset)$ and $(X_2, \emptyset)$ are all decision bireducts with the empty set as subset of attributes. In addition, taking into account the computations carried out in Example 23, we have that $[1]_{I(\{d\})} = \{1,2,6,8,14\}$ and $[3]_{I(\{d\})} = \{3,4,5,7,9,10,11,12,13\}$. As a consequence, comparing these subsets of objects with the decision bireducts $(X_1, \emptyset)$ and $(X_2, \emptyset)$, we obtain clearly a correspondence between the sets of objects X_i with $i \in \{1,2\}$ and the indiscernibility classes $[x]_{I(\{d\})}$ for all $x \in U$.

3.3 | U-decision bireducts

This section continues our study of decision bireducts taking into account an alternative definition, which also has been studied in previous works.[19,24,25] In addition, we will show some interesting results about this definition.

Definition 30. Given $X \subseteq U$ and $B \subseteq \mathcal{A}$, we say that the pair (X, B) is a U-decision bireduct if every pair $x \in X, y \in U$ are B-discernible when they are d-discernible, B is X-irreducible and X is B-inextensible with respect to this property.

Notice that, this definition is more restrictive than Definition 22 since each object $x \in X$ is compared with each object $y \in U$. As a consequence, in the decision subtables generated by U-decision bireducts there is no element triggering a contradiction in the original decision table. Now, we will clarify this definition in the following example.

Example 31. Consider the decision table given in Example 10 and the subset of attributes $B = \{O, T, H\} \subseteq \mathcal{A}$. As we mentioned in Example 23, the pair of objects 4,14 and 5,6 are the only objects d-discernible and B-indiscernible. As a consequence, none of these pairs of elements should be considered, because in the definition of U-decision bireduct every object in X is compared with any object in the universe U. Hence, none of them can appear in a U-decision bireduct. Therefore, according to Definition 30, the unique candidate to U-decision bireduct with B as subset of attributes is $(U\backslash\{4, 5, 6{,}14\}, B)$.

Clearly, all objects are B-discernible if they are d-discernible and the subset of objects is B-inextensible by construction. Following the same reasoning that the one given in Example 23, we can ensure that B is $U\backslash\{4, 5, 6{,}14\}$-irreducible. Therefore, according to Definition 30, $(U\backslash\{4, 5, 6{,}14\}, B)$ is a U-decision bireduct.

Continuing in the same line that the previous sections, we will show some technical properties related to U-decision bireducts.

Proposition 32. *Given (X, B) a U-decision bireduct, where $X \subseteq U$ and $B \subseteq \mathcal{A}$, the following equivalence holds:*

$$x \in X \textit{ if and only if } [x]_{I(B)} \subseteq [x]_{I(\{d\})}$$

Proof. Supposing that $x \in X$, we prove that $[x]_{I(B)} \subseteq [x]_{I(\{d\})}$. Since $x \in X$, for every $y \in U$, if x, y are d-discernible then they are B-discernible. Let $z \in [x]_{I(B)}$, then x, z are B-indiscernible and, as a consequence, they are d-indiscernible. Therefore, $z \in [x]_{I(\{d\})}$.

Now, suppose that $[x]_{I(B)} \subseteq [x]_{I(\{d\})}$ and $y \in U$ such that x, y are d-discernible. Then $y \notin [x]_{I(\{d\})}$. Therefore, $y \notin [x]_{I(B)}$ and x, y are B-discernible. Taking into account that X is B-inextensible, we can ensure that $x \in X$. □

It is convenient to emphasize the importance of this result. Specifically, we deduce that $x \in X$ if and only if there is no other object $y \in U$ such that x, y are B-indiscernible and d-discernible. Consequently, Proposition 32 proves that if there exists a U-decision bireduct with $B \subseteq \mathcal{A}$ as subset of attributes that is the unique U-decision bireduct with B as subset of attributes. In addition, from Proposition 32, we can deduce that if (X, B) is a U-decision bireduct, then $X = POS_B(\{d\})$ because $POS_B(\{d\}) = \{x \in U | [x]_{I(B)} \subseteq [x]_{I(\{d\})}\}$. In this case, we can compute U-decision bireducts in an easier way than decision bireducts, since Proposition 32 is more powerful than Proposition 24.

On the other hand, it is possible to obtain a stronger relationship between the positive region and U-decision bireducts, allowing the calculus of these bireducts through the positive region, as it is showed in Stawicki et al.[25]

Proposition 33 (Stawicki et al.[25]). *Let $(U, \mathcal{A}_d, \mathcal{V}_{\mathcal{A}_d}, \overline{\mathcal{A}_d})$ be a decision table such that $X \subseteq U$ and $B \subseteq \mathcal{A}$. Then (X, B) is a U-decision bireduct if and only if $X = POS_B(\{d\})$ and there is no subset $B' \subset B$ such that $POS_{B'}(\{d\}) = POS_B(\{d\})$.*

Analogously to decision bireducts, we introduce a result which relates the definitions of U-decision bireduct to d-reduct.

Proposition 34. *Given $B \subseteq \mathcal{A}$, if the pair (U, B) is a U-decision bireduct then $POS_B(\{d\}) = U$. Moreover, B is a d-reduct of $\mathcal{A}$.*

Proof. The proof is straightforwardly obtained from Proposition 33. □

Just like it occurs in Proposition 25, the reciprocal of the last property is not true in general. Coming back to Example 26, from Table 6, we deduce that $B = \{h, g\}$ is a d-reduct of $\mathcal{A}$. Therefore, there is no subset $B' \subset B$ such that $POS_{B'}(\{d\}) = POS_B(\{d\})$. Taking into account that $POS_B(\{d\}) = \{3, 4\}$ and applying Proposition 33, we obtain that $(\{3, 4\}, B)$ is the U-decision bireduct with B as subset of attributes.

Equivalent results to Proposition 16 and Corollary 17 for decision tables are deduced from Proposition 33.

Proposition 35. *Given $B \subseteq \mathcal{A}$, if B is a d-reduct of $\mathcal{A}$ then there exists $X \subseteq U$ such that (X, B) is a U-decision bireduct.*

Proof. The proof is straightforwardly obtained from Proposition 33. □

Corollary 36. *There exists $X \subseteq U$ such that $(X, \mathcal{A})$ is a U-decision bireduct if and only if $\mathcal{A}$ is a d-reduct of $\mathcal{A}$.*

Proof. The proof is straightforwardly obtained from Proposition 33. □

The reciprocal of Proposition 34 is true when a decision table satisfies that $POS_{\mathcal{A}}(\{d\}) = U$, as it is showed in Stawicki et al.[25]

Proposition 37 (Stawicki et al.[25]). *Let $(U, \mathcal{A}_d, \mathcal{V}_{\mathcal{A}_d}, \overline{\mathcal{A}_d})$ be a decision table such that $POS_{\mathcal{A}}(\{d\}) = U$ and $B \subseteq \mathcal{A}$. Then, the pair (U, B) is a U-decision bireduct if and only if B is a d-reduct of $\mathcal{A}$.*

Finally, we introduce the result which studies U-decision bireducts whose subset of attributes is $B = \emptyset$.

Proposition 38. *Let $X \subseteq U$, $B \subseteq \mathcal{A}$ and (X, B) be a U-decision bireduct. Suppose that there exist $x, y \in U$ such that $y \notin [x]_{I(\{d\})}$. Then, $B = \emptyset$ if and only if $X = \emptyset$.*

Proof. First of all, we suppose $B = \emptyset$. By Proposition 32, $X \neq \emptyset$ if and only if there exists $x \in U$ such that $[x]_{I(\emptyset)} \subseteq [x]_{I(\{d\})}$. However, $[x]_{I(\emptyset)} = U$ and, by hypothesis $y \notin [x]_{I(\{d\})}$, then we deduce that $[x]_{I(\{d\})} \neq U$. Then $X = \emptyset$.

Now, we suppose $X = \emptyset$. Then, by Proposition 32, $[x]_{I(B)} \nsubseteq [x]_{I(\{d\})}$ for all $x \in U$. On the other hand, if $B' \subseteq B$ then $[x]_{I(B)} \subseteq [x]_{I(B')} \nsubseteq [x]_{I(\{d\})}$. Therefore, since $[x]_{I(\emptyset)} \nsubseteq [x]_{I(\{d\})}$, by the irreducibility condition of B, this set must be the empty set. □

Hence, several properties of the three different classes of bireducts have been studied, which have shed more light on the notion of bireduct. Next section compares bireducts with another useful notion: value reduct.

4 | RELATION BETWEEN VALUE REDUCTS AND BIREDUCTS

This section is devoted to study the relationships among the different notions of value reducts and bireducts. The importance of this study lies on the comparison of two notions originally independent, so that the study is divided into three different parts depending on the different notions of bireduct in information tables and in decision tables.

4.1 | Relation between information value reducts and information bireducts

This first part focuses on information tables in order to connect information value reducts and information bireducts, that is, Definitions 5 and 11. In this section, an information table $(U, \mathcal{A}, \mathcal{V}_{\mathcal{A}}, \overline{\mathcal{A}})$ will be fixed. Next result studies the belonging of objects to information bireducts by using the notion of value reduct.

Proposition 39. *Let $x \in U$ and $B \subseteq \mathcal{A}$ be a value reduct of $\mathcal{A}$ for x. Then, for each $y_0 \in [x]_{I(\mathcal{A})}$ there exists (X_{y_0}, B) information bireduct such that $y_0 \in X_{y_0}$. Indeed, all information bireducts (X, B) satisfy that $[x]_{I(\mathcal{A})} \cap X \neq \emptyset$.*

Proof. Given $B \subseteq \mathcal{A}$ being B a value reduct of $\mathcal{A}$ for $x \in U$, we will prove that there exists (X_0, B) information bireduct with $y_0 \in X_0$ being $y_0 \in [x]_{I(\mathcal{A})}$. Now, we consider the set $X_0' = \{y_0\}$ and we add to X_0' all objects satisfying that they are B-discernible among them, so that the new obtained set X_0 be B-inextensible. In this way, we have that X_0 is maximal. Now, we will prove that B is X_0-irreducible by reductio ad absurdum. We suppose that there exists $B' \subset B$ such that, for all $y, z \in X_0$, we have that y, z are B'-discernible. Since B is a value reduct of $\mathcal{A}$ for x, we deduce that $[x]_{I(B')} \neq [x]_{I(B)}$. Specifically, $[x]_{I(B)} \subseteq [x]_{I(B')}$ but $[x]_{I(B')} \nsubseteq [x]_{I(B)}$. Therefore, there exists $w \in [x]_{I(B')}$ such that $w \notin [x]_{I(B)}$. In addition, as $y_0 \in [x]_{I(\mathcal{A})}$, we obtain that $[x]_{I(B')} = [y_0]_{I(B')}$ and $[x]_{I(B)} = [y_0]_{I(B)}$. We distinguish two cases:

- If $w \in X_0$, since y_0, w are B'-indiscernible, we obtain a contradiction.
- If $w \notin X_0$, since X_0 is B-inextensible by construction, there exists $z \in [w]_{I(B)}$ such that $z \in X_0$. Then the following chain holds:

$$z \in [w]_{I(B)} \subseteq [w]_{I(B')} = [x]_{I(B')} = [y_0]_{I(B')}$$

which lead us to conclude that y_0, z are B'-indiscernible, obtaining again a contradiction.

In any case, B is X_0-irreducible and, as a consequence, (X_0, B) is an information bireduct with $y_0 \in X_0$ being $y_0 \in [x]_{I(\mathcal{A})}$.

Now, we will prove that for each information bireduct (X, B) there exists $y \in [x]_{I(\mathcal{A})}$ such that $y \in X$. We will prove it by reductio ad absurdum. Given (X', B) an information bireduct, suppose that for each $y \in [x]_{I(\mathcal{A})}$, we obtain that $y \notin X'$. Then, given $y \in [x]_{I(\mathcal{A})}$, there exists $z \in X'$ such that y, z are B-indiscernible, that is, $z \in [y]_{I(B)}$. Since B is a value reduct of $\mathcal{A}$ for x, we deduce the equality $[x]_{I(B)} = [x]_{I(\mathcal{A})}$. As $y \in [x]_{I(\mathcal{A})}$, we have $[y]_{I(\mathcal{A})} = [x]_{I(\mathcal{A})}$ and $[y]_{I(B)} = [x]_{I(B)}$ too. Hence, the following chain of equalities holds:

$$[y]_{I(\mathcal{A})} = [x]_{I(\mathcal{A})} = [x]_{I(B)} = [y]_{I(B)} = [z]_{I(B)}$$

which implies that $z \in [x]_{I(\mathcal{A})}$, and as $z \in X'$, this fact contradicts the supposition considered on (X', B). □

In the particular case of $[x]_{I(\mathcal{A})} = \{x\}$, the last result is simplified as follows.

Corollary 40. *Given $B \subseteq \mathcal{A}$ and $x \in U$ such that $[x]_{I(\mathcal{A})} = \{x\}$ with B value reduct of $\mathcal{A}$ for x, then every information bireduct (X, B) satisfies that $x \in X$.*

Proof. The proof follows from Proposition 39 considering that $[x]_{I(\mathcal{A})} = \{x\}$. □

This corollary allows to ensure that, given a value reduct for a certain object x, then x belongs to all information bireducts whose subset of attributes is the aforementioned value reduct. This fact is illustrated by means of the following example.

Example 41. Coming back to Examples 10 and 13, we can ensure that $[x]_{I(\mathcal{A})} = \{x\}$ for all $x \in U$, so that we are in the environment of Corollary 40, as well as $B = \{O, T, W\}$ is a value reduct of $\mathcal{A}$ for the object 5 and $(U \backslash \{10,13\}, B)$, $(U \backslash \{3,10\}, B)$, $(U \backslash \{4,13\}, B)$ and $(U \backslash \{3, 4\}, B)$ are all the information bireducts, with B as subset of attributes. Clearly, we obtain that

$$U \backslash \{3, 4, 10,13\} = U \backslash \{10,13\} \bigcap U \backslash \{3,10\} \bigcap U \backslash \{4,13\} \bigcap U \backslash \{3, 4\}$$

and object 5 belongs to all information bireducts.

Next result shows the close relationship between the notions of value reducts and information bireducts. Specifically, we provide the required conditions to the subsets of objects of all information bireducts with B as subset of attributes, in order to guarantee the existence of a subset $B' \subseteq B$ being a value reduct of $\mathcal{A}$, and vice versa.

Proposition 42. *Given $B \subseteq \mathcal{A}$, $X_1, \dots, X_n \subseteq U$ and $x \in U$ such that $(X_1, B), \dots, (X_n, B)$ are all the information bireducts with B as subset of attributes. Then for each $i \in \{1, \dots, n\}$ there exists $y_i \in [x]_{I(\mathcal{A})}$ such that $y_i \in X_i$ if and only if there exists $B' \subseteq B$ such that B' is a value reduct of $\mathcal{A}$ for x.*

Proof. Supposing that for each X_i there exists $y_i \in [x]_{I(\mathcal{A})}$ such that $y_i \in X_i$, we will prove that there exists $B' \subseteq B$ such that B' is a value reduct of $\mathcal{A}$ for x. The proof will be done by reductio ad absurdum. We suppose that $B' \subseteq B$ is not a value reduct of $\mathcal{A}$ for x, for all $B' \subseteq \mathcal{A}$. Then $[x]_{I(B')} \neq [x]_{I(\mathcal{A})}$ and since $[x]_{I(\mathcal{A})} \subseteq [x]_{I(B')}$, there exists $z \in [x]_{I(B')}$ satisfying $z \notin [x]_{I(\mathcal{A})}$. In particular, considering $B' = B$, we obtain x, z are B-indiscernible, that is, y_i, z are B-indiscernible for all $y_i \in [x]_{I(\mathcal{A})}$. Therefore, necessarily $z \notin X_i$ for all $i \in \{1, \dots, n\}$. Hence, $z \notin \bigcup_{i=1}^{n} X_i$ and by Proposition 12, we have that $z \notin U$, which leads us to a contradiction.

Now, assuming that there exists $B' \subseteq B$ such that B' is a value reduct of $\mathcal{A}$ for x, we will prove that there exists $y_i \in [x]_{I(\mathcal{A})}$ such that $y_i \in X_i$, for all $i \in \{1, \dots, n\}$. By reductio ad absurdum we will suppose that there exists $i \in \{1, \dots, n\}$, we have that $y \notin X_i$ for all $y \in [x]_{I(\mathcal{A})}$. Then, given $y_i[x]_{I(\mathcal{A})}$, there exists $z \in X_i$ such that y_i, z are B-indiscernible, so $z \in [y_i]_{I(B)}$. Since $y_i \in [x]_{I(\mathcal{A})}$, it is obtained that $[x]_{I(B)} = [y_i]_{I(B)}$ and $[x]_{I(B')} = [y_i]_{I(B')}$. Hence, we obtain

$$[x]_{I(\mathcal{A})} \subseteq [x]_{I(B)} = [y_i]_{I(B)} \subseteq [y_i]_{I(B')} = [x]_{I(B')} = [x]_{I(\mathcal{A})}$$

That is due to monotony of equivalence classes and because B' is a value reduct of $\mathcal{A}$ for x. As a consequence, $[y_i]_{I(B)} = [x]_{I(\mathcal{A})}$ and then $z \in [x]_{I(\mathcal{A})}$, obtaining a contradiction with the hypothesis considered about X_i. □

Now, we include the last result in the particular case of $[x]_{I(\mathcal{A})} = \{x\}$. Specifically, we show that if we know all the information bireducts, associated with a fixed subset of attributes, we can delimit the study of value reducts for the objects belonging to the intersection of these information bireducts.

Corollary 43. *Let $B \subseteq \mathcal{A}$, $X_1, \dots, X_n \subseteq U$, $x \in U$ such that $[x]_{I(\mathcal{A})} = \{x\}$ and $(X_1, B), \dots, (X_n, B)$ all the information bireducts with B as subset of attributes. Then $x \in \bigcap_{i=1}^{n} X_i$ if and only if there exists $B' \subseteq B$ such that B' is a value reduct of $\mathcal{A}$ for x.*

Proof. The proof follows from Proposition 42 due to $[x]_{I(\mathcal{A})} = \{x\}$. □

On the other hand, next example will show that the condition $B' \subseteq B$ is indispensable in Corollary 43, that is, B could not be a value reduct of $\mathcal{A}$ for x, in general. As a consequence, the mentioned condition will be also essential in Proposition 42.

Example 44. Consider the information table given in Example 10 and the subset of attributes $B = \{O, T, W\} \subseteq \mathcal{A}$. In Example 13, we calculated all information bireducts with B as subset of attributes, obtaining that they are $(U \backslash \{10,13\}, B)$, $(U \backslash \{3,10\}, B)$, $(U \backslash \{4,13\}, B)$ and $(U \backslash \{3,4\}, B)$. Hence, it is easy to see that object 9 belongs to all of them.

On the other hand, from Table 2, we conclude that $\{O, T\}$ is a value reduct of $\mathcal{A}$ for the object 9. Therefore, $\{O, T, W\}$ is not a value reduct of $\mathcal{A}$ for the object 9 and so, we must include the condition $B' \subseteq B$ in Proposition 42 and in Corollary 43.

4.2 | Relation between decision value reducts and decision bireducts

This second part focuses on decision tables in order to relate d-value reducts and decision bireducts, that is, Definitions 9 and 22. From now on, a decision table $(U, \mathcal{A}_d, \mathcal{V}_{\mathcal{A}_d}, \overline{\mathcal{A}_d})$ will be fixed. First of all, we introduce a result which will be useful later.

Lemma 45. *Let $B \subseteq \mathcal{A}$ be a d-value reduct of $\mathcal{A}$ for $x \in U$ such that $[x]_{I(\mathcal{A})} \subseteq [x]_{I(\{d\})}$ and (X, B) be a decision bireduct. If $x \in X$ then $[x]_{I(B)} \subseteq X$.*

Proof. Given $y \in [x]_{I(B)}$, we will prove that $y \in X$ by reductio ad absurdum. Suppose that $y \notin X$. Then, there exists $z \in X$ such that y, z are d-discernible and B-indiscernible. Since B is a d-value reduct of $\mathcal{A}$ for x and $[x]_{I(\mathcal{A})} \subseteq [x]_{I(\{d\})}$, we obtain that $[x]_{I(B)} \subseteq [x]_{I(\{d\})}$. Hence, $y \in [x]_{I(\{d\})}$. As a consequence, $x, z \in X$, they are d-discernible and B-indiscernible, which leads us to a contradiction. Therefore, $[x]_{I(B)} \subseteq X$. □

Following the same scheme of the previous section, the following result studies the belonging of objects to decision bireducts by using the notion of d-value reduct.

Proposition 46. *Let $B \subseteq \mathcal{A}$ be a d-value reduct of $\mathcal{A}$ for $x \in U$ such that $[x]_{I(\mathcal{A})} \subseteq [x]_{I(\{d\})}$. Then, there exists a decision bireduct (X_0, B) such that $x \in X_0 \subseteq U$. Indeed, all decision bireducts (X, B) satisfy that $[x]_{I(B)} \subseteq X$.*

Proof. First of all, we will prove the existence of a decision bireduct (X_0, B) with $x \in X_0$. The strategy followed in this proof consists of defining a set of objects $X_0 \subseteq U$ such that (X_0, B) is a decision bireduct with $x \in X_0$. Consider $X_0' = \{x\}$. Since B is a d-value reduct of $\mathcal{A}$ for x with $[x]_{I(\mathcal{A})} \subseteq [x]_{I(\{d\})}$, we have for each $a \in B$ that there exists $z_a \in [x]_{I(B \backslash \{a\})}$ such that $z_a \notin [x]_{I(\{d\})}$. Now, consider $X_0'' = \{x, z_{a_1}, \dots, z_{a_{|B|}}\}$. As B is a d-value reduct of $\mathcal{A}$ for x and $z_{a_i} \notin [x]_{I(\{d\})}$ for all $i \in \{1, \dots, |B|\}$, we deduce that $z_{a_i} \notin [x]_{I(B)}$. Then x, z_{a_i} are d-discernible, B-discernible and $B \backslash \{a_i\}$-indiscernible, for all $i \in \{1, \dots, |B|\}$.

We suppose now that there exist $j, k \in \{1, \dots, |B|\}$ such that z_{a_j}, z_{a_k} are d-discernible and B-indiscernible. Hence,

$$z_{a_k} \in \left[z_{a_j}\right]_{I(B)} \subseteq \left[z_{a_j}\right]_{I(B \backslash \{a_j\})} = [x]_{I(B \backslash \{a_j\})}$$

Therefore, x, z_{a_k} are $B \backslash \{a_j\}$-indiscernible. By construction, x, z_{a_k} are also $B \backslash \{a_k\}$-indiscernible. As a consequence, we can consider $X_0'' \backslash \{z_{a_j}\}$ so that, for each $a \in B$, there exists $z_a \in X_0'' \backslash \{z_{a_j}\}$ such that x, z_a are d-discernible, B-discernible and $B \backslash \{a\}$-indiscernible, being B X_0''-irreducible. Let X_0''' be the set obtained from X_0'', applying the previous procedure to every pair of d-discernible and B-indiscernible objects z_{a_j}, z_{a_k}. Therefore, the new obtained set X_0''' satisfies that every pair of objects d-discernible are B-discernible and B is X_0'''-irreducible with respect to this property. Now, we can increase X_0''' keeping this property until the set is B-inextensible, being this last set X_0. As a consequence, (X_0, B) is a decision bireduct with $x \in X_0$.

Now, we will prove that for each decision bireduct (X, B) it obtains that $[x]_{I(B)} \subseteq X$. First of all, we will prove that $x \in X$ by reductio ad absurdum. Let (X', B) be a decision bireduct such that $x \notin X'$. As X' is B-inextensible, there exists $y \in X'$ such that x, y are d-discernible and B-indiscernible, and therefore $y \in [x]_{I(B)}$. Since B is a d-value reduct of $\mathcal{A}$ for x, we obtain that $[x]_{I(B)} \subseteq [x]_{I(\{d\})}$. Therefore $y \in [x]_{I(\{d\})}$. As a consequence, x, y are d-indiscernible, which contradicts that x, y be d-discernible. Therefore, $x \in X$ and applying Lemma 45, $[x]_{I(B)} \subseteq X$. □

Notice that the assumption $[x]_{I(\mathcal{A})} \subseteq [x]_{I(\{d\})}$ is not restrictive since otherwise the objects in the class $[x]_{I(\mathcal{A})}$ do not classify correctly (they have different decisions) and so, reducing attributes is meaningless. Indeed, in this case it is necessary to increase the number of attributes to obtain a better classification.

Hence, this proposition guarantees that given a d-value reduct for a certain object x, then x belongs to all decision bireducts whose subset of attributes is the aforementioned d-value reduct. This fact is exemplified below.

Example 47. Returning to Example 10, we have that $B = \{O, H\}$ is a d-value reduct of $\{O, T, H\}$ and a d-value reduct of $\mathcal{A}$ for the object 1. In addition, following a similar procedure to the one given in Example 23, we have that $(U \backslash \{4, 6\}, B)$, $(U \backslash \{6{,}14\}, B)$, $(U \backslash \{4, 5, 10\}, B)$ and $(U \backslash \{5{,}10{,}14\}, B)$ are all decision bireducts with $B = \{O, H\}$ as subset of attributes. Clearly, object 1 belongs to all of them, as Proposition 46 states.

To finish this section, it is important to emphasize that a similar result to Proposition 42 is not verified to decision bireducts. From the computations of Example 23, we have that $(U \backslash \{5{,}14\}, \{O, T, H\})$ and $(U \backslash \{4, 5\}, \{O, T, H\})$ are all decision bireducts with $\{O, T, H\}$ as subset of attributes. Hence, it is easy to see that the object 6 belongs to all of them.

On the other hand, from Table 4, we have that $\{O, W\}$ is a d-value reduct of $\mathcal{A}$ for the object 6 and the pair of objects 5,6 are d-discernible and $\{O, T, H\}$-indiscernible. Therefore, $\{O, T, H\}$ is not a d-value reduct of $\mathcal{A}$ for the object 6. As a consequence, the attribute W must belong to all d-value reducts of $\mathcal{A}$ for the object 6. Following an analogous reasoning with the pair of objects 6,7 and the subset of attributes $\{T, H, W\}$ we can conclude that the attribute O must belong to all d-value reducts of $\mathcal{A}$ for the object 6. In short, since both attributes, O and W, must belong to all d-value reducts of $\mathcal{A}$ for the object 6 and $\{O, W\}$ is a d-value reduct, we can conclude that $\{O, W\}$ is the unique d-value reduct of $\mathcal{A}$ for the object 6. Since the object 6 belongs to all decision bireducts with $\{O, T, H\}$ as subset of attributes, $\{O, W\}$ is the unique d-value reduct of $\mathcal{A}$ for the object 6 and $\{O, W\} \nsubseteq \{O, T, H\}$, we conclude that if an object x belongs to all decision bireducts with a fixed subset of attributes B, then it may not exist $B' \subseteq B$ with B' a d-value reduct of $\mathcal{A}$ for the object x. As a consequence, an analogous result to Proposition 42 for decision tables is not verified to decision bireducts.

4.3 | Relation between decision value reducts and U-decision bireducts

This section finishes our study relating decision value reducts and U-decision bireducts. First of all, we introduce a result which will be useful later.

Lemma 48. *Let (X, B) be a U-decision bireduct where $X \subseteq U$ and $B \subseteq \mathcal{A}$. Given $x \in U$, if $x \in X$ then $[x]_{I(B)} \subseteq X$.*

Proof. Given $y \in [x]_{I(B)}$, we will prove that $y \in X$. Since $x \in X$, by using Proposition 32, we obtain that $[x]_{I(B)} \subseteq [x]_{I(\{d\})}$. As a consequence, we obtain the following chain of equalities:

$$[y]_{I(B)} = [x]_{I(B)} \subseteq [x]_{I(\{d\})} = [y]_{I(\{d\})}$$

As a consequence, by using Proposition 32, we obtain that $y \in X$. Therefore, $[x]_{I(B)} \subseteq X$. □

Next result studies the belonging of objects to U-decision bireducts by using the notion of d-value reduct, obtaining a result similar to Proposition 46.

Proposition 49. *Let $B \subseteq \mathcal{A}$ be a d-value reduct of $\mathcal{A}$ for $x \in U$ such that $[x]_{I(\mathcal{A})} \subseteq [x]_{I(\{d\})}$. Then, there exists $X \subseteq U$, such that the pair (X, B) is a U-decision bireduct satisfying that $[x]_{I(B)} \subseteq X$.*

Proof. We define the set $X = \{y \in U | [y]_{I(B)} \subseteq [y]_{I(\{d\})}\}$. Notice that, $y \in X$ if and only if for each object $z \in U$ such that y, z are d-discernible, they also are B-discernible. Therefore, X is B-inextensible. Furthermore, $x \in X$ because B is a d-value reduct of $\mathcal{A}$ for x and $[x]_{I(\mathcal{A})} \subseteq [x]_{I(\{d\})}$. Finally, since B is a d-value reduct of $\mathcal{A}$ for x, for each $B' \subset B$ we obtain that $[x]_{I(B')} \nsubseteq [x]_{I(\{d\})}$. As a consequence, for each $B' \subset B$ there exists $y_{B'} \in U$ such that $x, y_{B'}$ are d-discernible and B'-indiscernible. Hence, B is X-irreducible. Since $x \in X$, (X, B) is a U-decision bireduct and applying Lemma 48, we obtain that $[x]_{I(B)} \subseteq X$. □

The following result also relates the notions of U-decision bireduct to d-value reduct, in particular, it studies the existence of d-value reducts for those objects which belong to U-decision bireducts. Notice that, this result is equivalent to Proposition 42 for decision tables, by using the notion of U-decision bireducts.

Proposition 50. *Let (X, B) be a U-decision bireduct with $X \subseteq U$, $B \subseteq \mathcal{A}$ and $x \in U$ such that $[x]_{I(\mathcal{A})} \subseteq [x]_{I(\{d\})}$. Then, $[x]_{I(B)} \subseteq X$ if and only if there exists $B' \subseteq B$ such that B' is a d-value reduct of $\mathcal{A}$ for x.*

Proof. Supposing that $[x]_{I(B)} \subseteq X$, we will prove that there exists $B' \subseteq B$ such that B' is a d-value reduct of $\mathcal{A}$ for x. We will prove it by reductio ad absurdum. Consider $x \in X$ and suppose that any subset of attributes $B' \subseteq B$ is not a d-value reduct of $\mathcal{A}$ for x. Then $[x]_{I(B')} \nsubseteq [x]_{I(\{d\})}$, for all $B' \subseteq B$. However, by Proposition 32, $[x]_{I(B)} \subseteq [x]_{I(\{d\})}$ which leads us to a contradiction when $B' = B$ is considered in the expression above.

Now, supposing that there exists $B' \subseteq B$ such that B' is a d-value reduct of $\mathcal{A}$ for x, we will prove that $[x]_{I(B)} \subseteq X$. Given $y \in [x]_{I(B)}$ we obtain

$$[y]_{I(B)} = [x]_{I(B)} \subseteq [x]_{I(B')} \subseteq [x]_{I(\{d\})} = [y]_{I(\{d\})}$$

where $[x]_{I(B)} \subseteq [x]_{I(B')}$ because $B' \subseteq B$ and $[x]_{I(B')} \subseteq [x]_{I(\{d\})}$, since B' is a d-value reduct of $\mathcal{A}$ for x. Therefore, by using Proposition 32, we obtain that $y \in X$. As a consequence, $[x]_{I(B)} \subseteq X$. □

It is important to observe that B could not be a d-value reduct of $\mathcal{A}$ for x, that is, the condition $B' \subseteq B$ is indispensable in Proposition 50 as we will show next.

Example 51. Consider the decision table given in Example 10 and the subset of attributes $B = \{O, T, H\} \subseteq \mathcal{A}$. In Example 31, we obtained that $(U \backslash \{4, 5, 6, 14\}, \{O, T, H\})$ is the U-decision bireduct with $\{O, T, H\}$ as subset of attributes. From now on, we will focus on the object 1, which belongs to that U-decision bireduct. From Table 3, we can conclude that the pair of objects 1,3 are d-discernible and $\{T, H, W\}$-indiscernible. Therefore, attribute O must belong to all d-value reducts of $\mathcal{A}$ for the object 1. Now, we compute all d-value reducts of $\mathcal{A}$ for the object 1.

On the one hand, the pair of objects 1,9 are d-discernible and $\{O\}$-indiscernible. Therefore, $\{O\}$ is not a d-value reduct of $\mathcal{A}$ for the object 1. From Table 4, we have that $B'_1 = \{O, H\}$ is a d-value reduct of $\mathcal{A}$ for the object 1.

On the other hand, from Table 3, it is easy to see that

$$[1]_{I(\{O,T\})} = \{1, 2\} \subseteq [1]_{I(\{d\})} = \{1, 2, 6, 8, 14\}$$

and, as a result, $B'_2 = \{O, T\}$ is a d-value reduct of $\mathcal{A}$ for the object 1.

Notice that, $\{O, W\}$ is not a d-value reduct of $\mathcal{A}$ for the object 1, since we also obtain that the pair of objects 1,9 are d-discernible and $\{O, W\}$-indiscernible from Table 3.

Therefore, B'_1 and B'_2 are all the d-value reducts of $\mathcal{A}$ for the object 1, being $B'_1, B'_2 \neq B$. Hence, B is not a d-value reduct of $\mathcal{A}$ for the object 1 and so, we must include the condition $B' \subseteq B$ in Proposition 50.

5 | CONCLUSIONS AND FURTHER WORK

This paper has delved into the study of bireducts, analyzing new properties about bireducts whose subsets of objects or attributes are extreme cases, that is, they are the whole set of objects or attributes contained in the information/decision table or the empty set. Furthermore, we have proven that the attribute set of an information bireduct provides a covering of the universe of objects into maximal consistent subsystems.

The second part of the paper is focused on the relationship between bireducts and value reducts in both information and decision tables, and contains interesting results of these two originally independent notions which really prove both are very well related. Furthermore, in order to clarify the content of the paper, we have accompanied all the notions and results with examples. This relationship will also be useful for obtaining more optimal decision rules extraction mechanisms, which will be analyzed in detail in the future.

Moreover, we will study the generalization of the results introduced in this paper to the fuzzy environment. Furthermore, we will analyze more relationships between bireducts and value reducts, and their relation and application to other frameworks, such as logic, formal concept analysis and non-linear relation equations.[16,27–29] In particular, we are also interested in the study of the relationship between decision rules and attribute implication, studied in formal concept analysis.

ACKNOWLEDGEMENTS

Partially supported by the 2014-2020 ERDF Operational Programme in collaboration with the State Research Agency (AEI) in project PID2019-108991GB-I00, and with the Department of Economy, Knowledge, Business and University of the Regional Government of Andalusia in project FEDER-UCA18-108612, and by the European Cooperation in Science & Technology (COST) Action CA17124.

CONFLICT OF INTERESTS

This work does not have any conflict of interests.

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How to cite this article: Chacón-Gómez F, Cornejo ME, Medina J, Ramírez-Poussa E. Value reducts and bireducts: a comparative study. *Math Meth Appl Sci*. 2022;1-20. doi:10.1002/mma.8597