Intertwining Relations, Commutativity and Orbits.

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To my beloved parents Kheira Belalia and Djelil ...

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Contents

Notation

$\mathbb{C}$ the set of complex numbers.
$\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ the Riemann sphere.
$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ the unit disk of $\mathbb{C}$.
$\partial\mathbb{D} = \{z \in \mathbb{C} : |z| = 1\}$ the unit circle.
$D(a, r) = \{z \in \mathbb{C} : |z - a| < r\}$ disk centered on $a \in \mathbb{C}$ of radius $r > 0$.
$\Pi = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ the upper half plane of the complex plane
$\alpha_p(z) = \frac{p-z}{1-\bar{p}z}$ the involution automorphism exchanging 0 and $p \in \mathbb{D}$.
$\mathcal{X}$ a Banach space.
$\mathcal{F}$ a Fréchet space.
H a Hilbert space.
$\mathcal{B}(\mathcal{X})$ the algebra of bounded linear operators on a Banach space $\mathcal{X}$.
$\sigma_p(A, \mathcal{X}) := \{\lambda \in \mathbb{C} : \exists x \in \mathcal{X} \quad x \neq 0 : Ax = \lambda x\}$ the point spectrum of the operator A on $\mathcal{X}$.
$\mathrm{Ext}(T, \mathcal{X}) := \{X \in \mathcal{B}(\mathcal{X}) : TX = \lambda XT \text{ for some } \lambda \in \mathbb{C}\}$ the extended spectrum of T on $\mathcal{X}$.
$H^p(\mathbb{D}) := H^p$ Hardy spaces
$\|f\|_p$ norm of $f \in H^p(\mathbb{D})$.
D_β weighted Dirichlet spaces
B the backward shift.
F the forward shift.
M_f multiplication operator by the fixed function f.
C_φ composition operator induced by the function φ.

Introduction

Let $\mathcal{X}$ be a complex Banach space and let $B(\mathcal{X})$ be the algebra of all bounded linear operators on $\mathcal{X}$. We say that a complex scalar λ is an extended eigenvalue of $T \in B(\mathcal{X})$ provided that there exists a non-zero bounded linear operator X on $\mathcal{X}$ such that $TX = \lambda XT$. In this case, X is called an extended eigenoperator for T corresponding to the extended eigenvalue λ. Equivalently, we can simply say that X λ-commutes with T. Throughout this manuscript, we shall denote by $\mathrm{Ext}(T, \mathcal{X})$ the set of the extended eigenvalues of $T \in B(\mathcal{X})$.

The study of extended eigenvalues of a bounded operator has its origin in the study of the Invariant Subspace Problem. Recall that this Functional Analysis famous open problem is stated as follows:

> Does every bounded linear operator T on a separable, infinite-dimensional Hilbert space H have a nontrivial invariant subspace? That is, do there exist a closed subspace $\mathcal{M}$ of H with $\mathcal{M} \neq \{0\}$ and $\mathcal{M} \neq H$ such that $T\mathcal{M} \subset \mathcal{M}$?

Using the Schauder fixed-point Theorem, V. I. Lomonosov [62] showed in 1973 that if a bounded operator (non-multiple of the identity) commutes with a compact operator on a Banach space, then it has a non-trivial hyperinvariant subspace. Recall that a subspace $\mathcal{M}$ is said to be hyperinvariant for an operator T if $T\mathcal{M} \subset \mathcal{M}$ and $X\mathcal{M} \subset \mathcal{M}$ for every bounded operartor X that commutes with T. Six years later, S. Brown [19] and the mathematicians H. W. Kim, R. L. Moore, and C. M. Pearcy showed independently in [48] that if A has a non-zero compact eigenoperator then A has a closed hyperinvariant (and thus an invariant) subspace on H.

One of the aims of our investigation is to study different questions related to the extended eigenoperators and extended eigenvalues of some bounded linear operators on Banach spaces (or eventually on Fréchet spaces). For instance, what is the extended-spectrum of composition operators on weighted Hardy spaces? What if we consider operators λ-commuting with the differentiation operator on the space of entire functions? Under which conditions are they hypercyclic?

This manuscript is structured as follows:

Chapter 1 is a preliminary chapter in which some basic definitions and some results related to bounded and unbounded operators, are introduced.

In Chapter 2, we provide a characterization of when the Cesáro means of higher-order are hypercyclic on Banach spaces. Moreover, we establish necessary conditions on the extended-spectrum of a bounded linear operator for convex-cyclicity. Then, we use these results to gather some examples of operators that are not convex-cyclic. At the end, we ask whether the latter result can be strengthened to show its non-Supercyclicity. We answer this question negatively by giving a counterexample. Finally, as we shall see in Chapter 4, this result is no longer true for continuous linear operators on Fréchet spaces.

In Chapter 3, we fully characterize the hypercyclicity of the extended eigenoperators of the differentiation operator D in the space of entire functions. In particular, if T is an extended eigenoperator of D associated with the extended eigenvalue λ, we show that T is hypercyclic if and only if the modulus of λ is greater than 1 and T is not a multiple of a composition operator induced by an affine endomorphism of $\mathbb{C}$.

Chapter 4 is mainly devoted to extend an investigation which begun in [55] on Hardy spaces. We shall focus in this chapter on the study of extended eigenvalues of linear fractional composition operators defined on weighted Hardy spaces. Depending on the classification of the linear fractional self-map on the unit disc, we compute the extended spectrum. The case when the self-map fixes one interior fixed point in the unit disk and another one outside of its closure is completely solved.

Chapter 5 deals with unbounded linear operators. The Theory of Unbounded Operators was mainly developed by J. von Neumann and M. Harvey Stone in the 1930s. This theory stands between Mathematics and Physics. For instance, it plays a major role in the Theory of Differential Equations as well as Quantum Mechanics. In this part of the thesis, it should be noticed that a bounded operator can be defined on a subspace of Hilbert space, unlike the previous chapters where a bounded operator was always defined on the entire Fréchet (or Banach) space.

Recall that if A and B are two linear operators with respective domains $D(A)$ and $D(B)$, then B is said to be an extension of A if $D(A) \subseteq D(B)$ and $Ax = Bx$, for all $x \in D(A)$. In this case, we shall write $A \subseteq B$. A bounded linear operator T on a Hilbert space is said to *intertwine* two linear operators A and B if $TA \subseteq BT$. Particularly, if $A = B$, then we say that T commutes with A. Certainly, one of the fundamental results related to these intertwining relations is the Fuglede Theorem. In [37], B.

Fuglede proved the following theorem: "*If an operator intertwines two normal operators, then it intertwines their adjoints*". Throughout Chapter 5, we shall investigate the following related conjecture which appeared in [64]:

> Let T be an operator and let $B \in B(H)$ be normal. Does $BT \subset TB^*$ imply $B^*T \subset TB$?

Observe that when $T \in B(H)$, then the previous conjecture is a Fuglede-Putnam version. Therefore, it is interesting to investigate this problem when T is an unbounded closed operator. Notice in the end that the conjecture is not covered by any of the known unbounded generalizations of Fuglede-Putnam Theorem.

Formal aspects

Resumen

La densidad de las órbitas y la conmutatividad salvo un factor multiplicativo de los operadores lineales acotados han resultado de gran interés para los teóricos de operadores durante las últimas décadas. Este interés proviene de la relación que existe entre el estudio de las propiedades orbitales y espectrales de los operadores lineales sobre espacios de Banach y el problema del subespacio invariante. Como consequencia, la investigación sobre la Teoría de la Hiperclicidad ha aumentado considerablemente. En este manuscrito, caracterizamos la hiperciclicidad de los medias de Cesàro de orden superior sobre espacios de Banach. Demostramos algunas condiciones suficientes sobre el espectro extendido de un operador lineal acotado que garantizan su no ciclicidad-convexa. La noción de ciclicidad-convexa fue introducida por H. Rezaei en [83] en 2013. Es una noción que garantiza la ciclicidad y a su vez es necesaria para la hiperciclicidad. Caracterizamos la hiperciclicidad de los operadores que conmutan, salvo factor multiplicativo, con el operador de diferenciación, en el espacio de las funciones enteras con la topologia de la convergencia uniforme sobre conjuntos compactos. Nuestros resultados son una extensión de algunos de los resultados más clásicos relacionados con el operador de diferenciación, es decir, los de G. Godefroy y J. H. Shapiro [39], y Aron y Markose [2]. A continuación, consideramos algunos operadores particulares, como los operadores de composición en espacios de Hardy ponderados. Estos operadores han sido estudiados intensamente por varios matemáticos en el espacio de Hardy, véase el reciente trabajo [54]. Aunque sabemos menos cosas sobre estos operadores en espacios de Hardy con pesos, hemos calculado por completo el espectro extendido de los operadores de composicion que son inducidos por una transformación bilineal que fija un punto interior del disco unitario y uno exterior de su cierre. En concreto, tratamos los casos elíptico, loxodrómico y un subcaso hiperbólico. Por último, pasamos al estudio de los operadores no acotados más generales. Después del artículo de von Neumann [99], la conmutatividad y las relaciones de entrelazamiento de los operadores no acotados han sido desarrolladas por muchos matemáticos. Entre estos matemáticos, citamos a Fuglede, cuyo teorema fue una mejora del teorema espectral para operadores normales. Finalmente, demostramos una nueva versión del Teorema de Fuglede para operadores normales no acotados.

Abstract

The density of orbits and commutativity up to a factor of bounded linear operators have become of great interest for Operator Theorists during the last decades. This interest comes from the relationship that exists between the study of orbits and spectral properties of linear operators on Banach spaces and the Invariant Subspace Problem. As a consequence, the research on the Theory of Hyperclicity has increased considerably. In this manuscript, we characterize the hypercyclicity of the Cesàro means of higher-order on Banach spaces. We prove some sufficient conditions on the extended-spectrum of a bounded linear operator that guarantee its non convex-cyclicity. The notion of convex-cyclicity, was introduced by H. Rezaei in 2013 (see [83]). It is a sufficient condition for cyclicity and a necessary condition for hypercyclicity. We characterize the hypercyclicity of operators commuting up to a factor with the differentiation operator in the space of entire functions equipped with the topology of uniform convergence for compact sets. Our results are an extension of some of the most classical results related to the differentiation operator, that is, the ones of G. Godefroy and J. H. Shapiro [39], and R. Aron and D. Markose [2]. Next, we consider some particular operators, such as composition operators in weighted Hardy spaces. These operators have been studied intensely by several mathematicians in the Hardy space, see the recent of [54]. Although we know fewer things about these operators in weighted Hardy spaces, we calculated the extended-spectrum of composition operators that are induced by a bilinear transformation that fixes an interior point of the unit disk and an exterior one of its closure. Namely, we treat the elliptic, loxodromic cases and a hyperbolic subcase. Finally, we continue to the study of the more general unbounded operators. After the paper of von Neumann [99], commutativity and intertwining relations of unbounded operators have been developed by many mathematicians. Among these mathematicians, we state Fuglede whose Theorem was an improvement of the Spectral Theorem for Normal Operators. We show a new version of the Fuglede Theorem for unbounded normal operators.

State of the art

In this section, we will discuss the research done before the publication of our articles [7, 9]. It is not our scope to write an exhaustive description of the current state of knowledge. However, we shall outline some ideas and comments on each chapter.

Chapter2: The notion of *Cesàro-hypercyclicity* appeared for the first time in León-Saavedra's paper [58] in 2002. Ever since, several researchers have been interested in the study of Cesàro-hypercyclic operators (see for instance [22, 25, 26, 33, 58]). In 2013, H. Rezaei introduced the concept of *convex-cyclicity* in [83]. The two properties are important as they guarantee cyclicity of bounded linear operators. Moreover, both notions are closely related as the Cesàro-hypercyclicity implies convex-cyclicity. Recently, convex-cyclic operators have been studied in [1, 34, 97]. León-Saavedra characterized the hypercyclicity of the powers of bounded linear operators in terms of the Cesàro-hypercyclicity. In 2016, T. Bermúdez-Bonilla and N. S. Feldman [12] characterized the hypercyclicity of the powers of a bounded linear operator in terms of the hypercyclicity of a certain convex means. In our work [9], we show that it is possible to characterize the Cesàro-hypercyclicity of higher-order of an operator $T \in B(\mathcal{X})$ using its powers.

Chapter3: The first theorems in the Theory of Hypercyclicity are certainly attributable to G. D. Birkhoff [16] and G. R. McClane [63] who proved, respectively, the hypercyclicity of the translation operator and the differentiation operator in the space of entire functions. Ever since, the interest in the hypercyclicity of linear continuous operators in Fréchet spaces has increased significantly. Surely, one of the most impressive works related to this topic is the one of G. Godefroy and J. H. Shapiro [39] which is cited more than 670 times. In their work, they proved that each non-scalar operator commuting with the differentiation operator is hypercyclic. In [8], we characterize when operators commuting up to a factor with the differentiation operator are hypercyclic, extending the results of G. Godefroy and J. H. Shapiro.

Chapter4: The notion of *extended eigenvalues* was introduced by A. Biswas, A. Lambert, and S. Petrovic in [17] in 2002. The investigation of extended eigenvalues has its beginning within the application of the famous theorem of Lomonosov and a fortiori within the study of the Invariant Subspace Problem. Much research has been done to compute the *set of all extended eigenvalues* of bounded linear operators on Banach spaces. This is what is usually called the computation of the *extended-spectrum.* In 2019, M. Lacruz, F. León-Saavedra, S. Petrovic, and L. Rodríguez-Piazza calculated in [54] the extended spectrum for composition operators in the classical Hardy space of analytic functions. In our work [10], we aim in the same direction in the more generalized weighted Hardy spaces. Although we lose a lot of data in the transition from the Hardy space to weighted Hardy spaces, we treated the elliptic and loxodromic cases and a hyperbolic nonautomorphic sub-case successfully. Currently, we are collecting the missing pieces to treat the parabolic and hyperbolic cases (our work is still under development).

Chapter5: In 1942, J. von Neuman proved in [99] that if the matrix B commutes with the matrix N, with N being normal (that is $BN = NB$ with $N^*N = NN^*$), then B commutes with N^*. A natural question was then raised by the same author who asked whether his result remained true in the case of infinite-dimensional spaces. In 1950, B. Fuglede answered this question affirmatively for a non-necessarily bounded normal operator N and a bounded operator B (see [37]). Since then, this result has been referred to as the Fuglede Theorem. Almost simultaneously, P. R. Halmos gave another proof in [41] where both of B and N were assumed to be bounded. In 1951, C. R. Putnam generalised the Fuglede Theorem to two normal operators (see [81]). Namely, if $B \in B(H)$ and N and M are non-necessarily bounded normal operators then B intertwines N and M if and only if it intertwines their adjoints N^* and M^*. That is:

$$BN \subseteq MB \quad \text{if and only if} \quad BN^* \subseteq M^*B.$$

There are different proofs of the Fuglede-Putnam Theorem besides the previously cited ones. For instance, M. Rosenblum provided a very nice and elegant proof in [86] based upon the Liouville Theorem. In 1959, S. K. Berberian observed that Fuglede's version was in fact equivalent to Putnam's, which was an interesting discovery. Ever since, much work has been done on these intertwining relations and several generalisations have been made in this sense (see [66, 68, 69, 76, 82] for instance). The majority of these generalizations seem to go in one direction. That is, towards weakening the normality hypothesis, regardless of the fact that the first version still has some uncharted territories. For example, in [64], M.H. Mortad et al. proposed a conjecture related to unbounded normal operators. It is our scope in [7] to treat this conjecture and to see in which cases it holds and in which cases it doesn't do.

Objectives

At the outset, we had some initial purposes that were completely achieved. Along the way, new problems related to our initial work arose. These problems were then our new aims. During our research, we planned few questions that we answered afterward in [7, 8, 9, 10]. Basically, we were expecting to

1 characterize the Cesàro-Hypercicylicity of higher order of bounded linear operators on separable Banach spaces.

2 compute the extended spectrum of linear fractional composition operators on weighted Hardy spaces.

3 characterize the hypercyclicity of operators commuting up to a factor with the differentiation operator on the space of entire functions.

4 answer an open question raised in [64] about intertwining relations for unbounded normal operators and related to Fuglede's theorem.

Methodology

During my training period, I have been able to attend several congresses, some of them are national and others are international. Through these congresses, I had the opportunity to meet many researchers in the field with whom I have discussed fruitfully my research problems. It is said that "From the discussion comes the light", and this is certainly true as I was able to shed light on many ideas I had in mind.

During these last 3 years, I have been able to assist the Functional Analysis seminars of the research group FQM257. These seminars have allowed me to broaden my knowledge on different disciplines of the subject. They were generally held weekly on the Puerto Real campus. During the pandemic, Zoom or Google Meet sessions were held.

During my stay in Oran, I had regular meetings with my supervisor Hichem. Our method of dealing with problems was generally as follows: either we tried to prove our results by providing a demonstration or we looked for counter-examples to prove the opposite. During our meetings, we also discussed Hichem's achievements: his books [72, 73] and recent articles. As a result, I was always up to date on the novelty of his research. In the 2019 winter, Hichem had presented seminars on the *Matrices of Unbounded Operators*. These seminars complemented my research work and helped me to have a better understanding of the behaviour of this class of operators. During these meetings, we also obtained some modest results on posinormal operators.

My supervisor Fernando and I had daily meetings during my stay in Jerez. Our work rhythm was intense and very exciting. Many questions were asked, and with multiple techniques, we tried to find answers to our interrogations. We could solve several problems while others are still open. During the past three years, Fernando has given interesting seminars, for instance *El fárrago de punto fijo* (2019) and *Las aventuras de Volterra* (2020). Fernando has also encouraged me to make oral and poster communications in the *II Workshop of Functional Analysis* and the *BYMAT* conferences.

Finally, for my research, I had to use the bibliographic databases *MathSciNet* and *Google Scholar*. I also used the social academic network *ResearchGate*. The shadow libraries *Sci-Hub* and *Library Genesis* have also been of great use to me.

Conclusions and impact

During our research, we were able to solve several problems related to the Theory of Bounded and Unbounded Operators. Mainly: :

1 We characterized when the Cesáro means of higher order are hypercyclic on Banach spaces. More precisely, we showed that a bounded linear operator T is (p) Cesàro-hypercyclic if and only if there exists a vector x such that $\frac{T^n(x)}{n^p}$ is dense in $\mathcal{X}$ (see Theorem 2.2.3).

2 We proved that if T is a bounded linear operator with arbitrary large extended eigenvalues then T cannot be convex-cyclic. (see Theorem 2.3.2)

3 On weighted Hardy spaces $\mathcal{H}^2(\beta)$, we computed the extended spectrum of composition operators that are induced by linear transformations that fix one interior fixed point of the unit disk and an other exterior one. To

be more explicit, if φ is an elliptic transformation with canonical form $\varphi(z) = \omega z$ for some $\omega \in \partial\mathbb{D} \setminus \{1\}$, then

$$\text{Ext}(C_\varphi, \mathcal{H}^2(\beta)) = \{\omega^n : n \in \mathbb{Z}\},$$

and if φ is a loxodromic or a hyerbolic nonautomorphism transformation with canonical form $\varphi(z) = c(z - p) + p$, $|c| < 1$, then

$$\text{Ext}(C_\varphi, \mathcal{H}^2(\beta)) = \{c^n : n \in \mathbb{Z}\},$$

(see Theorem 4.2.1 and Theorem 4.3.1 respectively).

4 We completely characterized when the extended eigenoperators of the differentiation operator D are hypercyclic on the space of entire functions. Accurately, if T is such that $DT = \lambda TD$ for some complex scalar λ, then T is hypercyclic if and only if $|\lambda| \geq 1$ and T is not a multiple of a composition operator induced by an affine endomorphism of $\mathbb{C}$.

5 We deduced that the operators λ-commuting with the adjoint of the Cesàro operator in H^2 are not hypercyclic (see Corollary 3.7.4).

6 We proved a new version of the Fuglede Theorem for unbounded operators. Namely, if $B \in B(H)$ is normal with a finite point spectrum and T is a non-necessarily bounded operator, then $BT \subset TB^*$ implies $B^*T \subset TB$ (see Theorem 5.1.1).

This thesis includes the following published articles:

- Ikram Fatima Zohra Bensaid, Souheyb Dehimi, Bent Fuglede, and Mohammed Hichem Mortad. The Fuglede theorem and some intertwining relations. Adv. Oper. Theory, 6(1):Paper No. 9, 8, 2021

- Ikram Fatima Zohra Bensaid, Fernando León-Saavedra, and María del Pilar Romero de la Rosa. Cesàro means and convex-cyclic operators. Complex Anal. Oper. Theory, 14(1):Art. 6, 8, 2020.

It also contains the submitted papers:

- Ikram Fatima Zohra Bensaid, Fernando León-Saavedra, and María Pilar Romero De La Rosa. Extended eigenvalues for composition operators on weighted Hardy spaces. Submitted, 2021.

- Ikram Fatima Zohra Bensaid, Manuel González, Fernando León-Saavedra, and María Pilar Romero De La Rosa. Hypercyclicity of operators that λ-commute with the differentiation operator on the space of entire functions. Submitted, 2020.

Chapter 1

Preliminaries

This chapter aims to gather some general known results related to bounded and unbounded operators. We shall divide it into six sections. The first section deals with linear fractional transformations on the extended complex-plane. The classification of these transformations on the unit disk is fundamental to our work in Chapter 4. The second section aims to introduce some classical spaces of analytic functions such as the Fréchet space of entire functions and weighted Hardy spaces. In the third section, we introduce composition operators induced by a linear fractional transformation of the unit disk. These operators are mainly considered in Chapters 4. In the fourth section, some properties of hypercyclic operators are given. This section aims specifically to recall the Hypercyclicity Criterion which will be discussed in Chapter 3. Finally, the fifth and last sections aim to give some known results on unbounded linear operators on Hilbert spaces.

1.1 Self-maps of the Unit Disk

In this section, we define linear fractional transformations on the extended complex plane and then we classify them on the unit disk. This classification is essential to our results in Chapter 4. Most of the results that we shall recall can be found in [94, Chapter 0] and/or [21] for instance.

The general form of a linear fractional transformation (also called a Möbius transformation) is given by

$$\varphi(z) = \frac{az+b}{cz+d} \quad \text{for all} \quad z \in \hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\},$$

where a, b, c and d are any complex numbers satisfying $ad - bc \neq 0$. If $ad = bc$, the linear fractional transformation (abbreviated LFT) defined above is a constant. It is easy to see that φ is always a bijective holomorphic function on the Riemann sphere $\hat{\mathbb{C}}$. Its inverse is given by the expression:

$$\varphi^{-1}(z) = \frac{dz-b}{-cz+a} \quad \text{for all} \quad z \in \hat{\mathbb{C}}.$$

Clearly φ^{-1} is also a Möbius transformation. In fact, the set of all LFTs on $\hat{\mathbb{C}}$ is a group under composition. Any Möbius transformation is determined by

its action on three points.
Recall now that automorphisms of the unit disk are of the form:

$$\alpha_p(z) = \lambda \frac{z-p}{\overline{p}z-1} \quad \text{with} \quad |\lambda| = 1 \quad \text{and} \quad |p| < 1. \tag{1.1}$$

and where p and λ are explicitly given by:

$$p = \varphi^{-1}(0) \quad \text{and,} \quad \lambda = -\frac{\varphi'(0)}{|\varphi'(0)|}.$$

Indeed, without loss of generality, we may assume that $ad - bc = 1$. In this case, any LFT can be expressed as:

$$\varphi(z) = \alpha \frac{z-\beta}{\gamma z - 1}, \quad \alpha \neq 0, \quad \beta\gamma \neq 1.$$

where $\alpha =$, $\beta =$ and $\gamma =$. To prove that φ has the form 1.1 , we only need to show that $\alpha = \lambda$ and that $\gamma = \overline{\beta}$ with $|\lambda| = 1$ and $|\beta| < 1$. We know that for all $z \in \mathbb{D}$ satisfying $|z| = 1$, we have $|\varphi(z)| = 1$. Thus, proceeding with some computations, we obtain:

$$\begin{aligned}
|\varphi(z)| = |\alpha| \frac{|z-\beta|}{|\gamma z - 1|} = 1 &\iff |\alpha|^2 |z-\beta|^2 = |\gamma z - 1|^2 \\
&\iff |\alpha|^2 (z-\beta)(\overline{z} - \overline{\beta}) = (\gamma z - 1)(\overline{\gamma z} - 1) \\
&\iff |\alpha|^2(1 + |\beta|^2) - 2|\alpha|^2 Re(\overline{\beta} z) = |\gamma|^2 + 1 - 2Re(\gamma z).
\end{aligned}$$

By identification, we have: $|\alpha|^2(1 + |\beta|^2) = |\gamma|^2 + 1$ and $\overline{\beta}|\alpha|^2 = \gamma$. It is clear that those two equations imply that either $|\alpha| = 1$ or $|\beta\alpha| = 1$. Indeed, if $|\beta\alpha| = 1$ then $\overline{\beta}|\alpha|^2 = \gamma$ would imply $|\beta\alpha|^2 = \beta\gamma = 1$, a contradiction. Hence, $|\alpha| = 1$, and:

$$\gamma = \overline{\beta}|\alpha|^2 = \overline{\beta}.1 = \overline{\beta}.$$

Obviously, $|\beta| < 1$, since

$$|\beta| = |\beta|.1 = |\beta||\alpha| = |\beta\alpha| < 1.$$

Now, observe that for $z \in \mathbb{D}$ we have

$$\varphi^{-1}(z) = \frac{z - \lambda p}{\overline{p}z - \lambda} \Longrightarrow \varphi^{-1}(0) = p.$$

The derivative of φ is given by:

$$\varphi'(z) = \lambda \frac{|p|^2 - 1}{(\overline{p}z - 1)^2}.$$

At the origin, we have;

$$\varphi'(0) = \lambda(|p|^2 - 1).$$

Since $|\lambda| = 1$, we obtain:

$$\frac{\varphi'(0)}{|\varphi'(0)|} = \lambda \frac{|p|^2 - 1}{||p|^2 - 1|},$$

but,

$$\frac{|p|^2 - 1}{||p|^2 - 1|} = -1.$$

Because:

$$|p| < 1 \Longrightarrow |p|^2 < |p| < 1 \Longrightarrow |p|^2 - 1 < 0.$$

Thus, $\lambda = -\frac{\varphi'(0)}{|\varphi'(0)|}$.

If $\lambda = 1$, we obtain the involution automorphism of the unit disk that exchanges the points p and 0.

1.1.1 Classification of linear fractional transformations

Recall that any Möbius transformation φ has either one double fixed point or two simple fixed points. Indeed, the equation $\varphi(z) = z$ is equivalent to solve:

$$cz^2 + (d - a)z - b = 0. \tag{1.2}$$

If $c \neq 0$, the solutions of the quadratic equation in (1.2) are the two fixed points:

$$\frac{a - d \pm \sqrt{(a - d)^2 + 4bc}}{2c}. \tag{1.3}$$

Now, if $c = 0$, then φ fixes ∞. The quadratic equation in (1.2) becomes an equation of one degree with solution:

$$z = \frac{b}{d - a}.$$

At this point, we observe that ∞ is the only fixed point if and only if:

$$a = d \text{ and } b \neq 0$$

and the transformation in this case can be expressed as:

$$\varphi(z) = z + \frac{b}{d}.$$

In the case when φ fixes one fixed point (of double multiplicity), we say that φ is parabolic on the Riemann sphere $\hat{\mathbb{C}}$. Otherwise, if φ is not parabolic then φ has two distinct fixed points α and β in $\hat{\mathbb{C}}$. In this case, we can see that φ is conjugate to $\psi(z) = \lambda z$ where λ is a complex scalar. That is, there exists a linear fractional transformation ϕ such that:

$$\psi = \phi \circ \varphi \circ \phi^{-1} \quad \text{with} \quad \psi(z) = \lambda z.$$

Indeed, let us consider $\phi(z) = \frac{z-\alpha}{z-\beta}$ then ϕ takes α to 0 and β to ∞. It is easy to see that $\psi := \phi \circ \varphi \circ \phi^{-1}$ fixes both 0 and ∞. Hence, ψ must have the form $\psi(z) = \lambda z$, where λ is some complex number.

Now, the classification of the linear fractional map φ depends on λ . This scalar is called a multiplier of φ.

Definition 1.1.1. *Suppose $\varphi \in LFT(\hat{\mathbb{C}})$ is neither parabolic nor the identity. Let $\lambda \neq 1$ be the multiplier of φ. Then φ is called:*

- *Elliptic if $|\lambda| = 1$.*
- *Hyperbolic if $\lambda > 0$.*
- *Loxodromic if φ is neither elliptic nor hyperbolic.*

Consequently, parabolic maps are conjugate to translations, elliptic maps to rotations, hyperbolic ones to positive dilations and loxodromic ones to complex dilations.

1.1.2 Classification of linear fractional self-maps on the unit disk

If φ is a linear fractional self-map of the unit disk, that is $\varphi(\mathbb{D}) \subset \mathbb{D}$ then the classification φ on $\mathbb{D}$ is based on the location of its fixed points. More precisely, we have the following theorem:

Theorem 1.1.2. *Let φ be a lineal fractional map of the unit disk $\mathbb{D}$ such that $\varphi(\mathbb{D}) \subset \mathbb{D}$. Then*

- *If φ is parabolic then its fixed point is in $\partial\mathbb{D}$.*
- *If φ is hyperbolic then one of the fixed points is in $\overline{\mathbb{D}}$ and the other one is outside $\mathbb{D}$.*
- *If φ is elliptic or loxodromic then one of the fixed points in $\mathbb{D}$ and the other one is outside $\overline{\mathbb{D}}$.*

The proof of the previous folklore-theorem can be found in [30, Proposition 4.47] for instance. In what comes, we will prove the parabolic case. Our proof is different from the one in [30].

Assume that φ is parabolic and that $p \in \mathbb{D}$ is its fixed point. Then, let $\psi := \alpha_p \circ \varphi \circ \alpha_p$, for $z \in \mathbb{D}$. It is easy to check that ψ fixes zero. Hence, ψ is holomorphic on $\mathbb{D}$ with $\psi(0) = 0$ and $|\psi| \leq 1$. Furtheremore,

$$\psi'(0) = \alpha_p'(p)\varphi'(p)\alpha_p'(0) = \varphi'(p).$$

Since φ is parabolic then $\varphi'(p) = 1$. Hence:

$$\psi'(0) = 1.$$

Clearly, all conditions of Schwarz lemma are satisfied. We deduce then that

$$\psi(z) = \lambda z \quad \text{with} \quad |\lambda| = 1.$$

This means that ψ is a rotation with two distinct fixed points: a contradiction. Thus, p cannot be in $\mathbb{D}$. In the same way, we show that p cannot be in $\hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ by the map $z \longmapsto \frac{1}{z}$. Indeed, suppose that $p \in \hat{\mathbb{C}} \setminus \mathbb{D}$ is the fixed point of φ. Then $\frac{1}{\overline{p}} \in \mathbb{D}$. Hence, the involution automorphism $\alpha_{\frac{1}{\overline{p}}}$ is well defined.
Let us consider $\psi := \alpha_{\frac{1}{\overline{p}}} \circ \varphi \circ \alpha_{\frac{1}{\overline{p}}}$, then $\psi(\mathbb{D}) \subset \mathbb{D}$. Besides, ψ fixes ∞. However, φ is parabolic then ψ is a translation. As ∞ is its unique fixed point then it has form:

$$\psi(z) = z + b$$

for some $b \in \mathbb{C} \setminus \{0\}$: a contradiction. There is no translation that takes the unit disk into itself. Therefore, p cannot be in $\hat{\mathbb{C}} \setminus \mathbb{D}$. Consequently, we deduce that $p \in \partial\mathbb{D}$.

We refer the readers to [94, Chapter 0] and to [21] for more details on linear fractional transformations.

1.2 Introduction to Some Function Spaces

In this section, we shall introduce some function spaces. We start here with the Fréchet space of entire functions:

1.2.1 Space of entire functions

Let $\mathbb{C}$ denote the complex plane and let $\mathcal{H}(\mathbb{C})$ be the space of entire functions on $\mathbb{C}$. Remember that each entire function f can be represented as a power series:

$$f(z) = \sum_{n \geq 0} a_n z^n,$$

where $(a_n)_{n\in\mathbb{N}}$ is a sequence of complex numbers and $z \in \mathbb{C}$. The series converges in the whole complex plane. Moreover, the coefficients $(a_n)_{n\in\mathbb{N}}$ are just Taylor's coefficients and are given by:

$$a_n = \frac{f^{(n)}(0)}{n!}, \quad \forall n \in \mathbb{N}.$$

Using Cauchy's estimates, it can be shown that for $0 < R < \infty$ and $z \in D(0, R)$ one has:

$$|a_n| \leq \frac{M_R}{R^n}, \quad \text{where} \quad M_R = \sup_{|z|=R} |f(z)|.$$

Now, let us focus on the topology of $\mathcal{H}(\mathbb{C})$. It is known that $\mathcal{H}(\mathbb{C})$ endowed with the topology of the uniform convergence on compact sets of $\mathbb{C}$ is a Fréchet space. We say that a sequence of entire functions $(f_k)_k$ converges to f in $\mathcal{H}(\mathbb{C})$ if and only if for all natural n, we have:

$$p_n(f_k - f) \to 0 \quad \text{as} \quad k \to \infty,$$

where $(p_n)_n$ is the following increasing sequence of seminorms:

$$p_n(f) := \sup_{|z|\leq n} |f(z)|.$$

The concept of normal families shall be used in Chapter 3. Let us recall the definition of this notion:

Definition 1.2.1. *Let $\mathcal{F}$ be a family of complex-valued functions on a planar domain Ω. $\mathcal{F}$ is called normal family if one the following situations is satisfied:*

1 Each sequence of elements of $\mathcal{F}$ has a subsequence which converges uniformly to some analytic function on compact sets of Ω.

2 Each sequence of elements of $\mathcal{F}$ has a subsequence which converges to ∞ uniformly on compact sets of Ω.

Some authors omit the second part of the previous definition. Now, we give an example of a normal family.

Example 1.2.2. *Set $\mathcal{F} = \{z^n\}$. Then $\mathcal{F}$ is a normal family on $\mathbb{D}$ because every subsequence of $\mathcal{F}$ converges uniformly (to 0) on compact sets of $\mathbb{D}$. On the other hand, $\mathcal{F}$ is also normal on $\mathbb{C} \setminus \overline{\mathbb{D}}$ because each subsequence converges to ∞ uniformly on compact sets of $\mathbb{C} \setminus \overline{\mathbb{D}}$.*

One of the most known results related to normal families is Montel's Theorem, which is stated as follows:

Theorem 1.2.3 (First version of Montel's Theorem)**.** *Let Ω be a domain in the complex plane. Let $\mathcal{F}$ be a family of holomorphic functions on Ω. If for each compact $K \subset \Omega$, there is a positive constant C_K such that:*

$$|f(z)| \leq C_K$$

for all $z \in K$ and $f \in \mathcal{F}$, then $\mathcal{F}$ is a normal family.

The proof of the latter can be obtained by applying the Arzelà-Ascoli Theorem. We refer to [90, p. 35] for more details on the proof. Now, we announce a stronger corollary of Montel's Theorem:

Theorem 1.2.4 (Second version of Montel's Theorem)**.** *Let $\mathcal{F}$ be a family of meromorphic functions on an open set Ω. Assume that $\mathcal{F}$ is not normal at z_0 for some $z_0 \in \Omega$. If $U \subset \Omega$ is a neighborhood of z_0 then*

$$\bigcup_{f\in\mathcal{F}} f(U)$$

is dense in $\mathbb{C}$.

The foregoing theorem can be proved using the first version of Montel's Theorem and the so-called Zalcman's Lemma. Other results related to normal families can be found in [65, 90] for instance.

1.2.2 Hardy spaces

In 1915, G. H. Hardy introduced some spaces of analytic functions on the unit disc in his paper [43]. It was until 1923 that the Hungarian mathematician F. Riesz [85] named these spaces after Hardy. Ever since, the so-called Hardy spaces have been studied intensely. In this section, we recall some standard properties of these spaces. We shall need these properties in the second part of Chapter 3. If the readers wish to broaden their knowledge about Hardy spaces, they may consult Duren's book [29] for instance.

Let $\mathbb{D}$ be the unit disk of the complex plane $\mathbb{C}$ and let $\mathcal{H}(\mathbb{D})$ be the collection of all holomorphic complex-valued functions on $\mathbb{D}$.

For $0 < p < \infty$, Hardy spaces which we denote by H^p, are defined as follows:

$$H^p := H^p(\mathbb{D}) = \Big\{ f \in \mathcal{H}(\mathbb{D}) : \lim_{r \longrightarrow 1^-} \int_0^{2\pi} |f(re^{it})|^p dt < \infty \Big\}.$$

It is not difficult to check that for any $1 \leq p < \infty$ these spaces equipped with the norm

$$\|f\|_p := \big(\lim_{r \longrightarrow 1^-} M_p(f,r) \big)^{1/p}, \quad (f \in H^p)$$

where

$$M_p(f,r) = \frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^p dt,$$

are Banach spaces. If p and q are such that $\frac{1}{p} + \frac{1}{q} = 1$, we say that p and q are conjugate. If, moreover, $p = 2$, then H^2 is a Hilbert space with respect to the inner product:

$$< f, g >_{H^2} = \sup_{0<r<1} \int_0^{2\pi} f(re^{i\theta}) \overline{g(re^{i\theta})} d\theta.$$

Since H^2 is isomorphic to $\ell^2(\mathbb{N})$, H^2 can be equivalently defined as following:

$$H^2 = \Big\{ f \in \mathcal{H}(\mathbb{D}), f(z) = \sum_{n \geq 0} a_n z^n \quad : \quad \sum_{n=0}^{\infty} |a_n|^2 < \infty \Big\}.$$

This equivalence is mainly due to the Identity of Parseval. What is interesting about H^2 is that any norm of a function $f \in H^2$ has also the integral representation $\|f\|_2^2 = \lim_{r \longrightarrow 1^-} M_2(f,r)$ and this has many advantages.

Now, let $H^\infty(\mathbb{D})$ be the collection of all holomorphic bounded complex-valued functions on the unit disk $\mathbb{D}$. The space H^∞ equipped with the norm

$$\|f\|_\infty := \sup_{|z|<1} |f(z)|, \quad (f \in H^\infty)$$

is also a Banach space and for any p and q such that $0 < p < q < \infty$, we have:

$$H^\infty \subset H^q \subset H^p.$$

The reverse inclusions are not true. Another interesting fact about Hardy spaces is their invariance under multiplication by a bounded function on $\mathbb{D}$, that is, if $f \in H^\infty$ and $g \in H^p$ then $fg \in H^p$.

1.2.3 Weighted Hardy spaces

For each sequence $\beta = (\beta_n)_{n\in\mathbb{N}}$ of positive numbers satisfying

$$\liminf_{n\to\infty}(\beta_n)^{1/n} \geq 1, \tag{1.4}$$

the weighted Hardy spaces $\mathcal{H}^2(\beta)$ are the Hilbert spaces of all holomorphic functions $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{H}(\mathbb{D})$ for which

$$\|f\|_\beta := \Big(\sum_{n=0}^{\infty} |a_n|^2 \beta_n^2\Big)^{1/2} < \infty.$$

The norm above is induced by the inner product:

$$\Big\langle \sum_{n=0}^{\infty} a_n z^n, \sum_{n=0}^{\infty} b_n z^n \Big\rangle = \sum_{n=0}^{\infty} a_n \overline{b_n} \beta_n^2.$$

If $\beta \equiv 1$, then $\mathcal{H}^2(1)$ is just the classical Hardy space $\mathcal{H}^2(\mathbb{D})$. If $\beta = (n+1)^\gamma$ for $\gamma \in \mathbb{R}$, the weighted Hardy spaces are just the weighted Dirichlet spaces $\mathcal{D}_\gamma$. Furthermore, the set $\{e_n : n \in \mathbb{N}\}$ where $e_n(z) = \frac{z^n}{\beta_n}$ forms an orthonormal basis of $\mathcal{H}^2(\beta)$.

1.3 Some Classical Operators

We start this section by gathering some known results about composition operators induced by linear fractional transformations on weighted Hardy spaces. Most of theses results can be found in [38, 94, 102].

1.3.1 Composition operators

Let $\varphi : \mathbb{D} \longrightarrow \mathbb{D}$ be a holomorphic self-map of the unit disk $\mathbb{D}$. Then φ induces a linear composition operator C_φ on $\mathcal{H}(\mathbb{D})$ defined by:

$$C_\varphi f := f \circ \varphi.$$

The operator C_φ is obviously continuous on $\mathcal{H}(\mathbb{D})$ while the boundedness of C_φ on the classical Hardy space $H^2(\mathbb{D})$ is non trivial. In fact, this boundedness is due to the non-trivial *Littlewood Subordination Principle*. In 1925, J. E. Littlewood [61] showed that $C_\varphi f \in H^2(\mathbb{D})$ whenever the function φ fixes the origin.

Theorem 1.3.1 (Littlewood's Subordination Principle)**.** *Let φ be a holomorphic self-map of the unit disk that fixes the origin then for each $f \in H^2(\mathbb{D})$, we have:*

$$C_\varphi f \in H^2(\mathbb{D}).$$

Furthermore, C_φ is a contraction on $H^2(\mathbb{D})$.

Litlewood's subordination principle proof. First, suppose that $f \in H^2(\mathbb{D})$ is polynomial, then $f \circ \varphi$ is bounded in $\mathbb{D}$. So, $f \circ \varphi \in H^2(\mathbb{D})$. Let $f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n$ with $\hat{f}(n) = \frac{f^{(n)}(0)}{n!}$ and let $B : H^2(\mathbb{D}) \to H^2(\mathbb{D})$ be the backward shift defined by $Bf(z) = \sum_{n=0}^{+\infty} \hat{f}(n+1) z^n$, for all $z \in \mathbb{D}$. Then, firstly, it can be checked that $f(z) = f(0) + zBf(z)$ for all$z \in \mathbb{D}$ and secondly that the equality $B^n f(0) = \hat{f}(n)$ holds for all $n \in \mathbb{N}$ and $z \in \mathbb{D}$. Since φ is a self-map, we can substitute z by $\varphi(z)$ in the first property of the backward shift, hence:

$$f(\varphi(z)) = f(0) + \varphi(z) Bf(\varphi(z)),$$

which we can write in terms of operators:

$$C_\varphi f = f(0) + M_\varphi C_\varphi B f.$$

Thus, we have:

$$\|C_\varphi f\|_2^2 \leq |f(0)|^2 + \|C_\varphi B f\|_2^2.$$

Substituting f by Bf, we obtain:

$$\|C_\varphi B f\|_2^2 \leq |Bf(0)|^2 + \|C_\varphi B^2 f\|_2^2,$$

Inductively, substituting f byBf, $B^2 f$..., we have:

$$\|C_\varphi B^n f\|_2^2 \leq |B^n f(0)|^2 + \|C_\varphi B^{n+1} f\|_2^2, \quad n \geq 0.$$

Summing and simplifying those last inequalities yields to :

$$\|C_\varphi f\|_2^2 \leq \sum_{k=0}^{n} |B^k f(0)|^2 + \|C_\varphi B^{n+1} f\|_2^2.$$

Since f is a polynomial of degree n, $B^{n+1} f = 0$. Hence, the last inequality becomes:

$$\|C_\varphi f\|_2^2 \leq \sum_{k=0}^{n} |B^k f(0)|^2.$$

Using the second property of the backward shift above we obtain:

$$\begin{aligned}\|C_\varphi f\|_2^2 &\leq \sum_{k=0}^{n} |B^k f(0)|^2 \\ &= \sum_{k=0}^{n} |\hat{f}(k)|^2 \\ &= \|f\|_2^2.\end{aligned}$$

Finally, by a limit procedure, we extend the above inequality for any f in $H^2(\mathbb{D})$. For all $z \in \mathbb{D}$, let:

$$f_n(z) = \sum_{k=0}^{n} \hat{f}(k) z^k$$

be the n-th partial sum of Taylor series of f. We see easily that:

$$\|f_n\|_2 \leq \|f\|_2,$$

and that:

$$\|f_n - f\|_2 \longrightarrow 0 \quad \text{as} \quad n \to \infty.$$

Furthermore, we know that H^2 convergence implies uniform convergence on $\mathbb{D}$. Hence, $f_n \longrightarrow f$ uniformly on each compact K of $\mathbb{D}$ and therefore $f_n \circ \varphi \longrightarrow f \circ \varphi$ on K too. Hence, $\|f_n \circ \varphi\|_2 \leq \|f_n\|_2$. For a fixed $0 < r < 1$, we have:

$$\begin{aligned}
M_2(f \circ \varphi, r) &= M_2(\lim_{n\to+\infty} f_n \circ \varphi, r) \\
&= \lim_{n\to+\infty} M_2(f_n \circ \varphi, r) \quad \text{(By uniform convergence.)} \\
&\leq \limsup_{n\to\infty} \|f_n \circ \varphi\|_2 \\
&\leq \limsup_{n\to\infty} \|f_n\|_2 \\
&\leq \|f\|_2.
\end{aligned}$$

Now, let r tend to 1, then we have finally the desired result:

$$\|f \circ \varphi\|_2 \leq \|f\|_2.$$

This completes the proof. □

Using the involution automorphism α_p of $\mathbb{D}$, J. V. Ryff [88] extended this result and showed that C_ψ is bounded on $H^2(\mathbb{D})$ where the self-map ψ does not necessarily fix the origin. In fact, linear fractional composition operators are bounded in all Hardy spaces H^p (see e.g. [21, 94]).

On weighted Dirichlet spaces, composition operators induced by linear fractional self-maps are also bounded. That is:

$$C_\varphi \in \mathcal{B}(\mathcal{D}_\gamma), \quad \forall \gamma \in \mathbb{R}.$$

This result is mainly due to P. Hurst [46] and N. Zorboska [101].

It is still an open problem to characterize the holomorphic selfmaps of the unit disk inducing bounded composition operators on weighted Hardy spaces $\mathcal{H}^2(\beta)$. In [102, 103, 104], N. Zorboska studied intensively composition operators on weighted Hardy spaces, obtaining results on boundedness, compactness and cyclicity. More recently, new striking results on the boundedness problem have been obtained in [56].

1.3.2 Multiplication operators

As one may notice in [54], multiplication operators are a very nice source of extended eigenoperators for composition operators on $\mathcal{H}^2(\mathbb{D})$. Unfortunately, this is not always the case on weighted Hardy spaces $\mathcal{H}^2(\beta)$. For instance, the multipliers of the particular weighted Dirichlet spaces $\mathcal{D}_\gamma$ are really not obvious to describe, specially when the parameter γ is strictly positive. Indeed, when

$\gamma < 0$, we already know that the multiplication operator M_f is bounded on $\mathcal{D}_\gamma$ whenever its symbol f belongs to $H^\infty(\mathbb{D})$. If $\gamma \geq 0$, another necessary condition is that $f \in \mathcal{D}_\gamma$. In both cases, we do see that the boundedness of the symbol f on the unit disk is mandatory. However, this is not sufficient and it does not ensure us that M_f is well defined on $\mathcal{D}_\gamma$ when γ is positive. In his paper [96], D. A. Stegenga gave a necessary and sufficient condition for a function f to be a multiplier of the classical Dirichlet space $\mathcal{D} = \mathcal{D}(\mathbb{D})$. Some years later, S. Axler and A. L. Shields [3] extended this result to Dirichlet spaces $\mathcal{D}(\Omega)$ and Bergman spaces $\mathcal{A}(\Omega)$ where Ω is an open connected set of the complex plane $\mathbb{C}$. R. Kerman and E. Sawyer [47] (among others) also gave a characterization using Carleson measures.

1.4 Hypercyclicity

During the last decades, a lot of mathematicians have been attracted to the study of hypercyclic operators. The study of these operators is related to dynamical systems of partial differential equations, Chaos Theory and much more. Probably, one of the major interests comes from the relation that exists between hypercyclic operators and the famous Invariant Subspace Problem:

> An operator T has no non-trivial closed invariant subspace on $\mathcal{X}$ if and only if every nonzero vector x of $\mathcal{X}$ is cyclic for T.

and:

> An operator T has no non-trivial closed invariant subset on $\mathcal{X}$ if and only if every nonzero vector x of $\mathcal{X}$ is hypercyclic for T.

Throughout this section, $\mathcal{X}$ is assumed to be a separable Fréchet space. Now, we recall the following basic definitions:

Definition 1.4.1. *A bounded linear operator T defined on a Fréchet space $\mathcal{X}$ is said to be:*

- *Cyclic if there exists a vector $x \in \mathcal{X}$ such that the linear span*
$$\mathrm{span}\{T^n x \,:\, n \geq 0\}$$
is dense in $\mathcal{X}$.
- *Hypercyclic if there exists $x \in \mathcal{X}$ such that the orbit*
$$\mathrm{Orb}(x, T) := \{T^n x \,:\, n \geq 0\}$$
is dense in $\mathcal{X}$.

If we deal with a sequence of bounded operators $\{T_n\}_{n \geq 0}$ then:

Definition 1.4.2. *We say that a sequence* $\{T_n\}_{n\geq 0}$ *of bounded operators defined on a Fréchet space* $\mathcal{X}$ *is hypercyclic if there exists a vector* $x \in \mathcal{X}$ *such that*

$$\{T_n x : n \geq 0\}$$

is dense in $\mathcal{X}$.

A good source of information about hypercyclic operators can be found in [6, 30] for instance.

1.4.1 Hypercyclicity Criterion

Since the first examples of hypercyclic operators due to G. D. Birkhoff [16] and G. R. MacLane [63], conditions implying hypercyclicity have been studied intensely. In this section, we will announce some of these and recall the well-known "Hypercyclicity Criterion".

There is no doubt that the *Hypercyclicity Criterion* was discovered from the one of C. Kitai. This last criterion appeared in Kitai's thesis in 1982 (see [50]). Recall that this criterion says that if there exist dense subsets X_0 and Y_0 of $\mathcal{X}$ and a map $S : Y_0 \to Y_0$ such that T^n and S^n converges to zero pointwise in X_0 and Y_0 respectively, and S is the right inverse of T in Y_0, then the operator T is hypercyclic. In fact, C. Kitai showed that T is mixing. Mixing property is stronger than hypercyclicity. We will not deal with that property in this manuscript.

What is surprising in Kitai's Criterion is that neither the linearity nor the continuity of the map S on Y_0 is needed. Some years later, Gethner and J. H. Shapiro rediscovered the Hypercyclicity Criterion of Kitai and replaced the sequence $(n)_n$ appearing in that criterion by an increasing sequence $(n_k)_k$ of positive integers. The following criterion is a slightly modified version of Kitai/Gethner-Shapiro Criterions and it is due to J. Bès and A. Peris (see [20]). This version only requires to $T^{n_k}S_{n_k}$ to converge pointwise to the identity.

Theorem 1.4.3 (Hypercyclicity Criterion)**.** *Let* $T \in \mathcal{B}(\mathcal{X})$. *If there exist dense subsets* X_0 *and* Y_0 *of* $\mathcal{X}$, *an increasing sequence* $(n_k)_k$ *of positive integers and maps* $S_{n_k} : Y_0 \to \mathcal{X}$ *such that:*

i $T^{n_k}x \to 0$, $\forall x \in X_0$.

ii $S_{n_k}y \to 0$, $\forall y \in Y_0$.

iii $T^{n_k}S_{n_k}y \to y$, $\forall y \in Y_0$.

Then T *is hypercyclic.*

Proof. First, let $\|.\|$ denote the $F-$ norm of $\mathcal{X}$. The proof of Hypercyclicity Criterion is based on the construction of a vector $x \in \mathcal{X}$ such that the orbit $\{T^n x : n = 0, 1, 2, \cdots\}$ is dense in $\mathcal{X}$. Indeed, by separability of $\mathcal{X}$, we can choose a sequence $(y_j)_{j\geq 1} \subset Y_0$ that is dense in $\mathcal{X}$ and we can show that there exist $x_j \in \mathcal{X}$ and positive integers k_j such that

$$x = x_1 + S_{n_{k_1}}y_1 + x_2 + S_{n_{k_2}}y_2 + x_3 + S_{n_{k_3}}y_3 + \cdots$$

is a hypercyclic vector for T. Indeed, we can construct the x_j, for $j \geq 1$ by recursion as follows:

$$\|x_j\| < \frac{1}{2^j} \quad \text{and} \quad \|T^{n_{k_l}} x_j\| < \frac{1}{2^j} \tag{1.5}$$

$$\|S_{n_{k_j}} y_j\| < \frac{1}{2^j} \quad \text{and} \quad \|T^{n_{k_l}} S_{n_{k_j}} y_j\| < \frac{1}{2^j} \tag{1.6}$$

$$\|T^{n_{k_j}} S_{n_{k_j}} y_j - y_j\| < \frac{1}{2^j} \quad \text{and} \quad \Big\|T^{n_{k_j}}\Big(\sum_{l=1}^{j-1}(x_l + S_{n_{k_l}} y_l) + x_j\Big)\Big\| < \frac{1}{2^j} \tag{1.7}$$

where $l = 1, ..., j-1$. The conditions *ii.* and *iii.* and the linearity of T ensure that the previous inequalities hold for $x_1 = 0$. Hence, the case $j = 1$ is true. For $j \geq 2$, we can suppose that $(x_i)_{i=1}^{j-1}$ and $(k_i)_{i=1}^{j-1}$ have been constructed. Since by assumption X_0 is dense in $\mathcal{X}$, there exist some x_j such that (1.5) holds and $\sum_{l=1}^{j-1}(x_l + S_{n_{k_l}} y_l) + x_j$ belongs to X_0. If we let j tend to infinity then our assumptions *i.*, *ii.* and *iii.* also ensure (1.6) and (1.7). Hence, the series $\sum_{j=1}^{\infty}(x_j + S_{n_{k_j}} y_j)$ is convergent and we can write:

$$\sum_{l=1}^{\infty}(x_l + S_{n_{k_l}} y_l) = \sum_{l=1}^{j-1}(x_l + S_{n_{k_l}} y_l) + x_j + S_{n_{k_j}} y_j + \sum_{l=j+1}^{\infty}(x_l + S_{n_{k_l}} y_l).$$

The linearity of T leads to:

$$T^{n_{k_j}} x = T^{n_{k_j}}\Big(\sum_{l=1}^{j-1}(x_l + S_{n_{k_l}} y_l) + x_j\Big) + T^{n_{k_j}} S_{n_{k_j}} y_j + \sum_{l=j+1}^{\infty} T^{n_{k_j}} x_l + \sum_{l=j+1}^{\infty} T^{n_{k_j}} S_{n_{k_l}} y_l.$$

Let $I_{n_{k_j}} = \|T^{n_{k_j}} x - y_j\|$ then we have:

$$I_{n_{k_j}} = \Big\|T^{n_{k_j}}\Big(\sum_{l=1}^{\infty}(x_l + S_{n_{k_l}} y_l) + x_j\Big) + T^{n_{k_j}} S_{n_{k_j}} y_j - y_j + \sum_{l=j+1}^{\infty} T^{n_{k_j}} x_l + \sum_{l=j+1}^{\infty} T^{n_{k_j}} S_{n_{k_l}} y_l\Big\|.$$

Using the inequalities (1.5), (1.6) and (1.7), we obtain

$$I_{n_{k_j}} \leq \frac{1}{2^j} + \frac{1}{2^j} + \sum_{l=j+1}^{\infty} \frac{1}{2^l} + \sum_{l=j+1}^{\infty} \frac{1}{2^l} = \frac{4}{2^j} \to 0 \quad \text{as} \quad j \to \infty.$$

Therefore, $\|T^{n_{k_j}} x - y_j\| \to 0$. the density of the $(y_j)_{j \geq 1}$ implies the density of the orbit of T generated by x. This completes the proof. □

At this stage, we should point out that Gethner-Shapiro version of Hypercyclicity Criterion is equivalent to the one of J. Bès and A. Peris (see [30, p. 81]). However, J. Bès and A. Peris version is easy to use.

Sufficient conditions ensuring the hypercyclicity of a sequence of bounded linear operators $(T_n)_{n \geq 1}$ are given in the following corollary:

Theorem 1.4.4 (Hypercyclicity Criterion for sequences)**.** *Let $\mathcal{X}$ and $\mathcal{Y}$ be two separable Fréchet spaces. For $n \geq 1$, let $T_n : \mathcal{X} \to \mathcal{Y}$ be a sequence of continuous linear operators. If there exist dense subsets $X_0 \subset \mathcal{X}$ and $Y_0 \subset \mathcal{Y}$, an increasing sequence $(n_k)_k$ of positive integers and maps $S_{n_k} : Y_0 \to \mathcal{X}$ such that:*

i $T_{n_k}x \to 0$, $\forall x \in X_0$.

ii $S_{n_k}y \to 0$, $\forall y \in Y_0$.

iii $T_{n_k}S_{n_k}y \to y$, $\forall y \in Y_0$.

Then the sequence $(T_n)_n$ is hypercyclic.

The proof of the previous theorem is similar to one of the Hypercyclicity Criterion and we omit it. Another consequence of the Hypercyclicity Criterion was given in [39] by G. Godefroy and J. H. Shapiro who proved the following result:

Corollary 1.4.5 (Eigenvalues Criterion)**.** *Let T be a continuous linear operator in a separable Frechet space $\mathcal{X}$. Assume that the subspaces:*

$$X_0 = \operatorname{span}\{x \in \mathcal{X};\quad Tx = \lambda x \quad \textit{for some} \quad \lambda \in \mathbb{K} \quad \textit{with} \quad |\lambda| < 1\}$$

and

$$Y_0 = \operatorname{span}\{x \in \mathcal{X};\quad Tx = \lambda x \quad \textit{for some} \quad \lambda \in \mathbb{K} \quad \textit{with} \quad |\lambda| > 1\}$$

are dense in $\mathcal{X}$. Then T is hypercyclic.

Proof. It suffices to apply the Hypercyclicity Criterion for the sequence of integers $(n)_{n\geq 0}$. Let $x \in X_0$, then x can be expressed as $x = \sum_{k=0}^{m} \alpha_k x_k$ where $Tx_k = \lambda_k x_k$ with $|\lambda_k| < 1$ and $\alpha_k \in \mathbb{K}$. Hence, by linearity we have: $Tx = \sum_{k=0}^{m} \alpha_k \lambda_k x_k$ and $T^n x = \sum_{k=0}^{m} \alpha_k (\lambda_k)^n x_k \to 0$ as $n \to \infty$. Similarly, each $y \in Y_0$ can be expressed as $y = \sum_{k=0}^{m} \beta_k y_k$ where $Ty_k = \mu_k y_k$ with $|\mu_k| > 1$ and $\beta_k \in \mathbb{K}$. Let $S_n : Y_0 \to \mathcal{X}$ be defined by $S_n y = \sum_{k=0}^{m} \beta_k \frac{1}{\mu_k^n} y_k$ then clearly $S_n \to 0$ as $k \to \infty$ and for all $n \geq 0$ and $y \in Y_0$ we have $T^n S_n y = y$. Hence, according to Hypercyclicity Criterion, T is hypercyclic.

□

At this point, we should point out that the Hypercyclicity Criterion is stronger than Eigenvalues Criterion. Indeed, there exist some bounded operators that satisfy Kitai's Criterion without satisfying the Eigenvalues one. For instance, hypercyclic operators with empty point spectrum, such as some unilateral weighted backward shifts, have that property (see [40, 59, 89]).

In 2001, a very interesting question was asked by F. León-Saavedra and A. Montes-Rodríguez in [59]: Does every hypercyclic operator satisfy the Hypercyclicity Criterion? J. Bès and A. Peris showed previously that an operator T satisfies the Hypercyclicity Criterion if and only if the direct sum $T \oplus T$ is hypercyclic (see [20]). In 2006, M. De La Rosa and C. Read proved in [27] the existence of a hypercyclic operator whose direct sum $T \oplus T$ is not hypercyclic. Thus, they answered the question of F. León-Saavedra and Montes-Rodríguez in the negative. Other interesting counterexamples were found later by F. Bayart and E. Matheron (see [4]).

1.4.2 MacLane's and Birkhoff's operators

Let $D : \mathcal{H}(\mathbb{C}) \to \mathcal{H}(\mathbb{C})$ denote the continuous linear differentiation operator defined by:

$$Df(z) = f'(z), \quad \forall z \in \mathbb{C}.$$

This operator is generally called MacLane's operator due to his paper [63]. In this section, we shall provide some results related to operators that commutes with D on $\mathcal{H}(\mathbb{C})$. That is, operators T satisfying

$$DT = TD, \tag{1.8}$$

in $\mathcal{H}(\mathbb{C})$. One non trivial example satisfying (1.8) is the translation operator T_a defined by

$$T_a f(z) = f(z+a), \quad \forall z \in \mathbb{C}.$$

for some fixed $a \in \mathbb{C}$. This operator is named after G. D. Birkhoff by means of his paper [16].

From another point of view, operators commuting with Birkhoff's operator are called, by definition, *convolution operators*. Thus, MacLane's operator is a simple example of a convolution operator. There is a big connection between these two operators. Indeed, for any $f \in \mathcal{H}(\mathbb{C})$ and any $z \in \mathbb{C}$, we have:

$$T_a f(z) = f(z+a) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z)}{n!} a^n = \sum_{n=0}^{\infty} \frac{(aD)^n}{n!} f(z) \tag{1.9}$$

In fact, we shall see later that an operator T commutes with MacLane's operator if and only it commutes with Birkhoff's operator.

We shall point out, in advance, that most of the following results (and their proofs) can be found in [30, Chapter 4] and/or in [6, Chapter 1]. First, let us recall the following definition:

Definition 1.4.6. *An entire function $\varphi \in \mathcal{H}(\mathbb{C})$ is said to be of exponential type provided that there exist two constants $\alpha, \beta > 0$ such that:*

$$|\varphi(z)| \leq \alpha e^{\beta |z|}, \quad \forall z \in \mathbb{C}.$$

Using Cauchy's estimates, it can be shown that an entire function $\varphi(z) = \sum_{n=0}^{\infty} a_n z^n$ is of exponential type if and only if

$$|a_n| \leq M \frac{R^n}{n!}, \quad \forall n \in \mathbb{N}$$

where $M, R > 0$ are some finite constants. Some simple examples of entire functions of exponential type are $e^{\alpha z}$, $\sin \alpha z$ and $\cos \alpha z$ with $\alpha \in \mathbb{C}$.

Functions of operators can be undefined, in general. The following proposition deals with functions of the differentiation operator D. It asserts that if φ is an entire function of exponential type then the operator $\varphi(D)$ is continuous in $\mathcal{H}(\mathbb{C})$.

Proposition 1.4.7. *Let $\varphi(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function of exponential type. Then for every entire function f the operator:*

$$\varphi(D)f = \sum_{n=0}^{\infty} a_n D^n f$$

is continuous on $\mathcal{H}(\mathbb{C})$.

Proof. Let $f \in \mathcal{H}(\mathbb{C})$ and $|z| \leq m$. Using Cauchy estimates, we know that

$$|a_n| = \left|\frac{f^{(n)}(z)}{n!}\right| \leq \frac{1}{m^n} \sup_{|z|\leq m} |f(z)|, \quad \forall n \in \mathbb{N}.$$

Moreover, since by assumption φ is of exponential type, there exist two constants $M, R > 0$ such that

$$|a_n| \leq M\frac{R^n}{n!}, \quad \forall n \in \mathbb{N}.$$

Hence, according to these estimations we get:

$$|a_n f^{(n)}(z)| \leq |a_n|\frac{n!}{m^n} \sup_{|z|\leq 2m} |f(z)| \leq M\frac{R^n}{n!}\frac{n!}{m^n} \sup_{|z|\leq 2m} |f(z)| = M\left(\frac{R}{m}\right)^n \sup_{|z|\leq 2m} |f(z)|.$$

If $\frac{R}{m} < 1$ then $\sum_{n=0}^{\infty} a_n f^{(n)} f(z)$ converges uniformly on $D(0, m)$. Thus, the series $\varphi(D)f = \sum_{n=0}^{\infty} a_n f^{(n)} f(z)$ is convergent in $\mathcal{H}(\mathbb{C})$. Moreover, for $\frac{R}{m} < 1$, the geometric series $\sum_{n=0}^{\infty} \left(\frac{R}{m}\right)^n$ converges to $\frac{1}{1-R/m}$ and we have:

$$p_m(\varphi(D)f) \leq Cp_{2m}(f)$$

where $C = \frac{M}{1-R/m}$. Hence, $\varphi(D)$ is a continuous operator on $\mathcal{H}(\mathbb{C})$. $\square$

In particular, since $\varphi(z) = e^{az}$ is of exponential type and taking into account the relation in (1.9), we deduce (by taking $b_n = a_n/n!$ in the previous proposition) that:

$$T_a = e^{aD}.$$

That is:

$$T_a = \varphi(D), \quad \text{where} \quad \varphi(z) = e^{az}.$$

The next theorem characterizes convolution operators. As we shall see in Chapter 3, this theorem shall be applied several times and will be of great importance.

Theorem 1.4.8. *Let $T \in \mathcal{B}(\mathcal{H}(\mathbb{C}))$. Then the following statements are equivalent:*

i. T commutes with the differentiation operator D.

ii. T commutes with each translation operator T_a, $a \in \mathbb{C}$.

iii. There exists an entire function φ of exponential type such that $T = \varphi(D)$.

Proof. Assume that T commutes with the differentiation operator D. Then we have:

$$TT_af = T\sum_{n=0}^{\infty}\frac{a_n}{n!}D^nf = \sum_{n=0}^{\infty}T\frac{a_n}{n!}D^nf = \sum_{n=0}^{\infty}\frac{a_n}{n!}D^n(Tf) = T_aTf.$$

To show the second implication, assume that T commutes with T_a. For each $f \in \mathcal{H}(\mathbb{C})$, consider the continuous map $f \mapsto Tf(0)$. By continuity, there exist a positive constant $M > 0$ and $R \in \mathbb{N}$ such that

$$|Tf(0)| \leq M \sup_{|z|\leq R}|f(z)|$$

For $n \in \mathbb{N} \cap \{0\}$, denote $e_n(z) = z^n$ and set $a_n = \frac{Tf(0)}{n!}$. Hence, we have:

$$|a_n| \leq M\frac{R^n}{n!}$$

which is equivalent to say that the entire function $\varphi(z) := \sum_{n=0}^{\infty}a_nz^n$ is of exponential type. According to Proposition , $\varphi(D)$ is a continuous operator in $\mathcal{H}(\mathbb{C})$. We have:

$$(\varphi(D)e_n)(0) = (\sum_{n=0}^{\infty}a_nD^nz^n)(0) = a_nn! = (Te_n)(0)$$

The density of the monomials e_n in $\mathcal{H}(\mathbb{C})$ allows us to conclude that:

$$(\varphi(D)f)(0) = (Tf)(0), \quad \forall f \in \mathcal{H}(\mathbb{C}).$$

Moreover, since the translation operator commutes with $\varphi(D)$, we have:

$$\varphi(D)f(z) = \varphi(D)T_zf(0) = T_z\varphi(D)f(0) = T_zTf(0) = TT_zf(0) = Tf(z)$$

That is:

$$\varphi(D)f = Tf \quad \forall f \in \mathcal{H}(\mathbb{C}),$$

or simply, $T = \varphi(D)$, as wanted.

Finally, assume that $T = \varphi(D)$ for some entire function φ of exponential type. The continuity of D in $\mathcal{H}(\mathbb{C})$ leads to:

$$TDf = \sum_{n=0}^{\infty}a_nD^nDf = \sum_{n=0}^{\infty}D(a_nD^nf) = D\sum_{n=0}^{\infty}a_nD^nf = DTf,$$

for any entire function f. Thus, T commutes with D and the third and last implication is proved.

□

Now, we shall announce a strong result ensuring the hypercyclicity of operators commuting with the differentiation operator D:

Theorem 1.4.9 (Godefroy-Shapiro Theorem)**.** *Let T be a continuous linear operator on $\mathcal{H}(\mathbb{C})$. Assume that T is not a scalar multiple of the identity such that:*

$$DT = TD$$

then T is hypercylic.

Proof. Assume that T is a continuous linear operator, non scalar multiple of the identity, that commutes with D. According to Proposition 1.4.8,there exists an entire function $\varphi(z) = \sum_{n=0}^{\infty} a_n z^n$ of exponential type such that $T = \varphi(D)$. Since T is not a scalar multiple of the identity, φ is nonconstant. For $\lambda \in \mathbb{C}$, consider the exponential function $e_\lambda(z) = e^{\lambda z}$, $z \in \mathbb{C}$. It is know that if a set $\Lambda \subset \mathcal{H}(\mathbb{C})$ has an accumulation point then the span$\{e_\lambda, \lambda \in \Lambda\}$ is dense in $\mathcal{H}(\mathbb{C})$ (see [30, p. 45]). Now, observe that we have:

$$\varphi(D)e_\lambda(z) = \sum_{n=0}^{\infty} a_n D^n e^{\lambda z} = \sum_{n=0}^{\infty} a_n \lambda^n e^{\lambda z} = \varphi(\lambda)e_\lambda(z),$$

that is:

$$Te_\lambda = \varphi(\lambda)e_\lambda.$$

Thus we have:

$$\mathrm{span}\{e_\lambda : |\varphi(\lambda)| < 1\} \subset \mathrm{span}\{f \in \mathcal{H}(\mathbb{C});\ Tf = \lambda f \text{ for some } \lambda \in \mathbb{C} \text{ with } |\lambda| < 1\}$$

Since the set $\{\lambda \in \mathbb{C} : |\varphi(\lambda)| < 1\}$ has an accumulation point, we deduce that span$\{e_\lambda : |\varphi(\lambda)| < 1\}$ is dense in $\mathbb{C}$. Therefore,

$$\mathrm{span}\{f \in \mathcal{H}(\mathbb{C});\ \ Tf = \lambda f \ \text{ for some } \ \lambda \in \mathbb{C} \ \text{ with } \ |\lambda| < 1\}$$

is dense in $\mathcal{H}(\mathbb{C})$. With similar arguments, we deduce that

$$\mathrm{span}\{f \in \mathcal{H}(\mathbb{C});\ \ Tf = \lambda f \ \text{ for some } \ \lambda \in \mathbb{C} \ \text{ with } \ |\lambda| > 1\}$$

is also dense in $\mathcal{H}(\mathbb{C})$. The Eigenvalues Criterion implies that T is hypercyclic, as wanted. □

It is worth of note to mention that Godefroy-Shapiro Theorem does not hold in Banach spaces, because contractions are not hypercyclic.

Our purpose now is to give a brief introduction to unbounded operators on Hilbert spaces. This class of operators appears naturally in Quantum Mechanics and it also plays a major role in PDEs.

In Section 1.5, we shall announce some basic definitions and results related to some subclasses of unbounded operators. That is, we introduce closed operators, self-adjoint operators, and normal ones and illustrate them with a few examples. Then, in Section 1.6, we shall see one of the fundamental results related to unbounded normal operators: the Fuglede-Putnam Theorem. The generalization of the latter constitutes our main investigation in [7]. Finally, we mention that most of the results that we will recall in the next sections appear in Schmudgen's Book [93] and/or J. Weidmann's book [100].

1.5 Unbounded Operators on Hilbert Spaces

We start by giving some basic definitions. Notice that some of these definitions might be different elsewhere.

1.5.1 Some basic definitions

Definition 1.5.1 (Linear operators). *Let H and K be two Hilbert spaces. A linear operator T from H into K is a linear mapping of a subspace $D(T)$ of H into K. That is, for all $x, y \in D(T)$ and all $\alpha, \beta \in \mathbb{C}$, the operator T satisfies*

$$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y).$$

The subspace $D(T)$ is called the domain of T.

If $D(T)$ is dense in H, we say that T is densely defined. The concept of domain plays an important role for unbounded operators. For instance, one particular domain is

$$D(T) = \{x \in H : Tx \in K\},$$

which is called the maximal domain. By considering the same symbol T on different domains, completely different operators can be obtained.
Now, recall that the subspace

$$\operatorname{ran} T := T(D(T)) = \{Tx : x \in D(T)\}$$

is called the range (or the image) of T and the subspace

$$\ker T = \{x \in D(T) : Tx = 0\}$$

is called the kernel (or the null space) of T.

Definition 1.5.2 (Bounded operator). *Let H and K be two Hilbert spaces. A linear operator $T : D(T) \subset H \to K$ is said to be bounded on $D(T)$ if there exists a constant $M > 0$ such that*

$$\|Tx\|_K \leq M\|x\|_{D(T)} \quad \forall x \in D(T).$$

If $D(T) = H$, then we shall write $T \in B(H, K)$. We then say that T is everywhere defined. In such a case, the quantity

$$\|T\| := \sup_{\|x\|_H \neq 0} \frac{\|Tx\|_K}{\|x\|_H}$$

is finite and is called the operator norm of T. If T is not bounded, we say that T is unbounded.

Example 1.5.3. *Consider the space ℓ^2 of all square summable sequences, equipped with the norm $\|x\|_2 = \left(\sum |x_i|^2\right)^{1/2}$ for $x = (x_i)_{i=0}^{\infty} \in \ell^2$. Let $M : \ell^2 \to \ell^2$ be such as:*

$$M(x_0, x_1, \cdots) = (x_0, 2x_1, 2^2 x_2, \cdots, 2^n x_n, \cdots).$$

The operator M is not bounded on ℓ^2. Indeed, there exists a non-zero vector $e_{n+1} = (0, 0, \cdots, 0, 1, 0, \cdots) \in \ell^2$, where the element 1 is in the $(n+1)th$ position; such that $\|e_{n+1}\|_2 = 1$ and $\|Me_{n+1}\|_2 = 2^n \to \infty$ as $n \to \infty$. Hence, M is unbounded on ℓ^2.

Another interesting example, for instance, is the multiplication operator M_φ on $L^2(\mathbb{R})$ where φ is a measurable function on $\mathbb{R}$. The boundedness of this operator depends on the behavior of φ on $\mathbb{R}$. It can be shown that $M_\varphi \in B(L^2(\mathbb{R}))$ if and only if φ is essentially bounded, that is, if there exists a constant $C > 0$ such that $|\varphi(x)| \leq C$ almost everywhere in $\mathbb{R}$. For a proof, see e.g. [100, p.54]. For more interesting properties of this operator, we refer to [73, Exercise 10.3.9] where this operator has been studied in detail.

1.5.2 Sum, product and extension of unbounded operators

Now, we introduce basic operations on linear operators:

Definition 1.5.4. *Let T and S be two linear operators.*

- *The operator αT is defined by*

$$(\alpha T)x = \alpha(Tx) \quad \text{for} \quad x \in D(\alpha T) = D(T).$$

- *We define the sum $T + S$ by*

$$(T + S)(x) = T(x) + S(x), \quad \text{for} \quad x \in D(S + T)$$

  where

$$D(T + S) = D(T) \cap D(S).$$

- *The product ST is defined by*

$$(ST)x = S(Tx) \quad \text{for} \quad x \in D(ST)$$

  where

$$D(ST) = \{x \in D(T) : Tx \in D(S)\}.$$

- *The operator T is called an extension of S (or S a restriction of T) if*

$$D(S) \subset D(T) \quad \text{and} \quad Tx = Sx \quad \text{for} \quad x \in D(S).$$

In this case, we write $S \subset T$ (or $S \supset T$).

Remark 1.5.5. *When dealing with unbounded operators, we usually encounter issues with their domains. In particular, it is quite conceivable to have $D(T^2) = \{0\}$ even when T is densely defined. For instance, it was shown in [73, Example 10.1.3] that if $\mathcal{C}_0^\infty(\mathbb{R})$ is the set of infinitely differentiable functions whose support is compact and $\mathcal{F}_0$ is the usual $L^2(\mathbb{R})$-Fourier transform restricted to $\mathcal{C}_0^\infty(\mathbb{R})$, that is $\mathcal{F}_0 f = \hat{f}$, then*

$$D(\mathcal{F}_0^2) = \{f \in \mathcal{C}_0^\infty(\mathbb{R}) : \hat{f} \in \mathcal{C}_0^\infty(\mathbb{R})\} = \{0\}.$$

For more sophisticated examples, see e.g. [23, 28, 91]. It is also likely to have $D(T) \cap D(S) = \{0\}$ even for some strong classes of operators. This follows from a famous von Neumann Theorem. More details and explicit constructions of such examples may be consulted in [51].

1.5.3 Closed operators

In the unbounded operator setting, closed operators may be perceived as the natural substitutes of the bounded ones. Before defining this class of operators, recall that the graph of a linear operator T from H into K is denoted by $\mathcal{G}(T)$ and is defined as follows:

$$\mathcal{G}(T) = \Big\{(x, Tx) : x \in D(T)\Big\} \subset H \times K.$$

Definition 1.5.6 (Closed operator)**.** *An operator T from H into K is said to be closed if its graph $\mathcal{G}(T)$ is closed in $H \times K$. That is, T is closed if and only if for all $(x_n)_n \subset D(T)$ if $x_n \to x$ and $Ax_n \to y$, then $x \in D(A)$ and $y = Ax$.*

Example 1.5.7 (c.f. [67])**.** *Let φ be a measurable function in $\mathbb{R}$ and let M_φ be the multiplication operator induced by φ on $L^2(\mathbb{R})$. We claim that M_φ is closed. Let $f_n \in D(M_\varphi)$ be such that $\|f_n - f\|_2 \to 0$ and $\|M_\varphi f_n - g\|_2 \to 0$. Since the space $(L^2(\mathbb{R}), \|.\|_2)$ is complete and both of (f_n) and (φf_n) are Cauchy, we have $f, g \in L^2(\mathbb{R})$. Moreover, we know that there exists a subsequence $(f_{n_k})_k \subset (f_n)_n$ such that $\lim_{k\to\infty} f_{n_k}(x) = f(x)$ for almost every x in $\mathbb{R}$ (see [87, 3.12 Theorem]). So, we have*

$$\lim_{k\to\infty} \varphi(x) f_{n_k}(x) = \varphi(x) f(x) \text{ for almost every } x \text{ in } \mathbb{R}.$$

Since $\varphi f_n \to g$ in $L^2(\mathbb{R})$, every subsequence $(\varphi f_{n_k}) \subset (\varphi f_n)$ converges to g in $L^2(\mathbb{R})$. Hence, (φf_{n_k}) too has an a.e. convergent subsequence to g. So, we have $\lim\limits_{j\to\infty} \varphi f_{n_{k_j}} = g$ and $\lim\limits_{k\to\infty} \varphi f_{n_k} = \varphi f$ a.e. in $\mathbb{R}$. Therefore $g = \varphi f$ a.e. in $\mathbb{R}$, and hence in $L^2(\mathbb{R})$. We also have $\varphi f \in L^2(\mathbb{R})$ as $g \in L^2(\mathbb{R})$. Thus, $f \in D(M_\varphi)$ and $g = \varphi f$ in $L^2(\mathbb{R})$, that is M_φ is closed.

By the Closed Graph Theorem, any closed linear operator between Banach spaces is bounded. Nevertheless, when the domain of T is not the whole space, there are many examples of closed operators which are not bounded, and bounded ones which are not closed (see e.g. [73]).

The sum and the product of two closed operators are not necessarily closed. Indeed, it suffices to consider a closed operator T whose domain $D(T)$ is non-closed, and to observe that neither $T - T = 0_{D(T)}$ nor $0_H T = 0_{D(T)}$ is closed, because if a bounded operator is defined on some domain D, then this operator is closed if and only if D is closed in H (see [100, Theorem 5.2.]).

1.5.4 Self-adjoint operators

An important subclass of closed operators is that of self-adjoint ones. Before we start, we point out that the notation $\langle .,. \rangle$ stands for the scalar product. Let T be a densely defined linear operator from $(H, \langle .,. \rangle_H)$ into $(K, \langle .,. \rangle_K)$. To define the adjoint of T, we shall consider the following domain:

$$D(T^*) = \{y \in K : \quad \exists u \in H : \langle Tx, y\rangle_K = \langle x, u\rangle_H \quad \forall x \in D(T)\}.$$

Observe that, according to Riesz Theorem, a necessary and sufficient condition for $y \in K$ to belong to $D(T^*)$ is the continuity of the linear functional $x \mapsto \langle Tx, y\rangle_K$ on $D(T)$. Since $\overline{D(T)}^H = H$, the vector u is uniquely determined by y. Hence, by setting $T^*y = u$, we see that the linear mapping T^* from K into H is well-defined and linear. The operator T^* is called the adjoint operator of T.

Definition 1.5.8 (Adjoint operator)**.** *Let T be a densely defined linear operator from $(H, \langle .,. \rangle_H)$ into $(K, \langle .,. \rangle_K)$. The operator T^* satisfying*

$$\langle Tx, y\rangle_K = \langle x, T^*y\rangle_H, \quad \text{for all} \quad x \in D(T), \quad y \in D(T^*)$$

is called the adjoint operator of T.

Definition 1.5.9. *Let T be a densely defined linear operator on a Hilbert space H. Say that*

- *T is symmetric if T^* is an extension of T, that is: $T \subset T^*$.*
- *T is self-adjoint if $T = T^*$.*

Example 1.5.10. *Consider the same operator as in Example 1.5.7. It was shown in [93] that M_φ is self-adjoint if and only if φ is real-valued almost everywhere.*

The next proposition provides some properties of the adjoint operation.

Proposition 1.5.11. *Let S and T be linear operators from H into K such that T is densely defined. Then:*

i *T^* is a closed linear operator from K into H.*

ii $(\mathrm{ran}(T))^{\perp} = \ker(T^*)$.

iii *If T^* is densely defined, then $T \subset (T^*)^* := T^{**}$.*

iv *If $S \subset T$, then $T^* \subset S^*$.*

v $(\lambda T)^* = \overline{\lambda} T^*$.

vi *If $S + T$ is densely defined, then $S^* + T^* \subset (S + T)^*$. Moreover, if $S \in B(H)$, then $(S + T)^* = S^* + T^*$.*

For a proof, see [93, Proposition 1.6]. It is known that $(ST)^* = T^*S^*$ whenever $S \in B(H, K)$ and $T \in B(K, L)$. However, this equality does not hold in general. In fact, while dealing with (unbounded) linear operators, the only thing we can be sure of is the inclusion $T^*S^* \subset (ST)^*$. The following proposition gives a condition on S for the reverse inclusion to hold.

Proposition 1.5.12. *Let S and T be linear operators from H into K and from K into L respectively. Assume that ST is densely defined then:*

i *If $D(S)$ is dense in K, then $T^*S^* \subset (ST)^*$.*

ii *If $S \in B(K)$, then $(ST)^* = T^*S^*$.*

The proof can be found in [93, p.10] for instance.

The next proposition relates symmetric operators to self-adjoint ones. It states that if a symmetric operator is an extension of a self-adjoint operator then the two operators must be equal. The proof is straightforward and we omit it.

Proposition 1.5.13. *Self-adjoint operators are maximally symmetric.*

1.5.5 Normal operators

Definition 1.5.14. *Let T be a densely defined linear operator on a Hilbert space H. T is said to be normal provided that T is closed and $T^*T = TT^*$.*

It is obvious that any self-adjoint operator is normal.

Remark 1.5.15. *If T is normal then T^* is normal too. However, the converse is not true in general. To see that it suffices to consider the identity operator I_D whose domain is non-closed. Hence I_D is not normal since it is not even closed. Nevertheless, its adjoint $(I_D)^* = I_H$ is normal.*

Proposition 1.5.16. *If T is closed and $T^*T \subset TT^*$, then T is normal.*

Proof. The closedness of T implies the self-adjointness of T^*T and TT^*. Since by assumption, $T^*T \subset TT^*$ and self-adjoint operators are maximally self-adjoint, Proposition 1.5.13 yields to the wanted result. □

Theorem 1.5.17. *Let N be a normal operator. Then*

i $D(N) = D(N^*)$.

ii $\|Nx\| = \|N^*x\|$, *for all* $x \in D(N)$.

iii N *is maximally normal.*

A proof of can be consulted in [87, 12.12 Theorem].

Corollary 1.5.18. *Let N be a normal operator on a Hilbert space H then* $\ker N = \ker N^*$.

Corollary 1.5.19. *Normal operators are maximally self-adjoint.*

1.5.6 Matrices of Unbounded Operators

In this section, we define matrices of operators and their adjoints.

Definition 1.5.20. *Let A, B, C and D be four unbounded operators, and let $D(A)$, $D(B)$, $D(C)$ and $D(D)$ be their respective domains. Then*

$$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

defines a matrix of operators on $D(T) = D(A) \cap D(C) \oplus D(B) \cap D(D)$. Moreover, for each $(x, y) \in D(T)$, we have

$$T \begin{pmatrix} x \\ y \end{pmatrix} := \begin{pmatrix} Ax + By \\ Cx + Dy \end{pmatrix}.$$

Matrices of unbounded operators could behave differently from their bounded counterparts. For instance, if we assume that all of the entries of an operator matrix T are unbounded then it might be thought that T too is unbounded. This is not the case as it can occur that T is bounded while none of its entries is.

When dealing with bounded entries, that is, when considering $A \in B(H)$, $B \in B(K, H)$, $C \in B(H, K)$ and $D \in B(K)$, the adjoint of the matrix operator T is given by:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^* = \begin{pmatrix} A^* & C^* \\ B^* & D^* \end{pmatrix}. \tag{1.10}$$

In particular, we notice that the anti-diagonal operator matrices

$$\begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & A^* \\ A & 0 \end{pmatrix}$$

are always self-adjoint.

However, if at least one of the entries of the matrix is not necessarily bounded and not everywhere defined, then Equality 1.10 does not hold in general. Fortunately, when the matrix is diagonal or anti-diagonal, we particularly have the following result:

Proposition 1.5.21. *Let A, B, C and D be unbounded densely defined operators and let* $T = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$ *and* $S = \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}$. *Then we have* $T^* = \begin{pmatrix} A^* & 0 \\ 0 & D^* \end{pmatrix}$ *and* $S^* = \begin{pmatrix} 0 & C^* \\ B^* & 0 \end{pmatrix}$. *Moreover,*

- *T is self-adjoint if and only if A and D are self-adjoint.*
- *S is self-adjoint if and only if B is closed and $B^* = C$.*

For a proof, see [98, Proposition 2.6.3] for example.

Corollary 1.5.22. *If A is closed, then* $T = \begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix}$ *is self-adjoint.*

1.6 On some Fuglede-Putnam Theorems

We start by giving the first version of the Fuglede Theorem [37] in an unbounded setting.

Theorem 1.6.1 (Fuglede Theorem)**.** *Let $B \in B(H)$ and let N be normal. If $BN \subseteq NB$, then $BN^* \subseteq N^*B$.*

C. R. Putnam proved in [81] that Fuglede Theorem holds for two normal operators, thus extending the result of B. Fugelde.

Theorem 1.6.2 (Fuglede-Putnam Theorem)**.** *Let $B \in B(H)$ and let N and M be normal operators. If $BN \subseteq MB$, then $BN^* \subseteq M^*B$.*

Before passing to the proof of Fuglede-Putnam Theorem, it is worth to mention that S. K. Berberian proved the equivalence between these two theorems in 1959. The idea was to apply Fuglede Theorem to the operator matrices:

$$\begin{pmatrix} N & 0 \\ 0 & M \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix},$$

then deduce the more general Fuglede-Putnam Theorem (see [11]).

The simplest and shortest proof of Fuglede-Putnam Theorem is due to M. Rosenblum [86].

Rosenblum's proof. First, we assume that N and M are bounded. By induction, it can be easily shown that

$$BN^n = M^nB \quad \text{for} \quad n = 1, 2, 3, \cdots$$

Hence, we have $e^{i\overline{\lambda}M}B = Be^{i\overline{\lambda}N}$; where e stands for the exponential function and $\overline{\lambda}$ is the conjugate of an arbitrary complex scalar λ . So, we can write

$$B = e^{i\overline{\lambda}M}Be^{-i\overline{\lambda}N}. \tag{1.11}$$

Now, notice that $\overline{\lambda}N + \lambda N^*$ and $\lambda M^* + \overline{\lambda}M$ are self-adjoint, so their exponentials are unitary. Hence, we have $\|e^{-i(\overline{\lambda}N+\lambda N^*)}\| = \|e^{i(\lambda M^*+\overline{\lambda}M)}\| = 1$. For a complex scalar λ, set

$$f(\lambda) = e^{i\lambda M^*}Be^{-i\lambda N^*}. \tag{1.12}$$

Using 1.11, we can write

$$f(\lambda) = e^{i\lambda M^*}Be^{-i\lambda N^*} = e^{i\lambda M^*}e^{i\overline{\lambda}M}Be^{-i\overline{\lambda}N}e^{-i\lambda N^*}$$

The normality of M and N entails :

$$f(\lambda) = e^{i(\lambda M^*+\overline{\lambda}M)}Be^{-i(\overline{\lambda}N+\lambda N^*)}$$

So, we have:

$$\|f(\lambda)\| \leq \|B\|$$

Thus, f is a bounded entire function. According to Liouville's Theorem, f is constant. That is:

$$f(\lambda) = f(0) = B, \quad \forall\lambda \in \mathbb{C}. \tag{1.13}$$

By 1.12, we obtain that

$$Be^{i\lambda N^*} = e^{i\lambda M^*}B.$$

Taking derivative with respect to λ, we obtain $iBN^*e^{i\lambda N^*=iM^*e^{i\lambda M^*}B}$. Finally, by setting $\lambda = 0$, we obtain the desired result for bounded operators. Now, assume that the operators N and M are possibly unbounded operators and that $D(N)$ and $D(M)$ are their respective domains. By Theorem 1.5.17, we know that $D(N) = D(N^*)$ and $D(M) = D(M^*)$. We have

$$D(BN^*) = D(N^*) = D(N) = D(BN)$$

Since by assumption $BN \subseteq MB$ and M is normal, we get

$$D(BN^*) = D(MB) = D(M^*B)$$

Let $a < \infty$ and let h_a be the characteristic function of $\overline{D}(0, a)$. It is known from the Spectral Theorem that $Nh_a(N)$ and $Mh_b(M)$ are normal bounded operators. Moreover, we have:

$$[h_b(M)Bh_a(N)]Nh_a(N) = Mh_b(M)[h_b(M)Bh_a(N)].$$

According to what has been shown previously in the bounded case, we get

$$[h_b(M)Bh_a(N)][Nh_a(N)]^* = [Mh_b(M)]^*[h_b(M)Bh_a(N)].$$

Leting $a \to \infty$ and then $b \to \infty$, we finally get $BN^* \subseteq M^*B$.

□

Fuglede-Putnam Theorem has many applications. For instance, some simple consequences of this theorem are the following corollaries:

Corollary 1.6.3. *Let $M, N \in B(H)$ be normal. If M and N are similar, then they are unitarily equivalent.*

Corollary 1.6.4. *Let N be an unbounded normal operator and let $B \in B(H)$. Then the following statements are equivalent:*

i $BN \subset NB$.

ii $BN^* \subset N^*B$.

iii $B^*N \subset NB^*$.

iv $B^*N^* \subset N^*B^*$.

Proof. Assume that $BN \subset NB$. Fuglede Theorem implies $BN^* \subset N^*B$ Taking the adjoint of both sides and using the fact that N is closed leads to $B^*N \subset NB^*$. Applying again Fuglede theorem gives $B^*N^* \subset N^*B^*$. Finally, since N is closed and B is bounded, taking the adjoint of both sides once more entails $BN \subset NB$. □

The sum and the product of two normal operators may not be normal. However, by adding a commutativity condition, we can establish their normality. The following result is proved in [75].

Corollary 1.6.5. *Let N and M be normal operators such that $M \in B(H)$. Assume that N commutes up to a factor with M, that is $MN \subset \lambda NM \neq 0$ for some scalar λ. Then NM is normal if and only if $|\lambda| = 1$.*

Remark 1.6.6. *For $\lambda = -1$, that is, when $NM \subset -MN$, the sum $N + M$ need not to be normal even though N and M are normal and bounded. A counterexample can be found in [71].*

The next result was shown in [70].

Corollary 1.6.7. *Let $N \in B(H)$ be normal and let M be a non-necessarily bounded normal operator such that $NM \subset MN$. Then the sum $N + M$ is normal.*

Chapter 2

Cesàro Means and Convex-Cyclic Operators.

The present chapter deals with convex-cyclicity and Cesàro-hypercyclicity of higher order of bounded linear operators on separable Banach spaces. In Section 2.1, we recall some basic definitions and some results related to generalized Cesàro-means and convex-cyclicity. In Section 2.2, we characterize when the Cesàro means of higher order are hypercyclic, that is, when an operator T is (p) Cesàro-hypercyclic for $p \in \mathbb{N} \cup \{0\}$. Next, in Section 2.3, we prove that if a bounded linear operator has arbitrary large extended eigenvalues then it cannot be convex-cyclic. We end this chapter by gathering some examples of non convex-cyclic operators. Specifically, we study composition operators on the Hardy space H^2 and the bilateral weighted backward shifts on $\ell^p(\mathbb{Z})$. Finally, we highlight that the above conditions on the extended eigenvalues are no longer true for an operator to be non-supercyclic.

The results appearing in this chapter are published in [9].

2.1 Background

In order to fix some ideas, we shall recall some definitions and results related to *Cesàro-hypercyclicity* of higher order and *convex-cyclicity*. In 2002, F. León-Saavedra introduced the notion of Cesàro-hypercyclicity in [58]. Since then, this property has been studied and investigated by many mathematicians and a lot of questions were asked. For instance, what is the relation between Hypercyclicity and Cesàro-hypercyclicity? In which cases the results for hypercyclic operators hold for Cesàro-hypercyclicity? Is there a Kitai Criterion's type for Cesàro-hypercyclicity? These questions among others were mostly answered in [22, 25, 26, 33, 58]. In 2013, the notion of convex-cyclicity was introduced by H. Rezaei in [83]. Convex-cyclicity is weaker than Cesàro-hypercyclicity but, equally, it has too many interesting properties. We refer to [1, 34, 97] for results related to this concept. In this section, we shall see more details on convex-cyclic operators and Cesàro hypercyclic operators of higher order.

2.1.1 Convex-cyclic operators

First, let us recall the notion of convex polynomials:

Definition 2.1.1. *A polynomial $p(z) = a_0 + a_1 z + \cdots + a_n z^n$ is said to be convex if $a_i \geq 0$, $\forall i = 0, ..., n$ and $a_0 + \cdots + a_n = 1$.*

Example 2.1.2. *For $z \in \mathbb{C}$, the polynomial $p(z) = \frac{1+z+z^2+...+z^n}{n+1}$ is convex in $\mathbb{C}$.*

Denote by $Cv[x]$ the semigroup of convex polynomials. Now, convex-cyclic operators are defined as follows:

Definition 2.1.3. *A bounded linear operator $T : \mathcal{X} \to \mathcal{X}$ is convex-cyclic if there exists $x \in \mathcal{X}$ such that*

$$\{p(T)x \,:\, p \in Cv[x]\}$$

is dense in $\mathcal{X}$.

Equivalently, $T \in B(\mathcal{X})$ is said to be convex-cyclic if there exists a vector $x \in \mathcal{X}$ such that the convex hull generated by the orbit of T under x is dense in $\mathcal{X}$. In such a case, x is called a convex-cyclic vector for T.

There are many examples of convex-cyclic operators. The next proposition appeared in [83] and was deduced from [89].

Proposition 2.1.4. *Let $\{e_n\}_{n\in\mathbb{N}}$ be the canonical basis of ℓ^2. Let T be the unilateral backward weighted shift defined in ℓ^2 by:*

$$Te_n = \omega_n e_{n-1}, \quad n \geq 1, \quad Te_0 = 0$$

where $\{\omega_n\}_{n\in\mathbb{N}}$ is a nonzero bounded sequence of weights. Then T is hypercyclic if and only if and only if there exists a sequence of positive integers $\{n_k\}_{k\in\mathbb{N}}$ such that

$$\lim_{k\to\infty} \prod_{i=0}^{n_k} \omega_i = \infty.$$

Example 2.1.5. *Let T be the unilateral backward weighted shift defined in ℓ^2 with the weight sequence $(2, 2, ..., 2, ...)$. Since $\lim_{n\to\infty} 2^n = \infty$, it comes that T is convex-cyclic.*

Convex-cyclicity is an intermediate property between hypercyclicity and cyclicity. Precisely, we have the following implications:

$$\text{Hypercyclicity} \implies \text{Convex-cyclicity} \implies \text{Cyclicity}.$$

Just like cyclicity, convex-cyclicity can occur in finite dimension (it suffices to consider convex polynomials of degree at most n of matrices). Let us denote by $\mathcal{M}_{n\times n}(\mathbb{K})$ the set of all square matrices of dimension $n \times n$ over a field $\mathbb{K}$.

In his article [83], H. Rezaei characterized when a finite square matrix is convex-cyclic in terms of its eigenvalues. In 2017, Feldman and McGuire corrected this characterization (see [34]). Namely, they proved the following: if $T \in \mathcal{M}_{n\times n}(\mathbb{R})$ then T is convex-cyclic on $\mathbb{R}^n$ if and only if T is cyclic and its (real and complex) eigenvalues are contained in $\mathbb{C} \setminus (\overline{\mathbb{D}} \cup \mathbb{R}^+)$. Moreover, if $T \in \mathcal{M}_{n\times n}(\mathbb{C})$ then T is convex-cyclic on $\mathbb{C}^n$ if and only if T is cyclic and its eigenvalues $\lambda_j \in \mathbb{C} \setminus (\overline{\mathbb{D}} \cup \mathbb{R})$ for all $1 \leq j \leq n$ with $\lambda_i \neq \overline{\lambda_j}$ for all $1 \leq i < j \leq n$.

In particular, if $T \in \mathcal{M}_{n\times n}(\mathbb{C})$ is diagonal then T is convex-cyclic if and only its eigenvalues $(\lambda_i)_{i=1}^{i=n}$ are distinct pure complex numbers located outside the closed unit disk. That is , $\mathrm{Im}\lambda_i \neq 0$ and $|\lambda_i| > 1$ for all $1 \leq i \leq n$. If we assume $T \in \mathcal{M}_{n\times n}(\mathbb{R})$ to be diagonal then A is convex-cyclic if and only if its eigenvalues are all distinct and $\lambda_i < -1$ for all $1 \leq i \leq n$.

Example 2.1.6. *Let*

$$A = \begin{pmatrix} -2 & 0 \\ 0 & -3 \end{pmatrix} \quad \textit{and} \quad B = \begin{pmatrix} 3-4i & 0 \\ 0 & 1+5i \end{pmatrix}$$

then A and B are convex-cyclic matrices on $\mathcal{M}_{2\times 2}(\mathbb{R})$ and $\mathcal{M}_{2\times 2}(\mathbb{C})$ respectively. Trivial examples of non convex-cyclic operators are the zero and the identity operators.

The powers of a convex-cyclic operator are in general not convex-cyclic. In 2014, F. León-Saavedra and Romero De La Rosa were the first ones to give a counterexample. In their article [60], they proved the existence of a convex-cyclic operator T on infinite dimensional spaces such that T^3 fails to be convex-cyclic. In 2016, Bermúdez, Bonilla and S. Feldman provided in [12] another example of an operator T such that T^2 is not convex-cyclic. Unlike the example of F. León-Saavedra and Romero De La Rosa, they required moreover to the adjoint of T to have an empty point spectrum .

In [83], H. Rezaei showed some necessary conditions for an operator $T \in B(\mathcal{X})$ to be convex-cyclic in an infinite dimensional separable Banach space $\mathcal{X}$. Precisely, he showed the following proposition:

Proposition 2.1.7. *Let $T \in B(\mathcal{X})$. If T is convex-cyclic then:*

i. $\|T\| > 1$.

ii. $\sup\{\|T^n\| : n \geq 1\} = +\infty$.

iii. $\sigma_p(T^*) \cap (\mathbb{D} \cup \mathbb{R}) = \emptyset$ *if $\mathcal{X}$ is a complex Banach space. And,*
$\sigma_p(T^*) \cap [-1, +\infty) = \emptyset$ *if $\mathcal{X}$ is a real Banach space.*

Another interesting necessary condition involving spectral properties of T is the following:

Theorem 2.1.8. *[83, Theorem 3.6] Let $\mathcal{X}$ be a complex separable Banach space and let $T \in B(\mathcal{X})$. If T is convex-cyclic then every component of $\sigma(T)$ must intersect $\mathbb{C} \setminus \mathbb{D}$.*

Since the point spectrum is included in the spectrum, we deduce the following:

Lemma 2.1.9. *Let $\mathcal{X}$ be a complex Banach space and let $T \in B(\mathcal{X})$. If T is convex-cyclic then $\sigma_p(T^*) \cap \overline{\mathbb{D}} = \emptyset$.*

In the Theory of Linear Chaos, Lemma 2.1.9 provides a relation between the density of orbits of an operator and its spectral properties. Precisely, Lemma 2.1.9 provides a necessary condition (that involves the point spectrum) of a bounded linear operator to be convex-cyclic. Similarly to this result, we will prove in Section 2.3 new conditions on the extended spectrum of an operator that guarantee its non convex-cyclicity (see Theorem 2.3.2).

2.1.2 Cesàro means of higher order

Generalized Cesàro means have been investigated by many mathematicians. If we consider powers of operators, it has been noticed in the recent results of Aleman-Suciu and Zemánek [1, 97] that the Cesàro means of higher order play an important role in Operator Ergodic Theory. Moreover, these means appear naturally in the Theory of Summability of power series (see [42, 80, 105]).

We start by recalling the following definitions:

Definition 2.1.10. *For $n, p \in \mathbb{N} \cup \{0\}$, the Cesàro means of order p of $T \in \mathcal{B}(\mathcal{X})$, denoted by $M_n^{(p)}(T)$, are defined as follows:*

$$M_0^{(p)}(T) = I, \quad M_n^0(T) = T^n$$

and for $n, p \in \mathbb{N}$:

$$M_n^{(p)}(T) := \frac{p}{(n+1)\cdots(n+p)} \sum_{j=0}^{n} \frac{(j+p+1)!}{j!} M_j^{(p-1)}(T).$$

In particular for $p = 1$, we denote

$$M_n(T) = M_n^{(1)}(T) = \frac{1}{n+1} \sum_{j=0}^{n} T^j.$$

Definition 2.1.11. *Let $p \in \mathbb{N} \cup \{0\}$. An operator $T \in B(\mathcal{X})$ is said to be (p)-Cesàro-hypercyclic, or Cesàro-hypercyclic of order (p), if there exists a vector $x \in \mathcal{X}$ such that the subset*

$$\{M_n^{(p)}(T)x \,:\, n \geq 0\}.$$

is a dense in $\mathcal{X}$.

It is obvious from Definitions 2.1.10 and 2.1.11 that Cesàro-hypercyclicity of order (0) is just hypercyclicity (see Definition 1.4.1). To follow the same path as in [58], Cesàro-hypercyclicity of order (1) shall simply be called Cesàro-hypercyclicity.

The following characterization is due to F. León-Saavedra and it provides a large number of examples of Cesàro-hypercyclic operators:

Proposition 2.1.12. *[58, Proposition 3.4] Let $\{e_n\}_{n\in\mathbb{Z}}$ be the canonical basis of $\ell^2(\mathbb{Z})$. Let T be the bilateral weighted shift defined in $\ell^2(\mathbb{Z})$ by:*

$$Te_n = \omega_n e_{n-1}$$

where $(\omega_n)_{n\in\mathbb{Z}}$ is a bounded weight sequence. Then T is Cesàro-hypercyclic if and only if there exists an increasing sequence (n_k) of positive integers such that for any integer q such that:

$$\lim_{k\to\infty} \frac{\prod_{i=1}^{n_k} \omega_{i+q}}{n_k} = \infty \quad \text{and} \quad \lim_{k\to\infty} \frac{\prod_{i=0}^{n_k-1} \omega_{q-i}}{n_k} = 0$$

Example 2.1.13. *[58, Example 3.6] Let T be the bilateral weighted shift defined in $\ell^2(\mathbb{Z})$ with the weight sequence:*

$$\omega_n = \begin{cases} 1 & \text{if } n < 0 \\ 2 & \text{if } n \geq 0. \end{cases}$$

Then we have:

$$\frac{\prod_{i=0}^{n-1} \omega_{q-i}}{n} = \frac{c}{n} \to 0, \quad \text{as } n \to \infty$$

where $c = \prod_{i=0}^{n-1} \omega_{q-i}$ is a finite constant and

$$\frac{\prod_{i=1}^{n} \omega_{i+q}}{n} = \begin{cases} \frac{2^n}{n} & \text{if } q > 0 \\ \frac{2^{n+q}}{n} & \text{if } q \leq 0 \end{cases}$$

tends to ∞ as $n \to \infty$. According to Proposition 2.1.12, T is Cesàro-hypercyclic.

Going back to the definition of the Cesàro means, observe that we have:

$$M_n(T) = q(T) \quad \text{where} \quad q(z) = \frac{1}{n+1}\sum_{k=0}^{n} z^k.$$

Since the polynomial q is convex, it turns out that every Cesàro-hypercyclic operator is convex-cyclic. Thus, the following implications hold:

$$\text{Cesàro-hyprcyclicity} \implies \text{Convex-cyclicity} \implies \text{Cyclicity}$$

At this point, one may wonder about the relation between hypercyclicity and Cesàro-hypercyclicity. F. León-Saavedra showed in [58] that neither hypercyclicity implies Cesàro-hypercyclicity, nor Cesàro-hypercyclicity implies hypercyclicity. In fact, on the one hand, he proved that every Cesàro-hypercyclic unilateral weighted shift is hypercyclic. On the other hand, he showed the existence of a bilateral weighted shift which is Cesàro-hypercyclic but without being hypercyclic (see Example 2.1.13). In the same paper [58], F. León-Saavedra showed that:

> T is Cesàro-hypercyclic if and only if the sequence $\frac{T^n}{n}$ is hypercyclic.

As a consequence, he provided a sufficient condition (involving the powers of the operator) for convex-cyclicity. More recently, Bermúdez-Bonilla and Feldman [12] have proved the following result:

> Let $c > 1$, then the convex means $\frac{c-1}{c^{n+1}-1}\sum_{k=0}^{n} c^{n-k}T^k$ is hypercyclic if and only if $\frac{T^n}{c^n}$ is hypercyclic.

As we shall see in the next section, it is possible to characterize when an operator $T \in B(\mathcal{X})$ is Cesàro-hypercyclic of higher order using the powers of T (see Theorem 2.2.3).

2.2 Cesàro-hypercyclic Operators of Higher Order

Let $p \in \mathbb{N}$ and consider a bounded linear operator T on a separable Banach space $\mathcal{X}$. In this section, we shall prove that T is (p)- Cesàro hypercyclic if and only if there exist $x \in \mathcal{X}$ such that the orbit $\{\frac{T^n}{n^p}(x)\}_{n\geq 0}$ is dense in $\mathcal{X}$. To prove this characterization, we shall need the following preliminary results:

Lemma 2.2.1. *Let T be an operator and let us assume that $\{\lambda_n\}_{n\geq 0}$ is a sequence of complex numbers which is not dense in $\mathbb{C}$. If there exists a vector $x \in \mathcal{X}$ such that $\{\lambda_n M_n^{(p)}(T)(x)\}_{n\geq 0}$ is dense in $\mathcal{X}$ $(p \geq 1)$ then $\operatorname{ran}(T - I)$ is dense in $\mathcal{X}$.*

Proof. Indeed, let $x \in \mathcal{X}$ be such that $\{\lambda_n M_n^{(p)}(T)x\}_{n\geq 0}$ is dense in $\mathcal{X}$. If $\operatorname{ran}(T - I)$ is not dense, choose $y_0 \in \ker(T^\star - I) \setminus \{0\}$. Since $M_n^{(p)}(T)$ are convex combinations of T, we have that $M_n^{(p)}(I) = I$, therefore

$$\begin{aligned}\{\langle \lambda_n M_n^{(p)}(T)x, y_0^\star \rangle\}_{n\geq 0} &= \{\lambda_n\langle x, M_n^{(p)}(T^\star)y_0^\star\rangle\}_{n\geq 0}\\ &= \{\langle x, M_n^{(p)}(I)y_0^\star\rangle\}_{n\geq 0} = \{\lambda_n\langle x, y_0^\star\rangle\}_{n\geq 0}\end{aligned}$$

which is not dense in $\mathbb{C}$, a contradiction. □

In fact, we will show later that if T is (p)-Cesàro hypercyclic $(p \geq 1)$ then $\sigma_p(T^\star) = \emptyset$. The following Lemma is the key of the proof of Theorem 2.2.3.

Lemma 2.2.2. *Let $\{\lambda_n\}_{n\geq 0}$ be a bounded sequence of complex numbers and $p \geq 1$. Then the sequence of operators $\{\lambda_n M_n^{(p)}(T)\}_{n\geq 0}$ is hypercyclic if and only if the sequence $\left\{\lambda_{n-1}\frac{M_n^{(p-1)}(T)}{n}\right\}_{n>0}$ is hypercyclic.*

Proof. Indeed, let us suppose that there exists $x \in \mathcal{X}$ such that $\{\lambda_n M_n^p(T)(x)\}_{n\geq 0}$ is dense in $\mathcal{X}$. Since $\operatorname{ran}(T-I)$ is dense, the subset

$$(T-I)\left(\{\lambda_n M_n^{(p)}(T)(x)\}_{n\geq 0}\right)$$

must be dense in $\mathcal{X}$. Then, by the well known formula

$$(T-I)M_n^{(p)}(T) = \frac{p}{n+1}\left(M_{n+1}^{(p-1)}(T) - I\right),$$

we obtain

$$(T-I)\left(\{\lambda_n M_n^{(p)}(T)(x)\}_{n\geq 0}\right) = \lambda_n \frac{p}{n+1} M_{n+1}^{(p-1)}(T)(x) - \frac{\lambda_n x}{n+1}. \tag{2.1}$$

Since the last term converges to 0, we obtain that the orbit $\frac{p}{n+1}M_{n+1}^{(p-1)}(T)(x)$ is dense, that is, the sequence $\left\{\lambda_{n-1}\frac{M_n^{(p-1)}(T)}{n}\right\}_{n>0}$ is hypercyclic as we desired.

Conversely, if $\left\{\lambda_{n-1}\frac{M_n^{(p-1)}(T)}{n}\right\}_{n>0}$ is hypercyclic, there exists some x such that the subset defined by the orbit of the second term in (2.1) is dense in $\mathcal{X}$. This implies that $(T-I)\left(\{\lambda_n M_n^{(p)}(T)(x)\}_{n\geq 0}\right)$ is dense. In other words, the sequence $\{\lambda_n M_n^{(p)}(T)\}_{n\geq 0}$ is hypercyclic, with hypercyclic vector $(T-I)x$. This completes the proof of the lemma. □

Theorem 2.2.3. *Let T be a bounded linear operator defined on a separable Banach space $\mathcal{X}$ and $p \in \mathbb{N}$. Then, T is (p)-Cesàro-hypercyclic if and only if there exists a vector x such that $\{\frac{T^n}{n^p}(x)\}_{n>0}$ is dense in $\mathcal{X}$.*

Proof. Assume that T is (p)- Cesàro hypercyclic, then there exists $x \in \mathcal{X}$ such that $\{M_n^{(p)}(T)x\}$ is dense in $\mathcal{X}$, then applying Lemma 2.2.2 we obtain that $\frac{1}{n}M_n^{(p-1)}(T)(x)$ is dense in $\mathcal{X}$. Applying again Lemma 2.2.2, we obtain that $\{\frac{1}{(n-1)(n)}M_n^{(p-2)}(T)(x)\}$ is dense in $\mathcal{X}$. Recursively, we obtain that $\{\frac{1}{(n-p)(n-p+1)\cdots(n)}T^n(x)\}_{n\geq p+1}$ must be dense in $\mathcal{X}$, or equivalently $\{\frac{T^n}{n^p}(x)\}_{n>0}$is dense in $\mathcal{X}$. □

Theorem 2.2.4. *Let T be a (p) Cesàro-hypercyclic operator, then $\sigma_p(T^\star) = \emptyset$.*

Proof. Indeed, let us suppose that there exists $x \in \mathcal{X}$ such that $\{\frac{T^n x}{n^p}\}$ is dense in $\mathcal{X}$. By way of contradiction, let us suppose that $y_0^\star \in \operatorname{Ker}(T^\star - \lambda I) \setminus \{0\}$ then the following subset should be dense in $\mathbb{C}$:

$$\left\langle \frac{T^n x}{n^p}, y_0^\star \right\rangle = \frac{\overline{\lambda}^n}{n^p}\langle x, y_0^\star\rangle$$

a contradiction. □

As it was mentioned before, the hypercyclicity of an operator does not imply its Cesàro means hypercyclicity and viceversa (see again Section 3 in [58]). In a similar manner, we can obtain examples of operators that are $(p-1)$-Cesàro hypercyclic and that are not (p)-Cesàro hypercyclic (and viceversa).

2.3 Convex-cyclic Operators and Extended Eigenvalues

Let λ be an extended eigenvalue of $T \in B(\mathcal{X})$, that is λ satisfies $TX = \lambda XT$ for some non-zero bounded linear operator X. Recall that the set of all extended eigenvalues of T is called the extended spectrum. We shall see in what follows some necessary conditions for an operator $T \in B(\mathcal{X})$ to be convex-cyclic. These conditions involve the extended spectrum of T.

Proposition 2.3.1. *Let $T \in B(\mathcal{X})$. Suppose that there exists an extended eigenvalue λ such that $\|T\| \geq |\lambda|$. Let X be the eigenoperator associated to λ and assume that X has right inverse. Then T is not convex-cyclic.*

Proof. Let R be the right inverse of X, then $p(T) = Xp(\lambda T)R$ which implies that if x is a convex cyclic vector for T, then the subset $\{p(\lambda T)Rx : \quad p \in Cv[x]\}$ must be dense in $\mathcal{X}$. A contradiction because this subset is bounded. $\square$

Theorem 2.3.2. *Let T be a bounded linear operator. Let us suppose that there exists an extended eigenvalue λ such that $\|T\| \leq |\lambda|$. Then T is not convex-cyclic.*

Proof. Let p be a convex polynomial. Since $\|T\| \leq \lambda$, the operator $p\left(\frac{1}{\lambda}T\right)$ is a contraction and since λ is an extended eigenvalue, we have the following equation:

$$p\left(\frac{1}{\lambda}T\right)X = Xp(T) \tag{2.2}$$

where X is an extended eigenoperator associated to the extended eigenvalue λ. Let y be a convex-cyclic vector for T, then we claim that $Xy \neq 0$. Indeed, if $Xy = 0$, then by (2.2) we observe that for every convex polynomial p, $Xp(T)y = 0$. That is: X is zero on a dense subset. Since X is continuous, we conclude that $X = 0$, a contradiction.

Let y be a convex cyclic vector for T, then we can suppose without loss of generality that $\|y\| = 1$, $\|X\| = 1$ and $Xy \neq 0$. Let us fix $\varepsilon > 0$, and let us consider $r > 1$ large enough such that $\|X(ry)\| > 1 + \varepsilon$. Since y is a convex cyclic vector, there exists a convex polynomial q such that $\|q(T)y - ry\| < \varepsilon$. Hence, we obtain:

$$\|Xq(T)y\| = \|Xq(T)y - Xry + Xry\| \geq \|X(ry)\| - \|Xp(T)y - X(ry)\| > 1.$$

On the other hand, if we look at the first term of the equation in (2.2), since $q\left(\frac{1}{\lambda}T\right)$ is a contraction, we have

$$\left\|q\left(\frac{1}{\lambda}T\right)Xy\right\| \leq 1$$

a contraction. $\square$

Since generalized Cesàro means are convex sums, Theorem 2.3.2 allows us to obtain a large number of examples of operators that are not convex-cyclic. Indeed, let us apply our result to some concrete examples of operators. We point out that the extended spectra of these operators are already computed in [53, 54, 55] and [57].

Example 2.3.3. *Bilateral weighted shifts. Let $(w_n)_{n\in\mathbb{Z}}$ be a bounded sequence. The bilateral weighted shift is defined on the canonical basis $\{e_n\}_{n\in\mathbb{Z}}$ as*

$$Be_k = w_k e_{k+1}, \quad k \in \mathbb{Z}.$$

It is known ([53]) that every bilateral weighted shift has one of the following extended spectra: a) $\overline{\mathbb{D}}$ b) $\mathbb{T}$ c) $\mathbb{C}\setminus\{0\}$ and d) $\mathbb{C}\setminus\mathbb{D}$.

Then, according to Theorem 2.3.2, if the extended spectra of B is $\mathbb{C}\setminus\{0\}$ or $\mathbb{C}\setminus\mathbb{D}$, then B is not convex-cyclic. Taking into account the relationship between the extended spectrum of B and its adjoint $B^\star$, we deduce that if the extended spectrum of B is $\overline{\mathbb{D}}$ then $B^\star$ is not convex-cyclic.

The case in which the extended spectrum of B is the unit circle $\mathbb{T}$, our result does not apply.

Example 2.3.4. *Cesàro operators. The discrete Cesàro operator is defined on the sequences spaces ℓ^p as*

$$C_0(x_1, x_2, \cdots) = (x_1, \frac{x_1+x_2}{2}, \frac{x_1+x_2+x_3}{3}, \cdots).$$

The extended spectrum of C_0 is the interval $[0,\infty)$ (see [55]). Therefore according to Theorem 2.3.2, the Cesàro operator C_0 is not convex cyclic on ℓ^p.

On the other hand, the continuous Cesàro operator $C_1 f(x) = \frac{1}{x}\int_0^x f(s)\,ds$ is hypercyclic on $L^p[0,1]$, $1 < p < \infty$ (see [57]) and since the extended spectrum is the subset $(0,1]$ ([55]), using Theorem 2.3.2, we obtain that its adjoint $C_1^\star$ is not convex-cyclic.

Example 2.3.5. *Composition operators. Let φ be a parabolic non-automorphism linear fractional self map of the unit disk. The symbol φ induces a bounded composition operator C_φ on the Hardy space $H^2(\mathbb{D})$. It was proved by Rezaei ([83] [Theorem 5.2]) that the operator C_φ is not convex-cyclic. Recently, F. León-Saavedra, M. Lacruz, L.R. Piazza and S. Petrovic computed in [54] the extended eigenvalues for composition operators induced by linear fractional self maps of the unit disk. When the symbol is parabolic non-automorphism, they obtained that the extended spectrum contains arbitrarily large and arbitrarily small values. Hence, applying Theorem 2.3.2, we obtain that C_φ and $C_\varphi^\star$ are not convex-cyclic.*

The conditions in Theorem 2.3.2 are not sufficient in order for an operator to be non supercyclic. The following example shows that there exist operators satisfying these conditions and yet they are supercyclic.

Example 2.3.6. *Unilateral weighted backward shifts. Let us consider an injective unilateral weighted backward shift B defined on the canonical basis of $\ell^p(\mathbb{N})$ or c_0 by*

$$Be_n = w_n e_{n-1}, \quad n \geq 1$$

and $Be_0 = 0$. The classical result by Hilden and Wallen ([44]) shows that every unilateral backward shift is supercyclic. On the other hand, the extended spectra of the unilateral backward shift (it was computed by S. Petrovic in [79]) is either $\mathbb{C} \setminus \{0\}$ or $\overline{\mathbb{D}} \setminus \{0\}$. In short, there are supercyclic operators with arbitrarily large extended eigenvalues.

2.4 Some Open Questions

The Cesàro means can be extended for real numbers $\alpha \in \mathbb{R} \setminus \{-1, -2, \cdots\}$ (see [31, 32]). If $\alpha \in \mathbb{R} \setminus \{-1, -2, \cdots\}$, then the numbers A_n^α are defined as follows:

$$A_n^\alpha = \frac{(\alpha+1)(\alpha+2)\cdots(\alpha+n)}{n!},$$

the following identity follows:

$$A_n^{\alpha+1} = \sum_{k=1}^{n} A_k^\alpha.$$

The Cesàro means of order α are defined as the convex combinations:

$$M_n^\alpha(T) = \frac{1}{A_n^\alpha} \sum_{k=1}^{n} A_{n-k}^{\alpha-1} T^k$$

and the following relations are still true:

$$A_{n+1}^\alpha - A_n^\alpha = A_{n+1}^{\alpha-1} \tag{2.3}$$

for all $\alpha \in \mathbb{R}$ and $n \geq 0$. And using the above equation, we obtain:

$$M_n^\alpha(T)(T-I) = \frac{\alpha}{\alpha+n} M_{n+1}^{\alpha-1}(T) - \frac{\alpha}{n+1} I.$$

It would be nice if we could extend Theorem 2.2.3 for Cesàro means with arbitrary real $\alpha \in \mathbb{R} \setminus \{-1, -2, \cdots\}$. In this direction, we should note that if p is a natural number then the successive finite differences (2.3) stabilize. On the other hand, if $\alpha \in \mathbb{R} \setminus \{-1, -2, -3, \cdots\}$, those successive finite differences do not stabilize. The ideas in the proof of Theorem 2.2.3 also yield the following result:

Theorem 2.4.1. *Let $\alpha \in \mathbb{R} \setminus \mathbb{Z}$, let $\beta = \alpha - [\alpha]$ where $[\alpha]$ denotes the integer part of α. Then, the Cesàro means $\{M_n^\alpha(T)\}$ are hypercyclic if and only if the sequence $\{\frac{M_n^\beta(T)}{n^{[\alpha]}}\}$ is hypercyclic.*

Chapter 3

Operators λ–commuting with the Differentiation Operator and Hypercyclicity.

An operator T acting on a separable F-space $\mathcal{X}$ is called hypercyclic if there exists $f \in \mathcal{X}$ such that the orbit $\{T^n f\}$ is dense in $\mathcal{X}$. In this chapter, we determine when an operator that λ-commutes with the operator of differentiation on the space of entire functions is hypercyclic, extending results by G. Godefroy and J. H. Shapiro [39] and R. M. Aron and D. Markose [2]. The results we obtained are collected in [8].

3.1 Background

The term λ-commuting was introduced by J.B. Conway and G. Prăjitură in [24]. More recently, a complex number λ is called an *extended eigenvalue* of an operator T if there exists a non-zero continuous operator X, which is called an *extended λ-eigenoperator of* T, such that $TX = \lambda XT$. Extended eigenvalues and extended eigenoperators are naturally born to improve V. Lomonosov's famous result on the invariant subspace problem ([19, 49, 62]) and their study is currently under development (see [52, 53]).

Let $\mathcal{H}(\mathbb{C})$ be the space of entire functions endowed with the topology of uniform convergence on compact subsets. G. D. Birkhoff ([16]) proved in 1929 that translation operators on $\mathcal{H}(\mathbb{C})$ are hypercyclic. In 1952, G. R. MacLane ([63]) proved the same result for the differentiation operator D on $\mathcal{H}(\mathbb{C})$. These results appear to be the first hypercyclicity theorems for operators.

In 1991, G. Godefroy and J. H. Shapiro (see [39]) unified Birkhoff's and MacLane's results by proving that each non-scalar operator that commutes with D is hypercyclic. The simplicity and beauty of this statement is striking, and it is worthy to note that there is no analogous result in the context of Banach spaces since contractions on these spaces are never hypercyclic. This result has been improved and extended in different directions, making [39] one of the most cited papers on hypercyclic operators. A step further in Godefroy and Shapiro's result arises with the following question:

> Suppose that T is an operator on $\mathcal{H}(\mathbb{C})$ which is an extended λ-eigenoperator of D; that is, $DT = \lambda TD$. Is T hypercyclic?

At first glance, this question seems difficult because there are examples of non-trivial extended eigenoperators of D which are not hypercyclic. The first one was discovered by L. Bernal and A. Montes (see [14]), who showed that the composition operator $C_{\lambda,b}f(z) = f(\lambda z+b)$ induced by the affine endomorphism $\varphi(z) = \lambda z + b$ is hypercyclic if and only if φ is a proper translation ($\lambda = 1$ and $b \neq 0$). It is easy to see that $C_{\lambda,b}$ is an extended λ-eigenoperator of D. From these facts, it can be suspected that there are many extended eigenoperators of D that are not hypercyclic. But, is there a non-trivial one that is hypercyclic? R. M. Aron and D. Markose answered this question affirmatively. Denoting $T_{\lambda,b}f(z) = f'(\lambda z + b)$, then $T_{\lambda,b}$ is an extended λ-eigenoperator of D, and $T_{\lambda,b}$ is hypercyclic if and only if $|\lambda| \geq 1$ (see [2, 35, 55]).

In this chapter, we fully characterize when an extended eigenoperator of D is hypercyclic by proving the following result:

Main Theorem. *Let T be an extended λ-eigenoperator of D, $\lambda \neq 1$. Then T is hypercyclic if and only if $|\lambda| \geq 1$ and T is not a multiple of the composition operator $C_{\lambda,b}$ induced by an affine endomorphism.*

This chapter is organized as follows. In Section 3.2, We show that an extended λ-eigenoperator T of D can be factorized as $T = R_\lambda \phi(D)$, where $R_\lambda f(z) = f(\lambda z)$ and ϕ is an entire function of exponential type. So we can study the hypercyclicity of T in terms of the properties of ϕ and λ. We also show that ϕ has no zeros if and only if $R_\lambda \phi(D)$ is a nonzero multiple of $C_{\lambda,b}$, and it is not hypercyclic in this case.

We divide the rest of the proof of the Main Theorem in cases (assuming that ϕ has an isolated zero) which are treated in successive sections:

3.3: $|\lambda| < 1$ and $\lambda^n = 1$ for some $n \in \mathbb{N}$;

3.4: $|\phi(0)| > 1$ and $|\lambda| \geq 1$;

3.5: $0 < |\phi(0)| \leq 1$ and $|\lambda| > 1$;

3.6: $0 < |\phi(0)| \leq 1$ and $|\lambda| = 1$;

3.7: $\phi(0) = 0$ and $|\lambda| \geq 1$.

Thus, the main result is obtained by considering different cases for the values of λ and $\phi(0)$, which share a similar flavour to recent studies on algebras of hypercyclic vectors for convolution operators by F. Bayart, J. Bès and coworkers [15, 5].

Each particular case is solved by a different method. Using some arguments borrowed from [39], we prove the cases $|\lambda| < 1$ and λ is a root of the unity. However, new ideas are needed to solve the rest of the cases.

In the case $|\lambda| \geq 1$ and $|\phi(0)| > 1$, we analyze the action of T on the exponentials e^{az}, and we show that T has a dense generalized kernel. Then we construct the right inverse required by the Hypercyclicity Criterion using the triangularity of T and a linear algebra argument.

When $|\lambda| > 1$ and $0 < |\phi(0)| \leq 1$, the operator $T = R_\lambda \phi(D)$ is not injective. So the right inverse needed to apply the Hypercyclicity Criterion is not unique, and the construction in the previous section does not provide a right inverse in this case. However, using the Pólya representation of an entire function, we obtain an integral representation of the powers of the operator which allows us to find a sequence of right inverses for the powers of the operator. With this sequence of right inverses and the Hypercyclicity Criterion we prove the desired result.

The case $0 < |\phi(0)| \leq 1$ and λ a irrational rotation is the most intriguing one. When λ is a root of unity, the problem can be solved using a result on powers of hypercyclic operators, but when λ is an irrational rotation the solution is different. In many cases; e.g., when $\phi(z)$ is a polynomial $p(z)$ or $\phi(z) = p(z)e^z$, we can deduce the result by standard arguments. However, as far as we know, these arguments cannot be used in the general case, and the problem requires an argument involving normal families. Montel's Theorem plays an important role in guaranteeing the universality of a family of functions on the complex plane: it is the key of the proof.

A different treatment is needed also in the case $\phi(0) = 0$, including the operator $T_{\lambda,b}$, which is different from those used in [2, 35, 55]. We need to refine the computations in the case $|\phi(0)| > 0$ and $|\lambda| \geq 1$ using the complex Volterra operator.

Along this chapter, ϕ will be a non-zero entire function of *exponential type:* there are constants $A, B > 0$ such that $|\phi(z)| \leq Ae^{B|z|}$ for all $z \in \mathbb{C}$. We will denote by $\operatorname{span} A$ the subspace generated by a subset A of a vector space.

We close this chapter by giving an example of a bounded linear operator on a Banach space such that all its related extended eigenoperators are not hypercyclic.

3.2 Factorization of Operators λ-commuting with the differentiation operator

Now, we give a result inspired by [55] that will be central in our discussion.

Proposition 3.2.1. *Let T be an operator on $\mathcal{H}(\mathbb{C})$. Then $DT = \lambda TD$ for some $0 \neq \lambda \in \mathbb{C}$ if an only if $T = R_\lambda \phi(D)$ with $R_\lambda f(z) = f(\lambda z)$ for $z \in \mathbb{C}$ and ϕ an entire function of exponential type.*

Proof. Suppose that $DT = \lambda TD$ with $\lambda \neq 0$. Given $f \in \mathcal{H}(\mathbb{C})$ and $\lambda \in \mathbb{C}\setminus\{0\}$, the operator $R_{1/\lambda}$ is an extended $(1/\lambda)$-eigenoperator of D, that is, $DR_{1/\lambda} = \frac{1}{\lambda}R_{1/\lambda}D$. Besides,

$$R_{1/\lambda}TDf = \frac{1}{\lambda}R_{1/\lambda}DTf = \frac{1}{\lambda}\lambda DR_{1/\lambda}T = DR_{1/\lambda}Tf.$$

Hence, $R_{1/\lambda}T$ commutes with D. By Proposition 5.2 in [39], there exists an entire function ϕ of exponential type such that $R_{1/\lambda}T = \phi(D)$. Since $R_{1/\lambda}$ is invertible with inverse R_λ, we deduce that $T = R_\lambda \phi(D)$. Conversely, if there exists an entire function ϕ of exponential type such that $T = R_\lambda\phi(D)$, since $D\phi(D) = \phi(D)D$, $DT = \lambda TD$. □

The following result takes care of the case in which ϕ has no zeros. Thus we can assume that ϕ has an isolated zero in the remaining sections.

Proposition 3.2.2. *Let $1 \neq \lambda \in \mathbb{C}$. Then $T = R_\lambda\phi(D)$ is a multiple of $C_{\lambda,b}$ if and only if ϕ has no zeros on $\mathbb{C}$. In this case T is not hypercyclic.*

Proof. If $\phi(z) \neq 0$ for all $z \in \mathbb{C}$, we can define the logarithm of $\phi(z)$ (see, e.g., p. 226 in [45]), and there exists an entire function g such that $\phi(z) = e^{g(z)}$. Since ϕ is entire of exponential type, $g(z) = az + b$ for some $a, b \in \mathbb{C}$. Thus $Tf(z) = e^b f(\lambda z + a)$, which is not hypercyclic when $\lambda \neq 1$ (see [14]). Indeed, set $c = a/(1-\lambda)$, the fixed point of the map $\lambda z + a$. If f is hypercyclic for T then, the orbit $T^n f(c)$ should be dense in $\mathbb{C}$. However the sequence $T^n f(c) = e^{nb} f(c)$ is either bounded (if $|e^b| \leq 1$) or diverges to infinity (if $|e^b| > 1$), a contradiction, which gives the desired result. □

Propositions 3.2.1 and 3.2.2 provide a way to study our problem by looking at the properties of ϕ and λ.

3.3 The cases $|\lambda| < 1$ and λ is a root of 1

In the first case we will show that $T = R_\lambda\phi(D)$ is not hypercyclic. At first glance, one may think that the cases of Fréchet spaces and Banach spaces are similar. However, using some ideas of [9], we will show that in the Banach space setting an extended λ-eigenoperator with $|\lambda| < 1$ is not hypercyclic, but this is no longer true for Fréchet spaces.

Proposition 3.3.1. *Let A and T be two operators on a Banach space. If T is an extended λ-eigenoperator of A and $|\lambda| < 1$ then T is not hypercyclic.*

Proof. Assume that T is hypercyclic. Then there exists $x \in \mathcal{X}$ such that $\{T^n x\}_{n\geq1}$ is dense in $\mathcal{X}$. Hence, $\{A^m T^n x\}_{n\geq1}$ is also dense in $A^m(\mathcal{X})$ for each $m \geq 1$. Since $|\lambda| < 1$, we can choose $m \geq 1$ such that $|\lambda|^m \|T\| \leq 1$. Observing that $A^m T^n = \lambda^{nm} T^n A^m$, we have:

$$\|A^m T^n x\| = |\lambda|^{nm} \|T^n A^m x\| \leq |\lambda|^{nm} \|T^n\| \|A^m x\| \leq \|A^m x\|.$$

Hence, we get a contradiction, and T cannot be hypercyclic. □

Proposition 3.3.1 is not true in Fréchet spaces:

Example 3.3.2. *For $|\lambda| > 1$, $T_{\lambda,b}D = (1/\lambda)DT_{\lambda,b}$. Hence D is an hypercyclic extended $(1/\lambda)$-eigenoperator of $T_{\lambda,b}$ with $|1/\lambda| < 1$.*

However, our case is not one of these examples:

Theorem 3.3.3. *If $|\lambda| < 1$ and T is an extended λ-eigenoperator of D then T is not hypercyclic.*

Proof. First, we give a representation of T similar to one in the proof of Proposition 5.2 in [39]. We consider $\Lambda \in \mathcal{H}(\mathbb{C})^*$ defined by $\Lambda f = Tf(0)$. By the Hahn-Banach theorem and the Riesz Representation theorem, there exists a complex Borel measure μ with compact support in $\mathbb{C}$ such that

$$\Lambda f = Tf(0) = \int f(w)\, d\mu(w)$$

for all $f \in \mathcal{H}(\mathbb{C})$. For each $\alpha \in \mathbb{C}$ we consider the translation operator τ_α defined by $\tau_\alpha f(z) = f(z+\alpha)$. Since

$$f(z+\alpha) = \sum_{k=0}^{\infty} f^{(k)}(z)\frac{\alpha^k}{k!} = \left(\sum_{k=0}^{\infty} \frac{\alpha^k}{k!} D^k\right) f(z)$$

we have $\tau_\alpha T = \left(\sum_{k=0}^{\infty} \frac{\alpha^k}{k!} D^k\right) T = T\left(\sum_{k=0}^{\infty} \frac{\lambda^k\alpha^k}{k!} D^k\right) = T\tau_{\lambda\alpha}$. Therefore

$$Tf(z) = (\tau_z Tf)(0) = (T\tau_{\lambda z} f)(0) = \int f(\lambda z + w)\, d\mu(w)$$

for each $f \in \mathcal{H}(\mathbb{C})$. Iterating the above equality we get:

$$T^n f(z) = \int \cdots \int f(\lambda^n z + \lambda^{n-1} w_1 + \cdots + w_n)\, d\mu(w_n) \cdots d\mu(w_1).$$

Thus, if the disc $D(0,R)$ contains the support of μ, since

$$|\lambda^n z + \lambda^{n-1} w_1 + \cdots + w_n| \leq M(|z|) = |\lambda^n||z| + \frac{1-|\lambda|^n}{1-|\lambda|} R,$$

for $|z| \leq r$ each element of the argument of f in the above integral lies in the disk $D(0, M(r))$. Hence, for $f \in \mathcal{H}(\mathbb{C})$ and $|z| \leq r$ we get:

$$|T^n f(z)| \leq \sup_{|z|=M(r)} |f(z)|\ \|\mu\|^n,$$

where $\|\mu\|$ denotes the total variation of μ.

Assume there exists $f \in \mathcal{H}(\mathbb{C})$ such that $\{T^n f\}_{n\geq 1}$ is dense in $\mathcal{H}(\mathbb{C})$. Since $|\lambda| < 1$, there exists $m \in \mathbb{N}$ such that $|\lambda|^m < 1/\|\mu\|$. Since D has dense range, $\{D^m T^n f\}_{n\geq 1}$ is dense in $\mathcal{H}(\mathbb{C})$. However, for $|z| \leq r$ we get

$$|D^m T^n f(z)| = |\lambda|^{mn} T^n (D^m f)(z)| \leq |\lambda|^{mn} \|\mu\|^n \max_{|z|\leq M(r)} |D^m f(z)|$$

which goes to 0 as $n \to \infty$, a contradiction. Thus T is not hypercyclic. □

When λ is a root of 1, the result of Godefroy and Shapiro for $\lambda = 1$ allows us to prove the following result.

Theorem 3.3.4. *If $\lambda^{n_0} = 1 \neq \lambda$ and $T = R_\lambda \phi(D)$ is not a multiple of $C_{\lambda,b}$ then T is hypercyclic.*

Proof. If $\lambda^{n_0} = 1$ then $R_\lambda^{n_0} = I$. Hence,

$$T^{n_0} f = \big(R_\lambda \phi(D)\big)^{n_0} f = \big(\phi(D)\phi(\lambda D)\cdots\phi(\lambda^{n_0-1}D)\big) f.$$

Thus, if $\Phi(z) = \prod_{j=0}^{n_0-1} \phi(\lambda^j z)$ is not constant, then $T^{n_0} = \Phi(D)$ is hypercyclic by [39, Theorem 5.1]. Hence T is hypercyclic.

If $\Phi = 0$ then $T = 0$, and if Φ is a nonzero constant function, then ϕ has no zeros, and T is a multiple of $C_{\lambda,b}$ by Proposition 3.2.2. □

3.4 The case $|\phi(0)| > 1$ and $|\lambda| \geq 1$

This case can be dealt with by using a standard argument.

Theorem 3.4.1. *Assume $|\phi(0)| > 1$, $|\lambda| \geq 1$, and λ is not a root of the unity. If $T = R_\lambda \phi(D)$ is not a multiple of $C_{\lambda,b}$ then T is hypercyclic.*

Proof. By Proposition 3.2.2, there exists $a \in \mathbb{C}$, $a \neq 0$, such that $\phi(a) = 0$. We consider the subset $X_0 = \operatorname{span}\{e^{(a/\lambda^n)z}\,;\, n \geq 0\}$.

Since $|\lambda| \geq 1$ and λ is not a root of the unity, the set $\{a/\lambda^n : n \in \mathbb{N}\}$ has an accumulation point in $\mathbb{C}$; hence X_0 is dense in $\mathcal{H}(\mathbb{C})$. On the other hand, since $T^n e^{(a/\lambda^k)z} = 0$ if $n > k$, T^n converges to zero pointwise on X_0.

We will construct a mapping S on a dense subset Y_0 such that $S^n y \to 0$ for all $y \in Y_0$, and $TS = \mathrm{Id}_{Y_0}$. First, observe that the subspace $\mathcal{P}_n$ of polynomials of degree less or equal than n is invariant under $T = R_\lambda \phi(D)$, and the action of T on $\mathcal{P}_n$ can be represented by a finite triangular matrix with diagonal entries $\phi(0)\lambda^k$, $k \geq 0$. Since $\lambda \neq 1$, T has $n+1$ different eigenvalues in $\mathcal{P}_n$. Thus, there exists a sequence $\{p_k : k \geq 0\}$ of polynomials with degree of p_k equal to k such that $Tp_k = \phi(0)\lambda^k p_k$ for all $k \geq 0$, and

$$Y_0 = \operatorname{span}\{p_k(z) \,:\, k \geq 0\}$$

is the subspace of polynomials, which is dense. We define $Sp_k = \frac{1}{\phi(0)\lambda^k} p_k$ and extend S to Y_0 by linearity. Since $|\phi(0)| > 1$, $S^n p_k \to 0$ as $n \to \infty$ for every $|\lambda| \geq 1$, hence $S^n y \to 0$ as $n \to \infty$ for all $y \in Y_0$. Therefore, the Hypercyclicity Criterion implies that T is hypercyclic. □

3.5 The case $0 < |\phi(0)| \leq 1$ and $|\lambda| > 1$.

In this case, if we use the inverse defined as in the previous section; that is $Sp_k = \frac{1}{\phi(0)\lambda^k} p_k$, then $|\phi(0)\lambda^n| > 1$ for $n > n_0$ for some n_0. This implies that $S^k p_n \to 0$ for all $n > n_0$. However on the subspace of polynomials of degree less or equal than n_0 we do not have convergence to zero.

It was pointed out in [13] that $\phi(D)$ is injective if and only if it is a multiple of $C_{\lambda,b}$. Thus our operator is not injective, so its right inverse is not unique, and we will overcome the obstacle by defining a different right inverse.

Let f be an entire function of exponential type $f(z) = \sum_{n=0}^{\infty} a_n z^n$. The Borel transform of f is defined as

$$Bf(z) = \sum_{n=0}^{\infty} \frac{n! a_n}{z^{n+1}}.$$

It is well known that $Bf(z)$ is analytic on $|z| > c$ for some $c > 0$. In particular, for the monomials $f_n(z) = z^n/n!$ we have $Bf_n(z) = 1/z^{n+1}$ which is analytic on $|z| > 0$.

Pólya representation of f (see [18] p. 78) asserts that if $Bf(z)$ is analytic on $|z| > c$ then for any $R > c$, we have

$$f(z) = \frac{1}{2\pi i} \oint_{|t|=R} e^{zt} Bf(t)\, dt.$$

Using this representation, if $\phi(D) = \sum_{n=0}^{\infty} \phi_n D^n$, then

$$\begin{aligned} R_\lambda \phi(D) f(z) &= R_\lambda \left(\sum_n \phi_n \frac{1}{2\pi i} \oint_{|t|=R} t^n e^{tz} Bf(t)\, dt \right) \\ &= R_\lambda \frac{1}{2\pi i} \oint_{|t|=R} \left(\sum_n \phi_n t^n \right) e^{zt} Bf(t)\, dt \\ &= \frac{1}{2\pi i} \oint_{|t|=R} \phi(t) e^{\lambda z t} Bf(t)\, dt. \end{aligned}$$

And iterating the above formula, we get:

$$\left(R_\lambda \phi(D)\right)^n f(z) = \frac{1}{2\pi i} \oint_{|t|=R} \phi(t)\phi(\lambda t) \cdots \phi(\lambda^{n-1} t) e^{\lambda^{n-1} z t} Bf(t)\, dt.$$

On the other hand, denoting $\omega = 1/\lambda$, if for some $R > c$ we define

$$S_1 f(z) = \frac{1}{2\pi i} \oint_{|t|=R} \frac{1}{\phi(\omega t)} e^{\omega z t} Bf(t)\, dt,$$

arguing as in the above computation of $R_\lambda \phi(D) f(z)$ we get $R_\lambda \phi(D) S_1 f = f$.

The next result will be needed to prove this case.

Proposition 3.5.1. *Let $P(z) = c(1 - z/a)$ with $c \neq 0 \neq a$. Then there exists a sequence (R_k) of positive numbers converging to ∞ such that for each $n \geq 0$,*

$$(L_k f_n)(z) = \frac{1}{2\pi i} \oint_{|t|=R_k} \frac{1}{P(\omega t) \cdots P(\omega^k t)} e^{\omega^k z t} Bf_n(t)\, dt \tag{3.1}$$

converges to zero, uniformly on compact subsets, as $k \to \infty$.

Proof. Since $|\omega| < 1$ the subset $X_0 = \operatorname{span}\{e^{a\omega^n z} : n \geq 0\}$ is dense in $H(\mathbb{C})$. Moreover for each $x_0 \in X_0$, $T^n x_0 = 0$ for n large enough. We choose $M_0 \geq 1$

such that $|P(z)| \geq 1$ for $|z| \geq M_0$, set $R_k = |\lambda|^k M_0$, define L_k on $f_n(z) = z^n/n!$ by

$$L_k f_n(z) = \frac{1}{2\pi i} \oint_{|t|=R_k} \frac{1}{P(\omega t)\cdots P(\omega^k t)} e^{\omega^k zt} Bf_n(t)\, dt.$$

If $|t| = R_k = |\lambda|^k M_0$ and $1 \leq j \leq k$, then $|\omega^j t| = |\lambda^{k-j}|M_0 \geq M_0$. Therefore $|P(\omega^j t) \geq 1$ and

$$|L_k f_n(z)| \leq \frac{1}{2\pi} 2\pi R_k\, e^{M_0|z|} \frac{1}{R_k^{n+1}} = \frac{e^{M_0|z|}}{R_k^n} \to 0$$

uniformly on compact subsets as $k \to \infty$. □

Theorem 3.5.2. *Suppose that ϕ vanishes at some $a \in \mathbb{C}$ and $0 < |\phi(0)| \leq 1$. If $|\lambda| > 1$ then $T = R_\lambda \phi(D)$ is hypercyclic.*

Proof. Again, since $|\omega| < 1$, the subset $X_0 = \operatorname{span}\{e^{a\omega^n z} : n \geq 0\}$ is dense in $H(\mathbb{C})$, and if $x_0 \in X_0$ then $T^n x_0 = 0$ for n large enough. Let n_0 be the first natural number satisfying $|\phi(0)\lambda^{n_0+1}| > 1$. The proof will be finished if we can define a sequence of mappings S_k on the monomials f_n ($n = 0, \ldots, n_0$) satisfying

1 $S_k f_n \to 0$ uniformly on compact subsets as $k \to \infty$, and

2 $(R_\lambda \phi(D))^k S_k f_n \to f_n$ uniformly on compact subsets as $k \to \infty$.

We denote $P(z) = \phi(0)(1 - z/a)$. First we define S_k on f_0. By Proposition 3.5.1, there exists a sequence of positive numbers $R_k \to \infty$ such that

$$L_k f_n n = \frac{1}{2\pi i} \oint_{|t|=R_k} \frac{1}{P(\omega t)\cdots P(\omega^k t)} e^{\omega^k zt} Bf_n(t)\, dt \to 0$$

uniformly on compact subsets as $k \to \infty$ for $n = 0, \ldots, n_0$.

Taking $S_k f_0 = L_k f_0$, we get $(R_\lambda \phi(D))^k S_k f_0 = f_0$. Indeed, denoting

$$\Phi_k(t) = \frac{\phi(\omega t)}{P(\omega t)} \cdots \frac{\phi(\omega^k t)}{P(\omega^k t)},$$

and observing that $\Phi_k(0) = 1$, we get

$$(R_\lambda \phi(D))^k S_k f_0(z) = \frac{1}{2\pi i} \oint_{|t|=R_k} \Phi_k(t) e^{zt} \frac{dt}{t} = 1,$$

where the last equality follows from the fact that the function $\Phi_k(t)e^{zt}$ is analytic: the integral is equal to the residue of $\Phi_k(t)e^{zt}(1/t)$, which is 1.

Next we define S_k on f_1. Since $\Phi_k(t)$ is an entire function and $\Phi_k(0) = 1$, we can write $\Phi_k(z) = \sum_{j=0}^{\infty} a_j^{(k)} t^k$ with $a_0^{(k)} = 1$ for all k. Also, the second term of the Cauchy product $\left(\sum_{j=0}^{\infty} \frac{z^j}{j!} t^j\right) \cdot \left(\sum_{j=0}^{\infty} a_j^{(k)} t^k\right)$ coincides with

$$\frac{1}{2\pi i} \oint_{|t|=R_k} \Phi_k(t) e^{zt} \frac{dt}{t^2} = z + a_1^{(k)}.$$

So defining

$$S_k f_1(z) = \frac{1}{2\pi i}\oint_{|t|=R_k} \Phi_k(t)e^{zt}\frac{dt}{t^2} - a_1^{(k)} S_k f_0,$$

we get $(R_\lambda\phi(D))^k S_k f_1 = f_1$, and $(S_k f_1)$ converges uniformly to zero on compact sets provided $(a_1^{(k)})$ is bounded. By the chain rule, $a_1^{(k)} = (\omega + \ldots + \omega^k)c$, where c is the derivative at zero of the function $\varphi(z)/P(z)$. Since $|\omega| < 1$ the sequence $(a_1^{(k)})$ is bounded.

Assume that $S_k f_j$ has already been defined for $0 \leq j < m$, satisfying $S_k f_j \to 0$ as $k \to \infty$ uniformly on compact subsets, $(R_\lambda\phi(D))^k S_k f_j = f_j$, and $(a_j^{(k)})_{k\in\mathbb{N}}$ bounded. Let us construct $S_k f_m$. Since

$$\begin{aligned} L_k f_m(z) &= \frac{1}{2\pi i}\oint_{|t|=R_k} \Phi_k(t)e^{zt}\frac{dt}{t^{m+1}} = c_m \\ &= \sum_{j=0}^{m} \frac{z^j}{j!} a_{m-j}^{(k)} = f_m(z) + \sum_{j=0}^{m-1} a_{m-j}^{(k)} f_j(z), \end{aligned}$$

defining

$$S_k f_m = L_k f_m - \sum_{j=0}^{m-1} a_{m-j}^{(k)} S_k f_j$$

we get $(R_\lambda\phi(D))S_k f_m = f_m$ by construction, and $S_k f_m \to 0$ uniformly on compact subsets provided the sequence $(a_m^{(k)})$ is bounded for all $k \in \mathbb{N}$, which follows directly by Leibniz rule. Indeed, denoting $\varphi(z) = \phi(z)/P(z)$,

$$\begin{aligned} |a_m^{(k)}| &= \left|[\varphi(\omega z), \ldots, \varphi(\omega^k z)]^{(m)}(0)\right| \\ &= \left|\sum_{h_1+\ldots+h_k=m} \binom{m}{h_1, \ldots, h_k} \prod_{t=1}^{k} (\varphi(\omega^t z))^{(h_t)}(0)\right| \\ &\leq \sum_{h_1+\ldots+h_k=m} \binom{m}{h_1, \ldots, h_k} \prod_{t=1}^{k} \omega^{th_t} |(\varphi^{(h_t)}(0))| \\ &\leq \mathrm{C}(\omega + \ldots + \omega^k)^m, \end{aligned}$$

where $C = \max_{j=1}^{m} |\varphi^j(0)|^m$. Since $\varphi(0) = 1$, we can construct a sequence of mappings S_k acting on f_n, $n = 0, \ldots, n_0$, satisfying all the requirements we desired, and this finishes the proof. □

3.6 The case $0 < |\phi(0)| \leq 1$ and $|\lambda| = 1$.

We take $\lambda = e^{2\pi i\theta}$ with θ an irrational number, since the case λ is a root of the unity has already been studied in Section 3.4, and we set $\omega = \lambda^{-1}$.

By Proposition 3.2.2 we can suppose that ϕ has a zero at some point $\alpha \neq 0$. To apply the Hypercyclicity Criterion, we consider the dense subset $X_0 = \mathrm{span}\,\{e^{\omega^n \alpha z} : n \geq 1\}$, where the powers of $T = R_\lambda\phi(D)$ are eventually zero, and we will find a dense subset of the form $Y_0 = \mathrm{span}\,\{e^{bz} : b \in U\}$, where U has a cluster point in $\mathbb{C}$, and a map $S : Y_0 \to Y_0$ such that S^n converges pointwise to zero on Y_0 and $TS = \mathrm{Id}_{Y_0}$.

Taking $Se^{bz} = \phi(\omega b)^{-1}e^{wbz}$ we get

$$S^n e^{bz} = \frac{1}{\phi(\omega b)\cdots\phi(\omega^n b)} e^{\omega^n bz},$$

so we only have to show that $S^n e^{bz} \to 0$ as $n \to \infty$ uniformly on compact sets. Since the modulus of the term $e^{\omega^n bz}$ on a compact subset can be controlled independently on n, it should be sufficient to show that

$$\phi(\omega b)\cdots\phi(\omega^n b) \to \infty,$$

pointwise on U. Suppose that $\phi(z) = p(z)$ is a monic polynomial (monic is for simplicity), thus $p(z) = (z-a_1)\cdots(z-a_n)$, let $R_0 = \max_{1\le j\le n}\{|a_j|, 2\}$. If $|b| > 2R_0$ we obtain that $|p(\mu b)| > R_0^k$ for all $|\mu| = 1$, hence

$$|p(\omega b)\cdots p(\omega^n b)| \ge R_0^{nk} \to \infty$$

as $n \to \infty$. Thus, we can choose $U = \{b \in \mathbb{C} : |b| > 2R_0\}$.

Suppose that ϕ has an exponential term, that is $\phi(z) = e^{az}p(z)$, and $p(z)$ is again a polynomial. Then

$$\phi(\omega b)\cdots\phi(\omega^n b) = e^{(\omega+\cdots+\omega^n)ab}p(\omega b)\cdots p(\omega^n b).$$

Since the product $p(\omega b)\cdots p(\omega^n b)$ goes to ∞ as $n \to \infty$ for $|b|$ big, we need to show that the first term of the product is bounded below. Writing

$$e^{(\omega+\cdots+\omega^n)ab} = e^{(1-\omega^n)\frac{\omega ab}{1-\omega}},$$

and selecting $b = (1-\omega)R/(a\omega)$ with R big enough we obtain:

$$|e^{(\omega+\cdots+\omega^n)ab}| = e^{(1-\cos(n\theta))R} \ge e^R.$$

Thus, it is sufficient to consider $U = \{(1-\omega)R/(a\omega) : R \ge 2R_0|a|\}$.

At first glance, one might think that the above ideas can be applied to prove the case in which ϕ has infinitely many zeros, simply by cutting the infinite product into a polynomial by a tail. However, to control the tail of the product, we must consider z away from the zeros of the tail. But at the same time, to get divergence of the iteration of the polynomial, we must choose z larger than the zeros of the polynomial. Since both requirements are not compatible, we need a new proof for the case of infinite zeros.

Let $r_k < r_{k+1}$ be the absolute values of two zeros of the function ϕ so that the set $A = \{z : r_k < |z| < r_{k+1}\}$ is free of zeros, and let us denote

$$f_n(z) = \phi(\omega z)\phi(\omega^2 z)\cdots\phi(\omega^n z).$$

Proposition 3.6.1. *Assume that there is $z_0 \in A$ such that $\limsup_n f_n(z_0) = \infty$. Then $T = R_\lambda\phi(D)$ is hypercyclic.*

Proof. Let $\{n_k\}$ be a subsequence such that $\lim_k f_{n_k}(z_0) = \infty$. By the Hypercyclicity Criterion, it is sufficient to consider the dense subset

$$Y_0 = \text{span}\,\{e^{\omega^k z_0 z} \;:\; k \geq 1\}$$

and find $\{n_k\}_k$ such that the sequence of iterates S^{n_k}, given by

$$S^{n_k} e^{bz} = \frac{1}{\phi(\omega b) \cdots \phi(\omega^{n_k} b)} e^{\omega^{n_k} bz},$$

converges to zero pointwise on Y_0. Indeed, let $K \subset A$ be a closed annulus containing z_0. Since ϕ does not vanish on K, denoting $m > 0$ the minimum and M the maximum of $|\phi(z)|$ on K, and fixing $l \in \mathbb{N}$, we get

$$\begin{aligned}
\left|S^{n_k}(e^{\omega^l z_0 z})\right| &= \left|\frac{1}{\phi(\omega^{l+1} z_0)\phi(\omega^{l+1} z_0) \cdots \phi(\omega^{n_k+l} z_0)} e^{\omega^{l+n_k} z_0 z}\right| \\
&= \left|\frac{\phi(\omega z_0)\phi(\omega^2 z_0) \cdots \phi(\omega^l z_0)}{\phi(\omega^{n_k+1} z_0)\phi(\omega^{n_k+2} z_0) \cdots \phi(\omega^{n_k+l} z_0)}\right| \left|\frac{1}{f_{n_k}(z_0)} e^{\omega^{l+n_k} z_0 z}\right| \\
&\leq \left(\frac{M}{m}\right)^l \left|\frac{1}{f_{n_k}(z_0)} e^{\omega^{l+n_k} z_0 z}\right| \to 0
\end{aligned}$$

uniformly on compact sets as $n_k \to \infty$, and we get the desired result. □

Proposition 3.6.1 reduces our problem of the hypercyclicity of $T = R_\lambda \phi(D)$, to find $z_0 \in U$ such that $\limsup f_n(z_0) = \infty$. In Theorem 3.6.3 we will show that such $z_0 \in U$ exists. This problem is connected with extremal behaviour of the family $\mathcal{F} = \{f_n\}_{n \geq 1}$. Specifically we are looking for an open subset G such that $\mathcal{F}$ is normal at no point of G. Then Montel's Theorem shows that $\mathcal{F}$ restricted to G is universal, therefore there exists $z_0 \in G$ such that $\{f_n(z_0)\}_{n \geq 1}$ is dense in $\mathbb{C}$. In particular, there exists a subsequence $\{n_k\}_k$ such that $\lim_{k \to \infty} f_{n_k}(z_0) = \infty$, as we wish to show.

We need the following result:

Lemma 3.6.2. *Let ω be an irrational rotation, let $r_k < r_{k+1}$ be the absolute values of two zeros of ϕ so that $A = \{z : r_k < |z| < r_{k+1}\}$ is free of zeros, and let $z_0 \in A$. If the family $\mathcal{F} = \{f_n(z)\}_n$ is uniformly bounded on a neighborhood of z_0 then $\mathcal{F}$ is uniformly bounded on $\overline{D(0, |z_0|)}$.*

Proof. Indeed, we select a closed ball B_0 centered at z_0 such that $\mathcal{F}$ is uniformly bounded on B_0. We can suppose that B_0 is strictly contained in the annulus $\{z : r_k < |z| < r_{k+1}\}$. Now we will show that $\mathcal{F}$ is uniformly bounded on $D(0, |z_0|)$. Indeed, since $\{w^n : n \geq 1\}$ is dense in $\partial\mathbb{D}$, by the compactness of $\partial D(0, |z_0|)$ there is an integer n_0 (see Figure 3.1) such that

$$\partial D(0, |z_0|) \subset B_0 \cup \omega B_0 \cup \cdots \cup \omega_0^n B_0.$$

Since $A = B_0 \cup \cdots \cup \omega^{n_0} B_0$ is strictly contained in $\{z : r_0 < |z| < r_1\}$ and ϕ does not vanish on A, let $m = \inf\{|\phi(z) : z \in A\} > 0$.

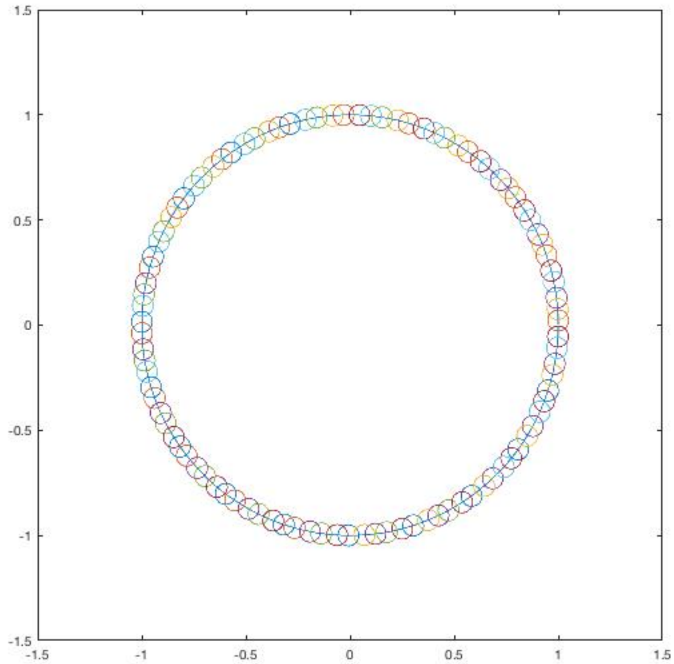

Figure 3.1: $\omega = e^{\sqrt{2}\pi i}$, and $n_0 = 100$.

By the modulus maximum principle, since A contains the boundary of $D(0, |z_0|)$, the family $\mathcal{F}$ is uniformly bounded on $D(0, |z_0|)$ provided $\mathcal{F}$ is uniformly bounded on $A = B_0 \cup \cdots \cup \omega^{n_0} B_0$. Since $\mathcal{F}$ is uniformly bounded on B_0, let M such that $|f_n(z)| < M$ for all $z \in B_0$ and $f_n \in \mathcal{F}$. Let us show that $\mathcal{F}$ is uniformly bounded on each $\omega^k B_0$, $1 \le k \le n_0$. Indeed each element in $\omega^k B_0$ have the form $\omega^k z$ with $z \in B_0$. Thus,

$$|f_n(\omega^k z)| = |\phi(\omega^{k+1} z) \cdots \phi(\omega^{k+n} z)| = \frac{|f_{n+k}(z)|}{|f_k(z)|} \le \frac{M}{m^k},$$

for all $\omega^k z \in \omega^k B$ and $f_n \in \mathcal{F}$. Therefore $\mathcal{F}$ is uniformly bounded on $D(0, |z_0|)$ as we wanted to show. □

Theorem 3.6.3. *Suppose that ω is an irrational rotation and $0 < |\phi(0)| \le 1$. Then $T = R_\lambda \phi(D)$ is hypercyclic.*

Proof. We will show that there exists z_0 such that $\limsup_n f_n(z_0) = \infty$. So the result follows from Proposition 3.6.1.

Case $|\phi(0)| = 1$. By way of contradiction, suppose that $\{f_n(z_0)\}_{n\ge1}$ is bounded for each z_0. We claim that there exists an open subset G such that $\mathcal{F} = \{f_n(z)\}$ is normal at no $z_0 \in G$. Indeed, let $r_0 < r_1$ be the radii of the two smallest circles centered at 0 containing zeros of ϕ. We select z_0 such that $r_0 < |z_0| < r_1$, and suppose that $\mathcal{F}$ is normal in z_0. Since the orbit $\{f_n(z_0)\}_{n\ge1}$ is bounded, we can select a closed ball B_0 centered at z_0, such that $\mathcal{F}$ is uniformly bounded on B_0. By Lemma 3.6.2, $\mathcal{F}$ is uniformly bounded on $\overline{D(0,|z_0|)}$. Now, by Montel's Theorem, there exists a subsequence f_{n_k} that converges uniformly on the compact subsets on $D(0,|z_0|)$ to some function f analytic on $D(0,|z_0|)$. Let a be a zero of ϕ of modulus r_0. Then, for each k, $\omega^{-k}a$ is a zero of f_n for $n \geq k$, hence $f(\omega^{-k}a) = 0$ for every k. Since $\{\omega^{-k}a\}_{k\geq 1}$ has an accumulation point, we get that $f = 0$. However $|f_n(0)| = 1$ for all n, a contradiction. Therefore $\mathcal{F}$ is normal at no point of the annulus $\{z : r_0 < |z| < r_1\}$.

Let G be a non-empty open set such that $\mathcal{F}$ is normal at no point $z \in G$. We claim that the family $\{f_n|_G\}$ is transitive; i.e., there exists $z_0 \in G$ such that $\{f_n(z_0)\}_{n\geq 1}$ is dense in $\mathbb{C}$. Indeed by Birkhoff's transitivity Theorem (see [30, Theorem 1.16]), it is sufficient to show that for each open subsets $U \subset G$ and $V \subset \mathbb{C}$ there exists n such that $f_n(U) \cap V \neq \emptyset$. By Montel's Theorem, since $\mathcal{F}$ is not normal on U, the subset $\cup_n f_n(U)$ is dense in the complex plane, thus $\cup_n f_n(U) \cap V \neq \emptyset$, therefore there exists n such that $f_n(U) \cap V \neq \emptyset$ as we desired. This fact contradicts the initial asumption and proves the result.

Case $0 < |\phi(0)| < 1$. As in the previous case, we suppose that for each z_0 the orbit $\{f_n(z_0)\}_{n\geq 1}$ is bounded. Let $0 < r_0 < r_1 < r_2 < \cdots$ be sequence of the radii of the circles centered at 0 containing zeros of ϕ. Since ϕ is an entire function of exponential type, by Hadamard's Theorem we can deduce that $\phi(z) = \phi(0)(1 - \frac{z}{r_0})\varphi(z)$ where $\varphi(z)$ is an entire function satisfying $\varphi(0) = 1$. Take k_0 large enough so that $|\phi(0)|\left(\frac{r_{k_0}}{r_0} - 1\right) > 1$. As in the previous case, we will show that if z_0 is in the annulus $r_{k_0} < |z| < r_{k_0+1}$, then $\mathcal{F}$ is not normal at z_0. Indeed, since $\{f_n(z_0)\}_{n\geq 0}$ is bounded, if $\mathcal{F}$ were normal at z_0, $\mathcal{F}$ would be uniformly bounded on a disk B_0 centered at z_0. By Lemma 3.6.2, $\mathcal{F}$ would be uniformly bounded on $\overline{D(0,|z_0|)}$ and there exists $M > 0$ such that $|f_n(z)| \leq M$ for all z with $|z| = |z_0|$, and for all $n \geq 1$.

We consider the functions $\Phi_n(z) = \varphi(\omega z)\cdots\varphi(\omega^n z)$. For $|z| = |z_0| > r_{k_0}$ we get $|\phi(z)| \geq |\varphi(z)|$, and thus $|\Phi_n(z)| \leq |f_n(z)| \leq M$ for $|z| = |z_0|$ and $n \geq 1$. This fact implies that $\Phi_n(z)$ converges uniformly on the compact subsets of $D(0,|z_0|)$ to a function $f(z)$ analytic on $D(0,|z_0|)$. Again, we get that f has an infinite numbers of zeros with an accumulation point in $\mathbb{C}$, therefore $f = 0$. However, $\Phi_n(0) = 1$ for all n, a contradiction. Thus $\mathcal{F}$ is normal at no point of the annulus $G = \{z : r_{k_0} < |z| < r_{k_0+1}\}$ and, arguing as in the previous case, we get $z_0 \in G$ such that $\{f_n(z_0)\}_{n\geq 1}$ is dense in $\mathbb{C}$, a contradiction with the initial asumption which proves the result. □

3.7 The case $\phi(0) = 0$ and $|\lambda| \geq 1$

The operator $T_{\lambda,b}$ of R. Aron and D. Markose is included in this case with $\phi(z) = ze^{bz}$.

Theorem 3.7.1. *Assume* $\phi(0) = 0$. *If* $|\lambda| \geq 1$ *then* $T = R_\lambda\phi(D)$ *is hypercyclic.*

Proof. We write $\phi(z) = z^m\psi(z)$ with $\psi(0) \neq 0$ and we denote by X_0 the set of complex polynomials $p(z)$. Note that $T^n p(z) = 0$ for $n > \deg(p)$.

Set $A_\lambda = R_\lambda\psi(D)$ so that $T = A_\lambda D^m$. Since $\psi(0) \neq 0$, as in the proof of Theorem 3.4.1, the subspace of polynomials of degree less or equal to n is invariant under the operator A_λ and the eigenvalues are simple on that subspace. We denote $p_0, p_1, \cdots p_k$ the polynomials of degree $\leq k$ which are the eigenvectors associated to $\psi(0)\lambda^k$, that is, $A_\lambda p_k = \psi(0)\lambda^k p_k$ for $k \geq 0$.

Let V be the complex Volterra operator defined by

$$Vf(z) = \int_0^z f(\xi)d\xi, \quad (z \in \mathbb{C}).$$

The equation $TV^m p_k = A_\lambda p_k = \psi(0)\lambda^k p_k$ gives us the key to construct the maps S_k required by the Hypercyclicity Criterion. Indeed, let us define

$$S_k p_n = \frac{V^{mk}p_n}{\lambda^m\lambda^{2m}\dots\lambda^{(k-1)m}(\psi(0)\lambda^n)^k},$$

and extend S_k to $Y_0 = \operatorname{span}\{p_k(z) \ : \ k \geq 0\}$ by linearity. Since $\frac{V^k 1}{\psi(0)^k} \to 0$ uniformly on compact sets as $k \to \infty$, we obtain that $\frac{1}{\psi(0)^k}V^{mk}p_n \to 0$ in $\mathcal{H}(\mathbb{C})$. Hence, since $|\lambda| \geq 1$,

$$|S_k(p_n)(z)| \leq \frac{|V^{mk}p_n(z)|}{|\psi(0)|^k} \to 0$$

uniformly on compact sets. To check that $T^k S_k = \mathrm{Id}_{Y_0}$, note that

$$T^k = A_\lambda D^m A_\lambda D^m \cdots A_\lambda D^m, \quad \text{(k times)}.$$

Since A_λ is an extended λ-eigenoperator of D, $D^m A_\lambda = \lambda^m A_\lambda D^m$. Therefore $T^k = \lambda^m\lambda^{2m}\cdots\lambda^{(k-1)m}A_\lambda^k D^{km}$, hence

$$T^k S_k p_n = T^k\left(\frac{V^{mk}p_n}{\lambda^m\lambda^{2m}\dots\lambda^{(k-1)m}(\psi(0)\lambda^n)^k}\right) = \frac{A_\lambda^k p_n}{\psi(0)^k\lambda^{nk}} = p_n,$$

and the Hypercyclicity Criterion implies that T is hypercyclic. □

We end this chapter by showing that there is a bounded linear operator on a Banach space which has no hypercyclic extended eigenoperators. This will be an application of Proposition 3.3.1.

Lemma 3.7.2. *Consider the operator $\mathcal{A} : H^q \longrightarrow H^q$, $q \geq 1$ defined by:*

$$Af(z) = \frac{1}{z-1}\int_1^z f(\xi)d\xi, \quad (z \in \mathbb{D})$$

then $\mathcal{A}$ does not have any hypercyclic extended eigenoperator in H^q.

The key for proving this lemma is a theorem that appeared in [55] and which uses the concept of *rich point spectrum.* We recall the readers that an operator $T \in B(\mathcal{X})$ is said to have a rich point spectrum if the interior of its point spectrum is non-empty and that for each open disk D of $\sigma_p(T)$, the family of the eigenvectors $\bigcup_{z\in D}\ker(T - z)$ is a total subset on $\mathcal{X}$.

Theorem 3.7.3. *[55, Theorem 3.2] Let $A \in B(\mathcal{X})$. Assume that A has a rich point spectrum and that $\sigma_p(A) = D(r, r)$ for some $r > 0$. If $\lambda \in Ext(A, \mathcal{X})$ then λ is real and $\lambda \in (0, 1]$*

The proof of Lemma 3.7.2 is straightforward. Indeed, observe that according to [78, Theorem B], the operator $\mathcal{A}$ has a rich point spectrum in H^q. Moreover, it is known from [84, 95] that:

$$\sigma_p(\mathcal{A}, H^q) = D\Big(\frac{q}{2(q-1)}, \frac{q}{2(q-1)}\Big).$$

Applying Theorem 3.7.3, we have:

$$\mathrm{Ext}(\mathcal{A}, H^q) \subset (0, 1].$$

Hence, by Proposition 3.3.1 , we deduce that no extended eigenoperator of $\mathcal{A}$ is hypercyclic.

Finally, observe that if $q = 2$, then $\mathcal{A}$ is known to be the adjoint of the Cesàro operator in H^2. Hence, we deduce particularly that:

Corollary 3.7.4. *The adjoint of the Cesàro operator in H^2 has no hypercyclic extended eigenoperators.*

Chapter 4

Extended Eigenvalues of Composition Operators.

In weighted Hardy spaces $\mathcal{H}^2(\beta)$, invariant under automorphisms, we completely compute the extended eigenvalues of composition operators C_φ induced by lineal fractional self-maps φ of the unit disk $\mathbb{D}$ with an interior fixed point in $\mathbb{D}$ and another one outside $\overline{\mathbb{D}}$. Such classes of transformations include elliptic and loxodromic cases and a hyperbolic nonautomorphic sub-case. Our work [10] can be seen as a continuation of the work [54] of M. Lacruz, F. León-Saavedra, S. Petrovic, and L. Rodríguez-Piazza who have completely calculated the extended spectra of C_φ in $\mathcal{H}^2(\mathbb{D})$.

4.1 Background

Recall that a complex number α is said to be an extended eigenvalue of C_φ if there exists a non-zero operator X such that $C_\varphi X = \alpha X C_\varphi$. We shall denote by $\operatorname{Ext}(C_\varphi, \mathcal{H}^2(\beta))$ the collection of such scalars α and we shall call it the *extended spectra* of C_φ in $\mathcal{H}^2(\beta)$.

In this chapter, we are interested in finding the extended-spectrum of bounded composition operators on $\mathcal{H}^2(\beta)$. At this step, we mention that it is still an open problem to characterize the holomorphic selfmaps of the unit disk inducing bounded composition operators on $\mathcal{H}^2(\beta)$. N. Zorboska [102, 103, 104] studied intensively composition operators on weighted Hardy spaces, obtaining results on boundedness, compactness and cyclicity. More recently, new striking results on the boundedness problem have been obtained in [56]. In our work, we will require on the space $\mathcal{H}^2(\beta)$ invariance under automorphism. We compute the extended-spectrum for bounded composition operators induced by linear fractional self-maps of the unit disk with an interior fixed point in $\mathbb{D}$ and another one outside $\overline{\mathbb{D}}$.

Finally, observe that several properties are lost in the transition from the Hardy space $\mathcal{H}^2(\mathbb{D})$ to the weighted Hardy spaces $\mathcal{H}^2(\beta)$. For instance, the authors in [54] used basic properties of analytic Toeplitz operators on the Hardy space that are no longer true on $\mathcal{H}^2(\beta)$. We overcome this difficulty by constructing some triangular operators that will replace the Toeplitz operators used in [54].

4.2 Composition Operators Induced by Elliptic Automorphism

Let C_φ be a composition operator induced by an elliptic self-map of the unit disk. Since $\mathcal{H}^2(\beta)$ is automorphism invariant we can assume without loss of generality that the fixed points of φ are 0 and ∞, and that φ has the form:

$$\varphi(z) = \omega z, \quad \omega \in \partial\mathbb{D} \setminus \{1\}. \tag{4.1}$$

That is, in particular, on weighted Hardy spaces, composition operators induced by elliptic self-maps of the unit disk, are bounded.

Theorem 4.2.1. *Assume that $\mathcal{H}^2(\beta)$ is automorphism invariant. If C_φ is composition operator on $\mathcal{H}^2(\beta)$ induced by an elliptic self-map φ of the unit disk then:*

$$\mathrm{Ext}(C_\varphi, \mathcal{H}^2(\beta)) = \{\omega^n : n \in \mathbb{Z}\}.$$

Proof. Let $\alpha \in \mathrm{Ext}(C_\varphi, \mathcal{H}^2(\beta))$ then there exists a non-zero bounded operator X on $\mathcal{H}^2(\beta)$ such that $C_\varphi X = \alpha X C_\varphi$. In particular, for all $m \in \mathbb{N}$, the functions z^m satisfy:

$$C_\varphi X z^m = \alpha X C_\varphi z^m = \alpha X z^m \circ \varphi(z) = \alpha \omega^m X z^m.$$

Clearly, $Xz^m \neq 0$ for some $m \in \mathbb{N}$. Otherwise, we would have $X \equiv 0$ on $\mathcal{H}^2(\beta)$. This contradicts the fact that X is an extended eigenoperator for C_φ. Hence, $Xz^m \neq 0$ and $\alpha\omega^m \in \sigma_p(C_\varphi, \mathcal{H}^2(\beta))$ (see [21, Theorem 7.1]). So, there exists $m_0 \in \mathbb{N}$ such that $\alpha\omega^m = \omega^{m_0}$. So $\alpha = \omega^n$ with $n = m_0 - m \in \mathbb{Z}$. Therefore, $\alpha \in \{\omega^n : n \in \mathbb{Z}\}$ and the direct inclusion is proved.

For the converse inclusion, let $e_k(z) = \frac{z^k}{\beta_k}$ be the orthonormal basis of $\mathcal{H}^2(\beta)$. We consider the forward shift operator F acting on the basis as follows:

$$Fe_k = e_{k+n} \quad \text{where} \quad n \in \mathbb{N}.$$

Observe now that for all natural k, we have:

$$C_\varphi F e_k = C_\varphi e_{k+n} = \omega^{k+n} e_{k+n},$$

and on the other hand,

$$F C_\varphi e_k = F(\omega^k e_k) = \omega^k e_{k+n}.$$

Hence, we obtain:

$$C_\varphi F e_k = \omega^n F C_\varphi e_k, \quad \forall k \in \mathbb{N}$$

which means that $\{\omega^n : n \in \mathbb{N}\} \subset \mathrm{Ext}(C_\varphi, \mathcal{H}^2(\beta))$.

We claim now that $\{\omega^{-n} : n \in \mathbb{N}\}$ is also a subset of $\mathrm{Ext}(C_\varphi, \mathcal{H}^2(\beta))$. Consider the backward shift operator $X = B^n$ that shifts back the coefficient

n-time to the left that is: $B^n e_k = e_{k-n}$ if $k \geq n$ and $B^n e_k = 0$ if $k < n$. X is clearly bounded, moreover, for all $z \in \mathbb{D}$, we have:

$$C_\varphi X e_k(z) = C_\varphi e_{k-n}(z) = \omega^{k-n} e_{k-n}(z).$$

On the other hand, we have:

$$\frac{1}{\omega^n} X C_\varphi e_k(z) = = \frac{\omega^k}{\omega^n} X e_k(z) = \omega^{k-n} e_{k-n}(z),$$

So, for all $k \geq n > 0$, we have:

$$C_\varphi X e_k = \frac{1}{\omega^n} X C_\varphi e_k. \tag{4.2}$$

It is clear that if $k < n < 0$, $C_\varphi X e_k = 0 = \frac{1}{\omega^n} X C_\varphi e_k$. Since e_k is an orthonormal basis, the Equality 4.2 is true for all functions in $\mathcal{H}^2(\beta)$. Hence, $C_\varphi B^n = \frac{1}{\omega^n} B^n C_\varphi$ and ω^{-n} is an extended eigenvalue of C_φ. Finally, for $n = 0$ and by taking $X = I_{\mathcal{H}^2(\beta)}$, we observe that $\omega^0 = 1 \in \mathrm{Ext}(C_\varphi, \mathcal{H}^2(\beta))$. Therefore, we obtain:

$$\{\omega^n : n \in \mathbb{Z}\} \subset \mathrm{Ext}(C_\varphi, \mathcal{H}^2(\beta)).$$

The desired equality is deduced. □

We stress here the difficulty to extend the results from the Hardy space to other analytic function spaces. To find extended eigenoperators the authors in [54] used the good properties of Toeplitz operators on $\mathcal{H}^2(\mathbb{D})$. For instance, M_z is bounded on $\mathcal{H}^2(\mathbb{D})$ however, these Toeplitz operators are not necessarily bounded on $\mathcal{H}^2(\beta)$.

4.3 Composition Operators Induced by Loxodromic/hyperbolic nonauotomorphism Transformations

We consider now composition operators C_φ induced by non-elliptic self-maps φ of $\mathbb{D}$ that fix an interior point of $\mathbb{D}$ and another one outside of its closure. In the frame of this configuration we cite the loxodromic case and a (sub)case of hyperbolic nonautomorphism. These two classes of maps behave quite similarly in terms of fixed points. Automorphism invariance of $\mathcal{H}^2(\beta)$ allow us to suppose without loss of generality that the exterior fixed point is $+\infty$. In such a case, φ has the form

$$\varphi(z) = k(z-p) + p \quad ; \quad |k| < 1$$

where $p \in \mathbb{D}$ is the interior fixed point.

The extended-spectrum of C_φ in this case is characterized in the next theorem:

Theorem 4.3.1. *Assume that $\mathcal{H}^2(\beta)$ is automorphism invariant. If C_φ is a bounded composition operator on $\mathcal{H}^2(\beta)$ induced by a non-elliptic self-map of the unit disk with a fixed point in $\mathbb{D}$ and another one outside of $\overline{\mathbb{D}}$, then:*

$$\operatorname{Ext}(C_\varphi, \mathcal{H}^2(\beta)) = \{k^n : n \in \mathbb{Z}\}$$

where $k = \varphi'(0)$.

Proof. An easy computation shows that the Köenigs map in such a case is $\sigma(z) = z - p$. That is, $C_\varphi \sigma(z) = k\sigma(z)$ with $0 < |\varphi'(0)| = |k| < 1$. Since the set $\{\sigma^n \ : \ n \geq 0\}$ is a total set in $\mathcal{H}^2(\beta)$, according to [54, Lemma 2.6], we obtain that $\operatorname{Ext}(C_\varphi, \mathcal{H}^2(\beta)) \subset \{k^n \ : n \in \mathbb{Z}\}$.

For the reverse inclusion, we first show that $\{k^n, n < 0\} \subset \operatorname{Ext}(C_\varphi, \mathcal{H}^2(\beta))$. For that, define the linear map

$$X\sigma^n = \begin{cases} \omega_n \sigma^{n-1} & n \geq 1 \\ 0 & n = 0 \end{cases}$$

which can be defined by linearity on a dense subset of $\mathcal{H}^2(\beta)$. In what follows, we will show that X can be extended to a bounded operator on $\mathcal{H}^2(\beta)$ for a suitable choice of the weight sequence $(\omega_n)_{n\in\mathbb{N}}$. Indeed, let us define X as the following triangular operator:

$$\begin{aligned} X(z^n) = X((z-p+p)^n)) &= \sum_{k=0}^{n} \binom{n}{k} p^{n-k} X(z-p)^k \\ &= \sum_{k=1}^{n} \binom{n}{k} p^{n-k} \omega_k (z-p)^{k-1} \end{aligned}$$

Set $0 < s < 1$ such that $|p| + s < 1$ and let us define $\omega_k = \frac{s^k}{\|(z-p)^{k-1}\|_\beta}$, $k \geq 1$. Since:

$$\begin{aligned} \|Xz^n\|_\beta &\leq \sum_{k=1}^{n} \binom{n}{k} |p|^{n-k} \omega_k \|(z-p)^{k-1}\|_\beta \\ &= \sum_{k=1}^{n} \binom{n}{k} |p|^{n-k} s^k = (|p| + s)^n - |p|^n \end{aligned}$$

we get that for any $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{H}^2(\beta)$:

$$\begin{aligned} \|X(f)\|_\beta &\leq \sum_{n=0}^{\infty} |a_n| \|X(z^n)\|_\beta \\ &\leq \left(\sum_{n=0}^{\infty} |a_n|^2 \beta_n^2 \right)^{1/2} \left(\sum_{n=0}^{\infty} \frac{\|X(z^n)\|_\beta^2}{\beta_n^2} \right)^{1/2} \\ &\leq \left(\sum_{n=0}^{\infty} \frac{[(|p| + s)^n - |p|^n]^2}{\beta_n^2} \right)^{1/2} \|f\|_\beta, \end{aligned}$$

Notice that, by using (1.4), the series $\sum_{n=0}^{\infty} \frac{[(|p|+s)^n - |p|^n]^2}{\beta_n^2}$ is convergent. Thus, we have proved that X is bounded on $\mathcal{H}^2(\beta)$ as desired.

Now, observe that $C_\varphi X^m = k^{-m} X^m C_\varphi$ for $m \geq 0$. Indeed, it is sufficient to verify the last equality on the total set $\{\sigma^n \ : \ n \geq 0\}$. On the subset $\{\sigma^n \ : \ n \geq m\}$, we have:

$$\begin{aligned} C_\varphi X^m \sigma^n &= C_\varphi(\omega_n \omega_{n-1} \ldots \omega_{n-m+1} \sigma^{n-m}) \\ &= \omega_n \omega_{n-1} \ldots \omega_{n-m+1} C_\varphi \sigma^{n-m} \\ &= \omega_n \omega_{n-1} \ldots \omega_{n-m+1} k^{n-m} \sigma^{n-m} \\ &= k^{-m} k^n \omega_n \omega_{n-1} \ldots \omega_{n-m+1} \sigma^{n-m} \end{aligned}$$

and on the other hand:

$$\begin{aligned} X^m C_\varphi \sigma^n &= X^m k^n \sigma^n \\ &= k^n X^m \sigma^n \\ &= k^n \omega_n \omega_{n-1} \ldots \omega_{n-m+1} \sigma^{n-m}. \end{aligned}$$

Notice that $X = 0$ on the subset $\{\sigma^n, : \ n < m\}$, hence,

$$C_\varphi X^m = k^{-m} X^m C_\varphi,$$

which implies that $k^m \in \operatorname{Ext}(C_\varphi, \mathcal{H}^2(\beta))$ for all $m \leq 0$.

Now, we shall show that $k^m \in \operatorname{Ext}(C_\varphi, \mathcal{H}^2(\beta))$ for all $m \geq 0$. Again let us define the map $X\sigma^n = \omega_n \sigma^{n+1}$, which can be extended by linearity on a dense subset. We shall prove that X is a bounded linear operator on $\mathcal{H}^2(\beta)$ for a suitable weight sequence $\{\omega_n\}$. Indeed, let us define X as the operator:

$$X(z^n) = X((z - p + p)^n) = \sum_{k=0}^{n} \binom{n}{k} p^{n-k} \omega_k (z-p)^{k+1}.$$

By setting $\omega_k = \frac{s^k}{\|(z-p)^{k+1}\|_\beta}$, for some $s > 0$ such that $|p| + s < 1$ we get:

$$\|X(z^n)\|_\beta \leq (|p| + s)^n,$$

which implies that for any $f \in \mathcal{H}^2(\beta)$

$$\begin{aligned} \|Xf\|_\beta &\leq \sum_{n=0}^{\infty} |a_n| \|X(z^n)\|_\beta \\ &= \left(\sum_{n=0}^{\infty} \frac{(|p|+s)^{2n}}{\beta_n^2} \right)^{1/2} \left(\sum_{n=0}^{\infty} |a_n|^2 \beta_n^2 \right)^{1/2} = C \|f\|_\beta \end{aligned}$$

which proves the boundedness of X on $\mathcal{H}^2(\beta)$.

Doing similarly the same process as before, it can be checked that $C_\varphi X^m \sigma^n = k^m X^m C_\varphi \sigma^n$ for $n \geq 1$. Hence, we have $k^m \in \operatorname{Ext}(C_\varphi, \mathcal{H}^2(\beta))$ for $m \geq 0$. Finally, we conclude that $\operatorname{Ext}(C_\varphi, \mathcal{H}^2(\beta)) = \{\varphi'(0)^n \ : n \in \mathbb{Z}\}$, as we wanted. $\square$

Remark 4.3.2. *We point out that it is unusual to obtain results on boundedness of* C_φ *without a restriction on the weight sequence* (β_n) *(see [56]). On the other hand, we remark that when the map has another fixed point configuration, the problem becomes a bit intractable. The existing results in the Hardy space depend on the good characterization of the Toeplitz operators in such spaces, which are not so nice for weighted Hardy spaces. Moreover, for some maps such as the parabolic nonautomorphic case, the proof in Hardy's space [54] is based on some specific results which are only true for the Hardy space. Namely, such an operator is similar to a multiplication operator in the space of Sobolev. Finally let us see that the proof of Theorem 4.3.1 uses the algebraic form of the Köenigs map. If* $\mathcal{H}^2(\beta)$ *is not automorphism invariant we can not apply the same trick.*

Chapter 5

An Unbounded Version of the Fuglede Theorem Related to Normal Operators

The Fuglede Theorem plays an important role when studying normal operators. The main application of this theorem is the fact that it weakens some assumptions in the statement of the Spectral Theorem for normal operators. Related to this theorem, M. Meziane and M.H. Mortad proposed the following conjecture in [64]:

Conjecture 5.0.1. *Let A be an operator (densely defined and closed if necessary) and let $B \in B(H)$ be normal. Then*

$$BA \subset AB^* \Longrightarrow B^*A \subset AB.$$

In this chapter, we will take a closer look at this conjecture. Specifically, in Section 5.1, we show that Conjecture 5.0.1 holds true in the case B has a finite pure point spectrum (see Theorem 5.1.1). Then, in Section 5.2, we show that the conjecture is not true even when we assume that A is self-adjoint and B is unitary (see Proposition 5.2.1). Finally, we provide a pair of a closed and self-adjoint unbounded operators which is not intertwined by any bounded or closed operator except the zero operator.

5.1 On a New Version of the Fuglede Theorem for Unbounded Operators

In this setion, we prove that the conjecture 5.0.1 holds true in case B has a finite pure point spectrum.

Theorem 5.1.1. *Let B be a bounded normal operator with a finite point spectrum and let A be an unbounded operator on a complex Hilbert space H. Let $f, g : \mathbb{C} \to \mathbb{C}$ be two functions. Then*

$$BA \subset Af(B) \Longrightarrow g(B)A \subset A(g \circ f)(B).$$

Proof. Using induction, we verify that $B^m A \subset A(f(B))^m$ for all $m \geq 0$. Therefore,

$$p(B)A \subset A(p \circ f)(B)$$

for any polynomial $p \in \mathbb{C}(X)$.

By assumption, B has a point spectrum with finitely many distinct eigenvalues $\lambda_j, j \in \{1, \cdots, n\}$, and corresponding eigenprojectors E_j adding up to the identity operator I, so $B = \sum_{j=1}^n \lambda_j E_j$ is the spectral representation of B. Now, using the Lagrange interpolation theorem, we find a polynomial p such that $p(\lambda_j) = g(\lambda_j)$ and $p(f(\lambda_j)) = g(f(\lambda_j))$ for all j. From $p(B)A \subset A(p \circ f)(B)$, we obtain

$$g(B)A = p(B)A \subset A(p \circ f)(B) = A(g \circ f)(B).$$

□

Corollary 5.1.2. *With A and B as in Theorem 5.1.1, we have*

$$BA \subset AB^* \Longrightarrow B^*A \subset AB.$$

Proof. It suffices to apply Theorem 5.1.1 to the functions $f, g : z \mapsto \overline{z}$, so that $g \circ f$ becomes the identity map on $\mathbb{C}$. □

A similar reasoning applies to establish the following consequence:

Corollary 5.1.3. *With A and B as in Theorem 5.1.1, we likewise have*

$$BA \subset AB \Longrightarrow B^*A \subset AB^*.$$

Using an idea by Berberian, we may generalize this result to the case of two normal operators whereby we obtain a Fuglede-Putnam style theorem.

Proposition 5.1.4. *Let B and C be bounded normal operators with a finite point spectrum and let A be an unbounded operator on a complex Hilbert space H. Then*

$$BA \subset AC \Longrightarrow B^*A \subset AC^*.$$

Proof. Define $\tilde{B}$ on $H \oplus H$ by $\tilde{B} = \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix}$ and let $\tilde{A} = \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}$ with $D(\tilde{A}) = H \oplus D(A)$. Since $BA \subset AC$, it follows that $\tilde{B}\tilde{A} \subset \tilde{A}\tilde{B}$ for $D(\tilde{B}\tilde{A}) = H \oplus D(A) \subset H \oplus D(AC) = D(\tilde{A}\tilde{B})$. Now, since B and C are normal, so is $\tilde{B}$. Finally, apply Corollary 5.1.3 to the pair $(\tilde{B}, \tilde{A})$ to get $\tilde{B}^*\tilde{A} \subset \tilde{A}\tilde{B}^*$ which, upon examining their entries, yields the required result. □

Corollary 5.1.5. *Let B and C be bounded normal operators with a finite point spectrum and let A be a densely defined operator on a complex Hilbert space H. Then*

$$BA \subset AC \Longrightarrow CA^* \subset A^*B.$$

Proof. Merely use the foregoing result, then take adjoints. □

5.2 Intertwining Relations and Counterexamples

One may wonder whether $BA \subset AB^*$ implies $B^*A \subset AB$ in the events of the self-adjointness of A and the normality of $B \in B(H)$? The next example says that this is untrue, thus providing a counterexample to Conjecture 5.0.1.

Proposition 5.2.1. *There is a unitary $B \in B(H)$ and a self-adjoint A with domain $D(A) \subset H$ such that $BA \subset AB^*$ but $B^*A \not\subset AB$.*

The proof is based on the following example (which appeared in [36]):

Example 5.2.2. There exists a unitary $U \in B(H)$ and a closed and symmetric T with domain $D(T) \subset H$ such that $UT \subset TU$ but $U^*T \not\subset TU^*$.

Now, we prove Proposition 5.2.1.

Proof. Consider a unitary $U \in B(H)$ and a densely defined closed T such that $UT \subset TU$ and $U^*T \not\subset TU^*$ (as in Example 5.2.2). Take

$$B = \begin{pmatrix} U & 0 \\ 0 & U^* \end{pmatrix} \text{ and } A = \begin{pmatrix} 0 & T \\ T^* & 0 \end{pmatrix}.$$

Then B is unitary on $B(H \oplus H)$ and A is self-adjoint with domain $D(T^*) \oplus D(T)$ (thanks to the closedness of T). Besides,

$$BA = \begin{pmatrix} 0 & UT \\ U^*T^* & 0 \end{pmatrix} \text{ and } AB^* = \begin{pmatrix} 0 & TU \\ T^*U^* & 0 \end{pmatrix}.$$

Since $UT \subset TU$, it follows by taking adjoints that $U^*T^* \subset T^*U^*$ making $BA \subset AB^*$. Since $U^*T \not\subset TU^*$ is equivalent to $UT^* \not\subset T^*U$, we get that

$$B^*A \not\subset AB$$

as $D(B^*A) \not\subset D(AB)$. □

Fuglede found (in [37]) a densely defined closed operator which does not commute with any bounded operator except scalar ones (i.e. those of the form αI where $\alpha \in \mathbb{C}$). The next results lie within the same scope but are with a different aim and different assumptions.

The example we are about to give is based upon a newly obtained densely defined closed operator (see [28] and [74]) B with domain $D(B) \subset L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$ which obeys:

$$D(B^2) = D(B^{*2}) = \{0\}.$$

Recall that M. Naĭmark was the first one who found an example of a closed symmetric operator S with $D(S^2) = \{0\}$ (see [77]). P. R. Chernoff then found a simpler closed, unbounded, densely defined, symmetric and semi-bounded operator T such that $D(T^2) = \{0\}$ (see [23]). It is worth noticing that K. Schmüdgen obtained in [91, 92] almost simultaneously with P. R. Chernoff that every unbounded self-adjoint operator possesses two closed symmetric restrictions S and T such that

$$D(S) \cap D(T) = \{0\} \text{ and } D(S^2) = D(T^2) = \{0\}.$$

To close this digression, notice that none of the counterexamples by P. R. Chernoff and K. Schmüdgen are helpful in the second part of the proof. Indeed, while these examples are quite strong they do not entail $D\left(T^{*2}\right) = \{0\}$ for the simple reason that if T is symmetric (and densely defined), i.e. $T \subset T^*$, then $T^*T \subset T^{*2}$. By the closedness of T, it follows that T^{*2} must be densely defined (as T^*T is self-adjoint, and in particular densely defined).

Example 5.2.3. There are a self-adjoint operator A and a densely defined closed operator B which are not intertwined by any (everywhere defined) bounded operator except the zero operator. Also, the same pair A and B in the opposite order cannot be intertwined either by any bounded operator except the zero operator.

Let $H = L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$. Let A be any unbounded self-adjoint operator with domain $D(A) \subset H$ and B be a densely defined closed operator such that

$$D(B^2) = D(B^{*2}) = \{0\}.$$

Let $T \in B(H)$. Then, clearly

$$TA \subset BT \Longrightarrow TA^2 \subset BTA \subset B^2T.$$

Hence

$$D(A^2) = D(TA^2) \subset D(B^2T) = \{x \in H : Tx \in D(B^2) = \{0\}\} = \ker T.$$

Since A^2 is densely defined, it follows that $\ker T = H$, that is, $T = 0$, as required.

Now, we pass to the second statement. Let $S \in B(H)$. Then plainly

$$SB \subset AS \Longrightarrow S^*A \subset B^*S^*.$$

As before, we obtain

$$S^*A^2 \subset B^{*2}S^*.$$

Similar arguments as above then yield $S^* = 0$ or simply $S = 0$, as needed

In the preceding example and in the case of the operator T, we did not really need to work on $L^2(\mathbb{R})$. In fact, any closed operator B such that $D(B^2) = \{0\}$ will do. Thanks to Schmüdgen's construction, there are plenty of them. Having made this observation, we may state the following examples:

Example 5.2.4. There are a self-adjoint operator A and a densely defined closed symmetric operator B (with $B \subset A$) which are not intertwined by any (everywhere defined) bounded operator except the zero operator.
Indeed, take any unbounded self-adjoint operator A, then consider any of its two closed symmetric restrictions and denote it by B (with $D(B^2) = \{0\}$). Finally, consider $T \in B(H)$ such that $TA \subset BT$. Then obtain $T = 0$ as carried out above.

Now, we consider the case where all operators involved are closed.

Example 5.2.5. There are two densely defined closed operators A and B which are not intertwined by any densely defined closed operator apart from the zero operator. Indeed, let A be a densely defined closed operator with domain $D(A)$ such that $A^2 = 0$ on $D(A^2) = D(A)$, (an explicit example is to consider e.g.

$$A = \begin{pmatrix} 0 & C \\ 0 & 0 \end{pmatrix}$$

where C is any unbounded closed operator with domain $D(C)$). Now, let B be a closed operator satisfying $D(B^2) = \{0\}$. Clearly

$$TA \subset BT \Longrightarrow TA^2 \subset B^2T.$$

But

$$D(TA^2) = \{x \in D(A^2) : A^2x = 0 \in D(T)\} = D(A)$$

and $D(B^2T) = \ker T$. Since $D(A) \subset \ker T \subset H$, upon passing to the closure (w.r.t. H), it follows that $\ker T = H$ because $\ker T$ is closed for T is closed. Therefore, $Tx = 0$ for all $x \in D(T)$. Accordingly, $T = 0$ everywhere, as coveted.

5.3 An Open Question

Easy arguments allow us to show that $BA = AB^*$ does imply that $B^*A = AB$ when B is unitary and A is any (unbounded) operator. If we further assume that A is self-adjoint, then $BA = AB^*$ signifies that BA is self-adjoint and $B^*A = AB$ means that B^*A is self-adjoint. A similar problem is: If $B \in B(H)$ is normal and A is (unbounded) self-adjoint, then does one have

$$BA \text{ is self-adjoint} \Longleftrightarrow B^*A \text{ is self-adjoint?}$$

Recall that the previous question has a positive answer when $A \in B(H)$. However, this is untrue in the case of two unbounded operators. To see this, consider the following example (which appear in [69])

$$Af(x) = (1 + |x|)f(x) \text{ and } Bf(x) = -i(1 + |x|)f'(x)$$

on their respective domains

$$D(A) = \{f \in L^2(\mathbb{R}) : (1 + |x|)f \in L^2(\mathbb{R})\}$$

and

$$D(B) = \{f \in L^2(\mathbb{R}) : (1 + |x|)f' \in L^2(\mathbb{R})\}$$

where the derivative is taken in the sense of distributions. It is known that B is normal on $D(B)$ and that its adjoint is given by:

$$B^*f(x) = -\mathrm{isgn}(x)f(x) - \mathrm{i}(1 + |x|)f'(x).$$

From [69], we know that $AB^* = BA$. Since $A^{-1} \in B(L^2(\mathbb{R}))$, it follows that $(BA)^* = AB^*$ making BA self-adjoint. However, B^*A is not self-adjoint as we do not even have $AB \subset B^*A$ (as maybe checked again in [69]).

Going back to the main question, observe that if BA is closed, then B^*A is necessarily closed (and conversely). Indeed, the normality of B gives

$$\|B^*Ax\| = \|BAx\|, \ \forall x \in D(B^*A) = D(BA) = D(A).$$

Hence, the graph norms of B^*A and BA coincide and hence the closedness of one implies the closedness of the other, and that's the best we have obtained so far.

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