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Reductions and Conservation Laws of a Generalized Third-Order PDE via Multi-Reduction Method

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Abstract: In this work, we consider a family of nonlinear third-order evolution equations, where two arbitrary functions depending on the dependent variable appear. Evolution equations of this type model several real-world phenomena, such as diffusion, convection, or dispersion processes, only to cite a few. By using the multiplier method, we compute conservation laws. Looking for traveling waves solutions, all the the conservation laws that are invariant under translation symmetries are directly obtained. Moreover, each of them will be inherited by the corresponding traveling wave ODEs, and a set of first integrals are obtained, allowing to reduce the nonlinear third-order evolution equations under consideration into a first-order autonomous equation.

Keywords: third-order partial differential equations; conservation laws; multi-reduction method; partial differential equations

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1. Introduction

Over the last years, the analysis of integrable equations to understand phenomena in the real world has drawn the attention of a significant part of the research community. In [1], Qiao and Liu considered the third-order partial differential equation (PDE) given by

$$u_t = \frac{1}{2}\left(\frac{1}{u^2}\right)_{xxx} - \frac{1}{2}\left(\frac{1}{u^2}\right)_x. \tag{1}$$

The authors showed that Equation (1) is completely integrable and its corresponding Lax pair and bi-Hamiltonian structure were stated. In [2], the authors considered the generalized equation

$$u_t = (g(u))_{xxx} + (f(u))_x. \tag{2}$$

In the following $u : I_x \times I_t \to \mathbb{R}$, with I_x and I_t suitable intervals in $\mathbb{R}$, is an analytic function of the spatial coordinate x and the time coordinate t, whereas $f, g : D \subseteq \mathbb{R} \to \mathbb{R}$ are arbitrary analytic functions of the dependent variable u.

They determined the subclasses of family (2) which are self-adjoint and quasi self-adjoint. Then, by using Ibragimov's formula [3], they found conservation laws for some equations belonging to class (2). Nevertheless, as we will explain in detail hereafter, Ibragimov's method does not guarantee that all conservation laws will be obtained, which motivates the present study.

Conservation laws present themselves in several branches of science. They allow to find the solution of problems which involve physical characteristics that remain unchanged over time in an isolated physical system. Conservation laws, which are physically important, possess conserved densities and fluxes with low differential orders, whereas

those involving higher order derivatives are usually related to integrability. Furthermore, the accuracy, existence, uniqueness and stability of numerical solutions of PDEs as well as the construction of exact solutions can be investigated through the use of conservation laws.

A conservation law for PDE (2) is a space–time divergence expression $D_t T + D_x X = 0$ which holds on all solutions of the given PDE, where T represents the conserved density and X represents the spatial flux. Both T and X are functions of t, x, and u, and partial derivatives of u. The pair (T, X) is called a conserved current, whereas D_t and D_x are total derivatives.

There are different methods to obtain conservation laws. For variational problems, Noether's theorem is the best known and important result [4]. This theorem establishes a one-to-one correspondence between each differentiable symmetry and the conservation laws of its Euler–Lagrange equations.

Nevertheless, several differential equations (DEs) with interesting physical and mathematical applications are not variational systems. This has motivated a great number of works over the last decades devoted to generalize Noether's theorem to non-variational DEs [5–13].

Noether's theorem is completely generalized by a direct method based on adjoint symmetries, whose theoretical basis framework was established in [14,15] and developed in a systematic way, including algorithms for the determination of conservation laws, in [5,16–18]. The direct method can be applied to any PDE and reduce the issue of constructing conservation laws to find sets of local multipliers.

A local multiplier is a function depending on the independent variables, dependent variables and at most a finite number of derivatives of the dependent variable of the considered PDE, in a manner that a divergence expression is obtained for any dependent variable, not only solutions of the given PDE, after multiplying the given PDE by the local multiplier.

In general, the set of multiplier determining equations consists of the adjoint of the symmetry determining equations to which further equations similar to Helmholtz conditions are added. Moreover, the set of multiplier determining equations can be solved analogously to the usual procedure for solving the set of symmetry determining equations. Each set of local multipliers yields local conservation laws of the considered PDE. Consequently, all local conservation laws admitted by a given PDE can be determined systematically, without requiring any extra restriction or the use of a specified ansatz. More importantly, conservation laws can be obtained by using an homotopy integral formula or by direct integration of the characteristic equation [15,17–20]. The situation for variational systems is especially interesting since the determining system for multipliers includes the determining system for symmetries. Therefore, each set of local multipliers yields a local symmetry, in evolutionary form, of the considered PDE. The general method given in [5,16–18] was extensively revised and further developed in [21]. On the other hand, in [22,23], the symmetry properties of conservation laws of PDEs were examined through the use of the multiplier method.

In the last few years, Ibragimov [3] promoted a conservation law formula which does not need the existence of a classical Lagrangian. This formula is based on the notion of adjoint equation for nonlinear DEs. Similarly, the concepts of self-adjoint, quasi self-adjoint, weak self-adjoint and nonlinear self-adjoint DEs were developed [24–26]. However, in [22], the equivalence between the formula given by Ibragimov for constructing conservation laws and a standard formula for the action of an infinitesimal symmetry on a conservation law multiplier was demonstrated. Furthermore, it was recently proved that Ibragimov's formula can yield trivial conservation laws [27]. Most importantly, this formula does not necessarily ensure that all nontrivial conservation laws are obtained, except if the action of the symmetry on the admitted conservation laws is transitive, a property which cannot be checked prior to determining all admitted conservation laws.

One of the most effective applications of symmetries is the construction of group-invariant solutions of nonlinear PDEs, i.e., interesting classes of solutions invariant under

a specific symmetry group which verify a reduced DE system with a fewer number of independent variables. However, in order to obtain the desired group-invariant solutions in an explicit form, one must solve this reduced DE system. For that purpose, it is necessary to obtain sufficient reductions of order using first integrals, which lead to a quadrature. A different approach arises when taking into account the relationship between symmetries and conservation laws. In [28–30], a double reduction method which can be applied to non-variational PDEs was proved. This method consists in finding a symmetry which leaves the conserved current for a local conservation law of the considered PDE invariant. In the case of a PDE with two independent variables, the invariance under the symmetry reduces the given PDE to an ordinary differential equation (ODE) and therefore, the subsequent reduction of the conserved current leads to a first integral of this ODE.

In [31], the double reduction method proposed in [28–30] was naturally extended to PDEs with any number of independent and dependent variables. Here, the reduced PDE involves one less independent variable, whereas the reduction of the corresponding conserved current leads to a conservation law of the reduced PDE.

In [32], an alternative and improved generalization of the double reduction method, called multi-reduction method, for PDEs in $k \geq 2$-independent variables which admit a symmetry algebra of dimension at least $k - 1$ was presented. This general method of symmetry reduction of PDEs with conservation laws provides an explicit algorithm to determine all symmetry-invariant conservation laws that will reduce to first integrals for the ODE that describes the symmetry-invariant solutions of the PDE.

In [33] a third-order PDE was analyzed from the point of view of point symmetries and reductions. In particular, the three- and four-dimensional solvable symmetry algebras admitted by the considered family depending on its arbitrary functions were determined. In this way, one is able to reduce the considered PDE into a first-order ODE. Nevertheless, this ODE does not admit an obvious quadrature. Therefore, by using solvable symmetry algebras, the reduction of the considered PDE to the quadrature can be only obtained for special forms of the functions involved. It turns out that these cases are included in family (2). In this paper, we show that taking into account symmetry-invariant conservation laws, a triple reduction in PDE (2) is obtained. Most importantly, the reduced ODE is separable, yielding a straightforward quadrature which gives the general traveling wave solution of PDE (2) for arbitrary $f(u)$ and $g(u)$. Thus, the results obtained in this paper on traveling wave solutions generalize those included in [33].

The goal of this paper is to study PDE (2) from the point of view of conservation laws in order to apply the multi-reduction method. In Section 2, the direct method presented in [5,16–18] is successfully applied to the generalized third-order PDE (2) and a complete classification of zero-order multipliers admitted by family (2) depending on not constant functions $f(u)$ and $g(u)$, with $f(u)$ and $g(u)$ both being non-linear, is determined. Furthermore, taking into account the zero-order multipliers obtained, we construct conservation laws of family (2). In Section 3, we determine those zero-order multipliers admitted by PDE (2) which are invariant under a traveling wave translation with arbitrary constant wave speed. From the corresponding invariant conservation laws, we obtain two functionally independent first integrals which are combined to yield a first-order autonomous equation whose general solution can be determined in implicit form. Therefore, the general solution of PDE (2) starting from a traveling wave reduction is found. Finally, in Section 4, we present the conclusions.

2. Conservation Laws

We recall same relevant elements on conservation laws; for more details, see, for example, [15,17–19].

Let $\mathfrak{E}$ be a PDE of order n, in the unknown function $u(x, t)$, with x and t being the spatial and the time coordinates, respectively. A conservation law is an expression

$$D_t T(t, x, u, u_t, u_x, \ldots) + D_x X(t, x, u, u_t, u_x, \ldots) = 0, \tag{3}$$

true for all solutions $u(x,t)$ of equation $\mathfrak{E}$. Here, T represents the conserved density and X the associated flux, while D_t and D_x given by

$$
\begin{aligned}
D_t &= \partial_t + u_t \partial_u + u_{tx} \partial_{u_x} + u_{tt} \partial_{u_t} + \dots, \\
D_x &= \partial_x + u_x \partial_u + u_{xx} \partial_{u_x} + u_{xt} \partial_{u_t} + \dots,
\end{aligned}
$$

denote the total derivatives with respect to t and x respectively.

A conserved current (T, X) is called trivial if the conservation law (3) of PDE $\mathfrak{E}$ holds for any smooth function $u(x,t)$.

Two conserved currents (T_1, X_1) and (T_2, X_2) are called equivalent if they differ by a trivial conserved current; we will use this in order to simplify the conserved currents that we construct in this section. Then, in the following, a conservation law of $\mathfrak{E}$ is an equivalence class of conservation laws of $\mathfrak{E}$.

In [18], it is proved that when an equation admits a Cauchy–Kovalevskaya form, every nontrivial (up to equivalence) local conservation law emerges from multipliers; moreover, there is a one-to-one correspondence between equivalence classes of conservation laws and multipliers, which do not depend on u_t and its differential consequences.

As the Equation (2) admits a Cauchy–Kovalevskaya form with respect to the independent variable t, this implies that for each multiplier Q there exists a nontrivial (up to equivalence) local conservation law (that is a conserved current (T, X)) such that the equation

$$
D_t T + D_x X = (u_t - (g(u))_{xxx} - (f(u))_x)Q, \tag{4}
$$

is true for all functions $u(t,x)$ (not just solutions of PDE (2)). Then multipliers Q are obtained by requiring that the divergence condition must be satisfied

$$
\frac{\delta}{\delta u}((u_t - (g(u))_{xxx} - (f(u))_x)Q) = 0, \tag{5}
$$

where $\frac{\delta}{\delta u} = \partial_u - D_x \partial_{u_x} - D_t \partial_{u_t} + D_x^2 \partial_{u_{xx}} + D_x D_t \partial_{u_{xt}} + D_t^2 \partial_{u_{tt}} + \dots$, denotes the Euler operator.

We consider the zero-order multipliers admitted by PDE (2) which have the form

$$
Q(x, t, u). \tag{6}
$$

Here, we consider $Q : I_x \times I_t \times \mathbb{R} \to \mathbb{R}$ an analytic function.

The corresponding Equation (5) splits into an overdetermined system linear in the unknown Q but that involves two arbitrary functions f and g of u. By solving these determining equations, and taking into account that f and g cannot be both linear functions, we obtain the following result.

Proposition 1. *The low-order multipliers of differential order zero (6) admitted by the generalized third-order PDE (2) with $f(u)$ and $g(u)$ not being linear functions simultaneously, are:*

$$
Q_1 = 1, \tag{7}
$$
$$
Q_2 = g(u). \tag{8}
$$

Additional multipliers are admitted in the following cases:

1. *If $f(u)$ and $g(u)$ satisfy the relation $g(u) = g_0 f'(u) + g_1$, then the additional multiplier is*

$$
Q_3 = x + f'(u)t. \tag{9}
$$

2. If $f(u)$ and $g(u)$ satisfy the relation $g(u) = -g_0 f(u) + g_1 u + g_2$, with $g_0 > 0$, then the additional multipliers are

$$Q_4 = \exp\left(\frac{g_0 x + g_1 t}{g_0^{3/2}}\right), \tag{10}$$

$$Q_5 = \exp\left(-\frac{g_0 x + g_1 t}{g_0^{3/2}}\right). \tag{11}$$

3. If $f(u)$ and $g(u)$ satisfy the relation $g(u) = g_0 f(u) + g_1 u + g_2$, with $g_0 > 0$, then the additional multipliers are

$$Q_6 = \sin\left(\frac{g_1 t - g_0 x}{g_0^{3/2}}\right), \tag{12}$$

$$Q_7 = \cos\left(\frac{g_1 t - g_0 x}{g_0^{3/2}}\right). \tag{13}$$

4. If $f(u) = f_0 e^{qu} + f_1 u + f_2$ and $g(u) = \tilde{g}_0 e^{qu} + \tilde{g}_1$, with $\frac{\tilde{g}_0}{f_0} < 0$, then the additional multipliers are Q_3, $Q_4\big|_{g_0=-\frac{\tilde{g}_0}{f_0}, g_1=-\frac{f_1 \tilde{g}_0}{f_0}}$ and $Q_5\big|_{g_0=-\frac{\tilde{g}_0}{f_0}, g_1=-\frac{f_1 \tilde{g}_0}{f_0}}$.

5. If $f(u) = f_0 e^{qu} + f_1 u + f_2$ and $g(u) = \tilde{g}_0 e^{qu} + \tilde{g}_1$, with $\frac{\tilde{g}_0}{f_0} > 0$, then the additional multipliers are Q_3, $Q_6\big|_{g_0=\frac{\tilde{g}_0}{f_0}, g_1=-\frac{f_1 \tilde{g}_0}{f_0}}$ and $Q_7\big|_{g_0=\frac{\tilde{g}_0}{f_0}, g_1=-\frac{f_1 \tilde{g}_0}{f_0}}$.

6. If $f(u) = f_0 u + f_1$, then the additional multipliers are Q_3 and

$$Q_8 = (x + f_0 t)^2. \tag{14}$$

7. If $f(u) = f_0 u + f_1$ and $g(u) = g_1 + g_0 e^{qu}$, then the additional multipliers are Q_3, Q_8 and

$$Q_9 = (x + f_0 t)^3 + 6q g_0 t e^{qu}. \tag{15}$$

In the above, $q \neq 0$, $f_0 \neq 0$, f_1, f_2, $g_0 \neq 0$, g_1, g_2, $\tilde{g}_0 \neq 0$ and $\tilde{g}_1$ are real arbitrary constants.

Proof. Divergence condition (5) in the unknown function $Q(x, t, u)$ leads to the following linear system of six determining equations

$$\begin{gathered}
g' Q_{xu} = 0, \quad g' Q_{xxu} = 0, \quad g' Q_{xuu} = 0, \\
g' Q_{uu} - g'' Q_u = 0, \\
f' Q_x - Q_t + g' Q_{xxx} = 0, \\
g' Q_{uuu} - g''' Q_u = 0.
\end{gathered} \tag{16}$$

Taking into account that $g'(u) \neq 0$, from the fourth equation of system (16), we obtain

$$Q(x, t, u) = p(x, t) + r(x, t) g(u), \tag{17}$$

where $p, r : I_x \times I_t \to \mathbb{R}$ are smooth functions which must be determined. Substituting the form of the multiplier (17) in the remaining equations, we obtain that $r(x, t) = r(t)$ and p, r, f and g are related by the following equation:

$$p_{xxx} g' + p_x f' - r_t g - p_t = 0. \tag{18}$$

We solve Equation (18) depending on functions f and g, which leads to different solutions for functions p and r, and consequently for solution Q. Moreover, every multiplier Q may be written in the form $\sum_{i=1}^{k} c_i Q_i$, with c_i being arbitrary real constants for some integer

k, where $\{Q_1, \ldots, Q_k\}$ represents a basis for the set of all zero-order multipliers admitted by PDE (2) depending on the form of f and g. This leads to the different cases listed in the present proposition. $\square$

Each zero-order multiplier Q_i, $i = 1, \ldots, 9$, given in Proposition 1, yields a corresponding conserved density T_i and flux X_i. The conserved current (T_i, X_i) can be constructed by integrating the characteristic Equation (4) or by using an homotopy formula [15,17–20]. The following result is obtained.

Theorem 1. *The conservation laws associated with the zero-order multipliers given in Proposition 1 which are admitted by the generalized third-order PDE (2) for arbitrary $f(u)$ and $g(u)$, not being linear functions simultaneously, are given by*

$$
\begin{aligned}
T_1 &= u, \\
X_1 &= -(g(u))_{xx} - f(u).
\end{aligned}
\tag{19}
$$

$$
\begin{aligned}
T_2 &= \int g(u)\, du, \\
X_2 &= -g(u)\left(g'(u)u_{xx} + g''(u)u_x^2\right) + \tfrac{1}{2}g'(u)^2 u_x^2 - \int g(u)f'(u)\, du.
\end{aligned}
\tag{20}
$$

The generalized third-order PDE (2) admits additional conservation laws associated with the zero-order multipliers given in Proposition 1 in the following cases:

- *If $f(u)$ and $g(u)$ satisfy the relation $g(u) = g_0 f'(u) + g_1$, then the additional conservation law admitted is*

$$
\begin{aligned}
T_3 &= xu + tf(u), \\
X_3 &= -g_0(tf'(u) + x)\left(f''(u)u_{xx} + f'''(u)u_x^2\right) \\
&\quad + g_0 f''(u)^2 u_x\left(1 + \frac{t}{2}f''(u)u_x\right) - \int f'(u)(tf'(u) + x)\, du.
\end{aligned}
\tag{21}
$$

- *If $f(u)$ and $g(u)$ satisfy the relation $g(u) = -g_0 f(u) + g_1 u + g_2$, with $g_0 > 0$, then the additional conservation laws admitted are*

$$
\begin{aligned}
T_4 &= \exp\left(\frac{g_0 x + g_1 t}{g_0^{3/2}}\right)u, \\
X_4 &= \exp\left(\frac{g_0 x + g_1 t}{g_0^{3/2}}\right)\left((g_0 f'(u) - g_1)\left(u_{xx} - \frac{1}{\sqrt{g_0}}u_x\right)\right. \\
&\quad \left. + g_0 f''(u)u_x^2 - \frac{g_1}{g_0}u\right).
\end{aligned}
\tag{22}
$$

$$
\begin{aligned}
T_5 &= \exp\left(-\frac{g_0 x + g_1 t}{g_0^{3/2}}\right)u, \\
X_5 &= \exp\left(-\frac{g_0 x + g_1 t}{g_0^{3/2}}\right)\left((g_0 f'(u) - g_1)\left(u_{xx} + \frac{1}{\sqrt{g_0}}u_x\right)\right. \\
&\quad \left. + g_0 f''(u)u_x^2 - \frac{g_1}{g_0}u\right).
\end{aligned}
\tag{23}
$$

- If $f(u)$ and $g(u)$ satisfy the relation $g(u) = g_0 f(u) + g_1 u + g_2$, with $g_0 > 0$, then the additional conservation laws admitted are

$$
\begin{aligned}
T_6 &= -\sin\left(\frac{g_1 t - g_0 x}{g_0^{3/2}}\right) u, \\
X_6 &= (g_0 f'(u) + g_1)\left(\sin\left(\frac{g_1 t - g_0 x}{g_0^{3/2}}\right) u_{xx} + \frac{1}{\sqrt{g_0}}\cos\left(\frac{g_1 t - g_0 x}{g_0^{3/2}}\right) u_x\right) \\
&\quad + g_0 \sin\left(\frac{g_1 t - g_0 x}{g_0^{3/2}}\right) f''(u) u_x^2 - \frac{g_1}{g_0}\sin\left(\frac{g_1 t - g_0 x}{g_0^{3/2}}\right) u.
\end{aligned}
\tag{24}
$$

$$
\begin{aligned}
T_7 &= -\cos\left(\frac{g_1 t - g_0 x}{g_0^{3/2}}\right) u, \\
X_7 &= (g_0 f'(u) + g_1)\left(\cos\left(\frac{g_1 t - g_0 x}{g_0^{3/2}}\right) u_{xx} - \frac{1}{\sqrt{g_0}}\sin\left(\frac{g_1 t - g_0 x}{g_0^{3/2}}\right) u_x\right) \\
&\quad + g_0 \cos\left(\frac{g_1 t - g_0 x}{g_0^{3/2}}\right) f''(u) u_x^2 - \frac{g_1}{g_0}\cos\left(\frac{g_1 t - g_0 x}{g_0^{3/2}}\right) u.
\end{aligned}
\tag{25}
$$

- If $f(u) = f_0 e^{qu} + f_1 u + f_2$ and $g(u) = \tilde{g}_0 e^{qu} + \tilde{g}_1$, with $\frac{\tilde{g}_0}{f_0} < 0$, then the additional conservation laws admitted are (T_3, X_3), $\left(T_4\big|_{g_0 = -\frac{\tilde{g}_0}{f_0}, g_1 = -\frac{f_1 \tilde{g}_0}{f_0}}, X_4\big|_{g_0 = -\frac{\tilde{g}_0}{f_0}, g_1 = -\frac{f_1 \tilde{g}_0}{f_0}}\right)$ and $\left(T_5\big|_{g_0 = -\frac{\tilde{g}_0}{f_0}, g_1 = -\frac{f_1 \tilde{g}_0}{f_0}}, X_5\big|_{g_0 = -\frac{\tilde{g}_0}{f_0}, g_1 = -\frac{f_1 \tilde{g}_0}{f_0}}\right)$.

- If $f(u) = f_0 e^{qu} + f_1 u + f_2$ and $g(u) = \tilde{g}_0 e^{qu} + \tilde{g}_1$, with $\frac{\tilde{g}_0}{f_0} < 0$, then the additional conservation laws admitted are (T_3, X_3), $\left(T_6\big|_{g_0 = \frac{\tilde{g}_0}{f_0}, g_1 = -\frac{f_1 \tilde{g}_0}{f_0}}, X_6\big|_{g_0 = \frac{\tilde{g}_0}{f_0}, g_1 = -\frac{f_1 \tilde{g}_0}{f_0}}\right)$ and $\left(T_7\big|_{g_0 = \frac{\tilde{g}_0}{f_0}, g_1 = -\frac{f_1 \tilde{g}_0}{f_0}}, X_7\big|_{g_0 = \frac{\tilde{g}_0}{f_0}, g_1 = -\frac{f_1 \tilde{g}_0}{f_0}}\right)$.

- If $f(u) = f_0 u + f_1$, then the additional conservation laws admitted are (T_3, X_3) and

$$
\begin{aligned}
T_8 &= -(x + f_0 t)^2 u, \\
X_8 &= 2g - 2(x + f_0 t)g' u_x + (x + f_0 t)^2 (g'' u_x^2 + g' u_{xx}).
\end{aligned}
\tag{26}
$$

- If $f(u) = f_0 u + f_1$ and $g(u) = g_1 + g_0 e^{qu}$, then the additional conservation laws admitted are (T_3, X_3), (T_8, X_8) and

$$
\begin{aligned}
T_9 &= \left(6 g_0 t e^{qu} + (x + f_0 t)^3\right) u, \\
X_9 &= g_0 e^{qu}\left(q(x + f_0 t)^3(u_{xx} + q u_x^2) - 3q(x + f_0 t)^2 u_x + 6(x + 2 f_0 t)\right) \\
&\quad + f_0(x + f_0 t)^3 u + 3 q^2 g_0^2 t e^{2qu}(2 u_{xx} + q u_x^2).
\end{aligned}
\tag{27}
$$

Here, we recall (T_i, X_i), $i = 1, \ldots, 9$, represents the conservation law obtained from the multiplier Q_i given in Proposition 1; $q \neq 0$, $f_0 \neq 0$, f_1, f_2, $g_0 \neq 0$, g_1, g_2, $\tilde{g}_0 \neq 0$ and $\tilde{g}_1$ are real arbitrary constants.

Proof. For each of the multipliers Q_i, $i = 1, \ldots, 9$, along with the form of the functions f and g, given in Proposition 1, the conserved currents (T_i, X_i) are obtained by applying integration by parts to the terms in $(u_t - (g(u))_{xxx} - (f(u))_x)Q$ in order to obtain a total time derivative $D_t T_i$ plus a total space derivative $D_x X_i$ [17]. Furthermore, since PDE (2) admits a Cauchy–Kovalevskaya form, there is a one-to-one correspondence between conserved currents (T_i, X_i) (up to equivalence) and multipliers Q_i. Therefore, the only conservation laws admitted by PDE (2) are linear combinations of the admitted conserved currents (T_i, X_i), depending on the form of functions f and g. Now we show a detail proof of the construction of conservation law (T_2, X_2). Let us consider $Q_2 = g(u)$ given in Proposition 1. The conserved current (T_2, X_2) can be derived in terms of the multiplier Q_2 by applying integration by parts to the terms in

$$
(u_t - (g(u))_{xxx} - (f(u))_x)Q_2,
\tag{28}
$$

to obtain a total time derivative $D_t T_2$ plus a total space derivative $D_x X_2$, which yields (T_2, X_2). Taking into account the coefficient of u_{xxx} in (28), we observe that

$$D_x(-gg'u_{xx}) = -gg'u_{xxx} - (g'^2 + gg'')u_x u_{xx}.$$

Consequently, expression (28) can be written as

$$gu_t + (g'^2 - 2gg'')u_x u_{xx} - gg'''u_x^3 - f'gu_x + D_x(-gg'u_{xx}).\tag{29}$$

Now, we take into account the coefficient of u_{xx} in (29) and observe that

$$D_x\left(\tfrac{1}{2}(g'^2 - 2gg'')u_x^2\right) = (g'^2 - 2gg'')u_x u_{xx} - gg'''u_x^3.$$

Therefore, expression (29) can be written as

$$gu_t - f'gu_x + D_x\left(\tfrac{1}{2}(g'^2 - 2gg'')u_x^2 - gg'u_{xx}\right).\tag{30}$$

We take into account the coefficient of u_x in (30) and observe that

$$D_x\left(-\int f'g\,du\right) = -f'gu_x.$$

Finally, the coefficient of u_t in (30) can be obtained by using $D_t(\int g\,du)$. Thus, we conclude that expression (30) can be written as

$$D_t\left(\int g\,du\right) + D_x\left(\tfrac{1}{2}(g'^2 - 2gg'')u_x^2 - gg'u_{xx} - \int f'g\,du\right),\tag{31}$$

which leads to the result presented in this theorem. The remaining conservation laws can be determined analogously. $\square$

3. Multi-Reduction Method

A great number of the interesting solutions of nonlinear PDEs are traveling wave solutions. In the last years, many direct methods were developed to obtain exact solutions of the reduced ODE by traveling wave reduction. The double reduction method proposed in [29,30] has been mostly applied to reduce a qth-order PDE with two independent variables, with conservation laws invariant under translations, to a $(q-1)$-order ODE. Recently, in [32], a new method was proposed. In this method, starting from a symmetry which is used for the reduction of a PDE, all the conservation laws that are invariant under this symmetry are directly obtained. Moreover, each of them is inherited by the reduced ODE and a set of first integrals are obtained, allowing further reductions in the ODE.

A traveling wave has the form

$$u(x,t) = U(x - ct)\tag{32}$$

where $c \neq 0$ is an arbitrary constant. Invariance of a PDE $G(t,x,u,u_t,u_x,\ldots) = 0$ under the translation symmetry

$$X = \partial_t + c\partial_x,\tag{33}$$

gives rise to traveling wave solutions, with

$$z = x - ct, \qquad u = U(z),\tag{34}$$

being the invariants.

Proposition 2. *The multipliers of the generalized third-order PDE (2) which are invariant under the translation symmetry (33), with c arbitrary constant, are Q_1 and Q_2, given respectively by (7) and (8).*

Proof. For each of the multipliers Q_i, $i = 1, \dots, 9$, the invariance condition with respect to the operator (33), $X(Q_i) = 0$, leads to the following constraint

$$Q_{it} + cQ_{ix} = 0. \tag{35}$$

It is straightforward to verify that the multipliers Q_1 and Q_2 satisfy condition (35). When $i = 3$, condition (35) is not verified when $f''(u) \neq 0$, while if f is linear, it is satisfied for a specific value of the constant c. Finally, for the multipliers Q_i, $i = 4, \dots, 7$, the constraint (35) is satisfied for different specific values of the constant c. Therefore, the only multipliers invariant under the translation symmetry (33), with c arbitrary constant, are Q_1 and Q_2. □

Proposition 3. *The conservation laws of the generalized Equation (2) which are invariant under the translation symmetry (33), with c arbitrary constant, are (T_1, X_1) and (T_2, X_2), given respectively by (20) and (21).*

Proof. The proof is immediately followed by taking into account that a conservation law is invariant under the translation symmetry (33) if, and only if, the corresponding multiplier is invariant with respect to the translation symmetry [32], and using the fact that from Proposition 2, only multipliers Q_1 and Q_2, given respectively by (7) and (8), are invariant under the translation symmetry (33) for c arbitrary constant. □

Substitution of the traveling wave expression (32) into PDE (2) yields a nonlinear third-order ODE

$$g'U''' + U'(3g''U'' + g'''(U')^2 + f' + c) = 0.$$

By using the two translation-invariant conservation laws, we obtain the following two first integrals

$$\Psi_1 := g'U'' + g''U'^2 + f + cU = C_1,$$

$$\Psi_2 := g(g'U'' + g''(U')^2) - \frac{1}{2}(g')^2(U')^2 + \int g(f' + c) \, du = C_2,$$

with C_1 and C_2 real arbitrary constants. By combining these first integrals, we have obtained a triple reduction in PDE (2) to a first-order autonomous equation

$$(U')^2 = -\frac{2}{(g')^2}\left(fg - \int g(f' + c) \, dU + cgU - C_1 g + C_2\right). \tag{36}$$

Equation (36) is of the form $U' = h(U)$ with

$$h(U) = \pm\sqrt{-\frac{2}{(g')^2}\left(fg - \int g(f' + c) \, dU + cgU - C_1 g + C_2\right)}.$$

Consequently, the general solution of ODE (36) is given in implicit form as

$$z = \int \frac{dU}{h(U)} + C_3,$$

where C_3 is a real arbitrary constant.

We recall that functions $f(u)$ and $g(u)$ cannot be both linear functions for obtaining local low-order conservation laws. However, conservation laws (T_1, X_1) and (T_2, X_2), which are invariant under the translation symmetry (33), are admitted by PDE (2) also

when $f(u)$ and $g(u)$ are both linear functions. Therefore, the above results are directly extended to all equations of class (2), with f and g being non-constant functions.

Consequently, we determined the general traveling wave solution of PDE (2) for arbitrary $f(u)$ and $g(u)$ by taking into account symmetry-invariant conservation laws, which yield a triple reduction in PDE (2) plus a final quadrature.

This result can be used to determine large families of solutions of PDE (2). For instance, by considering $C_1 = C_2 = 0$ in ODE (36), one can easily obtain exact solutions for numerous PDEs belonging to family (2), in particular, soliton solutions. Solitons are interesting solutions which exhibit both dispersive and nonlinear effects. They arise in several physical systems, e.g., shallow and deep water waves, tsunamis, optimal fiber signals or atmospheric waves, among others.

By way of example, let us consider $f(u) = -\dfrac{a}{n+1}u^{n+1} - \dfrac{b}{2n+1}u^{2n+1}$, $g(u) = -\mu u$, with a and b being not simultaneously zero, $\mu \neq 0$, $n \geq 1$, and arbitrary parameters, then PDE (2) becomes the generalized Gardner equation

$$u_t + au^n u_x + bu^{2n} u_x + \mu u_{xxx} = 0, \tag{37}$$

whose soliton solutions were previously determined in the existing literature by using different techniques. For $b = 0$ and $n = 1$, PDE (37) becomes the well-known Korteweg–De Vries equation. By considering $C_1 = C_2 = 0$, the separable first-order ODE (36) takes the form

$$(U')^2 = \frac{c}{\mu}U^2 - \frac{2a}{\mu(n+1)(n+2)}U^{n+2} - \frac{b}{\mu(n+1)(2n+1)}U^{2n+2}, \tag{38}$$

whose general solution is given by

$$U(z) = \left(\frac{c(n+1)(n+2)}{a\left(1 + \sqrt{1 + \frac{bc(n+1)(n+2)^2}{a^2(2n+1)}}\cosh\left(n\sqrt{\frac{c}{\mu}}(z+z_0)\right)\right)} \right)^{1/n}, \tag{39}$$

where z_0 is an arbitrary real constant. Undoing the change of variables (34), we finally obtain the analytical expression of the solitons admitted by PDE (37)

$$u(x,t) = \left(\frac{c(n+1)(n+2)}{a\left(1 + \sqrt{1 + \frac{bc(n+1)(n+2)^2}{a^2(2n+1)}}\cosh\left(n\sqrt{\frac{c}{\mu}}(x-ct+z_0)\right)\right)} \right)^{1/n}. \tag{40}$$

On the other hand, we consider $f(u) = -u^m$, $g(u) = -u^n$, then PDE (2) becomes the $K(m,n)$ equation [34]

$$u_t + (u^n)_{xxx} + (u^m)_x = 0. \tag{41}$$

Now, we obtain the same solutions for the particular cases of PDE (41) which were previously considered in Ref. [34] by using the different approach presented in this paper. For $m = 3$, $n = 2$ and $C_1 = C_2 = 0$, the separable first-order ODE (36) takes the form

$$(U')^2 = \frac{c}{3}U - \frac{1}{5}U^3, \tag{42}$$

whose general solution is given by

$$U(z) = -\frac{\sqrt[3]{6^2}}{3}\wp\left(\frac{\sqrt{15}\sqrt[3]{6}}{30}z + z_0, \frac{5\sqrt[3]{6^2}c}{3}, 0\right), \tag{43}$$

where $\wp(U; w_1, w_2)$ represents the Weierstrass elliptic function and z_0 is an arbitrary real constant. Undoing change of variables (34), we obtain

$$u(x,t) = -\frac{\sqrt[3]{6^2}}{3} \wp\left(\frac{\sqrt{15}\sqrt[3]{6}}{30}(x - ct) + z_0, \frac{5\sqrt[3]{6^2}c}{3}, 0\right).$$ (44)

For $m = 2$, $n = 3$ and $C_1 = C_2 = 0$, ODE (36) becomes

$$(U')^2 = \frac{c}{6} - \frac{2}{15}U,$$ (45)

whose general solution is given by

$$U(z) = -\frac{1}{30}z^2 + z_0 z - \frac{15z_0^2}{2} + \frac{5c}{4},$$ (46)

where z_0 is an arbitrary real constant. Undoing change of variables (34), we obtain

$$u(x,t) = -\frac{1}{30}(x - ct)^2 + z_0(x - ct) - \frac{15z_0^2}{2} + \frac{5c}{4}.$$ (47)

For $m = 2$, $n = 2$ and $C_1 = C_2 = 0$, ODE (36) becomes

$$(U')^2 = \frac{c}{3}U - \frac{1}{4}U^2,$$ (48)

whose general solution is given by

$$U(z) = \frac{2c}{3}\left(1 \pm \sin\left(\frac{z - z_0}{2}\right)\right),$$ (49)

with z_0 being an arbitrary real constant. Undoing the change of variables (34), we obtain

$$u(x,t) = \frac{2c}{3}\left(1 \pm \sin\left(\frac{x - ct - z_0}{2}\right)\right).$$ (50)

Finally, considering $m = n = 3$ and $C_1 = C_2 = 0$, ODE (36) becomes

$$(U')^2 = \frac{c}{6} - \frac{1}{9}U^2,$$ (51)

whose general solution is given by

$$U(z) = -\frac{\sqrt{6c}}{2}\sin\left(\frac{z - z_0}{3}\right),$$ (52)

with z_0 being an arbitrary real constant. Undoing the change of variables (34), we obtain

$$u(x,t) = -\frac{\sqrt{6c}}{2}\sin\left(\frac{x - ct - z_0}{3}\right).$$ (53)

On the other hand, in [33], taking into account three-dimensional solvable algebras, the general solution of PDE (2) starting from a traveling wave solution is obtained for some particular forms of f and g. However, in this section, we successfully obtain the general solution of PDE (2) starting from a traveling wave solution for arbitrary f and g, which certainly generalizes the above-mentioned results.

4. Conclusions

In the framework of the symmetry analysis, we studied a family of nonlinear third-order evolution equations, where two arbitrary functions depending on the dependent variable appear. This type of equation models several real-world phenomena, such as diffusion, convection, or dispersion processes. We are interested in the symmetry reduction in these PDEs with conservation laws by using the multi-reduction method. Then, firstly, by using the multipliers approach, we determined the conservation laws for this class. Of course, the number of conservation laws depends on the form of the arbitrary functions that appear in the PDEs, but we demonstrate that all equations of the family admit at least two conservation laws and additional conservation laws for special forms of the arbitrary functions. Both of the two conservation laws admitted by all equations of the class are invariant under translation symmetries as well as the third-order PDEs considered.

All the conservation laws that are invariant under translation symmetries were directly transformed to be inherited by the corresponding traveling-wave ODEs; then each of these conservation laws provided us a first integral of the reduced ODE. Finally, a combination of the two first integrals allowed us to reduce the nonlinear third-order evolution equations under consideration with two arbitrary functions of u into a first-order autonomous equation, whose general solution was obtained in implicit form.

These results generalize the results appearing in our recent paper [33], where, taking into account solvable algebras, we obtained a quadrature of PDE (2) only for some particular cases of f and g.

In future work, it is intended to determine the potential symmetries admitted by family (2) and use them to obtain new reductions and solutions.

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