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Characterizing affine $\mathcal{C}$-semigroups

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Abstract

Let $\mathcal{C} \subset \mathbb{N}^p$ be a finitely generated integer cone and $S \subset \mathcal{C}$ be an affine semigroup such that the real cones generated by $\mathcal{C}$ and by S are equal. The semigroup S is called $\mathcal{C}$-semigroup if $\mathcal{C} \setminus S$ is a finite set. In this paper, we characterize the $\mathcal{C}$-semigroups from their minimal generating sets, and we give an algorithm to check if S is a $\mathcal{C}$-semigroup and to compute its set of gaps. We also study the embedding dimension of $\mathcal{C}$-semigroups obtaining a lower bound for it, and introduce some families of $\mathcal{C}$-semigroups whose embedding dimension reaches our bound. In the last section, we present a method to obtain a decomposition of a $\mathcal{C}$-semigroup into irreducible $\mathcal{C}$-semigroups.

Keywords Affine semigroup · $\mathcal{C}$-semigroup · Embedding dimension · Gap of a semigroup · Generalized numerical semigroup · Irreducible semigroup

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1 Introduction

An affine semigroup $S \subset \mathbb{N}^p$ is called $\mathcal{C}_S$-semigroup if $\mathcal{C}_S \setminus S$ is a finite set where $\mathcal{C}_S \subset \mathbb{N}^p$ is the minimal integer cone containing it. These semigroups are a natural generalization of numerical semigroups, and several of their invariants can be generalized. For a given numerical semigroup G, it is well-known that $\mathbb{N} \setminus G$ is finite; in fact, $G \subset \mathbb{N}$ is a numerical semigroup if it is a submonoid of $\mathbb{N}$ and $\mathbb{N} \setminus G$ is finite (for topics related with numerical semigroups see [13] and the references therein). In general, it does not happen for affine semigroups.

$\mathcal{C}$-semigroups are introduced in [8], where the authors study several properties about them (for example, an extended Wilf's conjecture for $\mathcal{C}$-semigroups is given). These semigroups appear in different contexts: when the integer points in an infinite family of some homothetic convex bodies in $\mathbb{R}^p_{\geq}$ are considered (see, for instance, [9], [10] and the references therein), or when the non-negative integer solutions of some modular Diophantine inequality are studied (see [5]), et cetera. In case the cone $\mathcal{C}$ is $\mathbb{N}^p$, $\mathbb{N}^p$-semigroups are called generalized numerical semigroups and they were introduced in [6]. Recently, in [11] it is proved that the minimal free resolution of the associated algebra to any $\mathcal{C}$-semigroup has maximal projective dimension possible.

In this context, $\mathbb{N}^p$-semigroups are characterized in [3], but the general problem was opened, *given any affine semigroup S, how to detect if Sis or not a $\mathcal{C}_S$-semigroup?* The primary goal of this work is to determine the conditions that any affine semigroup given by its minimal set of generators has to verify to be a $\mathcal{C}_S$-semigroup. We solve this problem in Theorem 9, and in Algorithm 1 we provide a computational way to check it.

Another open problem is to compute the set of gaps of any $\mathcal{C}$-semigroup defined by its minimal generating set. We solve this problem by means of setting a finite subset of $\mathcal{C}$ containing all the gaps of a given $\mathcal{C}$-semigroup. Algorithm 2 computes the set of gaps of the given $\mathcal{C}$-semigroup.

In this paper, we also go in-depth to study the embedding dimension of $\mathcal{C}$-semigroups. In [8,Theorem 11], a lower bound of the embedding dimension of $\mathbb{N}^p$-semigroups is provided, and some families of $\mathbb{N}^p$-semigroups reaching this bound are given. Besides, in [8,Conjecture 12], it is proposed a conjecture about a lower bound for the embedding dimension of any $\mathcal{C}$-semigroup. In Sect. 5, we introduce a lower bound of the embedding dimension of any $\mathcal{C}$-semigroup, and some families of $\mathcal{C}$-semigroups whose embedding dimension is equal to this new bound.

An important problem in Semigroup Theory is to determine some decomposition of a semigroup into irreducible semigroups (for example, see [13,Chapter 3] for numerical semigroups, or its generalization for $\mathbb{N}^p$-semigroups in [2]). We propose an algorithm to compute a decomposition of any $\mathcal{C}$-semigroup into irreducible $\mathcal{C}$-semigroups.

The results of this work are illustrated with several examples. To this aim, we have used third-party software, such as Normaliz [4], and the libraries `CharacterizingAffineCSemigroup` and `Irreducible` [7] developed by the authors in Python [12].

The content of this work is organized as follows. Section 2 introduces the initial definitions and notations used throughout the paper, mainly related to finitely generated cones. In Sect. 3, a characterization of $\mathcal{C}$-semigroups is provided, and an algorithm to check if an affine semigroup is a $\mathcal{C}$-semigroup. Section 4 is devoted to give an algorithm to compute the set of gaps of a $\mathcal{C}$-semigroup. Section 5 makes a study of the minimal generating sets of $\mathcal{C}$-semigroups formulating explicitly a lower bound for their embedding dimensions. Finally, in Sect. 6 an algorithm for computing a decomposition of a $\mathcal{C}$-semigroup into irreducible $\mathcal{C}$-semigroups is presented.

2 Preliminaries

The sets of real numbers, rational numbers, integer numbers and the non-negative integer numbers are denoted by $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$ and $\mathbb{N}$, respectively. Given a subset A of $\mathbb{R}$, $A_{\geq}$ is the set of elements in A greater than or equal to zero. For any $n \in \mathbb{N}$, $[n]$ denotes the set $\{1, \ldots n\}$. Given an element x in $\mathbb{R}^n$, $||x||_1$ denotes the sum of the absolute value of its entries, that is, its 1-norm. In this paper we assume the set $\{\mathbf{e}_1, \ldots, \mathbf{e}_p\}$ is the canonical basis of $\mathbb{R}^p$.

For a non empty subset B of $\mathbb{R}^p_{\geq}$, we define the cone generated by B:

$$L(B) := \left\{ \sum_{i=1}^{n} \lambda_i \mathbf{b}_i \mid n \in \mathbb{N}, \{\mathbf{b}_1, \ldots, \mathbf{b}_n\} \subset B, \text{ and } \lambda_i \in \mathbb{R}_{\geq}, \forall i \in [n] \right\}.$$

Given a real cone $\mathcal{C} \subset \mathbb{R}^p_{\geq}$, it is well-known that $\mathcal{C} \cap \mathbb{N}^p$ is finitely generated if and only if there exists a rational point in each extremal ray of $\mathcal{C}$. Moreover, any subsemigroup of $\mathcal{C}$ is finitely generated if and only if there exists an element in the semigroup in each extremal ray of $\mathcal{C}$. A good monograph about rational cones and affine monoids is [1]. From now on, we assume that the integer cones considered in this work are finitely generated.

Definition 1 Given an integer cone $\mathcal{C} \subset \mathbb{N}^p$, an affine semigroup $S \subset \mathcal{C}$ is said to be a $\mathcal{C}$-semigroup if $\mathcal{C} \setminus S$ is a finite set. If the cone $\mathcal{C} = \mathbb{N}^p$, a $\mathcal{C}$-semigroup is called $\mathbb{N}^p$-semigroup.

Fix a finitely generated semigroup $S \subset \mathbb{N}^p$, we denote by $\mathcal{C}_S$ the integer cone $L(S) \cap \mathbb{N}^p$. Note that, if S is a $\mathcal{C}$-semigroup, the cone $\mathcal{C}$ is $\mathcal{C}_S$. Obviously, a unique cone corresponds to infinite different semigroups.

The cone $L(S)$ is a polyhedron and we denote by $\{h_1(x) = 0, \ldots, h_t(x) = 0\}$ the set of its supported hyperplanes. We suppose $L(S) = \{x \in \mathbb{R}^d_{\geq} \mid h_1(x) \geq 0, \ldots, h_t(x) \geq 0\}$. Unless otherwise stated, the considered coefficients of each $h_i(x)$ are integers and relatively primes.

Assume $L(S)$ has q extremal rays denoted by $\tau_1, \ldots, \tau_q$. Then, each τ_i is determined by the set of linear equations $H_i := \{h_{j_1^{(i)}}(x) = 0, \ldots, h_{j_{p-1}^{(i)}}(x) = 0\}$ where $J_i := \{j_1^{(i)} < \cdots < j_{p-1}^{(i)}\} \subset [t]$ is the index set of the supported hyperplanes containing τ_i. So, for each $i \in [q]$, there exists the minimal non-negative integer vector $\mathbf{a}_i$ such that $\tau_i = \{\lambda \mathbf{a}_i \mid \lambda \in \mathbb{R}_{\geq}\}$. The set $\{\mathbf{a}_1, \ldots, \mathbf{a}_q\}$ is a generating set of $L(S)$.

Note that a necessary condition for S to be a $\mathcal{C}_S$-semigroup is the set $\tau_i \cap (\mathcal{C}_S \setminus S)$ is finite for all $i \in [q]$.

From each extremal ray τ_i of $L(S)$, we define $\upsilon_i(\alpha)$ as the parallel line to τ_i given by the solutions of the linear equations $\bigcup_{j\in J_i}\{h_j(x) = \alpha_j\}$ where $\alpha = (\alpha_{j_1},\dots,\alpha_{j_{p-1}}) \in \mathbb{Z}^{p-1}$. For every integer point $P \in \mathbb{Z}^p$ and $i \in [q]$, there exists $\alpha \in \mathbb{Z}^{p-1}$ such that P belongs to $\upsilon_i(\alpha)$; if $P \in \mathcal{C}_S$, $\alpha \in \mathbb{N}^{p-1}$. We denote by $\Upsilon_i(P)$ the element $(h_{j_1^{(i)}}(P),\dots,h_{j_{p-1}^{(i)}}(P)) \in \mathbb{N}^{p-1}$ with $J_i = \{j_1^{(i)} < \cdots < j_{p-1}^{(i)}\}$, $P \in \mathcal{C}_S$ and $i \in [q]$. Note that for any $P \in \mathcal{C}_S$, $P \in \upsilon_i(\alpha)$ if and only if $\alpha = \Upsilon_i(P)$.

Since all the semigroups appearing in this work are finitely generated, from now on, we omit the term *affine* when affine semigroups are considered.

3 An algorithm to detect if a semigroup is a $\mathcal{C}$-semigroup

In this section, we study the conditions that a semigroup has to satisfy to be a $\mathcal{C}$-semigroup. This characterization depends on the minimal set of generators of the given semigroup.

Let $S \subset \mathbb{N}^p$ be the affine semigroup minimally generated by $\Lambda_S = \{\mathbf{s}_1,\dots,\mathbf{s}_q,\mathbf{s}_{q+1},\dots,\mathbf{s}_n\}$ and $\tau_1,\dots,\tau_q$ be the extremal rays of $L(S)$. Assume that for every $i \in [q]$, $\tau_i \cap (\mathcal{C}_S \setminus S)$ is finite and $\mathbf{s}_i$ is the minimum (respect to the natural order) element in Λ_S belonging to τ_i. We denote by $\mathbf{f}_i$ the maximal element in $\tau_i \cap (\mathcal{C}_S \setminus S)$ with respect to the natural order in $\mathbb{N}^p$. Recall that $\mathbf{a}_i$ is the minimal non-negative integer vector defining τ_i, and let $\mathbf{c}_i \in S$ be the element $\mathbf{f}_i + \mathbf{a}_i$. In case $\tau_i \cap (\mathcal{C}_S \setminus S) = \emptyset$, we fix $\mathbf{f}_i = -\mathbf{a}_i$. The elements $\mathbf{f}_i$ and $\mathbf{c}_i$ are a generalization on the semigroup $\tau_i \cap S$ of the concepts Frobenius number and conductor of a numerical semigroup; for numerical semigroups, the Frobenius number is the maximal natural number that is not in the semigroup, and the conductor is Frobenius number plus one (see [13,Chapter 1]). Hence, we call Frobenius element and conductor of the semigroup $\tau_i \cap S$ the elements $\mathbf{f}_i$ and $\mathbf{c}_i$, respectively. One easy but important property of S is for every $P \in S$, $P + \mathbf{c}_i + \lambda\mathbf{a}_i \in S$ for any $i \in [q]$ and $\lambda \in \mathbb{N}$.

Note that $\tau_i \cap \mathbb{N}^p$ is equal to $\{\lambda\mathbf{a}_i \mid \lambda \in \mathbb{N}\}$. So, there exists $S_i \subset \mathbb{N}$ such that $\tau_i \cap S = \{\lambda\mathbf{a}_i \mid \lambda \in S_i\}$. If we assume that $\tau_i \cap (\mathcal{C}_S \setminus S)$ is finite, it is easy to prove that S_i is a numerical semigroup.

Lemma 2 *The τ_i-semigroup $\tau_i \cap S$ is isomorphic to the semigroup $S_i = \{\lambda \in \mathbb{N} \mid \lambda\mathbf{a}_i \in S\}$. Moreover $\tau_i \cap (\mathcal{C}_S \setminus S)$ is finite if and only if S_i is a numerical semigroup.*

Proof Consider the isomorphism $\varphi : \tau_i \cap S \to S_i$ with $\varphi(\mathbf{w}):=\lambda$ such that $\mathbf{w} = \lambda\mathbf{a}_i$. The second statement holds since $\tau_i \cap (\mathcal{C}_S \setminus S) = \{\lambda\mathbf{a}_i \in \mathbb{N} \mid \lambda\mathbf{a}_i \notin S, \lambda \in \mathbb{N}\}$. □

Corollary 3 *Given the semigroup $\tau_i \cap S$, $\mathbf{f}_i$ is equal to $f\mathbf{a}_i$ and $\mathbf{c}_i = c\,\mathbf{a}_i$ where f and c are the Frobenius number and the conductor of the numerical semigroup S_i, respectively.*

To test whether $\tau_i \cap (\mathcal{C}_S \setminus S)$ is finite, the following result can be used.

Lemma 4 *Let $S \subset \mathbb{N}^p$ be a semigroup and τ be an extremal ray of $L(S)$ satisfying $\tau \cap \mathbb{N}^p = \{\lambda\mathbf{a} \mid \lambda \in \mathbb{N}\}$ with $\mathbf{a} \in \mathbb{N}^p$. Then, $\tau \cap (\mathcal{C}_S \setminus S)$ is finite if and only if* $\gcd(\{\lambda \mid \lambda\mathbf{a} \in \tau \cap \Lambda_S\}) = 1$.

Proof Assume that $\tau \cap (\mathcal{C}_S \setminus S)$ is finite and suppose that $\gcd(\{\lambda \mid \lambda \mathbf{a} \in \tau \cap \Lambda_S\}) = n \neq 1$. Hence, every element $\lambda \mathbf{a}$ with $\gcd(n, \lambda) = 1$ does not belong to S, and then $\tau \cap (\mathcal{C}_S \setminus S)$ is not finite.

Conversely if $\gcd(\{\lambda \mid \lambda \mathbf{a} \in \tau \cap \Lambda_S\}) = 1$, then the semigroup $S' = \{\lambda \in \mathbb{N} \mid \lambda \mathbf{a} \in S\}$ is a numerical semigroup. From the proof of Lemma 2, S' is isomorphic to $\tau \cap S$. Therefore, $\tau \cap (\mathcal{C}_S \setminus S)$ is finite. □

To introduce the announced characterization, we need to define some subsets of $L(S)$ and prove some of their properties. Associated to the integer cone $\mathcal{C}_S$, consider the sets $\mathcal{A}:=\{\sum_{i\in[q]} \lambda_i \mathbf{a}_i \mid 0 \le \lambda_i \le 1\} \cap \mathbb{N}^p$ and $\mathcal{D}:=\{\sum_{i\in[q]} \lambda_i \mathbf{s}_i \mid 0 \le \lambda_i \le 1\} \cap \mathbb{N}^p$.

Lemma 5 *Given $P \in \mathcal{C}_S$, there exist $Q \in \mathcal{A}$ and $\beta \in \mathbb{N}^q$ such that $P = Q + \sum_{i\in[q]} \beta_i \mathbf{a}_i$. Moreover, $\Upsilon_j(P) = \Upsilon_j(Q) + \sum_{i\in[q]} \beta_i \Upsilon_j(\mathbf{a_i})$ for every $j \in [q]$.*

Proof Since $P \in \mathcal{C}_S$, $P = \sum_{i\in[q]} \mu_i \mathbf{a}_i$ with $\mu_i \in \mathbb{Q}_{\geq}$. For each μ_i there exists $\lambda_i \in [0, 1)$ satisfying $\mu_i = \lfloor \mu_i \rfloor + \lambda_i$. Hence, $P = Q + \sum_{i\in[q]} \lfloor \mu_i \rfloor \mathbf{a}_i$ where $Q = \sum_{i\in[q]} \lambda_i \mathbf{a}_i = P - \sum_{i\in[q]} \lfloor \mu_i \rfloor \mathbf{a}_i \in \mathcal{A}$. Trivially, $\Upsilon_j(P)$ is equal to $\Upsilon_j(Q) + \sum_{i\in[q]} \beta_i \Upsilon_j(\mathbf{a}_i)$ for every $j \in [q]$. □

For every $i \in [q]$, consider $T_i \subset \mathbb{N}^{p-1}$ the semigroup generated by the finite set $\{\Upsilon_i(Q) \mid Q \in \mathcal{A}\}$ and let Γ_i be its minimal generating set. Note that the sets $\mathcal{A}$, T_i and Γ_i only depend on the cone $\mathcal{C}_S$, and $0 \in T_i$, since $\mathbf{a}_i \in \mathcal{A}$. The relationships between the elements in $\mathcal{C}_S$ and S, and the elements belonging to T_i and Γ_i are explicitly determined in the following results for each $i \in [q]$.

Lemma 6 *Let P be an element in $\mathcal{C}_S$ such that $P \in \upsilon_i(\alpha)$ for some $\alpha \in \mathbb{N}^{p-1}$, then $\alpha \in T_i$.*

Proof By definition, $P \in \upsilon_i(\alpha)$ means that $\alpha = \Upsilon_i(P)$. Using Lemma 5, $P = Q + \sum_{j\in[q]} \beta_j \mathbf{a}_j$ with $Q, \mathbf{a}_1, \ldots, \mathbf{a}_q \in \mathcal{A}$ and $\beta_1, \ldots, \beta_q \in \mathbb{N}$. Therefore, $\Upsilon_i(P) = \Upsilon_i(Q) + \sum_{j\in[q]} \beta_j \Upsilon_i(\mathbf{a}_j) \in T_i$. □

Corollary 7 *For every $\alpha \in T_i$, $\mathcal{C}_S \cap \upsilon_i(\alpha) \neq \emptyset$ if and only if $\mathcal{C}_S \cap \upsilon_i(\beta) \neq \emptyset$ for all $\beta \in \Gamma_i$.*

Proof Since $\Gamma_i \subset T_i$, if for all $\alpha \in T_i$, $\mathcal{C}_S \cap \upsilon_i(\alpha) \neq \emptyset$ then $\mathcal{C}_S \cap \upsilon_i(\beta) \neq \emptyset$ for all $\beta \in \Gamma_i$.

Assume that $\mathcal{C}_S \cap \upsilon_i(\beta) \neq \emptyset$ for all $\beta \in \Gamma_i$ and let α be an element in T_i. Then, there exist $\beta_1, \ldots, \beta_k \in \Gamma_i$, $\mu_1, \ldots, \mu_k \in \mathbb{N}$ and $Q_1, \ldots, Q_k \in \mathcal{A}$ such that $\alpha = \sum_{j\in[k]} \mu_j \beta_j$ and $\Upsilon_i(Q_j) = \beta_j$ for $j \in [k]$. Note that $P = \sum_{j\in[k]} \mu_j Q_j \in \mathcal{C}_S$ belongs to $\upsilon_i(\alpha)$. □

Corollary 8 *For every $\alpha \in T_i$, $S \cap \upsilon_i(\alpha) \neq \emptyset$ if and only if $S \cap \upsilon_i(\beta) \neq \emptyset$ for all $\beta \in \Gamma_i$.*

Proof Since $S \cap \upsilon_i(\beta) \neq \emptyset$ for all β, then there exists $Q_1, \ldots, Q_k \in \mathcal{A}$ such that $\Upsilon_i(Q_j) = \beta_j$ for $j \in [k]$. Thus, the proof of this corollary is analogous to the proof of Corollary 7. □

Note that if $P \in S \cap \upsilon_i(\alpha)$ for some $\alpha \in \mathbb{N}^{p-1}$ and $i \in [q]$, then $P + \mathbf{c}_i + \lambda \mathbf{a}_i \in S$ and $\Upsilon_i(P + \mathbf{c}_i + \lambda \mathbf{a}_i) = \alpha$ for all $\lambda \in \mathbb{N}$.

Now, we introduce a characterization of $\mathcal{C}$-semigroups. This characterization depends on the minimal generating set of the given semigroup. Besides, from its proof, we provide an algorithm for checking if a semigroup is a $\mathcal{C}$-semigroup (Algorithm 1). Note that most of the parts of Algorithm 1 can be parallelized at least in q stand-alone processes.

Theorem 9 *A semigroup S minimally generated by $\Lambda_S = \{\mathbf{s}_1, \ldots, \mathbf{s}_n\}$ is a $\mathcal{C}_S$-semigroup if and only if:*

1. *$\tau_i \cap (\mathcal{C}_S \setminus S)$ is finite for all $i \in [q]$.*
2. *$\Lambda_S \cap \upsilon_i(\alpha) \neq \emptyset$ for all $\alpha \in \Gamma_i$ and $i \in [q]$.*

Proof Let S be a $\mathcal{C}_S$-semigroup. Trivially, $\tau_i \cap (\mathcal{C}_S \setminus S)$ is finite for all $i \in [q]$. Now let $i \in [q]$ and $\alpha \in \Gamma_i$, we probe that $\Lambda_S \cap \upsilon_i(\alpha) \neq \emptyset$. Since $\alpha \in \Gamma_i$, there exists $Q \in \mathcal{A}$ such that $\alpha = \Upsilon_i(Q)$. Besides, $Q + \lambda \mathbf{a}_i \in \mathcal{C}_S$ and $\Upsilon_i(Q + \lambda \mathbf{a}_i) = \alpha$ for all $\lambda \in \mathbb{N}$. For some $\lambda \in \mathbb{N}$, $Q + \lambda \mathbf{a}_i$ has to be in S (S is $\mathcal{C}_S$-semigroup), that is to say, $Q + \lambda \mathbf{a}_i = \sum_{j \in [n]} \mu_j \mathbf{s}_j$ with $\mu_1, \ldots, \mu_n \in \mathbb{N}$. Therefore, $\alpha = \Upsilon_i(Q + \lambda \mathbf{a}_i) = \sum_{j \in [n]} \mu_j \Upsilon_i(\mathbf{s}_j)$. By Lemma 5, for all $j \in [n]$, $\mathbf{s}_j = Q_j + \sum_{k \in [q]} \beta_{jk} \mathbf{a}_k$ for some $Q_j \in \mathcal{A}$ and $\beta_{j1}, \ldots, \beta_{jq} \in \mathbb{N}$. So, $\alpha = \sum_{j \in [n]} \mu_j \Upsilon_i(Q_j + \sum_{k \in [q]} \beta_{jk} \mathbf{a}_k) = \sum_{j \in [n]} \mu_j \Upsilon_i(Q_j) + \sum_{j \in [n]} \sum_{k \in [q]} \mu_j \beta_{jk} \Upsilon_i(\mathbf{a}_k)$. Since α is a minimal generator of T_i, $\sum_{j \in [n]} \mu_j + \sum_{j \in [n]} \sum_{k \in [q] \setminus \{i\}} \mu_j \beta_{jk} = 1$. So $\beta_{jk} = 0$ for all $j \in [n]$ and for all $k \in [q] \setminus \{i\}$, and there exists $l \in [n]$ such that $\mu_l = 1$ and $\mu_j = 0$ for all $j \in [n] \setminus \{l\}$. Hence, there exists $\mathbf{s} \in \Lambda_S$ such that $\Upsilon_i(\mathbf{s}) = \alpha$ and then $\Lambda_S \cap \upsilon_i(\alpha) \neq \emptyset$.

Conversely, we assume that $\forall i \in [q]$ and $\forall \alpha \in \Gamma_i$, $\tau_i \cap (\mathcal{C}_S \setminus S)$ is finite and $\Lambda_S \cap \upsilon_i(\alpha) \neq \emptyset$ (recall that $\mathbf{c}_i = \mathbf{f}_i + \mathbf{a}_i$). Let Q be an element in $\mathcal{D}$. By Lemmas 5 and 6, $Q \in \alpha_i(\Upsilon_i(Q))$ and $\Upsilon_i(Q) \in T_i$. If $Q \in \tau_i$ for some $i \in [q]$, then $\upsilon_i(\Upsilon_i(Q)) = \tau_i$ and, by the first condition, $S \cap \upsilon_i(\Upsilon_i(Q)) \neq \emptyset$. If Q is not in any ray, by the second condition and Corollary 8, $S \cap \upsilon_i(\Upsilon_i(Q)) \neq \emptyset$. Therefore, for every $Q \in \mathcal{D}$, the line $\upsilon_i(\Upsilon_i(Q))$ includes a unique non zero minimum (respect 1-norm) point belonging to S. Denote by $\{\mathbf{m}_{i1}, \ldots, \mathbf{m}_{id_i}\}$ the set obtained from the union of above points for the different elements in $\mathcal{D}$ (some of these elements belong to Λ_S). Note that $\mathbf{m}_{ij} + \mathbf{c}_i + \lambda \mathbf{a}_i \in S$ for all $j \in [d_i]$ and $\lambda \in \mathbb{N}$. Consider $n_i := \max\{||\mathbf{m}_{i1} + \mathbf{c}_i||_1, \ldots, ||\mathbf{m}_{id_i} + \mathbf{c}_i||_1\}$, and $\mathbf{x}_i$ the minimum element (respect to the 1-norm) in $\tau_i \cap S$ such that $||\mathbf{x}_i||_1$ is greater than or equal to n_i. The set $\mathcal{D}_i := \mathcal{D} + \mathbf{x}_i$ satisfies that $\mathcal{D}_i \cap S = \mathcal{D}_i \cap \mathcal{C}_S = \mathcal{D}_i$. Consider $\mathbf{a} \in \mathbf{x}_i + \mathcal{C}_S$, proceeding as in the proof of Lemma 5, $\mathbf{a} = \mathbf{x}_i + P + \sum_{j \in [q]} \beta_j \mathbf{s}_j$ for some $P \in \mathcal{D}$ and $\beta_1, \ldots, \beta_q \in \mathbb{N}$, and hence, $\mathbf{x}_i + \mathcal{C}_S \subset S$. We define the bounded set $\mathcal{X} := \{\sum_{i \in [q]} \lambda_i \mathbf{x}_i \mid 0 \leq \lambda_i \leq 1\}$. Since $\mathbf{x}_i + \mathcal{C}_S \subset S$ for every $i \in [q]$ and $L(\mathcal{C}_S) = \{\sum_{i \in [q]} \lambda_i \mathbf{x}_i | \lambda_i \in \mathbb{R}_{\geq}\}$, $\mathcal{C}_S \setminus S \subset \mathcal{X}$. Therefore, S is a $\mathcal{C}_S$-semigroup. □

Example 10 illustrates Theorem 9 and Algorithm 1.

Example 10 Let $S \subset \mathbb{N}^3$ be the semigroup minimally generated by

$$\Lambda_S = \{(2, 0, 0), (4, 2, 4), (0, 1, 0), (3, 0, 0), (6, 3, 6), (3, 1, 1), (4, 1, 1), (3, 1, 2), (1, 1, 0), (3, 2, 3), (1, 2, 1)\}.$$

Algorithm 1: Test if a semigroup S is a $\mathcal{C}_S$-semigroup.

Input: The minimal generating set Λ_S of a semigroup $S \subset \mathbb{N}^p$.
Output: Check if S is a $\mathcal{C}_S$-semigroup.
begin
  $q \leftarrow$ number of extremal rays of $L(S)$;
  if $\tau_i \cap (\mathcal{C}_S \setminus S)$ *is not finite for some* $i \in [q]$ **then**
    return S is not a $\mathcal{C}_S$-semigroup.
  Compute the set $\{\mathbf{a}_1, \ldots, \mathbf{a}_q\}$ from $L(S)$;
  $\mathcal{A} \leftarrow \{\sum_{i\in[q]} \lambda_i \mathbf{a}_i \mid 0 \le \lambda_i \le 1\} \cap \mathbb{N}^p$;
  forall the $i \in [q]$ **do**
    $\Gamma_i \leftarrow$ the minimal generating set of T_i obtained from the finite set $\Upsilon_i(\mathcal{A})$;
  if $\Lambda_S \cap \upsilon_i(\alpha) \neq \emptyset$ *for all* $\alpha \in \Gamma_i$ *and* $i \in [q]$ **then**
    return S is a $\mathcal{C}_S$-semigroup.
  return S is not a $\mathcal{C}_S$-semigroup.

The cone $L(S)$ is $\langle(1,0,0),(2,1,2),(0,1,0)\rangle_{\mathbb{R}_\ge}$ and its supported hyperplanes are $h_1(x,y,z) \equiv 2y - z = 0$, $h_2(x,y,z) \equiv x - z = 0$ and $h_3(x,y,z) \equiv z = 0$. Recall $\mathcal{C}_S = L(S) \cap \mathbb{N}^3$. By $\mathbf{a}_1$, $\mathbf{a}_2$ and $\mathbf{a}_3$ we denote the vectors $(1,0,0)$, $(2,1,2)$ and $(0,1,0)$ respectively, and τ_1, τ_2 and τ_3 are the extremal rays with sets of defining equations $\{h_1(x,y,z) = 0, h_3(x,y,z) = 0\}$, $\{h_1(x,y,z) = 0, h_2(x,y,z) = 0\}$ and $\{h_2(x,y,z) = 0, h_3(x,y,z) = 0\}$, respectively. Hence, $S_1 = (\tau_1 \setminus \{(1,0,0)\}) \cap \mathbb{N}^3$, $S_2 = \tau_2 \setminus \{(2,1,2)\} \cap \mathbb{N}^3$ and $S_3 = \tau_3 \cap \mathbb{N}^3$, and the first condition in Theorem 9 holds.

The set $\mathcal{A}$ is equal to

$$\{(0,0,0),(0,1,0),(1,0,0),(1,1,0),(1,1,1),(2,1,1),(2,1,2),\\ (2,2,2),(3,1,2),(3,2,2)\}, \tag{1}$$

and

$$\begin{aligned}
\Upsilon_1(\mathcal{A}) &= \{(0,0),(0,2),(1,1),(2,0),(2,2)\},\\
\Upsilon_2(\mathcal{A}) &= \{(0,0),(0,1),(1,0),(1,1),(2,0),(2,1)\},\\
\Upsilon_3(\mathcal{A}) &= \{(0,0),(0,1),(0,2),(1,0),(1,1),(1,2)\}.
\end{aligned}$$

Therefore, $\Gamma_1 = \{(0,2),(1,1),(2,0)\}$ and $\Gamma_2 = \Gamma_3 = \{(0,1),(1,0)\}$.

Since $\Upsilon_1(\{(3,1,1),(3,1,2),(1,1,0)\}) = \Gamma_1$, $\Upsilon_2(\{(3,1,2),(3,2,3)\}) = \Gamma_2$, and $\Upsilon_3(\{(1,1,0),(1,2,1)\}) = \Gamma_3$, S satisfies the second condition in Theorem 9. Hence, S is a $\mathcal{C}_S$-semigroup.

By using our implementation of Algorithm 1, we can confirm that S is a $\mathcal{C}_S$-semigroup,

```
In [1]: IsCsemigroup([[2,0,0],[4,2,4],[0,1,0],[3,0,0],
         [6,3,6],[3,1,1],[4,1,1],[3,1,2],[1,1,0],[3,2,3],
[1,2,1]])
Out[1]: True
```

To finish this section, it should be pointed out that there exist some special cases of semigroups where Theorem 9 can be simplified: $\mathbb{N}^p$-semigroups and two-dimensional case.

Note that, if the integer cone $\mathcal{C}_S$ is $\mathbb{N}^p$, its supported hyperplanes are $\{x_1 = 0, \ldots, x_p = 0\}$. Moreover, since its extremal rays are the axes, $\tau_i \equiv \{\lambda \mathbf{e}_i \mid \lambda \in \mathbb{Q}_\geq\}$ is determined by the equations $\cup_{j\in[p]\setminus\{i\}}\{x_j = 0\}$, and for any canonical generator $\mathbf{e}$ of $\mathbb{N}^{p-1}$, there exists P in $\mathbb{N}^p$ such that $\Upsilon_i(P) = \mathbf{e}$. Furthermore, $\cup_{j\in[p]\setminus\{i\}}\{\Upsilon_i(\mathbf{e}_j)\}$ is the canonical basis of $\mathbb{N}^{p-1}$. Hence, $\Gamma_1 = \cdots = \Gamma_p$ is the canonical basis of $\mathbb{N}^{p-1}$. From previous considerations, the same characterization of $\mathbb{N}^p$-semigroups in [3,Theorem 2.8] is obtained from Theorem 9.

Corollary 11 *A semigroup S minimally generated by Λ_S is an $\mathbb{N}^p$-semigroup if and only if:*

1. *for all $i \in [p]$, the non null entries of the elements in $\tau_i \cap \Lambda_S$ are coprime, or $\mathbf{s}_i = \mathbf{e}_i$.*
2. *for all $i, j \in [p]$ with $i \neq j$, $\mathbf{e}_i + \lambda_j \mathbf{e}_j \in \Lambda_S$ for some $\lambda_j \in \mathbb{N}$.*

Focus on two dimensional case, note that the extremal rays and the supported hyperplanes of a cone are equal. Since for each extremal ray the coefficients of its defining linear equation are relatively primes, the linear equations $h_1(x, y) = 1$ and $h_2(x, y) = 1$ always have non-negative integer solutions. So, any semigroup $S \subset \mathbb{N}^2$ is a $\mathcal{C}_S$-semigroup if and only if $\tau_i \cap (\mathcal{C}_S \setminus S)$ is finite for $i = 1, 2$, and both sets $\Lambda_S \cap \{h_1(x, y) = 1\}$ and $\Lambda_S \cap \{h_2(x, y) = 1\}$ are non empty.

4 Set of gaps of $\mathcal{C}$-semigroups

This section gives an algorithm to compute the set of gaps of a $\mathcal{C}$-semigroup, i.e. the set $\mathcal{H}(S) = \mathcal{C}_S \setminus S$. This algorithm is obtained from Theorem 9. To introduce such an algorithm, let us start by redefining some objects used to prove that theorem.

Given S a $\mathcal{C}_S$-semigroup with q extremal rays, for any $i \in [q]$, let $\mathbf{c}_i$ be the conductor of the semigroup $\tau_i \cap S$. By Corollary 8, for any $\alpha \in \Upsilon_i(\mathcal{D})$ the intersection $\upsilon_i(\alpha) \cap S$ is not empty. Hence, set $\mathbf{m}_\alpha^{(i)}$ the element in $\upsilon_i(\alpha) \cap S$ with minimal 1-norm and $\alpha \in \Upsilon_i(\mathcal{D}) \setminus \{0\}$. Note that $\mathbf{m}_\alpha^{(i)} + \mathbf{c}_i + \lambda \mathbf{a}_i \in S$ for all $\lambda \in \mathbb{N}$. Let $n_i := ||\mathbf{c}_i||_1 + \max\left(\{||\mathbf{m}_\alpha^{(i)}||_1 \mid \alpha \in \Upsilon_i(\mathcal{D}) \setminus \{0\}\}\right)$, and $\mathbf{x}_i$ the minimal element in $\tau_i \cap S$ such that $||\mathbf{x}_i||_1$ is greater than or equal to n_i. The vector $\mathbf{x}_i$ can be computed as follows: let Q be the non-negative rational solution of the systems of linear equations $\{x_1 + \cdots + x_p = n_i, h_{j_1^{(i)}}(x) = 0, \ldots, h_{j_{p-1}^{(i)}}(x) = 0\}$ (recall that $h_{j_1^{(i)}}(x) = 0, \ldots, h_{j_{p-1}^{(i)}}(x) = 0$ are the equations defining τ_i), then $\mathbf{x}_i = \left\lceil \frac{||Q||_1}{||\mathbf{a}_i||_1} \right\rceil \mathbf{a}_i$.

By the proof of Theorem 9, $\mathcal{C}_S \setminus S \subset \mathcal{X}$, with $\mathcal{X} = \{\sum_{i\in[q]} \lambda_i \mathbf{x}_i \mid 0 \leq \lambda_i \leq 1\}$. Algorithm 2 shows the process to computed the set of gaps of S. Note that several of its steps can be computed in a parallel way.

We illustrate Algorithm 2 in the following example. Besides, we confirm our handmade computations by using our free software [7].

Algorithm 2: Computing the set of gaps of a $\mathcal{C}$-semigroup.

Input: The minimal generating set Λ_S of a $\mathcal{C}$-semigroup $S \subset \mathbb{N}^p$.
Output: Set of gaps of S.
begin
  $\mathcal{H} \leftarrow \emptyset$;
  $q \leftarrow$ number of extremal rays of $L(S)$;
  forall the $i \in [q]$ **do**
    $\mathbf{c}_i \leftarrow$ conductor of $\tau_i \cap S$;
  $\mathcal{D} \leftarrow \{\sum_{i\in[q]} \lambda_i \mathbf{s}_i \mid 0 \le \lambda_i \le 1\} \cap \mathbb{N}^p$;
  forall the $i \in [q]$ **do**
    $\Upsilon = \{\alpha_1, \ldots, \alpha_j\} \leftarrow \Upsilon_i(\mathcal{D}) \setminus \{0\}$;
    forall the $h \in [j]$ **do**
      $\mathbf{m}_h \leftarrow$ the element in $\upsilon_i(\alpha_h) \cap S$ with minimal 1-norm;
    $n \leftarrow ||\mathbf{c}_i||_1 + \max\left(\{||\mathbf{m}_1||_1, \ldots, ||\mathbf{m}_j||_1\}\right)$;
    $\mathbf{x}_i \leftarrow$ minimal element in $\tau_i \cap S$ with $n \le ||\mathbf{x}_i||_1$;
  $\mathcal{X} \leftarrow \{\sum_{i\in[q]} \lambda_i \mathbf{x}_i \mid 0 \le \lambda_i \le 1\} \cap \mathbb{N}^p$;
  while $\mathcal{X} \ne \emptyset$ **do**
    $Q \leftarrow \text{First}(\mathcal{X})$;
    if $Q \notin S$ **then**
      $\mathcal{H} \leftarrow \mathcal{H} \cup \{Q\}$
    $\mathcal{X} \leftarrow \mathcal{X} \setminus \{Q\}$;
  return $\mathcal{H}$ set of gaps of S.

Example 12 Consider the $\mathcal{C}_S$-semigroup S defined in example 10. So, $\mathbf{s}_1 = \mathbf{c}_1 = (2, 0, 0)$, $\mathbf{s}_2 = \mathbf{c}_2 = (4, 2, 4)$, $\mathbf{s}_3 = (0, 1, 0)$ and $\mathbf{c}_3 = (0, 0, 0)$. The set $\mathcal{D}$ is

$$\{(0,0,0),(0,1,0),(1,0,0),(1,1,0),(1,1,1),(2,0,0),(2,1,0),(2,1,1),\\ (2,1,2),(2,2,2),(3,1,1),(3,1,2),(3,2,2),(3,2,3),(4,1,2),(4,2,2),\\ (4,2,3),(4,2,4),(4,3,4),(5,2,3),(5,2,4),(5,3,4),(6,2,4),(6,3,4)\}.$$

For example, for the extremal ray τ_1, $\Upsilon_1(\mathcal{D})$ is the set

$$\{(0,0),(0,2),(0,4),(1,1),(1,3),(2,0),(2,2),(2,4)\},$$

and $\cup_{\alpha\in\Upsilon_1(\mathcal{D})\setminus\{0\}}\{\mathbf{m}_\alpha^{(1)}\}$ is

$$\{(0,1,0),(3,1,1),(3,1,2),(3,2,2),(3,2,3),(4,2,4),(4,3,4)\}$$

For τ_2 and τ_3,

$$\cup_{\alpha\in\Upsilon_2(\mathcal{D})\setminus\{0\}}\{\mathbf{m}_\alpha^{(2)}\} = \{(0,1,0),(3,1,2),(1,1,0),(3,2,3),(2,0,0),\\ (2,1,0),(6,3,5),(3,1,1)\}$$
$$\cup_{\alpha\in\Upsilon_3(\mathcal{D})\setminus\{0\}}\{\mathbf{m}_\alpha^{(3)}\} = \{(1,1,0),(1,2,1),(2,0,0),(2,3,1),(2,4,2),\\ (3,1,1),(3,1,2),(3,2,3),(4,2,2),(4,3,3),(4,2,4),(5,3,4),(6,2,4)\}$$

Then $n_1 = 13$, $n_2 = 24$ and $n_3 = 12$, and $\mathbf{x}_1 = (14, 0, 0)$, $\mathbf{x}_2 = (10, 5, 10)$ and $\mathbf{x}_3 = (0, 13, 0)$. Therefore, the set of gaps of S is,

$$\{(1,0,0),(1,1,1),(2,1,1),(2,1,2),(2,2,1),(2,2,2),(2,3,2),\\ (4,1,2),(4,2,3),(5,2,4),(5,3,5),(8,4,7)\}.$$

By using our implementation of Algorithm 2, we obtain the same gaps:

```
In [1]: ComputeGaps([[2,0,0],[4,2,4],[0,1,0],[3,0,0],
        [6,3,6],[3,1,1],[4,1,1],[3,1,2],[1,1,0],[3,2,3],
        [1,2,1]])
Out[1]: [[1,0,0], [1,1,1], [2,1,1], [2,1,2],
        [2,2,1], [2,2,2],[2,3,2], [4,1,2], [4,2,3],
        [5,2,4], [5,3,5], [8,4,7]]
```

5 Embedding dimension of $\mathcal{C}$-semigroups

In [8], it is proved that the embedding dimension of an $\mathbb{N}^p$-semigroup is greater than or equal to $2p$, and this bound holds. Furthermore, a conjecture about a lower bound of embedding dimension of any $\mathcal{C}$-semigroup is proposed. In this section, we determine a lower bound of the embedding dimension $e(S)$ of a given $\mathcal{C}$-semigroup S by studying its elements belonging to $\mathcal{A}$.

As in previous sections, let $\mathcal{C} \subset \mathbb{N}^p$ be a finitely generated cone and $\tau_1, \ldots, \tau_q$ its extremal rays. For any $i \in [q]$, $\mathbf{a}_i$ is the generator of $\tau_i \cap \mathbb{N}^p$, $\mathcal{A}$ is the finite set $\{\sum_{i\in[q]} \lambda_i \mathbf{a}_i \mid 0 \leq \lambda_i \leq 1\} \cap \mathbb{N}^p$ and Γ_i denotes the minimal generating set of the semigroup $T_i \subset \mathbb{N}^{p-1}$ generated by $\Upsilon_i(\mathcal{A})$. Given a $\mathcal{C}$-semigroup S, consider $\Lambda'_S:=\{\mathbf{s}_{t_1}, \ldots, \mathbf{s}_{t_k}\}$ the set of minimal generators of S belonging to $\mathcal{A}\setminus \cup_{i\in[q]} \tau_i$, and $M_l:=\{i \in [q] \mid \Upsilon_i(\mathbf{s}_{t_l}) \in \Gamma_i\}$ for $l \in [k]$.

The following result provides us with a lower bound for the embedding dimension of any $\mathcal{C}$-semigroup.

Proposition 13 *Given a $\mathcal{C}$-semigroup $S \subset \mathbb{N}^p$, then*

$$\mathrm{e}(S) \geq \sum_{i\in[q]} (\mathrm{e}(S_i) + \mathrm{e}(T_i)) + k - \sum_{i\in[k]} \sharp(M_i). \tag{2}$$

Proof From Theorem 9, for any $i \in [q]$, there exist $\mathrm{e}(S_i)$ minimal generators of S in τ_i. Moreover, for each element $\gamma \in \Gamma_i$, there is at least an element of Λ_S in $\upsilon_i(\gamma)$. Note that, for every $\mathbf{s} \in \Lambda_S \setminus \mathcal{A}$, there is no $\gamma \in \Gamma_i$ and $\gamma' \in \Gamma_j$ such that $\mathbf{s} \in \upsilon_i(\gamma) \cap \upsilon_j(\gamma')$, since for any $i, j \in [q]$, $\gamma \in \Gamma_i$ and $\gamma' \in \Gamma_j$, the intersection $\upsilon_i(\gamma) \cap \upsilon_j(\gamma')$ is empty or belongs to $\mathcal{A}$. However, if $\mathbf{s} \in \Lambda'_S$, then it is possible that $\mathbf{s}$ belongs to two (or more) different lines $\upsilon_i(\gamma)$ and $\upsilon_j(\gamma')$ with $\gamma \in \Gamma_i$ and $\gamma' \in \Gamma_j$ (in that case, $\upsilon_i(\gamma) \cap \upsilon_j(\gamma') = \{\mathbf{s}\}$). Thus, the value of $\sharp(M_l)$ indicates the number of different lines $\upsilon_i(\gamma_i)$ with $\gamma_i \in \Gamma_i$ to which $\mathbf{s}_{t_l} \in \Lambda'_S$ belongs. So, counting the minimal amount of elements needed to have at least one minimal generator in each line $\upsilon_i(\gamma)$ for each $\gamma \in \Gamma_i$ and $i \in [q]$, we have that the embedding dimension of S is greater than or equal to $\sum_{i\in[q]}(\mathrm{e}(S_i)+\mathrm{e}(T_i))+k-\sum_{i\in[k]}\sharp(M_i)$. □

Example 14 Consider the $\mathcal{C}_S$-semigroup S given in example 10. In that case, $\Lambda'_S = \{(3,1,2),(1,1,0)\}$, $\sharp(M_1)=2$ (i.e. $\Upsilon_i(3,1,2)\in\Gamma_i$ for $i=1,2$), and $\sharp(M_2)=2$ ($\Upsilon_1(1,1,0)\in\Gamma_1$ and $\Upsilon_2(1,1,0)\in\Gamma_3$). So, $\sum_{i\in[q]}(\mathrm{e}(S_i)+\mathrm{e}(T_i))+k-\sum_{i\in[k]}\sharp(M_i)=5+7+2-2-2=10$ that is smaller than $\mathrm{e}(S)=11$.

Given any bound, the first interesting question about it is if the bound is reached for some $\mathcal{C}$-semigroup. The answer is affirmative for (2), and this fact is formulated as follows.

Lemma 15 *Let $\mathcal{C}\subset\mathbb{N}^p$ be an integer cone generated by $\{\mathbf{a}_1,\ldots,\mathbf{a}_q\}$ and let $S_1,\ldots,S_q$ be the non proper numerical semigroups minimally generated by $\{n_1^{(i)},\ldots,n_{\mathrm{e}(S_i)}^{(i)}\}$ for each $i\in[q]$. Let $\Lambda''\subset\mathcal{C}\setminus\cup_{i\in[q]}\tau_i$ be a finite set satisfying:*

- *for every $\gamma\in\Gamma_i$ and $i\in[q]$, there exists a unique $\mathbf{d}\in\Lambda''$ such that $\Upsilon_i(\mathbf{d})=\gamma$,*
- *for every $\mathbf{d}\in\Lambda''$, $\Upsilon_i(\mathbf{d})\in\Gamma_i$ for some $i\in[q]$.*

Then, the embedding dimension of the $\mathcal{C}$-semigroup S generated by

$$\Lambda''\cup\bigcup_{i\in[q]}\{n_1^{(i)}\mathbf{a}_i,\ldots,n_{\mathrm{e}(S_i)}^{(i)}\mathbf{a}_i\}$$

is

$$\sum_{i\in[q]}(\mathrm{e}(S_i)+\mathrm{e}(T_i))+k-\sum_{i\in[k]}\sharp(M_i),$$

where k is the cardinality of $\Lambda'_S=\{\mathbf{s}_1,\ldots,\mathbf{s}_k\}$, the set of minimal generators of S belonging to $\mathcal{A}\setminus\cup_{i\in[q]}\tau_i$, and $M_l=\{i\in[q]\mid\Upsilon_i(\mathbf{s}_l)\in\Gamma_i\}$ for $l\in[k]$.

Proof By the hypothesis, there are exactly $\sum_{i\in[q]}\mathrm{e}(T_i)+k-\sum_{i\in[k]}\sharp(M_i)$ minimal generators of S outside its extremal rays, and $\sum_{i\in[q]}\mathrm{e}(S_i)$ belonging to its extremal rays. □

Example 16 Let $S\subset\mathbb{N}^3$ be the semigroup minimally generated by

$$\Lambda_S=\{(2,0,0),(4,2,4),(0,2,0),(3,0,0),(6,3,6),(0,3,0),(3,1,1),\\ (3,1,2),(1,1,0),(3,2,3),(1,2,1)\}.$$

Note that the cone $\mathcal{C}_S$ is the same as the cone in example 10. So, $\mathcal{A}$, Γ_1, Γ_2 and Γ_3 are the sets given in that example. For the semigroup S, $\Upsilon_1(\{(3,1,1),(3,1,2),(1,1,0)\})=\Gamma_1$, $\Upsilon_2(\{(3,1,2),(3,2,3)\})=\Gamma_2$ and $\Upsilon_3(\{(1,1,0),(1,2,1)\})=\Gamma_3$. Since $(1,1,0),(3,1,2)\in\mathcal{A}$, $\mathrm{e}(S)=11=6+7+2-2-2=\sum_{i\in[3]}(\mathrm{e}(S_i)+\mathrm{e}(T_i))+2-\sum_{i\in[2]}\sharp(M_i)$.

Fix a cone $\mathcal{C}$, studying the different possibilities to select sets of points $K \subset \mathcal{C}$ such that $\cup_{i\in[q]}\Gamma_i$ is the union of the minimal generating set of the semigroup given by $\cup_{Q\in K}\Upsilon_i(Q)$ (for i from 1 to q), we can state results like the following:

Corollary 17 *Let $S_1, \dots, S_q$ be the non proper numerical semigroups minimally generated by $\{n_1^{(i)}, \dots, n_{e(S_i)}^{(i)}\}$ for each $i \in [q]$, and $\Lambda'' \subset \mathcal{C}$ satisfying the hypothesis of Lemma 15. Thus, if $\Lambda'' \subset \mathcal{C} \setminus \mathcal{A}$, then the embedding dimension of the $\mathcal{C}$-semigroup generated by $\Lambda'' \cup \bigcup_{i\in[q]}\{n_1^{(i)}\mathbf{a}_i, \dots, n_{e(S_i)}^{(i)}\mathbf{a}_i\}$ is $\sum_{i\in[q]}(\mathrm{e}(S_i) + \mathrm{e}(T_i))$.*

Finally, we illustrate the above result with an example.

Example 18 Let $S \subset \mathbb{N}^3$ be the semigroup minimally generated by

$$\Lambda_S = \{(2, 0, 0), (4, 2, 4), (0, 2, 0), (3, 0, 0), (6, 3, 6), (0, 3, 0), (3, 1, 1), (4, 1, 2), (5, 2, 4), (2, 1, 0), (1, 2, 0), (3, 2, 3), (1, 2, 1)\}.$$

Again, the cone $\mathcal{C}_S$ is the cone appearing in example 10. Note that the elements $(2, 0, 0)$ and $(3, 0, 0)$ are in S_1, $(4, 2, 4)$ and $(6, 3, 6)$ belong to S_2, and $(0, 2, 0)$ and $(0, 3, 0)$ are in S_3. Moreover, $\Upsilon_1(\{(3, 1, 1), (4, 1, 2), (2, 1, 0)\}) = \Gamma_1$, $\Upsilon_2(\{(5, 2, 4), (3, 2, 3)\}) = \Gamma_2$, $\Upsilon_3(\{(1, 2, 0), (1, 2, 1)\}) = \Gamma_3$, and $\Lambda_S \setminus \cup_{i\in[q]}\tau_i \subset \mathcal{C}_S \setminus \mathcal{A}$. As previous corollary asserts, $\mathrm{e}(S) = 13 = 6 + 7 = \sum_{i\in[3]}(\mathrm{e}(S_i) + \mathrm{e}(T_i))$.

6 On the decomposition of a $\mathcal{C}$-semigroup in terms of irreducible $\mathcal{C}$-semigroups

We define the set of pseudo-Frobenius of a $\mathcal{C}$-semigroup S as $\mathrm{PF}(S) = \{\mathbf{a} \in \mathcal{H}(S) \mid \mathbf{a} + (S \setminus \{0\}) \subset S\}$ (recall that $\mathcal{H}(S) = \mathcal{C} \setminus S$), and the set of special gaps of S as $\mathrm{SG}(S) = \{\mathbf{a} \in \mathrm{PF}(S) \mid 2\mathbf{a} \in S\}$. Note that the elements $\mathbf{a}$ of $\mathrm{SG}(S)$ are those elements in $\mathcal{C} \setminus S$ such that $S \cup \{\mathbf{a}\}$ is again a $\mathcal{C}$-semigroup.

A $\mathcal{C}$-semigroup is $\mathcal{C}$-reducible (simplifying reducible) if it can be expressed as an intersection of two $\mathcal{C}$-semigroups containing it properly (see [11]). Equivalently, S is $\mathcal{C}$-irreducible (simplifying irreducible) if and only if $|\mathrm{SG}(S)| \leq 1$. A decomposition of a $\mathcal{C}$-semigroup S in terms of irreducible $\mathcal{C}$-semigroups is to express S as intersection of irreducible $\mathcal{C}$-semigroups. This definition generalizes the definitions of irreducible numerical semigroups (see [13]) and irreducible $\mathbb{N}^p$-semigroups (see [2]).

Our decomposition method into irreducible is based on adding to a $\mathcal{C}$-semigroup elements of $\mathrm{SG}(S)$. If we repeat this operation, we always reach an irreducible $\mathcal{C}$-semigroup or the cone $\mathcal{C}$. Since the set of gaps $\mathcal{H}(S)$ is finite, this process can be performed only a finite number of times. This allows us to state the following algorithm inspired by [13,Algorithm 4.49].

By definition, the set $\mathrm{SG}(S)$ is obtained from $\mathrm{PF}(S)$. If S is determined by its minimal generating set, then $\mathrm{PF}(S)$ can be computed from the set $\mathcal{H}(S)$ obtained with Algorithm 2, or using the two different ways given in [11,Corollary 9 and Example 10].

Example 19 Consider the $\mathcal{C}$-semigroup S given in examples 10 and 12. It is minimally generated by

Algorithm 3: Computing a decomposition into $\mathcal{C}$-semigroups.

Input: The minimal generating set Λ_S of a $\mathcal{C}$-semigroup $S \subset \mathbb{N}^p$.
Output: A decomposition of S into irreducible $\mathcal{C}$-semigroups.
begin
  $I \leftarrow \emptyset$;
  $C \leftarrow \{S\}$;
  while $C \neq \emptyset$ **do**
    $B \leftarrow \{S' \cup \{\mathbf{a}\} \mid S' \in C,\ \mathbf{a} \in \mathrm{SG}(S')\}$;
    $B \leftarrow B \setminus \{S' \in B \mid \exists \bar{S} \in I \text{ with } \bar{S} \subset S'\}$;
    $I \leftarrow I \cup \{S' \in B \mid S' \text{ is irreducible}\}$;
    $C \leftarrow \{S' \in B \mid S' \text{ reducible}\}$;
  return I.

$$\Lambda_S = \{(2,0,0), (4,2,4), (0,1,0), (3,0,0), (6,3,6), (3,1,1), (4,1,1), (3,1,2), (1,1,0), (3,2,3), (1,2,1)\},$$

with

$$\mathcal{H}(S) = \{(1,0,0), (1,1,1), (2,1,1), (2,1,2), (2,2,1), (2,2,2), (2,3,2), (4,1,2), (4,2,3), (5,2,4), (5,3,5), (8,4,7)\}.$$

Hence, $\mathrm{PF}(S) = \{(2,2,1), (2,3,2), (4,1,2), (8,4,7)\}$, and $\mathrm{SG}(S)$ is equal to $\mathrm{PF}(S)$.

Applying Algorithm 3 to S, we obtain the decomposition into six irreducible $\mathcal{C}$-semigroups, $S = S_1 \cap \cdots \cap S_6$ where

- $S_1 = \langle (3,0,0), (2,0,0), (1,1,0), (0,1,0), (4,1,1), (3,1,1), (3,1,2), (4,1,2), (1,2,1), (2,2,1), (2,2,2), (3,2,3), (4,2,4), (6,3,6) \rangle$;
- $S_2 = \langle (3,0,0), (2,0,0), (1,1,0), (0,1,0), (4,1,1), (3,1,1), (2,1,2), (3,1,2), (1,2,1), (2,2,1), (3,2,3) \rangle$;
- $S_3 = \langle (1,0,0), (0,1,0), (2,1,1), (3,1,2), (1,2,1), (3,2,3), (4,2,4), (5,3,5), (6,3,6) \rangle$;
- $S_4 = \langle (3,0,0), (2,0,0), (1,1,0), (0,1,0), (2,1,1), (1,1,1), (3,1,2), (4,1,2), (3,2,3), (4,2,4), (6,3,6) \rangle$;
- $S_5 = \langle (3,0,0), (2,0,0), (1,1,0), (0,1,0), (2,1,1), (1,1,1), (3,1,2), (3,2,3), (4,2,4), (5,2,4), (6,3,6) \rangle$;
- $S_6 = \langle (3,0,0), (2,0,0), (1,1,0), (0,1,0), (4,1,1), (3,1,1), (2,1,2), (3,1,2), (1,2,1), (3,2,3), (4,2,3) \rangle$;

To get these semigroups we have used our implementation in [7] by typing the following

```
Csemigroup([[2,0,0],[4,2,4],[0,1,0],[3,0,0],[6,3,6],
[3,1,1],[4,1,1],[3,1,2],[1,1,0],[3,2,3],[1,2,1]]).
DecomposeIrreducible()
```

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Declarations

Conflict of interests The authors declare that there is no conflict of interests regarding the publication of this paper.

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