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Congruencias y factorización como herramientas de reducción en el análisis de conceptos formales

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HACEN CONSTAR:

Que D. Roberto García Aragón, Graduado en Matemáticas, ha realizado en el Departamento de Matemáticas de la Universidad de Cádiz, bajo nuestra dirección, el trabajo de investigación correspondiente a su Tesis Doctoral titulado:

Congruencias y factorización como herramientas de reducción en el análisis de conceptos formales

Revisado el presente trabajo, estimamos que puede ser presentado al Tribunal que ha de juzgarlo. Y para que conste a efectos de lo establecido en el Real Decreto 99/2011, autorizamos la presentación de este trabajo en la Universidad de Cádiz.

Cádiz, a 1 de abril de 2022

Dr. Jesús Medina Moreno | Dra. Eloísa Ramírez Poussa

A mi familia,

Índice general

English Summary

Since its introduction at the beginning of the eighties by B. Ganter and R. Wille, Formal Concept Analysis (FCA) has been one of the most developed mathematical tools for data analysis. FCA is a mathematical theory that determines conceptual structures among datasets. In particular, the databases considered in this theory are called contexts and are composed of a set of objects, a set of attributes and a relationship between the sets.

The tools provided by FCA can properly manipulate data and extract relevant information from it. One of the most relevant and intensively developed research lines is the reduction of the set of attributes contained in these datasets, preserving the original information and removing the redundancy it may contain. Attribute reduction has also been studied in other environments, such as in Rough Set Theory, as well as in the different fuzzy generalizations of FCA and Rough Set Theory.

On the one hand, it has been shown that when an attribute reduction is carried out in a formal context, an equivalence relation is induced on the set of concepts obtained from the original context. This induced equivalence relation has a particularity, its equivalence classes have a join semilattice structure with a maximum element, i.e., in general, they could not have a closed algebraic structure.

In this thesis, we study how it is possible to enhance the attribute reduction by endowing equivalence classes with a closed algebraic struc-

ture. The notion of congruence achieves this purpose, however, the use of this kind of equivalence relation may cause a great loss of information due to the fact that the generated equivalence classes group many concepts. In order to address this issue, in this thesis a weakened notion of congruence is introduced and it is called local congruence. The local congruence gives rise equivalence classes with the structure of convex sublattice, being more flexible when it groups concepts but maintaining interesting properties from an algebraic point of view.

The results corresponding to the study of local congruences have been published in impact journals, specifically in three papers that are included in the Journal Citation Reports which have been included as Chapter 4, 5 and 6.

- Chapter 4. Reducing concept lattices by means of a weaker notion of congruence [6].
- Chapter 5. Identifying non-sublattice equivalence classes induced by an attribute reduction in FCA [7].
- Chapter 6. Impact of local congruences in variable selection from datasets [8].

A summary of these papers is introduced in Chapter 3 in which a general discussion of the main results and the application of local congruences is presented. In particular, the notion of local congruence is introduced together with an analysis of its properties, as well as the order which is defined on its equivalence classes in order to acquire an algebraic structure. In addition, we perform an extensive analysis of the impact generated by the use of local congruences in FCA on both, the formal context and the concept lattice. In this analysis we identify those equivalence classes of the relation induced by an attribute reduction, upon which the local congruence can be applied, realizing a different concept grouping in order to obtain convex sublattices. Furthermore, we carry out a study on the use of local congruences when in the considered attribute reduction all the unnecessary attributes of the context have been removed, obtaining interesting results. We present several mechanisms that can be applied to compute local congruences and we apply them to concept lattices, describing in detail the modifications that should be made to the formal context in order to provide a reduction method based on local congruences.

On the other hand, the factorization of matrices is another kind of technique that helps to reduce the complexity of the analysis of formal contexts. The procedures used to factorize can split a context into two or more formal subcontexts of smaller size, which can be studied separately more easily. Chapter 7 presents a preliminary study which has not been published in a journal yet, on the factorization of fuzzy formal contexts using modal operators. These modal operators have already been used to extract independent subcontexts from a classical formal context, when they exist, obtaining a factorization of the original context. In this thesis we also study several properties that help us to have a better understanding of how the decomposition of boolean data tables works, and afterwards we adapt these properties to the multi-adjoint framework. The study of these general properties in the multi-adjoint framework will be fundamental to obtain a procedure that allow us to factorize multi-adjoint formal contexts in the future. Thus, determining mechanisms for factorizing multi-adjoint contexts will be essential for the analysis and processing of large databases.

Resumen

Desde su introducción a principios de los años ochenta por B. Ganter y R. Wille, el Análisis de Conceptos Formales (FCA, de sus siglas en inglés) ha sido una de las herramientas matemáticas para el análisis de datos que más desarrollo ha experimentado. El FCA es una teoría matemática que determina estructuras conceptuales entre conjuntos de datos. En particular, las bases de datos se interpretan formalmente en esta teoría con la noción de contexto, que viene determinado por un conjunto de objetos, un conjunto de atributos y una relación entre ambos conjuntos.

Las herramientas que proporciona el FCA permiten manipular adecuadamente los datos y extraer información relevante de ellos. Una de las líneas de investigación con más importancia es la reducción del conjunto de atributos que contienen estos conjuntos de datos, preservando la información esencial y eliminando la redundancia que puedan contener. La reducción de atributos también ha sido estudiada en otros ambientes, como en la Teoría de Conjuntos Rugosos, así como en las distintas generalizaciones difusas de ambas teorías.

En el FCA, se ha demostrado que cuando se lleva a cabo una reducción de atributos de un contexto formal, se induce una relación de equivalencia sobre el conjunto de conceptos del contexto original. Esta relación de equivalencia inducida tiene una particularidad, sus clases de equivalencia tienen una estructura de semirretículo superior con un elemento máximo,

es decir, no forman estructuras algebraicas cerradas, en general.

En esta tesis estudiamos cómo es posible complementar las reducciones de atributos dotando a las clases de equivalencia con una estructura algebraica cerrada. La noción de congruencia consigue este propósito, sin embargo, el uso de este tipo de relación de equivalencia puede desembocar en una gran pérdida de información debido a que las clases de equivalencia agrupan demasiados conceptos. Para abordar este problema, en esta tesis se introduce una noción debilitada de congruencia que denominamos congruencia local. La congruencia local da lugar a clases de equivalencia con estructura de subretículo convexo, siendo más flexible a la hora de agrupar conceptos pero manteniendo propiedades interesantes desde un punto de vista algebraico.

Los resultados correspondientes al estudio de las congruencias locales se han publicado en revistas de impacto, concretamente en tres artículos que están recogidos en el *Journal Citation Reports* y que se han incluido en los Capítulos 4, 5 y 6.

- Capítulo 4. Reduciendo retículos de conceptos por medio de una noción debilitada de congruencia [6].

- Capítulo 5. Identificando las clases de equivalencias inducidas por una reducción de atributos en el FCA que no tienen estructura de retículo completo [7].

- Capítulo 6. Impacto de las congruencias locales en la selección de variables de bases de datos [8].

En el Capítulo 3 se presenta una discusión general de los principales resultados relativos al estudio y aplicación de las congruencias locales que se han obtenido a lo largo de la investigación desarrollada durante la tesis y que se incluyen en los artículos publicados [6, 7, 8]. En particular, se introduce la noción de congruencia local junto con un análisis de las propiedades que satisface, así como una relación de orden sobre el conjunto de las clases de equivalencia. Además, realizamos un análisis profundo del impacto que genera el uso de las congruencias locales en el FCA, tanto en el contexto formal como en el retículo de conceptos. En este análisis identificamos aquellas clases de equivalencia de la relación inducida por una reducción de atributos, sobre las cuales actuaría la congruencia local, realizando una agrupación de conceptos diferente para obtener subretículos convexos. Adicionalmente, llevamos a cabo un estudio sobre el uso de las congruencias locales cuando en la reducción de atributos considerada se han eliminado todos los atributos innecesarios del contexto, obtienen resultados interesantes. Presentamos diversos mecanismos que permiten calcular congruencias locales y aplicarlas sobre retículos de conceptos, detallando las modificaciones que se realizan sobre el contexto formal para proporcionar un método de reducción basado en congruencias locales.

Por otra parte, otra de las estrategias que nos permite reducir la complejidad del análisis de los contextos formales son los mecanismos de factorización. Los procedimientos utilizados para factorizar permiten dividir un contexto en dos o más subcontextos formales de menor tamaño, pudiéndose estudiar por separado más fácilmente. El Capítulo 7 presenta un estudio preliminar sobre la factorización de contextos formales difusos usando operadores modales, que no se ha publicado aún en una revista. Estos operadores modales ya han sido utilizados para extraer subcontextos independientes de un contexto formal clásico obteniéndose así una factorización del contexto original. En esta tesis estudiamos también diversas propiedades que nos ayudan a comprender mejor cómo funciona la descomposición de tablas de datos booleanos, para luego realizar una adaptación de dichas propiedades al marco de trabajo multiadjunto. El estudio de estas propiedades generales en el marco de trabajo multiadjunto será de gran relevancia para poder obtener en el futuro un procedimiento que nos permita factorizar contextos formales multiadjuntos. Por tanto, la obtención de mecanismos de factorización de contextos multiadjuntos será clave para el análisis y tratamiento de grandes bases de datos.

Capítulo 1

Contexto general de la tesis doctoral

1.1. Introducción y justificación de la unidad temática

En los tiempos que corren, la mayor parte de la información almacenada en el mundo se encuentra de forma digital y este proceso de digitalización que hemos experimentado durante las últimas décadas posee una característica de gran interés: la generación masiva de datos. Los datos generados en procesos digitales aumentan de manera exponencial y son considerados una materia prima de gran valor ya que pueden contener información de especial relevancia para instituciones y compañías. Como consecuencia, el poder manipular de manera eficiente grandes bases de datos extrayendo la información contenida en las mismas, es una de las líneas de investigación más atractiva e interesante tanto para la comunidad científica como para instituciones y empresas. Además, en muchas ocasiones, las bases de datos a tratar contienen información incompleta, incertidumbre o imprecisión. Por ello, se han desarrollado distintas herramientas matemáticas que posibilitan la modelización y procesamiento de información incompleta como la Teoría de Conjuntos Difusos (FST, de sus siglas en inglés) introducida por Lotfi A. Zadeh en los años sesenta, la Teoría de Conjuntos Rugosos (RST) propuesta por Pawlak en los ochenta o el Análisis de Conceptos Formales (FCA) desarrollado inicialmente por Ganter y Wille también a principios de los ochenta, entre otras.

Análisis de conceptos formales

El análisis de conceptos formales es una teoría para el análisis de datos que permite organizar y estudiar información, identificando estructuras conceptuales entre conjuntos de datos. Desde su introducción [52, 86], se ha estudiado de manera intensiva tanto desde un punto de vista teórico [4, 71, 72, 74, 89] como desde una perspectiva aplicada [2, 5, 12, 13, 32, 44, 50, 51, 62, 63, 68, 69, 76, 88]. Concretamente, se trata de una herramienta matemática para extraer fragmentos de información de bases de datos, las cuales se componen de un conjunto de atributos/propiedades A y un conjunto de objetos B que están relacionados entre sí mediante una relación $R \subseteq A \times B$. Dichos fragmentos se denominan *conceptos*, entre los cuales se puede establecer una jerarquía dando lugar a una estructura algebraica llamada *retículo de conceptos*.

Concretamente, el análisis de conceptos formales puede considerarse como un campo de la Teoría de Retículos con multitud de aplicaciones, ya que los fundamentos matemáticos en los que se basa se ubican en esta teoría destacada dentro de la Teoría de Orden. Además, se puede considerar que la teoría de retículos se inicia de la mano de George Boole en el siglo XIX, cuando en su intento de formalizar la lógica proposicional con técnicas algebraicas concibe la noción de álgebra booleana. Sin embargo, el mayor desarrollo de la teoría de retículos se lleva a cabo gracias a los trabajos de Garret Birkhoff. En estos, Birkhoff logró obtener un marco de trabajo general idóneo para adaptar distintas disciplinas matemáticas nunca antes vinculadas mediante los retículos. Para profundizar en esta teoría el lector puede consultar las siguientes referencias [27, 43, 57].

Además, otros elementos esenciales que encontramos en el FCA son las *conexiones de Galois* [45] y los *operadores de clausura* que juegan un papel fundamental para la construcción de retículos de conceptos y la determinación de dependencias entre atributos. Una conexión de Galois está formada por un par de funciones monótonas decrecientes que establecen una relación entre dos conjuntos parcialmente ordenados, permitiendo así compararlos con la finalidad de obtener nueva información de uno apoyándose en el otro. En [87], Wille declara que el origen natural de la noción de conexión de Galois viene dada por los conceptos, ya que estos se componen de una *extensión* (los objetos del concepto) y una *intensión* (los atributos del concepto), que se caracterizan mutuamente.

Posteriormente, se han desarrollado diferentes extensiones difusas del FCA. Una de ellas es la proporcionada por el marco multiadjunto [67, 69] que fue introducida con el objetivo de proporcionar un marco de trabajo general con capacidad de adaptar diferentes enfoques difusos existentes [14, 29, 54, 61]. Algunas de las ventajas del uso de este marco han sido estudiadas en varios trabajos como en [3, 36, 41]. Además, la filosofía del paradigma multiadjunto también se ha aplicado a otras áreas de conocimiento como, por ejemplo, a las Ecuaciones de Relaciones Difusas [34, 46], a la Programación Lógica [59, 70] o a los Conjuntos Rugosos Difusos [42, 65].

Métodos de reducción en el FCA

Uno de las principales dificultades que aparecen en el FCA es la complejidad computacional que tienen las técnicas matemáticas desarrolladas para el cálculo del retículo de conceptos asociado al contexto formal considerado (base de datos). Uno de los procedimientos para abordar este problema es encontrar mecanismos para reducir el número de atributos, preservando la información más importante del contexto. La eliminación de los datos redundantes no modifica la información contenida en el conjunto de datos, pero sí tiene un gran impacto a la hora de poder extraer la información, pues dificultan la obtención eficaz de información fundamental. Los atributos redundantes más comunes consisten en entradas repetidas, que pueden eliminarse sin coste alguno, o en variables dependientes, que pueden derivarse de las variables independientes (implicaciones de atributos), cuya detección es un tema de investigación atractivo en muchas áreas que se ocupan del análisis de datos [16, 22, 33, 48].

La detección de atributos u objetos (ir)relavantes de un contexto formal en el FCA se ha estudiado desde diferentes puntos de vista, por ejemplo, con el objetivo de obtener un retículo isomorfo al original [3, 17, 36, 41, 66, 73], con el de reducir eficientemente el tamaño del retículo de conceptos [1, 2, 4, 21, 30, 38, 39, 40, 75], considerando contextos con atributos positivos y negativos [11] o aplicando la filosofía utilizada en la RST [23, 25, 24], entre otros. En muchas ocasiones, realizar una correcta selección de las variables consideradas facilita el manejo de la información de manera considerable, pero también puede dar lugar a ciertas alteraciones de la información proporcionada que deben ser analizadas cuidadosamente.

Al mismo tiempo, otra estrategia aplicada al FCA, con el fin de reducir la complejidad del tratamiento de los datos, es la factorización de los contextos formales mediante el desarrollo de algoritmos que permiten obtener factores (tablas más pequeñas) de tablas booleanas [18, 20, 77]. Además, se han utilizado diferentes filosofías para abordar la factorización como, por ejemplo, la Teoría de la Posibilidad [49], cuyas herramientas permiten obtener subcontextos independientes a partir de un contexto formal, mediante los operadores modales que se pueden definir sobre el contexto. Adicionalmente, también podemos encontrar diferentes extensiones difusas de algoritmos de factorización, como las presentadas en [15, 19, 55].

1.2. Hipótesis, objetivos y análisis crítico de antecedentes

Como se comentó anteriormente, en los últimos años, se han llevado a cabo diversos estudios sobre mecanismos para la reducción del tamaño del contexto considerando diferentes estrategias. Una de las más interesantes, y sobre la cuál trabajamos en esta tesis, es la reducción de atributos. Tanto en la filosofía del FCA como de la RST se hace uso de la noción de reducto, que es el menor conjunto de información relevante del contexto considerado, cuyo cálculo permite descartar atributos redundantes.

Recientemente, en [24], se han presentado nuevos mecanismos para reducir contextos formales tanto clásicos como difusos basados en la filosofía de reducción de la RST, teoría matemática estrechamente relacionada con el FCA. Cuando se reduce el número de atributos de un contexto, usando tanto la filosofía propia de la RST como otra estrategia, se induce una relación de equivalencia en el conjunto de conceptos, es decir, sobre el retículo de conceptos. Además, las clases de equivalencia que se obtienen de dicha relación de equivalencia tienen una estructura algebraica de *semiretículo superior con elemento máximo*. Evidentemente, estas clases de equivalencia no poseen una estructura algebraica cerrada ya que el ínfimo de cada clase no tiene porqué pertenecer a esta. En caso de que dicho ínfimo perteneciera, se obtendría un *subretículo convexo* del retículo de conceptos original, con mejores propiedades que la estructura de semiretículo.

Este propósito se puede conseguir considerando la noción de relación de congruencia sobre retículos [28, 43, 57, 58]. Las relaciones de congruencia ya han sido estudiadas dentro del ámbito del FCA, por ejemplo, se pueden encontrar algunos trabajos que analizan el uso de relaciones de congruencia para la composición y descomposición de retículos tales como la descomposición Atlas [52], la descomposición subdirecta [79] o la construcción de duplicación invertida [78]. Incluso se ha llegado a estudiar el vínculo entre implicaciones y relaciones de congruencia [80] demostrando que estas últimas son apropiadas para trabajar con contextos de decisión formal inconsistentes [64]. Sin embargo, la noción de congruencia no se ha estudiado ampliamente dentro del ambiente del FCA debido fundamentalmente a que cuando son consideradas para reducir retículos de conceptos aparecen dos grandes inconvenientes: no son fáciles de calcular y pueden agrupar demasiados conceptos, traduciéndose en una posible pérdida de información. Todo ello se debe a lo restrictiva que son las propiedades que deben satisfacer las relaciones de congruencia. Por tanto, conseguir una relación con características similares a las congruencias, evitando las propiedades más restrictivas y manteniendo aquellas que proporcionan mejores resultados, permitiría conseguir mejores agrupaciones de conceptos, así como una mayor flexibilidad a la hora de trabajar con dichas particiones.

La idea es introducir una nueva relación de equivalencia debilitando la noción de congruencia y definirla sobre un retículo cualquiera para luego poder aplicarse de forma particular a los retículos de conceptos. Queremos dotar al conjunto formado por las clases de equivalencia de un orden, para obtener un conjunto parcialmente ordenado. Además, es interesante estudiar su aplicación en particular sobre retículos de conceptos para el caso clásico para posteriormente poder extenderlo al marco de trabajo difuso.

Con la perspectiva de aplicar esta nueva relación de equivalencia a retículos de conceptos, es importante estudiar el impacto que su uso tendría sobre el contexto formal asociado al retículo de conceptos, así como buscar una caracterización de sus clases de equivalencia y describir un procedimiento que detalle los cambios realizados para reducir el contexto mediante la aplicación de esta relación de equivalencia.

Por último, con respecto a la factorización de contextos, es importante conocer diferentes propiedades que puedan permitir desarrollar un mecanismo de factorización que simplifique el análisis de contextos en el marco de trabajo multiadjunto. Para ello, seguiremos la filosofía considerada en [49], donde utilizan los operadores de necesidad para obtener pares de subconjuntos de objetos y atributos que determinan subcontextos independientes. La idea es descubrir las propiedades que satisfacen estos pares con el objetivo de poder entender mejor el comportamiento del procedimiento de factorización (obtención de subcontextos independientes) en el entorno clásico y cómo estas propiedades se trasladarían al marco general multiadjunto, para poder finalmente ser capaces de obtener mecanismos que nos permitan factorizar contextos formales multiadjuntos.

Capítulo 2

Preliminares

En este capítulo se exponen algunas de las definiciones, notaciones y resultados que sirven de base para la comprensión y seguimiento de las definiciones y resultados presentados en esta tesis.

2.1. Retículos y semirretículos

En primer lugar, recordaremos las nociones de *retículo* y *retículo completo* ya que son las principales estructuras algebraicas consideradas a lo largo de este trabajo.

Recordar que un *conjunto parcialmente ordenado* es un par $(P, \leq)$, donde P es un conjunto arbitrario y $\leq$ es una relación de orden definida en P, es decir, una relación reflexiva, antisimétrica y transitiva. Para la definición de retículo y de retículo completo, supondremos conocidas también las nociones de ínfimo ($\wedge$) y supremo ($\vee$) tanto de elementos como de conjuntos.

Definición 2.1. *Un conjunto parcialmente ordenado, $(L, \leq)$, es un* retículo *si existe el supremo y el ínfimo de todo par de elementos del conjunto parcialmente* *ordenado. Si además, todo subconjunto de* L *tiene supremo e ínfimo, entonces* $(L, \leq)$ *es un* retículo completo.

Mencionar que un retículo también se puede denotar haciendo uso de su definición algebraica a partir de los operadores ínfimo y supremo, $(L, \vee, \wedge)$, en lugar de como conjunto ordenado. Un retículo (y un conjunto parcialmente ordenado, en general) se representa gráficamente mediante un *diagrama de Hasse*, cuya representación viene dada por un grafo por niveles, en el que los vértices representan los elementos y las aristas (dirigidas) el orden entre ellos, evitando las relaciones que puedan obtenerse por transitividad y los lazos que surgirían por la reflexividad.

Además, se recuerda a continuación las nociones particulares de estructuras algebraicas que están contenidas en las estructuras ya definidas. Comenzamos con la definición de subretículo.

Definición 2.2. *Sea* $(L, \vee, \wedge)$ *un retículo y* $M \subseteq L$ *un subconjunto no vacío. Entonces* $(M, \vee, \wedge)$ *es un* subretículo *de* $(L, \vee, \wedge)$*, si para cada* $a, b \in M$ *se satisface que* $a \vee b \in M$ *y* $a \wedge b \in M$.

Es posible que en un conjunto parcialmente ordenado se dé solo la existencia de supremos o solo la de ínfimos de sus subconjuntos. Este hecho da lugar a la siguiente definición.

Definición 2.3. *Un* semirretículo superior *(respectivamente* inferior*) es un conjunto parcialmente ordenado* $(L, \leq)$ *tal que para todo subconjunto no vacío* $K \subseteq L$ *existe el supremo (resp. ínfimo) de* K.

Además, puede ocurrir que un semirretículo superior (resp. inferior) esté contenido en un retículo, en este caso se denominaría *subsemirretículo superior (resp. inferior)*. Asimismo, un semirretículo superior puede convertirse en un retículo completo como se enuncia en el siguiente resultado.

Lema 2.1 ([43])**.** *Un semirretículo superior (resp. inferior)* $(L, \leq)$ *con un elemento mínimo (resp. máximo) es un retículo completo.*

Otra noción de la que se hará uso es la de retículo distributivo.

Definición 2.4. *Decimos que un retículo* $(L, \leq)$ *es* distributivo *si, para todo* $x, y, z \in L$*, se satisface que* $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.

Además, un retículo no distributivo se puede caracterizar a partir de dos retículos particulares.

Teorema 2.1 ([43])**.** *Un retículo* $(L, \leq)$ *es no distributivo si y solo si* M_3 *o* N_5 *es un subretículo de* $(L, \leq)$.

La siguiente noción jugará un papel fundamental ya que muchos de los resultados presentados en esta tesis hacen uso de ella.

Definición 2.5. *Dado un retículo* $(L, \leq)$*, tal que* $\wedge$*,* $\vee$ *son los operadores ínfimo y supremo. Un elemento* $x \in L$ *verificando que:*

1. *si* L *tiene un elemento máximo* $\top$*, entonces* $x \neq \top$.

2. *si* $x = x \wedge z$*, entonces* $x = y$ *o* $x = z$*, para todo* $y, z \in L$.

se denomina elemento ínfimo irreducible *de* L*. La condición 2 es equivalente a:*

2′. *Si* $x < y$ *y* $x < z$*, entonces* $x < y \wedge z$*, para todo* $y, z \in L$.

Por lo tanto, si x *es ínfimo irreducible, entonces no se puede expresar como ínfimo de elementos estrictamente mayores.*

Un elemento supremo irreducible *de* L *se define por dualidad.*

Por otra parte, la condición de cadena ascendente y descendente se introducen en la siguiente definición.

Definición 2.6. *Sea* $(P, \leq)$ *un conjunto (parcialmente) ordenado. Decimos que* P *satisface la* condición de cadena ascendente*, si dada cualquier secuencia* $x_1 \leq x_2 \leq \cdots \leq x_n \leq \ldots$ *de elementos de* P*, existe* $k \in \mathbb{N}$ *tal que* $x_l = x_{l+1}$*, para todo* $l \in \mathbb{N}$ *con* $k \leq l$*. La* condición de cadena descendente *se define de forma dual.*

Una noción muy interesante, con mucho interés en distintos campos de la matemática y, en particular, en herramientas matemáticas para el tratamiento de datos es la noción de *conexión de Galois*.

Definición 2.7. *Sean* $(P_1, \leq_1)$ *y* $(P_2, \leq_2)$ *dos conjuntos parcialmente ordenados y sean* $^{\downarrow}: P_1 \to P_2$ *y* $^{\uparrow}: P_2 \to P_1$. *El par* $(^{\uparrow}, ^{\downarrow})$ *es una* conexión de Galois antítona *entre* P_1 *y* P_2 *si:*

1. $^{\downarrow}$ *y* $^{\uparrow}$ *son decrecientes.*
2. $p_1 \leq_1 p_1^{\downarrow\uparrow}$, *para todo* $p_1 \in P_1$.
3. $p_2 \leq_2 p_2^{\uparrow\downarrow}$, *para todo* $p_2 \in P_2$.

Existe también otra versión de conexión de Galois, que a diferencia de la definida anteriormente que invierte el orden, esta lo preserva.

Definición 2.8. *Sean* $(P_1, \leq_1)$ *y* $(P_2, \leq_2)$ *dos conjuntos parcialmente ordenados y sean* $^{\downarrow}: P_1 \to P_2$ *y* $^{\uparrow}: P_2 \to P_1$. *El par* $(^{\uparrow}, ^{\downarrow})$ *es una* conexión de Galois isótona *entre* P_1 *y* P_2 *si:*

1. $^{\downarrow}$ *y* $^{\uparrow}$ *son crecientes.*
2. $p_1^{\downarrow\uparrow} \leq_1 p_1$, *para todo* $p_1 \in P_1$.
3. $p_2 \leq_2 p_2^{\uparrow\downarrow}$, *para todo* $p_2 \in P_2$.

Por último, un conjunto ordenado se puede encerrar en un retículo completo usando la *compleción de Dedekind-MacNeille*, que se asocia con la conexión de Galois $(\,^u,\,^l)$ que mostramos en la siguiente definición.

Definición 2.9. *Sea* $(P, \leq)$ *un conjunto ordenado. La* compleción de Dedekind-MacNeille *de* P *se define como:*

$$DM(P) = \{A \subseteq P \mid A^{ul} = A\}$$

donde las aplicaciones $^u\colon 2^P \to 2^P$ *y* $^l\colon 2^P \to 2^P$ *se definen para un subconjunto* $A \subseteq P$ *como:*

$$A^u = \{x \in P \mid a \leq x, \text{ para todo } a \in A\}$$
$$A^l = \{x \in P \mid x \leq a, \text{ para todo } a \in A\}$$

Nótese que $\mathrm{DM}(P)$ es un sistema de clausura del conjunto de partes de P. Luego, el ínfimo coincide con la intersección y el supremo es la clausura de la unión. Por lo tanto, el conjunto ordenado $(\mathrm{DM}(P), \subseteq)$ es un retículo completo.

2.2. Congruencias sobre retículos

En este apartado se introduce la noción de congruencia sobre un retículo, así como algunas características y propiedades esenciales para la comprensión de los resultados expuestos en esta tesis. En primer lugar, se presenta la definición de relación de equivalencia compatible con las operaciones ínfimo y supremo de la estructura algebraica sobre la que se define la relación, esto es, la noción de *congruencia*.

Definición 2.10. *Dado un retículo* $(L, \wedge, \vee)$*, una relación* θ *sobre* L *se dice que es una* congruencia *sobre* L *si es una relación de equivalencia compatible con el supremo y el ínfimo de* $(L, \wedge, \vee)$*, es decir, si*

$$(a, b) \in \theta \ \text{ y } \ (c, d) \in \theta \text{ implica que } (a \vee c, b \vee d) \in \theta \ \text{ y } \ (a \wedge c, b \wedge d) \in \theta$$

para todo $a, b, c, d \in L$*.*

El siguiente resultado proporciona una forma de expresar las congruencias en términos de la relación de equivalencia.

Lema 2.2. *Dado un retículo* $(L, \wedge, \vee)$ *se satisface que* *(i) Una relación de equivalencia θ sobre L es una congruencia si y solo si, para todo $a, b, c \in L$,*

$$(a, b) \in \theta \text{ implica que } (a \vee c, b \vee c) \in \theta \text{ y } (a \wedge c, b \wedge c) \in \theta.$$

(ii) Sea θ una congruencia sobre L y sean $a, b, c \in L$.

(a) Si $(a, b) \in \theta$ y $a \leq c \leq b$, entonces $(a, c) \in \theta$.

(b) $(a, b) \in \theta$ si y solo si $(a \wedge b, a \vee b) \in \theta$.

Las clases de equivalencia de una congruencia son subretículos convexos del retículo original y además son *cuadriláteros cerrados* [43]. Para recordar el significado de cuadrilátero cerrado se necesita en primer lugar el significado de lados opuestos de un cuadrilátero. Sea $(L, \leq)$ un retículo, θ una relación de equivalencia sobre L y supongamos que $\{a, b, c, d\}$ es un subconjunto de L compuesto de cuatro elementos formando un cuadrilátero, entonces a, b y c, d se dicen ser *lados opuestos del cuadrilátero* $\langle a, b; c, d\rangle$ (véase Figura 2.1) si $a \prec b$, $c \prec d$ y se cumple que:

$$(a \vee d = b \text{ y } a \wedge d = c) \text{ o bien } (b \vee c = d \text{ y } b \wedge c = a).$$

Figura 2.1: Lados opuestos de un cuadrilátero.

Por tanto, la noción de cuadrilátero cerrado obliga a que siempre que dados dos lados opuestos de un cuadrilátero a, b, c, d, satisfaciendo que el lado a, b pertenece a una clase de equivalencia, entonces el lado c, d pertenece a otra clase o a la misma clase de equivalencia.

El siguiente resultado introduce una caracterización de las congruencias que será fundamental para el propósito de esta tesis.

Teorema 2.2 ([43]). *Sea $(L, \wedge, \vee)$ un retículo y sea θ una relación de equivalencia sobre L. Entonces θ es una congruencia si y solo si*

(i) cada clase de equivalencia de θ es un subretículo de L,

(ii) cada clase de equivalencia de θ es convexa,

(iii) las clases de equivalencia de θ son cuadriláteros cerrados.

El conjunto de las congruencias definidas sobre un retículo L se denota como Con L y, ordenado por la inclusión de conjuntos, forma un retículo completo. La menor y mayor congruencia vienen dadas por $\theta_\perp = \{(a,a) \mid a \in L\}$ y $\theta_\top = \{(a,b) \mid a,b \in L\}$, respectivamente.

2.3. Análisis de conceptos formales

Esta tesis se enmarca principalmente en el FCA, considerando tanto el ambiente clásico como la generalización difusa proporcionada por el paradigma multiadjunto. Recordaremos definiciones básicas del FCA que ayudaran a comprender todos los resultados presentados en esta tesis.

Definición 2.11. *Un* contexto *es un triple (A, B, R) formado por un conjunto de atributos A, un conjunto de objetos B y una relación $R \subseteq A \times B$ tal que $(a,x) \in R$, si el objeto $x \in B$ posee el atributo $a \in A$.*

Además, si se considera un contexto, las aplicaciones ${}^\uparrow\colon 2^B \to 2^A$ y ${}^\downarrow\colon 2^A \to 2^B$, se definen para cada $X \subseteq B$ y $Y \subseteq A$ como:

$$X^\uparrow = \{a \in A \mid (a,x) \in R, \text{para todo } x \in X\} \quad (2.1)$$

$$Y^\downarrow = \{x \in B \mid (a,x) \in R, \text{para todo } a \in Y\} \quad (2.2)$$

Estos operadores se denominan *operadores de derivación* y forman una conexión de Galois antítona [43], esto nos lleva a la siguiente definición.

Definición 2.12. *Dado un contexto* (A,B,R) *y los operadores de derivación* $^{\uparrow}$ *y* $^{\downarrow}$*, decimos que un par* (X,Y)*, con* $X \subseteq B$ *y* $Y \subseteq A$*, es un* concepto *si se cumplen las igualdades* $X^{\uparrow} = Y$ *y* $Y^{\downarrow} = X$*.*

Dado un concepto (X,Y), el subconjunto de objetos X se denomina *extensión* y el subconjunto de atributos Y, *intensión*. El conjunto de extensiones e intensiones se denotan como $\mathrm{Ext}(A,B,R)$ e $\mathrm{Int}(A,B,R)$, respectivamente.

Adicionalmente, dado un par de conceptos (X_1,Y_1) y (X_2,Y_2), se dice que $(X_1,Y_1) \leq (X_2,Y_2)$ si $X_1 \subseteq X_2$ ($Y_2 \subseteq Y_1$, equivalentemente). El conjunto de todos los conceptos con esta relación de orden tiene estructura de retículo completo, se denomina *retículo de conceptos* y se denota por $\mathcal{C}(A,B,R)$ [43, 52].

Además, si un concepto se genera por un atributo $a \in A$, es decir, es de la forma $(a^{\downarrow}, a^{\downarrow\uparrow})$, entonces se llama *atributo concepto* (dualmente se define un *objeto concepto*). Los conjuntos de objetos y atributos que generan un determinado concepto C, se definen como sigue:

Definición 2.13. *Dado un contexto formal* (A,B,R)*, el retículo de conceptos asociado* $\mathcal{C}(A,B,R)$ *y un concepto* $C \in \mathcal{C}(A,B,R)$*, entonces el conjunto de* objetos generadores *de* C *se define como el conjunto:*

$$Obg(C) = \{b \in B \mid (b^{\uparrow\downarrow}, b^{\uparrow}) = C\}$$

Análogamente, el conjunto de atributos generadores *de* C *es:*

$$Atg(C) = \{a \in A \mid (a^{\downarrow}, a^{\downarrow\uparrow}) = C\}$$

Los operadores de derivación anteriormente presentados no son los únicos operadores que pueden definirse sobre un contexto formal, existen otros operadores modales que pueden definirse para cada subconjunto de objetos $X \subseteq B$ y $Y \subseteq A$ [47, 53, 90]. Sean las aplicaciones ${}^{\uparrow_\pi}\colon 2^B \to 2^A$ y ${}^{\downarrow^N}\colon 2^A \to 2^B$, definidas como:

$$X^{\uparrow_\pi} = \{a \in A \mid \text{ existe } b \in X, \text{ tal que } (a,b) \in R\}$$
$$Y^{\downarrow^N} = \{b \in B \mid \text{ para todo } a \in A, \text{ si } (a,b) \in R, \text{ entonces } a \in Y\}$$

se denominan *operador de posibilidad y necesidad*, respectivamente. Análogamente, se definen las aplicaciones ${}^{\uparrow_N}\colon 2^B \to 2^A$, ${}^{\downarrow^\pi}\colon 2^A \to 2^B$ como:

$$X^{\uparrow_N} = \{a \in A \mid \text{ para todo } b \in B, \text{ si } (a,b) \in R, \text{ entonces } b \in X\}$$
$$Y^{\downarrow^\pi} = \{b \in B \mid \text{ existe } a \in Y, \text{ tal que } (a,b) \in R\}$$

Los pares $({}^{\uparrow_\pi}, {}^{\downarrow^N})$ y $({}^{\uparrow_N}, {}^{\downarrow^\pi})$ forman conexiones de Galois isótonas y, por lo tanto, se pueden usar para obtener conceptos. Los retículos de conceptos obtenidos se denominan *retículo de conceptos orientados a propiedades* y *retículo de conceptos orientados a objetos* [31, 66].

2.4. Reducción de atributos en el FCA

Dado un contexto (A, B, R), si consideramos un subconjunto de atributos $Y \subseteq A$ y la relación R restringida a $Y \times B$, que denotamos como $R_{|Y\times B}$, el triple $(Y, B, R_{|Y\times B})$ es también un contexto formal que se puede interpretar como un *subcontexto* del original. Hay dos nociones destacadas sobre subconjuntos de atributos en el FCA que recordamos a continuación.

Definición 2.14. *Dado un contexto (A, B, R), si existe un subconjunto de atributos $Y \subseteq A$ tal que $\mathcal{C}(A, B, R) \cong \mathcal{C}(Y, B, R_{|Y\times B})$, entonces decimos que Y es un conjunto* consistente *de (A, B, R). Además, si $\mathcal{C}(Y \setminus \{y\}, B, R_{|Y\setminus\{y\}\times B}) \ncong \mathcal{C}(A, B, R)$, para todo $y \in Y$, entonces Y es un* reducto *de (A, B, R).*

También se recuerda la definición de los tres tipos de atributos, considerando la notación de [85] para denotar los subconjuntos de atributos correspondientes a cada tipo.

Definición 2.15. *Dado un contexto formal* (A, B, R) *y el conjunto* $\mathfrak{R}$ *de todos los reductos de* (A, B, R)*, entonces el conjunto de atributos* A *puede dividirse como:*

1. Atributos absolutamente necesarios $C_f = \bigcap\{Y \mid Y \in \mathfrak{R}\}$.
2. Atributos relativamente necesarios $K_f = (\bigcup\{Y \mid Y \in \mathfrak{R}\}) \backslash (\bigcap\{Y \mid Y \in \mathfrak{R}\})$.
3. Atributos absolutamente innecesarios $I_f = A \backslash (\bigcup\{Y \mid Y \in \mathfrak{R}\})$.

En esta tesis, usaremos la noción de atributo absolutamente innecesario y, concretamente, la siguiente caracterización introducida en [66].

Teorema 2.3 ([66]). *Dado un contexto formal* (A, B, R) *y el conjunto de elementos ínfimo irreducibles de* $\mathcal{C}(A, B, R)$*, denotado por* $M_F(A, B, R)$*, se obtiene las siguientes equivalencias:*

1. $a \in I_f$ *si y solo si* $(a^{\downarrow}, a^{\downarrow\uparrow}) \notin M_F(A, B, R)$.
2. $a \in K_f$ *si y solo si* $(a^{\downarrow}, a^{\downarrow\uparrow}) \in M_F(A, B, R)$ *y existe* $a_1 \in A$, $a_1 \neq a$, *tal que* $(a_1^{\downarrow}, a_1^{\downarrow\uparrow}) = (a^{\downarrow}, a^{\downarrow\uparrow})$.
3. $a \in C_f$ *si y solo si* $(a^{\downarrow}, a^{\downarrow\uparrow}) \in M_F(A, B, R)$ *y* $(a_1^{\downarrow}, a_1^{\downarrow\uparrow}) \neq (a^{\downarrow}, a^{\downarrow\uparrow})$, *para todo* $a_1 \in A$, $a_1 \neq a$.

Por último, recordaremos dos resultados sobre reducción en el FCA [24] que han sido fundamentales para la motivación de esta tesis. El primer resultado muestra que cuando reducimos el conjunto de atributos de un contexto formal, se induce una relación de equivalencia sobre el retículo de conceptos original.

Proposición 2.1 ([24]). *Dado un contexto formal* (A, B, R) *y un subconjunto de atributos* $D \subseteq A$*, entonces el conjunto definido como:*

$$\rho_D = \{((X_1, Y_1), (X_2, Y_2)) \mid (X_1, Y_1), (X_2, Y_2) \in \mathcal{C}(A, B, R), X_1^{\uparrow_D\downarrow} = X_2^{\uparrow_D\downarrow}\},$$

es una relación de equivalencia, donde $^{\uparrow_D}$ *denota el operador de derivación definido en la Expresión* (2.1)*, sobre el contexto reducido* $(D, B, R_{|D\times B})$.

El segundo resultado expone que cada clase de equivalencia de la relación de equivalencia inducida tiene estructura de semirretículo superior con elemento máximo.

Proposición 2.2 ([24]). *Dado un contexto* (A, B, R)*, un subconjunto de atributos* $D \subseteq A$ *y una clase de equivalencia* $[(X, Y)]_D$ *del conjunto cociente* $\mathcal{C}(A, B, R)/\rho_D$*, se cumple que la clase* $[(X, Y)]_D$ *es un semirretículo superior con elemento máximo* $(X^{\uparrow_D\downarrow}, X^{\uparrow_D\downarrow\uparrow})$.

En [25], se extendieron estos resultados al ambiente del FCA difuso de los retículos de conceptos multiadjuntos. En la siguiente sección presentamos un resumen de este marco de trabajo.

2.5. Análisis de conceptos formales difusos

En esta tesis también consideramos la generalización difusa del FCA dada por el marco multiadjunto [69]. A continuación, recordamos las generalizaciones difusas de las nociones anteriores al marco multiadjunto.

En primer lugar, es necesario recordar los operadores llamados triples adjuntos [35, 37].

Definición 2.16. *Sean* $(P_1, \leq_1)$, $(P_2, \leq_2)$, $(P_3, \leq_3)$ *tres conjuntos parcialmente ordenados y sean* $\&\colon P_1 \times P_2 \to P_3$, $\swarrow\colon P_3 \times P_2 \to P_1$, $\nwarrow\colon P_3 \times P_1 \to P_2$ *tres aplicaciones definidas con respecto a esos conjuntos, se dice que* $(\&, \swarrow, \nwarrow)$ *es un* triple adjunto *con respecto a* P_1, P_2, P_3 *si:*

$$x \leq_1 z \swarrow y \quad \text{si y solo si} \quad x \,\&\, y \leq_3 z \quad \text{si y solo si} \quad y \leq_2 z \nwarrow x \tag{2.3}$$

para todo $x \in P_1$*,* $y \in P_2$ *y* $z \in P_3$*. La condición* (2.3) *se denomina* propiedad de adjunción.

Además, para definir la noción de contexto multiadjunto, tenemos que fijar una estructura algebraica llamada marco multiadjunto.

Definición 2.17. *Un* marco multiadjunto *es una tupla* $(L_1, L_2, P, \&_1, \dots, \&_n)$*, donde* $(L_1, \preceq_1)$ *y* $(L_2, \preceq_2)$ *son retículos completos,* $(P, \leq)$ *es un conjunto parcialmente ordenado y* $(\&_i, \swarrow^i, \nwarrow_i)$ *es un triple adjunto con respecto a* L_1, L_2, P*, para todo* $i \in \{1, \dots, n\}$.

Una vez fijado un marco multiadjunto, podemos introducir la noción de contexto multiadjunto.

Definición 2.18. *Sea* $(L_1, L_2, P, \&_1, \dots, \&_n)$ *un marco multiadjunto. Un* contexto multiadjunto *es una tupla* (A, B, R, σ)*, con* A *y* B *conjuntos no vacíos (generalmente interpretados como atributos y objetos, respectivamente),* R *es una relación* P*-difusa* $R: A \times B \to P$ *y* $\sigma: A \times B \to \{1, \dots, n\}$ *es una aplicación que asocia cualquier elemento en* $A \times B$ *con algún triple adjunto del marco.*

Asimismo, podemos encontrar la generalización de los operadores de derivación ${}^{\uparrow_\sigma}: L_2^B \to L_1^A$ y ${}^{\downarrow^\sigma}: L_1^A \to L_2^B$ en el marco multiadjunto que se definen como:

$$\begin{aligned} g^{\uparrow_\sigma}(a) &= \inf\{R(a,b) \swarrow^{\sigma(a,b)} g(b) \mid b \in B\} \\ f^{\downarrow^\sigma}(b) &= \inf\{R(a,b) \nwarrow_{\sigma(a,b)} f(a) \mid a \in A\} \end{aligned}$$

para todo $g \in L_2^B$, $f \in L_1^A$ y $a \in A$, $b \in B$, donde L_2^B y L_1^A denotan el conjunto de aplicaciones $g: B \to L_2$ y $f: A \to L_1$, respectivamente. De forma equivalente, un *concepto multiadjunto* es un par $\langle g, f\rangle$, donde $g \in L_2^B$ es un subconjunto difuso de objetos y $f \in L_1^A$ es un subconjunto difuso de atributos, que satisfacen $g^{\uparrow_\sigma} = f$ y $f^{\downarrow^\sigma} = g$. Además, la noción de retículo de conceptos multiadjuntos se formaliza en la siguiente definición.

Definición 2.19. *El* retículo de conceptos multiadjunto *asociado a un marco multiadjunto* $(L_1, L_2, P, \&_1, \dots, \&_n)$ *y a un contexto multiadjunto* (A, B, R, σ)

dados, es el conjunto

$$\mathcal{M} = \{\langle g, f\rangle \mid g \in L_2^B, f \in L_1^A \text{ y } g^{\uparrow_\sigma} = f, f^{\downarrow^\sigma} = g\}$$

junto al orden definido por $\langle g_1, f_1\rangle \preceq \langle g_2, f_2\rangle$ *si y solo si* $g_1 \preceq_2 g_2$ *(equivalentemente* $f_2 \preceq_1 f_1$*), para todo* $\langle g_1, f_1\rangle, \langle g_2, f_2\rangle \in \mathcal{M}$.

De ahora en adelante, simplificaremos la notación escribiendo $^\uparrow$ y $^\downarrow$ en lugar de $^{\uparrow_\sigma}$ y $^{\downarrow^\sigma}$, respectivamente.

De manera similar a como ocurre en el ambiente clásico, los operadores $^\uparrow$ y $^\downarrow$ no son los únicos operadores que se puede definir en un contexto multiadjunto. Las definiciones clásicas de los operadores de necesidad y posibilidad también fueron generalizados en un marco de trabajo difuso [65] en el que se fijan dos retículos completos $(L_1, \preceq_1), (L_2, \preceq_2)$ y un conjunto parcialmente ordenado $(P, \leq)$. Un marco multiadjunto orientado a propiedades viene dado por $(L_1, L_2, P, \&_1, \dots, \&_n)$, donde $(\&_i, \swarrow^i, \nwarrow_i)$ es un triple adjunto con respecto a P, L_2, L_1 para todo $i \in \{1, \dots, n\}$. En esta álgebra multiadjunta, los operadores de necesidad y posibilidad son las aplicaciones $^{\downarrow^N}\colon L_1^A \to L_2^B$, $^{\uparrow_\pi}\colon L_2^B \to L_1^A$, definidas como

$$\begin{aligned} g^{\uparrow_\pi}(a) &= \sup\{R(a,b) \&_{\sigma(a,b)} g(b) \mid b \in B\} \\ f^{\downarrow^N}(b) &= \inf\{f(a) \nwarrow_{\sigma(a,b)} R(a,b) \mid a \in A\} \end{aligned}$$

para todo $a \in A, b \in B, g \in L_2^B$ y $f \in L_1^A$. Un marco multiadjunto orientado a objetos es la tupla $(L_1, L_2, P, \&_1, \dots, \&_n)$, donde $(\&_i, \swarrow^i, \nwarrow_i)$ es un triple adjunto con respecto a L_1, P, L_2 para todo $i \in \{1, \dots, n\}$. En esta álgebra estos operadores son las aplicaciones $^{\uparrow_N}\colon L_2^B \to L_1^A$, $^{\downarrow^\pi}\colon L_1^A \to L_2^B$, definidas como:

$$\begin{aligned} g^{\uparrow_N}(a) &= \inf\{g(b) \swarrow^{\sigma(a,b)} R(a,b) \mid b \in B\} \\ f^{\downarrow^\pi}(b) &= \sup\{f(a) \&_{\sigma(a,b)} R(a,b) \mid a \in A\} \end{aligned}$$

para todo $a \in A, b \in B, g \in L_2^B$ y $f \in L_1^A$. Los pares de operadores $(^{\uparrow_\pi}, ^{\downarrow^N})$ y $(^{\uparrow_N}, ^{\downarrow^\pi})$ forman conexiones de Galois isótonas.

Una vez fijado un marco multiadjunto podemos obtener un marco multiadjunto orientado a objetos y un marco multiadjunto orientado a propiedades considerando los órdenes duales en los retículos y el conjunto parcialmente ordenado del marco multiadjunto considerado. Para una información más detallada sobre la relación existente entre los marcos multiadjuntos, los orientados a objetos y los orientados a propiedades, véase [65]. Adicionalmente, un retículo de conceptos orientado a propiedades se denota como $\mathcal{M}_{\pi N}$, y un retículo de conceptos orientado a objetos como $\mathcal{M}_{N\pi}$.

Capítulo 3

Discusión conjunta de los resultados

En este capítulo se exponen los principales resultados de la investigación que se ha llevado a cabo a lo largo de esta tesis doctoral. En particular, se presenta un nuevo mecanismo útil para complementar las reducciones de atributos en el FCA que se ha llevado a cabo a través de relaciones de equivalencia y que puede aplicarse tanto en ambientes clásicos como difusos. Estos resultados han sido publicados en [6, 7, 8], donde se muestra que las nuevas nociones presentadas sirven para obtener estructuras algebraicas de gran interés y valor para la reducción de retículos de conceptos.

3.1. Congruencia local

En esta sección, introducimos la noción de congruencia local presentada en [6, 10], junto con las principales propiedades que tiene, algunas de ellas heredadas de las congruencias.

Sabemos que las clases de equivalencia inducidas por una reducción de atributos tienen estructura de semirretículos superiores, y por lo tanto, cada clase no tiene porqué ser una estructura cerrada. Debido a las bondades de poder obtener estructuras cerradas, surge la necesidad de generar grupos de conceptos mediante la definición de una nueva relación de equivalencia cuyas clases tengan una estructura cerrada, para complementar las reducciones dadas en el FCA. Por lo tanto, estamos interesados en que las clases formen subretículos del retículo original.

Las relaciones de congruencia cumplen con este propósito, pero tienen un inconveniente a la hora de aplicarlas. Como se ilustra en el Ejemplo 12 de [6], donde se utiliza la menor congruencia que contiene a la relación inducida, podemos ver que se obtienen agrupaciones que contienen muchos conceptos, lo que puede traducirse en una pérdida de información relevante. El resultado obtenido en dicho ejemplo se debe a lo restrictivas que son las condiciones que deben satisfacer las relaciones de congruencia, como la propiedad de cuadrilátero cerrado y la compatibilidad con el supremo e ínfimo. Estas propiedades provocan que las clases de equivalencia de una congruencia contengan numerosos elementos del retículo al que se aplica lo que, como hemos mencionado, puede conducir a la eliminación de información relevante. Este hecho evidencia la necesidad de proporcionar una noción debilitada de congruencia, la cual surge a partir de la eliminación de la propiedad de cuadrilátero cerrado y preservando las otras dos propiedades que aparecen en el Teorema 2.2.

Definición 3.1. *Dado un retículo $(L, \leq)$, decimos que una relación de equivalencia δ sobre L es una* congruencia local *si se satisfacen las siguientes propiedades:*

(i) cada clase de equivalencia de δ es un subretículo de L,

(ii) cada clase de equivalencia de δ es convexa.

Además, esta nueva noción la podemos caracterizar en términos de la relación de equivalencia δ a partir de sus clases de equivalencia.

Proposición 3.1. *Dado un retículo* $(L, \leq)$ *y una relación de equivalencia* δ *sobre* L*, la relación* δ *es una congruencia local sobre* L *si y solo si, para cada* $a, b, c \in L$*, se cumplen las siguientes propiedades:*

(i) Si $(a, b) \in \delta$ *y* $a \leq c \leq b$*, entonces* $(a, c) \in \delta$*.*

(ii) $(a, b) \in \delta$ *si y solo si* $(a \wedge b, a \vee b) \in \delta$*.*

Demostración. Véase la demostración de la Proposición 15 en [6]. □

Aunque esta nueva definición es una definición débil de la noción de congruencia, el nombre de congruencia débil ya ha sido usado anteriormente en otras publicaciones, como en [81, 82, 83, 84] para definir congruencias que no satisfacen la propiedad reflexiva, es decir, una congruencia débil es una relación simétrica, transitiva y compatible con los operadores ínfimo y supremo del retículo. Por este motivo, se ha considerado otro nombre más apropiado, como es el de congruencia local, y cuya justificación viene dada por el siguiente resultado.

Proposición 3.2. *Dado un retículo* $(L, \leq)$ *se satisface que una relación de equivalencia* δ *sobre* L *es una congruencia local si y solo si, para todo* $a, b, c \in L$*,*

$$(a, b) \in \delta \ \text{ y } \ a \wedge b \leq c \leq a \vee b \text{ implica que } (a \vee c, b \vee c) \in \delta \ \text{ y } \ (a \wedge c, b \wedge c) \in \delta.$$

Demostración. Se encuentra en la prueba de la Proposición 16 en [6]. □

A diferencia de la equivalencia dada en el Lema 2.2(i) donde el elemento c es arbitrario en L, en esta caracterización c es un elemento *local* acotado por $a \wedge b$ y $a \vee b$ lo que motiva el nombre de la noción considerada.

Cabe destacar que la congruencia local se define sobre un retículo L cualquiera y no sobre un retículo de conceptos. Esto origina dos líneas de trabajo: por un lado se estudiarán propiedades que la congruencia local tenga en un entorno general (sobre un retículo cualquiera) y por otro lado qué propiedades adicionales se obtienen si el retículo considerado es un retículo de conceptos. A lo largo de esta sección nos centraremos en estudiar sus propiedades generales.

3.1.1. El conjunto cociente de una congruencia local

Un problema que surge a la hora de construir las clases de equivalencia de una congruencia local es que no heredan la estructura de retículo de forma natural. Por lo tanto, una vez introducida la noción de congruencia local, en [6] se muestra un estudio de la estructura algebraica que se puede definir sobre el conjunto de todas las clases de equivalencia generadas por una congruencia local. El objetivo es conseguir definir un orden parcial, tal que el conjunto cociente proporcionado por una congruencia local junto con dicho orden parcial tenga la estructura de retículo completo. Para ello, debemos establecer una relación de orden entre las clases de equivalencia, para lo que se hace necesaria la siguiente definición.

Definición 3.2. *Sea $(L, \leq)$ un retículo y δ una congruencia local sobre L.*

*(i) Una sucesión de elementos de L, $(p_0, p_1, \ldots, p_n)$ con $n \geq 1$, recibe el nombre de δ-*sucesión *y se denota como $(p_0, p_n)_\delta$, si para cada $i \in \{1, \ldots, n\}$ se cumple $(p_{i-1}, p_i) \in \delta$ o bien $p_{i-1} \leq p_i$.*

*(ii) Si una δ-sucesión, $(p_0, p_n)_\delta$, satisface que $p_0 = p_n$, entonces recibe el nombre de δ-*ciclo*. Además, si el δ-ciclo satisface que $[p_0]_\delta = [p_1]_\delta = \cdots = [p_n]_\delta$ se denomina* cerrado*.*

Las nociones mostradas en la Definición 3.2 son ilustradas en la Figura 3.1 y la Figura 3.2, donde la triple línea vertical que conecta p_{i-1} con p_i significa que están relacionados bajo la relación de congruencia local considerada, es decir, $(p_{i-1}, p_i) \in \delta$. La línea simple indica que los dos elementos están conectados por medio de la relación de orden definida en el retículo.

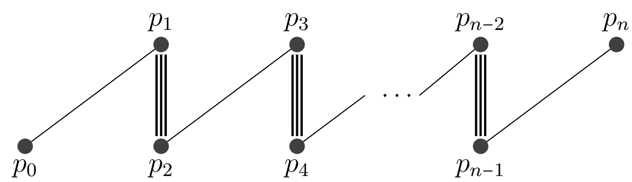

Figura 3.1: Ejemplo de δ-sucesión.

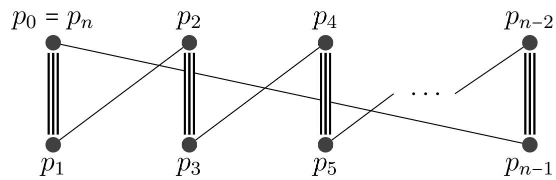

Figura 3.2: Ejemplo de δ-ciclo.

La siguiente definición muestra un primer paso para definir un orden parcial en el conjunto cociente de una congruencia local.

Definición 3.3. *Dado un retículo $(L, \leq)$ y una congruencia local δ sobre L, se define la relación binaria $\leq_\delta$ sobre L/δ como sigue:*

$$[x]_\delta \leq_\delta [y]_\delta \quad \textit{si existe una } \delta\textit{-sucesión } (x', y')_\delta$$

para un $x' \in [x]_\delta$ y un $y' \in [y]_\delta$.

Es conveniente mencionar que la relación binaria $\leq_\delta$ definida anteriormente satisface las propiedades reflexiva y transitiva, pero no se cumple la propiedad antisimétrica para cualquier congruencia local en general, véase el Ejemplo 20 de [6]. El principal problema reside en los δ-ciclos que se forman ya que la existencia de estos puede hacer que no se cumpla la propiedad antisimétrica. Por tanto, se trata de un preorden. En [6] enunciamos el siguiente resultado que establece la condición bajo la cual se satisface la propiedad antisimétrica, y por consiguiente, obtendríamos un orden parcial.

Teorema 3.1. *Dado un retículo* $(L, \preceq)$ *y una congruencia local* δ *sobre* L*, el preorden* $\preceq_\delta$ *dado en la Definición 3.3 es un orden parcial si y solo si todo* δ*-ciclo en* L *es cerrado.*

Demostración. Se halla en la prueba del Teorema 24 en [6]. □

Como consecuencia, este resultado nos permite tener un orden entre las clases de equivalencia bajo la condición necesaria y suficiente enunciada. No obstante, esta jerarquía no siempre tiene la estructura de retículo completo como ilustramos en el Ejemplo 25 en [6]. Por tanto, necesitamos estudiar más propiedades de las congruencias locales para asegurar que se pueda definir siempre una congruencia local de forma que el orden de la Definición 3.3 sea un orden parcial, por ejemplo que satisfaga que todo δ-ciclo sea cerrado. En concreto, es importante analizar la relación entre las distintas congruencias locales que se pueden definir sobre un mismo retículo.

3.1.2. Estructura algebraica de las congruencias locales

A continuación, nos centramos en el estudio de la estructura algebraica que forma el conjunto de todas las congruencias locales definidas sobre un retículo. En primer lugar, ya que las congruencias locales son relaciones de equivalencia, se puede hacer uso de la definición de inclusión de relaciones de equivalencia, que se recuerda a continuación.

Definición 3.4. *Sea* ρ_1 *y* ρ_2 *dos relaciones de equivalencia sobre un retículo* $(L, \preceq)$*. Se dice que la relación de equivalencia* ρ_1 *está incluida en* ρ_2*, y se denota como* $\rho_1 \sqsubseteq \rho_2$*, si para toda clase de equivalencia* $[x]_{\rho_1} \in L/\rho_1$ *existe una clase de equivalencia* $[y]_{\rho_2} \in L/\rho_2$ *tal que* $[x]_{\rho_1} \subseteq [y]_{\rho_2}$*.*

En adelante, el conjunto de todas las congruencias locales sobre un retículo $(L, \preceq)$ ordenado por la inclusión $\sqsubseteq$ se denotará como $(\text{LCon}\, L, \sqsubseteq)$. El siguiente resultado establece que el conjunto LCon L junto con la inclusión tiene estructura de retículo completo, y se caracteriza además el elemento mínimo y máximo del retículo.

Teorema 3.2. *Dado un retículo* $(L, \leq)$*, el conjunto* $(LCon\ L, \sqsubseteq)$ *es un retículo completo. Además, la menor y la mayor congruencia local vienen dadas por* $\delta_\perp = \{(a,a) \mid a \in L\}$ *y* $\delta_\top = \{(a,b) \mid a,b \in L\}$*, respectivamente.*

Demostración. Se encuentra en la demostración del Teorema 27 en [6]. □

Al ser las congruencias locales una noción debilitada de congruencia, el conjunto LCon L contiene a las congruencias que puedan definirse sobre L. Además, al igual que ocurre con las congruencias, podemos definir la menor congruencia local a partir de dos elementos dados del retículo.

Definición 3.5. *Dado un retículo* $(L, \leq)$ *y un par de elementos* $(a,b) \in L \times L$*, la* congruencia local principal generada por (a,b)*, y se denota como* $\delta_{(a,b)}$*, es la menor congruencia local que contiene a los elementos* a *y* b *en la misma clase de equivalencia, es decir,*

$$\delta_{(a,b)} = \bigwedge\{\delta \in LCon\ L \mid (a,b) \in \delta\}.$$

Se puede observar que la congruencia local principal para cualquier par de elementos siempre existe, ya que el conjunto (LCon $L, \sqsubseteq$) es un retículo completo. Además, las congruencias locales principales pueden ayudar directamente a encontrar la menor congruencia local que contenga una relación de equivalencia arbitraria definida sobre un retículo.

Teorema 3.3. *Dado un retículo* $(L, \leq)$ *y una relación de equivalencia* ρ *sobre* L*, la menor congruencia local que contiene a* ρ *es:*

$$\delta_\rho = \bigvee\{\delta_{(a,b)} \mid (a,b) \in \rho\}.$$

Demostración. Véase la demostración del Teorema 29 en [6]. □

En particular, el anterior resultado también se satisface si consideramos una congruencia local en lugar de una relación de equivalencia arbitraria por lo que permite caracterizar las congruencias locales a partir de las principales. Obsérvese también que este resultado será esencial para el procedimiento de reducción que se presentará a continuación, ya que nos permite encontrar una congruencia local que satisfaga que $(L/\delta, \preceq_\delta)$ sea un conjunto parcialmente ordenado, además de que sea la menor congruencia local, lo que proporciona el menor agrupamiento de conceptos posible.

3.1.3. Mecanismo de reducción de retículos de conceptos

Como se mencionó al inicio de esta sección, uno de los objetivos planteados al introducir la noción de congruencia local en el FCA es minimizar la cantidad de información perdida en comparación con el uso de congruencias. A continuación, mostramos un primer mecanismo de reducción de atributos que nos permite agrupar conceptos en subretículos convexos, obteniendo una jerarquía en forma de conjunto parcialmente ordenado, lo que equivale a calcular una congruencia local con todos los δ-ciclos cerrados, de acuerdo al Teorema 3.1.

Fijado un contexto formal, en un marco clásico o multiadjunto, cuyo conjunto de atributo es A y su retículo de conceptos asociado es isomorfo a un retículo $(L, \preceq)$; si una congruencia local δ definida sobre L satisface el Teorema 3.1, se podría concluir que $(L/\delta, \preceq_\delta)$ es un conjunto parcialmente ordenado. En caso contrario, podemos definir una relación de equivalencia ρ_δ sobre el conjunto cociente de la congruencia local como se muestra a continuación:

$$\rho_\delta = \{([x]_\delta, [y]_\delta) \in L/\delta \times L/\delta \mid [x]_\delta \preceq_\delta [y]_\delta \text{ y } [y]_\delta \preceq_\delta [x]_\delta\} \qquad (3.1)$$

Luego, el procedimiento continuaría de la siguiente manera:

- Sea un retículo de conceptos $(L, \preceq)$, un subconjunto de atributos $D \subseteq A$ y la relación de equivalencia ρ_D inducida por la reducción de atributos dada por D.
- Se calcula la menor congruencia local δ_D que contiene a ρ_D, utilizando el Teorema 3.3.
- Si δ_D tiene todos los δ_D-ciclos cerrados, entonces es la congruencia local que buscamos.
- Si δ_D tiene algún δ_D-ciclo que no es cerrado, se debe a que hay dos clases diferentes $[x]_{\delta_D}$, $[y]_{\delta_D}$ tales que $[x]_{\delta_D} \preceq_{\delta_D} [y]_{\delta_D}$ y $[y]_{\delta_D} \preceq_{\delta_D} [x]_{\delta_D}$.
- Luego, se define la relación de equivalencia como en (3.1) a la que llamamos ρ_{δ_D}. Observar que esta relación agrupa todas las clases de equivalencia que contienen elementos de un δ_D-ciclo en una única clase de equivalencia.
- Se debe comprobar si ρ_{δ_D} es una congruencia local o no. Si lo es, entonces es la congruencia local que buscamos. En caso contrario, volvemos a aplicar el Teorema 3.3 para obtener la menor congruencia local δ tal que $\delta_D \sqsubseteq \rho_{\delta_D} \sqsubseteq \delta$.

Este procedimiento se muestra en el Algoritmo 1 dado en [6] y un ejemplo de su aplicación se ha ilustrado en el Ejemplo 32 de [6]. Además, se puede observar en este algoritmo, que el conjunto de atributos D se puede haber obtenido a partir de cualquier mecanismo de reducción, como por ejemplo, un reducto propio de la RST [24, 25].

Por lo tanto, este procedimiento sirve para establecer un orden parcial entre las clases de equivalencia y, por consiguiente, para dotar al conjunto $(L/\delta, \preceq_\delta)$ de la estructura de conjunto parcialmente ordenado, siendo δ la menor congruencia local obtenida por este mecanismo de reducción.

Proposición 3.3. *Dado un retículo de conceptos $\mathcal{C}(A,B,R)$ y un subconjunto de atributos $D \subseteq A$, se satisface que el Algoritmo 1 en [6] proporciona la menor congruencia local δ que contiene la relación inducida ρ_D y que $(\mathcal{C}(A,B,R)/\delta, \leq_\delta)$ es un conjunto parcialmente ordenado.*

Demostración. Se encuentra en la prueba de la Proposición 31 en [6]. □

Este primer mecanismo de reducción presentado en [6] puede aplicarse tanto en el ambiente clásico como difuso del FCA, partiendo de la relación ρ_D que se define en el caso clásico en la Proposición 2.1 o de la dada para el caso difuso en [25]. En la siguiente sección nos centramos en el caso clásico y se mostrará un análisis más profundo sobre el impacto generado al aplicar una congruencia local.

3.2. Identificación de clases de equivalencias

En esta sección, analizamos la influencia de las congruencias locales en las reducciones de atributos en el FCA. En los ejemplos presentados en [9] se puede observar que la menor congruencia local que contiene a la relación de equivalencia inducida por una reducción de atributos no siempre coincide con la inducida. Este hecho implica que el uso de una congruencia local altera la reducción de atributos original. Por tanto, en el artículo [7], realizamos un estudio para conocer las condiciones que se deben satisfacer para que una relación de equivalencia inducida no proporcione subretículos convexos como clases de equivalencia, es decir, bajo qué condiciones las congruencias locales modifican la reducción original.

A lo largo de esta sección se fija un contexto formal (A,B,R) en una ambiente clásico y denotaremos como C_M al elemento máximo de una clase de equivalencia $[C]_D$, con $D \subseteq A$ y $C \in \mathcal{C}(A,B,R)$. Además, denotaremos como $\overline{C}$ al concepto asociado a la clase en el retículo de conceptos reducido, $\overline{C} \in \mathcal{C}(D, B, R_{|D\times B})$, con el objetivo de diferenciar cuándo nos referimos al retículo de conceptos reducido o cuándo al retículo de conceptos original. La notación utilizada para denotar el ínfimo de una clase de equivalencia se muestra en la siguiente definición, para simplificar las expresiones donde está involucrado.

Definición 3.6. *Dado un contexto (A, B, R), un subconjunto de atributos $D \subseteq A$, y una clase de equivalencia $[C]_D$, con $C \in \mathcal{C}(A, B, R)$, de la relación de equivalencia inducida, el ínfimo del subconjunto de conceptos $[C]_D$ se denota como C_m, es decir, $C_m = \bigwedge_{C_i \in [C]_D} C_i$.*

En [9], iniciamos un estudio preliminar sobre las consecuencias que tienen lugar cuando una clase de la relación de equivalencia inducida por una reducción de atributos no contiene al ínfimo. En otras palabras, identificar bajo qué condiciones se obtiene una clase que es un semirretículo superior. El siguiente resultado, publicado en [9], es un primer paso para identificar unívocamente este tipo de clase.

Proposición 3.4. *Sea (A, B, R) un contexto, $D \subseteq A$ un subconjunto de atributos y $[C]_D$ una clase de equivalencia de la relación de equivalencia inducida, con $C \in \mathcal{C}(A, B, R)$, que no es un subretículo convexo. Entonces una de las siguientes afirmaciones se satisface:*

- *Existe al menos un atributo $a \in D$ tal que $C_m = (a^{\downarrow}, a^{\downarrow\uparrow})$.*
- *Existe un concepto $C^* \in M_F(A, B, R)$ en una descomposición de ínfimo irreducibles $\{C_j \in M_F(A, B, R) \mid j \in J\}$ de C_m, tal que $C_{i_0} \nleq C^*$ para un concepto $C_{i_0} \in [C]_D$.*

Demostración. Véase la prueba de la Proposición 4 en [9]. □

Es importante mencionar que las dos afirmaciones del resultado anterior no son excluyentes, es decir, se puede encontrar un concepto que satisfaga ambas afirmaciones.

3.2.1. Caracterización del ínfimo de clases

De la Proposición 3.4 partimos de la hipótesis de que la clase considerada no es un subretículo convexo, puesto que las relaciones inducidas que tengan este tipo de clase (semirretículo superior con elemento máximo) se alterarán al aplicarles una congruencia local. Un elemento fundamental para determinar este tipo de clases es el ínfimo de estas, que como podemos observar, es el elemento común en las afirmaciones de la Proposición 3.4. Por tanto, nuestro objetivo es encontrar las condiciones que permitan caracterizar clases que no son subretículos convexos a través del ínfimo de dichas clases.

Uno puede pensar que el motivo de que el ínfimo de una clase no pertenezca a la clase se debe a que el ínfimo es un atributo concepto cuyo atributo generador se conserva después de realizar la reducción. No obstante, este hecho no es cierto en general y se deben cumplir ciertas condiciones adicionales como se expone en el siguiente teorema.

Teorema 3.4. *Sea (A, B, R) un contexto, $D \subseteq A$ un subconjunto finito de atributos y $C \in \mathcal{C}(A, B, R)$ un concepto tal que $C_j \in [C]_D$, para todo concepto C_j en cualquier descomposición de ínfimo irreducibles $\{C_j \in M_F(A, B, R) \mid j \in J\}$ de C_m. Si C_m no está en $[C]_D$, entonces existe un atributo $a \in D$ tal que $[C_m]_D = [(a^{\downarrow}, a^{\downarrow\uparrow})]_D$.*

Demostración. Es la demostración del Teorema 3 en [7]. □

Se puede observar que el Teorema 3.4 surge de la restricción de las hipótesis de la Proposición 3.4. Además, hay que destacar que la clase que considera el resultado anterior no impone ninguna condición en su estructura algebraica, aunque no es suficiente para caracterizar el ínfimo de las clases y uno de los motivos de que esto ocurra es la posible existencia de clases de equivalencia no triviales, es decir, clases que contienen más de un concepto. Sin embargo, solo nos interesa ver si los ínfimos de las clases no triviales pertenecen o no a las clases correspondientes, como caracterizamos en el siguiente teorema que es uno de los principales resultados que presentamos en [7].

Teorema 3.5. *Dado un contexto* (A, B, R), *un subconjunto de atributos* $D \subseteq A$, *y un concepto* $C \in \mathcal{C}(A, B, R)$ *tal que su clase de equivalencia* $[C]_D$ *por la relación de equivalencia inducida no es trivial, entonces* $C_m \notin [C]_D$ *si y solo si una de las siguientes afirmaciones se satisface.*

- *Existe al menos un atributo* $a \in D$ *tal que* $C_m = (a^{\downarrow}, a^{\downarrow\uparrow})$.
- *Existe un concepto* $C^* \in \mathcal{C}(A, B, R)$, *tal que* $C^* = (a^{*\downarrow}, a^{*\downarrow\uparrow})$ *con* $a^* \in D$, $C^* \notin [C]_D$ *y* $C_M \nleq C^*$. *Además,* $\overline{C^*}$ *está en una descomposición de ínfimo irreducibles* $\{\overline{C_j} \in M_F(D, B, R_{|D\times B}) \mid j \in J\}$ *de* $\overline{C_m}$.

Demostración. Se encuentra en la prueba del Teorema 4 en [7]. □

Como podemos observar, el Teorema 3.5 mejora la Proposición 3.4, ya que el concepto C^* debe ser generado por un atributo del subconjunto de atributos reducido D, y no necesita ser un concepto ínfimo irreducible para obtener una condición suficiente y necesaria que determine que el ínfimo de una clase no pertenezca a esta. Ilustramos una aplicación del Teorema 3.5 en el Ejemplo 2 de [7].

3.2.2. Reducción de atributos sin atributos innecesarios

Es interesante analizar la caracterización del ínfimo dada en el Teorema 3.5, cuando el subconjunto de atributos reducido considerado no contiene atributos innecesarios. Generalmente las reducciones de atributos en el FCA eliminan este tipo de atributos, pero también se puede considerar para cualquier estrategia de reducción de atributos fusionando el FCA y otros marcos de trabajo como la RST [23, 25, 26].

Fijarnos precisamente en los atributos innecesarios no es una casualidad. Ya podíamos intuir que la primera afirmación del Teorema 3.5 podía deberse a los atributos innecesarios y así es ciertamente, como enunciamos en el siguiente resultado.

Proposición 3.5. *Dado un contexto (A, B, R), un subconjunto de atributos $D \subseteq A$ y un concepto $C \in \mathcal{C}(A, B, R)$, si la clase de equivalencia $[C]_D$ de la relación inducida no es trivial y existe un atributo $a \in D$ tal que $C_m = (a^{\downarrow}, a^{\downarrow\uparrow})$, entonces $a \in I_f$.*

Demostración. Es la demostración de la Proposición 8 en [7]. □

Como consecuencia de este último resultado, podemos partir de un subconjunto de atributos $D \subseteq A \backslash I_f$ y de acuerdo a esta consideración, el Teorema 3.5 puede expresarse de la siguiente forma:

Corolario 3.1. *Dado un contexto (A, B, R), un subconjunto de atributos $D \subseteq A$ tal que $D \subseteq A \backslash I_f$ y un concepto $C \in \mathcal{C}(A, B, R)$, donde $[C]_D$ no es trivial, entonces $C_m \notin [C]_D$ si y solo si existe $C^* \in \mathcal{C}(A, B, R)$, tal que $C^* = (a^{*\downarrow}, a^{*\downarrow\uparrow})$ con $a^* \in D$, $C^* \notin [C]_D$, $C_M \nleq C^*$ y $\overline{C^*}$ está en una descomposición de ínfimo irreducibles $\{\overline{C_j} \in M_F(D, B, R_{|D \times B}) \mid j \in J\}$ de $\overline{C_m}$.*

Demostración. Se obtiene directamente del Teorema 3.5 y la Proposición 3.5. □

Como norma general, el concepto ínfimo irreducible $\overline{C^*}$ del retículo de conceptos reducido que aparece en el resultado anterior no tiene asociado un concepto ínfimo irreducible en el retículo de conceptos original. Cuando esto ocurre, nos damos cuenta que vuelven a aparecer los atributos innecesarios, ya que un concepto del retículo de conceptos original generado por un atributo innecesario (se puede obtener como ínfimos de conceptos ínfimo irreducibles), puede convertirse en un concepto ínfimo irreducible del retículo de conceptos reducido si se conserva el atributo innecesario en la reducción. Este suceso lo ilustramos en el Ejemplo 3 de [7]. Como consecuencia, podemos proporcionar una condición suficiente para asegurar la equivalencia entre conceptos ínfimo irreducibles del contexto original y del reducido.

Proposición 3.6. *Dado un contexto* (A, B, R)*, un subconjunto de atributos* $D \subseteq A \backslash I_f$ *y un concepto* $C \in \mathcal{C}(A, B, R)$*, tal que* $C = (a^{\downarrow}, a^{\downarrow\uparrow})$ *con* $a \in D$*, se cumple la siguiente equivalencia:*

$$C \in M_F(A, B, R) \text{ si y solo si } \overline{C} \in M_F(D, B, R_{|D \times B})$$

Demostración. Es la prueba de la Proposición 9 en [7]. □

Por lo tanto, partiendo del Corolario 3.1 y de la Proposición 3.6 podemos reescribir el Teorema 3.5 obteniendo así una caracterización basada principalmente en conceptos del retículo original.

Teorema 3.6. *Dado un contexto* (A, B, R)*, un subconjunto de atributos* $D \subseteq A \backslash I_f$*, una clase de equivalencia* $[C]_D$ *con* $C \in \mathcal{C}(A, B, R)$ *de la relación de equivalencia inducida, y el concepto* C_m*, entonces* $C_m \notin [C]_D$ *si y solo si existe un concepto* $C^* \in M_F(A, B, R)$ *en una descomposición de ínfimo irreducibles* $\{C_j \in M_F(A, B, R) \mid j \in J\}$ *de* C_m*, tal que* $C^* = (a^{*\downarrow}, a^{*\downarrow\uparrow})$*, con* $a^* \in D$ *y* $C_M \nleq C^*$*.*

Demostración. Véase la demostración del Teorema 5 en [7]. □

Los reductos en el FCA son un claro ejemplo de subconjuntos de atributos que no contienen atributos innecesarios [52, 66]. Además, si prestamos atención a la condición $C_M \nleq C^*$ podemos ver una relación con los retículos no distributivos. En particular, podemos determinar el tipo de retículo para el cual, después de aplicar cualquier reducción de atributos, obtenemos clases de equivalencia que son subretículos convexos, es decir, para cualquier clase de equivalencia de la relación inducida por una reducción de atributos, el concepto C_m pertenece a la clase.

Teorema 3.7. *Dado un contexto (A, B, R), cuyo retículo de conceptos es isomorfo a un retículo distributivo, $D \subseteq A \backslash I_f$, y una clase de equivalencia $[C]_D$, entonces $C_m \in [C]_D$.*

Demostración. Se encuentra en la demostración del Teorema 6 en [7]. □

El Teorema 3.6 simplifica la detección de retículos cuyas clases de equivalencias de una reducción de atributos no son subretículos convexos del retículo de conceptos original, por ejemplo, si usamos otro mecanismo basado en la RST [23, 25, 26] donde se pueden obtener clases que no son subretículos. Por otro lado, el Teorema 3.7 caracteriza las que proporcionan subretículos convexos.

Todos los resultados que hemos presentado en esta sección tienen un gran interés en la aplicación de congruencias locales, ya que caracterizan los casos en los que las clases no son subretículos convexos [9, 24] y, por tanto, qué clases se alteran al aplicar la congruencia local tras una reducción de atributos cualquiera.

3.3. Reducción de atributos mediante congruencias locales

En esta sección, se continua con la idea presentada en [9] de estudiar la aplicación de congruencias locales tras la reducción de atributos en contextos formales, además del impacto que las congruencias locales generan tanto en el contexto original como en el retículo de conceptos asociado. En [8] profundizamos un poco más en las estructuras asociadas con el contexto reducido y el conjunto cociente de conceptos generados por una reducción de atributos. Este estudio destaca la estrecha relación existente entre dichas estructuras. Además, se introduce un nuevo orden parcial para el conjunto cociente proporcionado por una congruencia local, para que su aplicación en retículos de conceptos sea más sencilla que la presentada en [6]. Por último, presentaremos la contribución principal del artículo [8], se trata de un procedimiento para la eliminación de conceptos de un retículo de conceptos basado en congruencias locales.

3.3.1. Conjunto parcialmente ordenado asociado a una congruencia local

En primer lugar, se define una relación de orden entre las clases de equivalencia inducidas por una reducción de atributos cuya relación de equivalencia inducida viene dada por ρ_D, introducida en la Proposición 2.1.

Proposición 3.7. *La relación $\sqsubseteq_D$, sobre el conjunto cociente $\mathcal{C}(A,B,R)/\rho_D$ asociado con un contexto (A,B,R), definida como $[(X_1,Y_1)]_D \sqsubseteq_D [(X_2,Y_2)]_D$ si ${X_1}^{\uparrow_D\downarrow} \subseteq {X_2}^{\uparrow_D\downarrow}$, para todo $[(X_1,Y_1)]_D, [(X_2,Y_2)]_D \in \mathcal{C}(A,B,R)/\rho_D$, es una relación de orden.*

Demostración. Se encuentra en la prueba de la Proposición 14 en [8]. □

Por lo tanto, $(\mathcal{C}(A,B,R)/\rho_D, \sqsubseteq_D)$ es un conjunto parcialmente ordenado. Además, podemos ver que dicho conjunto forma un retículo debido a la relación que guarda con el retículo de conceptos reducido, como presentamos en el siguiente resultado que mejora la Proposición 3.11 en [24].

Teorema 3.8. *Dado un contexto (A,B,R) y un subconjunto de atributos $D \subseteq A$, entonces el conjunto cociente generado por ρ_D y el retículo de conceptos reducido por D son isomorfos, es decir,*

$$(\mathcal{C}(A,B,R)/\rho_D, \sqsubseteq_D) \cong (\mathcal{C}(D,B,R_{|D\times B}), \leq_D)$$

donde $\leq_D$ es el orden en el retículo de conceptos original restringido al reducido.

Demostración. Véase la prueba del Teorema 15 en [8]. □

Recordemos que cada clase de equivalencia inducida por una reducción de atributos es un semirretículo superior con elemento máximo, por lo que el concepto $C_M = (X_M, Y_M)$ que definimos al inicio de la Sección 3.2 necesariamente pertenece a su clase. Ciertamente, este elemento máximo es $(X^{\uparrow_D\downarrow}, X^{\uparrow_D\downarrow\uparrow})$ por la Proposición 2.2 y, por lo tanto, $X^{\uparrow_D\downarrow} = X_M$, es decir, el orden definido en la Proposición 3.7 relaciona los elementos máximos de cada clase. Basándonos en esta idea, definimos una nueva relación sobre los elementos del conjunto cociente proporcionado por una congruencia local que resulta ser un orden parcial.

Teorema 3.9. *Dado un retículo completo $(L, \leq)$ y una congruencia local δ sobre L, la relación binaria definida como:*

$$[x]_\delta \leq_\delta [y]_\delta \quad \textit{si} \quad \bot_L \in [x]_\delta, \quad \textit{o} \quad x_M \leq y_M,$$

donde $y_M = \bigvee_{y_i \in [y]_\delta} y_i$, $x_M = \bigvee_{x_i \in [x]_\delta} x_i$ y $\bot_L$ es el elemento mínimo de $(L, \leq)$, es un orden parcial para L/δ.

Demostración. Véase la demostración del Teorema 16 en [8]. □

Esta relación de orden nos permite tener un orden parcial sin condiciones adicionales a diferencia de lo que ocurría con $\leq_\delta$ en el Teorema 3.1. Además, también nos permite identificar con mayor rapidez la relación entre clases de la congruencia local.

Podemos pensar que este orden definido en el Teorema 3.9 puede proporcionar un retículo completo, al igual que se obtenía para las clases de equivalencia inducidas en el Teorema 3.8. Sin embargo, cuando definimos el conjunto cociente de una congruencia local arbitraria sobre un retículo con el orden del Teorema 3.9 no podemos garantizar que dicho conjunto cociente sea un retículo. Un ejemplo de este hecho se encuentra en el Ejemplo 18 de [8]. Por lo tanto, $(\mathcal{C}(D, B, R)/\delta, \leq_\delta)$ no es un retículo en general, siendo δ una congruencia local cualquiera.

3.3.2. Eliminación de conceptos en retículos de conceptos

Si nos fijamos en la aplicación de las congruencias locales sobre el retículo de conceptos reducido en lugar del retículo original podemos observar que se pueden unir diferentes clases de equivalencia inducidas, lo que puede tener un notable impacto tanto en el retículo de conceptos como en el contexto reducido. La agrupación de conceptos por una congruencia local se puede ver como una eliminación de conceptos del retículo sobre el que se aplica, este hecho nos conduce a estudiar los efectos producidos por estas agrupaciones.

Debido a que las congruencias locales agrupan clases de equivalencia del retículo completo $(\mathcal{C}(A,B,R)/\rho_D, \sqsubseteq_D)$, estas pueden verse como conceptos de $(\mathcal{C}(D,B,R_{|D\times B}), \leq_D)$, usando el Teorema 3.8. Luego, si δ es una congruencia local, dada una clase $[C]_\delta \in \mathcal{C}(A,B,R)/\delta$ existe un conjunto de índices Λ_C tal que:

$$[C]_\delta = \bigcup\{[C_\lambda]_D \mid [C_\lambda]_D \in \mathcal{C}(A,B,R)/\rho_D, \lambda \in \Lambda_C\}$$

donde $\{[C_\lambda]_D \mid [C_\lambda]_D \in \mathcal{C}(A,B,R)/\rho_D, \lambda \in \Lambda_C\}$ es un subretículo convexo de $(\mathcal{C}(A,B,R)/\rho_D, \sqsubseteq_D)$. Luego, es posible que alguna clase $[C_\lambda]_D \in \mathcal{C}(A,B,R)/\rho_D$ con $\lambda \in \Lambda_C$ sea un elemento supremo irreducible, y por lo tanto, la congruencia local lo está agrupando con otra clase. Puesto que $[C_\lambda]_D$ es supremo irreducible, el conjunto de objetos que generan el concepto asociado a esta clase tiene que ser distinto del vacío, es decir, $\mathrm{Obg}([C_\lambda]_D) \neq \varnothing$. Por lo tanto, el aplicar la congruencia local es como si se eliminaran los elementos de ese conjunto de B para evitar el cálculo del concepto asociado a la clase $[C_\lambda]_D$ y que aparezca en el retículo de conceptos reducido. Como consecuencia de esta eliminación, otros conceptos del retículo también pueden ser eliminados como efecto colateral. Ilustramos este hecho con detalle en el Ejemplo 19 de [8].

A continuación, se analiza cómo la aplicación de la menor congruencia local, que contiene la relación de equivalencia inducida por una reducción de un contexto (A, B, R) por un subconjunto de atributos $D \subseteq A$, altera el retículo de conceptos asociado al contexto reducido $(D, B, R_{|D\times B})$. Es importante destacar que en [8] nos centramos en el contexto reducido, pero para simplificar la notación algunos resultados que presentamos aparecen con un contexto (A, B, R) cualquiera en lugar del contexto reducido. Se distinguen dos situaciones diferentes dependiendo de si el elemento eliminado (agrupado por la congruencia local) es o no supremo irreducible, examinando las modificaciones necesarias a realizar en el contexto asociado.

Eliminando elementos supremo irreducibles

En primer lugar, nos centramos en elementos supremos irreducibles y, además, de ahora en adelante se supone que el retículo de conceptos $(\mathcal{C}(A, B, R), \leq)$ satisface la condición de cadena ascendente, hecho que se verifica en retículos finitos. Como hemos mencionado anteriormente, la supresión de un concepto supremo irreducible mediante la eliminación de los objetos del conjunto de objetos que lo generan, puede hacer que otros conceptos dependientes del concepto eliminado también desaparezcan. Sin embargo, cuando extraemos únicamente un elemento supremo irreducible de un retículo completo, en general, no se altera la estructura de retículo, es decir, seguimos teniendo un retículo completo.

Lema 3.1. *Dado un retículo completo $(L, \wedge, \vee)$ que satisface la condición de cadena ascendente, y un elemento supremo irreducible $p \in L$, entonces $(L\backslash\{p\}, \wedge_p, \vee_p)$ es un subretículo de $(L, \wedge, \vee)$, donde $\wedge_p, \vee_p$ son la restricción de $\wedge, \vee$ a $L\backslash\{p\}$, y en particular, es un retículo completo.*

Demostración. Se encuentra en la prueba del Lema 20 en [8]. □

En el caso de retículos de conceptos, como se mostraba en el Ejemplo 19 de [8], el retículo de conceptos resultante de la eliminación de los objetos generadores del concepto supremo irreducible no coincide con el conjunto cociente de la menor congruencia que contiene a la relación de equivalencia inducida. Por tanto, nuestro objetivo es conseguir un nuevo retículo de conceptos eliminando el concepto supremo irreducible pero preservando el resto de conceptos.

Se va a detallar un procedimiento, introducido en [8], para modificar el contexto sobre cuyo retículo de conceptos asociado se aplica la congruencia local, para así obtener un nuevo retículo de conceptos isomorfo al retículo completo obtenido tras eliminar un elemento supremo irreducible.

Antes de nada, se establece la notación para ciertos elementos considerados en el procedimiento. Definimos el conjunto $\mathcal{T}(C_k) = \{C_i \mid C_i \in \mathcal{C}(A,B,R), C_k < C_i\}$ para un concepto $C_k \in \mathcal{C}(A,B,R)$ diferente del elemento máximo $(B, B^{\uparrow})$. Este conjunto no es vacío ya que contiene a $(B, B^{\uparrow})$, y además, existe su ínfimo porque $(\mathcal{C}(A,B,R), \leq)$ es un retículo completo. Este concepto ínfimo lo denotamos como C_k^t y los conceptos minimales de $\mathcal{T}(C_k)\backslash\{C_k^t\}$ como $C_k^{m_i}$, con i en un conjunto de índices Γ.

Ahora bien, supongamos que queremos eliminar un concepto supremo irreducible $C_j \in \mathcal{C}(A,B,R)$, podemos distinguir dos casos:

1. Si el concepto C_j no coincide con el ínfimo del conjunto $\mathcal{T}(C_j)$, es decir, $C_j \neq C_j^t$, entonces debemos distinguir dos situaciones:

    a) Si existe una descomposición de supremo irreducibles de C_j^t a la que C_j no pertenece, o bien, $\text{Obg}(C_j^t) \neq \varnothing$, entonces los elementos de $\text{Obg}(C_j)$ se eliminan de B. Obteniéndose un nuevo conjunto de objetos que denotamos por $B^* = B\backslash\text{Obg}(C_j)$ y una nueva relación que es simplemente la restricción al nuevo conjunto de objetos, es decir, $R^* = R_{|A\times B^*}$.

    b) En caso contrario, es decir, en el que toda descomposición de supremo irreducibles de C_j^t contiene a C_j y $\text{Obg}(C_j^t) = \varnothing$. Entonces se eliminan los objetos pertenecientes al conjunto $\text{Obg}(C_j)$

de B y se introduce un nuevo objeto b^* para evitar alteraciones colaterales. Obtenemos así un nuevo conjunto de objetos $B^* = \{b^*\} \cup (B\backslash\text{Obg}(C_j))$ y, además, consideramos una nueva relación $R^* \subseteq A \times B^*$ definida como $R^* = R_{|A\times(B\backslash\text{Obg}(C_j))} \cup \{(a, b^*) \mid a \in \text{Int}(C_j^t)\}$. De esta forma, preservamos el resto de conceptos.

2. Si $C_j = C_j^t$, debemos considerar los conceptos minimales $C_j^{m_i}$, con $i \in \Gamma$. Claramente, $C_j^{m_i} \neq C_j$ para todo $i \in \Gamma$, y procedemos a realizar el punto descrito anteriormente para cada uno de estos minimales obteniendo un subconjunto $\Gamma' \subseteq \Gamma$ que se define como $\Gamma' = \{i \in \Gamma \mid$ toda descomposición de supremo irreducibles de $C_j^{m_i}$ contiene a C_j y $\text{Obg}(C_j^{m_i}) = \varnothing\}$.

   De nuevo, volvemos a diferenciar dos casos similares a los descritos en el punto anterior:

   a) Si $\Gamma' = \varnothing$, entonces eliminamos los elementos de $\text{Obg}(C_j)$ de B. Obtenemos un nuevo conjunto de objetos $B^* = B\backslash\text{Obg}(C_j)$ y una nueva relación $R^* = R_{|A\times B^*}$.

   b) En caso contrario, obtenemos que $\Gamma' \neq \varnothing$, por lo que definimos un nuevo conjunto de objetos $B^* = \{b_i^* \mid i \in \Gamma'\} \cup (B\backslash\text{Obg}(C_j))$. Adicionalmente, consideramos una nueva relación $R^* \subseteq A \times B^*$ definida como $R^* = R_{|A\times(B\backslash\text{Obg}(C_j))} \cup \{(a, b_i^*) \mid a \in \text{Int}(C_j^{m_i}), i \in \Gamma'\}$

Este procedimiento se presenta en el Algoritmo 1 en [8] como un pseudocódigo. Como podemos observar, lo que obtenemos cuando aplicamos este mecanismo para eliminar un concepto supremo irreducible es un nuevo contexto cuyo retículo de conceptos es isomorfo a $(\mathcal{C}(A, B, R)\backslash\{C_j\}, \leq_j)$ que es un retículo completo por el Lema 3.1. Efectivamente, el mecanismo anterior determina el contexto deseado como demostramos en el siguiente resultado.

Teorema 3.10. *Dado un contexto (A, B, R) y un concepto supremo irreducible $C_j \in \mathcal{C}(A, B, R)$, se satisface que:*

$$(\mathcal{C}(A, B, R)\backslash\{C_j\}, \leq_j) \cong (\mathcal{C}(A, B^*, R^*), \leq^*),$$

donde (A, B^*, R^*) es el contexto obtenido por el Algoritmo 1 en [8], y $\leq_j$ es el orden definido de la restricción de la relación $\leq$ a $\mathcal{C}(A, B, R)\backslash\{C_j\}$.

Demostración. Véase la demostración del Teorema 21 en [8]. □

Hay que destacar que este procedimiento de eliminación de un concepto supremo irreducible puede aplicarse de manera secuencial cuando uno o más conceptos supremo irreducibles se deben eliminar. Además, los cambios realizados al contexto al que se aplica son los mínimos posibles para asegurar el isomorfismo en el Teorema 3.10. Por lo tanto, el mecanismo propuesto para caracterizar la influencia de eliminar un concepto supremo irreducible del retículo de conceptos en el contexto, proporciona el contexto con la mayor similitud posible al contexto original.

Eliminando elementos que no son supremo irreducibles

No siempre el tipo de concepto que agrupa una congruencia local es un supremo irreducible, por lo que también necesitamos explorar qué ocurre si dicho elemento no es supremo irreducible. En primera instancia, podemos pensar en los elementos duales a estos, es decir, los ínfimo irreducibles. Ciertamente, para este caso se satisfacen resultados duales y podemos presentar un resultado dual al Lema 3.1 para los ínfimo irreducibles, teniendo en cuenta el conjunto de atributos.

Lema 3.2. *Dado un retículo completo $(L, \wedge, \vee)$ que satisface la condición de cadena descendente, y un elemento ínfimo irreducible $q \in L$, entonces $(L\backslash\{q\}, \wedge_q, \vee_q)$ es un subretículo de $(L, \wedge, \vee)$, donde $\wedge_q, \vee_q$ denotan la restricción de $\wedge, \vee$ a $L\backslash\{q\}$ y, por tanto, también es un retículo completo.*

Demostración. Es la prueba del Lema 22 en [8]. □

Consecuentemente, para eliminar un concepto ínfimo irreducible podemos considerar un procedimiento dual al Algoritmo 1 en [8] en el que se eliminen atributos en lugar de objetos. Este nuevo mecanismo lo detallamos en el Algoritmo 2 en [8]. Además, podemos también introducir un resultado dual al Teorema 3.10.

Teorema 3.11. *Dado un contexto (A, B, R) y un concepto ínfimo irreducible $C_k \in \mathcal{C}(A, B, R)$, se satisface que:*

$$(\mathcal{C}(A, B, R)\backslash\{C_k\}, \leq_k) \cong (\mathcal{C}(A^*, B, R^*), \leq^*),$$

donde (A^*, B, R^*) es el contexto obtenido por el Algoritmo 2 en [8] y $\leq_k$ es el orden definido de la restricción de la relación $\leq$ a $\mathcal{C}(A, B, R)\backslash\{C_k\}$.

Demostración. Se encuentra en la demostración del Teorema 23 en [8]. □

En último lugar, nos queda considerar que el elemento a eliminar no sea ni supremo ni ínfimo irreducible. En este caso, surge una importante dificultad ya que eliminar este tipo de elemento puede provocar que se pierda la estructura de retículo, tal y como mostramos en el Ejemplo 24 de [8], obteniendo un conjunto parcialmente ordenado. Para solucionar este problema, tenemos que hacer uso de la compleción de Dedekind-MacNeille para recuperar la estructura de retículo. Efectivamente, el retículo completo que obtenemos de la compleción de Dedekind-MacNeille es isomorfo al retículo en el que hemos decidido quitar un elemento.

Lema 3.3. *Dado un retículo completo $(L, \leq)$ que satisface la condición de cadena ascendente y la descendente, y un elemento que no es ínfimo ni supremo irreducible $y \in L$, se satisface que:*

$$(DM(L\backslash\{y\}), \subseteq) \cong (L, \leq).$$

Demostración. Es la demostración del Lema 25 en [8]. □

Por lo tanto, la aplicación de la compleción de Dedekind-MacNeille solo será necesaria cuando eliminemos elementos del tipo que menciona el Lema 3.3, con el propósito de obtener un retículo isomorfo al original. Al igual que los otros dos procedimientos, podemos aplicar secuencialmente los resultados para la eliminación de diferentes conceptos, además el Algoritmo 1 y el Algoritmo 2 en [8] pueden ser combinados obteniendo así el resultado final presentado en [8] que resume todos los resultados anteriores sobre los procedimientos.

Teorema 3.12. *Dado un contexto (A, B, R) y un concepto $C \in \mathcal{C}(A, B, R)$, se satisface que:*

$$(DM(\mathcal{C}(A,B,R)\backslash\{C\}), \subseteq) \cong (\mathcal{C}(A^*, B^*, R^*), \leq^*),$$

donde (A^*, B^*, R^*) es el contexto obtenido por el Algoritmo 1 en [8] o el Algoritmo 2 en [8] o el original en el caso de que C no sea ínfimo ni supremo irreducible, y $\subseteq$ es el orden definido por la compleción de Dedekind-MacNeille.

Demostración. Véase la prueba del Teorema 27 en [8]. □

Por lo tanto, podemos aplicar el Teorema 3.12 sucesivamente para cada clase de una congruencia local que contenga más de un concepto reducido, caracterizando así la influencia de complementar una reducción de atributos con la aplicación de una congruencia local. Cabe destacar que tanto el Algoritmo 1 como el Algoritmo 2, ambos en [8], se pueden aplicar de manera independiente cuando el concepto a eliminar sea tanto supremo irreducible como ínfimo irreducible. Aunque obtenemos dos contextos diferentes los retículos de conceptos son isomorfos, por tanto, podemos decidir qué tipo de elementos preferimos modificar, atributos u objetos.

3.3.3. Reducción de conceptos por congruencia local

Por último, se muestra cómo el contexto reducido asociado a una reducción de atributos debe modificarse con el fin de obtener un retículo de conceptos isomorfo al retículo obtenido de la menor congruencia local que contiene a la relación de equivalencia inducida por una reducción de atributos. A este procedimiento lo denominamos *reducción de conceptos por congruencia local*.

Es evidente que cuando todas las clases en $(\mathcal{C}(A,B,R)/\rho_D, \sqsubseteq_D)$ son subretículos convexos, la congruencia local no modificará ninguna clase y, por lo tanto, la complementación del procedimiento de reducción de atributos no tiene sentido. No obstante, cuando al menos una de las clases no es un subretículo, entonces la congruencia local agrupará al menos dos clases inducidas. Destacar que el Teorema 3.3 caracteriza la menor congruencia local que contiene una relación de equivalencia dada, proporcionando un método para su cálculo. Además, recordemos que en la Sección 3.2 caracterizamos las clases inducidas que no eran subretículos convexos del original, por lo que esto trasladado a los elementos que agrupa una congruencia local nos permite determinar qué clases inducidas se unirán en una sola clase de la congruencia local.

Lema 3.4. *Sea (A,B,R) un contexto, $D \subseteq A$ un subconjunto de atributos y δ una congruencia local sobre $\mathcal{C}(D,B,R_{|D\times B})$. Si δ es la menor congruencia local tal que $\rho_D \sqsubset \delta$, entonces existe $[C]_\delta \in \mathcal{C}(D,B,R_{|D\times B})/\delta$ tal que hay al menos dos conceptos de $\mathcal{C}(D,B,R_{|D\times B})$ pertenecientes a $[C]_\delta$. En particular, hay dos conceptos $\overline{C_1}, \overline{C_2} \in [C]_\delta$ tales que $\overline{C_m} \neq \overline{C_1}$ y $\overline{C_m} = \overline{C_2}$, con $C_m = \bigwedge_{C_i \in [C_1]_D} C_i$. Además, todo concepto $\overline{C_j} \in \mathcal{C}(D,B,R_{|D\times B})$, con $\overline{C_m} \leq_D \overline{C_j} \leq_D \overline{C_1}$, satisface que $\overline{C_j} \in [C]_\delta$.*

Demostración. Directamente de la Definición 3.4, el Teorema 3.5 y la convexidad de las clases $[C]_\delta$. □

Algoritmo 3: Reducción de conceptos por congruencia local

Entrada: $(D, B, R_{|D\times B})$, ρ_D
Salida : (D^*, B^*, R^*)

Calculamos la menor congruencia local δ tal que $\rho_D \sqsubseteq \delta$;
Establecemos $(D^*, B^*, R^*) = (D, B, R_{|D\times B})$;
para cada *clase* $[C]_\delta$ *satisfaciendo el Lema 3.4* **hacer**
  Identificamos los conceptos $\overline{C_M}$ y $\overline{C_m}$ de la clase;
  Definimos el conjunto
  $E = \{\overline{C^*} \in (D, B, R_{|D\times B}) \mid \overline{C_m} \leq_D \overline{C^*} <_D \overline{C_M}\}$;
  para cada *concepto* $\overline{C^*} \in E$ **hacer**
    si $\overline{C^*}$ *es un concepto ínfimo irreducible de* $\mathcal{C}(D^*, B^*, R^*)$ **entonces**
      Usamos el Algoritmo 2 en [8] para eliminar $\overline{C^*}$;
      Nuevo contexto a considerar $(D^*, B^*, R^*) = (D', B^*, R')$;
      $E = E \backslash \{\overline{C^*}\}$;
    si no, si $\overline{C^*}$ *es un concepto supremo irreducible de* $\mathcal{C}(D^*, B^*, R^*)$ **entonces**
      Usamos el Algoritmo 1 en [8] para eliminar $\overline{C^*}$;
      Nuevo contexto a considerar $(D^*, B^*, R^*) = (D^*, B', R')$;
      $E = E \backslash \{\overline{C^*}\}$;
  si $E \neq \varnothing$ **entonces**
    para cada $\overline{C^*} \in E$ **hacer**
      Eliminamos $\overline{C^*}$ (ni supremo ni ínfimo irreducible) ;
      Usamos la compleción de Dedekind-MacNeille para obtener un retículo completo $\mathrm{DM}(\mathcal{C}(D^*, B^*, R^*) \backslash \{\overline{C^*}\})$;
      Nuevo contexto a considerar $(D^*, B^*, R^*) = (D^*, B^*, R^*)$;
devolver (D^*, B^*, R^*)

Recordemos que los conceptos $\overline{C_1}, \overline{C_2} \in [C]_\delta$ están relacionados con sus clases $[C_1]_D$, $[C_2]_D \in \mathcal{C}(A, B, R)/\rho_D$ por el isomorfismo obtenido en el Teorema 3.8 y, como consecuencia, se cumple que $C_m \notin [C_1]_D$ ($[C_1]_D$ no es un subretículo) y $C_m \in [C_2]_D$. Por lo tanto, como vimos en la Sección 3.2, la congruencia local no coincidirá con la relación inducida . En este caso, la congruencia local agrupará la clase de dicho ínfimo, $[C_m]_D$, y todas las otras clases comprendidas entre $[C]_D$ y $[C_m]_D$ como establece el Lema 3.4, cuyos conceptos reducidos asociados a estas clases son los que eliminamos mediante los procedimientos anteriormente descritos.

En el Algoritmo 3 se detalla el procedimiento que denominamos reducción de conceptos por congruencia local, aplicándolo al contexto reducido. Téngase en cuenta que los conceptos $\overline{C^*}$ del conjunto E en el Algoritmo 3 son aquellos que tienen que eliminarse debido a la convexidad de las clases de la congruencia local. Además, cuando un concepto $\overline{C^*}$ se extrae, el nuevo retículo de conceptos asociado al contexto obtenido por el Algoritmo 1 o el Algoritmo 2 en [8] tiene exactamente un concepto menos que el asociado al contexto que el algoritmo tiene como entrada. Por lo tanto, el resto de los conceptos del conjunto E siguen identificados en el nuevo contexto. De manera adicional, podemos eliminar todos los atributos y objetos que generarían el concepto eliminado en la Línea 17 del Algoritmo 3 ya que estos elementos serían innecesarios para el contexto.

Como consecuencia y de forma análoga al Teorema 3.12, obtenemos el siguiente isomorfismo.

Teorema 3.13. *Dado un contexto (A, B, R), un subconjunto de atributos $D \subseteq A$ y la menor congruencia local δ tal que $\rho_D \sqsubset \delta$, se satisface que:*

$$(DM(\mathcal{C}(D, B, R_{|D\times B})/\delta), \subseteq_\delta) \cong (\mathcal{C}(D^*, B^*, R^*), \leq^*),$$

donde (D^*, B^*, R^*) es el contexto obtenido por el Algoritmo 3 y $\subseteq_\delta$ es el orden definido por la compleción de Dedekind-MacNeille.

Demostración. Se obtiene directamente de aplicar el Teorema 3.12. □

Se puede observar que, en el Algoritmo 3, los dos algoritmos presentados en [8] se aplican secuencialmente para reducir el retículo de conceptos $(\mathcal{C}(D, B, R_{|D\times B}), \leq_D)$ con el fin de eliminar el concepto reducido agrupado por la congruencia local considerada. Por consiguiente, el Teorema 3.13 se verifica aplicando el Teorema 3.12, dado que proporciona un retículo de conceptos isomorfo al conjunto cociente resultante de aplicar la congruencia local. Asimismo, el uso de la compleción de Dedekind-MacNeille no siempre es necesario. Por lo tanto, cuando no se haga uso de la compleción de Dedekind-MacNeille, el orden $\subseteq_\delta$ se cambiaría por $\leq_\delta$ definido en el Teorema 3.9. De esta forma, el Teorema 3.13 asegura la estructura de retículo en el conjunto cociente proporcionado por la congruencia local utilizando los algoritmos. Como consecuencia de este resultado, obtenemos un nuevo contexto cuyo retículo de conceptos asociado es isomorfo al obtenido de la reducción de conceptos por congruencia local.

Capítulo 4

Reduciendo retículos de conceptos por medio de una noción debilitada de congruencia

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Reducing concept lattices by means of a weaker notion of congruence ☆

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Abstract

Attribute and size reductions are key issues in formal concept analysis. In this paper, we consider a special kind of equivalence relation to reduce concept lattices, which will be called local congruence. This equivalence relation is based on the notion of congruence on lattices, with the goal of losing as less information as possible and being suitable for the reduction of concept lattices. We analyze how the equivalence classes obtained from a local congruence can be ordered. Moreover, different properties related to the algebraic structure of the whole set of local congruences are also presented. Finally, a procedure to reduce concept lattices by the new weaker notion of congruence is introduced. This procedure can be applied to the classical and fuzzy formal concept analysis frameworks.

Keywords: Formal concept analysis; Size concept lattice reduction; Attribute reduction; Congruence relation; Fuzzy sets

1. Introduction

Formal Concept Analysis (FCA) is an exploratory data analysis technique, introduced by Ganter and Wille in [16], which has been widely studied from theoretical and applied perspectives. One of the key problems of formal concept analysis is to reduce the computational complexity of computing the complete lattice associated with the considered formal context (dataset). One procedure to address this problem is to find mechanisms to reduce the number of attributes, preserving the most important information contained in the context. Indeed, we can find many works which analyze different mechanisms that chase this goal [1,4,11–14,19,22,24,25].

Recently, in [7,8], the authors have presented novel mechanisms to reduce classical and fuzzy formal contexts based on the reduction philosophy considered in Rough Set Theory, which is another mathematical theory closely

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related to FCA [21,22]. In the aforementioned papers, the authors exposed that when the number of attributes of a context is reduced, an equivalence relation on the set of concepts of the original concept lattice is induced, both in the classical and fuzzy cases. In addition, they also showed that the resulting equivalence classes have the structure of join-semilattices with a maximum element. In the light of the results presented in [7,8], it is natural to ask how we could complement the introduced reduction mechanisms in order to ensure that the obtained equivalence classes are closed algebraic structures.

Specifically, we are interested in obtaining equivalence classes satisfying that they are convex sublattices of the original concept lattice. This target can be reached by considering the notion of congruence relation on lattices [9,15,17,18]. Although congruence relations within the environment of FCA have not been studied extensively, we can find some works that analyze the use of congruence relations within this mathematical theory. For example, congruence relations have been applied in lattice/context decomposition as Atlas decomposition [16], the subdirect decomposition [27] or the reverse doubling construction [26]. In addition, the links between implications and congruence relations have been analyzed in [28] and congruence relation have proved to be suitable to handle with inconsistent formal decision contexts [20].

However, a significant amount of information can be lost when congruence relations are considered to reduce concept lattices, due mainly to the restrictions imposed by the quadrilateral-closed property. In order to address this issue, in this paper we continue with the study presented in [5], in which we introduced a weaker notion of congruence by means of the elimination of the aforementioned restrictive property. We will analyze how the equivalence classes obtained from a local congruence can be ordered and so, if some hierarchy exists among the clusters provided by the local congruence. Then, since different local congruence relations can be defined on a concept lattice, we will go further to a meta level, studying the algebraic structure of the set of all local congruences that can be defined on a lattice and other interesting properties. Finally, based on the obtained results, a procedure to reduce concept lattices is presented by using local congruence relations. One of the advantages provided by this procedure is that it can be applied both in the classical and fuzzy generalizations of formal concept analysis. In this work, examples to illustrate the proposed procedure are also included. The introduced examples consider classical FCA, as well as the fuzzy generalization of this theory provided by the multi-adjoint framework [23]. These examples also confirm that the use of this kind of equivalence relations is more suitable for this task than the use of congruences, since the amount of lost information is minimized.

The paper is organized as follows: Section 2 reviews some preliminary notions related to lattice theory, congruences on lattices and formal concept analysis, which are necessary to follow this work. In Section 3 the notion of local congruence and several properties are introduced. A study on the ordering among the equivalence classes obtained from local congruences is included in Section 4. The properties related to the algebraic structure of the whole set of local congruences that can be defined on a lattice and to principal local congruences are given in Section 5. Section 6 presents a mechanism to reduce concept lattices based on the use of local congruences. The paper ends in Section 7, showing some conclusions and proposing diverse future challenges.

2. Preliminaries

In this section, some preliminary notions used in this work will be recalled and we will state the considered notation.

We will consider a lattice as an algebraic structure $(L, \wedge, \vee)$ and as an ordered set $(L, \preceq)$. It is well known that these two points of view are equivalent, since we can define the two operators infimum and supremum from the partial order and vice versa, see The Connecting Lemma in [15]. Therefore, we will write $(L, \wedge, \vee)$ or $(L, \preceq)$ indistinctly, depending on the most suitable point of view in each case.

In this work, we are interested in defining equivalence relations on complete lattices. We will write $(a, b) \in R$ with $a, b \in A$ to indicate that a and b are related under the binary relation R. Notice that an equivalence relation R on A gives rise to a partition of A, whose subsets are the equivalence classes of R. The set of all the equivalence classes of R is called *quotient set* and it is denoted as A/R. Equivalently, a partition of A gives rise to an equivalence relation whose equivalence classes are the subsets of the partition.

From this point forward if $\rho \subseteq A \times A$ is an equivalence relation on a set A, we will denote the equivalence class of an element $a \in A$ as $[a]_\rho = \{b \in A \mid (a, b) \in \rho\}$.

2.1. Congruence on lattices

This section introduces the notion of congruence on a lattice and some features which are essential to develop our work. First of all, we present the definition of equivalence relation that is compatible with the operation of the algebraic structure.

Definition 1. We say that an equivalence relation θ on a given lattice $(L, \wedge, \vee)$ is *compatible* with the supremum $\vee$ and the infimum $\wedge$ of the lattice if, for all $a, b, c, d \in L$,

$$(a, b) \in \theta \text{ and } (c, d) \in \theta$$

imply

$$(a \vee c, b \vee d) \in \theta \text{ and } (a \wedge c, b \wedge d) \in \theta$$

We can now state the definition of congruence on a lattice.

Definition 2. Given a lattice $(L, \wedge, \vee)$, we say that an equivalence relation on L, which is compatible with both the supremum and the infimum of $(L, \wedge, \vee)$ is a *congruence* on L.

Now, we introduce the notion of quotient lattice from a congruence based on the operations of the original lattice.

Definition 3. Given an equivalence relation θ on a lattice $(L, \wedge, \vee)$, two operators $\vee_\theta$ and $\wedge_\theta$ on the set of equivalence classes $L/\theta = \{[a]_\theta \mid a \in L\}$, for all $a, b \in L$, are defined as follows

$$[a]_\theta \vee_\theta [b]_\theta = [a \vee b]_\theta \text{ and } [a]_\theta \wedge_\theta [b]_\theta = [a \wedge b]_\theta.$$

$\vee_\theta$ and $\wedge_\theta$ are well defined on L/θ if and only if θ is a congruence.

When θ is a congruence on L, we call $\langle L/\theta, \vee_\theta, \wedge_\theta \rangle$ *the quotient lattice of L modulo* θ.

The following lemma is useful when calculating with congruences.

Lemma 4 *([15]). Given a lattice* $(L, \wedge, \vee)$ *we have that*

(i) *An equivalence relation* θ *on L is a congruence if and only if, for all* $a, b, c \in L$,

$$(a, b) \in \theta \textit{ implies } (a \vee c, b \vee c) \in \theta \textit{ and } (a \wedge c, b \wedge c) \in \theta.$$

(ii) *Let* θ *be a congruence on L and* $a, b, c \in L$.
  (a) *If* $(a, b) \in \theta$ *and* $a \preceq c \preceq b$, *then* $(a, c) \in \theta$.
  (b) $(a, b) \in \theta$ *if and only if* $(a \wedge b, a \vee b) \in \theta$.

The equivalence classes of a congruence are convex sublattices of the original lattice and besides are quadrilateral-closed. Let us recall the meaning of notion of quadrilateral-closed. Let $(L, \preceq)$ be a lattice, an equivalence relation θ on $(L, \preceq)$ and suppose that $\{a, b, c, d\}$ is a subset of L composed of four elements forming a quadrilateral, then a, b and c, d are said to be *opposite sides of the quadrilateral* $\langle a, b; c, d \rangle$ (see Fig. 1) if $a \prec b$, $c \prec d$ and either:

$$(a \vee d = b \text{ and } a \wedge d = c) \text{ or } (b \vee c = d \text{ and } b \wedge c = a).$$

Therefore, *quadrilateral-closed* means that whenever given two opposite sides of a quadrilateral a, b and c, d, satisfying that a, b belong to an equivalence class, then c, d belong to another or the same equivalence class, that is, if $a, b \in [x]_\theta$, with $x \in L$ then there exists $y \in L$ such that $c, d \in [y]_\theta$.

The following result introduces an interesting characterization of congruence which will be fundamental for the purpose of this paper.

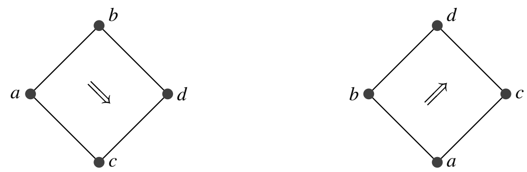

Fig. 1. Opposite sides of a quadrilateral.

Theorem 5 *([15]). Let $(L, \wedge, \vee)$ be a lattice and let θ be an equivalence relation on L. Then θ is a congruence if and only if*

(i) *each equivalence class of θ is a sublattice of L,*
(ii) *each equivalence class of θ is convex,*
(iii) *the equivalence classes of θ are quadrilateral-closed.*

The set of congruences on a lattice L, denoted as Con L, is a topped $\cap$-structure on $L \times L$. Hence Con L, ordered by inclusion, is a complete lattice. The least element and the greatest element are given by $\theta_\perp = \{(a, a) \mid a \in L\}$ and $\theta_\top = \{(a, b) \mid a, b \in L\}$, respectively.

Given a lattice $(L, \wedge, \vee)$ and two elements $a, b \in L$, the least congruence satisfying that a and b are related is denoted as $\theta_{(a,b)}$ and it is called the *principal congruence generated by* (a, b) and it is defined as follows

$$\theta_{(a,b)} = \bigwedge \{\theta \in \text{Con } L \mid (a, b) \in \theta\}.$$

Next lemma shows the importance of this definition.

Lemma 6 *([15]). Let $(L, \wedge, \vee)$ be a lattice and $\theta \in$ Con L. Then*

$$\theta = \bigvee \{\theta_{(a,b)} \mid (a, b) \in \theta\}.$$

Therefore, principal congruences factorize any congruence.

2.2. *Formal concept analysis*

Since equivalence relations will be considered in this work to reduce concept lattices, basic definitions of FCA are recalled in order to understand the motivation and results presented in this paper.

Definition 7. A *context* is a triple (A, B, R) with a set of attributes A, a set of objects B and a crisp relationship $R \subseteq A \times B$. We will write $R(a, b) = 1$ when $(a, b) \in R$ and $R(a, b) = 0$ when $(a, b) \notin R$.

Furthermore, if we consider a context, two mappings, ${}^\uparrow \colon 2^B \to 2^A$ and ${}^\downarrow \colon 2^A \to 2^B$, can be defined for each $X \subseteq B$ and $Y \subseteq A$ as:

$$X^\uparrow = \{a \in A \mid (a, x) \in R, \text{for all } x \in X\} \tag{1}$$

$$Y^\downarrow = \{x \in B \mid (a, x) \in R, \text{for all } a \in Y\} \tag{2}$$

These operators form a Galois connection [15], which leads us to the following definition.

Definition 8. Given a context (A, B, R) and the operators ${}^\uparrow$ and ${}^\downarrow$ defined above. If for a pair (X, Y) with $X \subseteq B$ and $Y \subseteq A$, the equalities $X^\uparrow = Y$ and $Y^\downarrow = X$ hold, then the pair (X, Y) is called *concept*.

Given a pair of concepts (X_1, Y_1) and (X_2, Y_2), we say that $(X_1, Y_1) \leq (X_2, Y_2)$ if $X_1 \subseteq X_2$ ($Y_2 \subseteq Y_1$, equivalently). The set of all concepts with this ordering relation has the structure of a complete lattice, it is called *formal concept lattice* and it is denoted as $\mathcal{C}(A, B, R)$ [15,16].

Now, we recall two results about reduction in FCA [7]. The first one shows that when we reduce the set of attribute of a formal context, an equivalence relation on the set of concepts of the original concept lattice is induced.

Proposition 9 *([7]). Given a context (A, B, R) and a subset $D \subseteq A$. The set $\rho_D = \{((X_1, Y_1), (X_2, Y_2)) \mid (X_1, Y_1), (X_2, Y_2) \in \mathcal{C}(A, B, R), X_1^{\uparrow_D\downarrow} = X_2^{\uparrow_D\downarrow}\}$ is an equivalence relation. Where $^{\uparrow_D}$ denotes the concept-forming operator, given in Expression (1), restricted to the subset of attributes $D \subseteq A$.*

The next result shows that every class of the equivalence relation defined above has the structure of a join semilattice with maximum element.

Proposition 10 *([7]). Given a context (A, B, R), a subset $D \subseteq A$ and a class $[(X, Y)]_D$ of the quotient set $\mathcal{C}(A, B, R)/\rho_D$. The class $[(X, Y)]_D$ is a join semilattice with maximum element $(X^{\uparrow_D\downarrow}, X^{\uparrow_D\downarrow\uparrow})$.*

Hence, we cannot ensure that the classes are sublattices of the original concept lattice, as it was shown in Example 3.10 of [7]. Therefore, it is interesting to study when these classes are sublattices and the properties of the obtained reduction. These results have been extended to the fuzzy FCA framework of multi-adjoint concept lattices in [8]. This framework was introduced by Medina, Ojeda-Aciego and Ruiz-Calviño in [23] with the main goal of presenting a general and flexible FCA framework based on the multi-adjoint philosophy. Multi-adjoint concept lattice generalizes different fuzzy extensions of FCA [3,6,10] and has widely been studied in diverse papers [2,13,14]. See the basic notions in [23].

In the multi-adjoint concept lattice framework a multi-adjoint frame $(L_1, L_2, P, \&_1, \dots, \&_n)$ needs to be fixed on which a context (A, B, R, σ) is defined and a concept lattice $\mathcal{M}(A, B, R, \sigma)$ is obtained. On this framework, the authors in [8] also proved that a reduction of the set of attributes induces an equivalence relation on $\mathcal{M}(A, B, R, \sigma)$, in which the equivalence classes are join-subsemilattices. This result will be recalled next, where $\uparrow_D$ and $\downarrow^D$ are the concept-forming operators associated with the subcontext $\mathcal{M}(D, B, R_{|D\times B}, \sigma)$, with $D \subseteq A$.

Proposition 11 *([8]). Let $D \subseteq A$ be a subset of attributes. The set $\rho_D = \{(\langle g_1, f_1\rangle, \langle g_2, f_2\rangle) \mid \langle g_1, f_1\rangle, \langle g_2, f_2\rangle \in \mathcal{M}(A, B, R, \sigma), g_1^{\uparrow_D\downarrow^D} = g_2^{\uparrow_D\downarrow^D}\}$ is an equivalence relation and every class $[\langle g, f\rangle]_D$ of $\mathcal{M}(A, B, R, \sigma)/\rho_D$ is a join-semilattice with maximum element $\langle g^{\uparrow_D\downarrow^D}, g^{\uparrow_D\downarrow^D\uparrow}\rangle$.*

Once we have recalled the required preliminary notions, the main contributions of this work are presented in the following section.

3. Weakening the notion of congruence

In this section, we present a weaker notion of congruence relation in order to complement reduction mechanisms in FCA. As we have recalled, any reduction of the set of attributes generates a partition in the set of concepts associated with the original context, where the obtained equivalence classes may not form sublattices of the original concept lattice. Our interest lies in generating groups of concepts with a closed algebraic structure by complementing the reductions given in FCA.

This goal can be achieved through the notion of congruence relation on lattices. Therefore, we will consider the use of congruence relations to reduce concept lattices, and we will analyze the obtained results. In particular, we are interested in the least congruence whose equivalence classes contain the equivalence classes induced by an attribute reduction of a context. Usually, this reduction is given by reducts which are minimal subsets of attributes preserving the information in the dataset. More details are included in [7].

Next, we illustrate the result of applying congruences through a practical example considered in [7]. In this example, we will show that the equivalence classes induced by a reduction procedure may be noticeably different from the ones provided by the least congruence containing the partition induced by the reduction, as a consequence, this difference would entail a relevant loss of information.

Example 12. Given the formal context (A, B, R) displayed in Table 1, where the set of objects in B are the planets of the Solar System together with the dwarf planet Pluto, that is $B = \{$Mercury (M), Venus (V), Earth (E), Mars (Ma),

Table 1
Relation of Example 12.

R	M	V	E	Ma	J	S	U	N	P
small size	1	1	1	1	0	0	0	0	1
medium size	0	0	0	0	0	0	1	1	0
large size	0	0	0	0	1	1	0	0	0
near sun	1	1	1	1	0	0	0	0	0
far sun	0	0	0	0	1	1	1	1	1
moon yes	0	0	1	1	1	1	1	1	1
moon no	1	1	0	0	0	0	0	0	0

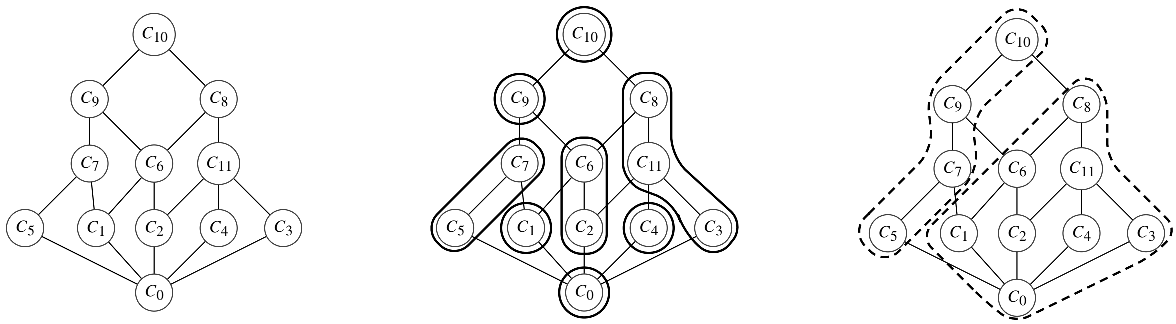

Fig. 2. The original concept lattice (left), the obtained reduction in [7] (middle) and the least congruence containing the previously reduction (right).

Jupiter (J), Saturn (S), Uranus (U), Neptune (N), Pluto (P)} and the set of attributes $A = \{$small size (ss), medium size (ms), large size (ls), near sun (ns), far sun, (fs), moon yes (my), moon no (mn)$\}$.

In the left side of Fig. 2, it is displayed the concept lattice from the given context. In [7], the rough set reduct $D_1 = \{$small size, medium size, near sun, moon yes$\}$ was considered to reduce the context. According to Proposition 9, this reduction makes that the concepts of the original concept lattice are grouped in equivalence classes which are represented in the middle of Fig. 2 by means of a Venn diagram. We know that each equivalence class has the structure of a join semilattice with maximum element as it is stated in Proposition 10.

We can bring together the reduction given in FCA and congruences, finding the least congruence such that each equivalence class induced by the reduction of the context is included in one equivalence class provided by the congruence relation. This least congruence is shown in the right side of Fig. 2 by means of a dashed Venn diagram. As we can see in Fig. 2, this congruence relation is composed of only two equivalence classes, since it has grouped too many concepts in each class. Consequently, in this case, the use of congruences entails a relevant loss of information, which is not convenient in any process of data analysis. □

The result obtained in the previous example reveals the necessity of a weaker notion of congruence removing the quadrilateral-closed property and preserving the other two properties in the characterization given in Theorem 5. This weaker notion is introduced in the following definition.

Definition 13. Given a lattice $(L, \preceq)$, we say that an equivalence relation δ on L is a *local congruence* if the following properties hold:

(i) each equivalence class of δ is a sublattice of L,
(ii) each equivalence class of δ is convex.

Remark 14. Clearly, the attribute reduction of the concept lattice introduced in Example 12 provides a local congruence (concept lattice in the middle of Fig. 2). Therefore, the introduced notion offers a better reduction than the one provided using congruences, aggregating as less concepts (information) as possible. Moreover, since in this particular case the reduction already produces equivalence classes that are convex sublattices, then the amount of lost information with this weaker notion of congruence is minimized as much as possible.

Although this new definition is a weak definition of the notion of congruence, the name of weak-congruence has already been used in the literature [29–32] in order to define congruences without the reflexivity property, that is, a weak congruence is a symmetry, transitivity and compatible relation. Therefore, another suitable name has been considered in this paper to the introduced general notion of congruence, whose justification will be introduced after the next direct characterization of Definition 13 in terms of the equivalence relation δ.

Proposition 15. *Given a lattice $(L, \preceq)$ and an equivalence relation δ on L, the relation δ is a local congruence on L if and only if, for each $a, b, c \in L$, the following properties hold:*

(i) *If $(a, b) \in \delta$ and $a \preceq c \preceq b$, then $(a, c) \in \delta$.*
(ii) *$(a, b) \in \delta$ if and only if $(a \wedge b, a \vee b) \in \delta$.*

Proof. The proof holds directly from the definition of local congruence. □

This result will be basic to introduce the following characterization, which generalizes Lemma 4(i) and motivates the considered notion.

Proposition 16. *Given a lattice $(L, \preceq)$ we have that an equivalence relation δ on L is a local congruence if and only if, for all $a, b, c \in L$, if $(a, b) \in \delta$ and $a \wedge b \preceq c \preceq a \vee b$, then we have that*

$$(a \vee c, b \vee c) \in \delta \text{ and } (a \wedge c, b \wedge c) \in \delta$$

Proof. Let us assume that δ is a local congruence on L and we consider $a, b, c \in L$ such that $(a, b) \in \delta$ and $a \wedge b \preceq c \preceq a \vee b$. Straightforwardly, by Proposition 15, we have that

$$(a, a \vee b) \in \delta \text{ and } (b, a \vee b) \in \delta. \tag{3}$$

In addition, since $a \wedge b \preceq c \preceq a \vee b$, by the supremum property, the following inequalities hold

$$a \vee (a \wedge b) \preceq a \vee c \preceq a \vee (a \vee b),$$

which is equivalent to

$$a \preceq a \vee c \preceq a \vee b.$$

Hence, as $(a, a \vee b) \in \delta$, by Proposition 15(i), we obtain that $(a, a \vee c) \in \delta$. Considering an analogous procedure to the previous one, we have that $(b, b \vee c) \in \delta$. Therefore, since $(a, a \vee c) \in \delta$ and $(b, b \vee c) \in \delta$, by the hypothesis $(a, b) \in \delta$ and the transitivity property of δ, we can assert that $(a \vee c, b \vee c) \in \delta$. Analogously, we have that $(a \wedge c, b \wedge c) \in \delta$.

Now, let us assume that δ is an equivalence relation on L, such that, if $(a, b) \in \delta$ and $a \wedge b \preceq c \preceq a \vee b$, then it satisfies that

$$(a \vee c, b \vee c) \in \delta \text{ and } (a \wedge c, b \wedge c) \in \delta$$

for all $a, b, c \in L$.

We will use Proposition 15 in order to prove that δ is a local congruence. If $(a, b) \in \delta$ and $a \preceq c \preceq b$ then, in particular, $a \wedge b \preceq a \preceq c \preceq b \preceq a \vee b$. Therefore, by hypothesis, we have that $(a \wedge c, b \wedge c) \in \delta$. Since $a \wedge c = a$ and $c \wedge b = c$, we obtain that $(a, c) \in \delta$. Hence, item (i) of Proposition 15 is satisfied.

In addition, for each $(a, b) \in \delta$ if we consider $c = a$, by hypothesis, we have that $(a \vee c, b \vee c) \in \delta$, that is, $(a, b \vee a) \in \delta$. Similarly, for $c = b$ we have that $(a \wedge b, b) \in \delta$. Hence, by the transitivity property of δ, we obtain that $(a \wedge b, a \vee b) \in \delta$. Following a similar reasoning, we can easily prove that, for all $a, b \in L$, if $(a \wedge b, a \vee b) \in \delta$ then $(a, b) \in \delta$. Therefore, item (ii) of Proposition 15 also holds.

Consequently, we can conclude that δ is a local congruence on L. □

Hence, the difference from the equivalence given in Lemma 4(i) is that in the aforementioned lemma, the element c is arbitrary in L, and in Proposition 16 is a *local* element bounded by $a \wedge b$ and $a \vee b$.

As we highlighted above, the particular algebraic structure of the equivalence classes is the most important property associated with the new notion. The set of these classes is formally defined next.

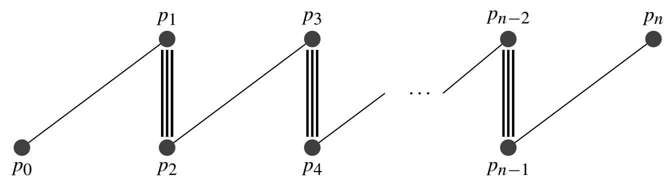

Fig. 3. Example of δ-sequence.

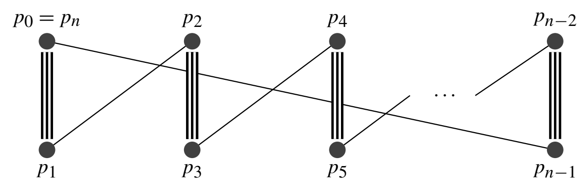

Fig. 4. Example of δ-cycle.

Definition 17. Let $(L, \preceq)$ be a lattice and δ a local congruence, the quotient set L/δ provides a partition of L, which is called *local congruence partition* (or *lc-partition* in short) of L and it is denoted as π_δ. The elements in the lc-partition π_δ are convex sublattices of L.

According to the previous definition, it can be noted that each equivalence class of the quotient set L/δ is a closed algebraic structure. Moreover, each local congruence relation univocally determines a local congruence partition and vice versa. As a consequence, both notions can be considered indistinctly.

In the following section, a formal definition of ordering among the classes of the quotient set of a local congruence will be studied.

4. The quotient set of a local congruence

Now, we focus on the equivalence classes of a quotient set provided by a local congruence. We are interested in studying how we can establish an ordering relation between these classes. The following definition will play a key role for this purpose.

Definition 18. Let $(L, \preceq)$ be a lattice and a local congruence δ on L.

(i) A sequence of elements of L, $(p_0, p_1, \ldots, p_n)$ with $n \geq 1$, is called a δ-*sequence*, denoted as $(p_0, p_n)_\delta$, if for each $i \in \{1, \ldots, n\}$ either $(p_{i-1}, p_i) \in \delta$ or $p_{i-1} \preceq p_i$ holds.
(ii) If a δ-sequence, $(p_0, p_n)_\delta$, satisfies that $p_0 = p_n$, then it is called a δ-*cycle*. In addition, if the δ-cycle satisfies that $[p_0]_\delta = [p_1]_\delta = \cdots = [p_n]_\delta$ is said to be *closed*.

The notions in Definition 18 are clarified in Fig. 3 and Fig. 4, where the triple vertical line that connects p_{i-1} with p_i means that they are related under the considered local congruence relation, that is, $(p_{i-1}, p_i) \in \delta$. The simple line indicates that the two elements are connected by means of the ordering relation defined on the lattice.

The following definition provides a first step to define a partial order on the quotient set provided by a local congruence.

Definition 19. Given a lattice $(L, \preceq)$ and a local congruence δ on L, we define a binary relation $\preceq_\delta$ on L/δ as follows:

$$[x]_\delta \preceq_\delta [y]_\delta \quad \text{if there exists a } \delta\text{-sequence } (x', y')_\delta$$

for some $x' \in [x]_\delta$ and $y' \in [y]_\delta$.

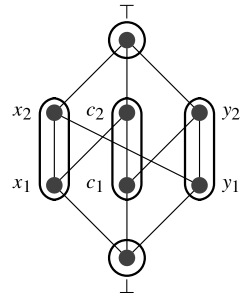

Fig. 5. Example where $\preceq_\delta$ is not a partial order.

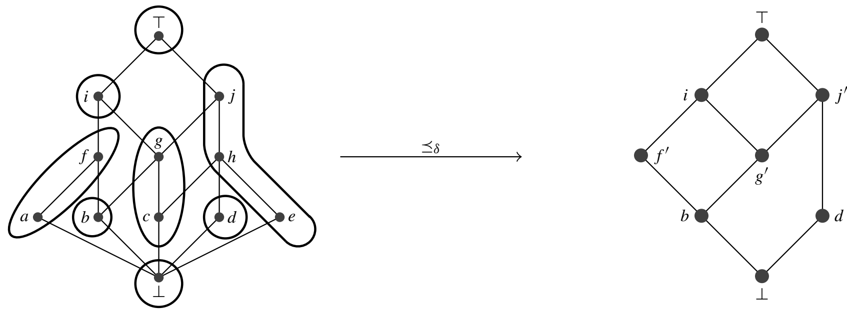

Fig. 6. A local congruence δ on L (left) and its quotient set $(L/\delta, \preceq_\delta)$ (right).

Notice that the relation $\preceq_\delta$ given in Definition 19 is a preorder. Clearly, by definition, $\preceq_\delta$ is reflexive and transitive. However, the relation $\preceq_\delta$ is not a partial order since the antisymmetry property does not hold, for any local congruence in general. In the following example, we show a case in which the previously defined relation $\preceq_\delta$ does not satisfy the antisymmetry property.

Example 20. Given the lattice $(L, \preceq)$ and the local congruence δ given in Fig. 5. We have that the equivalence classes of δ are $[\top]_\delta = \{\top\}$, $[x_1]_\delta = \{x_1, x_2\}$, $[c_1]_\delta = \{c_1, c_2\}$, $[y_1]_\delta = \{y_1, y_2\}$ and $[\bot]_\delta = \{\bot\}$. It is easy to check that these equivalence classes are convex sublattices of L. Moreover, we can observe that $[x_1]_\delta \preceq_\delta [y_1]_\delta$ since there exists a δ-sequence, $(x_1, y_2)_\delta = (x_1, c_2, c_1, y_2)$, and also $[y_1]_\delta \preceq_\delta [x_1]_\delta$ since there also exists a δ-sequence, $(y_1, x_2)_\delta = (y_1, x_2)$, but $[x_1]_\delta \neq [y_1]_\delta$ and thus $\preceq_\delta$ is not antisymmetric. □

There are certain cases in which the relation $\preceq_\delta$ is a partial order, depending on the local congruence.

Example 21. Let $(L, \preceq)$ be a lattice isomorphic to the concept lattice given in Example 12 and δ the local congruence shown in the left side of Fig. 6. In this case, the considered local congruence makes that the relation $\preceq_\delta$ satisfies the antisymmetry property and, consequently, the relation $\preceq_\delta$ is a partial order and the Hasse diagram of $(L/\delta, \preceq_\delta)$ can be given (right side of Fig. 6). □

The following results state different conditions under which the relation $\preceq_\delta$ is a partial order.

Proposition 22. *Given a lattice $(L, \preceq)$ and a local congruence δ, if for any two equivalence classes $[x]_\delta, [y]_\delta \in L/\delta$ there exists only one class $[c]_\delta \in L/\delta$ such that $[x]_\delta \preceq_\delta [c]_\delta \preceq_\delta [y]_\delta$ and $[y]_\delta \preceq_\delta [c]_\delta \preceq_\delta [x]_\delta$ satisfying that $x_1 \preceq c_1 \preceq y_1$ and $y_2 \preceq c_2 \preceq x_2$ with $x_1, x_2 \in [x]_\delta$, $c_1, c_2 \in [c]_\delta$ and $y_1, y_2 \in [y]_\delta$, then $[x]_\delta = [y]_\delta$.*

Proof. Let us consider two equivalence classes $[x]_\delta, [y]_\delta \in L/\delta$. Hence, there exists a class $[c]_\delta \in L/\delta$ such that $[x]_\delta \preceq_\delta [c]_\delta \preceq_\delta [y]_\delta$ and $[y]_\delta \preceq_\delta [c]_\delta \preceq_\delta [x]_\delta$ satisfying that $x_1 \preceq c_1 \preceq y_1$ and $y_2 \preceq c_2 \preceq x_2$ with $x_1, x_2 \in [x]_\delta$, $c_1, c_2 \in [c]_\delta$ and $y_1, y_2 \in [y]_\delta$. Then, we have that $[x]_\delta \preceq_\delta [c]_\delta$ and $[c]_\delta \preceq_\delta [x]_\delta$ and, since the classes of δ are sublattices of L, we also have that $x_1 \vee x_2$ and $c_1 \vee c_2$ exist and belong to the classes $[x]_\delta$ and $[c]_\delta$, respectively. In addition, we have that $x_1 \vee c_2 \preceq c_1 \vee c_2$ and $x_1 \vee c_2 \preceq x_1 \vee x_2$, thus $x_1 \vee c_2 \preceq (x_1 \vee x_2) \wedge (c_1 \vee c_2)$. Hence, $x_1 \preceq (x_1 \vee x_2) \wedge (c_1 \vee c_2) \preceq x_1 \vee x_2$ and $c_2 \preceq (x_1 \vee x_2) \wedge (c_1 \vee c_2) \preceq c_1 \vee c_2$, by the convexity of the classes we have that $(x_1 \vee x_2) \wedge (c_1 \vee c_2)$ belongs to both classes, which implies that both classes are just the same class: $[x]_\delta = [c]_\delta$.

We can proceed in an analogous way in order to prove that $[c]_\delta = [y]_\delta$. Hence, we obtain that $[x]_\delta = [c]_\delta = [y]_\delta$. □

From the previous proposition we obtain the following corollary.

Corollary 23. *Given a lattice $(L, \preceq)$ and a local congruence δ, if for any two equivalence classes $[x]_\delta, [y]_\delta \in L/\delta$ such that $[x]_\delta \preceq_\delta [y]_\delta$ and $[y]_\delta \preceq_\delta [x]_\delta$ satisfy that $x_1 \preceq y_1$ and $y_2 \preceq x_2$ with $x_1, x_2 \in [x]_\delta$ and $y_1, y_2 \in [y]_\delta$, then $[x]_\delta = [y]_\delta$.*

Proof. It is straightforwardly deduced from Proposition 22 taking the class $[c]_\delta$ as either the class $[x]_\delta$ or $[y]_\delta$. □

It is easy to see in Fig. 5 that $\preceq_\delta$ is not a partial order because of there exists a δ-cycle composed of elements belonging to different equivalence classes, i.e. the δ-cycle $(x_2, x_1, c_2, c_1, y_2, y_1, x_2)$. In order to avoid this problem, every δ-cycle must be contained in one single class, that is, every δ-cycle must be closed in the lattice, as the next result states.

Theorem 24. *Given a lattice $(L, \preceq)$ and a local congruence δ on L, the preorder $\preceq_\delta$ given in Definition 19 is a partial order if and only if every δ-cycle in L is closed.*

Proof. Let us assume that δ is a local congruence on L and that every δ-cycle in L is closed and let us prove that $\preceq_\delta$ is a partial order.

The reflexivity of $\preceq_\delta$ holds in a direct way.

Now, we prove the transitivity. If $[x]_\delta \preceq_\delta [y]_\delta$ and $[y]_\delta \preceq_\delta [z]_\delta$ for $[x]_\delta, [y]_\delta, [z]_\delta \in L/\delta$, then there exist two δ-sequences $(x', y_1)_\delta = (x', p_1, \ldots, p_n, y_1)$ and $(y_2, z')_\delta = (y_2, q_1, \ldots, q_m, z')$, with $x' \in [x]_\delta$, $y_1, y_2 \in [y]_\delta$ and $z' \in [z]_\delta$. Hence, there exists a δ-sequence $(x', z')_\delta = (x', p_1, \ldots, p_n, y_1, y_2, q_1, \ldots, q_m, z')$ satisfying the conditions of Definition 19. Thus, $[x]_\delta \preceq_\delta [z]_\delta$ and the relation $\preceq_\delta$ is transitive.

In order to prove that $\preceq_\delta$ is antisymmetric, we assume that $[x]_\delta \preceq_\delta [y]_\delta$ and $[y]_\delta \preceq_\delta [x]_\delta$, for some $x, y \in L$. Then there exist $x' \in [x]_\delta$, $y' \in [y]_\delta$, a δ-sequence $(x', y')_\delta = (x', p_1, \ldots, p_n, y')$ and a δ-sequence $(y', x')_\delta = (y', q_1, \ldots, q_m, x')$. Clearly, $(x', p_1, \ldots, y', q_1, \ldots, x')$ is a δ-cycle and since every δ-cycle is closed, we obtain $[x]_\delta = [y]_\delta$. Hence $\preceq_\delta$ is antisymmetric and thus a partial order.

Now, suppose that $\preceq_\delta$ is a partial order, if $(p_0, \ldots, p_n, p_0)$ is a δ-cycle of L (as it is showed in Fig. 4) then we have that

$$[p_0]_\delta \circledast_1 [p_1]_\delta \circledast_2 [p_2]_\delta \circledast_3 [p_3]_\delta \circledast_4 \cdots \circledast_{n-2} [p_{n-2}]_\delta \circledast_{n-1} [p_{n-1}]_\delta \circledast_n [p_n]_\delta \circledast_0 [p_0]_\delta$$

where $\circledast_i \in \{=, \preceq_\delta\}$ for all $i \in \{0, \ldots, n\}$. Since the chain begins and ends with the same element, and $\preceq_\delta$ is a partial order, we obtain that the δ-cycle is closed. □

As a consequence, under the assumption of the introduced necessary and sufficient condition, this result allows to order the convex sublattices (classes) obtained after the attribute reduction, which provide a hierarchization among the obtained concepts. The following example shows that this hierarchization does not form a complete lattice.

Example 25. Let $(L, \preceq)$ be the lattice given in the left side of Fig. 7 which is isomorphic to a concept lattice obtained from a formal context and δ the local congruence shown in the middle of Fig. 7. In this case, the considered local congruence makes that the relation $\preceq_\delta$ be a partial order.

However, the quotient set L/δ ordered with $\preceq_\delta$ does not form a lattice, as it is shown in the right side of Fig. 7, because the equivalence classes $[x]_\delta$ and $[y]_\delta$ have not got a supremum, that is, their least upper bound does not exist. □

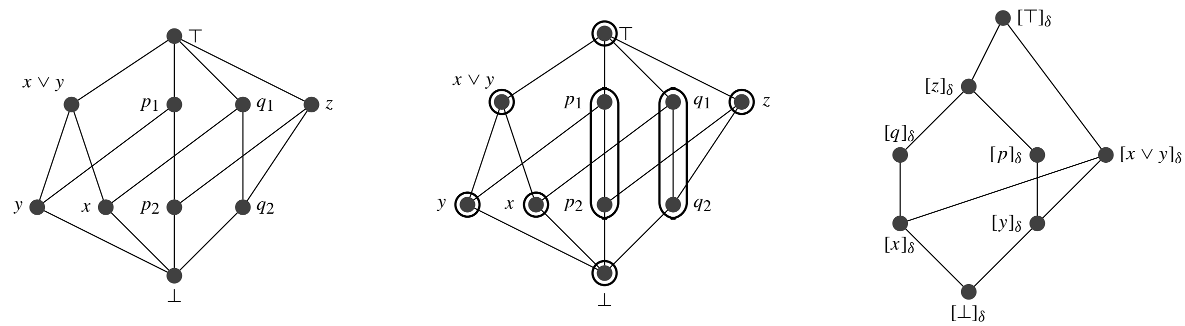

Fig. 7. The lattice $(L, \preceq)$ (left), the local congruence δ on L (middle) and its corresponding quotient set $(L/\delta, \preceq_\delta)$ with the ordering relation $\preceq_\delta$ (right).

Therefore, an ordering can be defined on the classes of a local congruence, when every δ-cycle in L is closed, which could not provide a complete lattice, but it is enough to produce a hierarchization among the computed reduced concepts. In order to ensure that a local congruence can always be computed, such as every δ-cycle is closed, more properties of local congruences need to be studied. Specifically, it is important to analyze the relationships among these new congruences.

5. Algebraic structure of local congruences on a lattice

In this section, we study the algebraic structure of the set of all local congruences defined on a lattice. First of all, we show that local congruences can be ordered by using the definition of inclusion of equivalence relations, which is recalled next.

Definition 26. Let ρ_1 and ρ_2 be two equivalence relations on a lattice $(L, \preceq)$. We say that the equivalence relation ρ_1 is included in ρ_2, denoted as $\rho_1 \sqsubseteq \rho_2$, if for every equivalence class $[x]_{\rho_1} \in L/\rho_1$ there exists an equivalence class $[y]_{\rho_2} \in L/\rho_2$ such that $[x]_{\rho_1} \subseteq [y]_{\rho_2}$.

We say that two equivalence relations, ρ_1 and ρ_2, are incomparable if $\rho_1 \not\sqsubseteq \rho_2$ and $\rho_2 \not\sqsubseteq \rho_1$.

From now on, the set of all local congruences on L ordered by the inclusion $\sqsubseteq$ will be denoted as $(\text{LCon } L, \sqsubseteq)$. First of all, we will show that the set $(\text{LCon } L, \sqsubseteq)$ is a complete lattice, proving that $(\text{LCon } L, \sqsubseteq)$ is a topped $\sqcap$-structure with a maximum element. In addition, the maximum and the minimum of the complete lattice $(\text{LCon } L, \sqsubseteq)$ are characterized.

Theorem 27. *Given a lattice $(L, \preceq)$, the set $(LCon\ L, \sqsubseteq)$ is a complete lattice. Moreover, the least and greatest element are given by $\delta_\bot = \{(a, a) \mid a \in L\}$ and $\delta_\top = \{(a, b) \mid a, b \in L\}$, respectively.*

Proof. Let us assume that $(L, \preceq)$ is a lattice and $(\text{LCon } L, \sqsubseteq)$ is the set of all local congruences. First of all, we need to prove that $(\text{LCon } L, \sqsubseteq)$ is a topped $\cap$-structure. Therefore, we consider a non-empty family of local congruence, that is, $\{\delta_i\}_{i \in I} \subseteq \text{LCon } L$ where I is a index set.

It is well known that the intersection of equivalence relations is an equivalence relation. Hence, $\bigcap_{i \in I} \delta_i$ is indeed an equivalence relation. Now, we prove that each equivalence class of the intersection is a convex sublattice. Let us consider an equivalence class $Z \in L/(\bigcap_{i \in I} \delta_i)$, hence there exist a family of equivalence classes $\{X_i \in L/\delta_i \mid i \in I\}$ such that $Z = \bigcap_{i \in I} X_i$. If we consider $a, b \in Z$, then we have that $a, b \in X_i$ for all $i \in I$ and, since each X_i is a convex sublattice of L, we have that $a \wedge b, a \vee b \in X_i$ for all $i \in I$. Therefore, $a \wedge b, a \vee b \in \bigcap_{i \in I} X_i = Z$, that is, Z is a sublattice of L. In addition, if $a \preceq b$ and we consider $c \in L$ such that $a \preceq c \preceq b$, then we have that $c \in X_i$ for all $i \in I$ since each X_i is convex. Therefore, $c \in \bigcap_{i \in I} = Z$, that is, Z is also convex. Thus, $\bigcap_{i \in I} \delta_i \in \text{LCon } L$, i.e., LCon L is a $\cap$-structure.

Now, we need to prove that LCon L has a maximum element. It is clear that the equivalence relation on L that relates all elements of L, that is, $\{(a, b) \mid a, b \in L\} = L \times L$, has convex sublattices of L as equivalence classes, hence $\{(a, b) \mid a, b \in L\} = L \times L \in \text{LCon } L$ and moreover, we cannot find another local congruence that contains it. Therefore, $\{(a, b) \mid a, b \in L\} = L \times L$ is the greatest local congruence and we denote it as $\delta_\top$. Thus, the set $(\text{LCon } L, \sqsubseteq)$ is a complete lattice.

In addition, it is clear that the least local congruence is the equivalence relation on L that only relates each element of L to itself, that is, $\delta_\perp = \{(a, a) \mid a \in L\}$. □

Next definition shows the notion of principal local congruence, which is the least local congruence that can be defined from two given elements of a lattice.

Definition 28. Given a pair of elements $(a, b) \in L \times L$, the *principal local congruence generated by* (a, b), denoted as $\delta_{(a,b)}$, is the least local congruence that contains the elements a and b in the same equivalence class, that is

$$\delta_{(a,b)} = \bigwedge \{\delta \in \text{LCon } L \mid (a, b) \in \delta\}$$

Note that, for every pair of elements $(a, b) \in L \times L$, the principal local congruence $\delta_{(a,b)}$ always exists since the set $(\text{LCon } L, \sqsubseteq)$ is a complete lattice.

Finally, the last theorem generalizes the characterization of congruences in terms of principal congruences (recalled in Lemma 6) for local congruences, considering an arbitrary equivalence relation.

Theorem 29. *Given a lattice $(L, \preceq)$ and an equivalence relation ρ, the least local congruence containing ρ is*

$$\delta_\rho = \bigvee \{\delta_{(a,b)} \mid (a, b) \in \rho\}$$

Proof. Let us assume that δ_ρ is the least local congruence containing an equivalence relation ρ and let us prove that δ_ρ is the least upper bound of the set $S = \{\delta_{(a,b)} \mid (a, b) \in \rho\}$. Due to $\rho \sqsubseteq \delta_\rho$, it is clear that $S \subseteq \{\delta_{(c,d)} \mid (c, d) \in \delta_\rho\}$ and, by Proposition 30, we have that $\delta_\rho = \bigvee \{\delta_{(c,d)} \mid (c, d) \in \delta_\rho\}$. Hence δ_ρ is an upper bound for S. Now, let us assume that δ'_ρ is an upper bound for S, which means that for all $(a, b) \in \rho$ then $\delta_{(a,b)} \sqsubseteq \delta'_\rho$. Therefore, by the supremum property we have that

$$\delta_\rho = \bigvee \{\delta_{(a,b)} \mid (a, b) \in \rho\} \sqsubseteq \delta'_\rho$$

which finishes the proof. □

In particular, the previous result is also satisfied when we consider a local congruence instead of an arbitrary equivalence relation.

Corollary 30. *Let $(L, \preceq)$ be a lattice and let δ a local congruence of $(LCon\ L, \sqsubseteq)$. Then*

$$\delta = \bigvee \{\delta_{(a,b)} \mid (a, b) \in \delta\}.$$

Proof. Straightforwardly from Theorem 29, considering a local congruence δ as the equivalence relation ρ. □

Note that this result will be very important in the reduction procedure in order to obtain a local congruence δ satisfying that $(L/\delta, \preceq_\delta)$ is a partial ordered set (poset), as we will show in the next section.

6. Reduction mechanism of concept lattices

This section will introduce an attribute reduction mechanism focused on grouping concepts in convex sublattices, having a hierarchy in form of a poset, which is equivalent by Theorem 24 to computing a local congruence with all δ-cycle in L being closed. In order to fulfill this last requirement we will use the following procedure from an arbitrary local congruence. Given a lattice $(L, \preceq)$ and a local congruence δ on L, if every δ-cycle in L is closed, then we already have that $(L/\delta, \preceq_\delta)$ is a poset. Otherwise, we can define an equivalence relation ρ on L/δ as

$$\rho_\delta = \{([x]_\delta, [y]_\delta) \in L/\delta \times L/\delta \mid [x]_\delta \preceq_\delta [y]_\delta \text{ and } [y]_\delta \preceq_\delta [x]_\delta\} \tag{4}$$

Algorithm 1: Reducing concept lattices by local congruences.

input : $\mathcal{C}(A, B, R)$, $D \subseteq A$
output: δ

1 Obtain the relation ρ_D associated with the attribute reduction given by D;
2 Compute the least local congruence δ_D containing ρ_D;
3 **if** *every* δ_D*-cycle is closed,* **then**
4     $\delta = \delta_D$
5 **else**
6   Compute ρ_{δ_D} by Equation (4);
7   **if** ρ *is a local congruence* **then**
8         $\delta = \rho_{\delta_D}$
9   **else**
10   Obtain the least local congruence δ_ρ such that $\delta_D \sqsubseteq \rho_{\delta_D} \sqsubseteq \delta_\rho$;
11         $\delta = \delta_\rho$
12 **return** δ

If there are two different equivalence classes $[x]_\delta$, $[y]_\delta$ such that $[x]_\delta \preceq_\delta [y]_\delta$ and $[y]_\delta \preceq_\delta [x]_\delta$, this means that there is a δ-cycle, $(x', x')_\delta$ or $(y', y')_\delta$ for some $x' \in [x]_\delta$, $y' \in [y]_\delta$. Therefore, the equivalence relation ρ_δ groups all the equivalence classes that contain elements in the δ-cycle in a unique equivalence class providing a new partition of L.

However, the equivalence relation ρ_δ may not be a local congruence. Since clearly $\delta \sqsubseteq \rho_\delta$, by Theorem 29, we can find the least local congruence $\bar{\delta}$ that contains the equivalence relation ρ_δ, that is, $\delta \sqsubseteq \rho_\delta \sqsubseteq \bar{\delta}$. Hence, every $\bar{\delta}$-cycle in L is closed and, by Theorem 24, $\preceq_{\bar{\delta}}$ is a partial order on the quotient set $L/\bar{\delta}$.

This procedure to ensure the ordering between the classes will be incorporate in the procedure to reduce concept lattices by local congruences, which is summarized in the following algorithm:

Notice that, the set D in Algorithm 1 can be given from any reduction mechanism. For example, it can be a rough set reduct [7,8]. Moreover, observe that the relation ρ_D was defined in the classical case in Proposition 9 and in the fuzzy case in Proposition 11.

This previous mechanism provides the desired reduction, as the following result shows.

Proposition 31. *Given a concept lattice $\mathcal{C}(A, B, R)$ and a subset of attributes $D \subseteq A$, then Algorithm 1 provides the least local congruence δ containing the induced relation ρ_D and $(\mathcal{C}(A, B, R)/\delta, \preceq_\delta)$ is a poset.*

Proof. Let us assume that we have a concept lattice $\mathcal{C}(A, B, R)$ and a partition of $\mathcal{C}(A, B, R)$ induced by an attribute reduction provided by $D \subseteq A$.

The starting point of the procedure (Line 1) is the computation of the equivalence relation associated with the attribute reduction given by the subset D, which is denoted as ρ_D.

In Line 2, by Theorem 29, we obtain the least local congruence containing ρ_D, which is denoted as δ_D. Hence, in particular, $\rho_D \sqsubseteq \delta_D$. From this local congruence, the relation $\preceq_{\delta_D}$ defined as in Definition 19 is a preorder. By Theorem 24, if every δ_D-cycle is closed (checked in Line 3), then $(\mathcal{C}(A, B, R)/\delta, \preceq_\delta)$ is a poset, that is, the required relation δ is δ_D (Line 4).

Otherwise, $\preceq_{\delta_D}$ is only a preorder and we consider the new equivalence relation ρ_{δ_D} defined in Equation (4). As a consequence, we have that $\delta_D \sqsubseteq \rho_{\delta_D}$. If ρ_{δ_D} is a local congruence, by the definition of ρ_{δ_D}, we have that every ρ_{δ_D}-cycle is closed and, according to Theorem 24, $\preceq_{\rho_{\delta_D}}$ is a partial order on $\mathcal{C}(A, B, R)/\rho_{\delta_D}$. In this case, $\delta = \rho_{\delta_D}$ is the least local congruence we are interested in (Lines 6-8). Otherwise, from Theorem 29, in Line 10 we obtain the least local congruence δ_ρ containing to ρ_{δ_D}, such that $\rho_{\delta_D} \sqsubseteq \delta_\rho$. Therefore, we have that every δ_ρ-cycle is closed by the definition of ρ_{δ_D}, and by Theorem 24 we obtain that $\preceq_{\delta_\rho}$ is a partial order on $\mathcal{C}(A, B, R)/\delta_\rho$. Thus, $\delta = \delta_\rho$ is the least local congruence we are looking for.

Consequently, from the procedure we obtain that $(\mathcal{C}(A, B, R)/\delta, \preceq_\delta)$ is a poset where δ is the least local congruence containing ρ_D. □

In the next example we will show the procedure described above.

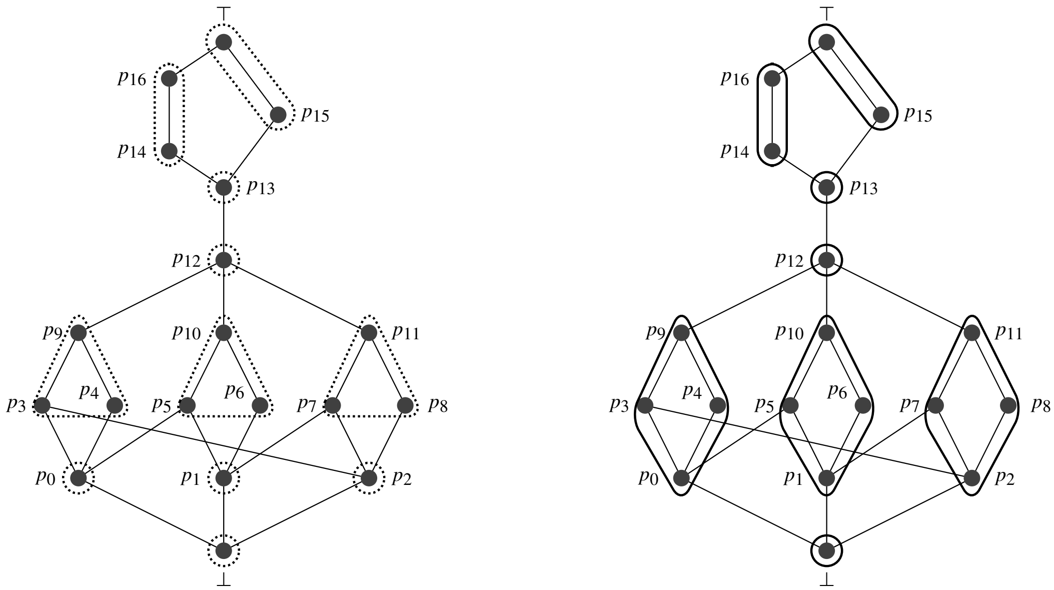

Fig. 8. Partition induced ρ_D (left) and local congruence δ_D (right) of Example 32.

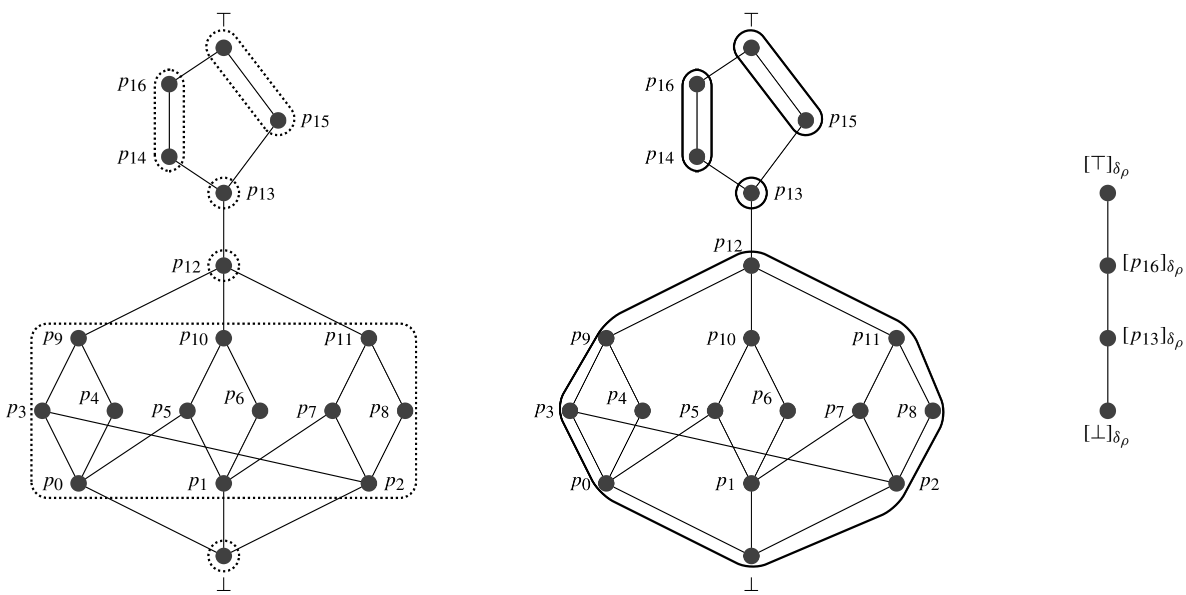

Fig. 9. The equivalence relation ρ on L/δ (left), the least local congruence δ_ρ containing ρ (middle) and the quotient set L/δ_ρ (right).

Example 32. Let us consider a context (A, B, R) and a subset of attributes $D \subseteq A$ such that after the reduction process we obtain the induced partition of the concept lattice displayed in the left side of Fig. 8. Thus, we consider the local congruence δ_D displayed in the right side of Fig. 8, it is easy to check that δ_D is indeed a local congruence and the least one containing the induced partition.

Considering the relation $\preceq_{\delta_D}$ given as in Definition 19, we can note that the δ_D-sequence, $(p_0, p_0)_{\delta_D} = (p_0, p_5, p_1, p_7, p_2, p_3, p_0)$, is in fact a δ_D-cycle in the lattice and it is not closed. Therefore, we define the equivalence relation $\rho = \{([x]_{\delta_D}, [y]_{\delta_D}) \in L/\delta_D \times L/\delta_D \mid [x]_{\delta_D} \preceq_{\delta_D} [y]_{\delta_D}$ and $[y]_{\delta_D} \preceq_{\delta_D} [x]_{\delta_D}\}$. The new partition of L provided by the equivalence relation ρ is shown in the left side of Fig. 9. We can observe that the equivalence relation ρ groups the classes of L/δ_D that contain elements in the δ_D-cycle into a single equivalence class. Moreover, it is also easy to observe that $\delta_D \sqsubseteq \rho$.

Table 2
Fuzzy relation R of Example 33.

R	b_1	b_2	b_3
a_1	1	0	0
a_2	0	0.5	0
a_3	0	0	1
a_4	0	0.5	1

C_i	Extent			Intent			
	b_1	b_2	b_3	a_1	a_2	a_3	a_4
0	0	0	0	1	1	1	1
1	1	0	0	1	0	0	0
2	0	0.5	0	0	1	0	1
3	0	0	1	0	0	1	1
4	1	1	1	0	0	0	0
5	0	1	0	0	0.5	0	0.5
6	0	0.5	1	0	0	0	1
7	0	1	1	0	0	0	0.5

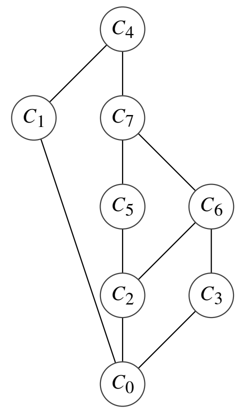

Fig. 10. Fuzzy concepts (left) and concept lattice (right) of the context associated with Table 2.

Now, we have to verify if the equivalence relation ρ is a local congruence, but we can observe that it is not, since the equivalence class that contains the δ_D-cycle is not a (convex) sublattice of L. Thus, we must find the least local congruence δ_ρ that contains the equivalence relation ρ.

In this case, the least local congruence that satisfies $\delta_D \sqsubseteq \rho \sqsubseteq \delta_\rho$ is the local congruence shown in the middle of Fig. 9. Hence, by Theorem 24, we have that $\preceq_{\delta_\rho}$ is a partial order on L/δ_ρ and the elements of the corresponding quotient set can be ranked. The ordered set $(L/\delta_\rho, \preceq_{\delta_\rho})$, is displayed in the right side of Fig. 9. It is important to note that the local congruence that we have finally obtained is not a congruence because the least congruence, containing the equivalence relation ρ, should include $p_{13}, p_{14}, p_{15}, p_{16}$ and $\top$ in the same class, in order to satisfy the quadrilateral-closed property. □

Now, we apply the proposed mechanism to reduce a concept lattice in a fuzzy formal concept framework. Specifically, the following example considers a fuzzy formal context studied in [8].

Example 33. The considered framework is $(L, L, L, \&_G^*)$, where the lattice $L = \{0, 0.5, 1\}$ and $\&_G^*$ is the discretization of the Gödel conjunctor defined on L. It is also considered a fuzzy context (A, B, R, σ), composed of three objects, $B = \{b_1, b_2, b_3\}$, four attributes $A = \{a_1, a_2, a_3, a_4\}$, the relation R shown in Table 2, and the mapping σ constantly $\&_G^*$. All concepts of this fuzzy context are listed in Fig. 10, where the corresponding concept lattice is illustrated as well.

In [8], authors obtained four different reducts to reduce the concept lattice. In this example, we will consider one of these reducts, specifically $D_1 = \{a_1, a_2\}$, to compute a local congruence of the reduced concept lattice obtained from this reduct.

The partition induced by D_1 is given in the left side of Fig. 11, and the corresponding reduced concept lattice is depicted in its right side. In this case, the least local congruence containing this partition is the partition itself, since each equivalence class is a convex sublattice of the original concept lattice. Moreover, this local congruence is not a congruence because it is not quadrilateral-closed, for example, C_0, C_2 and C_3, C_6 are opposite sides of the quadrilateral $\langle C_0, C_2; C_3, C_6\rangle$, the concepts C_3 and C_6 belong to one equivalence class, but C_0 and C_2 belong to different equivalence classes. Indeed, the least congruence containing the partition induced by the reduction of D_1 is the congruence with only one class containing all concepts. Thus, also in the fuzzy framework, local congruences offer more suitable reductions than the ones given by congruences. □

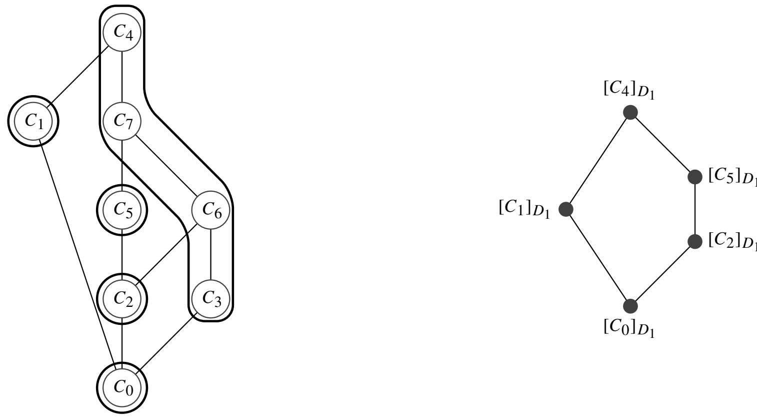

Fig. 11. Partition induced by the reduction (left) and concept lattice of the reduced context (right) considering the reduct D_1.

Therefore, the proposed reduction mechanism based on local congruences minimizes the amount of lost information with respect to the use of congruences, clustering the concepts in convex sublattices and forming a hierarchy among them.

7. Conclusions and future work

In this work, we have introduced a weaker notion of congruence, which has been called local congruence. We have analyzed how the elements of the quotient set generated by a local congruence can be ordered. Furthermore, we have proven that the algebraic structure of the set of local congruences is a complete lattice. We have also shown a characterization of local congruences in terms of its principal local congruences, as well as an extension of this characterization by considering any arbitrary equivalence relation. As a consequence, a procedure for computing the least local congruence containing a given equivalence relation has been presented. From this study, we have presented a new mechanism to reduce (fuzzy) concept lattices based on the notion of local congruence. Considering this reduction mechanism, we obtain a partition of the concepts of the original concept lattice satisfying that each equivalence class has the structure of a convex sublattice of the original concept lattice. In addition, we have shown that the consideration of local congruences to reduce concept lattices is more suitable than the consideration of congruences since a smaller amount of information is lost during the reduction process.

In the near future, more properties of the introduced procedure will be studied. For example, due to this reduction modifies the original partition given by the attribute reduction, it is important to analyze how it alters the formal context. In addition, we are interested in studying how an optimal reduct can be selected and the influence that this selection has on the complementary local congruence. Another important goal will be to apply this reduction procedure in real databases. Specifically, we would like to analyze the potential of the presented reduction mechanism in databases related to digital forensic analysis, in which we are leading the COST Action: DIGital FORensics: evidence Analysis via intelligent Systems and Practices (DigForASP).

Declaration of competing interest

None.

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Capítulo 5

Identificando las clases de equivalencias inducidas por una reducción de atributos en el FCA que no tienen estructura de retículo completo

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Identifying Non-Sublattice Equivalence Classes Induced by an Attribute Reduction in FCA

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Abstract: The detection of redundant or irrelevant variables (attributes) in datasets becomes essential in different frameworks, such as in Formal Concept Analysis (FCA). However, removing such variables can have some impact on the concept lattice, which is closely related to the algebraic structure of the obtained quotient set and their classes. This paper studies the algebraic structure of the induced equivalence classes and characterizes those classes that are convex sublattices of the original concept lattice. Particular attention is given to the reductions removing FCA's unnecessary attributes. The obtained results will be useful to other complementary reduction techniques, such as the recently introduced procedure based on local congruences.

Keywords: Formal Concept Analysis; equivalence relations; attribute reduction

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1. Introduction

Redundant data hinder the efficient acquisition of information from datasets. Obviously, the elimination of redundant data should not modify the information contained in a dataset. The most common redundant data consist of repeated entries, which can be removed without cost, or dependent variables, which can be derived from the independent variables, whose detection is an appealing research topic in many areas dealing with data analysis, such as Formal Concept Analysis (FCA). This mathematical theory was original developed in the 1980s by R. Wille and B. Ganter [1], and it has intensively been studied from a theoretical and applied point of view [2–12]. Two important features of FCA, in which the notion of Galois connection is fundamental [13–16], is that the information contained in a relational dataset can be described in a hierarchic manner by means of a complete lattice [17] and that dependencies between attributes can be determined [18–21], which is fundamental to applications. In both features, the removal of redundant data has a great impact.

The detection of (ir)relevant attributes or objects of a given formal context in FCA has been studied from different points of view, for example: in order to obtain a concept lattice isomorphic to the original one [22–26], to efficiently reduce the size of the concept lattice [8,27–32], to extensional stability [33], to consider contexts with positive and negative attributes [34], to apply the rough set philosophy [35–37], etc. Notice that the different mechanisms focused on attribute reduction can dually be adapted to object reduction.

In [36], it was demonstrated that attribute reductions of formal contexts induce equivalence relations whose equivalence classes have the structure of join-semilattices. In addition, in [38], local congruences were introduced as equivalence relations defined on lattices whose equivalence classes are sublattices of the original lattice. Therefore, local congruences were intended to complement the attribute reductions of formal contexts in order to ensure that the equivalence classes $[C]_D$ are sublattices of the original concept lattice. Due to a join-semilattice with the least element being a lattice, if the infimum $C_m = \bigwedge_{C_i \in [C]_D} C_i$ belongs to the equivalence class, we can assert that the class already is a sublattice. Obviously, in these cases, the use of local congruences, as a complementary mechanism to the attribute reduction, turns out to be unnecessary since they do not provide any modification in the classes, and so, it modifies neither the attributes nor the objects generating the concepts of these classes. Therefore, it is very interesting to characterize the required conditions in which these cases hold, which was precisely the main issue addressed in [39].

In this paper, we continue with the research line initiated in [39], improving the results introduced in that work. Specifically, in this paper, we show an enhanced version of Proposition 4 and Corollary 1 in [39], which characterize the infimum of the elements belonging to a non-singleton classes. In addition, due to the attribute reductions usually carried out in FCA tending to discard the set of unnecessary attributes from the formal context, we also analyze the characterization of the infimum of the induced equivalence classes when the considered attribute reduction does not contain unnecessary attributes. The fact of considering attribute reductions that do not contain unnecessary attributes allows us also to prove some interesting results. For example, we establish a sufficient condition to ensure an equivalence between meet-irreducible concepts in the reduced context and in the original one. Furthermore, under this consideration, we also prove that when the original concept lattice is isomorphic to a distributive lattice, the induced equivalence classes by the reduction are always sublattices. Finally, all the results presented in this work are accompanied by illustrative examples whose objective is to clarify all the introduced ideas.

The paper is structured as follows: Section 2 reviews some preliminary notions related to formal concept analysis and attribute reduction. In Section 3, the contributions of this paper are presented, the study on the equivalence classes induced by an attribute reduction.This section is divided into two parts: first, we study sufficient conditions to characterize the infimum of equivalence classes, and second, we carry out an analysis of the characterization when the considered subset of attributes in the reduction does not contain unnecessary attributes. Finally, Section 4 presents some conclusions and provides some prospects for future work.

2. Preliminaries

First of all, we recall some basic notions about formal concept analysis and attribute reduction. A context in FCA is a triple (A, B, R) where A is a set of attributes, B is a set of objects, and $R \subseteq A \times B$ is a relation, such that $(a, x) \in R$, if the object $x \in B$ possesses the attribute $a \in A$, and $(a, x) \notin R$, otherwise. The derivation operators are the mappings ${}^{\uparrow}\colon 2^B \to 2^A$ and ${}^{\downarrow}\colon 2^A \to 2^B$ defined for each $X \subseteq B$ and $Y \subseteq A$ as:

$$X^{\uparrow} = \{a \in A \mid \text{for all } x \in X, (a, x) \in R\} \quad (1)$$

$$Y^{\downarrow} = \{x \in B \mid \text{for all } a \in Y, (a, x) \in R\} \quad (2)$$

A concept in (A, B, R) is a pair $C = (X, Y)$, where $X \subseteq B$, $Y \subseteq A$, and satisfies that $X^{\uparrow} = Y$ and $Y^{\downarrow} = X$. The subset X is called the extent of the concept, and the subset Y is called the intent; they are denoted by $\mathfrak{E}(C)$ and $\mathfrak{I}(C)$, respectively. Furthermore, a concept generated by an attribute $a \in A$, that is $(a^{\downarrow}, a^{\downarrow\uparrow})$, is called an attribute concept.

In addition, the set of concepts is denoted by $\mathcal{C}(A, B, R)$ and is a complete lattice with the inclusion order on the left argument, that is for each $(X_1, Y_1), (X_2, Y_2) \in \mathcal{C}(A, B, R)$, we have $(X_1, Y_2) \leq (X_2, Y_2)$ if $X_1 \subseteq X_2$. $(\mathcal{C}(A, B, R), \leq)$ is called the concept lattice of the context (A, B, R). The meet $\wedge$ and join $\vee$ operators are defined by:

$$(X_1, Y_1) \wedge (X_2, Y_2) = (X_1 \wedge X_2, (Y_1 \vee Y_2)^{\downarrow\uparrow})$$

$$(X_1, Y_1) \vee (X_2, Y_2) = ((X_1 \vee X_2)^{\uparrow\downarrow}, Y_1 \wedge Y_2)$$

for all $(X_1, Y_1), (X_2, Y_2) \in \mathcal{C}(A, B, R)$.

Considering a subset of attributes $Y \subseteq A$ and the restriction relation $R_{|Y\times B} = R \cap (Y \times B)$, the triple $(Y, B, R_{|Y\times B})$ is also a formal context. There are two relevant notions regarding subsets of attributes in FCA that we recall below.

Definition 1. *Given a context (A, B, R), if there exists a subset of attribute $Y \subseteq A$ such that $\mathcal{C}(A, B, R) \cong \mathcal{C}(Y, B, R_{|Y\times B})$, then Y is called a consistent set of (A, B, R). Moreover, if $\mathcal{C}(Y\backslash\{y\}, B, R_{|Y\backslash\{y\}\times B}) \ncong \mathcal{C}(A, B, R)$, for all $y \in Y$, then Y is called a reduct of (A, B, R).*

Then, we can recall the definition of the three types of attributes considering the notation in [40] to denote the subsets of attributes.

Definition 2. *Given an index set Λ, a formal context (A, B, R), and the set $\{Y_i \mid Y_i$ is a reduct, $i \in \Lambda\}$ of all reducts of (A, B, R), the set of attributes A can be divided into the following three parts:*

1. *Absolutely necessary attributes $C_f = \bigcap_{i\in\Lambda} Y_i$.*
2. *Relatively necessary attributes $K_f = (\bigcup_{i\in\Lambda} Y_i)\backslash(\bigcap_{i\in\Lambda} Y_i)$.*
3. *Absolutely unnecessary attributes $I_f = A\backslash(\bigcup_{i\in\Lambda} Y_i)$.*

The set of attributes of the context is closely related to the meet-irreducible concepts, whose notion is recalled in the following definition.

Definition 3. *Given a lattice $(L, \preceq)$, such that $\wedge$ is the meet operator, and an element $x \in L$ verifying:*

1. *If L has a top element $\top$, then $x \neq \top$;*
2. *If $x = y \wedge z$, then $x = y$ or $x = z$, for all $y, z \in L$;*

x is called a meet-irreducible ($\wedge$-irreducible) element of L.

In particular, in this paper, we use the notion of the unnecessary attribute and, specifically, the following characterization introduced in [26].

Theorem 1. *Given a formal context (A, B, R) and the set of $\wedge$-irreducible elements of $\mathcal{C}(A, B, R)$, denoted by $M_F(A, B, R)$, the following equivalences are obtained:*

1. *$a \in I_f$ if and only if $(a^{\downarrow}, a^{\downarrow\uparrow}) \notin M_F(A, B, R)$.*
2. *$a \in K_f$ if and only if $(a^{\downarrow}, a^{\downarrow\uparrow}) \in M_F(A, B, R)$ and there exists $a_1 \in A$, $a_1 \neq a$, such that $(a_1^{\downarrow}, a_1^{\downarrow\uparrow}) = (a^{\downarrow}, a^{\downarrow\uparrow})$.*
3. *$a \in C_f$ if and only if $(a^{\downarrow}, a^{\downarrow\uparrow}) \in M_F(A, B, R)$ and $(a_1^{\downarrow}, a_1^{\downarrow\uparrow}) \neq (a^{\downarrow}, a^{\downarrow\uparrow})$, for all $a_1 \in A$, $a_1 \neq a$.*

With respect to attribute reductions in FCA, we recall the main results related to the induced equivalence relation on the set of concepts of the original concept lattice when we reduce the set of attributes of a formal context. For more detailed information, we refer the reader to [36,39]. The following proposition was proven in [36] for the classical setting of FCA.

Proposition 1 ([36])**.** *Given a context (A, B, R) and a subset $D \subseteq A$, the set $\rho_D = \{((X_1, Y_1), (X_2, Y_2)) \mid (X_1, Y_1), (X_2, Y_2) \in \mathcal{C}(A, B, R), X_1^{\uparrow_D\downarrow} = X_2^{\uparrow_D\downarrow}\}$ is an equivalence relation, where $^{\uparrow_D}$ denotes the concept-forming operator given in Expression (2), restricted to the subset of attributes $D \subseteq A$.*

Moreover, the authors also proved that each equivalence class of the induced equivalence relation has a structure of join-semilattice, and they also determined the maximum element.

Proposition 2 ([36]). *Given a context (A, B, R), a subset $D \subseteq A$, and a class $[(X, Y)]_D$ of the quotient set $\mathcal{C}(A, B, R)/\rho_D$, the class $[(X, Y)]_D$ is a join-semilattice with maximum element $(X^{\uparrow_D \downarrow}, X^{\uparrow_D \downarrow \uparrow})$.*

In addition, an ordering relation on the set of equivalence classes given by the relation ρ_D was defined in [41].

Proposition 3 ([41]). *On the quotient set $\mathcal{C}(A, B, R)/\rho_D$ associated with a context (A, B, R), the relation $\sqsubseteq_D$, defined as $[(X_1, Y_1)]_D \sqsubseteq_D [(X_2, Y_2)]_D$ if ${X_1}^{\uparrow_D \downarrow} \subseteq {X_2}^{\uparrow_D \downarrow}$, for all $[(X_1, Y_1)]_D, [(X_2, Y_2)]_D \in \mathcal{C}(A, B, R)/\rho_D$, is an ordering relation.*

The quotient set $\mathcal{C}(A, B, R)/\rho_D$ with the ordering relation $\sqsubseteq_D$ is closely related to the reduced concept lattice as shown in the following result presented in [41].

Theorem 2 ([41]). *Given a context (A, B, R) and a subset of attributes $D \subseteq A$, we have that the quotient set given by ρ_D and the reduced concept lattice by D are isomorphic, that is:*

$$(\mathcal{C}(A, B, R)/\rho_D, \sqsubseteq_D) \cong (\mathcal{C}(D, B, R_{|D \times B}), \leq_D)$$

where $\leq_D$ is the ordering in the original concept lattice restricted to the reduced one.

Next, the notation of the infimum of an equivalence class is given to simplify the expressions in which it is involved.

Definition 4. *Given a context (A, B, R), a subset of attributes $D \subseteq A$, and an equivalence class $[C]_D$, with $C \in \mathcal{C}(A, B, R)$, of the induced equivalence relation, the infimum of the subset of concepts $[C]_D$ is denoted by C_m, that is $C_m = \bigwedge_{C_i \in [C]_D} C_i$.*

The following result was presented in [39], and it establishes preliminary consequences whenever a class, of the equivalence relation induced by an attribute reduction, contains its infimum.

Proposition 4 ([39]). *Let (A, B, R) be a context, $D \subseteq A$ a subset of attributes, and $[C]_D$ an equivalence class of the induced equivalence relation, with $C \in \mathcal{C}(A, B, R)$, which is not a convex sublattice, then we have that one of the following statements is satisfied:*

- *There exists at least one attribute $a \in D$ such that $C_m = (a^{\downarrow}, a^{\downarrow\uparrow})$.*
- *There exists a concept $C^* \in M_F(A, B, R)$ in a meet-irreducible decomposition $\{C_j \in M_F(A, B, R) \mid j \in J\}$ of C_m, such that $C_{i_0} \not\leq C^*$ for a concept $C_{i_0} \in [C]_D$.*

We continue, in the following sections, this study exploring characterizations of the infimum of equivalence classes. Lastly, distributive lattices play an important role at the end of the paper. We recall their definition below.

Definition 5. *A lattice $(L, \preceq)$ is called distributive if, for all $x, y, z, \in L$,*

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z).$$

Distributive lattices offer interesting properties such as the uniqueness of meet and join-irreducible decompositions and the following result.

Proposition 5 ([17]). *Given a distributive lattice $(L, \preceq)$ and a meet-irreducible element p, if $\wedge_{i \in I} x_i \preceq p$, then there exists $i_0 \in I$, such that $x_{i_0} \preceq p$.*

3. Characterizing the Infimum of Classes

In this section, we continue with the study on the equivalence classes induced by a reduction of attributes presented in [39], improving the results introduced in it. From this point forward, a formal context (A, B, R) is fixed, and the maximum element of an equivalence class $[C]_D$, with $D \subseteq A$ and $C = (X, Y) \in \mathcal{C}(A, B, R)$, is denoted by $C_M = (X_M, Y_M)$. Notice that Proposition 2 characterizes C_M as $(X^{\uparrow_D\downarrow}, X^{\uparrow_D\downarrow\uparrow})$, and therefore, $X_M = X^{\uparrow_D\downarrow}$ is also the extent of a concept of the reduced concept lattice $\mathcal{C}(D, B, R_{|D\times B})$.

First of all, different technical results are introduced. Given a context (A, B, R) and any two concepts of its corresponding concept lattice $C_1 = (X_1, Y_1), C_2 = (X_2, Y_2) \in \mathcal{C}(A, B, R)$, it is known that if $C_1 < C_2$, then there exists $a_0 \in Y_1$, such that $a_0 \notin Y_2$. This attribute is completely determined when attribute concepts are considered.

Proposition 6. *Given a context (A, B, R), $C_1 \in \mathcal{C}(A, B, R)$, $D \subseteq A$, $a \in D$, and a concept $C_2 = (a^{\downarrow}, a^{\downarrow\uparrow})$, we have that:*

$$C_1 \not\leq C_2 \textit{ if and only if } a \notin \mathfrak{I}(C_1).$$

Proof. If $a \in \mathfrak{I}(C_1)$, then by the properties of the concept-forming operators, we obtain that $a^{\downarrow\uparrow} \subseteq \mathfrak{I}(C_1)^{\downarrow\uparrow} = \mathfrak{I}(C_1)$, which leads us to a contradiction with the hypothesis $C_1 \not\leq C_2$. Thus, $a \notin \mathfrak{I}(C_1)$.

By reduction ad absurdum, we assume that $C_1 \leq C_2$, then $\mathfrak{I}(C_2) \subseteq \mathfrak{I}(C_1)$, which implies that:

$$a \in a^{\downarrow\uparrow} = \mathfrak{I}(C_2) \subseteq \mathfrak{I}(C_1)$$

Hence, we obtain a contradiction with the hypothesis $a \notin \mathfrak{I}(C_1)$. □

In a similar way, the following proposition arises for equivalence classes induced by an attribute reduction.

Proposition 7. *Given a context (A, B, R), $C_1, C_2 \in \mathcal{C}(A, B, R)$ with $C_1 \leq C_2$, and $D \subseteq A$, we have that $C_1 \notin [C_2]_D$ if and only if there exists $a \in D$, such that $a \in \mathfrak{I}(C_1)$ and $a \notin \mathfrak{I}(C_2)$.*

Proof. We consider a context (A, B, R) and a subset of attribute $D \subseteq A$. Let us assume any two concepts $C_1, C_2 \in \mathcal{C}(A, B, R)$ such that $C_1 \leq C_2$. On the one hand, if $C_1 \notin [C_2]_D$, then we obtain straightforwardly that $C_1 < C_2$, and hence, there exists $a \in D$, such that $a \in \mathfrak{I}(C_1)$ and $a \notin \mathfrak{I}(C_2)$. On the other hand, if there exists an attribute $a \in D$ such that $a \in \mathfrak{I}(C_1)$ and $a \notin \mathfrak{I}(C_2)$, then $C_1 \notin [C_2]_D$ by the definition of ρ_D in Proposition 1. □

In Proposition 7, we chose the concept C_2 to represent the equivalence class $[C_2]_D$, but this class univocally determines a concept in the reduced concept lattice $\mathcal{C}(D, B, R_{|D\times B})$ by Theorem 2. In order to differentiate between classes and associated concepts in the reduced concept lattice, we denote the latter with a line over the concept, that is $\overline{C_2}$.

3.1. Characterizing the Infimum of Classes

The following property determines a sufficient condition to ensure that the equivalence class of the infimum element is generated by an attribute concept.

Theorem 3. *Let (A, B, R) be a context, a finite subset of attributes $D \subseteq A$, and $C \in \mathcal{C}(A, B, R)$ such that $C_j \in [C]_D$, for all concepts C_j in any meet-irreducible decomposition $\{C_j \in M_F(A, B, R) \mid j \in J\}$ of C_m. If C_m is not in $[C]_D$, then there exists an attribute $a \in D$ such that $[C_m]_D = [(a^{\downarrow}, a^{\downarrow\uparrow})]_D$.*

Proof. Since C_m is not in $[C]_D$, by Proposition 7, there exists $a_1 \in D$, such that $a_1 \in \mathfrak{I}(C_m)$ and $a_1 \notin \mathfrak{I}(C)$.

If $C_m = (a_1^{\downarrow}, a_1^{\downarrow\uparrow})$, we are finished. Otherwise, there exists $C_1 = (a_1^{\downarrow}, a_1^{\downarrow\uparrow})$, such that $C_m < C_1$. If $C_1 \in [C]_D$, then $a_1^{\downarrow\uparrow_D} = \mathfrak{I}(C)^{\downarrow\uparrow_D}$, and we obtain that:

$$a_1 \in a_1^{\downarrow\uparrow_D} = \mathfrak{I}(C)^{\downarrow\uparrow_D} \subseteq \mathfrak{I}(C)^{\downarrow\uparrow} = \mathfrak{I}(C)$$

which leads us to a contradiction. Therefore, we have that $C_1 \notin [C]_D$. Hence, in particular, C_1 cannot be meet-irreducible, since by hypothesis, in this case, C_1 should be in $[C]_D$. Therefore, we consider a meet-irreducible decomposition $\{C_j^1 \in M_F(A, B, R) \mid j \in J_1\}$ of C_1. Due to $C_m \leq C_1$, the meet-irreducible concepts C_j^1 are in a meet-irreducible decomposition of C_m, which implies by hypothesis that $C_j^1 \in [C]_D$ for all $j \in J_1$. Therefore,

$$[C_1]_D \sqsubset_D [C_j]_D = [C]_D$$

for all $j \in J_1$, where $\sqsubset_D$ is the ordering defined in Proposition 3. As a consequence, if $[C_m]_D = [C_1]_D$, then we are finished. Otherwise, we have that $[C_m]_D \sqsubset_D [C_1]_D \sqsubset_D [C]_D$. Thus, there exists $a_2 \in D \backslash \{a_1\}$, such that $a_2 \in \mathfrak{I}(C_m)$ and $a_2 \notin \mathfrak{I}(C)$. This process can be repeated, and due to D being finite, it must finish in an attribute $a \in D$ such that $[C_m]_D = [(a^{\downarrow}, a^{\downarrow\uparrow})]_D$. □

Note that Theorem 3 arises from the restriction of the hypotheses of Proposition 4. The following example is useful to illustrate the previous result. This example also inspects Corollary 1 presented in [39], and as a consequence, it also argues that Theorem 3 must be considered instead of this corollary.

Example 1. *We consider a context composed of the set of attributes $A = \{a_1, a_2, a_3, a_4\}$ and the set of objects $B = \{b_1, b_2, b_3\}$, related by $R\colon A \times B \to \{0, 1\}$, defined on Table 1, which has the concepts listed on Table 2. The associated concept lattice is given on the left side of Figure 1.*

Table 1. Relation of the context of Example 1.

R	b_1	b_2	b_3
a_1	1	1	0
a_2	1	0	1
a_3	0	1	1
a_4	0	0	1

Table 2. List of extents and intents of every concept of the context of Example 1.

C_i	Extent			Intent			
	b_1	b_2	b_3	a_1	a_2	a_3	a_4
0	0	0	0	1	1	1	1
1	1	0	0	1	1	0	0
2	0	1	0	1	0	1	0
3	0	0	1	0	1	1	1
4	1	1	0	1	0	0	0
5	1	0	1	0	1	0	0
6	0	1	1	0	0	1	0
7	1	1	1	0	0	0	0

From this context, we obtain the attribute concepts listed below, together with the induced equivalence classes obtained by removing attributes a_2 and a_3, that is considering only the subset of attributes $D = \{a_1, a_4\}$.

$$\begin{aligned} C_3 &= (a_4^{\downarrow}, a_4^{\downarrow\uparrow}) \\ C_4 &= (a_1^{\downarrow}, a_1^{\downarrow\uparrow}) \\ C_5 &= (a_2^{\downarrow}, a_2^{\downarrow\uparrow}) \\ C_6 &= (a_3^{\downarrow}, a_3^{\downarrow\uparrow}) \end{aligned} \qquad \begin{aligned} [C_0]_D &= \{C_0\} \\ [C_1]_D = [C_2]_D = [C_4]_D &= \{C_1, C_2, C_4\} \\ [C_3]_D &= \{C_3\} \\ [C_5]_D = [C_6]_D = [C_7]_D &= \{C_5, C_6, C_7\} \end{aligned}$$

The partition induced by such a reduction is shown on the right side of Figure 1. Notice that two of the obtained equivalence classes are not convex sublattices of the original concept lattice. The first one contains the concepts C_1, C_2, C_4, and the other one contains the concepts C_5, C_6, C_7. However, the reasons for which these classes are not convex sublattices are well differentiated.

Figure 1. Concept lattice of Example 1 (**left**) and the partition induced by the elimination of attributes a_2 and a_3 in Example 1 (**right**).

On the one hand, if we consider the equivalence class $[C_7]_D$, we have that the infimum of the concepts of this class is the concept C_3. Notice that the meet-irreducible decomposition of C_3 is $C_3 = C_5 \wedge C_6$, and both concepts C_5 and C_6 belong to $[C_7]_D$; this means that we are under the conditions given in Theorem 3. Since $C_3 \notin [C_7]_D$, we have that $[C_3]_D = [(a^{\downarrow}, a^{\downarrow\uparrow})]_D$, with $a \in D$. Specifically, in this case, we have that the concept C_3 is just generated by the attribute $a_4 \in D$. Notice that C_m is not always an attribute concept. For example, if we consider $D' = \{a_4\}$, then we obtain two classes: $[C_7]_{D'} = \{C_7, C_6, C_5, C_4, C_2, C_1\}$ and $[C_3]_{D'} = \{C_3, C_0\}$, where $C_0 = C_4 \wedge C_5 \wedge C_6$ and satisfying that C_4, C_5, C_6 belong to $[C_7]_{D'}$. Therefore, the hypotheses of Theorem 3 hold; indeed, $[C_0]_{D'} = [(a_4^{\downarrow}, a_4^{\downarrow\uparrow})]_{D'}$; however, C_0 is not an attribute concept.

On the other hand, if we consider the equivalence class $[C_4]_D$, we have that the infimum of the equivalence class $[C_4]_D$ is the concept C_0. In this case, the decomposition of C_0 is $C_0 = C_4 \wedge C_5 \wedge C_6$, we observe that there are two meet-irreducible concepts, C_5 and C_6, such that $C_5, C_6 \notin [C_4]_D$. Therefore, we cannot apply Theorem 3 since the hypothesis are not satisfied. Moreover, since the concept lattice $\mathcal{C}(A, B, R)$ is distributive, the condition that the meet-irreducible concepts of the decomposition are in the class is a required hypothesis in [39], Corollary 1. In addition, this corollary must also be corrected in its conclusion, since $[C_m]_D = [(a^{\downarrow}, a^{\downarrow\uparrow})]_D$ can only be ensured. Thus, Theorem 3 presents an improved version of Corollary 1 given in [39].

Next, we present one of the main results of this paper, which characterizes the infimum of the elements of non-singleton classes.

Theorem 4. *Given a context (A, B, R), a subset of attributes $D \subseteq A$, and a concept $C \in \mathcal{C}(A, B, R)$ such that its equivalence class $[C]_D$ of the induced equivalence relation is not a singleton, we have that $C_m \notin [C]_D$ if and only if one of the following statements is satisfied:*

- *There exists at least one attribute $a \in D$ such that $C_m = (a^{\downarrow}, a^{\downarrow\uparrow})$.*
- *There exists a concept $C^* \in \mathcal{C}(A, B, R)$, such that $C^* = (a^{*\downarrow}, a^{*\downarrow\uparrow})$ with $a^* \in D$, $C^* \notin [C]_D$, and $C_M \not\leq C^*$. Moreover, $\overline{C^*}$ is in a meet-irreducible decomposition $\{\overline{C_j} \in M_F(D, B, R_{|D\times B}) \mid j \in J\}$ of $\overline{C_m}$. Recall that the concept of the reduced concept lattices is denoted with an overline.*

Proof. Let us assume that we reduce the context (A, B, R) by a subset of attributes $D \subseteq A$. Given a concept $C \in \mathcal{C}(A, B, R)$, we consider the induced equivalence class $[C]_D$, which is not a singleton. The concept $C_m = \bigwedge_{C_i \in [C]_D} C_i$ does not necessarily belong to the class $[C]_D$ since $[C]_D$ is a join-semilattice by Theorem 2.

Therefore, if $C_m \notin [C]_D$, then we can distinguish two cases:

- If there exists $a_0 \in D$ such that $C_m = (a_0^{\downarrow}, a_0^{\downarrow\uparrow})$, the first statement holds.
- Otherwise, C_m is not generated by any attribute of D. On the one hand, since $C_m \notin [C]_D$, we have that $\overline{C_m} < \overline{C}$ and $\overline{C} = \overline{C_i}$ for all $C_i \in [C]_D$; applying Proposition 7 to the reduced context, we can assert that there exists at least one attribute $a^* \in D$ such that $a^* \in \mathfrak{I}(C_m)$ and $a^* \notin \mathfrak{I}(C_i)$ for all $C_i \in [C]_D$. On the other hand, there exists an attribute concept $C^* \in \mathcal{C}(A, B, R)$ such that $C^* = (a^{*\downarrow}, a^{*\downarrow\uparrow})$, which implies that $C_m \leq C^*$. Moreover, $C^* \notin [C]_D$, because $a^* \notin \mathfrak{I}(C_i)$ for all $C_i \in [C]_D$. If $C^* \in M_F(A, B, R)$, then the concept C^* is the required concept, and the second statement holds.
  If $C^* \notin M_F(A, B, R)$, we consider a meet-decomposition of C^* in the reduced concept lattice $\mathcal{C}(D, B, R_{|D\times B})$, that is $\overline{C^*} = \bigwedge_{j\in J} \overline{C_j}$, where $\overline{C_j} \in M_F(D, B, R_{|D\times B})$ for all $j \in J$. Since $C^* = (a^{*\downarrow}, a^{*\downarrow\uparrow})$ and $a^* \notin \mathfrak{I}(C_M)$, then by Proposition 6, we have that $\overline{C_M} \not\leq \overline{C^*}$. If $\overline{C_M} \leq \overline{C_j}$, for all $j \in J$, then, by the infimum property, we obtain that $\overline{C_M} \leq \bigwedge_{\overline{C_j}\in J} \overline{C_j} = \overline{C^*}$, which leads us to a contradiction. Therefore, there exists $j_0 \in J$, such that $\overline{C_{j_0}}$ is in a meet-decomposition in the reduced context of $\overline{C^*}$, with $\overline{C_M} \not\leq \overline{C_{j_0}}$. This last property implies that $C_{j_0} \notin [C_M]_D$ (since, otherwise, $\overline{C_M} = \overline{C_{j_0}}$) and $C_M \not\leq C_{j_0}$. Moreover, since $\overline{C_{j_0}}$ is a meet-irreducible concept of the reduced context, then there exists $a' \in D$, such that $\overline{C_{j_0}} = (a'^{\downarrow}, a'^{\downarrow\uparrow_D})$.
  Thus, C_{j_0} is the required concept in the second statement.

Now, we assume that one of the statements is satisfied, and we again consider two cases:

- There exists $a_0 \in D$ such that $C_m = (a_0^{\downarrow}, a_0^{\downarrow\uparrow})$, then since $[C]_D$ is not a singleton, we have that $C_m = \bigwedge_{C_i \in [C]_D} C_i < C$, and so, $a_0 \in \mathfrak{I}(C_m)$ and $a_0 \notin \mathfrak{I}(C)$. Therefore, by Proposition 7, we obtain that $C_m \notin [C]_D$.
- There exists a concept $C^* \in \mathcal{C}(A, B, R)$, such that $C^* = (a^{*\downarrow}, a^{*\downarrow\uparrow})$ with $a^* \in D$, $C_M \not\leq C^*$, and $\overline{C^*}$ is in a meet-irreducible decomposition $\{\overline{C_j} \in M_F(D, B, R_{|D\times B}) \mid j \in J\}$ of $\overline{C_m}$. Hence, by Proposition 6, we have that $a^* \in \mathfrak{I}(C^*)$ and $a^* \notin \mathfrak{I}(C_M)$. Due to $\overline{C^*}$ being in a meet-irreducible decomposition of $\overline{C_m}$, in particular, we have that $C_m \leq C^*$, which implies that $a^* \in \mathfrak{I}(C^*) \subseteq \mathfrak{I}(C_m)$. Thus, since $a^* \in \mathfrak{I}(C_m)$ and $a^* \notin \mathfrak{I}(C_M)$, by Proposition 7, we obtain that $C_m \notin [C_M]_D = [C]_D$.

□

Notice that, given $C^* = (a^{*\downarrow}, a^{*\downarrow\uparrow})$ with $a^* \in D$, if $C \not\leq C^*$, then a^* cannot belong to $\mathfrak{I}(C)$, and so, $C^* \notin [C]_D$. Hence, the hypothesis $C^* \notin [C]_D$ is not considered in the implication to prove $C_m \notin [C]_D$. Moreover, from the hypothesis that $\overline{C^*}$ is in a meet-irreducible decomposition $\{\overline{C_j} \in M_F(D, B, R_{|D\times B}) \mid j \in J\}$ of $\overline{C_m}$, only $C_m \leq C^*$ is used. These hypotheses are included in the characterization in order to collect as much as possible the consequences of $C_m \notin [C]_D$. The following corollary presents the reduced version.

Corollary 1. *Given a context (A, B, R), a subset of attributes $D \subseteq A$, and a concept $C \in \mathcal{C}(A, B, R)$ such that its equivalence class $[C]_D$ of the induced equivalence relation is not a singleton, then $C_m \notin [C]_D$ if and only if one of the following statements is satisfied:*

- *There exists at least one attribute $a \in D$ such that $C_m = (a^{\downarrow}, a^{\downarrow\uparrow})$.*
- *There exists a concept $C^* \in \mathcal{C}(A, B, R)$, such that $C^* = (a^{*\downarrow}, a^{*\downarrow\uparrow})$ with $a^* \in D$, $C_m \leq C^*$, and $C_M \not\leq C^*$.*

Notice that the previous corollary improves Proposition 4, since the concept C^* must be generated by an attribute of the reduced attribute subset D, and it does not need to be a meet-irreducible concept in order to obtain the equivalence. An application of Theorem 4 is illustrated in the following example.

Example 2. *We consider a context (A, B, R) whose Hasse diagram of its concept lattice $\mathcal{C}(A, B, R)$ is on the left side of Figure 2. Labels in the concept lattice indicate the mappings γ and μ of the fundamental theorem, that is the node labeled as a represents the concept $(a^{\downarrow}, a^{\downarrow\uparrow})$ for $a \in A$, and similarly, the node labeled as b represents the concept $(b^{\uparrow}, b^{\uparrow\downarrow})$ for $b \in B$.*

Now, if we consider a subset of attributes $D = \{a_1, a_4, a_5\}$, we obtain an induced partition, which is illustrated in the middle of Figure 2. We choose the class $[C_4]_D = \{C_2, C_3, C_4\}$, which is not a singleton, and therefore, we are under the conditions of Theorem 4. Thus, $C_m = C_0 \notin [C_4]_D$ if and only if one of the statements of Theorem 4 is satisfied. In this case, we have that $C_m \notin [C_4]_D$, and the second statement holds, as we show next.

We have that the concept $C_5 \in \mathcal{C}(A, B, R)$ is the attribute concept generated by a_5, $C_5 = (a_5^{\downarrow}, a_5^{\downarrow\uparrow})$, where $a_5 \in D$, and it satisfies that $C_5 \notin [C_4]_D$ and $C_M = C_4 \not\leq C_5$. Moreover, the meet-irreducible decomposition of $\overline{C_m}$ in the reduced concept lattice, shown on the right side of Figure 2, is the set $\{\overline{C_4}, \overline{C_5}\}$. Since $\overline{C_5}$ is in the meet-irreducible decomposition of $\overline{C_m}$, we can conclude that the second statement is satisfied.

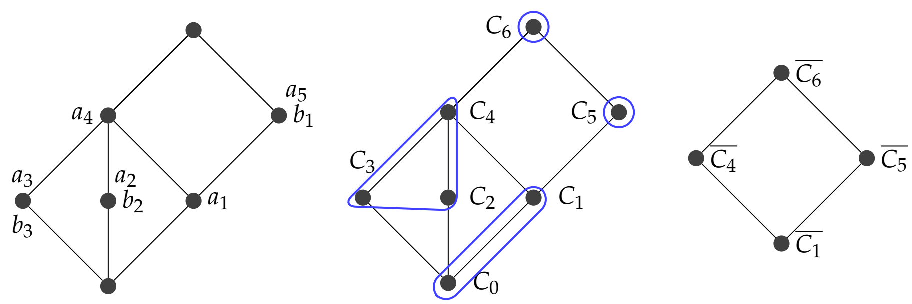

Figure 2. Concept lattices of Example 2.

3.2. Attribute Reduction without Unnecessary Attributes

Attribute reduction in FCA usually removes unnecessary attributes. Hence, this section analyzes the characterization when the set D does not contain unnecessary attributes. This study is interesting, for example, for any attribute reduction strategy merging FCA and other frameworks, such as rough set theory [35,42,43]. The first result shows that Statement 1 in Proposition 4 only arises when the context contains unnecessary attributes.

Proposition 8. *Given a context (A, B, R), a subset of attributes $D \subseteq A$, and a concept $C \in \mathcal{C}(A, B, R)$, if the equivalence class $[C]_D$ of the induced equivalence relation is not a singleton and there exists $a \in D$ such that $C_m = (a^{\downarrow}, a^{\downarrow\uparrow})$, then $a \in I_f$.*

Proof. Since $[C]_D$ is not a singleton, $C_m = \bigwedge_{C_i \in [C]_D} C_i$ and $C_m = (a^{\downarrow}, a^{\downarrow\uparrow})$ with $a \in D$, we have that C_m is not a $\wedge$-irreducible concept. Therefore, $a \in A$ generates a non-irreducible concept, and by Theorem 1, we obtain that $a \in I_f$. □

As a consequence of this result, Statement 1 in Theorem 4 and Statement 1in Corollary 1 cannot be satisfied when D does not contain any unnecessary attribute of A. Hence, according to this consideration, Theorem 4 can be written as follows.

Corollary 2. *Given a context (A, B, R), a subset of attributes $D \subseteq A$, such that $D \subseteq A \setminus I_f$, and a concept $C \in \mathcal{C}(A, B, R)$, where $[C]_D$ is not a singleton, then, $C_m \notin [C]_D$ if and only if there exists $C^* \in \mathcal{C}(A, B, R)$, such that $C^* = (a^{*\downarrow}, a^{*\downarrow\uparrow})$ with $a^* \in D$, $C^* \notin [C]_D$, $C_M \not\leq C^*$, and $\overline{C^*}$ is in a meet-irreducible decomposition $\{\overline{C_j} \in M_F(D, B, R_{|D\times B}) \mid j \in J\}$ of $\overline{C_m}$.*

In general, a meet-irreducible $\overline{C^*}$ in the reduced concept lattice does not have an associated meet-irreducible concept in the original concept lattice as the following example shows.

Example 3. *Let us consider the concept lattice on the left side of Figure 3, which is associated with a context $\mathcal{C}(A,B,R)$. If we select the subset of attributes $D = \{a_1, a_2, a_6\}$, then we obtain the induced partition illustrated in the middle of Figure 3. As we can see in the figure, there is a class, $[C_4]_D = \{C_1, C_2, C_4\}$, such that the concept $C_m = C_0 \notin [C_4]_D$. Therefore, by Theorem 4, there exists a concept satisfying the second statement of this theorem (the first statement is not satisfied since C_0 is not generated by any attribute).*

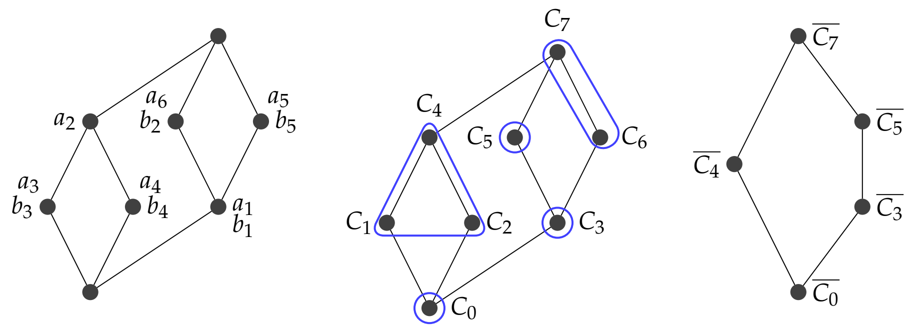

Figure 3. Concept lattices of Example 3.

Moreover, we have that the meet-irreducible decomposition of $\overline{C_m}$ is the set $\{\overline{C_3}, \overline{C_4}, \overline{C_5}\}$, as we can see in the reduced concept lattice induced by the attribute reduction depicted on the right side of Figure 3. In this case, the concept $\overline{C_3}$, which satisfies the conditions of the second statement of Theorem 4, is a meet-irreducible concept in the reduced concept lattice $\mathcal{C}(D,B,R_{|D\times B})$; however, the concept $C_3 = (a_1^{\downarrow}, a_1^{\downarrow\uparrow})$ is not a meet-irreducible concept in $\mathcal{C}(A,B,R)$.

The following result shows that removing unnecessary attributes provides a sufficient condition to ensure the equivalence between meet-irreducible concepts in the reduced context and in the original one.

Proposition 9. *Given a context (A,B,R), a subset of attributes $D \subseteq A \backslash I_f$, and a concept $C \in \mathcal{C}(A,B,R)$, such that $C = (a^{\downarrow}, a^{\downarrow\uparrow})$, with $a \in D$, the following equivalence holds:*

$$C \in M_F(A,B,R) \text{ if and only if } \overline{C} \in M_F(D,B,R_{|D\times B})$$

Proof. On the one hand, if we assume the concept C is a meet-irreducible concept in the original concept lattice, $C \in M_F(A,B,R)$, then taking into account that $a \in D$, the set of extents of the reduced concept lattice is included in the set of extents of the original, and if $C_1 \leq C_2$, then $\overline{C_1} \leq \overline{C_2}$ holds too; we have that $\overline{C} = (a^{\downarrow}, a^{\downarrow\uparrow_D})$ is also a meet-irreducible concept in the reduced concept lattice, $\overline{C} \in M_F(D,B,R_{|D\times B})$.

On the other hand, if we assume that $\overline{C} \in M_F(D,B,R_{|D\times B})$, then there exists $a \in D$ such that $\overline{C} = (a^{\downarrow}, a^{\downarrow\uparrow_D})$. Furthermore, since $a \notin I_f$, we have that $C = (a^{\downarrow}, a^{\downarrow\uparrow}) \in M_F(A,B,R)$ by Theorem 1. □

Notice that a_1 in Example 3 is an unnecessary attribute, which is the reason why C_3 is not a meet-irreducible element of the original concept lattice. Hence, as a consequence of the previous results, Theorem 4 can be rewritten as follows.

Theorem 5. *Given a context (A,B,R), a subset of attributes $D \subseteq A\backslash I_f$, an equivalence class $[C]_D$ with $C \in \mathcal{C}(A,B,R)$, of the induced equivalence relation, and the concept C_m, then $C_m \notin [C]_D$ if and only if there exists a concept $C^* \in M_F(A,B,R)$ in a meet-irreducible decomposition $\{C_j \in M_F(A,B,R) \mid j \in J\}$ of C_m, such that $C^* = (a^{*\downarrow}, a^{*\downarrow\uparrow})$ with $a^* \in D$ and $C_M \not\leq C^*$.*

Proof. The proof straightforwardly holds from Corollary 2 and Proposition 9. □

Therefore, if the subset $D \subseteq A$ contains no unnecessary attribute, which is the case of the reducts in FCA [1,26], the characterization is mainly based on the concepts of the original concept lattice instead of Theorem 4. This fact simplifies the detection of lattices whose equivalence classes of an attribute reduction are not convex sublattices of the original concept lattice. Moreover, this result also improves Proposition 4, showing that $D \subseteq A \backslash I_f$ must be included in the hypothesis of this proposition in order to obtain the equivalence.

Thus, the previous results and examples have a relevant interest for the application of local congruences, since they characterize the cases when the classes are not sublattices [36,39] and, so, what classes are affected when a local congruence is applied after an attribute reduction mechanism. In particular, we can determine the kind of lattices for which, after applying any attribute reduction, we obtain equivalence classes that are convex sublattices, that is for any class of any attribute reduction, the concept C_m belongs to the class. Based on these results, different particular cases are analyzed next.

Example 4. *The simplest non-linear concept lattice satisfying that "$C_m \in [C]_D$, for every class $[C]_D$ and attribute reduction $D \subseteq A$" is the one associated with the lattice $\mathcal{D}_1$, which is also denoted as M_2 [44] (left side of Figure 4) and without unnecessary attributes. If $[C]_D$ is a singleton, then clearly, $C_m \in [C]_D$. Otherwise, the only case in which $[C]_D$ does not contain C_m is when D implies that $[C]_D = \{C_M, C_1, C_2\}$. In this case, all meet-irreducible concepts in the decomposition of C_m belong to the class $[C]_D$, and by Theorem 3, we obtain that there exists $a \in D$, such that $[C_m]_D = [(a^{\downarrow}, a^{\downarrow\uparrow})]_D$, which implies in this particular case that $C_m = (a^{\downarrow}, a^{\downarrow\uparrow})$. Thus, by Proposition 8, we have that $a \in I_f$, which contradicts the hypothesis.*

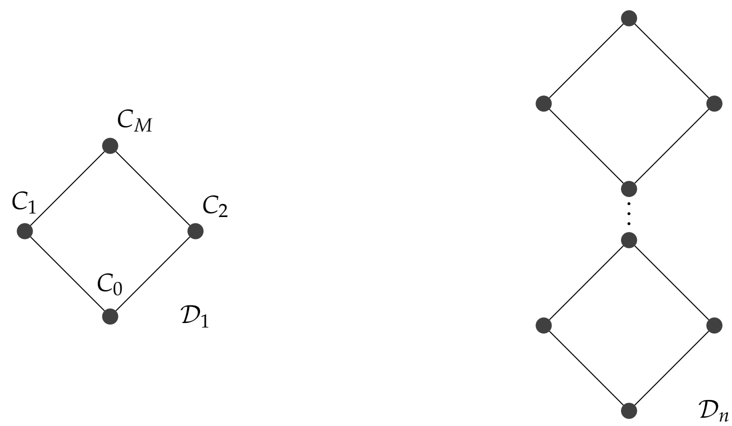

Figure 4. Concept lattices of Example 4.

The concatenation of this lattice (right side of Figure 4) also satisfies this statement as the following result shows.

Proposition 10. *Given a context (A, B, R), whose concept lattice is isomorphic to D_n, with $n \in \mathbb{N}$, $D \subseteq A \backslash I_f$, and a class $[C]_D$, we have that $C_m \in [C]_D$.*

Proof. Let us consider a context (A, B, R) whose associated concept lattice is isomorphic to D_n, with $n \in \mathbb{N}$. In addition, let us consider a subset of attributes $D \subseteq A$, a concept $C \in \mathcal{C}(A, B, R)$, and the equivalence class $[C]_D$ of the induced equivalence relation.

If $[C]_D$ is a singleton, then clearly, $C_m \in [C]_D$. Hence, we assume that $[C]_D$ is not a singleton. Since, by Proposition 8, C_m cannot be a meet-irreducible concept, we have that all meet-irreducible concepts in the decomposition of C_m belong to the class $[C]_D$, because of the shape of this lattice. Therefore, by Theorem 3, we have that $[C_m]_D = [(a^{\downarrow}, a^{\downarrow\uparrow})]_D$, with $a \in D$ and $a \notin \Im(C)$. If $C_m = (a^{\downarrow}, a^{\downarrow\uparrow})$, then by Proposition 8, we obtain that $a \in I_f$, which leads us to a contradiction. Otherwise, there exists a concept C^* such that

$C_m < C^* = (a^{\downarrow}, a^{\downarrow\uparrow})$ and $C^* \notin [C]_D$. As a consequence, by the shape of D_n, C^* must be a meet-irreducible concept in the meet-irreducible decomposition of C_m. Thus, we obtain a contradiction with the fact that all meet-irreducible concepts in the decomposition of C_m belong to the class $[C]_D$. □

The following example shows that the basic non-distributive lattices M_3 and N_5 do not satisfy the previous property.

Example 5. *We consider a context (A, B, R), where the Hasse diagram of its concept lattice $\mathcal{C}(A, B, R)$ is depicted on the left side of Figure 5, which is isomorphic to M_3, and it has no unnecessary attribute. If we carry out any attribute reduction on this particular context, we cannot ensure that every induced equivalence class obtained by the reduction is a convex sublattice. For instance, we consider the subset of attributes $D_{M_3} = \{a_3\}$, and therefore, the induced partition obtained by this attribute reduction is the Venn diagram depicted on the right side of Figure 5.*

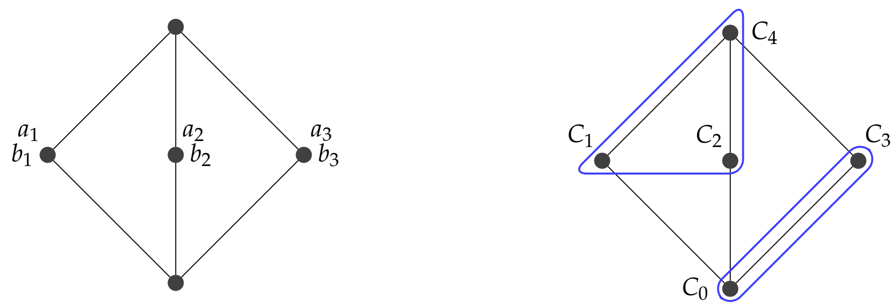

Figure 5. Concept lattice isomorphic to M_3 and induced partition by D_{M_3}.

We can notice that the infimum concept C_0 of the class $[C_4]_D = \{C_1, C_2, C_4\}$ does not belong to the class since there is a concept C_3 such that $C_3 = (a_3^{\downarrow}, a_3^{\downarrow\uparrow})$, with $a_3 \in D$, $C_m \leq C_3$ and $C_M \not\leq C_3$, that is Statement 2 of Corollary 1 is satisfied.

A similar case arises when a context (A, B, R), with concept lattice $\mathcal{C}(A, B, R)$ isomorphic to N_5 (left side of Figure 6) and without unnecessary attributes, is considered.

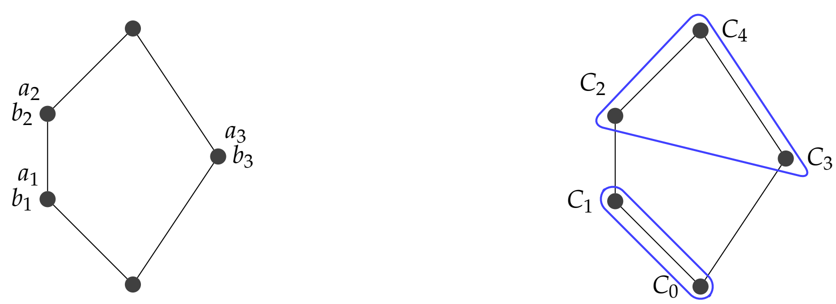

Figure 6. Concept lattice isomorphic to N_5 and induced partition by D_{N_5}.

If we consider the singleton $D_{N_5} = \{a_1\}$, we obtain the partition shown on the right side of Figure 6. In this case, Statement 2 of Corollary 1 also holds, because there is a concept C_1, such that $C_1 = (a_1^{\downarrow}, a_1^{\downarrow\uparrow})$ with $a_1 \in D$, $C_m \leq C_1$, and $C_M \not\leq C_1$. Thus, $C_m = C_0 \notin [C_4]_D = \{C_2, C_3, C_4\}$.

Finally, we prove that, in general, every distributive lattice satisfies the property.

Theorem 6. *Given a context (A, B, R), whose concept lattice is isomorphic to a distributive lattice, $D \subseteq A \backslash I_f$, and a class $[C]_D$, we have that $C_m \in [C]_D$.*

Proof. We proceed by reduction ad absurdum. Hence, we assume that $C_m \notin [C]_D$, and we get a contradiction.

From $C_m \notin [C]_D$, by Proposition 7, we have that there exists $a \in D$, such that $a \in \mathfrak{I}(C_m)$ and $a \notin \mathfrak{I}(C)$. Since $a \notin I_f$, by Proposition 8, there exists a concept C^* such that $C_m \leq C^* = (a^{\downarrow}, a^{\downarrow\uparrow})$ and $C^* \notin [C]_D$, which must be meet-irreducible, because otherwise, $a \in I_f$.

Moreover, by the definition of C_m, we have that:

$$C_m = \bigwedge_{C_i \in [C]_D} C_i = \bigwedge_{j \in J} C_j$$

where $\{C_j \in M_F(A,B,R) \mid j \in J\}$ is the unique meet-irreducible decomposition of C_m. The uniqueness arises because the concept lattice is distributive. Hence, in particular, the concept C^* belongs to this decomposition. Therefore, by Proposition 5, there exists $C_i \in [C]_D$, such that $C_i \leq C^*$, which implies that:

$$a \in \mathfrak{I}(C^*)^{\downarrow\uparrow_D} \subseteq \mathfrak{I}(C_i)^{\downarrow\uparrow_D} = \mathfrak{I}(C)^{\downarrow\uparrow_D}$$

which contradicts that $a \notin \mathfrak{I}(C)$. □

This result and the previous example are very interesting since they characterize the concept lattices providing convex sublattices for every attribute reduction. In addition, when the concept lattice is not distributive, we highlighted that many possibilities exist such that an attribute reduction provides equivalent classes, which are not sublattices of the original one. Moreover, Theorem 6 holds when the attribute reduction does not contain unnecessary attributes, as in FCA. If the reduction is given by another mechanism, such as based on the rough set theory philosophy [35,42,43], we can obtain classes that are not sublattices, as Example 3 shows. These facts also reinforce the necessity of studying mechanisms to lightly modify the equivalence relation given by the reduction in order to ensure that the classes are convex sublattices, as the new notion of local congruence [38,45] does.

4. Conclusions and Future Work

In this paper, we improve the results presented in [39], giving a characterization of the infimum of the elements belonging to a non-singleton class induced by an attribute reduction. Furthermore, we also found the characterization of these infimum elements when the considered attribute reduction does not contain unnecessary attributes, which is of special interest in FCA since attribute reductions usually discard this kind of attribute. We also introduced other interesting results in this framework. For example, we proved that the equivalence classes, induced by an attribute reduction on a distributive concept lattice, always have the structure of a convex sublattice. All the theoretical development carried out in this paper has a direct impact on the theory of local congruences [38].

In the future, we will study sufficient conditions on a (fuzzy) context in order to ensure that its concept lattice is distributive. This is an interesting problem, which has already attracted the attention of other researchers [46]. Moreover, the introduced results will be applied to real cases, such as the ones obtained from the COST Action DigForASP, which is focused on the application of artificial intelligence and automatic reasoning tools to digital forensics.

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Capítulo 6

Impacto de las congruencias locales en la selección de variables de bases de datos

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Impact of local congruences in variable selection from datasets ☆

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ABSTRACT

Formal concept analysis (FCA) is a useful mathematical tool for obtaining information from relational datasets. One of the most interesting research goals in FCA is the selection of the most representative variables of the dataset, which is called attribute reduction. Recently, the attribute reduction mechanism has been complemented with the use of local congruences in order to obtain robust clusters of concepts, which form convex sublattices of the original concept lattice. Since the application of such local congruences modifies the quotient set associated with the attribute reduction, it is fundamental to know how the original context (attributes, objects and relationship) has been modified in order to understand the impact of the application of the local congruence in the attribute reduction.

1. Introduction

The variable selection problem is a hot topic in many areas dedicated to data analysis. On many occasions, making a correct selection of the considered variables facilitates the management of the information in a considerable way, but it can also lead to certain changes in the provided information that must be analyzed in order to control them. This is one of the most appealing research lines within the theory of formal concept analysis.

Formal Concept Analysis (FCA) [1] is a mathematical theory to organize and analyze the information collected in a dataset, by means of the mathematical structure called concept lattice. Since its introduction [1], several mechanisms for variable selection have been intensively studied. One of the most researched lines deals with the reduction of the number of attributes, detecting the unnecessary ones and preserving the most important information of the considered formal context [2–11].

In [12,13], the authors proved that any attribute reduction of a formal context induces an equivalence relation on the set of concepts of the concept lattice. Moreover, the equivalence classes of the induced equivalence relation have the structure of a join-semilattice. This fact gave rise to the introduction of the definition of a new equivalence relation whose main goal was to lightly modify the equivalence relation induced by the attribute reduction, in order to provide new more robust equivalence classes grounded on the structure of convex sublattices. This new equivalence relation was called local congruence and the application to attribute reduction in FCA was initiated in [14,15].

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In addition,[16] studied that when the induced equivalence relation provided by an attribute reduction does not coincide with a local congruence, then the fact of using a local congruence to complement such a reduction had an influence on the original reduction.

In this paper, we continue with the idea highlighted in [16] of studying the impact of local congruences on concept lattices corresponding to formal contexts that have been previously reduced deleting a set of (unnecessary) attributes (attribute reduction). For that, a new partial ordering on the quotient set provided by a local congruence is introduced and analyzed. Moreover, the structures associated with the reduced context and the quotient set of concepts given by an attribute reduction have been compared in order to highlight the narrow relationship between them, which will be fundamental for simplifying and proving the rest of results of the paper.

Since the main transformation that a local congruence makes on the quotient set associated with the attribute reduction is to group different classes, this paper studies how the reduced context (attributes, objects and the relationship) needs to be modified (as less as possible) in order to obtain a complete lattice isomorphic to the one obtained after the grouping given by the local congruence. This study will be split into three different cases depending on the character of the element grouped by the local congruence, specifically, if it is join-irreducible, meet-irreducible or neither join nor meet-irreducible. Furthermore, due to the modification of the original context from the attribute reduction mechanism is well known, we can assert that this paper determines the precise modifications of an attribute reduction mechanism complemented by a local congruence makes on the original context. Hence, this study is fundamental for the traceability and knowledge of the information obtained from the proposed methodology for variable selection from datasets.

The paper is organized as follows: Section 2 recalls some necessary notions and results needed in the development of the contributions of the paper. In Section 3, we carried out the relationship between the reduced concept lattice and the quotient set associated with the attribute reduction. Moreover, it is defined and studied an ordering defined on the quotient set associated with a local congruence. The results obtained in Section 3 have led us to develop the analysis shown in Section 4, which is divided in two parts: the first one devoted to study the impact of eliminating a join-irreducible element of a concept lattice and the second one devoted to analyze the repercussion of eliminating other kind of elements. Section 5 summarizes the conclusions and prospect for future works.

2. Preliminaries

In this section, some preliminary notions and results used in this work will be recalled. In order to make this paper as self-contained as possible, the preliminary section is divided into three parts, the first one will be devoted to recall those necessary notions of FCA, the second one to those related to lattice theory and the last one to local congruences.

2.1. Formal concept analysis

In FCA a context is a triple (A, B, R) where A is a set of attributes, B is a set of objects and $R \subseteq A \times B$ is a relationship, such that $(a, x) \in R$ (also denoted as aRx), if the object $x \in B$ possesses the attribute $a \in A$. In addition, we call *derivation operators* to the mappings ${}^{\uparrow}\colon 2^B \to 2^A$ and ${}^{\downarrow}\colon 2^A \to 2^B$ defined for each $X \subseteq B$ and $Y \subseteq A$ as:

$$X^{\uparrow} = \{a \in A \mid \text{for all } x \in X, aRx\} \quad (1)$$

$$Y^{\downarrow} = \{x \in B \mid \text{for all } a \in Y, aRx\} \quad (2)$$

Taking into account the previous mappings, a *concept* is a pair (X, Y), with $X \subseteq B$ and $Y \subseteq A$ satisfying that $X^{\uparrow} = Y$ and $Y^{\downarrow} = X$. The subset X is called the *extent* of the concept and the subset Y is called the *intent*. The set of extents and intents are denoted by $\mathcal{E}(A, B, R)$ and $\mathcal{I}(A, B, R)$, respectively.

The whole set of concepts is denoted as $\mathcal{C}(A, B, R)$. The inclusion ordering on the left argument, $\leq$, provides $\mathcal{C}(A, B, R)$ with the structure of a complete lattice, which is called *concept lattice* of the context (A, B, R).

Furthermore, we need to recall the notion of meet(join)-irreducible element of a lattice.

Definition 1. Given a lattice $(L, \preceq)$, such that $\wedge$ is the meet operator, and an element $x \in L$ verifying

1. If L has a top element $\top$, then $x \neq \top$.
2. If $x = y \wedge z$, then $x = y$ or $x = z$, for all $y, z \in L$.

we call x *meet-irreducible ($\wedge$-irreducible) element* of L. Condition (2) is equivalent to

2′. If $x < y$ and $x < z$, then $x < y \wedge z$, for all $y, z \in L$.

The *join-irreducible ($\vee$-irreducible) element* of L is defined dually.

In addition, we will say that an *attribute-concept* is a concept generated by an attribute $a \in A$, that is $(a^{\downarrow}, a^{\downarrow\uparrow})$. Dually, an *object-concept* is defined as $(b^{\uparrow\downarrow}, b^{\uparrow})$ for $b \in B$. Moreover, the sets of objects and attributes that generate a concept are defined.

Definition 2. Given a formal context (A, B, R), the associated concept lattice $\mathcal{C}(A, B, R)$ and a concept $C \in \mathcal{C}(A, B, R)$, the set of *objects generating* C is defined as the set:

$$\text{Obg}(C) = \{b \in B \mid (b^{\uparrow\downarrow}, b^{\uparrow}) = C\}$$

Similarly, the set of *attributes generating* C is defined as the set:

$$\text{Atg}(C) = \{a \in A \mid (a^{\downarrow}, a^{\downarrow\uparrow}) = C\}$$

In addition, the sets Obg(C) and Atg(C) are always nonempty sets, for every join-irreducible and meet-irreducible concept C [7], respectively.

Proposition 3. *If C is a join-irreducible concept of $\mathcal{C}(A, B, R)$, then Obg(C) is a nonempty set. Equivalently, if C is a meet-irreducible concept of $\mathcal{C}(A, B, R)$, then Atg(C) is a nonempty set.*

Proof. The result straightforwardly arises from definition.

With respect to the reduction of the context in FCA, from the perspective of the set of objects, the classification of the objects based on the sets given in Definition 2, is given below. This result is dual to the one given for the set of attributes in [7]. For a more detailed information about the notions considered in the this result we refer the reader to [7].

Theorem 4. *Given an object $b \in B$, we have that*

- *b is an absolutely necessary object if and only if there exists a join-irreducible concept C of $(\mathcal{M}, \preceq)$, satisfying that $b \in Obg(C)$ and $card(Obg(C)) = 1$.*
- *b is a relatively necessary object if and only if b is not an absolutely necessary object and there exists a join-irreducible concept C with $b \in Obg(C)$ and $card(Obg(C)) > 1$, satisfying that $\big(B\backslash Obg(C)\big) \cup \{B\}$ is a consistent set.*
- *b is an absolutely unnecessary object if and only if, for any join-irreducible concept C, $b \notin Obg(C)$, or if $b \in Obg(C)$ then $\big(B\backslash Obg(C)\big) \cup \{b\}$ is not a consistent set.*

In addition, it is important to recall that when we reduce the set of attributes in a context, an equivalence relation on the set of concepts of the original concept lattice is induced. The following proposition was proved in [12] for the classical setting of FCA.

Proposition 5 (*[12]*)**.** *Given a context (A, B, R) and a subset $D \subseteq A$. The set $\rho_D = \{((X_1, Y_1), (X_2, Y_2)) \mid (X_1, Y_1), (X_2, Y_2) \in \mathcal{C}(A, B, R), X_1^{\uparrow_D\downarrow} = X_2^{\uparrow_D\downarrow}\}$ is an equivalence relation, where $^{\uparrow_D}$ denotes the concept-forming operator $X^{\uparrow_D} = \{a \in D \mid (a, x) \in R, \text{for all } x \in X\}$ restricted to the subset of attributes $D \subseteq A$.*

Notice that $^{\uparrow_D}$ and $^{\downarrow^D}$, respectively defined as above and as $Y^{\downarrow^D} = \{b \in B \mid (y, b) \in R, \text{for all } y \in Y\}$, for all $Y \subseteq D$, are the concept-forming operators of the reduced concept lattice. Notice that, since the set of object B is not modified, we have that $Y^{\downarrow^D} = Y^{\downarrow}$, for all $Y \subseteq D$.

In [12], the authors also proved that each equivalence class of the induced equivalence relation has a structure of join semilattice and they also characterized the maximum element.

Proposition 6 (*[12]*)**.** *Given a context (A, B, R), a subset $D \subseteq A$ and a class $[(X, Y)]_D$ of the quotient set $\mathcal{C}(A, B, R)/\rho_D$. The class $[(X, Y)]_D$ is a join semilattice with maximum element $(X^{\uparrow_D\downarrow}, X^{\uparrow_D\downarrow\uparrow})$.*

2.2. Lattice theory

In this paper, we will also make use of some well-known notions of lattice theory which are recalled below. The first notion is about the chain conditions on lattices.

Definition 7 (*[17]*)**.** Let $(L, \preceq)$ be a lattice. L is said to satisfy the *ascending chain condition*, denoted as ACC, if given any sequence $x_1 \preceq x_2 \preceq \cdots \preceq x_n \preceq \ldots$ of elements of L, there exists $k \in \mathbb{N}$ such that $x_k = x_{k+1} = \ldots$. The dual of the ascending chain condition is the *descending chain condition*, denoted as DCC.

The following result relates the chain conditions to the completeness of a lattice.

Theorem 8 (*[17]*)**.** *Let $(L, \preceq)$ be a lattice.*

(i) If L satisfies ACC, then for every non-empty subset $A \subseteq L$ there exists a finite subset $F \subseteq A$ such that $\bigvee A = \bigvee F$.
(ii) If L has a bottom element and satisfies ACC, then L is complete.
(iii) If L has no infinite chain, then L is complete.

On the other hand, an ordered set can be embedded in a complete lattice by using the Dedekind–MacNeille completion, which is associated with the Galois connection $(\,^u,\,^l)$, recalled in the following definition.

Definition 9 ([17]). Let $(P, \leq)$ be an ordered set. The *Dedekind–MacNeille completion* of P is defined as

$$\mathrm{DM}(P) = \{A \subseteq P \mid A^{ul} = A\}$$

where the mappings $^u : 2^P \to 2^P$ and $^l : 2^P \to 2^P$ are defined for a subset $A \subseteq P$ as

$$A^u = \{x \in P \mid a \leq x, \text{ for all } a \in A\}$$
$$A^l = \{x \in P \mid x \leq a, \text{ for all } a \in A\}$$

The ordered set $(\mathrm{DM}(P), \subseteq)$ is a complete lattice.

In addition, we can use the Dedekind–MacNeille completion to construct a complete lattice from the join-irreducible and meet-irreducible elements of a complete lattice as the following result states.

Theorem 10 ([17]). *Let $(L, \preceq)$ be a lattice with no infinite chains. Then*

$$L \cong DM(\mathcal{J}(L) \cup \mathcal{M}(L))$$

where $\mathcal{J}(L)$ and $\mathcal{M}(L)$ are the sets of join-irreducible and meet-irreducible elements of L, respectively. Moreover, $\mathcal{J}(L) \cup \mathcal{M}(L)$ is the smallest subset of L which is both join-dense and meet-dense in L.

2.3. Local congruences

The notion of local congruence arose with the goal of complementing attribute reduction in FCA. The purpose of local congruences is to obtain equivalence relations less-constraining than congruences [15] and with useful properties to be applied in size reduction processes of concept lattices. We recall the notion of local congruence next.

Definition 11. Given a lattice $(L, \preceq)$, we say that an equivalence relation δ on L is a *local congruence* if each equivalence class of δ is a convex sublattice of L.

The notion of local congruence in terms of the equivalence relation is given as follows.

Proposition 12 ([15]). *Given a lattice $(L, \preceq)$ and an equivalence relation δ on L, the relation δ is a local congruence on L if and only if, for each $a, b, c \in L$, the following properties hold:*

(i) If $(a, b) \in \delta$ and $a \preceq c \preceq b$, then $(a, c) \in \delta$.
(ii) $(a, b) \in \delta$ if and only if $(a \wedge b, a \vee b) \in \delta$.

Usually, we will look for a local congruence that contains a partition induced by an equivalence relation. When we say that a local congruence contain a partition provided by an equivalence relation, we are making use of the following definition of inclusion ordering of equivalence relations.

Definition 13. Let ρ_1 and ρ_2 be two equivalence relations on a lattice $(L, \preceq)$. We say that the equivalence relation ρ_1 *is included in* ρ_2, denoted as $\rho_1 \sqsubseteq \rho_2$, if for every equivalence class $[x]_{\rho_1} \in L/\rho_1$ there exists an equivalence class $[y]_{\rho_2} \in L/\rho_2$ such that $[x]_{\rho_1} \subseteq [y]_{\rho_2}$.

Once we have recalled the previous notions and results, next section will investigate the impact of local congruences on concept lattices when the considered concept lattices are associated with reduced contexts.

3. Attribute reduction quotient sets and local congruences

This section will begin showing, in an attribute reduction process, the narrow relationship between the quotient set associated with the attribute reduction (Proposition 5) and the reduced concept lattice. Then, an ordering relation will be defined and studied on the quotient set associated with a local congruence, which will be fundamental for the main goal of this paper.

3.1. Attribute reduction: quotient set versus reduced concept lattice

Since the equivalence classes induced by an attribute reduction are join-semilattices with maximum elements, for every equivalence class $[C]_D$, with $C = (X, Y) \in \mathcal{C}(A, B, R)$, the concept $C_M = (X_M, Y_M) = \bigvee_{C_i \in [C]_D} C_i$ necessarily belongs to $[C]_D$. Indeed, by Proposition 6, this maximum element is $(X^{\uparrow_D \downarrow}, X^{\uparrow_D \downarrow \uparrow})$ and so, $X_M = X^{\uparrow_D \downarrow}$, which implies that this extent is also the extent of a concept of the reduced concept lattice $\mathcal{C}(D, B, R_{|D \times B})$. Moreover, the least element of $\mathcal{C}(D, B, R_{|D \times B})$ is $(\varnothing^{\uparrow_D \downarrow}, \varnothing^{\uparrow_D})$, which correspond to the concept $(\varnothing^{\uparrow_D \downarrow}, \varnothing^{\uparrow_D \downarrow \uparrow})$ of the original context. Notice that $\mathcal{C}(D, B, R_{|D \times B})$ is a complete lattice and so, a join closed structure with a least element.

On the other hand, on the whole set of equivalence classes given by the relation ρ_D an ordering can be defined.

Proposition 14. *On the quotient set $\mathcal{C}(A,B,R)/\rho_D$ associated with a context (A,B,R), the relation $\sqsubseteq_D$, defined as $[(X_1,Y_1)]_D \sqsubseteq_D [(X_2,Y_2)]_D$ if ${X_1}^{\uparrow_D\downarrow} \subseteq {X_2}^{\uparrow_D\downarrow}$, for all $[(X_1,Y_1)]_D, [(X_2,Y_2)]_D \in \mathcal{C}(A,B,R)/\rho_D$, is an ordering relation.*

Proof. By Proposition 6, $\sqsubseteq_D$ is well defined. Moreover, from its definition, $\sqsubseteq_D$ is straightforwardly reflexive, antisymmetric and transitive. □

The following result shows the narrow relationship between the previously shown quotient set and the reduced concept lattice, which improves Proposition 3.11 in [12].

Theorem 15. *Given a context (A,B,R) and a subset of attributes $D \subseteq A$, we have that the quotient set given by ρ_D and the reduced concept lattice by D are isomorphic, that is*

$$(\mathcal{C}(A,B,R)/\rho_D, \sqsubseteq_D) \cong (\mathcal{C}(D,B,R_{|D\times B}), \leq_D)$$

where $\leq_D$ is the ordering in the original concept lattice restricted to the reduced one.

Proof. We will define two mappings $\varphi\colon \mathcal{C}(A,B,R)/\rho_D \to \mathcal{C}(D,B,R_{|D\times B})$ and $\psi\colon \mathcal{C}(D,B,R_{|D\times B}) \to \mathcal{C}(A,B,R)/\rho_D$, and we will prove that they preserve the ordering and that $\varphi\circ\psi$ and $\psi\circ\varphi$ are the corresponding identity mappings.

The mapping φ will be defined as $\varphi([(X,Y)]_D) = (X^{\uparrow_D\downarrow}, X^{\uparrow_D})$ for all $[(X,Y)]_D \in \mathcal{C}(A,B,R)/\rho_D$, and ψ is defined as $\psi(X,Y) = [(X,X^{\uparrow})]_D$ for all $(X,Y) \in \mathcal{C}(D,B,R_{|D\times B})$. φ is clearly well defined and ψ is also well defined by Proposition 6.

Given $(X,Y) \in \mathcal{C}(D,B,R_{|D\times B})$, we have that

$$\varphi\circ\psi(X,Y) = \varphi(\psi(X,Y)) = \varphi([(X,X^{\uparrow})]_D) = (X^{\uparrow_D\downarrow}, X^{\uparrow_D}) = (X,Y)$$

where the last equality holds because (X,Y) is a concept of the reduced concept lattice and so, it satisfies $X^{\uparrow_D\downarrow} = X^{\uparrow_D\downarrow^D} = X$.

In the other composition, we consider $[(X,Y)]_D \in \mathcal{C}(A,B,R)/\rho_D$ and we obtain

$$\psi\circ\varphi([(X,Y)]_D) = \psi(\varphi([(X,Y)]_D)) = \psi(X^{\uparrow_D\downarrow}, X^{\uparrow_D}) = [(X^{\uparrow_D\downarrow}, X^{\uparrow_D\downarrow\uparrow})]_D$$

and the last equivalence class is exactly equal to $[(X,Y)]_D$, by Proposition 6.

Finally, we will prove that φ and ψ are order-preserving. Given $[(X_1,Y_1)]_D, [(X_2,Y_2)]_D \in \mathcal{C}(A,B,R)/\rho_D$, such that $[(X_1,Y_1)]_D \sqsubseteq_D [(X_2,Y_2)]_D$, we have that ${X_1}^{\uparrow_D\downarrow} \subseteq {X_2}^{\uparrow_D\downarrow}$ by definition of $\sqsubseteq_D$ and consequently we have that

$$\varphi([(X_1,Y_1)]_D) = ({X_1}^{\uparrow_D\downarrow}, {X_1}^{\uparrow_D}) \leq_D ({X_2}^{\uparrow_D\downarrow}, {X_2}^{\uparrow_D}) = \varphi([(X_2,Y_2)]_D)$$

Now, given $(X_1,Y_1), (X_2,Y_2) \in \mathcal{C}(D,B,R_{|D\times B})$, such that $(X_1,Y_1) \leq_D (X_2,Y_2)$, we obtain that

$$\psi(X_1,Y_1) = [(X_1,{X_1}^{\uparrow})]_D \overset{(*)}{\sqsubseteq_D} [(X_2,{X_2}^{\uparrow})]_D = \psi(X_2,Y_2)$$

where $(*)$ holds because ${X_1}^{\uparrow_D\downarrow} = X_1^{\uparrow_D\downarrow^D} = X_1 \subseteq X_2 = X_2^{\uparrow_D\downarrow^D} = {X_2}^{\uparrow_D\downarrow}$. □

As a consequence, $(\mathcal{C}(A,B,R)/\rho_D, \sqsubseteq_D)$ is a complete lattice, which will be taken into account in the relationship with the local congruences.

3.2. The poset associated with a local congruence

Now, we define a new relationship on the elements of the quotient set provided by a local congruence, which turns out to be a partial order as we will prove in the following result.

Theorem 16. *Given a complete lattice $(L,\preceq)$ and a local congruence δ on L, the binary relation defined as follows:*

$$[x]_\delta \leq_\delta [y]_\delta \quad \text{if} \quad \bot_L \in [x]_\delta, \quad \text{or} \quad x_M \preceq y'$$

where $y' \in [y]_\delta$, $x_M = \bigvee_{x_i\in[x]_\delta} x_i$ and $\bot_L$ is the bottom of $(L,\preceq)$, is a partial order for L/δ.

Proof. We consider a lattice $(L,\preceq)$, a local congruence δ on L and the relation $\leq_\delta$ defined above. We are going to prove that the relation $\leq_\delta$ is a reflexive, antisymmetric and transitive relation. It is clear that it is reflexive. In order to prove the antisymmetry property, let us consider two classes $[x]_\delta, [y]_\delta \in L/\delta$ satisfying that $[x]_\delta \leq_\delta [y]_\delta$ and $[y]_\delta \leq_\delta [x]_\delta$, then we have to distinguish the following cases:

1. If $\bot_L \in [x]_\delta$ and $\bot_L \in [y]_\delta$, then $[x]_\delta = [y]_\delta$ since the equivalence classes are disjoint.
2. If $\bot_L \in [x]_\delta$ and $\bot_L \notin [y]_\delta$, then since $[x]_\delta \leq_\delta [y]_\delta$, we have that there exists $y' \in [y]_\delta$, such that $\bot_L \preceq y'$. On the other hand, since $[y]_\delta \leq_\delta [x]_\delta$, there exists $x' \in [x]_\delta$, such that $y_M \preceq x'$. Therefore,

$$\bot_L \preceq y' \preceq y_M \preceq x'$$

which implies that $[x]_\delta = [y]_\delta$, by the convexity of the equivalence classes provided by local congruences.

Table 1
Relation of Example 18.

R	b_1	b_2	b_3	b_4	b_5	b_6	b_7
a_1	1	1	1	1	1	0	0
a_2	1	1	0	1	0	0	0
a_3	1	1	1	0	1	0	0
a_4	0	1	1	0	1	1	0
a_5	0	1	1	0	0	1	1
a_6	1	0	0	0	0	0	0
a_7	0	0	1	0	0	0	0

3. If $\bot_L \notin [x]_\delta$ and $\bot_L \notin [y]_\delta$, then since $[x]_\delta \leq_\delta [y]_\delta$, we have that there exists $y' \in [y]_\delta$, such that $x_M \preceq y'$. On the other hand, since $[y]_\delta \leq_\delta [x]_\delta$, there exists $x' \in [x]_\delta$, such that $y_M \preceq x'$. Consequently,

$$x_M \preceq y' \preceq y_M \preceq x'$$

and, by the convexity of the classes, we can conclude that $[x]_\delta = [y]_\delta$.

Now, we consider three different classes $[x]_\delta, [y]_\delta, [z]_\delta \in L/\delta$ such that $[x]_\delta \leq_\delta [y]_\delta$ and $[y]_\delta \leq_\delta [z]_\delta$ in order to prove the transitivity property. We only have to distinguish the following two cases:

1. If $\bot_L \in [x]_\delta$, $\bot_L \notin [y]_\delta$ and $\bot_L \notin [z]_\delta$, then we have straightforwardly that $\bot_L \preceq z'$ for any $z' \in [z]_\delta$ by definition of infimum of the lattice. Therefore, $[x]_\delta \leq_\delta [z]_\delta$.
2. If $\bot_L \notin [x]_\delta$, $\bot_L \notin [y]_\delta$ and $\bot_L \notin [z]_\delta$, then since $[x]_\delta \leq_\delta [y]_\delta$ and $[y]_\delta \leq_\delta [z]_\delta$, we have that there exist $y' \in [y]_\delta$ and $z' \in [z]_\delta$ such that $x_M \preceq y'$ and $y_M \preceq z'$. Therefore,

$$x_M \preceq y' \preceq y_M \preceq z',$$

which implies that $x_M \preceq z'$, i.e. $[x]_\delta \leq [z]_\delta$.

Hence, we can conclude that the binary relation $\leq_\delta$ is a partial order. □

The condition $x_M \preceq y'$ given in Theorem 16 can clearly be more concrete, i.e., since it is satisfied that $y' \preceq y_M$ for all $y' \in [y]_\delta$, we have that $x_M \preceq y_M$ in particular, is satisfied. This fact is stated in the following result.

Corollary 17. *Given a complete lattice $(L, \preceq)$ and a local congruence δ on L, the partial order defined in Theorem 16 is equivalent to the following:*

$$[x]_\delta \leq_\delta [y]_\delta \quad \text{if} \quad \bot_L \in [x]_\delta \quad \text{or} \quad x_M \preceq y_M$$

where $y_M = \bigvee_{y_i \in [y]_\delta} y_i$.

Notice that the previous ordering is associated with the join-sublattice structure given by an attribute reduction (Proposition 6). Hence, since we could also include a bottom element, we could think that this ordering can provide a complete lattice. However, when we define the quotient set of an arbitrary local congruence δ on a lattice L with the partial order $\leq_\delta$, we do not always obtain that L/δ is a lattice.

Example 18. Let us consider a context (A, B, R) whose relation is given in Table 1. We also consider a local congruence δ on the associated concept lattice $\mathcal{C}(A, B, R)$ which is shown on the left side of Fig. 1.

If we define the quotient set $\mathcal{C}(A, B, R)/\delta$ with the partial order defined in Theorem 16, we obtain that $(\mathcal{C}(A, B, R)/\delta, \leq_\delta)$ is not a lattice but a poset because the infimum of each pair of elements of $(\mathcal{C}(A, B, R)/\delta, \leq_\delta)$ does not exist, for instance, the lower bounds of the concepts C_{12} and C_{13} are the concepts C_3, C_4 and C_1 whence C_3 and C_4 are minimal, as it can be observed in the right side of Fig. 1. Hence, the infimum of C_{12} and C_{13} does not exist. □

Therefore, $(\mathcal{C}(D, B, R)/\delta, \leq_\delta)$ is not a lattice in general. Moreover, local congruences can merge different equivalent classes, which can have a remarkable impact in the reduced context with a repercussion in the reduced concept lattice. Specifically, since δ is a local congruence whose equivalence classes contain the classes of $(\mathcal{C}(A, B, R)/\rho_D, \sqsubseteq_D)$, given $[C]_\delta \in \mathcal{C}(A, B, R)/\delta$, there exists an index set Λ_C, such that

$$[C]_\delta = \bigcup\{[C_\lambda]_D \mid [C_\lambda]_D \in \mathcal{C}(A, B, R)/\rho_D, \lambda \in \Lambda_C\}$$

where $\{[C_\lambda]_D \mid [C_\lambda]_D \in \mathcal{C}(A, B, R)/\rho_D, \lambda \in \Lambda_C\}$ is a convex sublattice of $(\mathcal{C}(A, B, R)/\rho_D, \sqsubseteq_D)$, in which $[C_M]_D$ is the greatest class and $[C_m]_D$, with $C_m = \bigwedge_{C_i \in [C]_D} C_i$ is the least one. Hence, it is possible that some class $[C_\lambda]_D \in \mathcal{C}(A, B, R)/\rho_D$ with $\lambda \in \Lambda_C$, be a join-irreducible element. Hence, the local congruence is grouping a join-irreducible class into another class. This fact can be reflected into the reduced context avoiding that the concept $[C_\lambda]_D$ appears in the reduced concept lattice. Since $[C_\lambda]_D$ is join-irreducible, $\mathrm{Obg}([C_\lambda]_D) \neq \varnothing$ and this set must be removed from B in order to avert the computation of this join-irreducible element. As a consequence of this deletion, other concepts can also be removed as a collateral effect, as the following example shows.

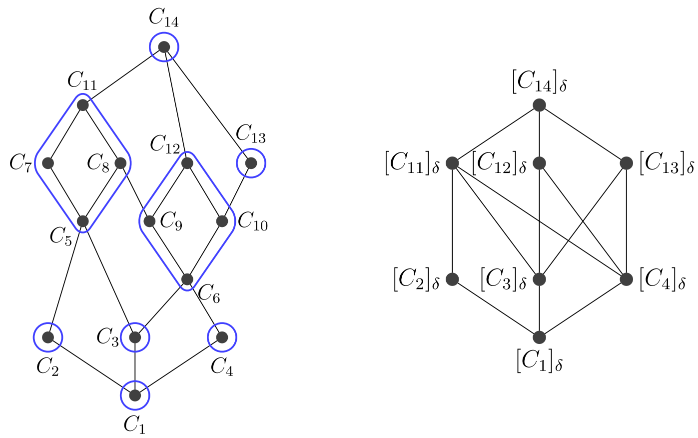

Fig. 1. The local congruence δ on $\mathcal{C}(A, B, R)$ of Example 18 (left) and its quotient set $(\mathcal{C}(A, B, R)/\delta, \leq_\delta)$ (right).

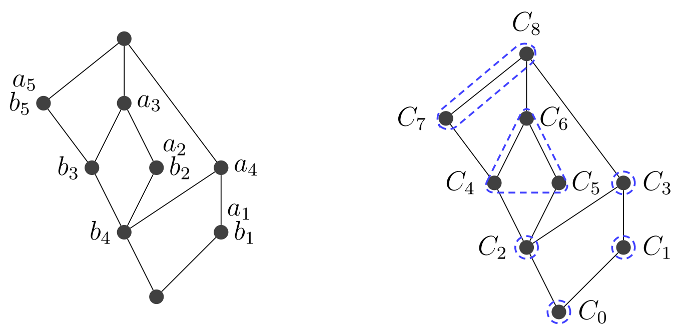

Fig. 2. The associated concept lattice $\mathcal{C}(A, B, R)$ (left) and the induced partition ρ_D (right) of Example 19.

Table 2
Relation of Example 19.

R	b_1	b_2	b_3	b_4	b_5
a_1	1	0	0	0	0
a_2	0	1	0	1	0
a_3	0	1	1	1	0
a_4	1	0	0	1	0
a_5	0	0	1	1	1

Example 19. We consider a context (A, B, R) whose relation is given in Table 2 and a subset of attribute $D = \{a_1, a_3, a_4\}$ from which we reduce the original context, that is, we remove the attributes a_2 and a_5. The associated concept lattice $\mathcal{C}(A, B, R)$ and the induced partition by the attribute reduction, ρ_D, are shown in Fig. 2, left and right sides respectively.

Now, we consider the least local congruence δ on the associated concept lattice $\mathcal{C}(A, B, R)$ containing the partition induced by the attribute reduction, which is illustrated on the left side of Fig. 3. Note that the class $[C_6]_\delta$ contains the class $[C_6]_D$, which is not a convex sublattice of the original concept lattice and it also contains the join-irreducible class $[C_2]_D$, as it is shown on the right side of Fig. 3, due to the equivalence class of δ has to be necessarily a convex sublattice.

Therefore, in order to avoid the computation of this concept, the reduced context must be modified removing $\mathrm{Obg}([C_2]_D) = \{b_4\}$ from B. However, this modification also affects to another concept. Specifically, since $[C_3]_D = [C_1]_D \vee [C_2]_D$ and C_3 is not generated by any object, this concept also disappears. Thus, grouping/removing join-irreducible elements by a local congruence can also have some impact in other concepts.

On the right side of Fig. 4, the quotient set of the least local congruence δ containing the equivalence relation associated with the reduction given by the subset $D \subseteq A$ is depicted, and the modified context $\mathcal{C}(D, B \backslash \{b_4\}, R_{|D \times B \backslash \{b_4\}})$ is shown on the left side. It is clear that both lattices are not isomorphic:

$$\mathcal{C}(D, B, R_{|D\times B}) \backslash \{[C_2]_D\} \ncong \mathcal{C}(D, B\backslash\{b_4\}, R_{|D\times B\backslash\{b_4\}}) \quad \square$$

As we have shown previously, the grouping of concepts by a local congruence can be seen as an elimination of concepts of the concept lattice, which has an impact on the context. The following section focuses on this issue.

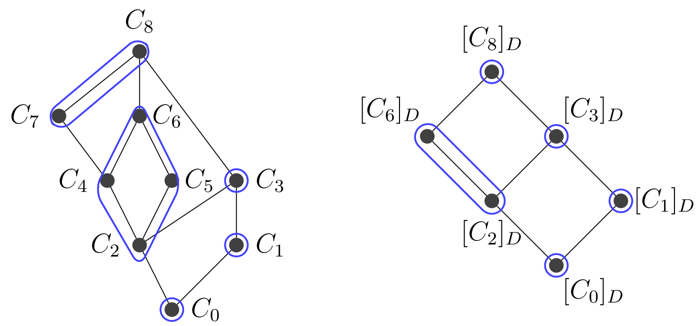

Fig. 3. The local congruence δ on $\mathcal{C}(A, B, R)$ of Example 19 (left) and its effect on $\mathcal{C}(A, B, R)/\rho_D$ (right).

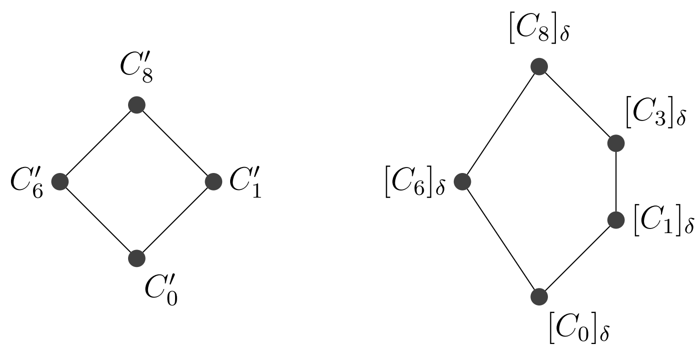

Fig. 4. The concept lattice $(\mathcal{C}(D, B\backslash\{b_4\}, R_{|D\times B\backslash\{b_4\}}), \leq_D)$ (left) and the quotient set $(\mathcal{C}(D, B, R)/\delta, \leq_\delta)$ (right) of Example 19.

4. Impact of removing elements in a concept lattices

In this section, we focus on the impact of local congruences that complement attribute reductions in formal contexts. Specifically, on the least local congruence containing the equivalence relation (partition) induced by the reduction of a context (A, B, R) by a subset $D \subseteq A$. Hence, we are interested in analyzing how the application of this local congruence disrupts the concept lattice associated with the reduced context $(D, B, R_{|D\times B})$.

Therefore, this section will study the necessary modifications to be done in a context, if some elements of the concept lattice are removed. For that purpose, we will distinguish two different situations described in the following two sections, depending on whether the removed element is a join-irreducible element or not.

Due to local congruences group equivalence classes of the complete lattice $(\mathcal{C}(A, B, R)/\rho_D, \sqsubseteq_D)$ and, by Theorem 15, they can be seen as elements in $(\mathcal{C}(D, B, R_{|D\times B}), \leq_D)$, this section will be focused on the reduced context. Moreover, in order to simplify the notation, we will simply write (A, B, R) instead of the reduced context.

4.1. Removing join-irreducible elements

First of all, we will study the necessary modifications in a context when join-irreducible elements need to be removed preserving the rest of concepts, included the ones "generated" by the removed join-irreducible element, e.g. preserving the concept whose join-decomposition contains the removed join-irreducible concept.

From now on, we will assume that the lattice $(\mathcal{C}(A, B, R), \leq)$ satisfies the ACC. As we commented above, the subtraction of a join-irreducible concept C, removing the objects in $\mathrm{Obg}(C)$, can also delete other concepts depending on it. However, the only subtraction of the join-irreducible element does not alter the structure of complete lattice.

Lemma 20. *Given a complete lattice $(L, \wedge, \vee)$ satisfying the ACC, and $p \in L$ a join-irreducible element. Then $(L\backslash\{p\}, \wedge_p, \vee_p)$ is a sublattice of $(L, \wedge, \vee)$, where $\wedge_p, \vee_p$ are the restriction of $\wedge, \vee$ to $L\backslash\{p\}$, and in particular a complete lattice.*

Proof. Given $X \subseteq L\backslash\{p\}$, in particular, we have that $X \subseteq L$. Therefore, $\bigvee X$ exists in L. If $\bigvee X \neq p$, then the supremum also exists in $L\backslash\{p\}$. Otherwise, if $\bigvee X = p$, by hypothesis and Theorem 8, there exists a finite subset $F \subseteq X$ such that $p = \bigvee X = \bigvee F$, which contradicts that p is a join-irreducible element due to $p \notin F$. Therefore, $(L\backslash\{p\}, \wedge_p, \vee_p)$ is a join-structure. Since the bottom of the lattice $(L, \wedge, \vee)$ is not a join-irreducible element, then it also belongs to $L\backslash\{p\}$. Thus, $(L\backslash\{p\}, \wedge_p, \vee_p)$ is a sublattice of $(L, \wedge, \vee)$. $\square$

Although the structure is preserved in a general lattice, in a concept lattice, removing a join-irreducible element implies the elimination of some objects of the context, which can have some impact in other concepts, as Example 19 shown. Now, we introduce a procedure to modify the original context in order to obtain a new concept lattice isomorphic to the complete lattice obtained after removing a join-irreducible element.

Given a concept $C_k \in \mathcal{C}(A, B, R)$, different from the top element, the set

$$\mathcal{T}(C_k) = \{C_i \mid C_i \in \mathcal{C}(A, B, R), C_k < C_i\}$$

is not empty, since it contains the top element $(B, B^{\uparrow})$, and there exists its infimum, because $(\mathcal{C}(A, B, R), \leq)$ is a complete lattice. This infimum concept will be denoted as C_k^t and the minimal elements of $\mathcal{T}(C_k)\backslash\{C_k^t\}$ as $C_k^{m_i}$, with i in an index set Γ.

Now, we analyze the associated context to the concept lattice obtained after removing a join-irreducible concept $C_j \in \mathcal{C}(A, B, R)$. In addition, we will denote the set of all join-decompositions of a concept $C \in \mathcal{C}(A, B, R)$ by $\mathfrak{J}(C)$.

1. If $C_j^t \neq C_j$ and
   (a) there exists a join-decomposition of C_j^t, which does not contain to C_j, then the elements $\text{Obg}(C_j)$ are removed from B, obtaining a new set of objects $B^* = B\backslash\text{Obg}(C_j)$.
   (b) all join-decompositions of C_j^t contain C_j. In this case, we consider again two cases:
      i. if $\text{Obg}(C_j^t) \neq \varnothing$, then the elements $\text{Obg}(C_j)$ are removed from B, obtaining a new object $B^* = B\backslash\text{Obg}(C_j)$.
      ii. if $\text{Obg}(C_j^t) = \varnothing$, we change the elements $\text{Obg}(C_j)$ in B by a new one b^*, that is, we remove the elements $\text{Obg}(C_j)$ from B, we define a new set of objects $B' = B\backslash\text{Obg}(C_j)$ and we add the new b^* to B', $B^* = \{b^*\} \cup (B\backslash\text{Obg}(C_j))$. Moreover, a new relation $R^* \subseteq A \times B^*$ is considered, which is defined as follows:

$$R^* = R_{|A\times B'} \cup \{(a, b^*) \mid a \in \mathcal{I}(C_j^t)\}$$

2. if $C_j^t = C_j$, we consider the minimal elements $C_j^{m_i}$, with $i \in \Gamma$. Clearly, $C_j^{m_i} \neq C_j$, for all $i \in \Gamma$, and we apply Step 1 to all these minimal concepts. Specifically, we consider the subset $\Gamma' \subseteq \Gamma$, defined as

$$\Gamma' = \{i \in \Gamma \mid \text{all join-decompositions of } C_j^{m_i} \text{ contain } C_j \text{ and } \text{Obg}(C_j^{m_i}) = \varnothing\}$$

   (a) if $\Gamma' = \varnothing$, then the elements $\text{Obg}(C_j)$ are removed from B, obtaining a new set of objects $B^* = B\backslash\text{Obg}(C_j)$.
   (b) otherwise, $\Gamma' \neq \varnothing$, and we define the new set of objects as $B^* = \{b_i^* \mid i \in \Gamma'\} \cup B'$, where $B' = B\backslash\text{Obg}(C_j)$. Moreover, a new relation $R^* \subseteq A \times B^*$ is considered, which is defined as follows:

$$R^* = R_{|A\times B'} \cup \{(a, b_i^*) \mid a \in \mathcal{I}(C_j^{m_i}), i \in \Gamma'\}$$

These steps are translated into Algorithm 1 in order to present a more simple an operational procedure.

Notice that the first step and Step 2a are merged in Line 3. Since C_j is join-irreducible, if $\text{Obg}(C_j)$ has only one element, it is an absolutely necessary object[1] and if $\text{Obg}(C_j)$ has more than one element, they are relatively necessary objects. Hence, all of them need to be removed in order to ensure that the concept C_j does not appear.

The mechanism above computes the required context, as the following result proves.

Theorem 21. *Given a context (A, B, R) and a join-irreducible concept $C_j \in \mathcal{C}(A, B, R)$, we have that*

$$(\mathcal{C}(A, B, R)\backslash\{C_j\}, \leq_j) \cong (\mathcal{C}(A, B^*, R^*), \leq^*),$$

where B^* and R^* are the set and the relation given by Algorithm 1, and $\leq_j$ is the ordering defined from the restriction of $\leq$ to $\mathcal{C}(A, B, R)\backslash\{C_j\}$.

Proof. The proof straightforwardly holds, if the intents of both set of concepts coincide. First of all, we will prove that the set of intents of concepts of $\mathcal{C}(A, B^*, R^*)$ is contained in the sets of intents of concepts of $\mathcal{C}(A, B, R)\backslash\{C_j\}$. For that purpose, let us consider an arbitrary concept $(X, Y) \in \mathcal{C}(A, B^*, R^*)$ and we will prove that Y is also the intent of a concept of $\mathcal{C}(A, B, R)\backslash\{C_j\}$. We differentiate the derivation operators of each context, that is, for the context (A, B^*, R^*) we will denote the derivation operators as $({}^{\uparrow_*}, {}^{\downarrow^*})$ and for (A, B, R) as $({}^{\uparrow}, {}^{\downarrow})$.

According to Algorithm 1, several cases should be distinguished:

1. If in the construction of (A, B^*, R^*) the condition given in Line 3 has been satisfied, then $B^* \subseteq B$ and, since the set of attributes is the same in both contexts, we have that $X^{\uparrow_*} = X^{\uparrow}$. Therefore, the following chain of inequalities holds:

$$Y^{\downarrow\uparrow} = X^{\uparrow_*\downarrow\uparrow} = X^{\uparrow\downarrow\uparrow} = X^{\uparrow} = X^{\uparrow_*} = Y$$

[1] We are considering dual notions of attribute reduction [18].

Algorithm 1: Removing a join-irreducible concept from a concept lattice

input : (A, B, R), C_j, $\mathcal{T}(C_j)$, Γ
output: (A, B^*, R^*)

1 Compute the infimum concept C_j^t of the set $\mathcal{T}(C_j)$;
2 **if** $C_j^t \neq C_j$, **then**
3   **if** *there exists* $\chi \in \mathfrak{J}(C_j^t)$ *such that* $C_j \notin \chi$ **or** $Obg(C_j^t) \neq \varnothing$ **then**
4     $B^* = B \backslash \mathrm{Obg}(C_j)$;
5     $R^* = R_{|A\times B^*}$;
6   **else**
7     $B' = B \backslash \mathrm{Obg}(C_j)$;
8     $B^* = \{b^*\} \cup B'$;
9     $R^* = R_{|A\times B'} \cup \{(a, b^*) \mid a \in \mathcal{I}(C_j^t)\}$;
10 **else**
11   Compute the minimal elements $C_j^{m_i}$ with $i \in \Gamma$;
12   Define the subset $\Gamma' \subseteq \Gamma$ as
13   $\Gamma' = \{i \in \Gamma \mid C_j \in \chi$ for all $\chi \in \mathfrak{J}(C_j^{m_i})$ and $\mathrm{Obg}(C_j^{m_i}) = \varnothing\}$;
14   **if** $\Gamma' = \varnothing$ **then**
15     $B^* = B \backslash \mathrm{Obg}(C_j)$;
16     $R^* = R_{|A\times B^*}$;
17   **else**
18     $B' = B \backslash \mathrm{Obg}(C_j)$;
19     $B^* = \{b_i^* \mid i \in \Gamma'\} \cup B'$;
20     $R^* = R_{|A\times B'} \cup \{(a, b_i^*) \mid a \in \mathcal{I}(C_j^t),\ i \in \Gamma'\}$;
21 **return** (A, B^*, R^*)

2. If in the construction of (A, B^*, R^*) the condition given in Line 6 has been satisfied, then $B^* = B' \cup \{b^*\}$ where $B' \subseteq B$. In this case, we have to differentiate three cases:

   2.1. If $Y \subseteq b^{*\uparrow_*}$, then we have that $b^* \in b^{*\uparrow_*\downarrow_*} \subseteq Y^{\downarrow_*} = X$ and so, the extent of the concept can be written as $X = X_0 \cup \{b^*\}$ where $X_0 \subseteq B'$ and, since the set of attributes is the same in both context, we have that $X_0^{\uparrow_*} = X_0^{\uparrow}$. Thus, $Y = X^{\uparrow_*} = (X_0 \cup \{b^*\})^{\uparrow_*} = X_0^{\uparrow_*} \cap b^{*\uparrow_*} = X_0^{\uparrow} \cap \mathcal{I}(C_j^t)$ by construction in Algorithm 1 and therefore,

   $$Y^{\downarrow\uparrow} = X^{\uparrow_*\downarrow\uparrow} = (X_0^{\uparrow} \cap \mathcal{I}(C_j^t))^{\downarrow\uparrow} \overset{(*)}{=} (X_0 \cup \mathcal{E}(C_j^t))^{\uparrow\downarrow\uparrow} = (X_0 \cup \mathcal{E}(C_j^t))^{\uparrow}$$

   The equality $(*)$ holds because C_j^t is a concept of (A, B, R) and therefore,

   $$(X_0 \cup \mathcal{E}(C_j^t))^{\uparrow} = X_0^{\uparrow} \cap \mathcal{I}(C_j^t) = X_0^{\uparrow_*} \cap b^{*\uparrow_*} = X^{\uparrow_*} = Y$$

   2.2. If $Y \nsubseteq b^{*\uparrow_*}$, then $X \subseteq B' \subseteq B$. Thus, the demonstration is analogous to the one given in the first case.

3. If in the construction of (A, B^*, R^*) the condition given in Line 14 has been satisfied, the proof follows an analogous reasoning to the one given in the first case.
4. If in the construction of (A, B^*, R^*) the condition given in Line 17 has been satisfied, the proof is analogous to one given in the second case, but considering $\{b_i^*\}$ instead of $\{b^*\}$.

Now, we will prove that the intents of concepts of $\mathcal{C}(A, B, R)\backslash\{C_j\}$ are also intents of concepts of $\mathcal{C}(A, B^*, R^*)$. For that purpose, let us consider a concept $(X, Y) \in \mathcal{C}(A, B, R)\backslash\{C_j\}$. Depending on the removed concept C_j, we can distinguish several cases considered in Algorithm 1:

1. If the condition given in Line 3 is satisfied, then $B^* \subseteq B$ and, since the set of attributes is the same in both context, we have that $X^{\uparrow_*} = X^{\uparrow}$. Therefore,

   $$Y^{\downarrow^*\uparrow_*} = X^{\uparrow\downarrow^*\uparrow_*} = X^{\uparrow_*\downarrow^*\uparrow_*} = X^{\uparrow_*} = X^{\uparrow} = Y$$

2. If the condition given in Line 6 is satisfied, then $B^* = B' \cup \{b^*\}$ where $B' \subseteq B$. In this case, we have to differentiate three cases:

2.1. If $Y \subseteq \mathcal{I}(C_j^t)$, then we can see the extent as $X = X_0 \cup \mathcal{E}(C_j^t)$. Thus, $Y = X^{\uparrow} = X_0^{\uparrow} \cap \mathcal{E}(C_j^t)^{\uparrow} = X_0^{\uparrow} \cap \mathcal{I}(C_j^t) = X_0^{\uparrow *} \cap \mathcal{I}(C_j^t)$ since $X_0 \subseteq B' \subseteq B^*$ and the set of attributes is the same in both context, we have that $X_0^{\uparrow} = X_0^{\uparrow *}$. Therefore,

$$Y^{\downarrow^* \uparrow_*} = X^{\uparrow\downarrow^*\uparrow_*} = (X_0^{\uparrow *} \cap \mathcal{I}(C_j^t))^{\downarrow^*\uparrow_*} \overset{(*)}{=} (X_0^{\uparrow *} \cap b^{*\uparrow_*})^{\downarrow^*\uparrow_*} = (X_0 \cup b^*)^{\uparrow_*\downarrow^*\uparrow_*}$$

The equality $(*)$ holds because $\mathcal{I}(C_j^t) = b^{*\uparrow_*}$ by construction in Algorithm 1. Therefore,

$$(X_0 \cup b^*)^{\uparrow_*\downarrow^*\uparrow_*} = (X_0 \cup b^*)^{\uparrow_*} = X_0^{\uparrow *} \cap b^{*\uparrow_*} = X_0^{\uparrow} \cap \mathcal{I}(C_j^t) = X^{\uparrow} = Y$$

2.2. If $Y \not\subseteq \mathcal{I}(C_j^t)$, then $X \subseteq B' \subseteq B$. Thus, the proof of this case is similar to the first case.

3. If the condition given in Line 14 is satisfied. The proof of this case is analogous to the one given in the first case.
4. If the condition given in Line 17 is satisfied. The proof of this case is analogous to the one given in the second case, but considering $\{b_i^*\}$ instead of $\{b^*\}$.

Therefore, we have proved that the intents are preserved and so, both lattices are isomorphic. □

This procedure can be applied sequentially, when more than one join-irreducible element need to be removed.

Moreover, by the notions of attribute and object reduction, the number of modified objects is the minimum one for ensuring the isomorphism in Theorem 21. Notice that all objects in $\mathrm{Obg}(C_j)$ have been removed in order to erase C_j and the minimum number of objects (only one per concept) have been introduced to preserve the concepts depending (in a join-decomposition) on C_j (see Example 19). Thus, the proposed mechanism, to characterize the impact of removing a join-irreducible concept from the concept lattice in the context, provides the closest context to the original one.

4.2. Removing non-join-irreducible elements

Moreover, we also need to inspect the possibility of removing a non-join-irreducible element, since the class $[C]_\delta$ can also includes this kind of elements. This section will be devoted to this issue.

In a general lattice $(L, \wedge, \vee)$, given a non-join-irreducible element $y \in L$, it can be meet-irreducible or not. Clearly, in the former case a dual result to Lemma 20 arises.

Lemma 22. *Given a complete lattice $(L, \wedge, \vee)$ satisfying the DCC, and $q \in L$ a meet-irreducible element. Then $(L\backslash\{q\}, \wedge_q, \vee_q)$ is a sublattice of $(L, \wedge, \vee)$, where $\wedge_q, \vee_q$ are the restriction of $\wedge, \vee$ to $L\backslash\{q\}$, and in particular a complete lattice.*

Proof. The proof is dual to the one given in Lemma 20. □

Therefore, if a meet-irreducible concept is removed, a dual procedure to Algorithm 1 can be done removing attributes instead of objects, which is detailed in Algorithm 2. Notice that the set $\mathfrak{M}(C_k)$ is the set of all meet-decompositions of a concept $C_k \in \mathcal{C}(A, B, R)$ and the set $\mathcal{S}(C_k)$ is the dual of the set $\mathcal{T}(C_k)$, that is, $\mathcal{S}(C_k) = \{C_i \mid C_i \in \mathcal{C}(A, B, R), C_i < C_k\}$. Also a dual result to Theorem 21 arises when a meet-irreducible element is removed from a general context.

Theorem 23. *Given a context (A, B, R) and a meet-irreducible concept $C_k \in \mathcal{C}(A, B, R)$, we have that*

$$(\mathcal{C}(A, B, R)\backslash\{C_k\}, \leq_k) \cong (\mathcal{C}(A^*, B, R^*), \leq^*),$$

where A^* and R^* is the set and the relation given by Algorithm 2, and $\leq_k$ is the ordering defined from the restriction of $\leq$ to $\mathcal{C}(A, B, R)\backslash\{C_k\}$.

Proof. The proof is dual to the one given to Theorem 21. □

Now, we will consider the case when the element to be removed is neither meet-irreducible nor join-irreducible. The following example shows that the structure of complete lattice does not hold.

Example 24. We consider a context (A, B, R) whose relation is given in Table 3 and a subset of attributes $D = \{a_1, a_2, a_3, a_4\}$, that is, we remove the attributes a_5 and a_6. The concept lattice $\mathcal{C}(A, B, R)$ and the induced partition by the attribute reduction, ρ_D, are shown in Fig. 5, left and right sides respectively.

Now, we consider the least local congruence δ on the concept lattice $\mathcal{C}(A, B, R)$ containing the partition induced by the attribute reduction, which is illustrated in the left side of Fig. 6. Note that the class $[C_8]_\delta$ contains the class $[C_8]_D$, which is not a convex sublattice of the original concept lattice and so, it also contains the class $[C_3]_D$, as it is shown in the middle of Fig. 6, due to the equivalence class of δ has to be necessarily a convex sublattice.

However, the class $[C_3]_D$ is neither a meet-irreducible nor a join-irreducible element of $\mathcal{C}(A, B, R)/\rho_D$ and if we merge it with $[C_8]_D$ as the local congruence δ does, then the lattice structure is broken. As a consequence, the quotient set $(\mathcal{C}(D, B, R)/\delta, \leq_\delta)$ is not a lattice but only a poset, as it is shown on the right side of Fig. 6. □

Algorithm 2: Removing a meet-irreducible concept of a concept lattice

input : (A, B, R), C_l, $\mathcal{S}(C_l)$, Γ
output: (A^*, B, R^*)

Compute the supremum concept C_l^s of the set $\mathcal{S}(C_l)$;
if $C_l^s \neq C_l$, **then**
  if *there exists* $\psi \in \mathfrak{M}(C_l^s)$ *such that* $C_l \notin \psi$ **or** $Atg(C_l^s) \neq \varnothing$ **then**
    $A^* = A \backslash \mathrm{Atg}(C_l)$;
    $R^* = R_{|A^* \times B}$;
  else
    $A' = A \backslash \mathrm{Atg}(C_l)$;
    $A^* = \{a^*\} \cup A'$;
    $R^* = R_{|A' \times B} \cup \{(a^*, b) \mid b \in \mathcal{E}(C_l^s)\}$;
else
  Compute the maximal elements $C_l^{s_i}$ with $i \in \Gamma$;
  Define the subset $\Gamma' \subseteq \Gamma$ as
  $\Gamma' = \{i \in \Gamma \mid C_l \in \psi$ for all $\psi \in \mathfrak{M}(C_l^{s_i})$ and $\mathrm{Atg}(C_l^{s_i}) = \varnothing\}$;
  if $\Gamma' = \varnothing$ **then**
    $A^* = A \backslash \mathrm{Atg}(C_l)$;
    $R^* = R_{|A^* \times B}$;
  else
    $A' = A \backslash \mathrm{Atg}(C_l)$;
    $A^* = \{a_i^* \mid i \in \Gamma'\} \cup A'$;
    $R^* = R_{|A' \times B} \cup \{(a_i^*, b) \mid b \in \mathcal{E}(C_l^s),\ i \in \Gamma'\}$;
return (A^*, B, R^*)

Table 3
Relation of Example 24.

R	b_1	b_2	b_3	b_4	b_5	b_6
a_1	1	0	0	0	0	0
a_2	0	1	0	0	0	0
a_3	1	1	1	0	1	0
a_4	1	1	0	1	0	0
a_5	1	1	0	0	1	0
a_6	1	1	1	0	0	1

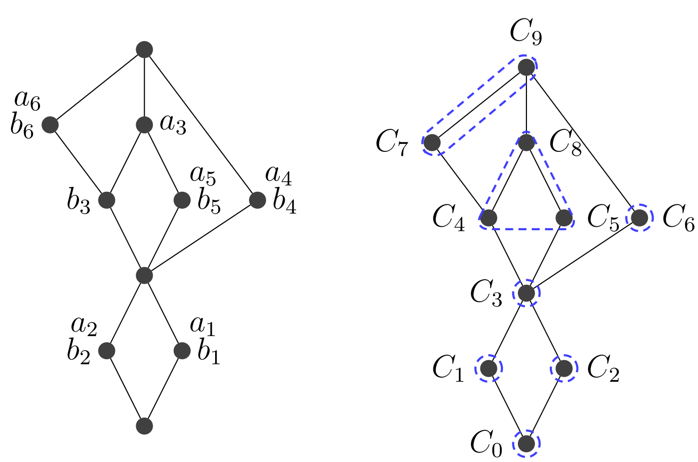

Fig. 5. The concept lattice $\mathcal{C}(A, B, R)$ (left) and the induced partition ρ_D (right) of Example 24.

Although the obtained poset is not a lattice, the Dedekind–MacNeille completion of the obtained poset provides the structure of a complete lattice. Indeed, this complete lattice is isomorphic to the lattice in which we have decided to remove a concept.

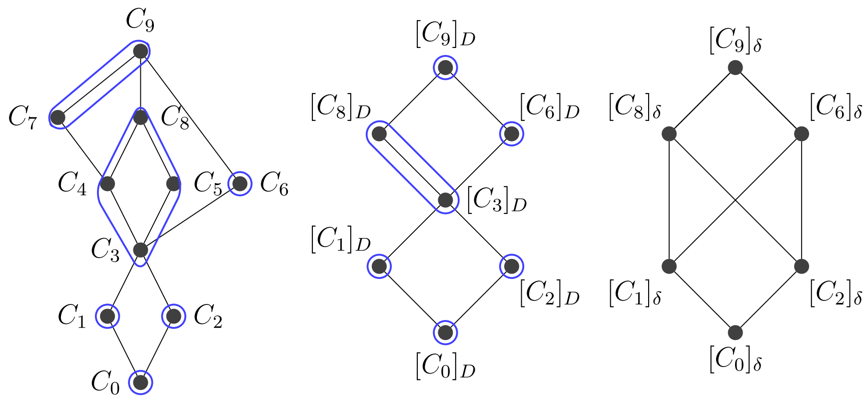

Fig. 6. The local congruence δ on $\mathcal{C}(A, B, R)$ (left), δ on $\mathcal{C}(A, B, R)/\rho_D$ (middle) and the quotient set $(\mathcal{C}(D, B, R)/\delta, \leq_\delta)$ of Example 24.

Lemma 25. *Given a complete lattice $(L, \preceq)$ satisfying the ACC and DCC, and a non-meet-irreducible and non-join-irreducible element $y \in L$. Then*

$$(DM(L\backslash\{y\}), \subseteq) \cong (L, \preceq)$$

Proof. Since $(L, \preceq)$ satisfies the ACC and DCC, then clearly $L\backslash\{y\}$ is join-dense and meet-dense (it contains the whole set of join and meet irreducible elements of L) and so, by Theorem 10, we obtain the isomorphism $(\mathrm{DM}(L\backslash\{y\}), \subseteq) \cong (L, \preceq)$. □

Therefore, since the attribute reduction procedure based on local congruences is also focused on obtaining a quotient set with the structure of a complete lattice, the Dedekind–MacNeille completion can be applied to achieve this challenge.

Proposition 26. *Given a context (A, B, R) and a non-join and non-meet-irreducible concept $C_i \in \mathcal{C}(A, B, R)$, we have that*

$$(DM(\mathcal{C}(A, B, R)\backslash\{C_i\}), \subseteq_i) \cong (\mathcal{C}(A, B, R), \leq),$$

where $\subseteq_i$ is the ordering defined by the Dedekind–MacNeille completion.

Proof. The proof straightforwardly follows from Lemma 25. □

Therefore, although a class with only this kind of concepts will be grouped in another class, no impact have in the concept lattice. Therefore, the application of the Dedekind–MacNeille completion is only needed to obtain a complete lattice isomorphic to the original one. The following theorem summarizes these results.

Similarly to the other procedures, this mechanism can be applied sequentially to different concepts and, moreover, Algorithms 1 and 2 can be interspersed.

Theorem 27. *Given a context (A, B, R) and a concept $C_k \in \mathcal{C}(A, B, R)$, we have that*

$$(DM(\mathcal{C}(A, B, R)\backslash\{C_k\}), \subseteq_k) \cong (\mathcal{C}(A^*, B^*, R^*), \leq^*),$$

where A^*, B^* and R^* are the sets and the relation given by either Algorithm 1 or Algorithm 2 or the original ones in case of C_k is neither a join nor a meet irreducible element, and $\subseteq_k$ is the ordering defined by the Dedekind–MacNeille completion.

Proof. The proof follows from Theorems 21 and 23, and the comment above concerning neither join nor meet irreducible elements. □

Since this theorem can be applied sequentially to the classes grouped by a local congruence, we have just characterized the impact of complementing an attribute reduction with the application of a local congruence. As a consequence, we can obtain a context from the original one which is isomorphic to the quotient set of the local congruence, considering the ordering defined in Theorem 16. It is also relevant to highlight that this mechanism computes the contexts more similar to the original one. Notice that either Algorithm 1 or Algorithm 2 can be applied to meet and join irreducible concepts and so, two different (although isomorphic) contexts arise. Therefore, the user can decide what kind of elements (attributes or objects) prefers to modify, and obtain the closest context to the original one, under this criterion.

5. Conclusions and future work

In this work, we have studied the impact of applying a local congruence on a concept lattice associated with a reduced context. We have proved that the quotient set generated by the equivalence relation induced by an attribute reduction is isomorphic to the concept lattice corresponding to the reduced context. However, we have seen that this fact does not hold for the quotient set generated by local congruences. We have also shown that the clustering carried out by a local congruence, after an attribute reduction, can have some impact in other concepts of the concept lattice. For that reason, we have studied the necessary modification to be done in a context when a concept of an arbitrary concept lattice needs to be removed. For this study, we have distinguished two types of elements in the lattice: join-irreducible and non-join-irreducible elements. We have proved that when we remove a join-irreducible element from a general complete lattice, the structure of complete lattice is preserved. Furthermore, it has been presented and proved a procedure for computing a modified context whose associated concept lattice is isomorphic to the original concept lattice when one of its join-irreducible concepts has been removed throughout the deletion of the objects generating that concept. In addition, dual results can be obtained for meet-irreducible concepts and an analogous procedure has been introduced when the removed element is a meet-irreducible concept. Finally, we have also analyzed the case when the removed element is neither a meet nor a join-irreducible element of the concept lattice, showing that, in this particular case, the Dedekind–MacNeille completion of the obtained poset is needed in order to provide the structure of a complete lattice.

In the near future, the introduced algorithms will be complemented with different attribute reduction mechanisms [6,8,19] and will be applied to real databases, such as the ones collected from our participation in the COST Action DigForASP. Moreover, the relationship between local congruences and attribute implications will be studied.

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Capítulo 7

Factorización de contextos formales

En este capítulo, presentamos un estudio preliminar sobre la factorización de contextos formales en el marco de trabajo multiadjunto por medio del operador de necesidad. En particular, nos centramos en el análisis de las propiedades que satisfacen los subcontextos independientes en el FCA para luego, ver cómo se comportan en el ambiente de los contextos formales multiadjuntos.

En [49], los autores utilizan uno de los operadores modales de la teoría de la posibilidad para caracterizar subcontextos independientes de un contexto establecido. En particular, se considera el par de operadores de necesidad, $({}^{\uparrow_N}, {}^{\downarrow^N})$ para descomponer la relación R de un contexto (A, B, R), mediante el cálculo de la siguiente intersección:

$$R^* = \bigcap\{(X \times Y) \cup (X^c \times Y^c) \mid X \subseteq B, Y \subseteq A, X^{\uparrow_N} = Y,\ Y^{\downarrow^N} = X\}$$

donde X^c e Y^c son los complementarios de X e Y, respectivamente. A partir del estudio presentado en [49], podemos resaltar algunas propiedades interesantes de un par de subconjuntos no vacíos $X \subseteq B$, $Y \subseteq A$, que satisface las siguientes igualdades:

$$X^{\uparrow_N} = Y \text{ y } Y^{\downarrow^N} = X \tag{7.1}$$

Al conjunto de estos pares se denotará como $\mathcal{C}_N$, es decir,

$$\mathcal{C}_N = \{(X,Y) \mid X \subseteq B, Y \subseteq A, X^{\uparrow_N} = Y,\ Y^{\downarrow^N} = X\}$$

A continuación, se presentan estas propiedades y se analizarán con el fin de conocer mejor cómo funciona la descomposición de tablas booleanas de datos en subcontextos independientes. Además, nos permitirá extender este procedimiento de factorización a entornos más generales, como la generalización difusa que ofrece el marco multiadjunto.

7.1. Subcontextos independientes en el FCA

De ahora en adelante, vamos a suponer que la tabla de datos (la relación del contexto) no tiene filas vacías ni atributos que posean todos los objetos del contexto. La primera propiedad que nos encontramos, relaciona las igualdades dadas en la Expresión (7.1) con los conceptos formales asociados al contexto.

Proposición 7.1. *Sea (A,B,R) un contexto formal y sea $(X,Y) \in \mathcal{C}_N$. Si $X^{\uparrow} \neq \varnothing$, entonces el par $(X, X^{\uparrow})$ es un concepto, es decir, $X^{\uparrow\downarrow} = X$.*

Demostración. Puesto que el par $(^{\uparrow}, ^{\downarrow})$ es una conexión de Galois antítona, se cumple que $X \subseteq X^{\uparrow\downarrow}$. Ahora vamos a probar que la otra contención también se satisface. Consideremos un $b_0 \in X^{\uparrow\downarrow} = \{b \in B \mid (a,b) \in R \text{ para todo } a \in X^{\uparrow}\}$, y probaremos que $b_0 \in X$. En particular, tomando cualquier $a_0 \in X^{\uparrow} = \{a \in A \mid (a,b) \in R \text{ para todo } b \in X\}$, obtenemos que $(a_0, b) \in R$, para todo $b \in X$. Además, se satisface que $X = X^{\uparrow_N\downarrow^N} = \{b \in B \mid \text{ si } (a,b) \in R \text{ entonces } a \in X^{\uparrow_N}\}$. Por consiguiente, $a_0 \in X^{\uparrow_N} = \{a \in A \mid$

si $(a,b) \in R$ entonces $b \in X\}$ y por lo tanto, como se verifica que $(a_0, b_0) \in R$ y $a_0 \in X^{\uparrow_N}$, podemos concluir que $b_0 \in X$. □

De forma dual, se puede obtener que el par $(Y^{\downarrow}, Y)$ es un concepto cuando se satisface $Y^{\downarrow} \neq \varnothing$.

En el siguiente resultado, se muestra cuándo el elemento máximo del subcontexto independiente es diferente del elemento máximo del retículo de conceptos original.

Proposición 7.2. *Sea (A,B,R) un contexto formal y sea $(X,Y) \in \mathcal{C}_N$. Si $X^{\uparrow} \neq \varnothing$, entonces no existe un concepto $(X_0, Y_0) \in \mathcal{C}(A,B,R)$ tal que $(X, X^{\uparrow}) < (X_0, Y_0) < (B, \varnothing^{\downarrow\uparrow})$.*

Demostración. Procedemos por reducción al absurdo. Supongamos que existe un concepto (X_0, Y_0) tal que $X \subset X_0 \subset B$ y $\varnothing^{\downarrow\uparrow} \subset Y_0 \subset X^{\uparrow}$. Esto significa que existe un objeto $b_0 \in B$ tal que $b_0 \in X_0 \backslash X$. Como $X_0 = Y_0^{\downarrow}$, se obtiene que $(a, b_0) \in R$ para todo $a \in Y_0$. Además, ya que $Y_0 \neq \varnothing$ (porque $\varnothing^{\downarrow\uparrow} \subset Y_0$), podemos considerar $a_0 \in Y_0$. Puesto que $Y_0 \subseteq X^{\uparrow} = \{a \in A \mid (a,b) \in R \text{ para todo } b \in X\}$, se satisface que $(a_0, b) \in R$, para todo $b \in X$. Ahora, dado $b' \in X$, como

$$b' \in X = X^{\uparrow_N\downarrow^N} = \{b \in B \mid \text{ si } (a,b) \in R \text{ entonces } a \in X^{\uparrow_N}\},$$

se deduce que $a_0 \in X^{\uparrow_N} = \{a \in A \mid \text{ si } (a,b) \in R \text{ entonces } b \in X\}$. Por lo tanto, de $a_0 \in X^{\uparrow_N}$ y $(a_0, b_0) \in R$, podemos concluir que $b_0 \in X$, lo que contradice la hipótesis. □

Consecuentemente, se ha probado que el elemento máximo del retículo de conceptos original es el concepto vecino superior del concepto obtenido en la Proposición 7.1. Un resultado equivalente para el elemento mínimo se puede enunciar por dualidad considerando el par $(Y^{\downarrow}, Y)$.

Ahora, vamos a analizar la relación que satisfacen los pares $(X, X^{\uparrow})$ y $(Y^{\downarrow}, Y)$, cuando son conceptos.

Proposición 7.3. *Dado un contexto formal* (A, B, R) *y un par* $(X, Y) \in \mathcal{C}_N$ *y las desigualdades* $X^{\uparrow} \neq \varnothing$ *y* $Y^{\downarrow} \neq \varnothing$*, entonces se cumple que el concepto generado por* $(X, X^{\uparrow})$ *es mayor que* $(Y^{\downarrow}, Y)$*, es decir,* $Y^{\downarrow} \subseteq X$*.*

Demostración. Consideremos un objeto $b_0 \in Y^{\downarrow} = \{b \in B \mid (a, b) \in R$ para todo $a \in Y\}$. Por lo tanto, si tomamos cualquier $a_0 \in Y$, se obtiene que $(a_0, b_0) \in R$ y puesto que $Y = Y^{\downarrow^N \uparrow_N} = \{a \in A \mid$ si $(a, b') \in R$ entonces $b' \in Y^{\downarrow^N}\}$, podemos concluir que $b_0 \in Y^{\downarrow^N} = X$. □

Es importante destacar que en todos los resultados anteriores se requiere que exista al menos un atributo en el subcontexto compartido por todos los objetos del contexto considerado, es decir, si consideramos el contexto $(X, Y, R_{|X \times Y})$ donde $R_{|X \times Y}$ denota la restricción de la relación R a los subconjuntos X e Y, se satisface que $X^{\uparrow} \neq \varnothing$ (equivalentemente $Y^{\downarrow} \neq \varnothing$ indica la existencia de al menos un objeto que posea todos los atributos del subcontexto). Estos requisitos garantizan que los retículos de conceptos asociados a subcontextos independientes no tengan ningún concepto en común, es decir, la factorización da lugar a una partición del retículo de conceptos original.

Hay que enfatizar que los pares $(X, X^{\uparrow})$ y $(Y^{\downarrow}, Y)$ no siempre son conceptos del retículo de conceptos asociado al contexto original. Esto ocurre cuando no se tiene en cuenta la condición $X^{\uparrow} \neq \varnothing$ (o $Y^{\downarrow} \neq \varnothing$). Con esta situación, podemos encontrar más de un concepto que satisfaga la Proposición 7.2 y la Proposición 7.3, es decir, existen elementos maximales que satisfacen estas propiedades (equivalentemente con los elementos minimales, cuando se cumple $Y^{\downarrow} = \varnothing$). Como consecuencia, cuando varios subcontextos independientes satisfacen esta condición, los elementos máximos de los retículos de conceptos asociados a estos subcontextos independientes coinciden, y además, coincide con el supremo del retículo de conceptos asociado al contexto original (de igual manera, cuando la condición $Y^{\downarrow} = \varnothing$ se cumple, los elementos mínimos de los retículos de conceptos coinciden y coinciden con el elemento ínfimo del retículo de conceptos original).

El siguiente ejemplo ilustra uno de los casos que acabamos de describir, así como muestra las propiedades que se han introducido.

Ejemplo 7.1. *Sea un contexto (A, B, R) compuesto por el conjunto de atributos $A = \{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8\}$ y el conjunto de objetos $B = \{b_1, b_2, b_3, b_4, b_5, b_6, b_7\}$, relacionados por $R: A \times B \to \{0, 1\}$, definida en la Tabla 7.1.*

R	b_1	b_2	b_3	b_4	b_5	b_6	b_7
a_1	0	1	1	1	0	0	0
a_2	0	0	0	1	0	0	0
a_3	1	0	0	0	0	0	0
a_4	0	0	0	0	0	1	1
a_5	0	1	1	0	0	0	0
a_6	0	0	0	0	1	1	0
a_7	0	0	1	0	0	0	0
a_8	0	0	0	0	1	1	1

Tabla 7.1: Relación del Ejemplo 7.1.

En primer lugar, tenemos que calcular todos los pares de $\mathcal{C}_N$. Estos pares, a excepción de los triviales, son los siguientes:

$$(X_1, Y_1) = (\{b_5, b_6, b_7\}, \{a_4, a_6, a_8\})$$
$$(X_2, Y_2) = (\{b_1\}, \{a_3\})$$
$$(X_3, Y_3) = (\{b_2, b_3, b_4\}, \{a_1, a_2, a_5, a_7\})$$

Si analizamos el primer par, (X_1, Y_1), vemos que $X_1^{\uparrow} = \{a_8\} \neq \varnothing$ y, además, $\{a_8\}^{\downarrow} = \{b_5, b_6, b_7\} = X$. Esto significa que $X^{\uparrow\downarrow} = X$, es decir, $(X_1, X_1^{\uparrow})$ es un concepto tal y como expone la Proposición 7.1. En particular, $(X_1, X_1^{\uparrow}) = C_8$ como se puede ver en la lista de conceptos que aparece en la Figura 7.1, junto con *el retículo de conceptos asociado al contexto. Como se puede observar, el concepto* C_8 *es estrictamente menor que el concepto* C_{10} *que es elemento máximo del retículo, es decir, el par* $(X_1, X_1^{\uparrow})$ *cumple también la Proposición 7.2. Por lo tanto, ya solo nos queda comprobar si el par* (X_1, Y_1) *satisface la Proposición 7.3. Para ello, necesitamos ver que* $Y_1^{\downarrow} \neq \varnothing$. *En efecto,* $Y_1^{\downarrow} = \{b_6\} \neq \varnothing$ *y además,* $Y_1^{\downarrow} = \{b_6\} \subseteq \{b_5, b_6, b_7\} = X$ *como enuncia la Proposición 7.3. Como conclusión, el par* (X_1, Y_1) *satisface las tres propiedades que hemos presentado, es decir, el par* (X_1, Y_1) *determina un bloque de conceptos que forma un subretículo del retículo de conceptos original delimitado por los conceptos* C_8 *y* C_1*, como se puede observar en la Figura 7.1.*

$C_0 = (\varnothing, A)$
$C_1 = (\{b_6\}, \{a_4, a_6, a_8\})$
$C_2 = (\{b_3\}, \{a_1, a_5, a_7\})$
$C_3 = (\{b_5, b_6\}, \{a_6, a_8\})$
$C_4 = (\{b_6, b_7\}, \{a_4, a_8\})$
$C_5 = (\{b_1\}, \{a_3\})$
$C_6 = (\{b_4\}, \{a_1, a_2\})$
$C_7 = (\{b_2, b_3\}, \{a_1, a_5\})$
$C_8 = (\{b_5, b_6, b_7\}, \{a_8\})$
$C_9 = (\{b_2, b_3, b_4\}, \{a_1\})$
$C_{10} = (B, \varnothing)$

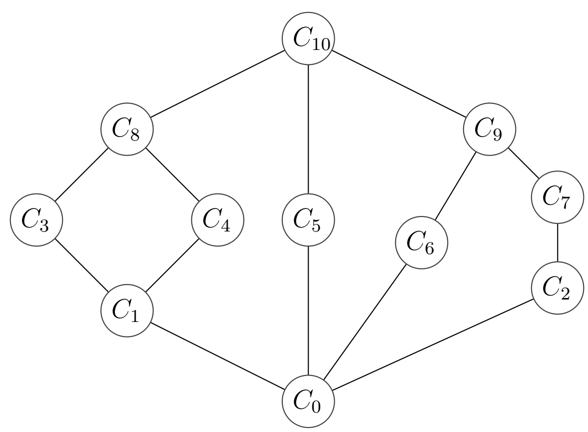

Figura 7.1: Lista de conceptos y retículo de conceptos del Ejemplo 7.1.

Por otra parte, si nos fijamos en el par (X_3, Y_3) *y seguimos un procedimiento similar al que hemos hecho con el par* (X_1, Y_1)*, podemos comprobar que* $(X_3, X_3^{\uparrow}) = (\{b_2, b_3, b_4\}, \{a_1\}) = C_9$ *y* C_9 *es menor estricto que* C_{10}*, es decir, satisface la Proposición 7.1 y la Proposición 7.2. No obstante, si calculamos la extensión de* Y_3 *obtenemos que* $Y_3^{\downarrow} = \varnothing$ *y, por lo tanto,* $(Y_3^{\downarrow}, Y_3)$ *no es un concepto y no satisface ninguna de las propiedades presentadas. Sin embargo, el par* (X_3, Y_3) *nos sigue delimitando un bloque de conceptos comprendido por* C_0*,* C_2*,* C_6*,* C_7 *y* C_9*. En* *otras palabras, que el par* $(Y_3^{\downarrow}, Y_3)$ *no sea un concepto significa que el elemento mínimo del retículo de conceptos asociado al subcontexto que genera* (X_3, Y_3) *es el mismo que el elemento mínimo del retículo de conceptos original.* □

En resumen, en los resultados anteriores estamos considerando el subretículo generado por el subcontexto independiente $(X, Y, R_{|X\times Y})$. En el caso de que solo se dé una de las dos condiciones, por ejemplo $X^{\uparrow} \neq \varnothing$, lo que genera el subcontexto independiente es un subretículo cuyo elemento máximo $(X, X^{\uparrow})$ no coincide con el elemento máximo del retículo de conceptos asociado al contexto original, evidentemente sin tener en cuenta el mínimo del retículo de conceptos original (que puede ser compartido con otro subcontexto independiente diferente). Análogamente, si $Y^{\downarrow} \neq \varnothing$, se puede obtener un subretículo cuyo elemento mínimo $(Y^{\downarrow}, Y)$ no coincide con el elemento mínimo del retículo de conceptos original.

7.2. Adaptación al FCA multiadjunto

A continuación, se realiza un estudio de las propiedades que se han presentado anteriormente sobre subcontextos independientes, trasladadas al marco de trabajo multiadjunto. Como se trata de un estudio preliminar, para simplificar los resultados pero sin pérdida de generalidad, consideraremos un marco multiadjunto con un solo triple adjunto. Por lo tanto, fijamos un marco multiadjunto $(L_1, L_2, P, \&, \swarrow, \nwarrow)$ y el contexto (A, B, R), donde la aplicación σ no aparece, ya que no es necesaria al considerar únicamente un solo triple adjunto.

En [65] se muestra cómo se puede trasladar un marco multiadjunto a un marco multiadjunto orientado a propiedades y a un marco multiadjunto orientado a objetos. Cada triple adjunto de cada marco multiadjunto se define en dominios diferentes, sin embargo, utilizando las relaciones de orden duales podemos trasladar los operadores de necesidad y posibilidad a los operadores de derivación del FCA como establece el Lema 2 y el Lema 3 en [65]. En esta sección, dado un triple adjunto con respecto a L_1, L_2, P, al triple adjunto obtenido al aplicar el Lema 1 de [65], que se usa en el marco multiadjunto orientado a propiedades o el triple adjunto obtenido al aplicar el Lema 1 de [65] que se utiliza en el marco multiadjunto orientado a objetos, los denotaremos por $(\&_p, \swarrow^p, \nwarrow_p)$ y $(\&_o, \swarrow^o, \nwarrow_o)$.

Además, vamos a considerar un par (g, f), con $g \in L_2^B$ y $f \in L_1^A$, que satisface las igualdades de la Expresión (7.1) pero considerando las versiones generalizadas de los operadores de necesidad para ambientes difusos, introducidos al final de la Sección 2.5. Específicamente, consideraremos el conjunto:

$$\mathcal{F}_N = \{(g, f) \mid g \in L_2^B, f \in L_1^A, f^{\downarrow^N} = g, g^{\uparrow_N} = f\}$$

En primer lugar, presentaremos algunas propiedades previas que satisface el par (g, f) en un contexto dado en un marco multiadjunto, con el objetivo de poder generalizar la Proposición 7.1. El siguiente resultado muestra la relación entre los operadores de posibilidad y necesidad aplicados al mismo conjunto difuso de objetos g del par considerado (g, f).

Proposición 7.4. *Dado un par de conjuntos difusos $(g, f) \in \mathcal{F}_N$, se cumple que $g^{\uparrow_\pi} \preceq_1 g^{\uparrow_N}$.*

Demostración. Consideremos $g \in L_2^B$. Se satisface que $g(b) = g^{\uparrow_N \downarrow^N}(b)$, donde $g^{\uparrow_N \downarrow^N}(b) = \text{ínf}\{\text{ínf}\{g(b') \swarrow^o R(a', b') \mid b' \in B\} \nwarrow_p R(a', b) \mid a' \in A\}$. Podemos sacar el ínfimo del consecuente de la implicación $\nwarrow_p$, por la Proposición 21 en [37]. Por lo tanto, se obtiene que $g(b) = \text{ínf}\{(g(b') \swarrow^o R(a', b')) \nwarrow_p R(a', b) \mid a' \in A, b' \in B\}$. Por consiguiente,

$$g(b) \preceq_1 (g(b') \swarrow^o R(a', b')) \nwarrow_p R(a', b),$$

para todo $a' \in A$ y $b' \in B$. Entonces, aplicando la propiedad de adjunción

(Expresión (2.3)), obtenemos que:

$$R(a',b)\&_p g(b) \preceq_1 g(b') \swarrow^o R(a',b').$$

Ahora, podemos tomar el supremo en el argumento de la izquierda de la desigualdad previa y el ínfimo en el argumento de la derecha, obteniendo:

$$\sup\{R(a',b)\&_p g(b) \mid b \in B\} \preceq_1 \text{ínf}\{g(b') \swarrow^o R(a',b') \mid b' \in B\}$$

Por lo tanto, $g^{\uparrow_\pi}(a') \preceq_1 g^{\uparrow_N}(a')$, para todo $a' \in A$. □

La siguiente propiedad relaciona el operador de cierre que surge de la composición de las aplicaciones del par $({}^{\uparrow_\pi}, {}^{\downarrow^N})$ con la composición de los operadores de necesidad ${}^{\uparrow_N}$ y ${}^{\downarrow^N}$.

Proposición 7.5. *Dado el par $(g,f) \in \mathcal{F}_N$, se verifica que $g^{\uparrow_\pi\downarrow^N} = g^{\uparrow_N\downarrow^N} = g$.*

Demostración. Por el Lema 2 en [65], se verifica que $({}^{\uparrow_\pi}, {}^{\downarrow^N})$ es una conexión de Galois antítona sobre $(L_1, \preceq_1^{op})$ y $(L_2, \preceq_2)$. Además, por la Proposición 7.4, obtenemos que $g^{\uparrow_\pi} \preceq_1 g^{\uparrow_N}$, es decir, $g^{\uparrow_N} \preceq_1^{op} g^{\uparrow_\pi}$. Si aplicamos el operador ${}^{\downarrow^N}$, entonces obtenemos que $g^{\uparrow_\pi\downarrow^N} \preceq_2 g^{\uparrow_N\downarrow^N}$, ya que ${}^{\downarrow^N}$ es decreciente. Adicionalmente, podemos obtener $g \preceq_2 g^{\uparrow_\pi\downarrow^N}$, puesto que la composición de las aplicaciones en el par $({}^{\uparrow_\pi}, {}^{\downarrow^N})$ es un operador de clausura y $g^{\uparrow_N\downarrow^N} = g$, donde la última igualdad se obtiene porque $(g,f) \in \mathcal{F}_N$. Por lo tanto, podemos concluir que

$$g \preceq_2 g^{\uparrow_\pi\downarrow^N} \preceq_2 g^{\uparrow_N\downarrow^N} = g.$$

□

Ahora, con la ayuda de los resultados enunciados anteriormente, podemos introducir una versión generalizada de la Proposición 7.1 en el marco multiadjunto. Resaltar que, en este caso, el concepto que generaría el par

(g, f) pertenece a un retículo de conceptos multiadjunto orientados a propiedades en lugar de un retículo de conceptos multiadjunto, como cabría esperar según los resultados obtenidos en el ambiente clásico.

Proposición 7.6. *Dado* $(g, f) \in \mathcal{F}_N$, *se cumple que* $\langle g, g^{\uparrow_\pi} \rangle \in \mathcal{M}_{\pi N}$.

Demostración. Todo concepto $\langle g, f \rangle$, con $g \in L_2^B, f \in L_1^A$, de un retículo de conceptos multiadjunto orientado a propiedades debe satisfacer $f^{\downarrow^N} = g$ y $g^{\uparrow_\pi} = f$. Por la Proposición 7.5, obtenemos que el par $\langle g, g^{\uparrow_\pi} \rangle$ verifica que $g^{\uparrow_\pi \downarrow^N} = g$. Por lo tanto, esto es equivalente a que se cumplan las dos igualdades que hemos mencionado anteriormente.

Por consiguiente, podemos concluir que el par $\langle g, g^{\uparrow_\pi} \rangle$ es un concepto del retículo de conceptos multiadjuntos orientado a propiedades. □

En esta ocasión, no hace falta imponer ninguna condición adicional en la hipótesis del resultado anterior, por lo que significa que el par $\langle g, g^{\uparrow_\pi} \rangle$ es siempre un concepto, a condición de que existan los subconjuntos g y f, y este no sea ninguno de los pares triviales que satisfacen las igualdades de la Expresión (7.1), es decir, que no sea (B, A) ni $(\varnothing, \varnothing)$.

Ahora bien, si nos fijamos en la Proposición 7.3 podemos apreciar que la propiedad $Y^{\downarrow} \subseteq X$ hace que el par (X, Y) determine un bloque de conceptos, es decir, un subretículo del retículo de conceptos original. Sin embargo, dentro del marco de trabajo multiadjunto, la propiedad $f^{\downarrow} \preceq_2 g$ no se cumple en general. Este hecho lo ilustramos en el siguiente ejemplo.

Ejemplo 7.2. *Consideremos el marco multiadjunto* $([0,1]_4, [0,1]_4, [0,1]_4, \leq, \leq, \leq, \&_P^*)$, *donde* $[0,1]_4$ *denota la partición del intervalo* $[0,1]$ *en cuatro partes iguales, y el operador* $\&_P^*$ *es la discretización del conjuntor producto, definido para todo* $a, b \in [0,1]_4$, *como:*

$$\&_P^*(a, b) = \frac{\lceil 4 \cdot a \cdot b \rceil}{4}$$

donde $\lceil\_\rceil$ *es la función parte entera superior. Las correspondientes implicaciones residuadas* $\swarrow_P^*, \nwarrow_P^*$: $[0,1]_4 \times [0,1]_4 \to [0,1]_4$ *se definen como:*

$$c \swarrow_P^* b = \frac{\lfloor 4 \cdot (c \leftarrow b) \rfloor}{4}, \; c \nwarrow_P^* a = \frac{\lfloor 4 \cdot (c \leftarrow a) \rfloor}{4}$$

donde $\lfloor\_\rfloor$ *es la función parte entera inferior y* $\leftarrow: [0,1] \times [0,1] \to [0,1]$ *es la implicación residuada de la t-norma producto, definida para todo* $b, c \in [0,1]$ *como:*

$$c \leftarrow b = \begin{cases} 1 & \textit{si } b \leq c \\ \dfrac{c}{b} & \textit{en otro caso} \end{cases}$$

El contexto multiadjunto considerado viene dado por el conjunto de atributos $A = \{a_1, a_2, a_3\}$*, el conjunto de objetos* $B = \{b_1, b_2, b_3\}$ *y la relación* $R : A \times B \to [0,1]_4$ *definida en la Tabla 7.2.*

R	b_1	b_2	b_3
a_1	0.5	0.5	1
a_2	0.25	1	0
a_3	0	0.75	0.25

Tabla 7.2: Relación difusa del Ejemplo 7.2.

Mencionar que, cuando $L_1 = L_2 = P$*, el marco multiadjunto, el multiadjunto orientado a propiedades y el multiadjunto orientado a objetos coinciden. Entonces, se están considerando los mismos triples adjuntos en todos estos marcos de trabajo [65].*

La lista de los conceptos multiadjuntos y el retículo de conceptos multiadjuntos asociado se muestra en la Figura 7.2.

$C_0 = \langle\{b_2/0.5\},\{a_1/1.0,a_2/1.0,a_3/1.0\}\rangle$
$C_1 = \langle\{b_1/0.25,b_2/0.5\},\{a_1/1.0,a_2/1.0\}\rangle$
$C_2 = \langle\{b_2/0.75\},\{a_1/0.5,a_2/1.0,a_3/1.0\}\rangle$
$C_3 = \langle\{b_2/0.5,b_3/0.25\},\{a_1/1.0,a_3/1.0\}\rangle$
$C_4 = \langle\{b_1/0.5,b_2/0.5\},\{a_1/1.0,a_2/0.5\}\rangle$
$C_5 = \langle\{b_1/0.25,b_2/1.0\},\{a_1/0.5,a_2/1.0\}\rangle$
$C_6 = \langle\{b_1/0.5,b_2/1.0\},\{a_1/0.5,a_2/0.5\}\rangle$
$C_7 = \langle\{b_1/0.5,b_2/0.5,b_3/1.0\},\{a_1/1.0\}\rangle$
$C_8 = \langle\{b_1/1.0,b_2/1.0\},\{a_1/0.5,a_2/0.25\}\rangle$
$C_9 = \langle\{b_1/1.0,b_2/1.0,b_3/1.0\},\{a_1/0.5\}\rangle$
$C_{10} = \langle\{b_2/1.0\},\{a_1/0.5,a_2/1.0,a_3/0.75\}\rangle$
$C_{11} = \langle\{b_2/0.75,b_3/0.25\},\{a_1/0.5,a_3/1.0\}\rangle$
$C_{12} = \langle\{b_2/1.0,b_3/0.25\},\{a_1/0.5,a_3/0.75\}\rangle$
$C_{13} = \langle\{b_2/1.0,b_3/0.5\},\{a_1/0.5,a_3/0.5\}\rangle$
$C_{14} = \langle\{b_2/1.0,b_3/1.0\},\{a_1/0.5,a_3/0.25\}\rangle$
$C_{15} = \langle\{b_2/0.5,b_3/0.5\},\{a_1/1.0,a_3/0.5\}\rangle$
$C_{16} = \langle\{b_2/0.5,b_3/1.0\},\{a_1/1.0,a_3/0.25\}\rangle$

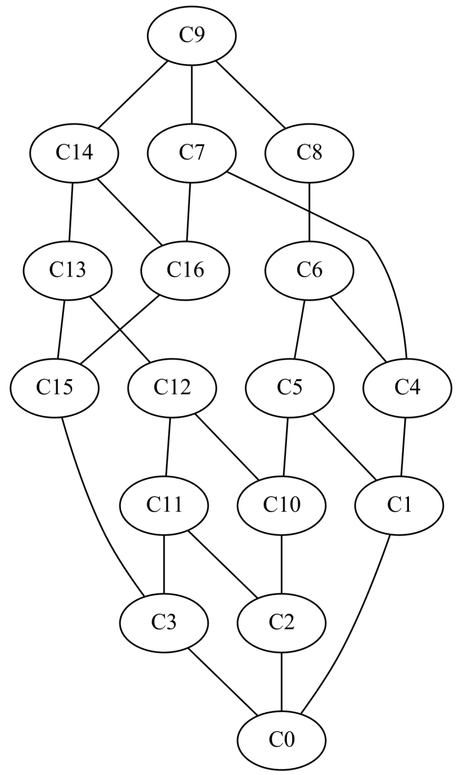

Figura 7.2: Lista de conceptos multiadjuntos y retículo de conceptos multiadjunto del Ejemplo 7.2.

Ahora, consideremos los siguientes conjuntos difusos de atributos y objetos:

$$f(a_1)=0.25 \qquad f(a_2)=0.5 \qquad f(a_3)=0.5$$
$$g(b_1)=0.5 \qquad g(b_2)=0.5 \qquad g(b_3)=0.25$$

Para poder probar que $f^{\downarrow} \le g$ *no se cumple, primeramente tenemos que demostrar que los subconjuntos difusos* g *y* f *satisfacen* $g^{\uparrow_N} = f$ *y* $f^{\downarrow^N} = g$*. Luego, vamos a calcular* $g^{\uparrow_N}$ *y* $f^{\downarrow^N}$*:*

$$g^{\uparrow_N}(a_1)=0.25 \qquad g^{\uparrow_N}(a_2)=0.5 \qquad g^{\uparrow_N}(a_3)=0.5$$
$$f^{\downarrow^N}(b_1)=0.5 \qquad f^{\downarrow^N}(b_2)=0.5 \qquad f^{\downarrow^N}(b_3)=0.25$$

Por tanto, se satisface que $g^{\uparrow_N} = f$ *y* $f^{\downarrow^N} = g$*, para todo* $a \in A$ *y* $b \in B$*. Ahora, podemos calcular* $f^{\downarrow}$*, esto es,* $f^{\downarrow}(b_1) = 0$*,* $f^{\downarrow}(b_2) = 1$ *y* $f^{\downarrow}(b_3) = 0$*.*

Podemos observar que $f^{\downarrow}(b_1) = 0 \leq 0.5 = g(b_1)$ *y* $f^{\downarrow}(b_3) = 0 \leq 0.25 = g(b_3)$*, pero* $f^{\downarrow}(b_2) = 1 \nleq 0.5 = g(b_2)$*. De hecho, si calculamos todos los pares* (g_i, f_i) *tales que* ${g_i}^{\uparrow_N} = f_i$ *y* ${f_i}^{\downarrow^N} = g_i$ *con* $g_i \in L_2^B$*,* $f_i \in L_1^A$*, obtenemos los siguientes pares excluyendo a los triviales:*

$$
\begin{aligned}
(g_1, f_1) &= (\{b_1/0.5, b_2/0.25, b_3/0.25\}, \{a_1/0.25, a_2/0.25, a_3/0.25\})\\
(g_2, f_2) &= (\{b_1/0.5, b_2/0.5, b_3/0.25\}, \{a_1/0.25, a_2/0.5, a_3/0.5\})\\
(g_3, f_3) &= (\{b_1/1, b_2/0.25, b_3/0.5\}, \{a_1/0.5, a_2/0.25, a_3/0.25\})\\
(g_4, f_4) &= (\{b_1/1, b_2/0.5, b_3/0.5\}, \{a_1/0.5, a_2/0.5, a_3/0.5\})\\
(g_5, f_5) &= (\{b_1/1, b_2/0.5, b_3/0.75\}, \{a_1/0.75, a_2/0.5, a_3/0.5\})\\
(g_6, f_6) &= (\{b_1/1, b_2/0.5, b_3/1\}, \{a_1/1, a_2/0.5, a_3/0.5\})\\
(g_7, f_7) &= (\{b_1/1, b_2/0.75, b_3/0.5\}, \{a_1/0.5, a_2/0.75, a_3/1\})\\
(g_8, f_8) &= (\{b_1/1, b_2/0.75, b_3/0.75\}, \{a_1/0.75, a_2/0.75, a_3/1\})\\
(g_9, f_9) &= (\{b_1/1, b_2/0.75, b_3/1\}, \{a_1/1, a_2/0.75, a_3/1\})\\
(g_{10}, f_{10}) &= (\{b_1/1, b_2/1, b_3/0.5\}, \{a_1/0.5, a_2/1, a_3/1\})\\
(g_{11}, f_{11}) &= (\{b_1/1, b_2/1, b_3/0.75\}, \{a_1/0.75, a_2/1, a_3/1\})
\end{aligned}
$$

Considerando cualquier par (g_i, f_i) *que aparece en la lista anterior, esto es, con* $i \in \{1, \ldots, 11\}$*, podemos fijarnos en su subconjunto difuso* g_i *y buscar el concepto multiadjunto con la extensión más grande posible que sea menor que* g_i*. Análogamente, hacemos lo mismo para su subconjunto difuso* f_i*, buscando el concepto multiadjunto con la menor intensión que sea mayor que* f_i*. Sin embargo, no siempre se obtiene un bloque de conceptos multiadjuntos del retículo de conceptos multiadjuntos a diferencia de lo ocurrido en el marco clásico, donde sí se obtiene* *un subretículo. Por ejemplo, si consideramos el par* (g_4, f_4)*, obtenemos que:*

$$Ext(C_{15}) = \{b_2/0.5, b_3/0.5\} \leq \{b_1/1, b_2/0.5, b_3/0.5\} = g_4,$$
$$Ext(C_4) = \{b_1/0.5, b_2/0.5\} \leq \{b_1/1, b_2/0.5, b_3/0.5\} = g_4,$$
$$f_4 = \{a_1/0.5, a_2/0.5, a_3/0.5\} \leq \{a_1/0.5, a_2/1.0, a_3/0.75\} = Int(C_{10}).$$

Como podemos observar en el retículo de conceptos, no obtenemos un bloque de conceptos. En particular, los conceptos multiadjuntos C_4, C_{15} *y* C_{10} *son incomparables.* □

Por lo tanto, si calculamos todos los pares (g, f) en un contexto multiadjunto dado, no obtenemos bloques independientes del retículo de conceptos multiadjuntos, es decir, la existencia de pares (g, f) que satisfagan las igualdades dadas en la Expresión (7.1) en el marco multiadjunto, no garantiza que haya subcontextos independientes.

Capítulo 8

Conclusiones y trabajo futuro

A continuación se detallan las conclusiones obtenidas de los estudios realizados, así como las de cada uno de los artículos publicados en las revistas que respaldan la presente tesis doctoral.

8.1. Conclusiones alcanzadas

Fundamentalmente, esta tesis doctoral se ha centrado en dos estrategias distintas para abordar la reducción de contextos formales, que es una de las líneas con mayor relevancia dentro de la teoría del FCA, tanto en un ambiente clásico como difuso.

Hemos realizado un estudio en profundidad sobre las congruencias locales como estrategia para complementar técnicas de reducción a través de relaciones de equivalencia. Este estudio se sustenta en unos sólidos fundamentos matemáticos que nos han permitido caracterizar el impacto que puede llegar a generar la aplicación de una congruencia local en un contexto formal. Además, proporcionan robustez a los procedimientos obtenidos.

Por otro lado, la segunda estrategia se ha llevado a cabo a través del estudio de la tabla de datos proporcionada por la relación del contexto, estudiando las propiedades que se verifican cuando es posible extraer subtablas (subcontextos) independientes. Se ha mostrado un estudio inicial sobre las propiedades de la factorización de contextos en un ambiente clásico del FCA y se ha analizado el comportamiento de las propiedades obtenidas en un marco de trabajo multiadjunto.

Debido al gran interés de obtener técnicas de reducción de elementos redundantes de bases de datos, los resultados que se han desarrollado en esta tesis son de gran utilidad en una gran diversidad de sectores. Los procedimientos presentados en esta tesis ayudan a sintetizar la información contenida en bases de datos, permitiendo así un mejor manejo de la información y tratamiento de los datos relevantes.

Conclusiones alcanzadas en *Reducing concept lattices by means of a weaker notion of congruence* publicado en *Fuzzy Sets and Systems* (Q1)

En este trabajo hemos introducido una noción debilitada de congruencia, que se ha denominado congruencia local. Hemos analizado cómo se pueden ordenar los elementos del conjunto cociente generado por una congruencia local. Además, hemos demostrado que la estructura algebraica del conjunto de congruencias locales es un retículo completo. También hemos mostrado una caracterización de las congruencias locales en términos de sus congruencias locales principales, así como una extensión de esta caracterización considerando cualquier relación de equivalencia arbitraria. Como consecuencia, se ha presentado un procedimiento para calcular la menor congruencia local que contiene una relación de equivalencia dada. A partir de este estudio, hemos presentado un nuevo mecanismo de reducción de retículos de conceptos (difusos) basado en la noción de congruencia local. Considerando este mecanismo de reducción, obtenemos una partición de los conceptos del retículo de conceptos original que satisface que cada clase de equivalencia tiene la estructura de un subretículo convexo del retículo de conceptos original. Además, hemos demostrado que la consideración de congruencias locales para reducir retículos de conceptos es más adecuada que la consideración de congruencias, ya que se pierde una menor cantidad de información durante el proceso de reducción.

Conclusiones alcanzadas en *Identifying Non-Sublattice Equivalence Classes Induced by an Attribute Reduction in FCA* publicado en *Mathematics* (Q1)

En este artículo, hemos seguido con el análisis iniciado en [9] consiguiendo resultados mejorados y de gran relevancia, ya que ayudan a conocer el impacto generado por una congruencia local. Hemos introducido una caracterización del ínfimo de los elementos pertenecientes a una clase no trivial, inducida por una reducción de atributos que contenga más de un único concepto. También hemos analizado la caracterización de estos elementos cuando la reducción de atributos considerada no contiene atributos innecesarios, lo cual es de especial interés en el FCA ya que las reducciones de atributos suelen descartar este tipo de atributos. No considerar atributos innecesarios nos ha permitido encontrar otros resultados interesantes en este marco. Hemos establecido una condición suficiente para asegurar una equivalencia entre los conceptos ínfimo irreducibles del retículo de conceptos reducido y el retículo original. Además, bajo esta consideración, hemos demostrado que cuando el retículo de conceptos original es isomorfo a un retículo distributivo, entonces las clases de equivalencia inducidas por una reducción de atributos siempre tienen estructura de subretículo. Todo el desarrollo teórico que hemos realizado en este trabajo tiene un impacto directo en la teoría de las congruencias locales [6].

Conclusiones alcanzadas en *Impact of local congruences in variable selection from datasets* publicado en *Journal of Computational and Applied Mathematics* (Q1)

En esta publicación, hemos estudiado el impacto de la aplicación de una congruencia local sobre un retículo de conceptos asociado a un contexto reducido. Hemos demostrado que el conjunto cociente generado por la relación de equivalencia inducida por una reducción de atributos es isomorfo al retículo de conceptos correspondiente al contexto reducido. Además, hemos probado que este hecho no se cumple para el conjunto cociente generado por congruencias locales. También hemos demostrado que el agrupamiento realizado por una congruencia local, después de una reducción de atributos, puede tener impacto en otros conceptos del retículo de conceptos. Por ello, hemos estudiado las modificaciones necesarias a realizar en un contexto cuando se requiere eliminar un concepto de un retículo de conceptos arbitrario. Para este estudio, hemos distinguido diferentes tipos de elementos en el retículo: elementos supremo irreducibles, elementos ínfimo irreducibles y otros elementos arbitrarios. Hemos demostrado que si se elimina un elemento supremo irreducible o un ínfimo irreducible de un retículo completo general, podemos obtener un subretículo que tiene, en particular, la estructura de retículo completo.

Adicionalmente, se ha presentado y demostrado un procedimiento para calcular un contexto modificado cuyo retículo de conceptos asociado es isomorfo al retículo de conceptos original cuando se ha eliminado uno de sus conceptos supremo irreducible mediante la eliminación de los objetos que generan ese concepto. También se pueden obtener resultados duales para los conceptos ínfimo irreducibles y se ha introducido un procedimiento análogo cuando el elemento eliminado es un concepto ínfimo irreducible. Por último, hemos analizado el caso en el que el elemento eliminado no es ni un ínfimo ni un supremo irreducible del retículo de conceptos, mostrando que, en este caso particular, la compleción de Dedekind-MacNeille del conjunto parcialmente ordenado es necesaria para proporcionar la estructura de retículo completo.

8.2. Perspectiva

En un futuro próximo, se estudiarán más propiedades de la noción de congruencia local sobre retículo de conceptos difusos, intentado extender los resultados obtenidos al marco de trabajo multiadjunto. Además, nos interesa estudiar la relación del impacto de las reducciones por congruencias locales y las implicaciones de atributos.

También nos interesa estudiar las condiciones suficientes sobre un contexto (difuso) para garantizar que su retículo de conceptos sea distributivo, ya que, como hemos mostrado en el Teorema 3.7, en este tipo de retículos se pueden obtener directamente clases que son subretículos convexos. Este es un problema interesante, que ya ha atraído la atención de otros investigadores [56]. Además, los algoritmos introducidos se complementarán con diferentes mecanismos de reducción de atributos [3, 40, 60].

Un objetivo importante es estudiar la complejidad computacional y viabilidad de los algoritmos y procedimientos de reducción basados en congruencia locales de cara a implementarlos en un software, para su posterior uso en bases de datos reales. En concreto, nos gustaría analizar el potencial de los mecanismos de reducción presentados en bases de datos relacionadas con el análisis forense digital, debido a nuestra participación en la Acción COST: *DIGital FORensics: evidence Analysis via intelligent Systems and Practices* (DigForASP), que se lidera desde la Universidad de Cádiz por el grupo de investigación al que pertenecezco FQM-406.

Además, continuaremos con el estudio de la factorización de contextos que se ha iniciado en esta tesis doctoral. Con el objetivo de obtener un mecanismo que nos permita factorizar contextos formales multiadjuntos, analizaremos nuevos métodos de factorización de contextos como, por ejemplo, a través de las Ecuaciones de Relaciones Difusas.

Por último, otro objetivo que nos planteamos es obtener mecanismos que nos permitan agregar la información proporcionada por los factores o subcontextos obtenidos tras el proceso de factorización, evitando la pérdida de información relevante durante el proceso, es decir, obteniendo un retículo de conceptos isomorfo al obtenido a partir del contexto original. Para ello, necesitaremos estudiar en profundidad el impacto de la agregación de nuevos elementos a un contexto difuso.

Anexos
Otras publicaciones relacionadas

Anexo A

A weakened notion of congruence to reduce concept lattices

A Weakened Notion of Congruence to Reduce Concept Lattices

Roberto G. Aragón, Jesús Medina, and Eloísa Ramírez-Poussa

Abstract This paper addresses the problem of attribute and size reduction of concept lattices in formal concept analysis. The reduction of the number of attributes in a formal context produces a partition on the set of concepts of the concept lattice. In this work, we introduce a weaker notion of congruence relation, called local congruence. This less restrictive kind of congruence guarantees that each subset of the partition forms a closed algebraic substructure, aggregating as few concepts as possible and preserving the main information.

Keywords Formal concept analysis · Size reduction · Attribute reduction · Congruence

1 Introduction

The large amount of relevant information collected in databases have aroused the interest in developing mathematical tools to manage, obtain and treat knowledge systems. One of these tools is the mathematical theory of Formal Concept Analysis (FCA) [9]. Since real databases usually contain unnecessary information, one of the most appealing topics within FCA is related to reduction mechanisms. We can find many works which analyze methods with the goal of reducing the number of attributes in contexts [1, 2, 4–7, 10–15]. Recently, [3] has presented a mechanism

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to reduce formal contexts based on the philosophy of reduction considered in Rough Set Theory, which is a theory closely related to FCA. In this previously mentioned study, the authors showed that when we reduce the number of attributes of a context, we are inducing an equivalence relation on the set of concepts of the original concept lattice. In addition, they also proved that the structure of the resulting equivalence classes are join-semilattices. This is the main result that has motivated the study presented in this work.

In this paper, we are interested in complementing the research given in [3]. Specifically, we are interested in the study of mechanisms to reduce concept lattices by means of equivalence classes in the concepts, satisfying that these classes are closed algebraic substructures, specifically, convex sublattices of the original concept lattice, that is, subsets of concepts closed by the supremum and infimum and that any element between two concepts of a class must also be in the class. This target is achieved by the notion of congruence relation on lattices [8]. However, the reductions based on congruence relations lead to a significant loss of information, as we will show in this work. Therefore, in order to overtake this drawback, we will introduce a new and weaker notion of congruence, which we have called local congruence. We see that the obtained results from the consideration of local congruence relations improve the ones provided by congruence relations and achieve the main goal of the paper.

2 Preliminaries

First of all, we recall the basic definitions in FCA [9] needed to understand the results which have motivated the presented study. In FCA, a *context* is a triple (A, B, R) with a set of attributes A, a set of objects B and a crisp relationship $R\colon A \times B \to \{0, 1\}$, such that $R(a, x) = aRx = 1$, if $a \in A$ and $x \in B$ are related, and $R(a, x) = 0$, otherwise. In addition, considering a context, two mappings, ${}^{\uparrow}\colon 2^B \to 2^A$ and ${}^{\downarrow}\colon 2^A \to 2^B$, can be defined for each $X \subseteq B$ and $Y \subseteq A$ as:

$$X^{\uparrow} = \{a \in A \mid \text{for all } x \in X, aRx\} \tag{1}$$

$$Y^{\downarrow} = \{x \in B \mid \text{for all } a \in Y, aRx\} \tag{2}$$

These operators form a Galois connection [8]. Therefore, if the equalities $X^{\uparrow} = Y$ and $Y^{\downarrow} = X$ hold, for a pair (X, Y), with $X \subseteq B$ and $Y \subseteq A$, this pair is called *concept*. For each pair of concepts $(X_1, Y_1), (X_2, Y_2) \in \mathcal{C}(A, B, R)$, if $X_1 \subseteq X_2$, then $(X_1, Y_1) \leq (X_2, Y_2)$.

The set of all the concepts with the previously defined ordering relation has the structure of a complete lattice, is called *formal concept lattice* and it is denoted as $\mathcal{C}(A, B, R)$ [8, 9]. The next result shows that when we reduce the set of attribute, an equivalence relation on the set of concepts of the original concept lattice is induced.

Proposition 1 ([3]) *Given a context* (A, B, R) *and a subset* $D \subseteq A$. *The set* $R_E = \{((X_1, Y_1), (X_2, Y_2)) \mid (X_1, Y_1), (X_2, Y_2) \in \mathcal{C}(A, B, R), X_1^{\uparrow_D \downarrow} = X_2^{\uparrow_D \downarrow}\}$ *is an* *equivalence relation. Where* $^{\uparrow_D}$ *denotes the concept-forming operator, given in Expression* (1)*, restricted to the subset of attributes* $D \subseteq A$.

Every class of the previously defined equivalence relation is a join semilattice with maximum element, as the following result shows.

Proposition 2 ([3]) *Given a context* (A, B, R)*, a subset* $D \subseteq A$ *and a class* $[(X, Y)]_D$ *of the quotient set* $\mathcal{C}(A, B, R)/R_E$*. The class* $[(X, Y)]_D$ *is a join semilattice with maximum element* $(X^{\uparrow_D\downarrow}, X^{\uparrow_D\downarrow\uparrow})$.

Now, we will introduce the notion of congruence on a lattice and some properties which are essential to develop our work. First of all, before introducing the notion of congruence we need to recall the following definition.

Definition 1 We say that an equivalence relation θ on a given lattice $(L, \preceq)$ is *compatible* with join and meet if, for all $a, b, c, d \in L$,

$$a \equiv b \ (\text{mod } \theta) \ \text{ and } \ c \equiv d \ (\text{mod } \theta)$$

imply $a \vee c \equiv b \vee d \ (\text{mod } \theta)$ and $a \wedge c \equiv b \wedge d \ (\text{mod } \theta)$.

Below we recall the definition of congruence on a lattice.

Definition 2 Given a lattice $(L, \preceq)$, we say that an equivalence relation on L, which is compatible with both join and meet is a *congruence* on L.

We will write $a \equiv b \ (\text{mod } \theta)$ or $(a, b) \in \theta$ to indicate that a and b are related under the congruence θ. Given a lattice L and a subset of L composed of four elements $\{a, b, c, d\}$, if $a < b$, $c < d$ and either ($a \vee d = b$ and $a \wedge d = c$) or ($b \vee c = d$ and $b \wedge c = a$), then a, b and c, d are said to be *opposite sides of the quadrilateral*. In addition, we say that the equivalence classes provided by a congruence are *quadrilateral-closed* if whenever given two opposite sides of a quadrilateral a, b and c, d, satisfying that a, b belong to one equivalence class X, then $c, d \in Y$ for another equivalence class Y.

Theorem 1 ([8]) *Let* $(L, \preceq)$ *be a lattice and let* θ *be an equivalence relation on* L. *Then* θ *is a congruence if and only if:*

(i) each equivalence class of θ *is a sublattice of* L,
(ii) each equivalence class of θ *is convex,*
(iii) the equivalence classes of θ *are quadrilateral-closed.*

The set of congruences on a lattice L, denoted as Con L, is a topped $\cap$-structure on $L \times L$. Hence Con L, together with the inclusion ordering, is a complete lattice. The bottom element and the top element are given by $\{(a, a) \mid a \in L\}$ and $\{(a, b) \mid a, b \in L\}$, respectively.

Table 1 Relation of Example 1

R	M	V	E	Ma	J	S	U	N	P
small size	1	1	1	1	0	0	0	0	1
medium size	0	0	0	0	0	0	1	1	0
large size	0	0	0	0	1	1	0	0	0
near sun	1	1	1	1	0	0	0	0	0
far sun	0	0	0	0	1	1	1	1	1
moon yes	0	0	1	1	1	1	1	1	1
moon no	1	1	0	0	0	0	0	0	0

3 Local congruences

As we mentioned previously, reductions of attributes in FCA generate a partition in the set of concepts of the original context whose equivalence classes may not necessarily form sublattices. At certain times, it could be interesting that the reduction generates groups of concepts with the structure of sublattices in order to have more homogeneous classes with respect to the ordering among the concepts and so, on a complete algebraic structure. In this section, we will see how we can complete the reduction given in FCA to satisfy the mentioned structural property.

First of all, we show the result of applying congruence relations on the concept lattice, in a practical example. In particular, we want to find the smallest congruence that contains the equivalence classes of the reduction obtained from FCA. We illustrate this idea by means of the following example, which was considered in [3].

Example 1 We consider the formal context (A, B, R) displayed in Table 1, where the set of objects in B are the planets of the Solar System together with the dwarf planet Pluto, that is $B = \{$Mercury (M), Venus (V), Earth (E), Mars (Ma), Jupiter (J), Saturn (S), Uranus (U), Neptune (N), Pluto (P)$\}$ and the set of attributes $A = \{$small size (ss), medium size (ms), large size (ls), near sun (ns), far sun, (fs), moon yes (my), moon no(mn)$\}$.

The concept lattice obtained from this context is given in the left side of Fig. 1. In [3], following the steps of reduction mechanism, it was carried out a reduction from the subset of attributes $D_1 = \{$small size, medium size, near sun, moon yes$\}$. Using such a reduction, the concepts of the original concept lattice are grouped in equivalence classes which are represented in the middle of Fig. 1. As it was mentioned in Proposition 2, each equivalence class has the structure of a join semilattice with maximum element.

We will show how we can bring together the reduction given in FCA and congruences. We find the smallest congruence such that each equivalence class induced by the reduction of the context, is included in one equivalence class provided by the congruence relation. This congruence is shown in the right side of Fig. 1. As we can

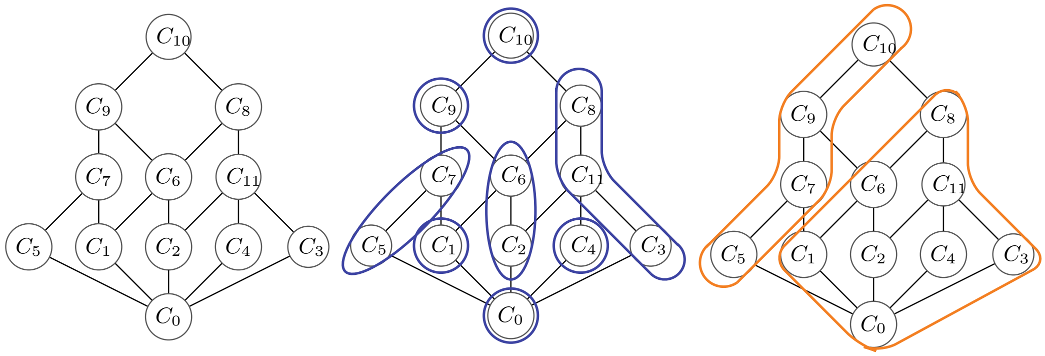

Fig. 1 The original concept lattice (left), the obtained reduction in [3] (middle) and the smallest congruence containing the previously reduction (right)

see in Fig. 1, this congruence relation has grouped many concepts in the same class and, consequently a lot of information is lost.

The result obtained in the previous example highlights the necessity of a weaker notion of congruence. This novel notion is presented in the following definition.

Definition 3 Given a lattice $(L, \preceq)$, we say that an equivalence relation δ on L is a *local congruence* if

(i) each equivalence class of δ is a sublattice of L,
(ii) each equivalence class of δ is convex.

Note that Definition 3 is similar to Theorem 1 by removing its third condition. Since δ is an equivalence relation, this definition provides a partition on L.

Definition 4 Let $(L, \preceq)$ be a lattice and δ a local congruence, the quotient set L/δ provides a partition of L, which is called *local congruence partition* (or *lc-partition* in short) of L and it is denoted as π_δ. The elements in the lc-partition π_δ are convex sublattices of L.

The definition of local congruence makes that the equivalence classes of the quotient set L/δ are closed algebraic structures. A direct characterization of the notion of local congruence in terms of the equivalence relation δ, is shown in the following result.

Proposition 3 *Given a lattice $(L, \preceq)$ and an equivalence relation δ on L, the relation δ is a local congruence on L if and only if, for each $a, b, c \in L$, the following properties hold:*

(i) If $(a, b) \in \delta$ and $a \preceq c \preceq b$, then $(a, c) \in \delta$.
(ii) $(a, b) \in \delta$ if and only if $(a \wedge b, a \vee b) \in \delta$.

Next, we define the inclusion of two local congruences taking into account the associated quotient sets.

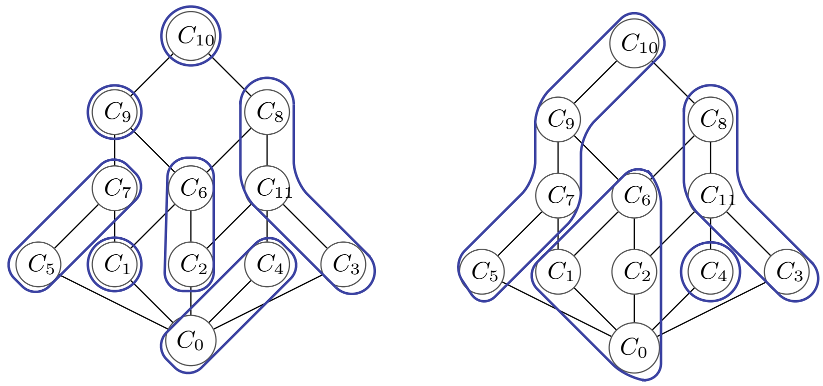

Fig. 2 Two local congruences containing the reduction from Fig. 1

Definition 5 Let $(L, \preceq)$ be a lattice and δ_1, δ_2 two local congruences on L. We say that the local congruence δ_1 is less than δ_2, denoted as $\delta_1 \sqsubseteq \delta_2$, if for every equivalence class $X \in L/\delta_1$ there exists an equivalence class $Y \in L/\delta_2$ such that $X \subseteq Y$.

All the previous notions and results are very important in order to obtain our main goal of this paper, that is, to find out the best reduction given by aggregating the concepts through closed sublattices of the original concept lattice, loosing as little information as possible. Hence, from the partition given by the reduction obtained from [3] in join-semilattices, we compute the infimum of the local congruences greater that it and so, we obtain the smaller partition on sublattices of L containing the given one.

Example 2 Returning to Example 1, there exist different local congruences containing the partition given by the reduction obtained from [3]. For example, Fig. 2 presents two of them which are incomparable and provide more granular partitions (are lesser than) than the one given by the usual definition of congruence (right diagram in Fig. 1). Hence, they offer a better reduction than the one provided using congruences, aggregating as less information as possible. Since this reduction already produces sublattices, the least local congruence containing it is itself. Thus, we can assert that the use of local congruences is the best notion to capture the main goal of the paper.

4 Conclusions and Future Work

In this work, we introduce a weaker definition of congruence relation which can be applied to the problem of attribute and size reduction in FCA. Its main properties are analyzed, highlighting that the elements in the lc-partition are convex sublattices of the original concept lattice This study will be continued in the future including complementary properties.

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Anexo B

On the hierarchy of equivalence classes provided by local congruences

On the Hierarchy of Equivalence Classes Provided by Local Congruences

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Abstract. In this work, we consider a special kind of equivalence relations, which are called local congruences. Specifically, local congruences are equivalence relations defined on lattices, whose equivalence classes are convex sublattices of the original lattices. In the present paper, we introduce an initial study about how the set of equivalence classes provided by a local congruence can be ordered.

Keywords: Congruence · Local congruence · Concept lattice · Ordering relation

1 Introduction

The notion of local congruence arose in an attempt to weaken the conditions imposed in the definition of a congruence relation on a lattice, with the goal of taking advantage of different properties of these relations with respect to attribute reduction in formal concept analysis [11,17,21].

Formal concept analysis (FCA) is a theory of data analysis that organizes the information collected in a considered dataset, by means of the algebraic structure of a complete lattice. Moreover, this theory also offers diverse mechanisms for obtaining, handling and relating (by attribute implications) information from datasets. One of the most interesting mechanisms is attribute reduction. Its main goal is the selection of the main attributes of the given dataset and detecting the unnecessary ones to preserve the estructure of the complete lattice.

In [4,5], the authors remarked that when a reduction of the set of attributes in the dataset is carried out, an equivalence relation is induced. This induced equivalence relation satisfies that the generated equivalence classes have the structure of a join-semilattice. Inspired by this fact, the original idea given in [1] was to complement these studies by proposing the use of equivalence relations

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containing the induced equivalence relation and satisfying that the generated equivalence classes be convex sublattices of the original lattice.

For example, congruence relations [6,10,12,13] hold the previously exposed requirements. In addition, congruence relations have already been applied to the framework of FCA [11,15,18–20]. Nevertheless, in [2] was proved that congruence relations are not suitable to complement the reductions in FCA, since the constraints imposed by this kind of equivalence relation entail a great loss of information. This reason is the main justification to weaken the notion of congruence relation, appearing the definition of local congruence. These new equivalence relations are also defined on lattices and only require that the equivalence classes be convex sublattices of the original lattice. The use of local congruences considerably reduces the problem of the loss of information.

However, the appearance of local congruences uncovers new open problems that require answers. One of these open problems is to provide an ordering relation on the set of equivalence classes, that is, on the quotient set associated with the local congruence. This is the main issue addressed in this paper. First of all, we will show that the usually considered ordering relations on the set of equivalence class of a congruence relation, cannot be used for local congruences. Then, we will define a new binary relation on lattices which turns out to be a pre-order when it is used to establish a hierarchy on the equivalence classes provided by a local congruence. Finally, we will also state under what conditions this pre-order is a partial order.

The paper is organized as follows: Sect. 2 recalls some preliminary notions used throughout of the paper. Section 3 presents the study of the hierarchy among the equivalence classes provided by local congruences. The paper finishes with some conclusions and prospects for future works, which are included in Sect. 4.

2 Preliminaries

In this section, we recall basic notions used in this paper. The first notion is related to a special kind of equivalence relation on lattices, which are called congruence relations.

Definition 1 **([10]).** *Given a lattice $(L, \preceq)$, we say that an equivalence relation θ on L is a* congruence *if, for all $a_0, a_1, b_0, b_1 \in L$,*

$$(a_0, b_0) \in \theta,\ (a_1, b_1) \in \theta \text{ imply that } (a_0 \vee a_1, b_0 \vee b_1) \in \theta,\ (a_0 \wedge a_1, b_0 \wedge b_1) \in \theta.$$

where $\wedge$ and $\vee$ are the infimum and the supremum operators.

Now, we recall the notion of quotient lattice from a congruence, based on the operations of the original lattice.

Definition 2 **([10]).** *Given an equivalence relation θ on a lattice $(L, \preceq)$, the operators infimum and supremum, $\vee_\theta$ and $\wedge_\theta$, can be defined on the set of equivalence classes $L/\theta = \{[a]_\theta \mid a \in L\}$ for all $a, b \in L$, as follows:*

$$[a]_\theta \vee_\theta [b]_\theta = [a \vee b]_\theta \text{ and } [a]_\theta \wedge_\theta [b]_\theta = [a \wedge b]_\theta.$$

$\vee_\theta$ and $\wedge_\theta$ are well defined on L/θ if and only if θ is a congruence.

When θ is a congruence on L, the tuple $(L/\theta, \vee_\theta, \wedge_\theta)$ is called quotient lattice *of L* modulo *θ.*

Now, let us suppose that $\{a, b, c, d\}$ is a subset of a given lattice $(L, \preceq)$. Then, the pairs a, b and c, d are said to be *opposite sides* of the quadrilateral $(a, b; c, d)$ if $a < b,\ c < d$ and either:

$$(a \vee d = b \text{ and } a \wedge d = c) \text{ or } (b \vee c = d \text{ and } b \wedge c = a).$$

In addition, we say that the equivalence classes provided by an equivalence relation are *quadrilateral-closed* if whenever given two opposite sides of a quadrilateral $(a, b; c, d)$, such that $a, b \in [x]_\theta$, with $x \in L$ then there exists $y \in L$ such that $c, d \in [y]_\theta$. This notion leads us to the following result which is a characterization of the congruence notion in terms of their equivalence classes and plays a key role in the definition of local congruences as we will show later (more detailed information on the characterization and the notions involved in this result can be found in [10]).

Theorem 1 ([10]). *Let $(L, \preceq)$ be a lattice and θ an equivalence relation on L. Then, θ is a congruence if and only if*

(i) each equivalence class of θ is a sublattice of L,
(ii) each equivalence class of θ is convex,
(iii) the equivalence classes of θ are quadrilateral-closed.

With the goal of obtaining a less-constraining equivalence relations than congruences, but preserving some interesting properties satisfied by this kind of equivalence relations, the notion of local congruence arose [2] in the framework of attribute reduction in FCA [7–9,11,16], focused on providing an optimal reduction on FCA from the application of Rough Set techniques [4,5,14]. This notion is recalled in the following definition and mainly consist in the elimination of a restriction (last item) in the previous theorem.

Definition 3. *Given a lattice $(L, \preceq)$, we say that an equivalence relation δ on L is a* local congruence *if the following properties hold:*

(i) each equivalence class of δ is a sublattice of L,
(ii) each equivalence class of δ is convex.

Next section studies how we can define an ordering relation between the equivalence classes obtained from a local congruence.

3 Ordering Classes of Local Congruences

In this section, we are interested in studying ordering relations for local congruences. This fact is fundamental for establishing a proper hierarchy among the classes of concepts obtained after the reduction in FCA [3–5].

The set of equivalence classes of a congruence on a lattice L can be ordered by a partial order $\preceq_\theta$ which is defined, for all $a, b \in L$, by means of the operators $\vee_\theta$ and $\wedge_\theta$ presented in Definition 2, as follows:

$$[a]_\theta \preceq_\theta [b]_\theta \quad \text{if} \quad [a]_\theta = [a]_\theta \wedge_\theta [b]_\theta \quad \text{or} \quad [b]_\theta = [a]_\theta \vee_\theta [b]_\theta \tag{1}$$

This ordering relation cannot be used for local congruences since local congruences are not compatible with either supremum or infimum, that is, the operators $\vee_\theta$ and $\wedge_\theta$ could not be well defined when the considered relation is a local congruence due to they do not satisfy the quadrilateral-closed property unlike congruences. In the next example, we illustrate this fact.

Example 1. Let us consider the lattice $(L, \preceq)$ shown in the left side of Fig. 1, and the local congruence δ, highlighted by means of a Venn diagram, given in the right side of Fig. 1.

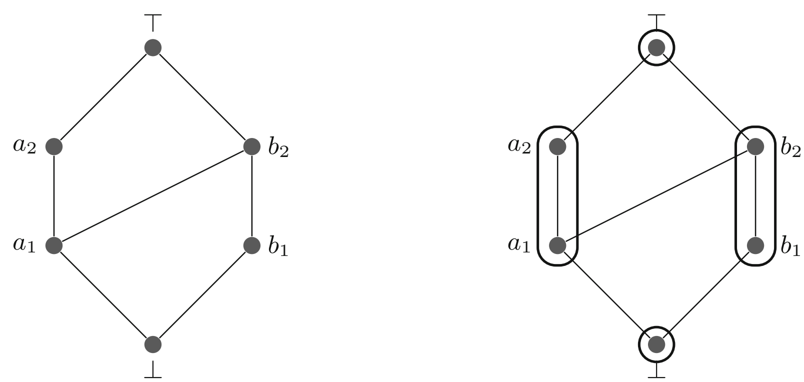

Fig. 1. Lattice (left) and local congruence (right) of Example 1.

It is easy to see that the considered local congruence δ provides four different equivalence classes which are listed below:

$$\begin{aligned}[\top]_\delta &= \{\top\} \\ [a_1]_\delta = [a_2]_\delta &= \{a_1, a_2\} \\ [b_1]_\delta = [b_2]_\delta &= \{b_1, b_2\} \\ [\bot]_\delta &= \{\bot\}\end{aligned}$$

We can observe that $a_1, \bot$ and b_1, b_2 are opposite sides, but a_1 and $\bot$ are not in the same equivalence class, which means that the equivalence classes of δ are not quadrilateral-closed. As a consequence, the infimum and supremum operators described in Expression (1) are not well defined. For example, we have that

$$\begin{aligned}[a_2]_\delta \wedge_\delta [b_1]_\delta = [a_2 \wedge b_1]_\delta = [\bot]_\delta \\ [a_2]_\delta \wedge_\delta [b_1]_\delta = [a_2 \wedge b_2]_\delta = [a_1]_\delta\end{aligned}$$

and clearly $[\bot]_\delta \neq [a_1]_\delta$. Therefore, the ordering $\preceq_\delta$ cannot be defined on local congruences. □

A property of the ordering relation, shown in Expression (1), was shown in [10], which provides another possibility of defining an ordering on the set of local congruences.

Proposition 1 **([10]).** *Let θ be a congruence on a lattice $(L, \preceq)$ and let $[a]_\theta$ and $[b]_\theta$ be equivalence classes of L/θ. Then, the binary relation $\leq$ defined on L/θ as: $[a]_\theta \leq [b]_\theta$, if there exist $a' \in [a]_\theta$ and $b' \in [b]_\theta$, for all $a' \preceq b'$, is an ordering relation.*

Clearly, the relation $\leq$ is the associated ordering relation with the algebraic lattice $(L/\theta, \vee_\theta, \wedge_\theta)$. Consequently, we cannot use either this alternative definition in the equivalence classes of a local congruence. In the following example, we show a case where the application of this ordering relation for a local congruence does not satisfies the transitivity property.

Example 2. We will consider the lattice $(L, \preceq)$ and the local congruence δ both given in Fig. 2. As we can observe, the local congruence provides five different equivalence classes:

$$
\begin{aligned}
[\top]_\delta &= \{\top\} \\
[a_1]_\delta = [a_2]_\delta &= \{a_1, a_2\} \\
[b_1]_\delta = [b_2]_\delta &= \{b_1, b_2\} \\
[c_1]_\delta = [c_2]_\delta &= \{c_1, c_2\} \\
[\bot]_\delta &= \{\bot\}
\end{aligned}
$$

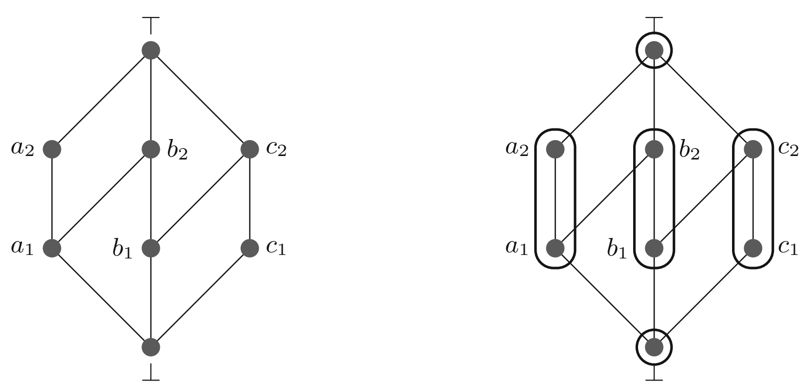

Fig. 2. Lattice (left) and local congruence (right) of Example 2.

If we try to order the equivalence classes of δ using the ordering relation described in Proposition 1, we obtain that $[a_1]_\delta \leq [b_1]_\delta$, since $a_1 \preceq b_2$, and

$[b_1]_\delta \leq [c_1]_\delta$ because $b_1 \preceq c_2$. Nevertheless, we can see that $[a_1]_\delta$ is not lesser than $[c_1]_\delta$ because neither a_1 nor a_2 are lesser than c_1 or c_2 in L. Therefore, the ordering relation defined in Proposition 1 is not transitive for local congruences in general and thus, it is not a partial order for local congruences. □

As we have seen in the previous example, the ordering relation defined in Proposition 1 cannot be used either to order the equivalence classes obtained from local congruences. However, the underlying idea of the ordering relation of Proposition 1 can be considered to define a more suitable ordering relation for being applied on local congruences. In order to achieve this goal, we formalize some notions presented in [6], which are related to the ordering of elements in the quotient set provided from equivalence relations defined on posets. The following notion is related to the way in which two elements of the original lattice can be connected via the local congruence.

Definition 4. *Let $(L, \preceq)$ be a lattice and a local congruence δ on L.*

*(i) A sequence of elements of L, $(p_0, p_1, \ldots, p_n)$ with $n \geq 1$, is called a δ-*sequence*, denoted as $(p_0, p_n)_\delta$, if for each $i \in \{1, \ldots, n\}$ either $(p_{i-1}, p_i) \in \delta$ or $p_{i-1} \preceq p_i$ holds.*
*(ii) If a δ-sequence $(p_0, p_n)_\delta$ satisfies that $p_0 = p_n$, then it is called a δ-*cycle*. In addition, if the δ-cycle satisfies that $[p_0]_\delta = [p_1]_\delta = \cdots = [p_n]_\delta$, then we say that the δ-cycle is *closed.

With the notions of Definition 4, we present a new binary relation on local congruences in the following definition.

Definition 5. *Given a lattice $(L, \preceq)$ and a local congruence δ on L, we define a binary relation $\preceq_\delta$ on L/δ as follows:*

$$[x]_\delta \preceq_\delta [y]_\delta \quad \text{if there exists a } \delta\text{-sequence } (x', y')_\delta$$

for some $x' \in [x]_\delta$ and $y' \in [y]_\delta$.

Now, we go back to Example 2 in order to illustrate this relation.

Example 3. Returning to Example 2, we want to establish a hierarchy among the equivalence classes depicted in Fig. 2 by means of the relation given in Definition 5. By considering this definition, it is clear that $[a_1]_\delta \preceq_\delta [b_1]_\delta$ and $[b_1]_\delta \preceq_\delta [c_1]_\delta$. In addition, we can observe that, in this case, we also have that $[a_1]_\delta \preceq_\delta [c_1]_\delta$ since there exists a δ-sequence that connects one element of the class $[a_1]_\delta$ with another element of the class $[c_1]_\delta$, this δ-sequence is shown below:

$$(a_1, c_2)_\delta = (a_1, b_2, b_1, c_2), \quad \text{since} \quad a_1 \preceq b_2, \quad (b_2, b_1) \in \delta \quad \text{and} \quad b_1 \preceq c_2$$

Therefore, the relationship among the elements in the quotient set L/δ given by $\preceq_\delta$ are shown in Fig. 3. □

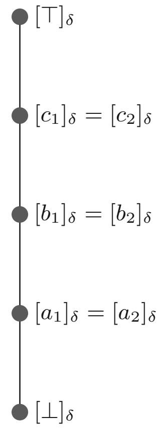

Fig. 3. Hasse diagram of the relation among the elements in L/δ of Example 3.

Observe that the binary relation $\preceq_\delta$ given in Definition 5 is a pre-order. Evidently, by definition, $\preceq_\delta$ is reflexive and transitive. Now, we present an example in which the previously defined relation $\preceq_\delta$ does not hold the antisymmetry property and, consequently, it cannot be used to establish an ordering among the equivalent classes obtained from a local congruence.

Example 4. Let us consider the lattice $(L, \preceq)$ and the local congruence δ given in Fig. 4. The equivalence classes provided by δ are:

$$[\top]_\delta = \{\top\}$$
$$[a_1]_\delta = [a_2]_\delta = [a_3]_\delta = [a_4]_\delta = \{a_1, a_2, a_3, a_4\}$$
$$[b_1]_\delta = [b_2]_\delta = \{b_1, b_2\}$$
$$[c_1]_\delta = [c_2]_\delta = \{c_1, c_2\}$$
$$[\bot]_\delta = \{\bot\}$$

.

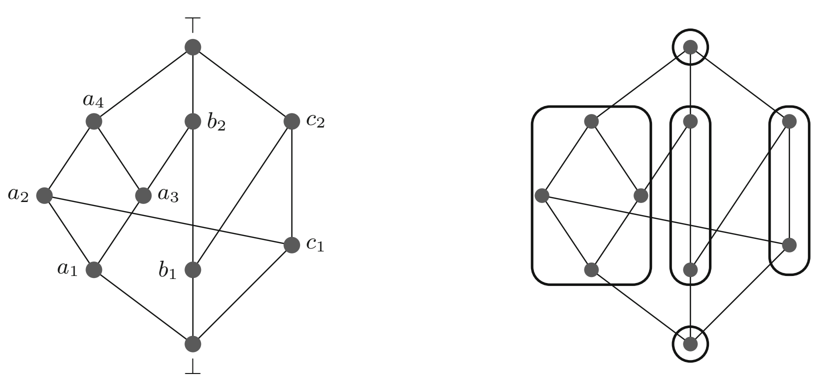

Fig. 4. Lattice and local congruence of Example 4.

If we try to establish a hierarchy among the equivalence classes using the binary relation given in Definition 5, we obtain that $[c_1]_\delta \preceq_\delta [a_1]_\delta$ since there exists another δ-sequence that connects c_1 with a_2:

$$(c_1, a_2)_\delta = (c_1, a_2) \quad \text{since} \quad c_1 \preceq a_2$$

In addition, we also have that $[a_1]_\delta \preceq_\delta [c_1]_\delta$, because there exists a δ-sequence that connects the elements a_3 and c_2:

$$(a_3, c_2)_\delta = (a_3, b_2, b_1, c_2) \quad \text{since} \quad a_3 \preceq b_2, (b_2, b_1) \in \delta \quad \text{and} \quad b_1 \preceq c_2$$

Therefore, we have that $[a_1]_\delta \preceq_\delta [c_1]_\delta$ and $[c_1]_\delta \preceq_\delta [a_1]_\delta$, but these classes are not equal. Thus, the antisymmetry property does not hold and, as a consequence, the obtained equivalent classes from the considered local congruence cannot be ordered by means of the considered binary relation. □

As we have seen in the previous example, the preorder $\preceq_\delta$ is not a partial order since the antisymmetry property is not satisfied for any local congruence, in general. Therefore, it is important to study sufficient conditions to ensure that $(L/\delta, \preceq_\delta)$ is a poset. The following result states a condition under which the binary relation of Definition 5 is a partial order on local congruences.

Theorem 2. *Given a lattice $(L, \preceq)$ and a local congruence δ on L, the preorder $\preceq_\delta$ given in Definition 5 is a partial order if and only if either no δ-cycle exists or every δ-cycle of elements in L is closed.*

Since no δ-cycle of elements in L exists with respect to the local congruences in Examples 1 and 2, we can ensure that the obtained quotient sets, together with the binary relation $\preceq_\delta$, are posets in both examples. The following example shows a local congruence on the lattice L given in Example 4, such as $(L/\delta, \preceq_\delta)$ is a poset.

Example 5. On the lattice $(L, \preceq)$ of Example 4, the quotient set L/δ_1 given by local congruence δ_1 depicted in the right side of Fig. 5, together with the binary relation defined in Definition 5, is a poset.

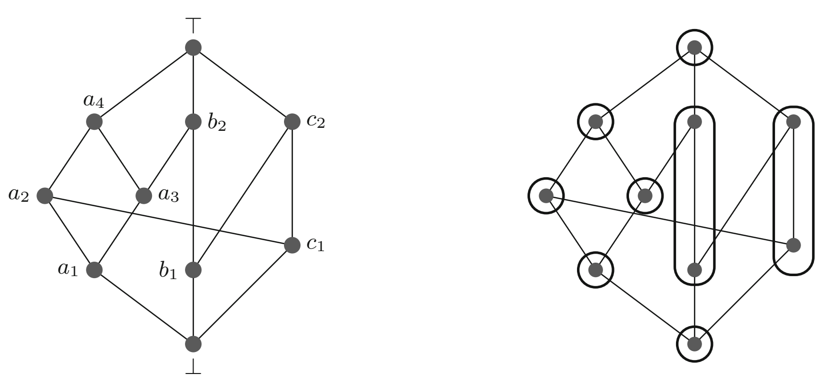

Fig. 5. Local congruence δ_1 (right) of Example 5 on the lattice of Example 4 (left).

We can ensure that because no δ_1-cycle exists. The right side of Fig. 6 shows an equivalence relation that contains the δ-cycle of Example 4 in one equivalence class. Therefore, the least local congruence, called δ_2, is the one that groups all the elements in a single class and, as a consequence, the δ-cycle is closed. Therefore, by Theorem 2, the pair $(L/\delta_2, \preceq_{\delta_2})$ it is also a poset. $\square$

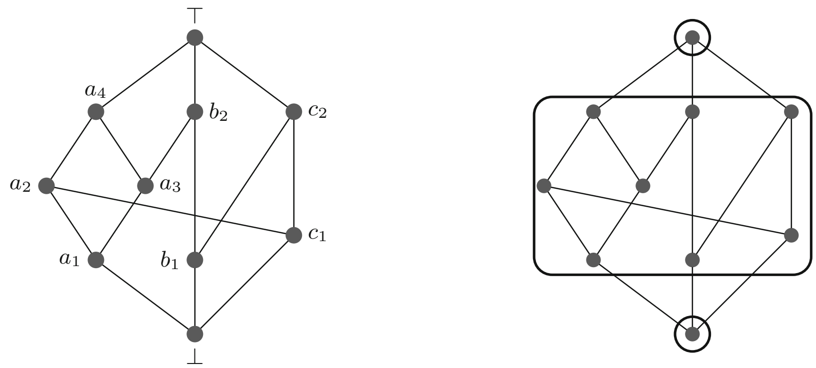

Fig. 6. Equivalence relation (right) of Example 5 on the lattice of Example 4 (left).

4 Conclusions and Future Work

In this paper, we have introduced an initial study about different ways of establish a hierarchy among the equivalence classes provided by local congruences. We have analyzed the results of applying the usually considered ordering relations on the quotient set of congruences, obtaining that these ordering relations are not suitable to be used on local congruences. Based on the underlying philosophy of one characterization of the ordering relation used for congruences, we have defined a new binary relation on the equivalence classes obtained from a local congruences. We have also proven that this binary relation is a preorder. Moreover, we have stated a sufficient condition on the lattice in which the local congruence is defined, in order to guarantee that this preorder is actually a partial order. All the ideas presented throughout this study have been illustrated by means of diverse examples.

As future work, we are interested in continuing this study and defining another binary relation, which will be a partial order on the equivalence classes of any local congruence. Furthermore, we will apply this type of equivalence relations in practical problems, such as in tasks related to the reduction of concept lattices in the framework of formal concept analysis.

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Anexo C

Impact of local congruences in attribute reduction

Impact of Local Congruences in Attribute Reduction

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Abstract. Local congruences are equivalence relations whose equivalence classes are convex sublattices of the original lattice. In this paper, we present a study that relates local congruences to attribute reduction in FCA. Specifically, we will analyze the impact in the context of the use of local congruences, when they are used for complementing an attribute reduction.

Keywords: Formal Concept Analysis · Size reduction · Attribute reduction · Local congruence

1 Introduction

Formal Concept Analysis (FCA) is a mathematical framework to analyze datasets introduced by Ganter and Wille in eighties [12]. The main goals of FCA are the following: to obtain the knowledge from data, to represent the obtained knowledge by means of the mathematical structure called concept lattice and to discover dependencies in data. The applied potential of FCA has encouraged the development of different generalizations.

One of the most intensively studied research lines by the research community of FCA in the last years, consists on decreasing the number of attributes of a dataset, preserving the information provided by the dataset [1,2,7,8,10,11,13–18]. In [6], authors proved that every reduction of attributes of a formal context induces an equivalent relation whose equivalent classes are join-semilattices. In [3], local congruences were introduced and applied to this attribute reduction. Local congruences are equivalence relations on lattices whose equivalence classes are convex sublattices. The idea in [3] was to find the least local congruence containing the equivalent relation induced by an attribute reduction of a formal context, in order to group the concepts of the original concept lattice using closed structures.

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Sometimes, the induced equivalent relation by a reduction of the context is already a local congruence but sometime it is not. In the latter case, the fact of using a local congruence that contains the induced equivalence relation has an influence on the original reduction. In this paper, we present an initial study about the relationship between local congruences and the induced equivalent relation by an attribute reduction of a formal context. This study provides a first step to know the influence that this special kind of equivalence relations has on the reduction procedure. We will include several examples to illustrate the obtained result.

2 Preliminaries

We need to recall some basic notions used in this work. In order to present the preliminary notions as clearly as possible, we will divide this section into two parts. The first one will be devoted to recall those necessary notions of FCA and the second one to those related to local congruences.

2.1 Formal Concept Analysis

In FCA a context is a triple (A, B, R) where A is a set of attributes, B is a set of objects and $R\colon A \times B \to \{0, 1\}$ is a relationship, such that $R(a, x) = aRx = 1$, if the object $x \in B$ possesses the attribute $a \in A$, and $R(a, x) = 0$, otherwise. In addition, we call *concept-forming operators* to the mappings ${}^{\uparrow}\colon 2^B \to 2^A$ and ${}^{\downarrow}\colon 2^A \to 2^B$ defined for each $X \subseteq B$ and $Y \subseteq A$ as:

$$X^{\uparrow} = \{a \in A \mid \text{for all } x \in X, aRx\} \quad (1)$$

$$Y^{\downarrow} = \{x \in B \mid \text{for all } a \in Y, aRx\} \quad (2)$$

Taking into account the previous mappings, a *concept* is a pair (X, Y), with $X \subseteq B$ and $Y \subseteq A$ satisfying that $X^{\uparrow} = Y$ and $Y^{\downarrow} = X$. The subset X is called the *extent* of the concept and the subset Y is called the *intent*. The set of extents and intents are denoted by $\mathfrak{E}(A, B, R)$ and $\mathfrak{I}(A, B, R)$, respectively.

In addition, all the concepts together with the inclusion ordering on the left argument has the structure of a complete lattice, which is called *concept lattice* and it is denoted as $\mathcal{C}(A, B, R)$.

From now on, we will say that an *attribute-concept* is a concept generated by an attribute $a \in A$, that is $(a^{\downarrow}, a^{\downarrow\uparrow})$.

On the other hand, we need to recall the notion of meet-irreducible element of a lattice.

Definition 1. *Given a lattice $(L, \preceq)$, such that $\wedge$ is the meet operator, and an element $x \in L$ verifying*

1. *If L has a top element $\top$, then $x \neq \top$.*
2. *If $x = y \wedge z$, then $x = y$ or $x = z$, for all $y, z \in L$.*

we call x meet-irreducible ($\wedge$-irreducible) element *of* L. *Condition* (2) *is equivalent to*

$2'$. *If* $x < y$ *and* $x < z$, *then* $x < y \wedge z$, *for all* $y, z \in L$.

On the other hand, with respect to the attribute reduction in FCA, it is important to recall that when we reduce the set of attributes in a context, an equivalence relation on the set of concepts of the original concept lattice is induced. The following proposition was proved in [6] for the classical setting of FCA and it is recalled below.

Proposition 1 ([6]). *Given a context* (A, B, R) *and a subset* $D \subseteq A$. *The set* $R_E = \{((X_1, Y_1), (X_2, Y_2)) \mid (X_1, Y_1), (X_2, Y_2) \in \mathcal{C}(A, B, R), X_1^{\uparrow_D\downarrow} = X_2^{\uparrow_D\downarrow}\}$ *is an equivalence relation. Where* $^{\uparrow_D}$ *denotes the concept-forming operator* $X^{\uparrow_D} = \{a \in D \mid \text{for all } x \in X, (a, x) \in R\}$ *restricted to the subset of attributes* $D \subseteq A$.

In [6], the authors also proved that each equivalence class of the induced equivalence relation has a structure of join semilattice.

Proposition 2 ([6]). *Given a context* (A, B, R), *a subset* $D \subseteq A$ *and a class* $[(X, Y)]_D$ *of the quotient set* $\mathcal{C}(A, B, R)/R_E$. *The class* $[(X, Y)]_D$ *is a join semilattice with maximum element* $(X^{\uparrow_D\downarrow}, X^{\uparrow_D\downarrow\uparrow})$.

2.2 Local Congruences

The notion of local congruence arose with the goal of complementing attribute reduction in FCA. The purpose of local congruences is to obtain equivalence relations less-constraining than congruences [3] and with useful properties to be applied in size reduction processes of concept lattices. We recall the notion of local congruence in the next definition.

Definition 2. *Given a lattice* $(L, \preceq)$, *we say that an equivalence relation* δ *on* L *is a* local congruence *if each equivalence class of* δ *is a convex sublattice of* L.

The notion of local congruence can be characterized in terms of the equivalence relation, as the following result shows.

Proposition 3. *Given a lattice* $(L, \preceq)$ *and an equivalence relation* δ *on* L, *the relation* δ *is a local congruence on* L *if and only if, for each* $a, b, c \in L$, *the following properties hold:*

(i) If $(a, b) \in \delta$ *and* $a \preceq c \preceq b$, *then* $(a, c) \in \delta$.
(ii) $(a, b) \in \delta$ *if and only if* $(a \wedge b, a \vee b) \in \delta$.

Usually, we will look for a local congruence that contains a partition induced by an equivalence relation. When we say that a local congruence contain a partition provided by an equivalence relation, we are making use of the following definition of inclusion of equivalence relations.

Definition 3. *Let* ρ_1 *and* ρ_2 *be two equivalence relations on a lattice* $(L, \preceq)$. *We say that the equivalence relation* ρ_1 *is included in* ρ_2, *denoted as* $\rho_1 \sqsubseteq \rho_2$, *if for every equivalence class* $[x]_{\rho_1} \in L/\rho_1$ *there exists an equivalence class* $[y]_{\rho_2} \in L/\rho_2$ *such that* $[x]_{\rho_1} \subseteq [y]_{\rho_2}$.

3 Analyzing Local Congruences

In this section, we will present an initial study about the role of local congruences when they are used along or together with other mechanisms to attribute reduction. In particular, we will analyze the relationship between local congruences and the induced equivalence relation by an attribute reduction from the perspective of the attribute of the context as well as from the meet-irreducible elements of the concept lattices. We are interested in discovering under what conditions the induced equivalence relation is a local congruence. We are also interested in analyzing the influence of the use of local congruence in the reduction of attributes, when the induced equivalence relation is not a local congruence.

Firstly, in the first example we will illustrate the main idea of this study.

Example 1. Let us consider a formal context (A, B, R) composed of the attributes $A = \{a_1, a_2, a_3\}$ and the objects $B = \{b_1, b_2, b_3\}$, related by a relationship $R \subseteq A \times B$, which is shown in the left side of Table 1, together with the list of concepts which appears in the right side of the same table. The associated concept lattice is displayed in the left side of Fig. 1.

Table 1. Relation and list of concepts of the context of Example 1.

R	b_1	b_2	b_3
a_1	1	0	1
a_2	0	1	1
a_3	0	0	1

C_i	Extent			Intent		
	b_1	b_2	b_3	a_1	a_2	a_3
0	0	0	1	1	1	1
1	1	0	1	1	0	0
2	0	1	1	0	1	0
3	1	1	1	0	0	0

In order to analyze the influence of local congruences in the reduction of the set of attributes of the considered context, we include a list in which we show the attribute that generates each concept of the concept lattice:

$$C_0 = (a_3^{\downarrow}, a_3^{\downarrow\uparrow})$$
$$C_1 = (a_1^{\downarrow}, a_1^{\downarrow\uparrow})$$
$$C_2 = (a_2^{\downarrow}, a_2^{\downarrow\uparrow})$$

If we consider, for example, the subset $D_1 = \{a_2, a_3\}$ to carry out the reduction of the set of attributes, that is, we remove the attribute a_1, we obtain a partition of the concept lattice induced by the reduction that is highlighted by means of a dashed Venn diagram in the middle of Fig. 1. We obtain that the concepts C_1 and C_3 are grouped in the same class whereas the concepts C_0 and C_2 provide two different classes composed of a single concept each one. Therefore, according to Proposition 2, we can see that the obtained equivalence classes are

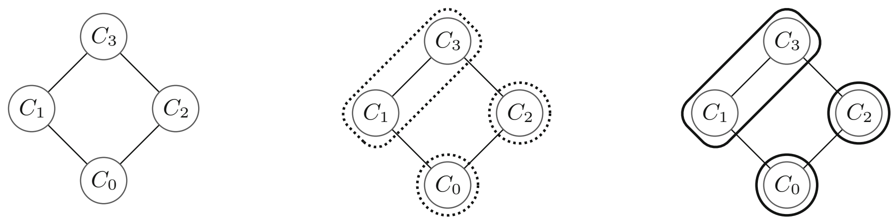

Fig. 1. Concept lattice of Example 1 (left), the partition induced by the subset D_1 (center) and the least local congruence containing the partition (right).

join semilattices. Indeed, all classes are convex sublattices of the original concept lattice.

As a consequence, the least local congruence containing such a reduction is the induced partition itself as it is shown in the right side of Fig. 1, where the local congruence is highlighted by means of a Venn diagram. In other words, the induced equivalence relation by the reduction is already a local congruence and, as a consequence, the consideration of local congruences does not alter the attribute reduction originally carried out on the set of attributes.

However, if the user decides to remove the attributes a_1 and a_2, that is, only the subset of attributes $D_2 = \{a_3\}$ is considered, the induced partition by the reduction is shown in the left side of Fig. 2.

Fig. 2. The partition induced by the elimination of the attributes a_1 and a_2 of Example 1 (left) and the least local congruence containing the induced partition (right).

The equivalence classes induced by the the reduction are the following:

$$[C_0]_{D_2} = \{C_0\}$$
$$[C_1]_{D_2} = [C_2]_{D_2} = [C_3]_{D_2} = \{C_1, C_2, C_3\}$$

In this case, the obtained equivalence classes are non-trivial join-semilattices since the concepts C_1, C_2 and C_3 do not form a convex sublattice of the original concept lattice. In this case, the infimum of the equivalence class $[C_1]_{D_2}$ is de concept C_0, which has not been included in $[C_1]_{D_2}$. This concept is not in $[C_1]_{D_2}$ because it is generated from the attribute a_3, which means that this attribute differences concept C_0 from the rest. Therefore, if this attribute is not removed in the reduction procedure, then it continues differentiating this concept from the rest and it cannot be in the same class of the rest.

If we compute the least local congruence containing the equivalence relation above, it groups all concepts in a single class, that is, the local congruence includes the infimum of the concepts C_1, C_2 and C_3, that is, the concept C_0, in the equivalence class $[C_1]_{D_2}$. This local congruence is depicted in the right side of Fig. 2. Clearly, this local congruence does not coincides with the equivalence relation induced by the attribute reduction which entails certain consequences with respect to the initial attribute reduction, since the inclusion of the concept C_0 in the equivalence class $[C_1]_{D_2}$, can be seen as a kind of elimination of the attribute a_3 (since the attribute a_3 generates the concept C_0). □

The previous example has shown different possibilities of applying local congruences for complementing an attribute reduction process. Hence, we have that some times the obtained equivalence relations is already a local congruence and other cases is not. In particular, we have seen a case that when the infimum of an induced equivalence class is generated by an attribute, which has not been removed during the reduction process, proper join semilattices arise and the induced equivalence relation is not a local congruence. In the following example, we will analyze another possible situations we can find when the set of attributes is reduced.

Example 2. We will consider a context composed of the set of attributes $A = \{a_1, a_2, a_3, a_4\}$ and the set of objects $B = \{b_1, b_2, b_3\}$, related by $R : A \times B \to \{0, 1\}$, defined on the left side of Table 2 together with the list of the corresponding concepts which appear in the right side of the same table. The associated concept lattice is given on the left side of Fig. 3.

Table 2. Relation and list of concepts of the context of Example 2.

R	b_1	b_2	b_3
a_1	1	1	0
a_2	1	0	1
a_3	0	1	1
a_4	0	0	1

C_i	Extent			Intent			
	b_1	b_2	b_3	a_1	a_2	a_3	a_4
0	0	0	0	1	1	1	1
1	1	0	0	1	1	0	0
2	0	1	0	1	0	1	0
3	0	0	1	0	1	1	1
4	1	1	0	1	0	0	0
5	1	0	1	0	1	0	0
6	0	1	1	0	0	1	0
7	1	1	1	0	0	0	0

From this context we obtain the following attribute-concepts:

$$C_3 = (a_4^{\downarrow}, a_4^{\downarrow\uparrow})$$
$$C_4 = (a_1^{\downarrow}, a_1^{\downarrow\uparrow})$$
$$C_5 = (a_2^{\downarrow}, a_2^{\downarrow\uparrow})$$
$$C_6 = (a_3^{\downarrow}, a_3^{\downarrow\uparrow})$$

For instance, if we are interested in considering the subset of attributes $D_1 = \{a_1, a_3\}$ and we carry out the corresponding reduction (removing the attributes a_2 and a_4), we obtain the partition induced by D_1, which is shown in the middle of Fig. 3. Once again, as in Example 1, the equivalence classes obtained from the reduction considering the subset D_1 are convex sublattices of the original concept lattice. Therefore, the least local congruence that contains such a reduction is the induced equivalence relation itself, as it can be seen in the right side of Fig. 3. Consequently, local congruences do not modify the considered attribute reduction.

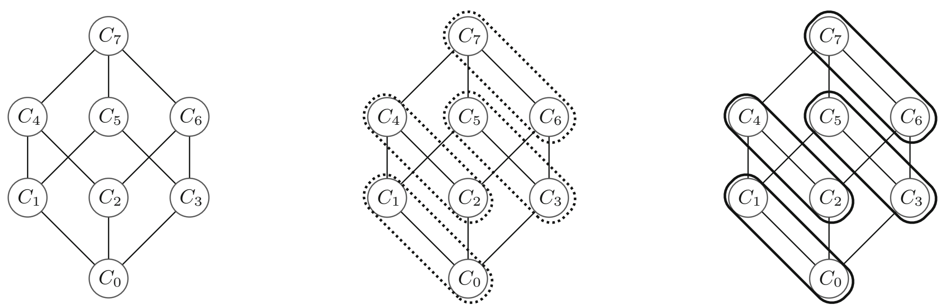

Fig. 3. Concept lattice of Example 2 (left), the partition induced by the subset D_1 (center) and the least local congruence containing the induced partition (right).

Now, if the attributes a_2 and a_3 are removed, i.e., only the subset of attributes $D_2 = \{a_1, a_4\}$ is considered, then the partition induced by the reduction is shown in the left side of Fig. 4 and the induced equivalence classes are listed below.

$$[C_0]_{D_2} = \{C_0\}$$
$$[C_1]_{D_2} = [C_2]_{D_2} = [C_4]_{D_2} = \{C_1, C_2, C_4\}$$
$$[C_3]_{D_2} = \{C_3\}$$
$$[C_5]_{D_2} = [C_6]_{D_2} = [C_7]_{D_2} = \{C_5, C_6, C_7\}$$

Notice that two of the obtained equivalence classes are not convex sublattices of the original concept lattice. The first one contains the concepts C_1, C_2, C_4 and the other one contains the concepts C_5, C_6, C_7. However, the reasons that make these classes are not convex sublattices are well differentiated.

On the one hand, with respect to the equivalence class of the concept $[C_5]_{D_2}$ we find a similar situation than the one shown in Example 1, that is, the infimum of the equivalence class $[C_5]_{D_2}$ is the concept C_3 which is generated from attribute a_4 that has not been removed in the reduction of the context.

On the other hand, the infimum of the equivalence class $[C_1]_{D_2}$ is the concept C_0 which is not generated by any attribute of the context. Nevertheless, $C_0 \notin [C_1]_{D_2}$ since in the decomposition of meet-irreducible concepts of the concept C_0, that is $C_0 = C_4 \wedge C_5 \wedge C_6$, we can find two meet-irreducible concepts C_5 and C_6 satisfying that $C_5, C_6 \notin [C_1]_{D_2}$.

In this case, the least local congruence whose equivalence classes contain the induced partition can be seen in the right side of Fig. 4. In this figure we have that the local congruence includes the infimum of the equivalence classes $[C_1]_{D_2}$ and $[C_5]_{D_2}$ in their respective classes. Thus, the least local congruence provides two different equivalence classes.

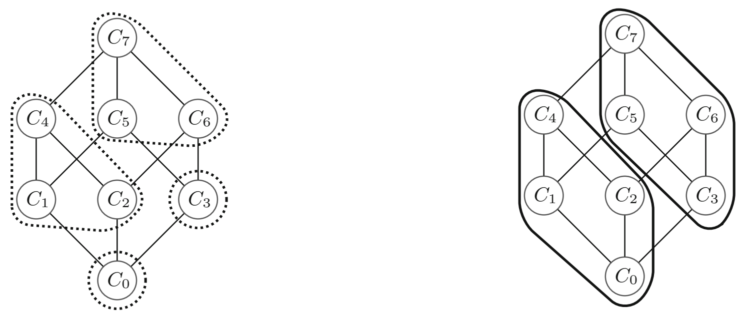

Fig. 4. The partition induced by the elimination of attributes a_2 and a_3 in Example 2 (left) and the least local congruence containing the induced partition (right).

Now, we will analyze how this local congruence influences in the reduction of the attributes. We can see that the inclusion of the concept C_3 in the equivalence $[C_5]_{D_2}$, is equivalent to the elimination of attribute a_4. We can also observe that the intension of the concept C_0 includes attribute a_4 which is ignored when C_0 is introduced in the equivalence class $[C_5]_{D_2}$. Hence, in spite of the reduction of the context was carried out originally from the elimination of attributes a_2 and a_3, somehow the consideration of the local congruence implies the elimination of attribute a_4. □

From the previous examples, we deduce that when the induced equivalence relation does not provide convex sublattices as equivalence classes, the use of local congruence relations alters the original attribute reduction, increasing the number of attributes to be removed. Moreover, it would be interesting to highlight these attributes, record its relationship with the removed attributes and the impact in attribute implications [4,5,9,19].

Next result relates the equivalence relations induced by an attribute reduction with the attributes-concepts and the meet-irreducible elements of the concept lattice. Due to the closely relation between the $\wedge$-irreducible concepts and the set of attributes of the context. This result summarizes the influence of local congruences in the attribute reduction of relational datasets.

Proposition 4. *Given a context (A, B, R), a subset of attributes $D \subseteq A$, an equivalence class $[C]_D$, with $C \in \mathcal{C}(A, B, R)$, of the induced equivalence relation which is not a convex sublattice and the concept $C' = \bigwedge_{C_i \in [C]_D} C_i$. Then, one of the following statements is satisfied:*

- *There exists at least one attribute $a \in A$ such that $C' = (a^{\downarrow}, a^{\downarrow\uparrow})$.*
- *There exists a concept $C^* \in M_F(A, B, R)$ in a meet-irreducible decomposition $\{C_j \in M_F(A, B, R) \mid j \in J\}$ of C', such as $C_{i_0} \not\leq C^*$ for a concept $C_{i_0} \in [C]_D$.*

Proof. Let us assume that we reduce the context (A, B, R), by considering a subset of attributes $D \subseteq A$, and that given $C \in \mathcal{C}(A, B, R)$, the induced equivalence class $[C]_D$ is not a convex sublattice of the original concept lattice.

Therefore, although by Proposition 2 the class $[C]_D$ is a join-semilattice, the concept $C' = \bigwedge_{C_i \in [C]_D} C_i$ is not in $[C]_D = \{C_1, \ldots, C_n\}$. Now, we will distinguish two cases:

(i) If there exists $a_0 \in A$ such that $C' = (a_0^{\downarrow}, a_0^{\downarrow\uparrow})$, the first statement holds.
(ii) Otherwise, let $\{C_j \in M_F(A, B, R) \mid j \in J\}$ be a meet-irreducible decomposition of C', that is, $C' = \bigwedge_{j \in J} C_j$. If there exists $j_0 \in J$ and $i_0 \in \{1, \ldots, n\}$, such as $C_{i_0} \not\leq C_{j_0}$ for all $C_i \in [C]_D$, we finish the proof. Otherwise, we have that $C_i \leq C_j$ for all $C_i \in [C]_D$ and $j \in J$. As a consequence, the set $\{C_j \in M_F(A, B, R) \mid j \in J\}$ is in the meet-irreducible decomposition of every concept in $[C]_D$, in particular in the maximum element of the class, denoted as C_M. Hence, we have that

$$C' < C_M \leq \bigwedge_{j \in J} C_j = C'$$

which leads us to a contradiction.

It is important to mention that the items exposed in the previous result are not exclusive, that is, we can find a concept C' satisfying simultaneously both conditions of the previous result. In this situation, this fact means that the intent of the concept C' has at least two different attributes, a_0 and a_1 such that these attributes do not belong to the intent of any concept $C_i \in [C]_D$ for all $i \in I$.

Notice also that the requirement "$C_{i_0} \not\leq C^*$ for a concept $C_{i_0} \in [C]_D$." in the second condition can be rewritten as C_{i_0} and C^* are incomparable, or $C^* < C_{i_0}$. This last inequality detects a possible non-distributivity lattice and discover the following consequences of Proposition 4.

Corollary 1. *Let (A, B, R) be a context where its concept lattice $\mathcal{C}(A, B, R)$ is distributive, $D \subseteq A$ a subset of attributes and $C \in M_F(A, B, R)$. If $C' = \bigwedge_{C_i \in [C]_D} C_i$ is not in $[C]_D$, then there exists an attribute $a \in A$ such that $C' = (a^{\downarrow}, a^{\downarrow\uparrow})$.*

These results show that the application of local congruences offers an advance and complemented procedure to reduce concept lattices, selecting and removing appropriate new attributes.

4 Conclusions and Future Work

In this paper, we have addressed an initial study about the relation between the equivalence classes provided by both an attribute reduction and the least local congruence containing such a reduction in FCA. In particular, we have analyzed more in detail the cases when the induced equivalence relation does not provide convex sublattices as equivalence classes and the behavior of the local congruence when we use it in these cases. As a consequence, we have observed that the use of local congruence relations modifies the subset of unconsidered attributes. Moreover, we have stated conditions on the attribute-concepts and the meet-irreducible elements of the concept lattice associated with a context in order to detect when an equivalence class is not a convex sublattice. All ideas presented in this paper have been illustrated by means of different examples.

As future work, we are interested in continuing the study of influence of local congruences in the attribute reduction of a dataset. For example, we will analyze the relationship of the use of local congruences with attribute implications and how the removed attributes can be recovered from the set of attribute implications associated with the context. Furthermore, we will explore the ideas presented in this paper in the fuzzy framework of the multi-adjoint concept lattices.

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Anexo D

Characterization of the infimum of classes induced by an attribute reduction in FCA

Characterization of the Infimum of Classes Induced by an Attribute Reduction in FCA

Roberto G. Aragón, Jesús Medina, and Eloísa Ramírez-Poussa

Abstract Attribute reduction is a topic of interest in data analysis. In particular, in formal concept analysis attribute reductions are associated with equivalence relations defined on concept lattices. In this paper, we study the equivalence relations induced by attribute reductions with the goal of characterizing when the equivalence classes are not convex sublattices of the original concept lattice.

Keywords Formal concept analysis · Equivalence relations · Attribute reduction

1 Introduction

Formal Concept Analysis (FCA) is based on the lattice theory [9] and in which the notion of Galois connection is fundamental [10, 14, 15]. Two of the main features of this mathematical framework is that a hierarchy of the information contained in a dataset can be given and dependencies between different sets of attributes of the context can be determined as well. In order to efficiently obtain information, in many cases, redundant information must be removed or/and the size of the concept lattice needs to be reduced. Both problems have been studied in diverse papers [1, 2, 6, 8, 11–13, 16].

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It has been proved that every attribute reduction induces an equivalence relation whose equivalence classes have the structure of join-semilattices [7]. In order to acquire equivalence classes with the structure of sublattices under an attribute reduction procedure, local congruences were introduced [4, 5] and applied after the reduction. This procedure must be complemented with the study of necessary and sufficient conditions on the concept lattice in order to characterize when these classes are not sublattices, in which the infimum of the elements in each class plays a fundamental role since the class is a sublattice if and only if this element is in the class. This research line began in [3] in which different examples and an interesting result were presented. In this paper, we continue this study providing diverse results focused on the characterization of the infimum element of a non-sublattice equivalence class of the equivalence relation obtained after an attribute reduction. In particular, these results provide an improved version of Corollary 1 given in [3]. Moreover, we relate these infimum elements to the unnecessary attributes of the formal context.

2 Preliminaries

In this section, we will recall some basic notions about formal concept analysis and attribute reduction. A context in FCA is a triple (A, B, R) where A is a set of attributes, B is a set of objects and $R \subseteq A \times B$ is a relationship, such that $(a, x) \in R$ (denoted as aRx), if the object $x \in B$ possesses the attribute $a \in A$, and $(a, x) \notin R$, otherwise. The *derivation operators* are the mappings ${}^{\uparrow}\colon 2^B \to 2^A$ and ${}^{\downarrow}\colon 2^A \to 2^B$ defined for each $X \subseteq B$ and $Y \subseteq A$ as:

$$X^{\uparrow} = \{a \in A \mid \text{for all } x \in X, aRx\} \tag{1}$$

$$Y^{\downarrow} = \{x \in B \mid \text{for all } a \in Y, aRx\} \tag{2}$$

Given the previous mappings, a *concept* is a pair $C = (X, Y)$, with $X \subseteq B$ and $Y \subseteq A$ satisfying that $X^{\uparrow} = Y$ and $Y^{\downarrow} = X$. The subset X is called the *extent* of the concept and the subset Y is called the *intent*, and they are denoted by $\mathfrak{E}(C)$ and $\mathfrak{I}(C)$, respectively.

The whole set of concepts is denoted as $\mathcal{C}(A, B, R)$ and, with the inclusion ordering on the left argument, provides an ordering $\leq$, such that $(\mathcal{C}(A, B, R), \leq)$ is a complete lattice, called *concept lattice* of the context (A, B, R). In addition, a concept generated by an attribute $a \in A$, that is $(a^{\downarrow}, a^{\downarrow\uparrow})$, is called an *attribute-concept*. Now, we recall the notion of meet-irreducible element of a lattice.

Definition 1 Given a lattice $(L, \preceq)$, such that $\wedge$ is the meet operator, and an element $x \in L$ verifying

1. If L has a top element $\top$, then $x \neq \top$.
2. If $x = y \wedge z$, then $x = y$ or $x = z$, for all $y, z \in L$.

The set of attributes of the context are closely related to the meet-irreducible concepts. In particular, we will use in this paper the notion of unnecessary attribute and, in particular, the following characterization introduced in [16].

Theorem 1 *Given a formal context* (A, B, R)*, an attribute* $a \in A$ *and the set of* $\wedge$*-irreducible elements of* $\mathcal{C}(A, B, R)$*, denoted as* $M_F(A, B, R)$*. Then, a is an absolutely unnecessary attribute,* $a \in I_f$*, if and only if* $(a^{\downarrow}, a^{\downarrow\uparrow}) \notin M_F(A, B, R)$*.*

With respect to the attribute reduction in FCA, we recall the main results related to the induced equivalence relation on the set of concepts of the original concept lattice when we reduce the set of attributes in a context. For more detailed information about the notions and results related to attribute reduction in FCA, we refer the reader to [3, 7]. The following proposition was proved in [7] for the classical setting of FCA.

Proposition 1 ([7]) *Given a context* (A, B, R) *and a subset* $D \subseteq A$*. The set* $\rho_D = \{((X_1, Y_1), (X_2, Y_2)) \mid (X_1, Y_1), (X_2, Y_2) \in \mathcal{C}(A, B, R), X_1^{\uparrow_D\downarrow} = X_2^{\uparrow_D\downarrow}\}$ *is an equivalence relation. Where* $^{\uparrow_D}$ *denotes the concept-forming operator given in Expression (2), restricted to the subset of attributes* $D \subseteq A$*.*

Moreover, the authors also proved that each equivalence class of the induced equivalence relation has a structure of join-semilattice and they also determined the maximum element.

Proposition 2 ([7]) *Given a context* (A, B, R)*, a subset* $D \subseteq A$ *and a class* $[(X, Y)]_D$ *of the quotient set* $\mathcal{C}(A, B, R)/\rho_D$*. The class* $[(X, Y)]_D$ *is a join-semilattice with maximum element* $(X^{\uparrow_D\downarrow}, X^{\uparrow_D\downarrow\uparrow})$*.*

Finally, the next result was presented in [3] and it relates the equivalence relations induced by an attribute reduction to the attributes-concepts and the meet-irreducible elements of the concept lattice.

Proposition 3 ([3]) *Given a context* (A, B, R)*, a subset of attributes* $D \subseteq A$*, an equivalence class* $[C]_D$*, with* $C \in \mathcal{C}(A, B, R)$*, of the induced equivalence relation which is not a convex sublattice and the concept* $C_m = \bigwedge_{C_i \in [C]_D} C_i$*. Then, one of the following statements is satisfied:*

- *There exists at least one attribute* $a \in D$ *such that* $C_m = (a^{\downarrow}, a^{\downarrow\uparrow})$*.*
- *There exists a concept* $C^* \in M_F(A, B, R)$ *in a meet-irreducible decomposition* $\{C_j \in M_F(A, B, R) \mid j \in J\}$ *of* C_m*, such as* $C_{i_0} \not\leq C^*$ *for a concept* $C_{i_0} \in [C]_D$*.*

Notice that, C_m is the infimum of the subset of concepts $[C]_D$. Next, the two possible cases of C_m will be analyzed in depth.

3 Characterizing the Infimum of Classes

In this section, we continue with the study on the equivalence classes induced by a reduction of attributes presented in [3], and in particular with Proposition 3, since it is primordial to know when the infimum C_m of $[C]_D$ belongs to the class, due to a join-semilattice with a least element is a lattice.

From now on, a context (A, B, R) will be fixed. Moreover, the maximum element of a class $[C]_D$, with $C = (X, Y) \in \mathcal{C}(A, B, R)$, which Proposition 2 characterized as $(X^{\uparrow_D\downarrow}, X^{\uparrow_D\downarrow\uparrow})$, will be denoted by $C_M = (X_M, Y_M)$. Notice that, since $X_M = X^{\uparrow_D\downarrow}$, then this extent is also the extent of a concept of the reduced concept lattice $\mathcal{C}(D, B, R_{|D\times B})$.

Now, we determine a sufficient condition to ensure that the equivalence class of the infimum element of an induced equivalence class that is not a convex sublattice is generated by an attribute.

Theorem 2 *Let (A, B, R) be a context, a finite subset of attributes $D \subseteq A$, and $C \in \mathcal{C}(A, B, R)$ such that $C_j \in [C]_D$, for all concept C_j in a meet-irreducible decomposition $\{C_j \in M_F(A, B, R) \mid j \in J\}$ of $C_m = \bigwedge_{C_i\in[C]_D} C_i$. If C_m is not in $[C]_D$, then there exists an attribute $a \in D$ such that $[C_m]_D = [(a^\downarrow, a^{\downarrow\uparrow})]_D$.*

Note that Theorem 2 arises from the restriction of the hypotheses of Proposition 3. The following example will be useful to illustrate the previous result. This example also inspects Corollary 1 presented in [3] and, as a consequence, it also argues that Theorem 2 must be considered instead of this corollary.

Example 1 We will consider a context composed of the set of attributes $A = \{a_1, a_2, a_3, a_4\}$ and the set of objects $B = \{b_1, b_2, b_3\}$, related by $R\colon A \times B \to \{0, 1\}$, defined on the left side of Table 1 together with the list of the corresponding concepts which appear on the right side of the same table. The associated concept lattice is given on the left side of Fig. 1.

From this context, we obtain the attribute-concepts listed below, together with the induced equivalence classes obtained by removing attributes a_2 and a_3, that is, considering only the subset of attributes $D = \{a_1, a_4\}$.

$$C_3 = (a_4^\downarrow, a_4^{\downarrow\uparrow}) \qquad [C_0]_D = \{C_0\}$$
$$C_4 = (a_1^\downarrow, a_1^{\downarrow\uparrow}) \qquad [C_1]_D = [C_2]_D = [C_4]_D = \{C_1, C_2, C_4\}$$
$$C_5 = (a_2^\downarrow, a_2^{\downarrow\uparrow}) \qquad [C_3]_D = \{C_3\}$$
$$C_6 = (a_3^\downarrow, a_3^{\downarrow\uparrow}) \qquad [C_5]_D = [C_6]_D = [C_7]_D = \{C_5, C_6, C_7\}$$

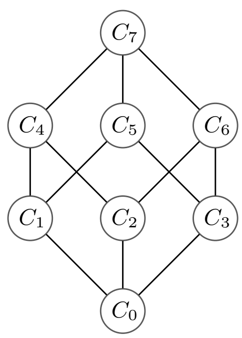
The partition induced by such a reduction is shown on the right side of Fig. 1. Notice that two of the obtained equivalence classes are not convex sublattices of the original concept lattice. The first one contains the concepts C_1, C_2, C_4 and the other one contains the concepts C_5, C_6, C_7. However, the reasons that make these classes are not convex sublattices are well differentiated.

On the one hand, if we consider the equivalence class $[C_7]_D$, we have that the infimum of the concepts of this class is the concept C_3. Notice that the meet-irreducible decomposition of C_3 is $C_3 = C_5 \wedge C_6$ and both concepts C_5 and C_6 belong to $[C_7]_D$, this means that we are under the conditions given in Theorem 2. Since $C_3 \notin [C_7]_D$, we have that $[C_3]_D = [(a^\downarrow, a^{\downarrow\uparrow})]_D$, with $a \in D$. Specifically, in this case we have that the concept C_3 is just generated by the attribute $a_4 \in D$. Notice that the C_m is not always an attribute concept. For example, if we consider $D' = \{a_4\}$, then we obtain

Table 1 Relation and list of concepts of the context of Example 1

R	b_1	b_2	b_3
a_1	1	1	0
a_2	1	0	1
a_3	0	1	1
a_4	0	0	1

C_i	Extent			Intent			
	b_1	b_2	b_3	a_1	a_2	a_3	a_4
0	0	0	0	1	1	1	1
1	1	0	0	1	1	0	0
2	0	1	0	1	0	1	0
3	0	0	1	0	1	1	1
4	1	1	0	1	0	0	0
5	1	0	1	0	1	0	0
6	0	1	1	0	0	1	0
7	1	1	1	0	0	0	0



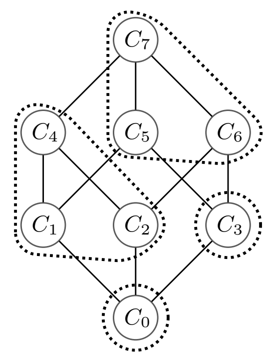


Fig. 1 Concept lattice of Example 1 (left) and the partition induced by the elimination of attributes a_2 and a_3 in Example 1 (right)

two classes: $[C_7]_{D'}$ and $[C_3]_{D'}$, satisfying that $C_0 = C_4 \wedge C_5 \wedge C_6$, which belong to $[C_7]_{D'}$. Therefore, the hypotheses of Theorem 2 hold, indeed, $[C_0]_{D'} = [(a_4^{\downarrow}, a_4^{\downarrow\uparrow})]_{D'}$, however C_0 is not an attribute concept.

On the other hand, if we consider the equivalence class $[C_4]_D$, we have that the infimum of the equivalence class $[C_4]_D$ is the concept C_0. In this case, the decomposition of C_0 is $C_0 = C_4 \wedge C_5 \wedge C_6$, we observe that there are two meet-irreducible concepts, C_5 and C_6, such that $C_5, C_6 \notin [C_4]_D$. Therefore we cannot apply Theorem 2 since the hypothesis are not satisfied. Moreover, since the concept lattice $\mathcal{C}(A, B, R)$ is distributive, the condition that the meet-irreducible concepts of the decomposition be in the class is the required condition in [3, Corollary 1]. Moreover, we only can ensure $[C_m]_D = [(a^{\downarrow}, a^{\downarrow\uparrow})]_D$. Thus, Theorem 2 presents an improved version of Corollary 1 given in [3]. □

The following result shows that statement 1 in Proposition 3 only arises when the context contains unnecessary attributes.

Proposition 4 *Given a context (A, B, R), a subset of attributes $D \subseteq A$, a concept $C \in \mathcal{C}(A, B, R)$, the equivalence class $[C]_D$ of the induced equivalence relation is not a singleton and the concept $C_m = \bigwedge_{C_i \in [C]_D} C_i$. If there exists $a \in D$ such that $C_m = (a^{\downarrow}, a^{\downarrow\uparrow})$, then $a \in I_f$.*

As a consequence, we can characterize the equivalence classes that are not convex sublattices of the original concept lattice, if we do not take into consideration the absolutely unnecessary attributes of the original context, reaching further than Proposition 3.

Theorem 3 *Given a context (A, B, R), a subset of attributes $D \subseteq A \backslash I_f$, an equivalence class $[C]_D$ with $C \in \mathcal{C}(A, B, R)$, of the induced equivalence relation and the concept $C_m = \bigwedge_{C_i \in [C]_D} C_i$. Then, $C_m \notin [C]_D$ if and only if there exists a concept $C^* \in M_F(A, B, R)$ in a meet-irreducible decomposition $\{C_j \in M_F(A, B, R) \mid j \in J\}$ of C_m, such that $C^* = (a^{*\downarrow}, a^{*\downarrow\uparrow})$ with $a^* \in D$, and $C_M \nleq C^*$.*

The previous results have a relevant interest for the application of local congruences, since they characterize the cases when the classes are not sublattices [3, 7] and so, what classes are affected when a local congruence is applied after an attribute reduction mechanism.

4 Conclusions and Future Work

In this paper, we have characterized the equivalence classes induced by an attribute reduction in FCA. Specifically, we have introduced a sufficient condition in order to ensure that the class of the infimum element is the class of an attribute concept of an attribute in D. We have also shown that if this infimum element is an attribute concept, then the considered attribute is an unnecessary attribute. Finally, we have stated a characterization of the equivalence classes that are not sublattices of the original concept lattice, which is very interesting, for example, when local congruences complement an attribute reduction process. Since local congruences group different classes for obtaining convex sublattices, from these results we can know the impact of the application of a local congruence in the original context.

As a future research line, we are interested in the study of the impact of local congruences on the considered context, when they are used to complement an attribute reduction mechanism.

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