Bull Braz Math Soc, New Series (2023) 54:4
https://doi.org/10.1007/s00574-022-00322-8

The Buchweitz Set of a Numerical Semigroup

Shalom Eliahou[1,2] · **Juan Ignacio García-García**[3,5] · **Daniel Marín-Aragón**[3] · **Alberto Vigneron-Tenorio**[4,5]

Received: 14 June 2022 / Accepted: 13 October 2022

Abstract

Let $A \subset \mathbb{Z}$ be a finite subset. We denote by $\mathcal{B}(A)$ the set of all integers $n \geq 2$ such that $|nA| > (2n-1)(|A|-1)$, where $nA = A + \cdots + A$ denotes the n-fold sumset of A. The motivation to consider $\mathcal{B}(A)$ stems from Buchweitz's discovery in 1980 that if a numerical semigroup $S \subseteq \mathbb{N}$ is a Weierstrass semigroup, then $\mathcal{B}(\mathbb{N}\setminus S) = \emptyset$. By constructing instances where this condition fails, Buchweitz disproved a longstanding conjecture by Hurwitz (Math Ann 41:403–442, 1893). In this paper, we prove that for any numerical semigroup $S \subset \mathbb{N}$ of genus $g \geq 2$, the set $\mathcal{B}(\mathbb{N}\setminus S)$ is finite, of unbounded cardinality as S varies.

Keywords Weierstrass numerical semigroup · Gapset · Additive combinatorics · Sumset growth · Freiman's $3k-3$ theorem

Mathematics Subject Classification 11P70 · 20M14 · 14H55

To the memory of our friend and colleague Fernando Torres (1961–2020).

✉ Juan Ignacio García-García
ignacio.garcia@uca.es

Shalom Eliahou
eliahou@univ-littoral.fr

Daniel Marín-Aragón
daniel.marin@uca.es

Alberto Vigneron-Tenorio
alberto.vigneron@uca.es

1 Univ. Littoral Côte d'Opale, UR 2597-LMPA-Laboratoire de Mathématiques Pures et Appliquées Joseph Liouville, 62100 Calais, France

2 CNRS, FR2037, Calais, France

3 Departamento de Matemáticas, Universidad de Cádiz, 11510 Puerto Real, Cádiz, Spain

4 Departamento de Matemáticas, Universidad de Cádiz, 11406 Jerez de la Frontera, Cádiz, Spain

5 INDESS (Instituto Universitario para el Desarrollo Social Sostenible), Universidad de Cádiz, 11406 Jerez de la Frontera, Cádiz, Spain

Published online: 06 December 2022

1 Introduction

Denote $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ and $\mathbb{N}_+ = \mathbb{N}\setminus\{0\} = \{1, 2, 3, \ldots\}$. For $a, b \in \mathbb{Z}$, let $[a, b[= \{z \in \mathbb{Z} \mid a \le z < b\}$ and $[a, \infty[= \{z \in \mathbb{Z} \mid a \le z\}$ denote the integer intervals they span. A *numerical semigroup* is a subset $S \subseteq \mathbb{N}$ containing 0, stable under addition and with finite complement in $\mathbb{N}$. Equivalently, it is a subset of $\mathbb{N}$ of the form $S = \langle a_1, \ldots, a_n\rangle = \mathbb{N}a_1 + \cdots + \mathbb{N}a_n$ where $\gcd(a_1, \ldots, a_n) = 1$. The set $\{a_1, \ldots, a_n\}$ is then called a *system of generators* of S, and the smallest such n is called the *embedding dimension* of S.

For a numerical semigroup S, its corresponding *gapset* is the complement $G = \mathbb{N}\setminus S$, its *genus* is $g = |G|$, its *multiplicity* is $m = \min S^*$ where $S^* = S\setminus\{0\}$, its *Frobenius number* is $f = \max(\mathbb{Z}\setminus S)$ and its *conductor* is $c = f + 1$. Thus $[c, \infty[\subseteq S$ and c is minimal for this property. Finally, the *depth* of S is $q = \lceil c/m \rceil$.

Given a finite subset $A \subset \mathbb{N}$, we denote by $nA = A + \cdots + A$ the *n-fold sumset of A*. See Sect. 2 for more details.

Definition 1.1 Let $A \subset \mathbb{Z}$ be a finite subset. We associate to A the function $\beta = \beta_A : \mathbb{N}_+ \to \mathbb{Z}$ defined for all $n \ge 1$ by

$$\beta_A(n) = |nA| - (2n - 1)(|A| - 1).$$

Notation 1.2 *We denote by* $\mathcal{B}(A)$ *the* positive support *in* $2 + \mathbb{N}$ *of the function* β_A, *i.e.*

$$\mathcal{B}(A) = \{n \ge 2 \mid \beta_A(n) \ge 1\}.$$

For instance, $2 \in \mathcal{B}(A)$ if and only if $|2A| \ge 3|A| - 2$. Interestingly, the failure of this condition, namely the inequality $|2A| \le 3|A| - 3$, is the key hypothesis of the famous Freiman's $3k - 3$ Theorem in additive combinatorics (Freiman 1959).

Example 1.3 If $|A| = 0$ or 1, then $\mathcal{B}(A)$ is infinite. Indeed, if $A = \emptyset$, then $|nA| = 0$ and so $\beta_\emptyset(n) = 2n - 1$ for all $n \ge 1$. Thus $\mathcal{B}(\emptyset) = 2 + \mathbb{N}$ in that case. Similarly, if $|A| = 1$, then $\beta_A(n) = 1$ for all $n \ge 1$. So here again $\mathcal{B}(A) = 2 + \mathbb{N}$.

In sharp contrast, Theorem 3.3 below states that *if* $S \subset \mathbb{N}$ *is a numerical semigroup of genus* $g \ge 2$, *then* $\mathcal{B}(\mathbb{N}\setminus S)$ *is finite*.

Example 1.4 Let $S = \langle 3, 7\rangle$. Then $\mathbb{N}\setminus S = \{1, 2, 4, 5, 8, 11\}$ and $\beta_{\mathbb{N}\setminus S}(n) = 0$ for all $n \ge 2$ as easily seen. In particular, $\mathcal{B}(\mathbb{N}\setminus S) = \emptyset$.

More generally, it was shown in Komeda (1998) and Oliveira (1991) that $\beta_{\mathbb{N}\setminus S}(n) = 0$ for all *symmetric* numerical semigroups S of multiplicity $m \ge 3$ and all $n \ge 2$. We shall not use this result below, but instead give a short self-contained proof of an immediate consequence, namely that $\mathcal{B}(\mathbb{N}\setminus S)$ is empty in that case.

In fact, $\mathcal{B}(\mathbb{N}\setminus S)$ is empty in most cases. Indeed, Buchweitz discovered in 1980 that the condition $\mathcal{B}(\mathbb{N}\setminus S) = \emptyset$ is necessary for S to be a Weierstrass semigroup. By constructing instances where this condition fails, Buchweitz (1980) was able to negate the longstanding conjecture by Hurwitz (1893) according to which all numerical semigroups of genus $g \geq 2$ are Weierstrass semigroups. His first counterexample was $S = \langle 13, 14, 15, 16, 17, 18, 20, 22, 23 \rangle$, with corresponding gapset

$$G = \mathbb{N} \backslash S = [1, 12] \cup \{19, 21, 24, 25\}$$

of cardinality 16. Then $2G = [2, 50] \backslash \{39, 41, 47\}$, so that $|2G| = 46$ and $\beta_G(2) = 46 - 3 \cdot 15 = 1$, implying $2 \in \mathcal{B}(G)$ and thus impeding S to be a Weierstrass semigroup. For more information on Buchweitz's condition and Weierstrass semigroups, see e.g. Eisenbud and Harris (1987) and Kaplan and Ye (2013).

Here are the contents of this paper. In Sect. 2, we recall a result of Nathanson in additive combinatorics and we use it to study the asymptotic behavior of the function $\beta_A(n)$. In Sect. 3, we introduce the Buchweitz set of a numerical semigroup and we prove our main results. Section 4 concludes the paper with open questions on the possible shapes of the sets $\mathcal{B}(A)$.

2 Sumset Growth

Given finite subsets A, B of a commutative monoid $(M, +)$, we denote as usual

$$A + B = \{a + b \mid a \in A, b \in B\},$$

the *sumset* of A, B, and $2A = A + A$. More generally, if $n \geq 2$, we denote $nA = A + (n-1)A$, where $1A = A$. The set nA is called the n-fold sumset of A.

A classical question in additive combinatorics is, how does $|nA|$ grow with n? Here we only consider the case $M = \mathbb{Z}$. We shall need the following result of Nathanson (1996, Theorem 1.1).

Theorem 2.1 *Let $A_0 \subset \mathbb{N}$ be a finite subset of cardinality $k \geq 2$, containing 0 and such that $\gcd(A_0) = 1$. Let $a_0 = \max(A_0)$. Then there exist integers c, d and subsets $C \subseteq [0, c-2]$, $D \subseteq [0, d-2]$ such that*

$$nA_0 = C \sqcup [c, a_0 n - d] \sqcup (a_0 n - D)$$

for all $n \geq \max\{(|A_0| - 2)(a_0 - 1)a_0, 1\}$.

As pointed out in Nathanson (1996), the hypotheses $0 \in A_0$ and $\gcd(A_0) = 1$ are not really restrictive. Indeed, for any finite set $A \subset \mathbb{Z}$ with $|A| \geq 2$, the simple transformation $A \mapsto A_0 = (A - \alpha)/d$, where $\alpha = \min(A)$ and $d = \gcd(A - \alpha)$, yields a set A_0 satisfying these hypotheses and such that $|nA_0| = |nA|$ for all n. In view of our applications to gapsets, we shall need the following version.

Corollary 2.2 *Let $A \subset \mathbb{N}_+$ be a finite subset containing $\{1, 2\}$. Let $a = \max(A)$. Then there is an integer $b \leq 1$ such that*

$$|nA| = (a-1)n + b$$

for all $n \geq (|A| - 2)(a - 2)(a - 1)$.

Proof Set $A_0 = A - 1$ and $a_0 = a - 1$. Then A_0 contains $\{0, 1\}$, hence it satisfies the hypotheses of Theorem 2.1. Using the same notation, its conclusion implies

$$|nA_0| = a_0 n + b \tag{1}$$

for all $n \geq \max\{(|A_0| - 2)(a_0 - 1)a_0, 1\}$, where

$$b = |C| + |D| - c - d + 1.$$

Note that $b \leq 1$ since $|C| \leq \max(0, c - 1)$, $|D| \leq \max(0, d - 1)$. The desired statement follows from (1) since $|nA| = |nA_0|$ for all $n \geq 0$. □

2.1 Asymptotic Behavior of $\beta_A(n)$

We now study the evolution of $\beta_A(n)$ as n grows.

Theorem 2.3 *Let $A \subset \mathbb{N}_+$ be a finite set containing $\{1, 2\}$. Let $f = \max(A)$ and $g = |A|$. Then*

$$\lim_{n\to\infty} \beta_A(n) = \begin{cases} -\infty & \text{if } f \leq 2g - 2, \\ +\infty & \text{if } f \geq 2g. \end{cases}$$

Finally if $f = 2g - 1$, then $\beta_A(n)$ is constant and nonpositive for n large enough.

Proof By Corollary 2.2, we have $|nA| = (f - 1)n + b$ for some integer $b \leq 1$ and for n large enough. Hence

$$\begin{aligned} \beta_A(n) &= (f - 1)n + b - (2n + 1)(g - 1) \\ &= (f - 2g + 1)n + b + 1 - g \end{aligned}$$

for n large enough. The claims for $f \leq 2g - 2$ and $f \geq 2g$ follow. If $f = 2g - 1$, then $\beta_A(n) = b + 1 - g \leq 0$ for n large enough, since $b \leq 1$ and $g \geq 2$. □

Corollary 2.4 *Let $A \subseteq \mathbb{N}_+$ be a finite set containing $\{1, 2\}$. Let $f = \max(A)$ and $g = |A|$. Then $\mathcal{B}(A)$ is finite if and only if $f \leq 2g - 1$.*

Proof If $f \geq 2g$, then $\lim_{n\to\infty} \beta_A(n) = \infty$ by the theorem, whence $\beta_A(n) \geq 1$ for all large enough n. Thus $\mathcal{B}(A)$ is infinite in this case. If $f \leq 2g - 1$, the theorem implies $\beta_A(n) \leq 0$ for n large enough, whence $\mathcal{B}(A)$ is finite in that case. □

3 Application to Numerical Semigroups

Definition 3.1 Let $S \subseteq \mathbb{N}$ be a numerical semigroup. We define the *Buchweitz set of* S as $\mathrm{Buch}(S) = \mathcal{B}(\mathbb{N} \setminus S)$. Explicitly, setting $G = \mathbb{N} \setminus S$, we have

$$\begin{aligned} \mathrm{Buch}(S) &= \{n \geq 2 \mid |nG| > (2n - 1)(|G| - 1)\} \\ &= \{n \geq 2 \mid \beta_G(n) \geq 1\}. \end{aligned}$$

In this section, we first prove that $\mathrm{Buch}(S)$ is *finite* for all numerical semigroups S of genus $g \geq 2$. We then show, by explicit construction, that the cardinality of $\mathrm{Buch}(S)$ may be arbitrarily large.

3.1 Finiteness of Buch(S)

We start with a well known inequality linking the Frobenius number and the genus of a numerical semigroup.

Proposition 3.2 *Let $S \subset \mathbb{N}$ be a numerical semigroup with Frobenius number f and genus $g \geq 1$. Then $f \leq 2g - 1$.*

Proof Let $x \in S \cap [0, f]$. Then $f - x \notin S$ since S is stable under addition and $x + (f - x) = f \notin S$. Hence, the map $x \mapsto f - x$ induces an injection

$$S \cap [0, f] \hookrightarrow \mathbb{N} \setminus S.$$

Since $|S \cap [0, f]| = (f + 1) - g$, it follows that $f \leq 2g - 1$, as claimed. □

Recall that S is said to be *symmetric* if $|S \cap [0, f]| = |\mathbb{N} \setminus S|$, i.e. if $f = 2g - 1$. A classical result of Sylvester states that any numerical semigroup of the form $S = \langle a, b \rangle$ with $\gcd(a, b) = 1$ is symmetric.

Theorem 3.3 *Let $S \subseteq \mathbb{N}$ be a numerical semigroup of genus $g \geq 2$. Then* $\mathrm{Buch}(S)$ *is finite.*

Proof Let $G = \mathbb{N} \setminus S$. Then $\mathrm{Buch}(S) = \mathcal{B}(G)$ by definition. We have $g = |G| \geq 2$. Let $f = \max(G)$ be the Frobenius number of S. Let $m = \min(S \setminus \{0\})$ be the multiplicity of S. Then $m \geq 2$ since $g \geq 2$, and $[1, m - 1] \subseteq G$.

Assume first $m \geq 3$. Then $\{1, 2\} \subseteq G$. Hence Corollary 2.4 applies, and since $f \leq 2g - 1$ by Proposition 3.2, it yields that $\mathcal{B}(G)$ is finite, as desired.

Assume now $m = 2$. Then $S = \langle 2, b \rangle$ with b odd and $b \geq 5$ since $|G| \geq 2$. At this point, we might conclude the proof right away using what is known in the symmetric case (Komeda 1998; Oliveira 1991). However, for the convenience of the reader, let us give a short self-contained argument. We have $G = \{1, 3, \ldots, b - 2\}$, i.e. all odd numbers from 1 to $b - 2$. Hence $G - 1 = \{0, 2, \ldots, b - 3\}$ and $\gcd(G - 1) = 2$. Set $A = (G - 1)/2 = [0, k]$, where $k = (b - 3)/2$. For all $n \geq 1$, we have

$$|nG| = |nA| = |nk + 1| = n(|G| - 1) + 1.$$

Therefore $\beta_G(n) = -(n - 1)|G| + n$, whence $\beta_G(n) \leq 0$ for all $n \geq 2$. It follows that $\mathcal{B}(G) = \emptyset$ and we are done. □

3.2 Unboundedness of | Buch(S)|

We show here, by explicit construction, that $|\mathrm{Buch}(S)|$ may be arbitrarily large.

Proposition 3.4 *For any integer $b \geq 3$, there exists a numerical semigroup S such that* $\mathrm{Buch}(S) = [2, b]$.

Proof Let $k = b-2$, and let S be the numerical semigroup of multiplicity $m = 6k+15$ and depth $q = 2$ whose corresponding gapset $G = \mathbb{N}\setminus S$ is given by

$$G = [1, m-1] \sqcup \{2m-7, 2m-5, 2m-2, 2m-1\}. \tag{2}$$

We claim that $\mathrm{Buch}(S) = [2, k+2]$. Indeed, we will show a more precise statement, namely

$$\beta_G(n) = \begin{cases} 1 & \text{if } n = 2, \\ 2 & \text{if } 3 \leq n \leq k+2, \\ -6(n-k-3) & \text{if } n \geq k+3. \end{cases}$$

Let $A = (2m-1) - G$. Then $\beta_G(n) = \beta_A(n)$ since $|nG| = |nA|$ for all $n \geq 1$. We have

$$A = [0, 1] \sqcup \{4, 6\} \sqcup [m, 2(m-1)].$$

Let us compute $2A$ and $3A$. We obtain

$$\begin{aligned} 2A &= [0, 2] \sqcup [4, 8] \sqcup \{10, 12\} \sqcup [m, 4(m-1)], \\ 3A &= [0, 14] \cup \{16, 18\} \cup [m, 6(m-1)]. \end{aligned}$$

In general, we have

$$nA = ([0, 6n-4] \sqcup \{6n-2, 6n\}) \cup [m, 2n(m-1)] \tag{3}$$

for all $n \geq 3$, as easily verified by induction on n.

Let us determine $|nA|$ for all $n \geq 1$. Note first that *the union in* (3) *is disjoint if and only if* $6n+1 \leq m$. Moreover, as $m = 6k+15$, we have

$$6n+1 \leq m \iff n \leq k+2.$$

In contrast, if $n \geq k+3$, i.e. if $6n-3 \geq m$, then the union in (3) collapses to a single interval and we get

$$nA = [0, 2n(m-1)].$$

Summarizing, we have

$$|nA| = \begin{cases} m+3 & \text{if } n = 1, \\ 3m+7 & \text{if } n = 2, \\ (2n-1)(m-1)+6n-1 & \text{if } 3 \leq n \leq k+2, \\ 2n(m-1)+1 & \text{if } n \geq k+4. \end{cases}$$

The stated formula for $\beta_G(n) = \beta_A(n) = |nA| - (2n-1)(|A|-1)$ follows. Hence $\mathrm{Buch}(S) = [2, k+2]$, as claimed. $\square$

This family of numerical semigroups was inspired by the *PF-semigroups* introduced in García-García et al. (2021).

3.3 More Intervals

What are the possible shapes of $\mathrm{Buch}(S)$ when S varies? We do not know in general. By Proposition 3.4, any finite integer interval I with $|I| \geq 2$ and $\min(I) = 2$ may be realized as $I = \mathrm{Buch}(S)$ for some numerical semigroup S. Here we present families of numerical semigroups S realizing as $\mathrm{Buch}(S)$ all finite integer intervals I with $|I| \geq 2$ and $\min(I) \in \{3, 4, 5, 6\}$.

Proposition 3.5 *Let $k \geq 1$. Let S be the numerical semigroup of multiplicity $m = 6k + 19$ and depth $q = 2$ whose corresponding gapset $G = \mathbb{N} \setminus S$ is given by*

$$G = [1, m-1] \sqcup \{2m-7, 2m-6, 2m-2, 2m-1\}. \tag{4}$$

Then $\mathrm{Buch}(S) = [3, k+3]$.

Proof Let again $A = (2m-1) - G = [0, 1] \sqcup \{5, 6\} \sqcup [m, 2(m-1)]$. We then have

$$\begin{aligned} 2A &= [0, 2] \sqcup [5, 7] \sqcup [10, 12] \sqcup [m, 4(m-1)], \\ 3A &= [0, 3] \sqcup [5, 8] \sqcup [10, 13] \sqcup [15, 18] \sqcup [m, 6(m-1)], \\ 4A &= [0, 24] \cup [m, 8(m-1)]. \end{aligned}$$

It follows that $nA = [0, 6n] \cup [m, 2n(m-1)]$ for all $n \geq 4$. In particular, if $6n \geq m$ then $nA = [0, 2n(m-1)]$. Therefore,

$$\beta_G(n) = \beta_A(n) = \begin{cases} 0 & \text{if } n = 2, \\ 1 & \text{if } n = 3, \\ 4 & \text{if } 4 \leq n \leq k+3, \\ 6k - 6n + 22 & \text{if } n \geq k+4. \end{cases}$$

Hence $\mathrm{Buch}(S) = [3, k+3]$, as claimed. $\square$

Proposition 3.6 *For $k \geq 1$ and $i \in \{1, 2, 3\}$, let S_i be the numerical semigroup with $G_i = \mathbb{N} \setminus S_i$ given by*

$$\begin{aligned} G_1 &= [1, m_1 - 1] \sqcup \{2m_1 - 6, 2m_1 - 2, 2m_1 - 1\}, \\ G_2 &= [1, m_2 - 1] \sqcup \{2m_2 - 10, 2m_2 - 4, 2m_2 - 3, 2m_2 - 2\}, \\ G_3 &= [1, m_3 - 1] \sqcup \{2m_3 - 10, 2m_3 - 9, 2m_3 - 2\}, \end{aligned}$$

where $m_1 = 4k + 22$, $m_2 = 7k + 44$ and $m_3 = 5k + 55$, respectively. Then

$$\mathrm{Buch}(S_1) = [4, k+4], \quad \mathrm{Buch}(S_2) = [5, k+5], \quad \mathrm{Buch}(S_3) = [6, k+6].$$

Proof Similar to the proofs of Propositions 3.4 and 3.5. We omit it here. □

Having realized all finite integer intervals I with $|I| \geq 2$ and $\min(I) \in [2, 6]$ as $I = \mathrm{Buch}(S)$ for a suitable numerical semigroups S, is it possible to do the same for all finite integer intervals I with $\min(I) \geq 7$? We do not know in general. But here is a particular case where $\min(I)$ can be arbitrarily large. It is based on a family of numerical semigroups found in Komeda (1998).

Proposition 3.7 *For any integer $k \geq 1$, there is a numerical semigroup S such that* $\mathrm{Buch}(S) = [7 + 2k, 7 + 4k]$.

Proof For $k \geq 1$, let S be the numerical semigroup minimally generated by the set $T_1 \cup T_2 \cup T_3$, where

$$\begin{aligned} T_1 &= [44 + 27k + 4k^2, 79 + 51k + 8k^2],\\ T_2 &= [81 + 51k + 8k^2, 84 + 53k + 8k^2],\\ T_3 &= [87 + 53k + 8k^2, 87 + 54k + 8k^2]. \end{aligned}$$

The corresponding gapset $G = \mathbb{N} \setminus S$ is then given by

$$G = [1, 43 + 27k + 4k^2] \cup \{80 + 51k + 8k^2, 85 + 53k + 8k^2, 86 + 53k + 8k^2\}.$$

Let $A = (86 + 53k + 8k^2) - G$. Then

$$A = [0, 1] \sqcup \{6 + 2k\} \sqcup [43 + 26k + 4k^2, 85 + 53k + 8k^2],$$

of cardinality $|A| = 46 + 27k + 4k^2$. The n-fold sumsets of A are then given by

$$\begin{aligned} nA &= [0, n] \cup \left(\bigcup_{i=1}^{n} [i(6 + 2k), i(6 + 2k) + n - i] \right)\\ &\quad \cup \left[43 + 26k + 4k^2, (85 + 53k + 8k^2)n \right]. \end{aligned} \tag{5}$$

- Assume first $2 \leq n < 6 + 2k$. In this case, we have

$$\begin{aligned} 0 &< n < 6 + 2k < 6 + 2k + n - 1 < \cdots < (n - 1)(6 + 2k)\\ &< (n - 1)(6 + 2k) + n - 1 < n(6 + 2k) < (7 + 2k)(6 + 2k) + 1\\ &= 43 + 26k + 4k^2 < (85 + 53k + 8k^2)n. \end{aligned}$$

Thus, all the sets appearing in (5) are disjoint and the cardinality of nA is equal to

$$\begin{aligned} &(n + 1) + \sum_{i=1}^{n} i + ((8k^2 + 53k + 85)n - (4k^2 + 26k + 43) + 1)\\ &= -41 - 26k - 4k^2 + (173n)/2 + 53kn + 8k^2n + n^2/2. \end{aligned}$$

Thus,

$$\begin{aligned}\beta_G(n) &= (-41 - 26k - 4k^2 + (173n)/2 + 53kn + 8k^2n + n^2/2)\\ &\quad -(4k^2 + 27k + 46 - 1)(2n - 1)\\ &= (4 + k) - \left(\frac{7}{2} + k\right)n + \frac{1}{2}n^2 \end{aligned} \tag{6}$$

for every $n \in [2, 5 + 2k]$.

The only difference between the case $n = 6 + 2k$ and the previous one is that the sets $[0, n]$ and $[6+2k, 6+2k+n-1]$ have a nonempty intersection, equal to $\{6+2k\}$. Replacing n by $6 + 2k$ and subtracting one, we obtain $\beta_G(6 + 2k) = 0$.

- Assume now $6 + 2k < n \leq 11 + 4k$. The sequence of sets $[0, n]$ and

$$[6+2k, 6+2k+n-1], \ldots, [(n-5-2k)(6+2k), (n-5-2k)(6+2k)+(5+2k)]$$

verifies that the intersection of any two consecutive terms is nonempty. Moreover, their union is the interval $[0, (n - 5 - 2k)(6 + 2k) + (5 + 2k)]$ whose cardinality is equal to $(n-5-2k)(6+2k)+(5+2k)+1$. For $i = n-4-2k, \ldots, 6+2k$ the intervals are disjoint with all the others sets appearing in the expression (5); the cardinality of the union of these sets is equal to $\sum_{i=n-2k-5}^{2k+5} i$. For every $i = 7 + 2k, \ldots, n$ the intersection

$$[i(6 + 2k), i(6 + 2k) + n - i] \cap [43 + 26k + 4k^2, (8k^2 + 53k + 85)n]$$

is nonempty, except for $n = 7 + 2k$. Since $(7 + 2k)(6 + 2k) = 42 + 26k + 4k^2$, the set

$$\left(\bigcup_{i=7+2k}^{n} [i(6 + 2k), i(6 + 2k) + n - i]\right) \cup \left[43 + 26k + 4k^2, (85 + 53k + 8k^2)n\right]$$

is equal to $[42 + 26k + 4k^2, (85 + 53k + 8k^2)n]$, and the cardinality of this set is $\left(\left(8k^2 + 53k + 85\right) n - 4k^2 - 26k - 42\right) + 1$. Putting all the above together, we have that if $6 + 2k < n \leq 11 + 4k$, the set nA has cardinality equal to

$$\begin{aligned}&((n - 5 - 2k)(6 + 2k) + (5 + 2k) + 1)\\ &\quad + \sum_{i=n-2k-5}^{2k+5} i + \left(\left(8k^2 + 53k + 85\right) n - 4k^2 - 26k - 42\right) + 1\\ &= -65 - 46k - 8k^2 + (193n)/2 + 57kn + 8k^2n - n^2/2,\end{aligned}$$

and therefore

$$\begin{aligned}\beta_G(n) &= -65 - 46k - 8k^2 + (193n)/2 + 57kn + 8k^2n - n^2/2\\ &\quad -(2n-1)(46 + 27k + 4k^2 - 1)\\ &= (-20 - 19k - 4k^2) + \left(3k + \frac{13}{2}\right)n - \frac{n^2}{2} \end{aligned} \tag{7}$$

for every $n \in [6 + 2k, 11 + 4k]$.

• Finally, assume $11 + 4k < n$. The set

$$[0, n] \cup \left(\bigcup_{i=1}^{6+2k} [i(6 + 2k), i(6 + 2k) + n - i]\right) \cup [43{+}26k{+}4k^2, (85{+}53k{+}8k^2)n]$$

is equal to $[0, (85 + 53k + 8k^2)n]$ and the remaining intervals are contained in this union. So we have $nA = [0, (85 + 53k + 8k^2)n]$ and therefore

$$\begin{aligned}\beta_G(n) &= -\left(4k^2 + 27k + 46 - 1\right)(2n - 1) + \left(8k^2 + 53k + 85\right)n + 1\\ &= (4k^2 + 27k + 46) - (k + 5)n \end{aligned} \tag{8}$$

for every $n > 11 + 4k$.

Combining $\beta_G(6 + 2k) = 0$ with the formulation of (6), (7) and (8) for $\beta_G(n)$, we get the following formulas:

$$\beta_G(n) = \begin{cases} (4 + k) - (\frac{7}{2} + k)n + \frac{1}{2}n^2 & \text{if } 2 \le n < 6 + 2k,\\ 0 & \text{if } n = 6 + 2k,\\ (-20 - 19k - 4k^2) + (3k + \frac{13}{2})n - \frac{n^2}{2} & \text{if } 6 + 2k < n \le 11 + 4k,\\ (4k^2 + 27k + 46) - (k + 5)n & \text{if } 11 + 4k < n. \end{cases}$$

Let $k \ge 1$ be fixed. For $2 \le n < 6 + 2k$, the formula of $\beta_G(n)$ is a degree two polynomial in n with positive leading coefficient such that $\beta_G(2) = -1 - k < 0$ and $\beta_G(5{+}2k) = -1{-}k < 0$. We have therefore $\beta_G(n) < 0$ for every $n = 2, \ldots, 5{+}2k$.

If $n > 11 + 4k$, we now have that $\beta_G(n)$ is a degree one polynomial with negative leading coefficient and such that $\beta_G(12 + 4k) = -14 - 5k < 0$. So $\beta_G(n) < 0$ for every $n > 11 + 4k$.

Finally, if $6{+}2k < n \le 11{+}4k$ the function $\beta_G(n)$ is a degree two polynomial in n with negative leading coefficient. As in addition $\beta_G(7{+}2k) = 1{+}k$, $\beta_G(7{+}4k) = 1$ and $\beta_G(8 + 4k) = -k$, the only positive values that we have in this part are for $n \in [7 + 2k, 7 + 4k]$.

Since $\beta_G(n) \le 0$ except for $n \in [7 + 2k, 7 + 4k]$, the set Buch(S) is equal to $[7 + 2k, 7 + 4k]$. □

4 Concluding Remarks

The current knowledge on the structure of $\mathcal{B}(A)$ for finite subsets $A \subset \mathbb{Z}$ is very scarce, even for gapsets. Do they have some special shape or property? We end this paper with three questions based on the few currently available observations.

Question 4.1 *Let $A \subset \mathbb{Z}$ be a finite subset, or more specifically a gapset. Is the set $\mathcal{B}(A)$ always an interval of integers?*

Question 4.2 *Even more so, is the function $\beta_A(n)$ unimodal?*

Question 4.3 *In sharp contrast with the above questions, let $T \subset 2 + \mathbb{N}$ be any finite subset. Does there exist a finite subset $A \subset \mathbb{N}$, or more specifically a gapset, such that $\mathcal{B}(A) = T$?*

Acknowledgements Part of this paper was written during a visit of the first-named author to the Universidad de Cádiz (Spain) which was partially supported by *Ayudas para Estancias Cortas de Investigadores* (EST2019-039, Programa de Fomento e Impulso de la Investigación y la Transferencia en la Universidad de Cádiz). The second, third and fourth-named authors were partially supported by Junta de Andalucía research groups FQM-366 and FQM-343, and by the project MTM2017-84890-P (MINECO/FEDER, UE).

Funding Funding for open access publishing: Universidad de Cádiz/CBUA.

References

Buchweitz, R.-O.: On Zariski's criterion for equisingularity and non-smoothable monomial curves. Preprints 115s, Universität Hannover, Institut für Mathematik (1980)

Eisenbud, D., Harris, J.: Existence, decomposition, and limits of certain Weierstrass points. Invent. Math. **87**, 495–515 (1987)

Freiman, G.A.: The addition of finite sets. I. (Russian). Izv. Vysš. Učebn. Zaved. Matematika **6**(13), 202–213 (1959)

García-García, J.I., Marín-Aragón, D., Torres, F., Vigneron-Tenorio, A.: On reducible non-Weierstrass semigroups. Open Math. **19**(1), 1134–1144 (2021)

Hurwitz, A.: Über algebraischer Gebilde mit eindeutigen Transformationen in sich. Math. Ann. **41**, 403–442 (1893)

Kaplan, N., Ye, L.: The proportion of Weierstrass semigroups. J. Algebra **373**, 377–391 (2013)

Komeda, J.: Non-Weierstrass numerical semigroups. Semigroup Forum **57**, 157–185 (1998)

Nathanson, M.B.: Additive Number Theory, Inverse Problems and the Geometry of Sumsets. Graduate Texts in Mathematics, vol. 165. Springer, New York (1996)

Oliveira, G.: Weierstrass semigroups and the canonical ideal of non-trigonal curves. Manuscr. Math. **71**, 431–450 (1991)