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Using slacks-based model to solve inverse DEA with integer intervals for input estimation

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Abstract

This paper deals with an inverse data envelopment analysis (DEA) based on the non-radial slacks-based model in the presence of uncertainty employing both integer and continuous interval data. To this matter, suitable technology and formulation for the DEA are proposed using arithmetic and partial orders for interval numbers. The inverse DEA is discussed from the following question: if the output of DMU_o increases from Y_o to β_o, such the new DMU is given by (α_o^*, β) belongs to the technology, and its inefficiency score is not less than t-percent, how much should the inputs of the DMU increase? A new model of inverse DEA is offered to respond to the previous question, whose interval Pareto solutions are characterized using the Pareto solution of a related multiple-objective nonlinear programming (MONLP). Necessary and sufficient conditions for input estimation are proposed when output is increased. A functional example is presented on data to illustrate the new model and methodology, with continuous and integer interval variables.

Keywords Data envelopment analysis (DEA) · Inverse DEA · Integer interval · Multiple-objective nonlinear programming (MONLP) · Slacks-based model

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1 Introduction

Data envelopment analysis is a practical non-parametric methodology to measure the efficiency of Decision Making Units (DMUs) by consuming inputs to produce outputs. DEA method was first proposed by Charnes et al. (1978), developed by Banker et al. (1984).

Also, some researchers have considered the applications of DEA, for example, Hadi-Vencheh et al. (2018) studied sustainable airline operations. They utilized a modified slack-based measure model to account for CO_2 emissions. Yousefi and Hadi-Vencheh (2016) compared three techniques to investigate six sigma optimized projects. They used Analytic Hierarchy Process (AHP), the Technique for Order Preference by Similarity to an Ideal Solution (TOPSIS), and Data Envelopment Analysis (DEA). Finally, as DEA is a good indicator for evaluating the optimized units they opted DEA.

Also, Tan et al. (2021) considered hotel performance. They investigated the role of information entropy in feedback processes between input and output management as well as evaluated the level of super efficiency with negative values and liquidity variables.

The concept of the inverse DEA model is firstly introduced by Zhang and Cui (1999). They study the input increases of a DMU are evaluated for its given output increases under the CCR efficiency fixed constraints. Inverse DEA is formally studied by Wei et al. (2000). They considered the first question in inverse DEA (output-estimation). "If the inputs of DMU_o increase, how much should the outputs of DMU_o increase to preserve the efficiency score of DMU_o?" Wei et al. (2000) proposed a linear programming problem when DMU_o is weakly efficient and a multiple-objective linear programming (MOLP) problem when DMU_o is inefficient to answer this question. The second question in inverse DEA (input-estimation) was considered by Hadi-Vencheh and Foroughi (2006). "If the outputs of DMU_o increase, how much should the outputs of DMU_o increase to preserve the efficiency score of DMU_o?"

Input-estimation and output-estimation were studied by Jahanshahloo et al. (2004), provided that DMU_o maintains or improves the efficiency score. Also, both questions were investigated under inter-temporal dependence by Jahanshahloo et al. (2015). The third question in inverse DEA (input–output estimation) is considered by Jahanshahloo et al. (2014). "If the inputs and outputs of DMU_o increase, how much should the inputs and outputs of DMU_o increase to preserve the efficiency score of DMU_o?" This question was answered only for the efficient DMU_o. They applied MOLP for input–output estimation. In addition to these, Chen and Wang (2021) studied the limitation of inputs and outputs in the inverse DEA method under variable returns to scale (VRS), because the inverse DEA method often has no feasible solution under VRS. Also, Chen et al. (2021) applied inverse DEA to the transportation science which is one of the most popular applications of DEA and inverse DEA. They introduced an objective constraints to extend an inverse DEA method with undesirable output to find the optimal realization path.

Most of the studies was done on radial inverse DEA. When slacks are of importance, radial inverse DEA may mislead to answer questions in inverse DEA.

Therefore, some researchers try to consider inverse DEA based on non-radial models. To the best of our knowledge, Jahanshahloo et al. (2014) introduced a non-radial inverse DEA based on the Enhanced Russel model. They assume that the efficiency scores of each dimension remain unchanged. Then Zhang and Cui (2020) proposed a non-radial inverse DEA model, supposing that the overall efficiency score remains unchanged, covering all radial and non-radial measures that are monotonous. In other words, they introduced a basic form of all inverse DEA models because monotonicity is one of the main properties of DEA measures.

Regarding integer DEA, Lozano and Villa (2006) firstly proposed integer DEA. Directional Distance Function (DDF), super-efficiency, flexible measures or congestion are type of advanced DEA models. Integer DEA has much application, for example, hotel performance, sports, and transportation.

Regarding interval DEA, there have also been many types of research, for example, radial multiplier formulations, additive imprecise DEA approaches, FDH interval DEA models, non-radial, non-oriented imprecise DEA approaches, ideal point approaches, inverted DEA approaches, interval DEA with negative data, flexible measure interval DEA approaches, and common weights imprecise DEA approaches. Manufacturing industry, banks and bank branches, power plants are the applications of interval values.

In this paper, we extend our previous work in Arana-Jiménez et al. (2021) from interval integer DEA to integer interval inverse DEA. To the best of our knowledge, there are a few literature that address inverse DEA with imprecise data, for instance, Hadi-Vencheh et al. (2014) and Ghobadi (2021) proposed Inverse DEA under interval data, They considered only continuous data while we use integer and continuous. Also, there is only one publication about integer inverse DEA. Shinto and Sushama (2019) considered inverse DEA with integer restriction while we apply inverse DEA to integer interval data. As previously mentioned, the closet existing non-radial inverse DEA is Zhang and Cui (2020), which is different from our approach. While they consider crisp input/output, we study uncertainty in data. Also, while they use continuous data, we apply hybrid scenario, containing both continuous and integer data. Therefore, the contribution of this research is vast.

The aim contribution of this paper is to consider prevailing methods with non-radial slacks-based measure, which has more properties than radial models, on integer interval framework. We consider the following question: "If the output of DMU_o increases such that its inefficiency score is not less than t-percent, how much should the input of DMU_o increase?" To answer this question, we propose, and apply a non-radial inverse DEA model involving integer and continuous interval data.

The structure of the paper is as follows. In Sect. 2, the basic ideas of the inverse DEA and slacks-based inverse DEA model are reviewed. Section 3, some concepts on integer intervals are introduced. The concepts in Sect. 4 are used to propose some theoretical extensions of inverse DEA with integer intervals. Necessary and sufficient conditions for input estimation are proposed when output is increased. In Sect. 5, numerical examples are presented. Finally, Sect. 6 indicates some conclusions.

2 Inverse DEA models with crisp data

Let us assume a set of N DMUs in which each DMU_j, $j \in J = \{1, \dots, N\}$, consume M inputs $X_j = (x_{1j}, \dots, x_{Mj}) \in \mathbb{R}^M$ to produces S outputs $Y_j = (y_{1j}, \dots, y_{Sj}) \in \mathbb{R}^S$. In the classic (Charnes et al., 1978) DEA model, the production possibility set (PPS) or technology, defined by T, satisfies in the following axioms:

(A1) Envelopment: $(X_j, Y_j) \in T$, for all $j \in J$.
(A2) Free disposability: $(X, Y) \in T, (X', Y') \in \mathbb{R}^{M+S}, X' \geqq X, Y' \leqq Y \Rightarrow (X', Y') \in T$.
(A3) Convexity: $(X, Y), (X', Y') \in T$, then $\lambda(X, Y) + (1 - \lambda)(X', Y') \in T$, for all $\lambda \in [0, 1]$.
(A4) Scalability: $(X, Y) \in T \Rightarrow (\lambda X, \lambda Y) \in T$, for all $\lambda \in \mathbb{R}_+$.

According to the minimum extrapolation principle in Banker et al. (1984), the DEA PPS, which contains all the feasible input–output bundles, is the intersection of all the sets that satisfy axioms (A1)-(A4) and can be defined as

$$T_{DEA} = \left\{ (X, Y) \in \mathbb{R}_+^{M+S} : X \geqq \sum_{j=1}^{N} \lambda_j X_j, Y \leqq \sum_{j=1}^{N} \lambda_j Y_j, \lambda_j \geq 0, \forall j \right\}.$$

Let us recall that a DMU_o is said to be efficient if and only if for any $(X, Y) \in T_{DEA}$ such that $X \leqq X_o$ and $Y \geqq Y_o$, then $(X, Y) = (X_o, Y_o)$. This can be got solving the following normalized slacks-based DEA model.

$$\begin{aligned} \text{(DEA)} \quad I^*(X_o, Y_o) = \text{Max} \sum_{i=1}^{M} \frac{s_i^x}{x_{io}} + \sum_{r=1}^{S} \frac{s_r^y}{y_{ro}} \\ \text{s.t.} \sum_{j=1}^{N} \lambda_j x_{ij} \leq x_{io} - s_i^x, \quad i = 1, \dots, M, \\ \sum_{j=1}^{N} \lambda_j y_{rj} \geq y_{ro} + s_r^y, \quad r = 1, \dots, S, \\ \lambda_j \geq 0, \quad j = 1, \dots, N, \\ s_i^x, s_r^y \geq 0, \quad i = 1, \dots, M, \quad r = 1, \dots, S. \end{aligned} \tag{1}$$

Where λ_j, $j = 1, \dots, N$, are the intensity variables used for defining the corresponding efficient target of DMU_o. The inefficiency measure $I^*(X_o, Y_o)$ is units invariant and non-negative. Furthermore, a DMU_o is efficient if and only if $I^*(X_o, Y_o) = 0$.

Now, the following question is considered based on investigations carried out in previous literature. If the outputs of DMU_o increase, how much should the inputs of the DMU_o increase to decrease the inefficiency score of DMU_o to the amount of t-percent. The aim of the question is to calculate the minimum increase of input (α_o^*) if the output of DMU_o increase from Y_o to $\beta_o = Y_o + \triangle Y_o$, where $\triangle Y_o \gneqq 0$ provided that the inefficiency score of DMU_o decrease to the amount of t-percent. In fact,

$$\alpha_o^* = (\alpha_{1o}^*, \alpha_{2o}^*, ..., \alpha_{Mo}^*)^t = X_o + \triangle X_o, \qquad \triangle X_o \gneqq 0.$$

Furthermore, we consider that the new *DMU* belongs to the technology. For the sake of simplicity, assume that the new *DMU* represents DMU_o. After modification of inputs and outputs, the following model is presented to estimate the inefficiency of the new *DMU*:

$$\begin{aligned} \text{(DEA)} \quad I^*(\alpha_o^*, \beta_o) = \text{Max} \sum_{i=1}^{M} \frac{s_i^x}{\alpha_{io}^*} + \sum_{r=1}^{S} \frac{s_r^y}{\beta_{ro}} & \\ \text{s.t.} \sum_{j=1}^{N} \lambda_j x_{ij} \le \alpha_{io}^* - s_i^x, \quad i = 1, \dots, M, & \\ \sum_{j=1}^{N} \lambda_j y_{rj} \ge \beta_{ro} + s_r^y, \quad r = 1, \dots, S, & \\ \lambda_j \ge 0, \quad j = 1, \dots, N, & \\ s_i^x, s_r^y \ge 0, \quad i = 1, \dots, M, \ \ r = 1, \dots, S. & \end{aligned} \tag{2}$$

Definition 1 (1) If the optimal values of the model (1) and (2) are equal, it is said to be the inefficiency score remains unchanged; that is, $I^*(\alpha_o^*, \beta_0) = I^*(X_o, Y_o)$.

(2) If the optimal values of the model (1) are less than model (2), it is said to be the inefficiency score decrease to the amount of t-percent; that is, $I^*(\alpha_o^*, \beta_0) = (1-t)I^*(X_o, Y_o)$.

To solve the above question, the following *MONLP* model is considered:

$$\begin{aligned} \text{(MONLP)} \quad \text{Min } (\alpha_{1o}, ..., \alpha_{Mo}) & \\ \text{s.t.} \sum_{j=1}^{N} \lambda_j x_{ij} \le \alpha_{io} - s_i^x, \quad i = 1, \dots, M, & \\ \sum_{j=1}^{N} \lambda_j y_{rj} \ge \beta_{ro} + s_r^y, \quad r = 1, \dots, S, & \\ \sum_{i=1}^{M} \frac{s_i^x}{\alpha_{io}} + \sum_{r=1}^{S} \frac{s_r^y}{\beta_{ro}} = (1-t)I^* & \\ \alpha_{io} \ge x_{io}, \quad i = 1, \dots, M, & \\ \lambda_j \ge 0, \quad j = 1, \dots, N, & \\ s_i^x, s_r^y \ge 0, \quad i = 1, \dots, M, \ \ r = 1, \dots, S. & \end{aligned} \tag{3}$$

Where I^* is the optimal value of problem (1) and $0 \le t \le 1$, note that when $t = 1$, $I^*(\alpha_o^*, \beta_o) = 0$, which means the new *DMU* is efficient and when $t = 0$, $I^*(\alpha_o^*, \beta_o) = I^*(X_o, Y_o)$. Therefore, when t increases, the inefficiency score decreases.

Definition 2 (see Zhang and Cui (2020)). Let $(\lambda^*, \alpha_0^*, s^{x*}, s^{y*})$ be a feasible solution to the problem (3). $(\lambda^*, \alpha_0^*, s^{x*}, s^{y*})$ is said to be a Pareto (efficient) solution to the problem (3) if there isn't feasible solution $(\lambda, \alpha_0, s^x, s^y)$ of (3) such that $\alpha_{io} \le \alpha_{io}^*$ for all $i = 1, 2, ..., M$ and $\alpha_{io} < \alpha_{io}^*$ for at least one i.

Definition 3 (see Zhang and Cui (2020)). Let $(\lambda^*, \alpha_0^*, s^{x*}, s^{y*})$ be a feasible solution to the problem (3). $(\lambda^*, \alpha_0^*, s^{x*}, s^{y*})$ is said to be a weakly Pareto (weakly efficient) solution to the problem (3) if there isn't feasible solution $(\lambda, \alpha_0, s^x, s^y)$ of (3) such that $\alpha_{io} \le \alpha_{io}^*$ for all $i = 1, 2, ..., M$.

There are different methods to generate weakly Pareto (weakly efficient) solutions of *MOLP* and *MONLP*. One of the most usual methods is weighted sum problems (see Arana-Jiménez (2010) and Arana-Jiménez and Antczak (2017)). Following formulation is this type of optimization problem. Given MONLP (3) and $w = (w_1, w_2, \cdots, w_M) \in \mathbb{R}^M$, $w_i > 0$, $\sum_{i=1}^{M} w_i = 1$, We define the related sum problem as follows.

$$
\begin{aligned}
\text{(MONLP)w} \quad \text{Min} \; & \sum_{i=1}^{M} w_i \alpha_{io} \\
\text{s.t.} \; & \sum_{j=1}^{N} \lambda_j x_{ij} \le \alpha_{io} - s_i^x, \quad i = 1, \dots, M, \\
& \sum_{j=1}^{N} \lambda_j y_{rj} \ge \beta_{ro} + s_r^y, \quad r = 1, \dots, S, \\
& \sum_{i=1}^{M} \frac{s_i^x}{\alpha_{io}} + \sum_{r=1}^{S} \frac{s_r^y}{\beta_{ro}} = (1-t) I^*, \\
& \alpha_{io} \ge x_{io}, \quad i = 1, \dots, M, \\
& \lambda_j \ge 0, \quad j = 1, \dots, N, \\
& s_i^x, s_r^y \ge 0, \quad i = 1, \dots, M, \; r = 1, \dots, S.
\end{aligned} \tag{4}
$$

Theorem 1 *Assume that $I^*(X_o, Y_o)$ be the inefficiency score of DMU_o under the monotonous measure in the problem* (1) *and the outputs of DMU_o are increased from Y_0 to $\beta_0 = Y_0 + \triangle Y_o (\triangle Y_o \leqq 0)$.*

(1) *Let $(\lambda^*, \alpha_o^*, s^{x*}, s^{y*})$ be a Pareto solution to the problem* (3) *then inefficiency score of the DMU_o under new inputs and outputs decrease to the amount of t-percent.*
(2) *Conversely, let $(\lambda^*, \alpha_o^*, s^{x*}, s^{y*})$ be a feasible solution to the problem* (3). *If the inefficiency score of the new DMU decreases to the amount of t-percent, then $(\lambda^*, \alpha_o^*, s^{x*}, s^{y*})$ must be a Pareto solution to the problem* (3).

Note that there is a similar resulting. If the input of DMU_o increases, how much should the output of DMU_o increase to decrease to the amount of t-percent the inefficiency score of DMU_o. In other words, we calculate $I^*(\alpha_o, \beta_o^*)$.

3 Notation and preliminaries on integer intervals

In this paper, in order to present uncertainty on the production possibility set by modelling the corresponding inequality relationship using partial orders on fuzzy intervals, we introduce the following notations and results.

Let $\mathbb{R}$ be the real number set. We denote by $\mathcal{K}_C = \left\{ \left[\underline{a}, \overline{a}\right] \mid \underline{a}, \overline{a} \in \mathbb{R} \text{ and } \underline{a} \leq \overline{a} \right\}$ the family of all bounded intervals in $\mathbb{R}$ and $\mathcal{K}_{C_+} \subseteq \mathcal{K}_C$ is the set of non-negative bounded intervals in $\mathbb{R}$, $\mathcal{K}_{C_+} = \left\{ \left[\underline{a}, \overline{a}\right] \mid \underline{a}, \overline{a} \in \mathbb{R} \text{ and } 0 \leq \underline{a} \leq \overline{a} \right\}$. Usual arithmetic between intervals is the following (see, for instance, (Stefanini & Arana-Jiménez, 2019) and the bibliography therein).

Definition 4 Let $A = [\underline{a}, \overline{a}] \in \mathcal{K}_C$, $B = [\underline{b}, \overline{b}] \in \mathcal{K}_C$.

- Addition: $A + B := \{a + b \mid a \in A, b \in B\} = [\underline{a} + \underline{b}, \overline{a} + \overline{b}]$,
- Opposite value: $-A = \{-a : a \in A\} = [-\underline{a}, -\overline{a}]$,
- Multiplication: $A \cdot B := \{a \cdot b \mid a \in A, b \in B\} = [min(AB), max(AB)]$, where $AB = \{\underline{a} \cdot \underline{b}, \underline{a} \cdot \overline{b}, \overline{a} \cdot \underline{b}, \overline{a} \cdot \overline{b}\}$,
- Multiplication by scalar: for any λ,

$$\lambda \cdot A := \begin{cases} [\lambda \cdot \underline{a}, \lambda \cdot \overline{a}] & \lambda \geq 0, \\ [\lambda \cdot \overline{a}, \lambda \cdot \underline{a}] & \lambda < 0. \end{cases}$$

Apt and Zoeteweij (2004) defined some arithmetic operations on integer intervals. Recently, Arana-Jiménez et al. (2021) have extended them and established a new notation, as following.

Let $\mathbb{Z}$ be the integer set. Given $\underline{a}, \overline{a} \in \mathbb{Z}$, $\underline{a} \leq \overline{a}$, we say that $[\underline{a}, \overline{a}]_{\mathbb{Z}} = \{a \in \mathbb{Z} : \underline{a} \leq a \leq \overline{a}\}$ is an integer interval in $\mathbb{Z}$ We denote by $\mathcal{K}_{\mathbb{Z}} = \left\{ \left[\underline{a}, \overline{a}\right]_{\mathbb{Z}} \mid \underline{a}, \overline{a} \in \mathbb{Z} \text{ and } \underline{a} \leq \overline{a} \right\}$ the set of bounded integer interval and $\mathcal{K}_{Z_+} \subseteq \mathcal{K}_Z$ is the set of non-negative bounded integer intervals in $\mathbb{Z}$, that is, $\mathcal{K}_{Z_+} = \left\{ \left[\underline{a}, \overline{a}\right] \mid \underline{a}, \overline{a} \in \mathbb{Z} \text{ and } 0 \leq \underline{a} \leq \overline{a} \right\}$.

Definition 5 Let $A = [\underline{a}, \overline{a}] \in \mathcal{K}_{\mathbb{Z}}$, $B = [\underline{b}, \overline{b}] \in \mathcal{K}_{\mathbb{Z}}$.

- Addition: $[\underline{a}, \overline{a}]_{\mathbb{Z}} + [\underline{b}, \overline{b}]_{\mathbb{Z}} = [\underline{a} + \underline{b}, \overline{a} + \overline{b}]_{\mathbb{Z}}$,
- Subtraction: $[\underline{a}, \overline{a}]_{\mathbb{Z}} - [\underline{b}, \overline{b}]_{\mathbb{Z}} = [\underline{a} - \overline{b}, \overline{a} - \underline{b}]_{\mathbb{Z}}$,
- Multiplication: $[\underline{a}, \overline{a}]_{\mathbb{Z}} \cdot [\underline{b}, \overline{b}]_{\mathbb{Z}} = [min(AB), max(AB)]_{\mathbb{Z}}$, where $AB = \{\underline{a} \cdot \underline{b}, \underline{a} \cdot \overline{b}, \overline{a} \cdot \underline{b}, \overline{a} \cdot \overline{b}\}$.
- Multiplication by scalar: for any integer λ,

$$\lambda \cdot A := \begin{cases} [\lambda \cdot \underline{a}, \lambda \cdot \overline{a}]_{\mathbb{Z}} & \lambda \geq 0, \\ [\lambda \cdot \overline{a}, \lambda \cdot \underline{a}]_{\mathbb{Z}} & \lambda < 0. \end{cases}$$

Example 1 Consider the following examples of the above operations for integer intervals. $[4,5]_{\mathbb{Z}} + [-1,2]_{\mathbb{Z}} = [3,7]_{\mathbb{Z}}$, $[-4,5]_{\mathbb{Z}} - [-1,2]_{\mathbb{Z}} = [-6,4]_{\mathbb{Z}}$, $[2,4]_{\mathbb{Z}} \cdot [4,6]_{\mathbb{Z}} = [8,24]_{\mathbb{Z}}$, $3 \cdot [2,4]_{\mathbb{Z}} = [6,12]_{\mathbb{Z}}$. It can be seen that the arithmetic operations for integer intervals defined above always produce integer intervals.

It is also useful to define the continuous extension of an integer interval $[\underline{a}, \overline{a}]_{\mathbb{Z}}$ as $C([\underline{a}, \overline{a}]_{\mathbb{Z}}) = [\underline{a}, \overline{a}]$. Conversely, given $\underline{a} \leq \overline{a}$ with $\underline{a}, \overline{a} \in \mathbb{Z}$, we define the integer projection of $[\underline{a}, \overline{a}] \in \mathcal{K}_C$ as $\mathbb{Z}([\underline{a}, \overline{a}]) = [\underline{a}, \overline{a}]_{\mathbb{Z}} \in \mathcal{K}_{\mathbb{Z}}$; and in this case, it is said that $[\underline{a}, \overline{a}] \in \mathcal{K}_{C \to \mathbb{Z}}$. In other words, $\mathcal{K}_{C \to \mathbb{Z}}$ is the set of intervals whose endpoints are integer. Note also that $\mathbb{Z}(C([\underline{a}, \overline{a}]_{\mathbb{Z}})) = [\underline{a}, \overline{a}]_{\mathbb{Z}}$.

With respect to partial order relationship between integer intervals, Arana-Jiménez et al. (2021) have proposed an adaptation of LU-fuzzy partial orders on intervals.

Definition 6 Given two intervals $A = [\underline{a}, \overline{a}], B = [\underline{b}, \overline{b}] \in \mathcal{K}_C$, we say that:

(i) $[\underline{a}, \overline{a}] \leqq [\underline{b}, \overline{b}]$ if and only if $\underline{a} \leq \underline{b}$ and $\overline{a} \leq \overline{b}$.
(ii) $[\underline{a}, \overline{a}] \prec [\underline{b}, \overline{b}]$ if and only if $\underline{a} < \underline{b}$ and $\overline{a} < \overline{b}$.

Definition 7 Given two integer intervals $A = [\underline{a}, \overline{a}]_{\mathbb{Z}}, B = [\underline{b}, \overline{b}]_{\mathbb{Z}} \in \mathcal{K}_{\mathbb{Z}}$, we say that:

(i) $[\underline{a}, \overline{a}]_{\mathbb{Z}} \leqq [\underline{b}, \overline{b}]_{\mathbb{Z}}$ if and only if $\underline{a} \leq \underline{b}$ and $\overline{a} \leq \overline{b}$.
(ii) $[\underline{a}, \overline{a}]_{\mathbb{Z}} \prec [\underline{b}, \overline{b}]_{\mathbb{Z}}$ if and only if $\underline{a} < \underline{b}$ and $\overline{a} < \overline{b}$.

In a similar manner, we define the relationships $A \geqq B$ and $A \succ B$ for intervals and integer intervals, which really means $B \leqq A$ and $B \prec A$, respectively. Note that, for the sake of simplicity, we use the same symbols of partial orders to compare intervals in $\mathcal{K}_C$ as to compare integer intervals in $\mathcal{K}_{\mathbb{Z}}$. Furthermore, in the next section, to define the corresponding DEA technology, we will need to relate intervals and integer intervals. To this matter, We will use the properties that $[\underline{a}, \overline{a}]_{\mathbb{Z}} \subseteq [\underline{a}, \overline{a}] \cap \mathbb{Z}$ for all $\underline{a} \leq \overline{a}$ with $\underline{a}, \overline{a} \in \mathbb{Z}$, as well as that given $\underline{a} \leq \overline{a}, \underline{b} \leq \overline{b}$ with $\underline{a}, \overline{a}, \underline{b}, \overline{b} \in \mathbb{Z}$, then $[\underline{a}, \overline{a}]_{\mathbb{Z}} \leqq (\prec) [\underline{b}, \overline{b}]_{\mathbb{Z}}$ if and only if $C([\underline{a}, \overline{a}]_{\mathbb{Z}}) = [\underline{a}, \overline{a}] \leqq (\prec) [\underline{b}, \overline{b}] = C([\underline{b}, \overline{b}]_{\mathbb{Z}})$.

4 Inverse DEA models with integer and continuous interval data

In this section, the non-radial slacks-based model is extended to an integer interval framework, which is considered by Arana-Jiménez et al. (2021). in other words, we provide the question, which is mentioned in previous sections, in the presence of integer interval data using a non-radial slacks-based model.

Let us assume a set of N DMUs, $j \in J = \{1, \ldots, N\}$, in which each DMU_j consumes M inputs denoted by $X_j = (x_{1j}, \ldots, x_{Mj}) \in (\mathcal{K}_{\mathbb{Z}^+})^M$, with $x_{ij} = [\underline{x_{ij}}, \overline{x_{ij}}]_{\mathbb{Z}} \in \mathcal{K}_{\mathbb{Z}+}$ for $i \in \{1, \ldots, M\}$ to produces S outputs denoted by $Y_j = (y_{1j}, \ldots, y_{Sj}) \in (\mathcal{K}_{\mathbb{Z}^+})^S$, with $y_{rj} = [\underline{y_{rj}}, \overline{y_{rj}}]_{\mathbb{Z}} \in \mathcal{K}_{\mathbb{Z}+}$ for $r \in \{1, \ldots, S\}$. Their continuous extensions are $C(X_j) = \left(C(x_{1j}), \ldots, C(x_{Mj})\right) \in (\mathcal{K}_{C^+})^M$ and $C(Y_j) = \left(C(y_{1j}), \ldots, C(y_{Sj})\right) \in (\mathcal{K}_{C^+})^S$.

Let us consider the following axioms, which are corresponding to (A1)-(A4) in Section 2, but considering integer fuzzy inputs and outputs and utilizing the corresponding partial order introduced in Definitions 6 and 7:

- (B1) *Envelopment:* $(X_j, Y_j) \in T$, *for all* $j \in J$.
- (B2) *Free disposability:* $(X, Y) \in T$, $(X', Y') \in (\mathcal{K}_{\mathbb{Z}^+})^{M+S}$, such that $X' \geqq X,\ Y' \leqq Y \ \Rightarrow\ (X', Y') \in T.$.
- (B3) *Convexity:* $(X, Y), (X', Y') \in T$, $\alpha \in [0, 1]$, *such that* $\alpha(C(X), C(Y)) + (1-\alpha)(C(X'), C(Y')) \in (\mathcal{K}_{C\to Z})^{M+S} \Rightarrow (X'', Y'') = \mathbb{Z}(\alpha(C(X), C(Y)) + (1-\alpha)(C(X'), C(Y'))) \in T$.
- (B4) *Scalability:* $(X, Y) \in T$, $\alpha \geq 0$, and $\alpha(C(X), C(Y)) \in (\mathcal{K}_{c\to z})^{M+S} \Rightarrow (X'', Y'') = \mathbb{Z}(\alpha(C(X), C(Y))) \in T$.

Theorem 2 *Under axioms (B1), (B2), (B3) and (B4), the interval production possibility set that results from the minimum extrapolation principle is* $T_{IFDEA} = \left\{(X, Y) \in (\mathcal{K}_{\mathbb{Z}^+})^{M+S} : C(X) \geqq \sum_{j=1}^{N} \lambda_j C(X_j), C(Y) \leqq \sum_{j=1}^{N} \lambda_j C(Y_j), \lambda_j \geq 0, \forall j\right\}$.

After the characterization result for the T_{IIDEA} given in Theorem 2, the following integer interval DEA (IIDEA) model, which is a slacks-based measure of inefficiency, can be extended from the non-radial slacks-based model.

$$
\begin{aligned}
\text{(IIDEA)}\ II^*(X_o, Y_o) = \quad & \text{Max} \quad \sum_{i=1}^{M} \frac{\underline{s_i^x} + \overline{s_i^x}}{\underline{x_{io}} + \overline{x_{io}}} + \sum_{r=1}^{S} \frac{\underline{s_r^y} + \overline{s_r^y}}{\underline{y_{ro}} + \overline{y_{ro}}} \\
& \text{s.t.} \quad \sum_{j=1}^{N} \lambda_j C(x_{ij}) \leqq C(x_{io}) - C(s_i^x), \quad i = 1, \ldots, M, \\
& \qquad \sum_{j=1}^{N} \lambda_j C(y_{rj}) \geqq C(y_{ro}) + C(s_r^y), \quad r = 1, \ldots, S, \\
& \qquad \lambda_j \geq 0, \quad j = 1, \ldots, N, \\
& \qquad s_i^x, s_r^y \in \mathcal{K}_{\mathbb{Z}+}, \quad i = 1, \ldots, M,\ r = 1, \ldots, S,
\end{aligned}
\tag{5}
$$

where inputs x_{ij} and outputs y_{rj} belong to $\mathcal{K}_{\mathbb{Z}}$, i.e.,

$$
\begin{aligned}
x_{ij} &= [\underline{x_{ij}}, \overline{x_{ij}}]_{\mathbb{Z}}, \quad i = 1, \ldots, M,\ j = 1, \ldots, N, \\
y_{rj} &= [\underline{y_{rj}}, \overline{y_{rj}}]_{\mathbb{Z}}, \quad r = 1, \ldots, S,\ j = 1, \ldots, N.
\end{aligned}
$$

A feasible solution for (IIDEA) is denoted by $(\boldsymbol{s}^{x*}, \boldsymbol{s}^{y*}, \lambda^*)$, where $\boldsymbol{s}^{x*} = (s_1^{x*}, \ldots, s_M^{x*}) \in (\mathcal{K}_z)^M$, $\boldsymbol{s}^{y*} = (s_1^{y*}, \ldots, s_S^{y*}) \in (\mathcal{K}_z)^S$, and $\lambda^* = (\lambda_1^*, \ldots, \lambda_N^*) \in \mathbb{R}^N$. Moreover, (IIDEA) model will deal directly without any ranking function. Also, its objective function is a real number, i.e. $II(X_o, Y_o) \in \mathbb{R}$.

Definition 8 A DMU_o is considered to be efficient if and only if $(x, y) \in T_{IFDEA}$, $x \leqq X_o$ and $y \geqq Y_o$ implies $(x, y) = (X_o, Y_o)$.

Theorem 3 *If DMU_o is efficient, then $II(X_o, Y_o) = 0$.*

Arana-Jiménez et al. (2021) extended the previous axioms, interval production possibility set, and result to the hybrid data scenario, that is, with integer and continuous integer data. The extended and corresponding non-radial slacks-based model is the following:

$$
\begin{aligned}
\text{(HIDEA)}\ II^*(X_o, Y_o) = \quad & \text{Max} \quad \sum_{i=1}^{M} \frac{\underline{s_i^x} + \overline{s_i^x}}{\underline{x_{io}} + \overline{x_{io}}} + \sum_{r=1}^{S} \frac{\underline{s_r^y} + \overline{s_r^y}}{\underline{y_{ro}} + \overline{y_{ro}}} \\
& \text{s.t.} \quad \sum_{j=1}^{N} \lambda_j C(x_{ij}) \leqq C(x_{io}) - C(s_i^x), \quad i \in O^{XI}, \\
& \sum_{j=1}^{N} \lambda_j x_{ij} \leqq x_{io} - s_i^x, \quad i \in O^{XNI}, \\
& \sum_{j=1}^{N} \lambda_j C(y_{rj}) \geqq C(y_{ro}) + C(s_r^y), \quad r \in O^{YI}, \\
& \sum_{j=1}^{N} \lambda_j y_{rj} \geqq y_{ro} + s_r^y, \quad r \in O^{YNI}, \\
& \lambda_j \geq 0, \quad j = 1, \ldots, N, \\
& s_i^x, s_r^y \in \mathcal{K}_{\mathbb{Z}+}, \quad i \in O^{XI}, \ r \in O^{YI}, \\
& s_i^x, s_r^y \in \mathcal{K}_{C+}, \quad i \in O^{XNI}, \ r \in O^{YNI},
\end{aligned}
\tag{6}
$$

with O^{XI} and O^{XNI} the index sets for integer input variables and continuous input variables, respectively, O^{YI} and O^{YNI} the index sets for integer output variables and continuous output variables, respectively, with $XI + XNI = M$, $YI + YNI = S$, $O^X = O^{XI} \cup O^{XNI} = \{1, \ldots, M\}$, $O^Y = O^{YI} \cup O^{YNI} = \{1, \ldots, S\}$. Let us write the above model in parameterized form as follows:

$$
\begin{aligned}
\text{(PHIDEA)}\quad II^*(X_o, Y_o) = \text{Max} & \sum_{i=1}^{M} \frac{\underline{s_i^x} + \overline{s_i^x}}{\underline{x_{io}} + \overline{x_{io}}} + \sum_{r=1}^{S} \frac{\underline{s_r^y} + \overline{s_r^y}}{\underline{y_{ro}} + \overline{y_{ro}}} \\
\text{s.t.} & \sum_{j=1}^{N} \lambda_j \underline{x_{ij}} \leq \underline{x_{io}} - \overline{s_i^x}, \quad i \in O^X, \\
& \sum_{j=1}^{N} \lambda_j \overline{x_{ij}} \leq \overline{x_{io}} - \underline{s_i^x}, \quad i \in O^X, \\
& \sum_{j=1}^{N} \lambda_j \underline{y_{rj}} \geq \underline{y_{ro}} + \underline{s_r^y}, \quad r \in O^Y, \\
& \sum_{j=1}^{N} \lambda_j \overline{y_{rj}} \geq \overline{y_{ro}} + \overline{s_r^y}, \quad r \in O^Y, \\
& \underline{s_i^x} \leq \overline{s_i^x}, \quad i \in O^X, \\
& \underline{s_r^y} \leq \overline{s_r^y}, \quad r \in O^Y, \\
& \lambda_j \geq 0, \quad j = 1, \ldots, N, \\
& \underline{s_i^x}, \overline{s_i^x}, \underline{s_r^y}, \overline{s_r^y} \in \mathbb{Z}_+, \quad i \in O^{XI}, \ r \in O^{YI}, \\
& \underline{s_i^x}, \overline{s_i^x}, \underline{s_r^y}, \overline{s_r^y} \geq 0, \quad i \in O^{XNI}, \ r \in O^{YNI}.
\end{aligned} \tag{7}
$$

The first four sets of constraints are just the corresponding transformation of the inputs/outputs constraints from the model (6), with regard to the partial order relation for integer interval numbers, considering in Definition 7. The two last constraints certify the integer and continuous slacks. Therefore, it is not difficult to derive the following proposition, which establishes the relationship between the (HIDEA) and (PHIDEA) solutions.

Proposition 1 $(\boldsymbol{s}^{x*}, \boldsymbol{s}^{y*}, \lambda^*)$ *with* $\boldsymbol{s}^{x*} \in (\mathcal{K}_{\mathbb{Z}_+})^{XI} * (\mathcal{K}_C)^{XNI}, XI + XNI = M$, $\boldsymbol{s}^{y*} \in (\mathcal{K}_{\mathbb{Z}_+})^{YI} * (\mathcal{K}_C)^{YNI}, YI + YIN = S$ *and* $\lambda^* \in \mathbb{R}_+^N$ *is an optimal solution of* (*HIDEA*) *if and only if its corresponding components or parameterization* $(\underline{s_1^{x*}}, \overline{s_1^{x*}}, \ldots, \underline{s_M^{x*}}, \overline{s_M^{x*}}, \underline{s_1^{y*}}, \overline{s_1^{y*}}, \ldots, \underline{s_S^{y*}}, \overline{s_S^{y*}} \lambda_1^*, \ldots, \lambda_N^*)$, *with* $\lambda_j^* \in \mathbb{R}_+, \ j = 1, \ldots, N$, $\underline{s_i^{x*}}, \overline{s_i^{x*}}, \underline{s_r^{y*}}, \overline{s_r^{y*}} \in \mathbb{Z}_+$ *for* $i \in O^{XI}, r \in O^{YI}$ *and* $\underline{s_i^{x*}}, \overline{s_i^{x*}}, \underline{s_r^{y*}}, \overline{s_r^{y*}} \in \mathbb{R}_+$ *for* $i \in O^{XNI}, r \in O^{YNI}$, *is an optimal solution of* (*PHIDEA*).

In this new framework with integer and continuous interval data, we reconsider the inverse DEA concept from the classic concept under continuous crisp data discussed in Section 2. It is known that, in general, given a real number, it is not guaranteed that one can attain such a real number utilizing an arithmetic combination of a finite collection of integer numbers. The latter makes that, in general, given β_0 an increase of a Y_0, there exists no α_0 an increase of X_0 such that inefficiency $II^*(X_0, Y_0)$ or a given

t-percent of it is attained, i.e., $II^*(\alpha_0^*, \beta_0) = (1-t)II^*(X_0, Y_0)$. Furthermore, transformations of a formulation of DEA problems via change of variables are, in general, not consistent with the integer condition of the original variables; that is, the result of a transformed integer variable is not necessarily an integer. In this regard, if one follows the procedure proposed by Zhang and Cui (2020) applied to our hybrid DEA model using a variable, with the division between variables, then an integer variable becomes a non necessarily integer variable. These remarks make us approach the question of inverse DEA as follows. The aim of the question is to estimate the minimum increase of input, (α_o^*), if the output of DMU_o increases from Y_o to β_o, such the new DMU is given by (α_o^*, β) belongs to the technology, and its inefficiency score of is not less than t-percent. Here, $\alpha_o^* = (\alpha_{1o}^*, \alpha_{2o}^*, ..., \alpha_{Mo}^*) \in (\mathcal{K}_{Z+})^{XI} * (\mathcal{K}_C)^{XNI}$, $\alpha_o^* \geqq X_0$, $\beta_o^* = (\alpha_{1o}^*, \beta_{2o}^*, ..., \beta_{So}^*) \in (\mathcal{K}_{Z+})^{YI} * (\mathcal{K}_C)^{YNI}$, $\beta_0 \geqq Y_0$.. After these previous considerations, the following slacks-based model estimate the inefficiency of the new DMU:

$$
\begin{aligned}
\text{(PHIDEA)}\quad II^*(\alpha_o^*, \beta_o) = \ & \text{Max} \sum_{i=1}^{M} \frac{\underline{s}_i^x + \overline{s}_i^x}{\underline{\alpha}_{io}^* + \overline{\alpha}_{io}^*} + \sum_{r=1}^{S} \frac{\underline{s}_r^y + \overline{s}_r^y}{\underline{\beta}_{ro} + \overline{\beta}_{ro}} \\
& \text{s.t.} \sum_{j=1}^{N} \lambda_j \underline{x}_{ij} \le \underline{\alpha}_{io}^* - \overline{s}_i^x, \quad i \in O^X, \\
& \sum_{j=1}^{N} \lambda_j \overline{x}_{ij} \le \overline{\alpha}_{io}^* - \underline{s}_i^x, \quad i \in O^X, \\
& \sum_{j=1}^{N} \lambda_j \underline{y}_{rj} \ge \underline{\beta}_{ro} + \underline{s}_r^y, \quad r \in O^Y, \\
& \sum_{j=1}^{N} \lambda_j \overline{y}_{rj} \ge \overline{\beta}_{ro} + \overline{s}_r^y, \quad r \in O^Y, \\
& \underline{s}_i^x \le \overline{s}_i^x, \quad i \in O^X, \\
& \underline{s}_r^y \le \overline{s}_r^y, \quad r \in O^Y, \\
& \lambda_j \ge 0, \quad j = 1, \dots, N, \\
& \underline{s}_i^x, \overline{s}_i^x, \underline{s}_r^y, \overline{s}_r^y \in \mathbb{Z}_+, \quad i \in O^{XI}, \ r \in O^{YI}, \\
& \underline{s}_i^x, \overline{s}_i^x, \underline{s}_r^y, \overline{s}_r^y \ge 0, \quad i \in O^{XNI}, \ r \in O^{YNI}.
\end{aligned} \tag{8}
$$

To solve integer interval problem, the following (IP) problem is established:

$$
\begin{aligned}
\text{(IP)}\quad & \text{Min} \quad (\alpha_{1o}, , ..., \alpha_{Mo}) \\
& \text{s.t.} \quad (\alpha_o, \beta_o) \in T, \\
& \qquad \alpha_o \geqq X_o, \\
& \qquad II^*(\alpha_o, \beta_o) \ge (1-t)II^*.
\end{aligned} \tag{9}
$$

Definition 9 Let $\alpha_o^* \in (\mathcal{K}_{Z+})^{XI} * (\mathcal{K}_C)^{XNI}$ be a feasible solution to the problem (9). It is said to be an interval Pareto solution to the problem (9) if there isn't feasible solution α_o of (9) such that $\alpha_o \leqq \alpha_o^*, \alpha_o \neq \alpha_o^*$.

Definition 10 Let $\alpha_o^* \in (\mathcal{K}_{Z+})^{XI} * (\mathcal{K}_C)^{XNI}$ be a feasible solution to the problem (9). It is said to be an interval weakly Pareto solution to the problem (9) if there isn't feasible solution α_o of (9) such that $\alpha_o \leqq \alpha_o^*$.

After parametrization of (IP), the following (MONLP) problem is established:

$$
\begin{aligned}
\text{(MONLP)}\quad & \text{Min } (\underline{\alpha_{1o}}, \overline{\alpha_{1o}}, \ldots, \underline{\alpha_{Mo}}, \overline{\alpha_{Mo}}) \\
& \text{s.t.} \sum_{j=1}^{N} \lambda_j \underline{x_{ij}} \leq \underline{\alpha_{io}} - \overline{s_i^x}, \quad i \in O^X, \\
& \sum_{j=1}^{N} \lambda_j \overline{x_{ij}} \leq \overline{\alpha_{io}} - \underline{s_i^x}, \quad i \in O^X, \\
& \sum_{j=1}^{N} \lambda_j \underline{y_{rj}} \geq \underline{\beta_{ro}} + \underline{s_r^y}, \quad r \in O^Y, \\
& \sum_{j=1}^{N} \lambda_j \overline{y_{rj}} \geq \overline{\beta_{ro}} + \overline{s_r^y}, \quad r \in O^Y, \\
& \sum_{i=1}^{M} \frac{\underline{s_i^x} + \overline{s_i^x}}{\underline{\alpha_{io}} + \overline{\alpha_{io}}} + \sum_{r=1}^{S} \frac{\underline{s_r^y} + \overline{s_r^y}}{\underline{\beta_{ro}} + \overline{\beta_{ro}}} \geq (1-t)II^*, \\
& \underline{\alpha_{io}} \geq \underline{x_{io}}, \quad i \in O^X, \overline{\alpha_{io}} \geq \overline{x_{io}}, \quad i \in O^X, \\
& \underline{s_i^x} \leq \overline{s_i^x}, \quad i \in O^X, \\
& \underline{s_r^y} \leq \overline{s_r^y}, \quad r \in O^Y, \\
& \underline{\alpha_{io}} \leq \overline{\alpha_{io}}, \quad i \in O^X, \\
& \lambda_j \geq 0, \quad j = 1, \ldots, N, \\
& \underline{s_i^x}, \overline{s_i^x}, \underline{s_r^y}, \overline{s_r^y} \in \mathbb{Z}_+, \quad i \in O^{XI}, \ r \in O^{YI}, \\
& \underline{s_i^x}, \overline{s_i^x}, \underline{s_r^y}, \overline{s_r^y} \geq 0, \quad i \in O^{XNI}, \ r \in O^{YNI}, \\
& \underline{\alpha_{io}}, \overline{\alpha_{io}} \in \mathbb{Z}_+, \quad i \in O^{XI}, \\
& \underline{\alpha_{io}}, \overline{\alpha_{io}} \geq 0, \quad i \in O^{XNI},
\end{aligned}
\tag{10}
$$

where II^* is the optimal value of problem (7) and $0 \leq t \leq 1$.

Given MONLP (10) and $w = (w_1, w_2, \cdots, w_{2M}) \in \mathbb{R}^{2M}$, $w_i > 0$, $\sum_{i=1}^{2M} w_i = 1$, We introduce the following related sum problem.

$$
\begin{aligned}
\text{(MONLP)w}\quad \text{Min}\ & \sum_{i=1}^{2M} w_i \alpha_{io} \\
\text{s.t.}\ & \sum_{j=1}^{N} \lambda_j \underline{x_{ij}} \le \underline{\alpha_{io}} - \overline{s_i^x}, \quad i \in O^X, \\
& \sum_{j=1}^{N} \lambda_j \overline{x_{ij}} \le \overline{\alpha_{io}} - \underline{s_i^x}, \quad i \in O^X, \\
& \sum_{j=1}^{N} \lambda_j \underline{y_{rj}} \ge \underline{\beta_{ro}} + \underline{s_r^y}, \quad r \in O^Y, \\
& \sum_{j=1}^{N} \lambda_j \overline{y_{rj}} \ge \overline{\beta_{ro}} + \overline{s_r^y}, \quad r \in O^Y, \\
& \sum_{i=1}^{M} \frac{\underline{s_i^x} + \overline{s_i^x}}{\underline{\alpha_{io}} + \overline{\alpha_{io}}} + \sum_{r=1}^{S} \frac{\underline{s_r^y} + \overline{s_r^y}}{\underline{\beta_{ro}} + \overline{\beta_{ro}}} \ge (1-t)II^*, \\
& \underline{\alpha_{io}} \ge \underline{x_{io}}, \quad i \in O^X, \\
& \overline{\alpha_{io}} \ge \overline{x_{io}}, \quad i \in O^X, \\
& \underline{s_i^x} \le \overline{s_i^x}, \quad i \in O^X, \\
& \underline{s_r^y} \le \overline{s_r^y}, \quad r \in O^Y, \\
& \underline{\alpha_{io}} \le \overline{\alpha_{io}}, \quad i \in O^X, \\
& \lambda_j \ge 0, \quad j = 1, \dots, N, \\
& \underline{s_i^x}, \overline{s_i^x}, \underline{s_r^y}, \overline{s_r^y} \in \mathbb{Z}_+, \quad i \in O^{XI}, \ r \in O^{YI}, \\
& \underline{s_i^x}, \overline{s_i^x}, \underline{s_r^y}, \overline{s_r^y} \ge 0, \quad i \in O^{XNI}, \ r \in O^{YNI}, \\
& \underline{\alpha_{io}}, \overline{\alpha_{io}} \in \mathbb{Z}_+, \quad i \in O^{XI}, \\
& \underline{\alpha_{io}}, \overline{\alpha_{io}} \ge 0, \quad i \in O^{XNI}.
\end{aligned} \tag{11}
$$

Let us pointed out that the previous problem is a mixed- integer nonlinear optimization problem, which is NP-hard, in general. To deal with it and compute examples (following), on the one hand, we include penalties on integer variables in the objective function, following a proposal used in Arana-Jiménez and Salles (2017) and Le Thi (2020), among others. Then, we apply the R-package called "nloptr", which used methods based on gradients to provide a solution. From now on, and for the sake of simplicity, we use a similar notation to refer to vector interval solutions of (IP) and their parameterizations as real vector solutions of (MONLP). In this regard, for instance, $\boldsymbol{\alpha}_o = (\underline{\alpha_{1o}}, \overline{\alpha_{1o}}, ..., \underline{\alpha_{Mo}}, \overline{\alpha_{Mo}})$ can be interpreted as a vector of intervals or as a vector of real numbers, depending on the problem at hand. The inequality relationships are used according to the previous interpretation, being $\leqq$ for intervals, and $\leqq$ for vectors of real numbers, for instance.

The following theorem represents the relationship between the (IP) and (MONLP) solutions.

Theorem 4 *$\boldsymbol{\alpha}_o^* \in (\mathcal{K}_{\mathbb{Z}_+})^{XI} * (\mathcal{K}_C)^{XNI}, XI + XNI = M$ is an interval Pareto solution of (IP) if and only if there exist $\boldsymbol{\lambda}^* \in \mathbb{R}_+^N$, $\boldsymbol{s}^{x*} \in (\mathcal{K}_{\mathbb{Z}_+})^{XI} * (\mathcal{K}_C)^{XNI}, XI + XNI = M$ and $\boldsymbol{s}^{y*} \in (\mathcal{K}_{\mathbb{Z}_+})^{YI} * (\mathcal{K}_C)^{YNI}, YI + YNI = S$ such that the corresponding parameterization of $(\boldsymbol{\lambda}^*, \boldsymbol{\alpha}_o^*, \boldsymbol{s}^{x*}, \boldsymbol{s}^{y*})$, $(\lambda_1^*, \dots, \lambda_N^*, \underline{\alpha_{1o}^*}, \overline{\alpha_{1o}^*}, \dots, \underline{\alpha_{Mo}^*}, \overline{\alpha_{Mo}^*}, \underline{s_1^{x*}}, \overline{s_1^{x*}}, \dots, \underline{s_M^{x*}}, \overline{s_M^{x*}}, \underline{s_1^{y*}}, \overline{s_1^{y*}}, \dots, \underline{s_S^{y*}}, \overline{s_S^{y*}})$, is a Pareto solution of (MONLP).*

Proof (i) Suppose that $\boldsymbol{\alpha}_o^*$ is an interval Pareto solution of (IP). It implies that, if one considers the related optimization problem to calculate $II^*(\alpha_o^*, \beta_o)$, then there exist $\boldsymbol{\lambda}^* \in \mathbb{R}_+^N$, $\boldsymbol{s}^{x*} \in (\mathcal{K}_{\mathbb{Z}_+})^{XI} * (\mathcal{K}_C)^{XNI}, XI + XNI = M$ and $\boldsymbol{s}^{y*} \in (\mathcal{K}_{\mathbb{Z}_+})^{YI} * (\mathcal{K}_C)^{YNI}, YI + YNI = S$ such that

$$II^*(\alpha_o^*, \beta_o) = \sum_{i=1}^{M} \frac{\underline{s_i^{x*}} + \overline{s_i^{x*}}}{\underline{\alpha_{io}^*} + \overline{\alpha_{io}^*}} + \sum_{r=1}^{S} \frac{\underline{s_r^{y*}} + \overline{s_r^{y*}}}{\underline{\beta_{ro}} + \overline{\beta_{ro}}} \geq (1-t)II^*.$$

The latter means that $(\boldsymbol{\lambda}^*, \boldsymbol{\alpha}_o^*, \boldsymbol{s}^{x*}, \boldsymbol{s}^{y*})$, in its parameterization form, is a feasible solution of (MONLP). Now, reasoning by contradiction, suppose that$(\boldsymbol{\lambda}^*, \boldsymbol{\alpha}_o^*, \boldsymbol{s}^{x*}, \boldsymbol{s}^{y*})$ is not a Pareto solution of (MONLP), which implies that there exists $(\boldsymbol{\lambda}^{**}, \boldsymbol{\alpha}_o^{**}, \boldsymbol{s}^{x**}, \boldsymbol{s}^{y**})$ a feasible solution of (MONLP) such that $\alpha_o^{**} \leqq \alpha_o^*, \alpha_o^{**} \neq \alpha_o^*$. Therefore, $(\alpha_o^{**}, \beta_o) \in T, \alpha_o^{**} \geqq X_o$ and

$$II^*(\alpha_o^{**}, \beta_o) = \sum_{i=1}^{M} \frac{\underline{s_i^{x**}} + \overline{s_i^{x**}}}{\underline{\alpha_{io}^{**}} + \overline{\alpha_{io}^{**}}} + \sum_{r=1}^{S} \frac{\underline{s_r^{y**}} + \overline{s_r^{y**}}}{\underline{\beta_{ro}} + \overline{\beta_{ro}}} \geq (1-t)II^*.$$

In consequence, we have that α_o^{**} is a feasible solution of (IP), with $\alpha_o^{**} \leqq \alpha_o^*, \alpha_o^{**} \neq \alpha_o^*$, what is a contradiction with α_o^* is an interval Pareto solution of (IP).

(ii) Suppose that $(\lambda_1^*, \dots, \lambda_N^*, \underline{\alpha_{1o}^*}, \overline{\alpha_{1o}^*}, \dots, \underline{\alpha_{Mo}^*}, \overline{\alpha_{Mo}^*}, \underline{s_1^{x*}}, \overline{s_1^{x*}}, \dots, \underline{s_M^{x*}}, \overline{s_M^{x*}}, \underline{s_1^{y*}}, \overline{s_1^{y*}}, \dots, \underline{s_S^{y*}}, \overline{s_S^{y*}})$ is Pareto solution of (MONLP). From the problem (10), we derive

$$II^*(\alpha_o^*, \beta_o) = \sum_{i=1}^{M} \frac{\underline{s_i^{x*}} + \overline{s_i^{x*}}}{\underline{\alpha_{io}^*} + \overline{\alpha_{io}^*}} + \sum_{r=1}^{S} \frac{\underline{s_r^{y*}} + \overline{s_r^{y*}}}{\underline{\beta_{ro}} + \overline{\beta_{ro}}} \geq (1-t)II^*.$$

Then $(\alpha_o^*, \beta_o) \in T$, $\alpha_o^* \geqq X_o$, that is, α_o^* is a feasible solution of (IP). Proceeding by contradiction, suppose α_o^* is not an interval Pareto solution of (IP), i.e. there exists α_o^{**} feasible for (IP), with $\alpha_o^{**} \leqq \alpha_o^*$, $\alpha_o^{**} \neq \alpha_o^*$. Since α_o^{**} is feasible solution of (IP), it implies that there exists $(\lambda_1^{**}, \dots, \lambda_N^{**}, \underline{\alpha_{1o}^{**}}, \overline{\alpha_{1o}^{**}}, \dots, \underline{\alpha_{Mo}^{**}}, \overline{\alpha_{Mo}^{**}}, \underline{s_1^{x**}}, \overline{s_1^{x**}}, \dots,$

$\underline{s}^{x**}_M, \overline{s^{x**}_M}, \underline{s}^{y**}_1, \overline{s^{y**}_1}, \ldots, \underline{s}^{y**}_S, \overline{s^{y**}_S}$) feasible solution of (MONLP), with $\alpha^{**}_o \leqq \alpha^*_o$, $\alpha^{**}_o \neq \alpha^*_o$, what is a contradiction with α^*_o Pareto solution of (MONLP). □

As a consequence of the previous theorem, we have the following one that shows that the above integer interval (MONLP) can be used for input level estimation.

Theorem 5 *Assume that* II^* *is the inefficiency score of* DMU_o *in the problem* (7) *and the output of* DMU_o *are increased from* Y_0 *to* $\beta_0 = (\underline{\beta_{1o}}, \overline{\beta_{1o}}, \underline{\beta_{2o}}, \overline{\beta_{2o}}, ..., \underline{\beta_{So}}, \overline{\beta_{So}}) = Y_0 + \triangle Y_o, \triangle Y_o \gneqq 0$.

(1) *Let* $(\lambda^*_1, \ldots, \lambda^*_N, \underline{\alpha^*_{1o}}, \overline{\alpha^*_{1o}}, \ldots, \underline{\alpha^*_{Mo}}, \overline{\alpha^*_{Mo}}, \underline{s^{x*}_1}, \overline{s^{x*}_1}, \ldots, \underline{s^{x*}_M}, \overline{s^{x*}_M}, \underline{s^{y*}_1}, \overline{s^{y*}_1}, \ldots, \underline{s^{y*}_S}, \overline{s^{y*}_S})$ *be a Pareto solution to the problem* (10), *then the inefficiency score of* DMU_o *under new inputs and outputs is not less than t-percent.*

(2) *Conversely, if the new* DMU_o *belongs to the technology, and the inefficiency score of the new* DMU_o *is not less than t-percent, then there exist* $\lambda^*, \boldsymbol{s}^{x*}, \boldsymbol{s}^{y*}$ *such that* $(\lambda^*_1, \ldots, \lambda^*_N, \underline{\alpha^*_{1o}}, \overline{\alpha^*_{1o}}, \ldots, \underline{\alpha^*_{Mo}}, \overline{\alpha^*_{Mo}}, \underline{s^{x*}_1}, \overline{s^{x*}_1}, \ldots, \underline{s^{x*}_M}, \overline{s^{x*}_M}, \underline{s^{y*}_1}, \overline{s^{y*}_1}, \ldots, \underline{s^{y*}_S}, \overline{s^{y*}_S})$ *is a feasible solution for* (*MONLP*). *Furthermore, if any decrease in the input* α^*_o *of the new* DMU_o *in the Pareto sense makes not fulfill the previous conditions, then it follows that* α^*_o *is a Pareto solution of* (*MONLP*).

Proof If $(\lambda^*_1, \ldots, \lambda^*_N, \underline{\alpha^*_{1o}}, \overline{\alpha^*_{1o}}, \ldots, \underline{\alpha^*_{Mo}}, \overline{\alpha^*_{Mo}}, \underline{s^{x*}_1}, \overline{s^{x*}_1}, \ldots, \underline{s^{x*}_M}, \overline{s^{x*}_M}, \underline{s^{y*}_1}, \overline{s^{y*}_1}, \ldots, \underline{s^{y*}_S}, \overline{s^{y*}_S})$ is a Pareto solution of the problem (MONLP), then by Theorem 4 it follows that $(\lambda^*, \boldsymbol{\alpha}^*_o, \boldsymbol{s}^{x*}, \boldsymbol{s}^{y*})$ is interval Pareto solution of (IP), and then $II^*(\alpha^*_o, \beta_o) \geq (1-t)II^*$. Therefore, (1) is proof. Conversely, if the inefficiency score of DMU_o is not less than t-percent, $II^*(\alpha^*_o, \beta_o) \geq (1-t)II^*$, it means that (α^*_o, β_o) is feasible for (IP), and there exist $\lambda^*, \boldsymbol{s}^{x*}, \boldsymbol{s}^{y*}$ such that $(\lambda^*_1, \ldots, \lambda^*_N, \underline{\alpha^*_{1o}}, \overline{\alpha^*_{1o}}, \ldots, \underline{\alpha^*_{Mo}}, \overline{\alpha^*_{Mo}}, \underline{s^{x*}_1}, \overline{s^{x*}_1}, \ldots, \underline{s^{x*}_M}, \overline{s^{x*}_M}, \underline{s^{y*}_1}, \overline{s^{y*}_1}, \ldots, \underline{s^{y*}_S}, \overline{s^{y*}_S})$ is a feasible solution of (MONLP). Furthermore, since (α^*_o, β_o) is feasible for (IP) and there aren't t $\alpha^{**}_o \leqq \alpha^*_o$, $\alpha^{**}_o \neq \alpha^*_o$ then α^*_o is an interval Pareto solution of (IP), and then, by Theorem 4, is a Pareto solution of (MONLP). □

5 Numerical experiments

In this section, we introduce a problem that contains both integer and continuous variables. The data set which comes from Zhang and Cui (2020) are shown in Table 1. There are 12 DMUs. Every DMU consume three inputs and produce two outputs. The first input and the second output are continuous, and the other data are integer. Firstly, we calculate the inefficiency score of the model (7). It is indicated in Table 2. Then due to the dependency between DEA and MONLP, we can relate inverse DEA mode into single objective programming by means of weighted problems.

To illustrate the example, the result is shown for three values for DMU_1 and DMU_2 in Tables 3 and 4, respectively. In Table 3, we increase the output of DMU_1 from $Y_1 = ([67, 67], [751, 751])$ to $\beta_1 = ([80, 85], [780, 850])$ and put $t = 0.3$. After solving the model (10) by using weighted sum problem, $w = (0.2, 0.3, 0.1, 0.2, 0.1, 0.1)$, we can get $\alpha_1^* = ([350.00, 350.11], [47, 47], [13, 13])$. According to the the model (8), $II(\alpha_1^*, \beta_1) = 1.01$ which is not less than $(1 - t)II^*(X_1, Y_1) = 0.994$. Also, if we increase from $Y_1 = ([67, 67], [751, 751])$ to $\beta_1 = ([70, 73], [760, 770])$ and put $t = 0.3$, a Pareto solution for MONLP will be $\alpha_1^* = ([350.00, 350.00], [47, 47], [13, 13])$, which means the inefficiency score is not less than $(1 - t)II^*(X_1, Y_1) = 0.994$. In addition, again we increase from $Y_1 = ([67, 67], [751, 751])$ to $\beta_1 = ([67, 70], [760, 765])$ and put $t = 0.3$ and get $\alpha_1^* = ([350.00, 350.00], [47, 47], [13, 13])$ that is not less than $(1 - t)II^*(X_1, Y_1) = 0.994$. Also, in Table 4, we consider the problem for the outputs of DMU_2 and get new inputs. For example, when we increase $Y_2 = ([70, 76], [608, 620])$ to $\beta_2 = ([80, 85], [620, 630])$, we calculate $\alpha_2^* = ([304.73, 304.75], [35, 38], [14, 14])$ that the inefficiency score of new DMU is not less than $(1 - t)II^*(X_2, Y_2) = 0.259$. Also, after changing $Y_2 = ([70, 76], [608, 620])$ to $\beta_2 = ([72, 78], [619, 625])$, we get $\alpha_2^* = ([298.00, 299.11], [35, 38], [14, 14])$ which $II^*(\alpha_2^*, \beta_2)$ is not less than $(1 - t)II^*(X_2, Y_2) = 0.259$. Finally, we increase $Y_2 = ([70, 76], [608, 620])$ to $\beta_2 = ([75, 78], [610, 622])$, and a pareto solution will be $\alpha_2^* = ([298.00, 299.11], [35, 38], [14, 14])$ which the inefficiency score of new DMU is not less than $(1 - t)II^*(X_2, Y_2) = 0.259$.

As a summary of the method, we have followed to get the inefficiency score of the new DMU $II^*(\alpha^*, \beta)$, first, we calculate the inefficiency score of $II^*(X_o, Y_o)$. Then, we get the value of α in the model $(MONLP)_w$. And finally, we obtain the inefficiency score of new DMU $II^*(\alpha^*, \beta)$. The result shows the inefficiency score of new DMU under new input and output is not less than t-percent. As a limitation of this method, we point out the role of the election of w to get α, although this is normal since we are dealing with a multiobjective optimization problem.

6 Conclusions

In this paper, we present a new inverse DEA problem on the non-radial slacks-based model with integer and continuous data set. The main question on inverse DEA on the input estimation has been discussed. in this regard, we use Pareto solutions of the MONLP to determine sufficient and necessary conditions of input estimation. It is shown that in this new framework, with integer and continuous interval data, it is not guaranteed when Y_o increase to β_o, there is an increase of X_o such that $II^*(\alpha_o, \beta_o) = (1 - t)I^*(X_o, Y_o)$, what happens with crisp data. This is of difference between crisp and interval data. Therefore, the method can be applied to increase inputs for a slacks-based model such that the inefficiency score of DMU_o is not less than t-percent. Necessary and sufficient conditions are established for each DMU with integer and interval variables. The present work establishes the first response to inverse DEA under integer interval-type uncertainty on data, which is an important step to address a future study under fuzzy data. Another potential research direction

Table 1 Data of 12 DMUs

DMU (j)	1	2	3	4	5	6	7	8	9	10	11	12
x_{1j}	[350, 350]	[298, 298]	[420, 424]	[281, 281]	[301, 301]	[360, 360]	[540, 540]	[276, 276]	[300, 350]	[444, 444]	[323, 323]	[444, 444]
x_{2j}	[39, 39]	[25, 28]	[31, 31]	[16, 16]	[16, 16]	[29, 29]	[15, 21]	[33, 33]	[25, 25]	[61, 67]	[25, 25]	[64, 64]
x_{3j}	[9, 9]	[8, 8]	[7, 7]	[9, 9]	[4, 7]	[17, 17]	[10, 10]	[5, 5]	[5, 5]	[6, 6]	[3, 7]	[3, 9]
y_{1j}	[67, 67]	[70, 76]	[75, 75]	[70, 70]	[75, 75]	[83, 83]	[70, 81]	[78, 78]	[75, 75]	[74, 74]	[25, 25]	[104, 104]
y_{2j}	[751, 751]	[608, 620]	[584, 584]	[665, 665]	[442, 449]	[1070, 1070]	[457, 457]	[583, 595]	[1074, 1074]	[1072, 1072]	[350, 350]	[1199, 1199]

Table 2 Results of inefficiency score of slacks-based model ($II^*(X_o, Y_o)$) and slacks

DMU (j)	1	2	3	4	5	6	7	8	9	10	11	12
$II^*(X_o, Y_o)$	1.42	0.37	1.66	0.0	0.0	0.0	0.0	0.0	0.0	1.02	2.68	0.0
s_1^x	[0.0, 50.00]	[0.0, 9.51]	[4.00, 60.00]	[0.0, 0.0]	[0.0, 0.0]	[0.0, 0.0]	[0.0, 0.0]	[0.0, 0.0]	[0.0, 0.0]	[24.00, 84.00]	[65.24, 89.12]	[0.0, 0.0]
s_2^x	[14, 14]	[0, 0]	[1, 1]	[0, 0]	[0, 0]	[0, 0]	[0, 0]	[0, 0]	[0, 0]	[31, 31]	[0, 0]	[0, 0]
s_3^x	[4, 4]	[2, 3]	[1, 1]	[0, 0]	[0, 0]	[0, 0]	[0, 0]	[0, 0]	[0, 0]	[0, 0]	[0, 0]	[0, 0]
s_1^y	[8, 8]	[0, 0]	[15, 15]	[0, 0]	[10, 10]	[10, 10]	[0, 0]	[0, 0]	[0, 0]	[16, 16]	[32, 32]	[0, 0]
s_2^y	[323.00, 323.00]	[28.37, 28.37]	[704.80, 704.80]	[0.0, 0.0]	[0.0, 0.0]	[0.0, 0.0]	[0.0, 0.0]	[0.0, 0.0]	[0.0, 0.0]	[216.80, 216.80]	[407.58, 407.58]	[0.0, 0.0]

Table 3 The results of (MONLP) and inefficiency score ($II^*(\alpha^*, \beta)$) for DMU_1 and slacks with $w = (0.2, 0.3, 0.1, 0.2, 0.1, 0.1)$ and $t = 0.3$

Optimal value, slacks and inputs	$\beta_1 = ([80, 85], [780, 850])$	$\beta_1 = ([70, 73], [760, 770])$	$\beta_1 = ([67, 70], [760, 765])$
α_1	([350.00, 350.11], [47, 47], [13, 13])	([350.00, 350.00], [47, 47], [13, 13])	([350.00, 350.00], [47, 47], [13, 13])
$II^*(\alpha^*, \beta)$	1.01	1.58	1.63
optimal value	191.76	191.70	191.74
s_1^x	[0.00, 6.00]	[31.15, 34.74]	[30.80, 32.53]
s_2^x	[1, 1]	[1, 1]	[1, 1]
s_3^x	[3, 6]	[3, 6]	[3, 6]
s_1^y	[1, 2]	[1, 2]	[1, 2]
s_2^y	[1.00, 2.00]	[17.12, 17.30]	[20.55, 21.55]

Table 4 The results of (MONLP) and inefficiency score ($II^*(\alpha^*, \beta)$) for DMU_2 and slacks with $w = (0.2, 0.3, 0.1, 0.2, 0.1, 0.1)$ and $t = 0.3$

Optimal value, slacks and inputs	$\beta_2 = ([80, 85], [620, 630])$	$\beta_2 = ([72, 78], [619, 625])$	$\beta_2 = ([75, 78], [610, 622])$
α_2	([304.73, 304.75], [35, 38], [14, 14])	([298.00, 299.11], [35, 38], [14, 14])	([298.00, 298.97], [35, 38], [14, 14])
$II^*(\alpha^*, \beta)$	0.65	0.89	0.91
optimal value	176.05	165.19	165.06
s_1^x	[0.00, 0.00]	[0.00, 2.00]	[0.00, 2.00]
s_2^x	[1, 1]	[1, 1]	[1, 1]
s_3^x	[2, 3]	[2, 3]	[2, 3]
s_1^y	[1, 2]	[1, 2]	[1, 2]
s_2^y	[1.00, 2.00]	[1.00, 2.00]	[1.00, 2.00]

would be non-radial inverse DEA with negative and undesirable integer and continuous interval data, which will lead our future research.

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