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Hypercyclicity of operators that λ-commute with the differentiation operator on the space of entire functions ☆

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An operator T acting on a separable F-space $\mathcal{X}$ is called hypercyclic if there exists $f \in \mathcal{X}$ such that the orbit $\{T^n f\}$ is dense in $\mathcal{X}$. Here we determine when an operator that λ-commutes with the operator of differentiation on the space of entire functions is hypercyclic, extending results by G. Godefroy and J. H. Shapiro [16] and R. M. Aron and D. Markose [1].

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1. Introduction

A (continuous, linear) operator T defined on a separable F-space $\mathcal{X}$ is said to be *hypercyclic* if there exists $f \in \mathcal{X}$ such that $\{T^n f\}_{n\geq 1}$ is dense in $\mathcal{X}$. We refer to [17] and [4] for further information about hypercyclic operators.

The term λ-commuting was introduced by J.B. Conway and G. Prăjitură in [14]. More recently, a complex number λ is called an *extended eigenvalue* of an operator T if there exists a non-zero continuous operator X, which is called an *extended λ-eigenoperator of T*, such that $TX = \lambda XT$. Extended eigenvalues and extended eigenoperators are naturally born to improve V. Lomonosov's famous result on the invariant subspace problem ([13,19,24]) and their study is currently under development (see [21,26]).

Let $\mathcal{H}(\mathbb{C})$ be the space of entire functions endowed with the topology of uniform convergence on compact subsets. G. D. Birkhoff ([10]) proved in 1929 that translation operators on $\mathcal{H}(\mathbb{C})$ are hypercyclic. In 1952, G. R. MacLane ([25]) proved the same result for the differentiation operator D on $\mathcal{H}(\mathbb{C})$. These results appear to be the first hypercyclicity theorems for operators.

In 1991, G. Godefroy and J. H. Shapiro (see [16]) unified Birkhoff's and MacLane's results by proving that each non-scalar operator that commutes with D is hypercyclic. The simplicity and beauty of this statement is striking, and it is worthy to note that there is no analogous result in the context of Banach spaces since contractions on these spaces are never hypercyclic. This result has been improved and extended in different directions, making [16] one of the most cited papers on hypercyclic operators. A step further in Godefroy and Shapiro's result arises with the following question:

> Suppose that T is an operator on $\mathcal{H}(\mathbb{C})$ which is an extended λ-eigenoperator of D; that is, $DT = \lambda TD$. Is T hypercyclic?

At first glance, this question seems difficult because there are examples of non-trivial extended eigenoperators of D which are not hypercyclic. The first one was discovered by L. Bernal and A. Montes (see [7]), who showed that the composition operator $C_{\lambda,b}f(z) = f(\lambda z + b)$ induced by the affine endomorphism $\varphi(z) = \lambda z + b$ is hypercyclic if and only if φ is a proper translation ($\lambda = 1$ and $b \neq 0$). It is easy to see that $C_{\lambda,b}$ is an extended λ-eigenoperator of D. From these facts, it can be suspected that there are many extended eigenoperators of D that are not hypercyclic. But, is there a non-trivial one that is hypercyclic? R. M. Aron and D. Markose answered this question affirmatively. Denoting $T_{\lambda,b}f(z) = f'(\lambda z + b)$, then $T_{\lambda,b}$ is an extended λ-eigenoperator of D, and $T_{\lambda,b}$ is hypercyclic if and only if $|\lambda| \geq 1$ (see [1,15,23]).

Along the paper, ϕ will be a non-zero entire function of *exponential type:* there are constants $A, B > 0$ such that $|\phi(z)| \leq Ae^{B|z|}$ for all $z \in \mathbb{C}$. We will denote by span A the subspace generated by a subset A of a vector space.

In this article, we fully characterize when an extended λ-eigenoperator of D is hypercyclic. Our main result characterizes the $\lambda-$ commutant of D on $\mathcal{H}(\mathbb{C})$ and summarizes all the results on hypercyclicity and mixing properties. Let us recall that an operator T is called *topologically mixing* if for any non empty open subsets U, V, there exists $N \in \mathbb{N}$ such that $T^n(U) \cap V \neq \emptyset$ for all $n \geq N$ (equivalently for any subsequence of natural numbers $(n_k) \subset \mathbb{N}$ there exists a function f such that $\{T^{n_k} f\}_{k\geq 1}$ is dense in $H(\mathbb{C})$). Our Main Theorem is then stated as follows:

Main Theorem. *An operator $T : H(\mathbb{C}) \to H(\mathbb{C})$ is an extended λ-eigenoperator of D if and only if T factors as $T = R_\lambda \phi(D)$, where $R_\lambda f(z) = f(\lambda z)$ is the dilation operator and ϕ is an entire function of exponential type.*

Moreover, the following statements are equivalent for $T = R_\lambda \phi(D)$ when $\lambda \neq 1$:

- **a)** *The operator T is hypercyclic.*
- **b)** *The operator T is topologically mixing.*
- **c)** *The zero-set of ϕ is non-trivial (i.e., $\phi^{-1}(0) \neq \emptyset, \mathbb{C}$) and $|\lambda| \geq 1$.*

The paper is organized as follows. In Section 2, we introduce some preliminary results, including the Hypercyclicity Criterion of C. Kitai [20], in a version formulated by J. Bès and A. Peris in [9]. We show that an extended λ-eigenoperator T of D can be factorized as $T = R_\lambda \phi(D)$, where $R_\lambda f(z) = f(\lambda z)$ and ϕ is an entire function of exponential type. So we can study the hypercyclicity of T in terms of the properties of ϕ and λ. We also show that ϕ has no zeros if and only if $R_\lambda \phi(D)$ is a nonzero multiple of $C_{\lambda,b}$, and it is not hypercyclic in this case.

We divide the rest of the proof of the Main Theorem in cases (assuming that ϕ has an isolated zero) which are treated in successive sections:

3: $|\lambda| < 1$ and $\lambda^n = 1$ for some $n \in \mathbb{N}$;
4: $|\phi(0)| > 1$ and $|\lambda| \geq 1$;
5: $0 < |\phi(0)| \leq 1$ and $|\lambda| > 1$;
6: $0 < |\phi(0)| \leq 1$ and $|\lambda| = 1$;
7: $\phi(0) = 0$ and $|\lambda| \geq 1$.

Thus, the main result is obtained by considering different cases for the values of λ and $\phi(0)$, which share a similar flavor to recent studies on algebras of hypercyclic vectors for convolution operators by F. Bayart, J. Bès and coworkers [8,2].

Each particular case is solved by a different method. Using some arguments borrowed from [16], we prove the cases when $|\lambda| < 1$ and when λ is a root of the unity. However, new ideas are needed to solve the rest of the cases.

In the case when $|\lambda| \geq 1$ and $|\phi(0)| > 1$, we analyze the action of T on the exponentials e^{az}, and we show that T has a dense generalized kernel. Then we construct the right inverse required by the Hypercyclicity Criterion using the triangularity of T and a linear algebra argument.

When $|\lambda| > 1$ and $0 < |\phi(0)| \leq 1$, the operator $T = R_\lambda \phi(D)$ is not injective. So the right inverse needed to apply the Hypercyclicity Criterion is not unique, and the construction in the previous section does not provide a right inverse in this case. However, using the Pólya representation of an entire function, we obtain an integral representation of the powers of the operator which allows us to find a sequence of right inverses for the powers of the operator. With this sequence of right inverses and the Hypercyclicity Criterion we prove the desired result.

The case $0 < |\phi(0)| \leq 1$ and λ an irrational rotation is the most intriguing one. When λ is a root of unity, the problem can be solved using a result on powers of hypercyclic operators, but when λ is an irrational rotation the solution is different. In many cases; e.g., when $\phi(z)$ is a polynomial $p(z)$ or $\phi(z) = p(z)e^z$, we can deduce the result by standard arguments. However, as far as we know, these arguments cannot be used in the general case, and the problem requires an argument involving normal families. Montel's Theorem plays an important role in guaranteeing the universality of a family of functions on the complex plane: it is the key of the proof.

A different treatment is needed also in the case $\phi(0) = 0$, including the operator $T_{\lambda,b}$, which is different from those used in [1,15,23]. We need to refine the computations in the case $|\phi(0)| > 0$ and $|\lambda| \geq 1$ using the complex Volterra operator.

2. Some preliminary results

We will need the following version of the Hypercyclicity Criterion formulated by J. Bés and A. Peris in [9]. Although the original Kitai-Gethner-Shapiro criterion and the Bés-Peris version are equivalent (see [17] p. 81), the latter is easier to use in practice.

Theorem 2.1 *(Hypercyclicity Criterion). Let T be an operator on an F-space $\mathcal{X}$ satisfying the following conditions: there exist X_0 and Y_0 dense subsets of $\mathcal{X}$, a sequence (n_k) of non-negative integers, and (not necessarily continuous) mappings $S_{n_k} : Y_0 \to \mathcal{X}$ so that:*

- **i)** $T^{n_k} \to 0$ *pointwise on* X_0.
- **ii)** $S_{n_k} \to 0$ *pointwise on* Y_0.
- **iii)** $T^{n_k} S_{n_k} \to Id_{Y_0}$ *pointwise on* Y_0.

Then the operator T is hypercyclic.

Observe that if T satisfies the Hypercyclicity Criterion for the full sequence of natural numbers, then T is topologically mixing.

In many cases, we obtain that the operators we study are *chaotic* in the Devaney sense, that is: they are hypercyclic and have a dense set of periodic points (see [17, Section 1.2]). Moreover, we will prove that these operators are frequently hypercyclic in several cases. Recall that an operator T defined on an F-space $\mathcal{X}$ is said to be *frequently hypercyclic* if there exists some $f \in \mathcal{X}$ such that for any non-empty subset U, the subset $\{n \in \mathbb{N} \, : \, T^n f \in U\}$ has a positive lower density in $\mathbb{N}$, that is:

$$\liminf_{N\to\infty} \frac{\text{card}\{n \leq N, : \, T^n f \in U\}}{N+1} > 0.$$

To obtain frequently hypercyclicity, we will use the following sufficient condition discovered by F. Bayart and S. Grivaux (see [3]). The next version was formulated by A. Bonilla and KG. Grosse-Erdmann in [12]:

Theorem 2.2 *(Frequently Hypercyclicity Criterion). Let T be an operator on an F-space $\mathcal{X}$ if there is a dense subset X_0 and a map $S : X_0 \to X_0$ such that, for any $x_0 \in X_0$*

i) $\sum_{n=0}^{\infty} T^n x_0$ *converges unconditionally.*
ii) $\sum_{n=0}^{\infty} S^n x_0$ *converges unconditionally.*
iii) $TSx_0 = x_0$.

Then T is frequently hypercyclic.

Next, we give a result inspired by [22] that will be central in our discussion.

Proposition 2.3. *Let T be an operator on $\mathcal{H}(\mathbb{C})$. Then $DT = \lambda TD$ for some $0 \neq \lambda \in \mathbb{C}$ if and only if $T = R_\lambda \phi(D)$ with $R_\lambda f(z) = f(\lambda z)$ for $z \in \mathbb{C}$ and ϕ an entire function of exponential type.*

Proof. Suppose that $DT = \lambda TD$ with $\lambda \neq 0$. Given $f \in \mathcal{H}(\mathbb{C})$ and $\lambda \in \mathbb{C} \setminus \{0\}$, the operator $R_{1/\lambda}$ is an extended $(1/\lambda)$-eigenoperator of D, that is, $DR_{1/\lambda} = \frac{1}{\lambda} R_{1/\lambda} D$. Besides,

$$R_{1/\lambda} TDf = \frac{1}{\lambda} R_{1/\lambda} DTf = \frac{1}{\lambda} \lambda DR_{1/\lambda} T = DR_{1/\lambda} Tf.$$

Hence, $R_{1/\lambda}T$ commutes with D. By Proposition 5.2 in [16], there exists an entire function ϕ of exponential type such that $R_{1/\lambda}T = \phi(D)$. Since $R_{1/\lambda}$ is invertible with inverse R_λ, we deduce that $T = R_\lambda \phi(D)$. Conversely, if there exists an entire function ϕ of exponential type such that $T = R_\lambda \phi(D)$, since $D\phi(D) = \phi(D)D$, $DT = \lambda TD$. □

The following result deals with the case in which ϕ has no zeros. Thus we can assume that ϕ has an isolated zero in the remaining sections.

Proposition 2.4. *Let $1 \neq \lambda \in \mathbb{C}$. Then $T = R_\lambda \phi(D)$ is a multiple of $C_{\lambda,b}$ for some scalar b if and only if ϕ has no zeros on $\mathbb{C}$. In this case T is not hypercyclic.*

Proof. If $\phi(z) \neq 0$ for all $z \in \mathbb{C}$, we can define the logarithm of $\phi(z)$ (see, e.g., p. 226 in [18]), and there exists an entire function g such that $\phi(z) = e^{g(z)}$. Since ϕ is entire of exponential type, $g(z) = az + b$ for some $a, b \in \mathbb{C}$. Thus $Tf(z) = e^b f(\lambda z + a)$, which is not hypercyclic when $\lambda \neq 1$ (see [7]). Indeed, set $c = a/(1-\lambda)$, the fixed point of the map $\lambda z + a$. If f is hypercyclic for T then, the orbit $T^n f(c)$ should be dense in $\mathbb{C}$. However the sequence $T^n f(c) = e^{nb} f(c)$ is either bounded (if $|e^b| \leq 1$) or diverges to infinity (if $|e^b| > 1$), a contradiction. Conversely, if $R_\lambda \phi(D) = \mu C_{\lambda,b}$ for some scalars μ and b, then

$$\phi(D) = R_\lambda^{-1} \mu C_{\lambda,b} = \mu C_{1,b}.$$

So, $\phi(z) = \mu e^{bz}$ for all $z \in \mathbb{C}$. Clearly, ϕ has no zeros on $\mathbb{C}$, and this finishes the proof. □

Propositions 2.3 and 2.4 provide a way to study our problem by looking at the properties of ϕ and λ.

3. The cases $|\lambda| < 1$ and λ is a root of 1

In the first case, we will show that $T = R_\lambda \phi(D)$ is not hypercyclic. At first glance, one may think that the cases of Fréchet spaces and Banach spaces are similar. However, using some ideas of [5], we will show that in the Banach space setting, an extended λ-eigenoperator with $|\lambda| < 1$ is not hypercyclic. This is no longer true for Fréchet spaces.

Proposition 3.1. *Let A and T be two operators on a Banach space. If T is an extended λ-eigenoperator of A and $|\lambda| < 1$ then T is not hypercyclic.*

Proof. Assume that T is hypercyclic. Then there exists $x \in \mathcal{X}$ such that $\{T^n x\}_{n \geq 1}$ is dense in $\mathcal{X}$. Hence, $\{A^m T^n x\}_{n \geq 1}$ is also dense in $A^m(\mathcal{X})$ for each $m \geq 1$. Since $|\lambda| < 1$, we can choose $m \geq 1$ such that $|\lambda|^m \|T\| \leq 1$. Observing that $A^m T^n = \lambda^{nm} T^n A^m$, we have:

$$\|A^m T^n x\| = |\lambda|^{nm} \|T^n A^m x\| \leq |\lambda|^{nm} \|T^n\| \|A^m x\| \leq \|A^m x\|.$$

Hence, we get a contradiction, and T cannot be hypercyclic. □

Proposition 3.1 is not true in Fréchet spaces:

Example 3.2. Let us recall the operator $T_{\lambda,b} f = f'(\lambda z + b)$ introduced by Aron and Markose. For $|\lambda| > 1$, $T_{\lambda,b}$ is hypercyclic. On the other hand $T_{\lambda,b} D = (1/\lambda) D T_{\lambda,b}$. Hence D is a hypercyclic extended $(1/\lambda)$-eigenoperator of $T_{\lambda,b}$ with $|1/\lambda| < 1$.

However, our case is not one of these examples:

Proposition 3.3. *If $|\lambda| < 1$ and T is an extended λ-eigenoperator of D then T is not hypercyclic.*

Proof. First, we give a representation of T similar to one in the proof of Proposition 5.2 in [16]. We consider $\Lambda \in \mathcal{H}(\mathbb{C})^*$ defined by $\Lambda f = Tf(0)$. By the Hahn-Banach theorem and the Riesz Representation theorem, there exists a complex Borel measure μ with compact support in $\mathbb{C}$ such that

$$\Lambda f = Tf(0) = \int f(w)\,d\mu(w)$$

for all $f \in \mathcal{H}(\mathbb{C})$. For each $\alpha \in \mathbb{C}$ we consider the translation operator τ_α defined by $\tau_\alpha f(z) = f(z+\alpha)$. Since

$$f(z+\alpha) = \sum_{k=0}^{\infty} f^{(k)}(z)\frac{\alpha^k}{k!} = \left(\sum_{k=0}^{\infty} \frac{\alpha^k}{k!} D^k\right) f(z)$$

we have $\tau_\alpha T = \left(\sum_{k=0}^{\infty} \frac{\alpha^k}{k!} D^k\right) T = T\left(\sum_{k=0}^{\infty} \frac{\lambda^k \alpha^k}{k!} D^k\right) = T\tau_{\lambda\alpha}$. Therefore

$$Tf(z) = (\tau_z Tf)(0) = (T\tau_{\lambda z} f)(0) = \int f(\lambda z + w)\,d\mu(w)$$

for each $f \in \mathcal{H}(\mathbb{C})$. Iterating the above equality we get:

$$T^n f(z) = \int \cdots \int f(\lambda^n z + \lambda^{n-1} w_1 + \cdots + w_n)\,d\mu(w_n)\cdots d\mu(w_1).$$

Thus, if the disc $D(0,R)$ contains the support of μ, since

$$|\lambda^n z + \lambda^{n-1} w_1 + \cdots + w_n| \le M(|z|) = |\lambda^n||z| + \frac{1-|\lambda|^n}{1-|\lambda|}R,$$

for $|z| \le r$ each element of the argument of f in the above integral lies in the disk $D(0, M(r))$. Hence, for $f \in \mathcal{H}(\mathbb{C})$ and $|z| \le r$ we get:

$$|T^n f(z)| \le \sup_{|z|=M(r)} |f(z)|\ \|\mu\|^n,$$

where $\|\mu\|$ denotes the total variation of μ.

Assume there exists $f \in \mathcal{H}(\mathbb{C})$ such that $\{T^n f\}_{n\ge1}$ is dense in $\mathcal{H}(\mathbb{C})$. Since $|\lambda| < 1$, there exists $m \in \mathbb{N}$ such that $|\lambda|^m < 1/\|\mu\|$. Since D has dense range, $\{D^m T^n f\}_{n\ge1}$ is dense in $\mathcal{H}(\mathbb{C})$. However, for $|z| \le r$ we get

$$|D^m T^n f(z)| = |\lambda|^{mn} T^n (D^m f)(z)| \le |\lambda|^{mn} \|\mu\|^n \max_{|z|\le M(r)} |D^m f(z)|$$

which goes to 0 as $n \to \infty$, a contradiction. Thus T is not hypercyclic. □

When λ is a root of 1, the result of Godefroy and Shapiro for $\lambda = 1$ allows us to prove the following result.

Proposition 3.4. *If $\lambda^{n_0} = 1 \neq \lambda$ and for some scalar b the operator $T = R_\lambda \phi(D)$ is not a multiple of $C_{\lambda,b}$ then T is hypercyclic.*

Proof. If $\lambda^{n_0} = 1$ then $R_\lambda^{n_0} = I$. Hence,

$$T^{n_0} f = \big(R_\lambda \phi(D)\big)^{n_0} f = \big(\phi(D)\phi(\lambda D) \cdots \phi(\lambda^{n_0-1} D)\big) f.$$

Thus, if $\Phi(z) = \prod_{j=0}^{n_0-1} \phi(\lambda^j z)$ is zero then $T = 0$, and if Φ is a nonzero constant function, then ϕ has no zeros, and T is a multiple of $C_{\lambda,b}$ by Proposition 2.4. On the other hand if $\Phi(z)$ is not constant, then $T^{n_0} = \Phi(D)$ is hypercyclic by [16, Theorem 5.1]. □

Remark 3.5. The previous proof entails a stronger result: the operator T in Proposition 3.4 is frequently hypercyclic, chaotic, and topologically mixing. Indeed, from [12], we have that $T^{n_0} = \Phi(D)$ is frequently hypercyclic, chaotic, and topologically mixing whenever Φ is non-constant. The following claim leads straight to the conclusion.

Claim. *If a power T^{n_0} of an operator T in a F-space $\mathcal{X}$ is (i) frequently hypercyclic, (ii) chaotic or (iii) topologically mixing, then the operator T has the same property.*

Proof of the Claim. The case (i) follows from [17, Theorem 9.27]. If x_0 is periodic for T^{n_0} then by definition x_0 is also periodic for T, therefore (ii) follows directly. Finally to show (iii), let U, V be two non-empty open subsets, then since the rank of T is dense, we get that $T^{-j}(U)$ is an non-empty open subset for each $j = 0, \cdots, n_0$. Now, since T^{n_0} is topologically mixing, there exist $N_0, N_1, \cdots, N_{n_0}$ such that

$$T^{mn_0}(V) \cap T^{-j}(U) \neq \emptyset$$

for all $m \geq N_j$, $j = 0, \cdots, n_0$. Hence,

$$T^{mn_0+j}(V) \cap U \neq \emptyset$$

for all $n \geq N_j$. Let $N = \max\{N_0, \cdots, N_{n_0}\}$. For all, $m \geq N(n_0 + 1)$, we obtain that $m = nn_0 + j$ for some $0 \leq j < n_0$, $n > N \geq N_j$ therefore

$$T^m(V) \cap U = T^{nn_0+j}(V) \cap U \neq \emptyset,$$

which means that T is topologically mixing as desired.

4. The case $|\phi(0)| > 1$ and $|\lambda| \geq 1$

This case can be dealt with by using a standard argument.

Proposition 4.1. *Assume $|\phi(0)| > 1$, $|\lambda| \geq 1$, and λ is not a root of the unity. If $T = R_\lambda \phi(D)$ is not a multiple of $C_{\lambda,b}$ then T is topologically mixing.*

Proof. By Proposition 2.4, there exists $a \in \mathbb{C}$, $a \neq 0$, such that $\phi(a) = 0$. We consider the subset $X_0 = \operatorname{span}\{e^{(a/\lambda^n)z}\,;\, n \geq 0\}$.

Since $|\lambda| \geq 1$ and λ is not a root of the unity, the set $\{a/\lambda^n : n \in \mathbb{N}\}$ has an accumulation point in $\mathbb{C}$; hence X_0 is dense in $\mathcal{H}(\mathbb{C})$. On the other hand, since $T^n e^{(a/\lambda^k)z} = 0$ if $n > k$, T^n converges to zero pointwise on X_0.

We will construct a mapping S on a dense subset Y_0 such that $S^n y \to 0$ for all $y \in Y_0$, and $TS = \mathrm{Id}_{Y_0}$. First, observe that the subspace $\mathcal{P}_n$ of polynomials of degree less or equal than n is invariant under $T = R_\lambda \phi(D)$, and the action of T on $\mathcal{P}_n$ can be represented by a finite triangular matrix with diagonal entries $\phi(0)\lambda^k$, $k \geq 0$. Since $\lambda \neq 1$, T has $n+1$ different eigenvalues in $\mathcal{P}_n$. Thus, there exists a sequence $\{p_k : k \geq 0\}$ of polynomials with degree of p_k equal to k such that $Tp_k = \phi(0)\lambda^k p_k$ for all $k \geq 0$, and

$$Y_0 = \operatorname{span}\{p_k(z) \,:\, k \geq 0\}$$

is the subspace of polynomials, which is dense. We define $Sp_k = \frac{1}{\phi(0)\lambda^k} p_k$ and extend S to Y_0 by linearity. Since $|\phi(0)| > 1$, $S^n p_k \to 0$ as $n \to \infty$ for every $|\lambda| \geq 1$, hence $S^n y \to 0$ as $n \to \infty$ for all $y \in Y_0$. Therefore, the Hypercyclicity Criterion implies that T is hypercyclic. Moreover, we have shown that the Hypercyclicity Criterion is satisfied for the full sequence of natural numbers, therefore the operator T is topologically mixing. □

Remark 4.2. Note that when $|\phi(0)| > 1$ and $|\lambda| > 1$, the proof of the above result yields the frequent hypercyclicity of T. As a consequence, T is also chaotic and topologically mixing. Let us check that T satisfies the Frequent Hypercyclicity Criterion. For that, set $e_k(z) = e^{a/\lambda^k}$, with $a \in \phi^{-1}(0)$. Since $a\lambda^k$ has an accumulation point $X_0 = \operatorname{span}\{e^{(a/\lambda^n)z}\,;\, n \geq 0\}$ is dense in $H(\mathbb{C})$. Since $T^n e^{a/\lambda^k} = 0$ for $n > k$, it follows that if $f \in X_0$

$$\sum_{n=0}^{\infty} T^n f$$

is a finite sum, therefore it converges unconditionally for any $f \in X_0$. Let us define the linear mapping

$$Se_k(z) = \frac{1}{\phi\left(\frac{a}{\lambda^k}\right)} e_{k+1}.$$

The linear mapping S satisfies that $TS = I$ on X_0. It remains to show that

$$\sum_{n=0}^{\infty} S^n f$$

converges unconditionally for any $f \in X_0$, and this fact follows if we show that for any $R > 0$ and $k \geq 0$

$$\sum_{n=0}^{\infty} \rho_R(S^n e_k),$$

converges for every seminorm $\rho_R(f) = \max_{|z|\leq R} |f(z)|$. Indeed, since

$$\left|\phi\left(\frac{a}{\lambda^r}\right)\right| \longrightarrow |\phi(0)| > 1, \quad \text{as} \;\; r \to \infty,$$

there exist $\varepsilon > 0$ and $r_0 \in \mathbb{N}$ such that for any $r \geq r_0$ $\left|\phi\left(\frac{a}{\lambda^r}\right)\right| > 1 + \varepsilon$. Thus,

$$\begin{aligned}\sum_{n=r_0+k}^{\infty} \rho_R(S^n e_k) &= \sum_{n=r_0+k}^{\infty} \rho_R\left(\frac{1}{\phi\left(\frac{a}{\lambda^{k+1}}\right)\phi\left(\frac{a}{\lambda^{k+2}}\right)\cdots\phi\left(\frac{a}{\lambda^{k+n}}\right)} e^{\frac{a}{\lambda^{k+n}}z}\right) \\ &= \sum_{n=r_0+k}^{\infty} \frac{1}{\left|\phi\left(\frac{a}{\lambda^{k+1}}\right)\phi\left(\frac{a}{\lambda^{k+2}}\right)\cdots\phi\left(\frac{a}{\lambda^{k+n}}\right)\right|}\rho_R\left(e^{\frac{a}{\lambda^{k+n}}z}\right) \\ &\leq C_1 \sum_{n=r_0+k}^{\infty} \frac{1}{\left|\phi\left(\frac{a}{\lambda^{r_0+1}}\right)\cdots\phi\left(\frac{a}{\lambda^{k+n}}\right)\right|} e^{\frac{|a|R}{|\lambda|^{k+n}}} \\ &\leq C_2 \sum_{n=r_0+k}^{\infty} \frac{1}{(1+\varepsilon)^{k+n-r_0}} < \infty,\end{aligned}$$

for some constants C_1 and C_2 which are independent of n. Hence T is frequently hypercyclic

5. The case $0 < |\phi(0)| \leq 1$ and $|\lambda| > 1$

In this case, if we use the inverse defined as in the previous section; that is $Sp_k = \frac{1}{\phi(0)\lambda^k} p_k$, then $|\phi(0)\lambda^n| > 1$ for $n > n_0$ for some n_0. This implies that $S^k p_n \to 0$ for all $n > n_0$. However on the subspace of polynomials of degree less or equal than n_0 we do not have convergence to zero.

It was pointed out in [6] that $\phi(D)$ is injective if and only if it is a multiple of $C_{\lambda,b}$. Thus our operator T is not injective, so its right inverse is not unique, and we will overcome the obstacle by defining a different right inverse.

Let f be an entire function of exponential type $f(z) = \sum_{n=0}^{\infty} a_n z^n$. The Borel transform of f is defined as

$$Bf(z) = \sum_{n=0}^{\infty} \frac{n! a_n}{z^{n+1}}.$$

It is well known that $Bf(z)$ is analytic on $|z| > c$ for some $c > 0$. In particular, for the monomials $f_n(z) = z^n/n!$ we have $Bf_n(z) = 1/z^{n+1}$ which is analytic on $|z| > 0$.

Pólya representation of f (see [11] p. 78) asserts that if $Bf(z)$ is analytic on $|z| > c$ then for any $R > c$, we have

$$f(z) = \frac{1}{2\pi i} \oint_{|t|=R} e^{zt} Bf(t)\, dt.$$

Using this representation, if $\phi(D) = \sum_{n=0}^{\infty} \phi_n D^n$, then

$$\begin{aligned} R_\lambda \phi(D) f(z) &= R_\lambda \left(\sum_n \phi_n \frac{1}{2\pi i} \oint_{|t|=R} t^n e^{tz} Bf(t)\, dt \right) \\ &= R_\lambda \frac{1}{2\pi i} \oint_{|t|=R} \left(\sum_n \phi_n t^n \right) e^{zt} Bf(t)\, dt \\ &= \frac{1}{2\pi i} \oint_{|t|=R} \phi(t) e^{\lambda zt} Bf(t)\, dt. \end{aligned}$$

And iterating the above formula, we get:

$$\left(R_\lambda \phi(D)\right)^n f(z) = \frac{1}{2\pi i} \oint_{|t|=R} \phi(t)\phi(\lambda t) \cdots \phi(\lambda^{n-1} t) e^{\lambda^n zt} Bf(t)\, dt.$$

On the other hand, denoting $\omega = 1/\lambda$, if for some $R > c$ we define

$$S_1 f(z) = \frac{1}{2\pi i} \oint_{|t|=R} \frac{1}{\phi(\omega t)} e^{\omega zt} Bf(t)\, dt, \tag{1}$$

arguing as in the above computation of $R_\lambda \phi(D) f(z)$ we get $R_\lambda \phi(D) S_1 f = f$. Let us point out that $R > c$ is chosen such that $\phi(z)$ has no zeros on the circle $|z| = R|\lambda|$, because the zeros of ϕ are isolated. That is, the integral in equation (1) is well defined.

The next result will be needed to prove this case.

Proposition 5.1. *Let $P(z) = c(1 - z/a)$ with $c \neq 0 \neq a$. Then there exists a sequence (R_k) of positive numbers converging to ∞ such that for each $n \geq 0$,*

$$(L_k f_n)(z) = \frac{1}{2\pi i} \oint_{|t|=R_k} \frac{1}{P(\omega t) \cdots P(\omega^k t)} e^{\omega^k zt} Bf_n(t)\, dt \tag{2}$$

converges to zero, uniformly on compact subsets, as $k \to \infty$.

Proof. Since $|\omega| < 1$ the subset $X_0 = \text{span}\{e^{a\omega^n z} : n \geq 0\}$ is dense in $H(\mathbb{C})$. Moreover for each $x_0 \in X_0$, $T^n x_0 = 0$ for n large enough. We choose $M_0 \geq 1$ such that $|P(z)| \geq 2$ for $|z| \geq M_0$, set $R_k = |\lambda|^k M_0$, define L_k on $f_n(z) = z^n/n!$ by

$$L_k f_n(z) = \frac{1}{2\pi i} \oint\limits_{|t|=R_k} \frac{1}{P(\omega t)\cdots P(\omega^k t)} e^{\omega^k z t} B f_n(t)\, dt.$$

If $|t| = R_k = |\lambda|^k M_0$ and $1 \leq j \leq k$, then $|\omega^j t| = |\lambda^{k-j}| M_0 \geq M_0$. Therefore $|P(\omega^j t)| \geq 2$ and

$$|L_k f_n(z)| \leq \frac{1}{2\pi} 2\pi R_k\, e^{M_0|z|} \frac{1}{2^k R_k^{n+1}} = \frac{e^{M_0|z|}}{2^k R_k^n} \to 0$$

uniformly on compact subsets as $k \to \infty$. □

Proposition 5.2. *Suppose that ϕ vanishes at some $a \in \mathbb{C}$ and $0 < |\phi(0)| \leq 1$. If $|\lambda| > 1$ then $T = R_\lambda \phi(D)$ is hypercyclic.*

Proof. Again, since $|\omega| < 1$, the subset $X_0 = \text{span}\{e^{a\omega^n z} : n \geq 0\}$ is dense in $H(\mathbb{C})$, and if $x_0 \in X_0$ then $T^n x_0 = 0$ for n large enough. Let n_0 be the first natural number satisfying $|\phi(0)\lambda^{n_0+1}| > 1$. The proof will be finished if we can define a sequence of mappings S_k on the monomials f_n $(n = 0, \ldots, n_0)$ satisfying

1. $S_k f_n \to 0$ uniformly on compact subsets as $k \to \infty$, and
2. $(R_\lambda \phi(D))^k S_k f_n \to f_n$ uniformly on compact subsets as $k \to \infty$.

We denote $P(z) = \phi(0)(1 - z/a)$. First we define S_k on f_0. By Proposition 5.1, there exists a sequence of positive numbers $R_k \to \infty$ such that

$$L_k f_n(z) = \frac{1}{2\pi i} \oint\limits_{|t|=R_k} \frac{1}{P(\omega t)\cdots P(\omega^k t)} e^{\omega^k z t} B f_n(t)\, dt \to 0$$

uniformly on compact subsets as $k \to \infty$ for $n = 0, \ldots, n_0$.

Taking $S_k f_0 = L_k f_0$, we get $(R_\lambda \phi(D))^k S_k f_0 = f_0$. Indeed, denoting

$$\Phi_k(t) = \frac{\phi(\omega t)}{P(\omega t)} \cdots \frac{\phi(\omega^k t)}{P(\omega^k t)},$$

and observing that $\Phi_k(0) = 1$, we get

$$(R_\lambda \phi(D))^k S_k f_0(z) = \frac{1}{2\pi i} \oint\limits_{|t|=R_k} \Phi_k(t) e^{zt} \frac{dt}{t} = 1,$$

where the last equality follows from the fact that the function $\Phi_k(t)e^{zt}$ is analytic: the integral is equal to the residue of $\Phi_k(t)e^{zt}(1/t)$, which is 1.

Next we define S_k on f_1. Since $\Phi_k(t)$ is an entire function and $\Phi_k(0) = 1$, we can write $\Phi_k(z) = \sum_{j=0}^{\infty} a_j^{(k)} t^j$ with $a_0^{(k)} = 1$ for all k. Also, the second term of the Cauchy product $\left(\sum_{j=0}^{\infty} \frac{z^j}{j!} t^j\right) \cdot \left(\sum_{j=0}^{\infty} a_j^{(k)} t^j\right)$ coincides with

$$\frac{1}{2\pi i} \oint\limits_{|t|=R_k} \Phi_k(t)e^{zt}\frac{dt}{t^2} = z + a_1^{(k)}.$$

So defining

$$S_k f_1(z) = \frac{1}{2\pi i} \oint\limits_{|t|=R_k} \Phi_k(t)e^{\omega^k zt}\frac{dt}{t^2} - a_1^{(k)} S_k f_0,$$

we get $(R_\lambda \phi(D))^k S_k f_1 = f_1$, and $(S_k f_1)$ converges uniformly to zero on compact sets provided $(a_1^{(k)})$ is bounded. By the chain rule, $a_1^{(k)} = (\omega + \ldots + \omega^k)c$, where c is the derivative at zero of the function $\varphi(z)/P(z)$. Since $|\omega| < 1$ the sequence $(a_1^{(k)})$ is bounded.

Assume that $S_k f_j$ has already been defined for $0 \le j < m$, satisfying $S_k f_j \to 0$ as $k \to \infty$ uniformly on compact subsets, $(R_\lambda \phi(D))^k S_k f_j = f_j$, and $(a_j^{(k)})_{k\in\mathbb{N}}$ bounded. Let us construct $S_k f_m$. Since

$$\begin{aligned} L_k f_m(z) &= \frac{1}{2\pi i} \oint\limits_{|t|=R_k} \Phi_k(t)e^{zt}\frac{dt}{t^{m+1}} = c_m \\ &= \sum_{j=0}^{m} \frac{z^j}{j!} a_{m-j}^{(k)} = f_m(z) + \sum_{j=0}^{m-1} a_{m-j}^{(k)} f_j(z), \end{aligned}$$

here c_m is the term m of the Cauchy product $\left(\sum_{j=0}^{\infty} \frac{z^j}{j!} t^j\right) \cdot \left(\sum_{j=0}^{\infty} a_j^{(k)} t^j\right)$. Thus, defining

$$S_k f_m = L_k f_m - \sum_{j=0}^{m-1} a_{m-j}^{(k)} S_k f_j$$

we get $(R_\lambda \phi(D)) S_k f_m = f_m$ by construction, and $S_k f_m \to 0$ uniformly on compact subsets provided the sequence $(a_m^{(k)})$ is bounded for all $k \in \mathbb{N}$, which follows directly by Leibniz rule. Indeed, denoting $\varphi(z) = \phi(z)/P(z)$,

$$\begin{aligned} |a_m^{(k)}| &= \frac{1}{m!} \left| [\varphi(\omega z), \ldots, \varphi(\omega^k z)]^{(m)}(0) \right| \\ &= \frac{1}{m!} \left| \sum_{h_1+\ldots+h_k=m} \binom{m}{h_1, \ldots, h_k} \prod_{t=1}^{k} (\varphi(\omega^t z))^{(h_t)}(0) \right| \end{aligned}$$

$$\leq \frac{1}{m!} \sum_{h_1+\ldots+h_k=m} \binom{m}{h_1,\ldots,h_k} \prod_{t=1}^{k} |\omega|^{th_t} |(\varphi^{(h_t)}(0))|$$
$$\leq \frac{\mathrm{C}}{m!}(|\omega| + \ldots + |\omega|^k)^m,$$

where $C = \max_{j=0}^{m} |\varphi^j(0)|$. Since $\varphi(0) = 1$, we can construct a sequence of mappings S_k acting on f_n, $n = 0, \ldots, n_0$, satisfying all the requirements we desired, and this finishes the proof. □

Remark 5.3. Notice that the proof of Proposition 5.2 provides that when $|\lambda| > 1$ and $0 < |\phi(0)| \leq 1$, the operator $T = R_\lambda \phi(D)$ satisfies the Hypercyclicity Criterion for the full sequence of natural numbers. Hence, T is topologically mixing.

Remark 5.4. Following the proof of Proposition 5.2, we can show that $T = R_\lambda \phi(D)$ is frequently hypercyclic whenever $0 < |\phi(0)| < 1$ and $|\lambda| = 1$ an irrational rotation. Indeed, let us check that T satisfies the Frequent Hypercyclicity Criterion. For that, consider X_0 the subset of polynomials. Since $0 < |\phi(0)| < 1$ and λ is an irrational rotation, we see that T restricted to the polynomials of degree n, is a triangular operator with different diagonal entries $\phi(0)\lambda^k$, $k = 0, \cdots, n$. If $\{p_0, \cdots p_n\}$ are the eigenvectors of this matrix then

$$\|T^m p_k\| = |\phi(0)|^m \to 0.$$

Thus, for each k the series

$$\sum_{m=0}^{\infty} T^m p_k(z) = \sum_{m=0}^{\infty} \phi(0)^m p_k$$

is unconditionally convergent.

On the other hand, inductively, we can construct the sequence of mappings S_k acting on $f_n(z) = z^n/n!$ as in the proof of Proposition 5.2. That is, $S_k f_0 = L_k f_0$ here

$$L_k f_n(z) = \frac{1}{2\pi i} \oint_{|t|=R_k} \frac{1}{P(\omega t) \cdots P(\omega^k t)} e^{\omega^k z t} B f_n(t)\, dt$$

where $P(z) = \phi(0)(1 - z/a)$ and $R_k = M_0$ is chosen such that

$$|L_k f_n(z)| \leq \frac{e^{M_0|z|}}{2^k M_0^n},$$

and

$$S_k f_m = L_k f_m - \sum_{j=0}^{m-1} a_{m-j}^{(k)} S_k f_j$$

for some constants $a_m^{(k)}$ which satisfy:

$$|a_m^{(k)}| \leq \frac{C}{m!}(|w| + \cdots + |\omega|^k)^m = C\frac{k^m}{m!}.$$

Now we see that $T^k S_k f_m = f_m$ by construction, and $S_k f_m(z)$ converges quickly to zero. Indeed

$$|S_k f_0(z)| = |L_k f_0| \leq C\frac{e^{M_0|z|}}{2^k M_0^0}$$

and

$$|S_k f_1(z)| \leq C\left(\frac{1}{2^k M_0^0} + \frac{1}{2^k M_0^1}\right)e^{M_0|z|}$$

For each m,

$$|S_k f_m(z)| \leq C\left(\frac{1}{2^k M_0^m} + \sum_{j=0}^{m-1}\frac{k^{m-j}}{2^k M_0^j}\right)e^{M_0|z|}$$

as $k \to \infty$. Therefore, for each m, the series

$$\sum_{k=0}^{\infty} S_k f_m$$

is unconditionally convergent. Thus T is frequently hypercyclic.

6. The case $0 < |\phi(0)| \leq 1$ and $|\lambda| = 1$

We take $\lambda = e^{2\pi i\theta}$ with θ an irrational number, since the case λ is a root of the unity has already been studied in Section 3, and we set $\omega = \lambda^{-1}$. Let us show the following result:

Proposition 6.1. *Let λ be a scalar of modulus one and that is not a root of unity. Let ϕ be entire and of exponential type, and so that it is not a scalar multiple of an exponential (i.e., not of the form $\phi(z) = ae^{bz}$, $(z \in \mathbb{C})$ where a and b are fixed scalars). Then $T = R_\lambda\phi(D)$ is topologically mixing.*

To do this, we will distinguish the cases in which $\phi^{-1}(\{0\})$ is finite (Proposition 6.2) or $\phi^{-1}(\{0\})$ is infinite (Proposition 6.4). The case where $\phi^{-1}(\{0\})$ is empty was settled with Proposition 2.4. We show that T is topologically mixing whenever $\phi^{-1}(\{0\})$ is non-empty. Standard arguments are used to prove Proposition 6.2. However, the proof of Proposition 6.4 follows by using a normal families argument. Specifically, the proof follows by showing that a certain family of maps acts transitively on the complex plane. The following consequence of Montel's Theorem is needed:

Corollary. *(Montel's Theorem). Let us suppose that $\mathcal{F}$ is a family of meromorphic functions defined on an open subset D. If $z_0 \in D$ is such that $\mathcal{F}$ is not normal at z_0 and $z_0 \in U \subset D$, then*

$$\bigcup_{f\in\mathcal{F}} f(U)$$

is dense for any non-empty neighborhood U of z_0.

Proposition 6.2. *If $\phi^{-1}(\{0\})$ is finite (and non-empty), then T satisfies the Hypercyclicity Criterion for the full sequence $(n_k) = (k)$.*

Proof. Set $\omega = 1/\lambda$ and fix $\alpha \in \phi^{-1}(\{0\})$. To apply the Hypercyclicity Criterion, we consider the dense subset

$$X_0 = \operatorname{span}\{e^{\omega^n \alpha z} \,:\, n \geq 1\},$$

where the powers of $T = R_\lambda \phi(D)$ are eventually zero.

To see that T satisfies the Hypercyclicity Criterion with respect to a sequence (n_k) it is sufficient to find a subset of the form $Y_0 = \operatorname{span}\{e^{bz} \,:\, b \in U\}$ for some non empty open subset U satisfying:

- **i)** U has a cluster point in $\mathbb{C}$.
- **ii)** $\omega U \subset U$
- **iii)** $\phi(\omega b)\phi(\omega^n b)\cdots\phi(\omega^{n_k} b) \to \infty$ as $k \to \infty$.

Indeed, since U has a cluster point in $\mathbb{C}$, Y_0 is a dense subset. And if we consider $S : Y_0 \to Y_0$ defined by $Se^{bz} = \frac{1}{\phi(\omega b)} e^{\omega b z}$, we get that S^{n_k} converges pointwise to zero on Y_0 and $TS = \mathrm{Id}_{Y_0}$.

Since ϕ is of exponential type and $\phi^{-1}(\{0\})$ is finite we can suppose that $\phi(z) = e^{az}p(z)$, for some $a \in \mathbb{C}$ and for some non constant polynomial $p(z)$.

Since $|p(z)| \to \infty$ as $|z| \to \infty$ we select $R > 0$ so that $|p(z)| \geq 2$ for all $|z| \geq R$. Let us consider $U = \{b \in \mathbb{C} \,:\, |b| \geq R\}$. Clearly U satisfy i) and ii). To show iii) for the full sequence of natural numbers, let us observe that

$$\begin{aligned} |\phi(\omega b)\cdots\phi(\omega^n b)| &= |e^{(\omega+\cdots+\omega^n)ab}||p(\omega b)\cdots p(\omega^n b)| \\ &\geq e^{\operatorname{Re}\frac{\omega-\omega^{n+1}}{1-\omega}} 2^n \\ &\geq e^{\frac{-2|a|R}{|1-\omega|}} 2^n \end{aligned}$$

which diverges to ∞ as $n \to \infty$. That is, T satisfies de Hypercyclicity Criterion for the full sequence of natural numbers as desired. $\square$

At first glance, one might think that the above ideas can be applied to prove the case in which ϕ has infinitely many zeros, simply by cutting the infinite product into a polynomial by a tail. However, to control the tail of the product, we must consider z away from the zeros of the tail. But at the same time, to get divergence of the iteration of the polynomial, we must choose z larger than the zeros of the polynomial. Since both requirements are not compatible, we need a new proof for the case of infinite zeros.

Lemma 6.3. *Let $0 < r < R$, and let ϕ be an entire function of exponential type that has no zero on the annulus:*

$$A = \{z \in \mathbb{C} \, : \, r < |z| < R\}$$

and which has a zero of modulus r. Let $\omega = e^{2\pi\theta i}$ with θ an irrational scalar. For each $n \in \mathbb{N}$ let

$$g_n(z) = \phi(\omega z)\phi(\omega^2 z) \cdots \phi(\omega^n z) \quad (z \in \mathbb{C}).$$

Let (n_k) such that $n_k \to \infty$ and $z_0 \in A$ so that

$$\mathcal{G} = \{g_k\}_{k \geq 1}$$

is normal at z_0 and $\sup_{k\geq 1} |g_{n_k}(z_0)| < \infty$. Then $\mathcal{G}$ is uniformly bounded on $D(0, |z_0|)$.

Proof. By assumption, there exists $\varepsilon > 0$ such that $D(z_0, \varepsilon) \subset A$ and $\mathcal{G}$ is uniformly bounded on $D(z_0, \varepsilon)$. Therefore, there exists $M > 0$ such that for each $k \geq 1$ whenever $|\omega^{-j} z - \omega^{-j} z_0| = |z - z_0| \leq \varepsilon$ we have

$$\begin{aligned}
M &\geq |g_{n_k}(z)| \\
&= |\phi(\omega z)\phi(\omega^2 z) \cdots \phi(\omega^{n_k} z)| \\
&= |\phi(\omega^{1+j}(\omega^{-j} z))\phi(\omega^{2+j}(\omega^{-j} z)) \cdots \phi(\omega^{n_k+j}(\omega^{-j} z))| \\
&= |\phi(\omega(\omega^{-j} z)) \cdots \phi(\omega^{n_k}(\omega^{-j} z))| \cdot \frac{|\phi(\omega^{n_k+1}(\omega^{-j} z)) \cdots \phi(\omega^{n_k+j}(\omega^{-j} z))|}{|\phi(\omega(\omega^{-j} z)) \cdots \phi(\omega^j(\omega^{-j} z))|}
\end{aligned}$$

Thus, for each z such that $|z - \omega^{-j} z_0| < \varepsilon$ and $k \geq 1$ we obtain

$$\begin{aligned}
M &\geq |g_{n_k}(z)| \cdot \frac{|\phi(\omega^{n_k+1} z) \cdots \phi(\omega^{n_k+j}(z)|}{|\phi(\omega z) \cdots \phi(\omega^j z)|} \\
&\geq C^j |g_{n_k}(z)|
\end{aligned}$$

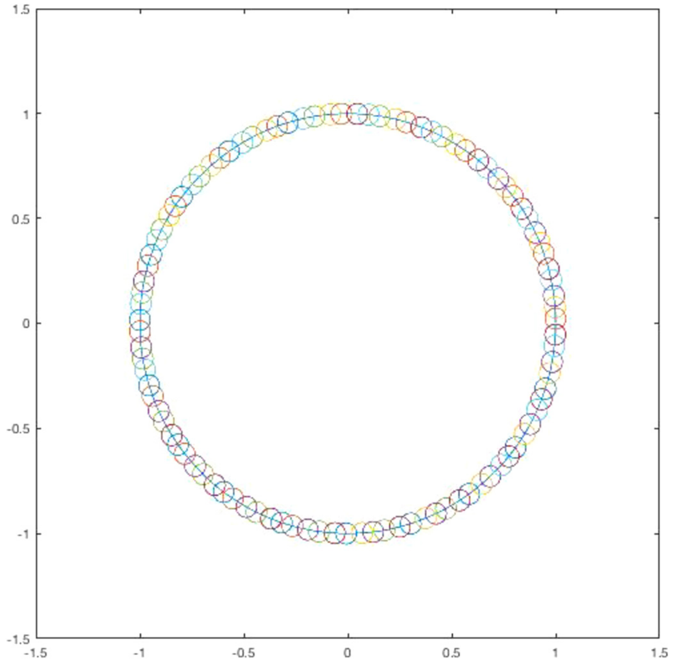

Fig. 1. $\omega = e^{\sqrt{2}\pi i}$, and $m = 100$.

where $C = \frac{\min\{|\phi(z): |z|=|z_0|\}}{\max\{|\phi(z)|: |z|=|z_0|\}} \in (0,1)$.

Hence $\mathcal{G}$ is uniformly bounded on $D(\omega^{-j}z_0, \varepsilon)$ for each $j \geq 1$. By the compactness of $C(0,|z_0|)$ there exists m such that

$$C(0,|z_0|) \subset \bigcup_{j=1}^{m} D(\omega^{-j}z_0, \varepsilon).$$

Therefore $\mathcal{G}$ is uniformly bounded on the above finite union, and hence on $C(0,|z_0|)$. By the Maximum Modulus Principle, $\mathcal{G}$ is uniformly bounded on $D(0,|z_0|)$ as desired (see Fig. 1). □

Proposition 6.4. *Suppose $\phi^{-1}(\{0\})$ is infinite. Then for each sequence (n_k) converging to ∞, there exists a subsequence (n_{k_j}) such that T satisfies the Hypercyclicity Criterion with respect to (n_{k_j}). In particular, $T = R_\lambda\phi(D)$ is topologically mixing.*

Proof. Let $0 < r_0 < r_1 < r_2 < \cdots$ be the radii of all those circles centered at 0 that contain zeroes of ϕ. Since $\phi^{-1}(\{0\})$ has no accumulation point in $\mathbb{C}$ we may assume that $r_n \to \infty$ as $n \to \infty$. Let $a \in \phi^{-1}(\{0\})$ with $|a| = r_0$. By Hadamard's Theorem we have

$$\phi(z) = \phi(0)\left(1 - \frac{z}{a}\right)\varphi(z)$$

with φ of exponential type satisfying $\varphi(0) = 1$. Now, for each $n \in \mathbb{N}$ let f_n and g_n the functions defined by

$$f_n(z) = \phi(\omega z)\phi(\omega^2 z) \cdots \phi(\omega^n z)$$
$$g_n(z) = \varphi(\omega z)\varphi(\omega^2 z) \cdots \varphi(\omega^n z).$$

Let $R > 0$ be large enough so that

$$\left(\frac{|z|}{r_0} - 1\right)|\phi(0)| > 1 \qquad \text{for} \quad |z| > R.$$

Let k_0 large enough so that $r_k > R$ for $k \geq k_0$. Then for $|z| > R$ we have

$$\begin{aligned} |\phi(z)| &= |\phi(0)\left(1 - \frac{z}{a}\right)\varphi(z)| \\ &\geq \left(\frac{|z|}{|a|} - 1\right)|\phi(0)||\varphi(z)| \\ &> |\varphi(z)|. \end{aligned}$$

So for each $n \in \mathbb{N}$ and $|z| > R$ we have

$$|f_n(z)| = |\phi(\omega z) \cdots \phi(\omega^n z)| > |\varphi(\omega z) \cdots \varphi(\omega^n z)| = |g_n(z)|.$$

It suffices to show the following claim.

Claim 1. *There exists* $z_0 \in A_{r_{k_0}} = \{z \in \mathbb{C} : r_{k_0} < |z| < r_{k_0+1}\}$ *such that* $\limsup_{k\to\infty} |g_{n_k}(z_0)| = \infty$.

Indeed, suppose Claim 1 holds. Then there exists a subsequence (n_{k_l}) of (n_k) such that

$$|f_{n_{k_l}}(z_0)| \geq |g_{n_{k_l}}(z_0)| \to \infty,$$

as $l \to \infty$. Then $\Lambda = \{\omega^j z_0 : j \geq 1\}$ satisfies $\omega\Lambda \subset \Lambda$ and has an accumulation point in $\mathbb{C}$, therefore

$$Y_0 = \text{span}\{e^{bz} : b \in \Lambda\}$$

is dense in $\mathcal{H}(\mathbb{C})$. Moreover, the linear mapping $S : Y_0 \longrightarrow Y_0$ defined by $S(e^{bz}) := \frac{1}{\phi(\omega b)} e^{\omega bz}$ $(b \in \Lambda)$ clearly satisfies that $TS = I$ on Y_0 and

$$S^{n_{k_l}} \to 0$$

pointwise on Y_0. Indeed, to see the latter it suffices to show that for each $b \in \Lambda$ we have $S^{n_{k_l}}(e^{bz}) \to 0$ as $l \to \infty$. So let $b = \omega^r z_0 \in \Lambda$. Let us denote by $0 < m_1 < m_2$ the minimum and maximum of $|\phi|$ on the circle $C(0, |z_0|)$. Then for any $R > 0$ and $|z| \leq R$ we have

$$\begin{aligned} \left|S^{n_{k_l}}(e^{bz})\right| &= \left|\frac{1}{\phi(\omega b)\phi(\omega^2 b)\cdots\phi(\omega^{n_{k_l}} b)} e^{\omega^{n_{k_l}} bz}\right| \\ &\leq \left|\frac{C_R}{\phi(\omega^{l+1} z_0)\phi(\omega^{l+2} z_0)\cdots\phi(\omega^{n_{k_l}+l} z_0)}\right| \\ &\leq \frac{C_R}{|f_{n_{k_l}}(z_0)|}\left(\frac{m_2}{m_1}\right)^r \end{aligned}$$

where $C_R = \sup_{|z|\leq R|z_0|} |e^z|$. Therefore $S^{n_{k_l}}(e^{bz}) \to 0$ as $l \to \infty$, and T satisfies the Hypercyclicity Criterion with respect to the sequence (n_{k_l}) as we wanted to prove. So it remains to show Claim 1.

Proof of Claim 1. By means of contradiction, suppose $\mathcal{G} = \{g_{n_k}\}_{k\geq 1}$ is pointwise bounded on $A_{r_{k_0}}$. Notice first that $\mathcal{G}$ is normal at no point of $A_{r_{k_0}}$. To see this, suppose that $\mathcal{G}$ is normal at a point $z_0 \in A_{r_{k_0}}$ then by Lemma 6.3 $\mathcal{G}$ should be uniformly bounded on $D(0, |z_0|)$, and by Montel's Theorem (see [27, p. 35]) there exists a holomorphic function g on $D(0, |z_0|)$ and a subsequence (n_{k_j}) of (n_k) such that

$$g_{n_{k_j}} \to g \quad as \quad j \to \infty \tag{3}$$

locally uniformly on $D(0, |z_0|)$. So pick a root b of ϕ with $|b| = r_{k_0}$. Then for each $l \in \mathbb{N}$ and $n > l$ we have $g_n(w^{-l}b) = 0$.

Thus by (3) we have that g vanishes at each $w^{-l}b$ for ($l \geq 1$), which implies that g is identically zero. However $|g_n(0)| = 1$ for each $n \geq 1$, hence $|g(0)| = \lim_{j\to\infty} |g_{n_{k_j}}(0)| = 1$, a contradiction.

So $\mathcal{G}$ is normal at no point of $A_{r_{k_0}}$. Hence by a consequence of Montel's Theorem, for each non-empty open subset U of $A_{r_{k_0}}$ the set

$$\bigcup_{k=1}^{\infty} g_{n_k}(U)$$

is dense in $\mathbb{C}$. That is, for each non-empty open subsets U of $A_{r_{k_0}}$ and V of $\mathbb{C}$ there exists $k \in \mathbb{N}$ such that $g_{n_k}(U) \cap V \neq \emptyset$. Since $A_{r_{k_0}}$ has no isolated points and is homeomorphic to a complete metric space, by Birkhoff's Transitivity Theorem there exists $z_0 \in A_{r_{k_0}}$ such that $\{g_{n_k}(z_0) : k \in \mathbb{N}\}$ is dense in $\mathbb{C}$, which contradicts that $\mathcal{G}$ is pointwise bounded on $A_{r_{k_0}}$. So Claim 1 holds, and the proof of Proposition 6.4 is now complete. □

7. The case $\phi(0) = 0$ and $|\lambda| \geq 1$

The operator $T_{\lambda,b}$ of Aron and Markose is included in this case with $\phi(z) = ze^{bz}$.

Theorem 7.1. *Assume $\phi(0) = 0$. If $|\lambda| \geq 1$ then $T = R_\lambda \phi(D)$ is hypercyclic.*

Proof. We write $\phi(z) = z^m \psi(z)$ with $\psi(0) \neq 0$ and we denote by X_0 the set of complex polynomials $p(z)$. Note that $T^n p(z) = 0$ for $n > \deg(p)$.

Set $A_\lambda = R_\lambda \psi(D)$ so that $T = A_\lambda D^m$. Since $\psi(0) \neq 0$, as in the proof of Proposition 4.1, the subspace of polynomials of degree less or equal to n is invariant under the operator A_λ and the eigenvalues are simple on that subspace. We denote $p_0, p_1, \cdots p_k$ the polynomials of degree $\leq k$ which are the eigenvectors associated to $\psi(0)\lambda^k$, that is, $A_\lambda p_k = \psi(0)\lambda^k p_k$ for $k \geq 0$.

Let V be the complex Volterra operator defined by

$$Vf(z) = \int_0^z f(\xi) d\xi, \quad (z \in \mathbb{C}).$$

The equation $TV^m p_k = A_\lambda p_k = \psi(0)\lambda^k p_k$ gives us the key to construct the maps S_k required by the Hypercyclicity Criterion. Indeed, let us define

$$S_k p_n = \frac{V^{mk} p_n}{\lambda^m \lambda^{2m} \cdots \lambda^{(k-1)m} (\psi(0)\lambda^n)^k},$$

and extend S_k to $Y_0 = \text{span}\{p_k(z) \ : \ k \geq 0\}$ by linearity. Since $\frac{V^k 1}{\psi(0)^k} \to 0$ uniformly on compact sets as $k \to \infty$, we obtain that $\frac{1}{\psi(0)^k} V^{mk} p_n \to 0$ in $\mathcal{H}(\mathbb{C})$. Hence, since $|\lambda| \geq 1$,

$$|S_k(p_n)(z)| \leq \frac{|V^{mk} p_n(z)|}{|\psi(0)|^k} \to 0$$

uniformly on compact sets. To check that $T^k S_k = \text{Id}_{Y_0}$, note that

$$T^k = A_\lambda D^m A_\lambda D^m \cdots A_\lambda D^m, \quad \text{(k times)}.$$

Since A_λ is an extended λ-eigenoperator of D, $D^m A_\lambda = \lambda^m A_\lambda D^m$. Therefore $T^k = \lambda^m \lambda^{2m} \cdots \lambda^{(k-1)m} A_\lambda^k D^{km}$, hence

$$T^k S_k p_n = T^k \left(\frac{V^{mk} p_n}{\lambda^m \lambda^{2m} \cdots \lambda^{(k-1)m} (\psi(0)\lambda^n)^k} \right) = \frac{A_\lambda^k p_n}{\psi(0)^k \lambda^{nk}} = p_n,$$

and the Hypercyclicity Criterion implies that T is hypercyclic. □

Remark 7.2. Here again, when $\phi(0) = 0$, we can show that T is frequently hypercyclic by applying the Frequent Hypercyclicity Criterion. Therefore, T is also chaotic and topologically mixing.

Let us finish with a question. As the reader can see, in many cases scattered parts of the manuscript (Remarks 3.5, 4.2, 5.4 and 7.2) it has been established the frequent hypercyclicity of $T = R_\lambda \phi(D)$. There exist only one case unsolved.

Problem 7.3. Suppose λ is an irrational rotation and $|\phi(0)| \geq 1$. Is $T = R_\lambda \phi(D)$ frequently hypercyclic?

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