Research Article

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On reducible non-Weierstrass semigroups

https://doi.org/10.1515/math-2021-0111
received April 11, 2021; accepted September 20, 2021

Abstract: Weierstrass semigroups are well known along the literature. We present a new family of non-Weierstrass semigroups which can be written as an intersection of Weierstrass semigroups. In addition, we provide methods for computing non-Weierstrass semigroups with genus as large as desired.

Keywords: additive combinatorics, Buchweitz semigroup, numerical semigroup, pseudo-Frobenius number, Weierstrass points, Weierstrass semigroup

MSC 2020: 14H55, 11P70, 20M14

†To the memory of our friend and colleague Fernando Torres (1961–2020).

1 Introduction

A *numerical semigroup* H is an additive submonoid of the non-negative integers $\mathbb{N}$ whose complement $G(H) = \mathbb{N}\setminus H$, the *set of gaps* of H, is finite; its cardinality $g(H) = \#G(H)$ is the *genus* of H. The elements of H will be referred to as the *non-gaps* of H. A suitable reference for the background on numerical semigroups that we assume is the book in [1].

In the theory of Weierstrass points one associates a numerical semigroup $H(P)$ with any point P of a complex (projective, irreducible, non-singular, algebraic) curve $\mathcal{X}$ in such a way that its genus coincides with the genus g of the underlying curve, see, e.g., [2, III.5.3]. This semigroup is called the *Weierstrass semigroup* at P, and it is the set of pole orders at P of regular functions on $\mathcal{X}\setminus\{P\}$. We have that the set of gaps of $H(P)$ equals $\{1,\ldots,g\}$ for all but finitely many points P which are the so-called *Weierstrass points* of the curve; they carry a lot of information about the curve, see, e.g., [2, III.5.11] and [3].

A numerical semigroup H is called *Weierstrass* if there is a pointed curve $(\mathcal{X}, P)$ such that $H = H(P)$. Around 1893, Hurwitz [4] asked about the characterization of Weierstrass semigroups; long after that, in 1980, Buchweitz [5,6] pointed out the following simple combinatorial criterion. Let $\mathcal{S}(g)$ denote the collection of numerical semigroups of genus g, and for $n \geq 2$ an integer let

$$\mathcal{B}_n(g) = \{H \in \mathcal{S}(g) : \ \#G_n(H) > (2n-1)(g-1)\} \tag{1}$$

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be the subcollection of *n-Buchweitz semigroups* of genus g, being $G_n(H)$ the n-fold sum of elements of the set of gaps of H. Then by the Riemann-Roch theorem each element of $\mathcal{B}_n(g)$ is non-Weierstrass (see [7, p. 122] for further historical information). Although Buchweitz also pointed out that $\mathcal{B}_n(g) \neq \emptyset$ for each n and large g, Kaplan and Ye [8, Theorem 5] noticed that

$$\lim_{g\to\infty} \frac{\#\mathcal{B}(g)}{\#\mathcal{S}(g)} = 0,$$

where $\mathcal{B}(g) = \bigcup_{n\geq 2}\mathcal{B}_n(g)$; in particular, $\lim_{g\to\infty}\#\mathcal{B}_2(g)/\#\mathcal{S}(g) = 0$ which is a result suggested by previous numerical computations in Komeda's paper [9, §2]. This shows in particular that it is difficult to give explicit examples of semigroups in $\mathcal{B}_2(g)$ for large g.

The paper addresses two natural questions in the theory of numerical semigroups: (1) to determine whether the set of Weierstrass semigroups is closed under intersection and (2) to construct large explicit families of semigroups which are not Weierstrass.

With the idea of studying Weierstrass semigroups and having methods to find 2-Buchweitz semigroups, we introduce the concept of PF-semigroup. A numerical semigroup is a PF-semigroup if its multiplicity is the difference between its genus and its type plus one, and its set of pseudo-Frobenius numbers coincides with its set of gaps greater than its multiplicity. These semigroups are the key to prove that the set of Weierstrass semigroups is not closed under intersection.

Inspired by the Schubert index definition (see [9]), we construct families of PF-semigroups using sequences of positive integer numbers. These sequences are used to calculate subfamilies of 2-Buchweitz PF-semigroups, in particular, to generate semigroups of this type with large genera. The main result of this work (Theorem 5.5) gives us an operation such that from two sequences we obtain new sequences describing again families of 2-Buchweitz PF-semigroups.

The content of this work is organized as follows. In Section 2, we introduce the concepts of PF-semigroup, g-semigroup, and n-Buchweitz semigroup, and study some of their properties. Section 3 is devoted to the decomposition of PF-semigroups as the intersection of Weierstrass semigroups. Finally, in Sections 4 and 5, we represent the PF-semigroups using the differences of their pseudo-Frobenius numbers and study conditions on those sets to obtain 2-Buchweitz PF-semigroups. Theorem 5.5 will give us a method for joining these sequences to get 2-Buchweitz PF-semigroups with genus as large as desired.

2 Preliminaries and results

Given a numerical semigroup H, the minimal set, according to inclusion, $\{h_1,\dots,h_p\}$ with $h_1 < \dots < h_p$ such that $H = \langle h_1,\dots,h_p\rangle = \left\{\sum_{i=1}^{p} x_i h_i \,|\, x_i \in \mathbb{N}\right\}$ is called the *minimal system of generators* of H. The element h_1 is called the *multiplicity* of H and it is denoted by $m(H)$.

The *pseudo-Frobenius numbers* of H are the elements of $\mathrm{PF}(H) = \{x \in G(H) \,|\, \forall h \in H\backslash\{0\}, x + h \in H\}$. The cardinality of this set is the *type* of H, denoted by $t(H)$. Note that $\max G(H) = \max \mathrm{PF}(H)$. This number is called the *Frobenius number* of H and denoted by $\mathrm{Fb}(H)$. It is trivial to check that if $x > \mathrm{Fb}(H)$ then $x \in H$. The number $c(H) = \mathrm{Fb}(H) + 1$ is the *conductor* of H.

Definition 2.1. A numerical semigroup H is a PF-semigroup if $G(H)$ is the set $\{1,\dots,g(H) - t(H)\} \sqcup \mathrm{PF}(H)$ and $\min \mathrm{PF}(H) > m(H)$.

Remark 2.2. Note that if H is a PF-semigroup then $m(H) = g(H) - t(H) + 1$.

Lemma 2.3. *Every* PF*-semigroup H has Frobenius number odd, namely* $\mathrm{Fb}(H) = 2g(H) - 2t(H) + 1$.

Proof. Let H be a PF-semigroup, $t = t(H)$, $g = g(H)$, and $G(H) = \{1,\dots,g - t\} \sqcup \mathrm{PF}(H)$.

Since $m(H) = g - t + 1$, we assume that there exists $h \in G(H)$ such that $h > 2g - 2t + 1$. The element $2m(H) = 2g - 2t + 2$ belongs to H, so $h > 2g - 2t + 2$. In this case, $h = d(g - t + 1) + r$ where the integers

d and r satisfy that $d \geq 2$ and $r \in [1, g - t]$. Thus, $h = (d - 1)(g - t + 1) + g - t + 1 + r$. Note that $(d - 1)(g - t + 1) \in H$ and as $h \notin H$, then $g - t + 1 + r \in G(H) \backslash \mathrm{PF}(H)$. That is, H is not a PF-semigroup. We can affirm every integer greater than or equal to $2g - 2t + 2$ belongs to H.

Now, since $g - t + h \geq 2g - 2t + 2$ for every $h \in H \backslash \{0, g - t + 1\}$ and $g - t \in G(H) \backslash \mathrm{PF}(H)$, $2g - 2t + 1 = (g - t) + (g - t + 1)$ is a gap of H. Therefore, $\mathrm{Fb}(H) = 2g - 2t + 1$. □

The condition $\mathrm{Fb}(H) = 2g(H) - 2t(H) + 1$ from Lemma 2.3 does not imply that H is a PF-semigroup. For example, the semigroup $\langle 5, 6, 14 \rangle$ satisfies $\mathrm{Fb}(H) = 2g(H) - 2t(H) + 1$ but it is not a PF-semigroup.

Definition 2.4. (See [10]) We say that a numerical semigroup is irreducible if it cannot be expressed as an intersection of two numerical semigroups containing it properly.

Definition 2.5. A g-semigroup is a numerical semigroup with genus g and multiplicity equal to g.

Note that, by [11, Theorem 14.5], all g-semigroups are Weierstrass.

Given G the set of gaps of a numerical semigroup H, we denote by $G_n(H)$ the set $\{g_1 + \cdots + g_n \mid g_1, \ldots, g_n \in G\}$. Note that the set $G_n(H)$ is known as $nG = G + G + \cdots + G$ in additive combinatorics.

As in Section 1, we define:

Definition 2.6. A numerical semigroup H is called an n-Buchweitz semigroup for some $n \geq 2$ if the cardinality of $G_n(H)$ is strictly greater than $(2n - 1)(g(H) - 1)$.

Several papers study n-Buchweitz semigroups because it is known that these semigroups are non-Weierstrass (see [5] or [8]). For example, in [9, p. 161], we find a computational result showing that there are no 2-Buchweitz semigroups with genus strictly smaller than 16, the genus of Buchweitz's original example.

3 Irreducible decomposition of PF-semigroups

The decomposition of a numerical semigroup as the intersection of irreducible numerical semigroups has been studied in several papers (see [10] and references therein). In this section, a decomposition of PF-semigroups by means of irreducible Weierstrass semigroups is introduced. We use this decomposition to show that the set of Weierstrass semigroups is not closed under intersection.

Lemma 3.1. *Let H be a PF-semigroup with type t and set of gaps $G(H) = \{1, \ldots, g(H) - t\} \sqcup \mathrm{PF}(H)$. Then there exist $H_1, \ldots, H_t$ irreducible g_i-semigroups of genus g_i, respectively, such that $H = H_1 \cap \cdots \cap H_t$.*

Proof. Assume that $\mathrm{PF}(H) = \{f_1, \ldots, f_t = \mathrm{Fb}(H)\}$ with $f_1 < \cdots < f_t$. For $i \in \{1, \ldots, t\}$, we define the g_i-semigroup H_i given by the set of gaps $G(H_i) = \{1, 2, \ldots, \lfloor f_i/2 \rfloor, f_i\}$.

In order to prove that $H = H_1 \cap \cdots \cap H_t$, we show that $G(H) = G(H_1) \cup \cdots \cup G(H_t)$. By construction, both sets of gaps are equal for every element greater than or equal to f_1.

Moreover, $\bigcup_{i=1}^{t} G(H_i) = \{1, \ldots, \lfloor f_t/2 \rfloor\} \cup \{f_1, \ldots, f_t\}$, and since $\lfloor f_t/2 \rfloor = \lfloor \mathrm{Fb}(H)/2 \rfloor = (\mathrm{Fb}(H) - 1)/2 = g(H) - t$, $G(H) = \bigcup_{i=1}^{t} G(H_i)$. □

Since every g-semigroup is Weierstrass, we obtain the following result.

Corollary 3.2. *Any PF-semigroup is the intersection of Weierstrass semigroups.*

Example 3.3. Let H and H' be the semigroups given by the set of gaps

$$G(H) = \{1,\ldots,g-4, 2g-13, 2g-11, 2g-8, 2g-7\}$$

and

$$G(H') = \{1,\ldots,g-3, 2g-29, 2g-9, 2g-5\}.$$

We have that both families of semigroups are 2-Buchweitz and 4-Buchweitz semigroups for genus g greater than or equal to 16 and 99, respectively (see [12, Example 1]). These semigroups can be decomposed as Lemma 3.1:

1. $H = H_1 \cap \cdots \cap H_4$ where
   (a) $G(H_1) = \{1,\ldots,g-7\} \cup \{2g-13\}$,
   (b) $G(H_2) = \{1,\ldots,g-6\} \cup \{2g-11\}$,
   (c) $G(H_3) = \{1,\ldots,g-4\} \cup \{2g-8\}$,
   (d) and $G(H_4) = \{1,\ldots,g-4\} \cup \{2g-7\}$.
2. $H' = H_1' \cap H_2' \cap H_3'$ where
   (a) $G(H_1') = \{1,\ldots,g-15\} \cup \{2g-29\}$,
   (b) $G(H_2') = \{1,\ldots,g-5\} \cup \{2g-9\}$,
   (c) and $G(H_3') = \{1,\ldots,g-3\} \cup \{2g-5\}$.

Note that, in general, the intersection of g_i-semigroups with genus g_i is not a PF-semigroup. For example, fixing a set of g_i-semigroups with genus g_i where the maximum of its Frobenius numbers is even, its intersection is not a PF-semigroup.

From Example 3.3, we obtain that intersection of Weierstrass semigroups is not necessarily Weierstrass, so we can affirm that the set of Weierstrass semigroups is not closed under intersection.

Corollary 3.4. *The set of Weierstrass semigroups is not closed under intersection.*

It is possible for some intersections to be a Weierstrass semigroup. Consider the semigroups H_1, H_2, and H_3 given by the gapsets $\{1, 2, 3, 6\}$, $\{1, 2, 3, 4, 8\}$, and $\{1, 2, 3, 4, 9\}$, respectively. These semigroups are 4-semigroup and 5-semigroups, respectively. Their intersection is the PF-semigroup $H = \langle 5, 7, 11, 13\rangle$. The semigroups H_1, H_2, and H_2, as well as H are Weierstrass because their multiplicity are smaller than or equal to 5 (see [13, p. 42]).

4 Computing 2-Buchweitz PF-semigroups

In this section, we focus on the study of PF-semigroups H with genus g, type t, multiplicity $m = g - t + 1$, and gapset $G(H) = \{1,\ldots,g-t, 2m-a_1,\ldots,2m-a_t\}$, where $a_1,\ldots,a_t \in \mathbb{N}$ satisfy $a_1 > a_2 > \cdots > a_{t-1} > a_t = 1$. Recall that for a set A, $2A$ denotes $\{a+b | a, b \in A\}$.

Theorem 4.1. *Let* $A = \{a_1, a_2,\ldots,a_{t-1}, a_t\} \subset \mathbb{N}$ *with* $a_1 > a_2 > \cdots > a_{t-1} > a_t = 1$, $g, t \in \mathbb{Z}$ *such that* $t \geq 3$ *and* $g \geq 2a_1 + t - 1$, *and let* H *be the semigroup with gapset* $G(H) = \{1,\ldots,g-t, 2m-a_1,\ldots,2m-a_t\}$. *Then,* $\#(2A) > 3(t-1)$ *if and only if* H *is a* 2*-Buchweitz* PF*-semigroup.*

Proof. Assume that $\#(2A) > 3(t-1)$. We first prove that H is a PF-semigroup. Note that for every $h \in \{m, m+1,\ldots,2m-1\}$, $h + m > 2m - 1$, and then $h + m \in H$. Thus, the elements $2m - a_1,\ldots, 2m - a_{t-1}$, and $2m - a_t$ are pseudo-Frobenius numbers of H. We must now show the remaining gaps $1,\ldots, g-t$ are not in PF(H). For all $k \in \{1,\ldots,m-a_1\}$, $2m - a_1 - k \in H$, and $k + 2m - a_1 - k = 2m - a_1 \in G(H)$. That means $\{1,\ldots,m-a_1\} \subset G(H)\backslash \mathrm{PF}(H)$. Analogously, for every $k \in \{a_1,\ldots,g-t\}$, using that $2m - 1 - k \in H$, we obtain $k + 2m - 1 - k = 2m - 1 \in G(H)$, and then $\{a_1,\ldots,g-t\} \subset G(H)\backslash \mathrm{PF}(H)$. By the hypothesis $g \geq 2a_1 + t - 1$, we get $\{1,\ldots,m-a_1\} \cup \{a_1,\ldots,g-t\} = \{1, 2, \ldots, g-t\} \neq \emptyset$ and $m < 2m - a_1$. Therefore, H is a PF-semigroup.

Table 1: Number of numerical semigroups (NS) and 2-Buchweitz semigroups (2-BS) compared with number of 2-Buchweitz PF-semigroups (2-BPFS) up to genus 35

Genus	NS	2-BS	2-BPFS	Genus	NS	2-BS	2-BPFS
16	4,806	2	2	26	770,832	3,591	1,497
17	8,045	6	3	27	1,270,267	6,584	2,655
18	13,476	15	10	28	2,091,030	11,871	4,555
19	22,464	31	19	29	3,437,839	20,987	7,745
20	37,396	67	35	30	5,646,773	37,598	13,450
21	62,194	145	72	31	9,266,788	66,330	23,108
22	103,246	293	146	32	15,195,070	116,501	38,944
23	170,963	542	257	33	24,896,206	203,300	64,873
24	282,828	1,053	469	34	40,761,087	353,978	110,576
25	467,224	1,944	795	35	66,687,201	615,762	187,966

Now we prove that H is 2-Buchweitz. We describe explicitly the set $G_2(H)$ (recall that $m = g - t + 1$),

$$\begin{aligned} G_2(H) \;=\; & \{2,\ldots,2m-2\} \cup \{2m-a_1+1,\ldots,3m-a_1-1\} \\ & \cup \{2m-a_2+1,\ldots,3m-a_2-1\} \cup \cdots \\ & \cup \{2m-a_{t-1}+1,\ldots,3m-a_{t-1}-1\} \cup \{2m,\ldots,3m-2\} \\ & \cup \{4m-2a_1, 4m-a_1-a_2,\ldots,4m-a_{t-1}-1, 4m-2\}. \end{aligned} \tag{2}$$

Since $t \geq 3$, then $a_1 \geq 3$ and $2m - a_1 + 1 \leq 2m - 2$. Also, using the condition $g \geq 2a_1 + t - 1$, we have that $m \geq 2a_1 > a_1 + 1 > a_2 + 1 > \cdots > a_t + 1$, and then $2m - a_{i+1} + 1 \leq 3m - a_i - 1 \leq 3m - a_{i+1} - 1 < 4m - 2a_1$ for every $i \in \{1,\ldots,t-1\}$. Therefore, $G_2(H) = \{2, 3,\ldots,3m-3, 3m-2\} \sqcup \{4m-2a_1, 4m-a_1-a_2,\ldots,4m-2\}$, and its cardinality is $3m - 3 + \#(2A)$. Since, $\#(2A) > 3(t-1)$, H is a 2-Buchweitz semigroup.

Conversely, assume that H is 2-Buchweitz PF-semigroup. We have that $\#G_2(H) = 3m - 3 + \#(2A)$. Since H is 2-Buchweitz PF-semigroup, $m = g - t + 1$ and $3m - 3 + \#(2A) > 3(g-1) = 3m + 3(t-1)$. Therefore, $\#(2A) > 3(t-1)$. □

Remark 4.2. In Theorem 4.1, the condition $t \geq 3$ is a necessary condition. Consider H the PF-semigroup of type 2 with gapset $G(H) = \{1,\ldots,g-2, 2m-a_1, 2m-1\}$. By equality (2), $\#(G_2(H)) = 3m - 3 + \#(\{2, 1+a_1, 2a_1\}) = 3m \not> 3m = 3(g-1)$. From now on, we assume that the type is strictly greater than 2.

Table 1 indicates that the number of 2-Buchweitz semigroups that are also PF-semigroups, at least up to genus 35, is quite high. The code used for computing this table is found in Appendix A. It is implemented in GAP [14] and uses the library [15].

5 Difference sequences

In [9], the concept of Schubert index is defined: given a numerical semigroup with gap-set $\{l_1 < l_2 < \cdots < l_g\}$, for any $i = 0,\ldots, g-1$, set $\alpha_i = l_{i+1} - i - 1$, the tuple $(\alpha_0,\ldots,\alpha_{g-1})$ is called the Schubert index associated with the semigroup. Moreover, [9, Proposition 2.2] introduces some 2-Buchweitz semigroups that are PF-semigroups. For example, case (3) of that proposition studies the semigroup of genus $g = q + p + 2$ and Schubert index $\alpha = (0,\ldots,0, q-2p, q-2p, q-2p+1,\ldots,q-2p+p-1, q-2p+p-1) \in \{0\}^q \times \mathbb{N}^{g-q}$, with $q \geq 4p$ and $p \geq 3$ $(q, p \in \mathbb{N})$. Its gap-set is $\{1,\ldots q, 2q-2p+1, 2q-2p+2, 2q-2p+2+2\cdot 1,\ldots,2q-2p+2+2(p-1) = 2q, 2q+1\}$. Since $q + 1 + (2q-2p+1) = 3q - 2p + 2 \geq 2q + 4p - 2p + 2 > 2q + 2$, $(q+1) + \{2q-2p+1, 2q-2p+2, 2q-2p+2+2\cdot 1,\ldots,2q-2p+2+2(p-1) = 2q, 2q+1\}$ is contained in the semigroup, the elements in $\{2q-2p+1, 2q-2p+2, 2q-2p+2+2\cdot 1,\ldots,2q-2p+2+2(p-1) = 2q, 2q+1\}$ are pseudo-Frobenius numbers. If we consider any $j \in \{2p+1,\ldots,q\}$, then $2q + 1 = j + (2q+1) - j$, and the elements in

$\{2p+1,\dots,q\}$ are not pseudo-Frobenius numbers. Something similar happens to the numbers in $\{1,\dots,2p\}$. In this case, $2q-2p = j + (2q-2p) - j$ for all $j \in \{1,\dots,2p\}$. That is to say, the semigroup of case (3) in Proposition 2.2 is a PF-semigroup. Analogously, it can be proved that cases (1) and (2) are PF-semigroups.

Given a difference sequence $\mathbf{d} = (d_1,\dots,d_{t-1}) \in (\mathbb{N}\setminus\{0\})^{t-1}$, we consider the numerical semigroup with gap-set $\{1,\dots,g-t, 2m-1-\sum_{i=1}^{t-1} d_i < \dots < 2m-1-d_{t-2}-d_{t-1} < 2m-1-d_{t-1} < 2m-1\}$. For a PF-semigroup H with $G(H) = \{1,\dots,g-t, 2m-a_1,\dots,2m-a_t\}$, the set $G(H)$ is equal to $\{1,\dots,g-t, 2m-1-\sum_{i=1}^{t-1} d_i,\dots, 2m-1-d_{t-1}, 2m-1\}$ where $d_i = a_i - a_{i+1}$ for $i = 1,\dots,t-1$.

For instance, the difference sequence of the above-mentioned example of case (3) in [9] is $\mathbf{d} = (1,2,\dots,2,1) \in \mathbb{N}^{p+1}$, and the difference sequences for cases (1) and (2) are $(2,\dots,2,3,1)$ and $(1,3,2,\dots,2)$, respectively. For a PF-semigroup with gap-set $\{1,\dots,g-t, 2m-1-\sum_{i=1}^{t-1} d_i,\dots,2m-1-d_{t-1}, 2m-1\}$, the relation between the difference sequence $\mathbf{d} = (d_1,\dots,d_{t-1}) \in (\mathbb{N}\setminus\{0\})^{t-1}$ and its Schubert index $\alpha = (\alpha_0,\dots,\alpha_{g-1})$ is the following: for any non-negative integer $i \le g-t-1$, $\alpha_i = 0$, and for any $i \in \{g-t+1,\dots, g-1\}$, $d_{i-g+t} = \alpha_i - \alpha_{i-1} + 1$.

Definition 5.1. Given a difference sequence $\mathbf{d} = (d_1,\dots,d_{t-1}) \in (\mathbb{N}\setminus\{0\})^{t-1}$, we say that $\mathbf{d}$ is a (g,t)-Buchweitz sequence if there exists a 2-Buchweitz PF-semigroup H with genus g, type t, and gap-set $G(H) = \{1,\dots, g-t, 2m-1-\sum_{i=1}^{t-1} d_i,\dots,2m-1-d_{t-1}, 2m-1\}$.

Many (g,t)-Buchweitz sequences can be constructed. The next example provides one.

Example 5.2. Consider the difference sequence $\mathbf{d} = (7,1,2,1)$ and H the PF-semigroup with genus g and $G(H) = \{1,\dots,g-5\} \cup \{2g-20, 2g-13, 2g-12, 2g-10, 2g-9\}$. The set $G_2(H)$ is

$$\begin{aligned}
G_2(H) &= \{2,\dots,2g-10\} \cup \{2g-19,\dots,3g-25\} \cup \{4g-40\}\\
&\quad \cup \{2g-12,\dots,3g-18\} \cup \{4g-33, 4g-26\}\\
&\quad \cup \{2g-11,\dots,3g-17\} \cup \{4g-32, 4g-25, 4g-24\}\\
&\quad \cup \{2g-9,\dots,3g-15\} \cup \{4g-30, 4g-23, 4g-22, 4g-20\}\\
&\quad \cup \{2g-8,\dots,3g-14\} \cup \{4g-29, 4g-22, 4g-21, 4g-19, 4g-18\}\\
&= \{2,\dots,3g-14\} \cup \{4g-40, 4g-33, 4g-32, 4g-30, 4g-29\}\\
&\quad \cup \textstyle\bigcup_{i=0}^{8} \{4g-26+i\}.
\end{aligned}$$

Note that if $3g-14 \le 4g-40$, then $g \ge 26$. If this occurs, then the cardinality of the set $G_2(H)$ is greater than or equal to $3g-2$ and $3g-2 > 3(g-1)$. Thus, $\mathbf{d}$ is a $(g,5)$-Buchweitz sequence for every $g \ge 26$.

The following result determines the conditions that a given sequence must satisfy for being a (g,t)-Buchweitz sequence for each integer large enough g. Moreover, we prove that the reverse of a (g,t)-Buchweitz sequence is also a (g,t)-Buchweitz sequence.

Corollary 5.3. *Let* $\mathbf{d} = (d_1,\dots,d_{t-1}) \in (\mathbb{N}\setminus\{0\})^{t-1}$ *and* $\mathbf{d}' = (d_{t-1}, d_{t-2},\dots,d_1) \in (\mathbb{N}\setminus\{0\})^{t-1}$. *For every integer* $g \ge 2\sum_{i=1}^{t-1} d_i + t + 1$, *the difference sequences* $\mathbf{d}$ *and* $\mathbf{d}'$ *are* (g,t)*-Buchweitz sequences if and only if* $\#\left\{\sum_{i=p}^{t-1} d_i + \sum_{j=q}^{t-1} d_j \mid p, q \in \{1,2,\dots,t-1\}\right\} > 3(t-1)$.

Proof. Apply Theorem 4.1 taking $A = \{1+\sum_{i=1}^{t-1} d_i,\dots, 1+d_{t-2}+d_{t-1}, 1+d_{t-1}, 1\}$ for $\mathbf{d}$, and $A = \{1+\sum_{i=1}^{t-1} d_i, \dots, 1+d_2+d_1, 1+d_1, 1\}$ for $\mathbf{d}'$. □

Note that we obtain an algorithm for checking the existence of 2-Buchweitz PF-semigroups for a fixed difference sequence from the previous result. The sketch of this algorithm is Algorithm 1.

Algorithm 1: Algorithm to check if a difference sequence is the sequence of a family of 2-Buchweitz PF-semigroups.

Input: $\mathbf{d} = (d_1, \dots, d_{t-1}) \in (\mathbb{N} \setminus \{0\})^{t-1}$
Output: An integer $g \in \mathbb{N}$ such that $\mathbf{d}$ is a (g', t)-Buchweitz sequence for every integer $g' \geq g$, or 0 if such integer g does not exist.

```
begin
  g ← 0;
  a ← 1;
  A ← {a};
  for i ← t − 1 to 1 do
    A ← {a + d_i} ∪ A;
    a ← {a + d_i};
  if #(2A) > 3(t − 1) then
    g ← 2 ∑_{i=1}^{t−1} d_i + t + 1;
  return g;
```

We illustrate Corollary 5.3 and Algorithm 1 with some easy examples.

Example 5.4. Fix the difference sequence $\mathbf{d} = (1, 3, 3, 2)$, the integer g obtained from Algorithm 1 is $g = 24$. Thus, the semigroups with genus greater than or equal to 24 and associated sequence $\mathbf{d}$ are 2-Buchweitz PF-semigroups.

Obtaining this type of difference sequence is really simple, just by random trial one stumbles onto many of them. Some examples are the following: $(1, 4, 3)$ for $g \geq 21$, $(2, 4, 3)$ for $g \geq 23$, etc.

Given two Buchweitz sequences, one can construct a new Buchweitz sequence. This result allows us to obtain 2-Buchweitz semigroups with genus and type as large as wanted.

Theorem 5.5. *Let* $\mathbf{d} = (d_1, \dots, d_{t-1}) \in (\mathbb{N}\setminus\{0\})^{t-1}$ *and* $\mathbf{d}' = (d'_1, \dots, d'_{h-1}) \in (\mathbb{N}\setminus\{0\})^{h-1}$ *be two* (g, t)*-Buchweitz and* (g', h)*-Buchweitz sequences, respectively, satisfying* Corollary 5.3. *For every integer* $k \geq 1$, *if* $d'_{h-1} > k$, *then*

$$\mathbf{d}'' = (d'_1, \dots, d'_{h-1}, k, d_1, \dots, d_{t-1})$$

is a $(g'', t + h)$*-Buchweitz sequence for every integer* $g'' \geq g + g' + 2k - 1$.

Proof. Let $A = \{a_1, \dots, a_{t-1}, a_t = 1\} \subset \mathbb{N}$ and $A' = \{a'_1, \dots, a'_{h-1}, a'_h = 1\} \subset \mathbb{N}$ (with $a_1 > \cdots > a_{t-1} > 1$ and $a'_1 > \cdots > a'_{h-1} > 1$) be the sets such that $d_i = a_i - a_{i+1}$ for $i = 1, \dots, t-1$, and $d'_i = a'_i - a'_{i+1}$ for $i = 1, \dots, h-1$. Note that if $\mathbf{d}'' = (d''_1, \dots, d''_{h+t-1})$, then $d''_i = a''_i - a''_{i+1}$ for every $i = 1, \dots, h+t-1$, with $a''_i = a_{i-h}$ for $i = h+1, \dots, t+h$, and $a''_i = a'_i + a_1 + k - 1$ for $i = 1, \dots, h$.

Define $A'' = \{a''_1, \dots, a''_{h+t} = 1\}$. We have that

$$A'' = \{a'_1 + a_1 + k - 1, \dots, a'_{h-1} + a_1 + k - 1, k + a_1, a_1, \dots, a_{t-1}, a_t = 1\}.$$

Since $g'' \geq g + g' + 2k - 1$, we obtain that $g'' \geq 2\sum_{i=1}^{h-1} d'_i + h + 1 + 2\sum_{i=1}^{t-1} d_i + t + 1 + 2k - 1 = 2a''_1 + h + t - 1$.

To determine whether $\mathbf{d}''$ is a Buchweitz sequence, we study the cardinality of $2A''$. Note that $2A = \{2 = 2a_t < 1 + a_{t-1} < \cdots < 2a_1\}$, $2A' = \{2 = 2a'_h < 1 + a'_{h-1} < 1 + a'_{h-2} < \cdots < 2a'_1\}$, and the following elements are all in $2A''$:

$$\begin{aligned}2 = 2a_t < 1 + a_{t-1} < \cdots < 2a_1 < \cdots < 2a_1 + k < \cdots < 2a_1 + 2(k-1) + 2 < 2a_1 + 2(k-1) + 1 + a'_{h-1} \\ < \cdots < 2a_1 + 2(k-1) + 2a'_1.\end{aligned}$$

Thus, $B = 2A \sqcup (2(a_1 + k - 1) + 2A') \subset 2A''$ and $\#(2A'') \geq 3(t-1) + 1 + 3(h-1) + 1$. Note that $2a_1 + k \in 2A'' \backslash B$.

Let $a'_{h-1} + k - 1 + 2a_1 \in 2A''$, we know that $a'_{h-1} + k - 1 + 2a_1 < 2a_1 + 2(k-1) + 1 + a'_{h-1}$. Since $d'_{h-1} > k$, then $a'_{h-1} + k - 1 + 2a_1 > 2a_1 + 2k$ and $a'_{h-1} + k - 1 + 2a_1 \in 2A'' \backslash B$. Hence, $2A \sqcup (2(a_1 + k - 1) + 2A') \sqcup \{2a_1 + k, a'_{h-1} + k - 1 + 2a_1\} \subset 2A''$ and therefore $\#(2A'') \geq 3(h-1) + 3(t-1) + 4 = 3(h+t-1) + 1 > 3(h+t-1)$. By Theorem 4.1, the semigroup associated with $\mathbf{d}''$ is a 2-Buchweitz PF-semigroup for every integer $g'' \geq g + g' + 2k - 1$. □

The condition $d'_{h-1} > k$ in Theorem 5.5 cannot be removed. For $k = 1$ and every large enough genera g and g', consider the $(g, 5)$-Buchweitz sequence $(1, 2, 2, 1)$ and the $(g', 5)$-Buchweitz sequence $(2, 3, 1, 1)$. For every genus, the numerical semigroups associated with the sequences $(1, 2, 2, 1, \mathbf{1}, 1, 2, 2, 1)$ and $(2, 3, 1, 1, \mathbf{1}, 1, 2, 2, 1)$ are not 2-Buchweitz.

From the examples of Buchweitz sequences obtained from Algorithm 1 and their associated 2-Buchweitz semigroups, and by using Theorem 5.5, it is easy to generate several 2-Buchweitz semigroups with large genera and types.

Example 5.6. If we take the difference sequences $(2, 4, 3)$ and $(1, 4, 3)$ from Example 5.4, we know that the following sequences are (g, t)-Buchweitz

$$\begin{aligned}
(2, 4, 3) &\quad \text{for } g \geq 23,\\
(1, 4, 3, \mathbf{2}, 2, 4, 3) &\quad \text{for } g \geq 48,\\
(1, 4, 3, \mathbf{2}, 1, 4, 3, 2, 2, 4, 3) &\quad \text{for } g \geq 73,\\
(1, 4, 3, \mathbf{2}, 1, 4, 3, 2, 1, 4, 3, 2, 2, 4, 3) &\quad \text{for } g \geq 98,\\
&\vdots
\end{aligned}$$

In the same way, we can use other difference sequences to get 2-Buchweitz semigroups with genera and types as large as we wish.

Acknowledgements: The authors thank the referees for their helpful observations.

Funding information: Part of this paper was written during a visit of Fernando Torres to the Universidad de Cádiz (Spain); his visit was partially supported by *Ayudas para Estancias Cortas de Investigadores* (EST2018-Ř0, Programa de Fomento e Impulso de la Investigación y la Transferencia en la Universidad de Cádiz). Fernando Torres was partially supported by CNPq/Brazil (Grant 310623/2017-0). Juan Ignacio García-García, Daniel Marín-Aragón, and Alberto Vigneron-Tenorio were partially supported by Junta de Andalucía research groups FQM-343 and FQM-366, and by the project MTM2017-84890-P (MINECO/FEDER, UE).

Author contributions: All authors have accepted responsibility for the entire content of this manuscript and approved its submission.

Conflict of interest: Authors state no conflict of interest.

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Appendix
A GAP Code

```
# Input: two lists of integers
# Returns the set A+B
sumAB := function(A,B)
  local a,C;
  C := Set([]);
  for a in A do
    C := Union(C, a + B);
  od;
  return C;
end;

# Input: a list of integers
# Returns nA, that is, A+...+A (n-times)
nA:=function(n,A)
  if(n=1) then
    return A;
  elif(n=2) then
    return sumAB(A,A);
  else
    return sumAB(A,nA(n-1,A));
  fi;
end;

# Input: a numerical semigroup S
# Returns true if the semigroup S is 2-Buchweitz
is2Bw := function(S)
  local gaps;
  gaps := GapsOfNumericalSemigroup(S);
  return Length(nA(2,gaps))>(2*2-1)*(Length(gaps)-1);
end;

# Input: a numerical semigroup S
# Returns true is S is a PF-numerical semigroup
isPFNS := function(S)
  local gapsOfS, multiplicidadS, frS, pFr, m;
  gapsOfS := GapsOfNumericalSemigroup(S);
  frS := FrobeniusNumberOfNumericalSemigroup(S);
  m := MultiplicityOfNumericalSemigroup(S);
  pFr := PseudoFrobenius(S);
  if(frS=2*m-1 and ForAll(pFr,x-> m<x and x<2*m)) then
    return true;
  fi;
  return false;
end;
```

```
# Input: g, a natural number
# Returns: the set of numerical semigroups that are 2-Buchweitz
# with genus g
list2BS := function(g)
  return Filtered( NumericalSemigroupsWithGenus(g),
    x-> is2Bw(x));
end;

# Input: g, a natural number
# Returns: the set of numerical semigroups that are 2-Buchweitz
# and PF-semigroups with genus g
list2BPFS := function(g)
  return Filtered( NumericalSemigroupsWithGenus(g),
    x-> isPFNS(x) and is2Bw(x));
end;

# An example: computation of the set of 2-Buchweitz
# PF-semigroups with genus 21
g:=21;Length(list2BS(g));Length(list2BPFS(g));
```