

Mathematics and Online Education

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Mathematics and Online Education

- A brief overview of distance learning

Partially supported by *Fundação para a Ciência e a Tecnologia* (Portugal) project CAMGSD UIDB/MAT/04459/2020



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A brief overview of distance learning

The *best* way to learn Mathematics is by teacher guidance in the time-honored judicious mixture of:

- explanation and orientation by the teacher,
- self-study of the theory,
- training by solving progressively more challenging exercises,
- resolution of problems,
- work in individual and group projects,
- evaluation(s) of progress

Preferably, these items should be developed in a **face-to-face** situation between teacher and students and of students among themselves.

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When this is not possible, **distance learning** educational situations are the alternative.



A brief overview of distance learning

Phases of distance learning:

- (i) Post office phase (1728, then 19th to late 20th Century)
- (ii) Radio and telephone days (from 1930s to early 21st Century)
- (iii) Television era (from late 1960s to early 21st Century)
- (iv) Internet epoch (from the 1990s onward)
 - email
 - e-learning platforms
 - MOOCs
 - videoconference systems
 - autonomous systems (using AI)
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Historically, earlier phases used to survive in more recent ones.

Presently, only phase (iv) exists in any significant way.



Learning mathematics online

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...infusing face-to-face learning with some of the positive characteristics of contemporary Distance Learning !



Learning mathematics online

For distance learning **mathematics** the positive characteristics of contemporary Distance Learning are, among possibly others, the following:

- permanent access to discussions and explanations posted in forums
- automatic feedback and correction of exercises
- staged progress of knowledge acquisition (e.g., through gamification)
- repeatability of verbal explanations (videos)
- easy access to materials (software, books, papers)
- possibility of synchronous interaction with colleagues and teachers (chat rooms, videoconf)



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Additionally to these, there is the time honoured characteristic of distance learning: spatial and temporal flexibility.



An example from Lisbon: *Universidade Aberta*

Universidade Aberta (UAb) is the only state owned university in Portugal exclusively dedicated to distance learning.

UAb currently has 13 mathematicians and offers programmes in Mathematics at the levels of: **life long learning** (2), **undergraduate** (2), **master** (2), **doctoral** (1), and several undergraduate and graduate service courses in UAb study programmes.



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We take opportunity of the flexibility of distance learning also to get collaboration of colleagues outside the UAb for teaching specific courses in postgraduate programmes. For instance, the PhD programmes in Applied Mathematics and Modelling and the now closed Computational Algebra have had, along its several editions, teaching collaborators from: **Canada**, **France**, **Greece**, **Spain**, **United Kingdom**, and **United States**.



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A few other occasional collaborations of colleagues from Portuguese universities and polytechnical schools also take place.



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Online teaching at UAb takes place on the e-learning platform MOODLE.



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Evaluation depend on the level of the course:

- in undergraduate programmes individual assignments along the semestre are complemented with a final examination (usually on-site; a pilot experiment is under way to make it also online only).
- in postgrad (master and doctoral) programmes there are individual and group assignments that can be complemented by a face-to-face (or zoom) final discussion.



An example from Lisbon: *Universidade Aberta*

The following printscreens illustrate the way a course in the PhD programme is run.

Arriving at the Ordinary Differential Equations course one sees. . .



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The screenshot shows the Moodle interface for the course "Equações Diferenciais Ordinárias 2020". The top navigation bar includes the AbERTA logo, "Home", "Dashboard", "My courses", and "Edit mode" (checked). The left sidebar lists various course elements under "General" and "Topic 1". The main content area displays the course title in Portuguese and English, the year "(2020 / 2021)", and a phase plane diagram with labels for "sink", "source", and "limit cycle". Below the diagram, a welcome message addresses the student. The right sidebar contains a search forum section, a "Latest announcements" section with a note that no announcements have been posted yet, and an "Upcoming events" section with a note that there are no upcoming events.

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Information of what is to be expected is given in the Learning Contract.

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The screenshot shows the user interface of the Aberta Learning Management System. At the top, there is a navigation bar with 'Home', 'Dashboard', and 'My courses'. A sidebar on the left contains a menu with 'General' and 'Topic 1' sections. The main content area is titled '23025_20_01 / Learning Contract' and features a 'Learning Contract' header with a sub-menu including 'Book', 'Settings', 'Import chapter', and 'More'. Below this, a 'Contract' section is visible. A prominent blue banner reads 'ABERTA Classe Virtual'. The main text area contains the following information:
Academic Year: 2020/21
Learning Unit Code: **23025**
Learning Unit Name: **Equações Diferenciais Ordinárias / Ordinary Differential Equations**
Instructors: Fernando Pestana da Costa
A 'Table of contents' sidebar on the right lists the course structure, starting with '1. Learning Contract' and '2. Goals', followed by numbered sections up to 10.3.4. Navigation buttons like 'Next' and 'Nex ?' are present.

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Work is split into a number of topics, with indication of what to study and local resources.



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The screenshot shows the course interface for 'Topic 7: Planar Systems' on the Universidade Aberta platform. The top navigation bar includes 'Home', 'Dashboard', and 'My courses'. A left sidebar lists various course activities, with 'Topic 7' currently selected. The main content area displays the following information:

- Topic 7**
- Theme 7: Planar Systems** (January 18 - 22)
- Main bibliography:**
 - Barreira & Valls:** sections 7.1-7.2
- Additional bibliography:**
 - Hale:** section II.1
 - Hartman:** sections VII.4-VII.10
- A list of resources for the theme:
 - Teorema de Poincaré-Bendixson, by M. Viana (in Portuguese)
 - Exercise sheet for Theme 7
 - Discussion Forum for Theme 7
 - Activity A, part 7 (downloading & uploading device)

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The screenshot displays the 'Discussion Forum for Theme 2' interface on the AbERTA platform. The top navigation bar includes 'Home', 'Dashboard', and 'My courses'. The forum title is 'Discussion Forum for Theme 2', with sub-navigation for 'Forum', 'Settings', 'Advanced grading', 'Subscriptions', 'Reports', and 'More'. A search bar is present with the text 'Search forums' and a search icon. A button labeled 'Add discussion topic' is also visible.

The main content area features a table of discussions:

Discussion	Started by	Last post	Replies
☆ Hortman 4.2	Tomás Tavar... 17 Nov 2020	Tomás Tavares 18 Nov 2020	2
☆ Relatório	David Gil 15 Nov 2020	David Gil 15 Nov 2020	1
☆ Barreira 1.16	Vinicius Sant... 12 Nov 2020	Vinicius Santos 13 Nov 2020	2
☆ Barreira 1.7	David Gil 8 Nov 2020	Vinicius Santos 13 Nov 2020	22
☆ Hale 2.1	Vinicius Sant... 13 Nov 2020	David Gil 13 Nov 2020	2
☆ Barreira 1.10	Vinicius Sant... 8 Nov 2020	Vinicius Santos 10 Nov 2020	9

The left sidebar contains a list of course materials, including 'Exercise sheet for Them...', 'Discussion Forum for Th...', and 'Activity A, part 1 (downl...'. The 'Discussion Forum for Th...' item is highlighted.

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A thread in a discussion forum typically looks as follows

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The screenshot shows a forum thread on the Universidade Aberta website. The top navigation bar includes the logo, 'Home', 'Dashboard', and 'My courses'. The left sidebar lists various forum topics, with 'Discussion Forum for Th...' highlighted. The main content area shows a post by 'Mara Jacinto' titled 'Re: Barreira 2.3' from December 5, 2020. The post includes a greeting to the professor and a mathematical derivation. The derivation starts with the sum from k=0 to infinity of (sum from l=0 to k of (At)^l * (-As)^{k-l}) / (k-l)!. This is simplified to the sum from l=0 to infinity of (At)^l / l! * sum from k=l to infinity of (-As)^{k-l} / (k-l)!. The final result is the product of sum from k=0 to infinity of (At)^k / k! and sum from k=0 to infinity of (-As)^k / k!.

Re: Barreira 2.3
by Mara Jacinto - Sábado, 5 Dezembro 2020, 9:10 PM

Caro professor

A minha ideia era:

$$\begin{aligned} & \sum_{k=0}^{+\infty} \left(\sum_{l=0}^k \frac{(At)^l (-As)^{k-l}}{l! (k-l)!} \right) = \\ &= \sum_{l=0}^{+\infty} \frac{(At)^l}{l!} \sum_{k=l}^{+\infty} \frac{(-As)^{k-l}}{(k-l)!} = \\ &= \sum_{k=0}^{+\infty} \frac{(At)^k}{k!} \sum_{k=0}^{+\infty} \frac{(-As)^k}{k!} \end{aligned}$$

Cumprimentos
Mara Jacinto

Permalink Show parent Edit Split Delete Reply

Re: Barreira 2.3
by Fernando Pestana da Costa - Domingo, 6 Dezembro 2020, 10:50 AM

Cará Mara,

Sim, é isso: a primeira igualdade corresponde a trocar a ordem dos somatórios de $\sum_k \sum_\ell$ para $\sum_\ell \sum_k$ (que é válido porque a s'erie (dupla) é absolutamente convergente); e a segunda igualdade é apenas uma mudança de variável $k - \ell \mapsto k$ no integral em k e por fim uma mudança de

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What can we expect for the near future?



What next?

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Presently the fashion is Artificial Inteligence (e.g. **ChatGPT** and the like.)



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Chat GPT *will* clearly be a challenge to online teaching of mathematics, with opportunities and with (many!) drawbacks, namely for the **evaluation** process.

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Chat GPT *will* clearly be a challenge to online teaching of mathematics, with opportunities and with (many!) drawbacks, namely for the **evaluation** process.

However the use of ChatGPT for the **production of mathematical answers** by students is currently limited by its lack of accuracy in less than elementary duties.



What next?

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In [5] (“ChatGPT: A revolutionary tool for teaching and learning mathematics”), which is based on interviews, many potentialities and problems are raised, as well as hopes for future developments. I highlight just one phrase (in page 6) that shocked me as killing the *present* usefulness of ChatGPT for production of mathematical text:

“It is also necessary to cross-check the generated responses with other sources to confirm their accuracy and reliability.”



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“It is also necessary to cross-check the generated responses with other sources to confirm their accuracy and reliability.”

(So what's the point?!)



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“Contrary to many positive reports in the media about GPT-4 and ChatGPT’s exam-solving abilities (a potential case of selection bias), their overall mathematical performance is well below the level of a graduate student. Hence, if your goal is to use ChatGPT to pass a graduate-level math exam, you would be better off copying from your average peer!”



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FC factorize $x^5 - 4x^4 + x^2 - 34$

< 2 / 2 >



To factorize the polynomial $x^5 - 4x^4 + x^2 - 34$, you can start by factoring out the greatest common factor (GCF) from all terms. In this case, the GCF is 1. Then, you can look for further factorization:

$$x^5 - 4x^4 + x^2 - 34$$

$$= x^4(x - 4) + x^2 - 34$$

Now, let's focus on the remaining terms, $x^2 - 34$. This is a difference of squares, which can be factored as follows:


$$x^2 - 34 = (x + \sqrt{34})(x - \sqrt{34})$$

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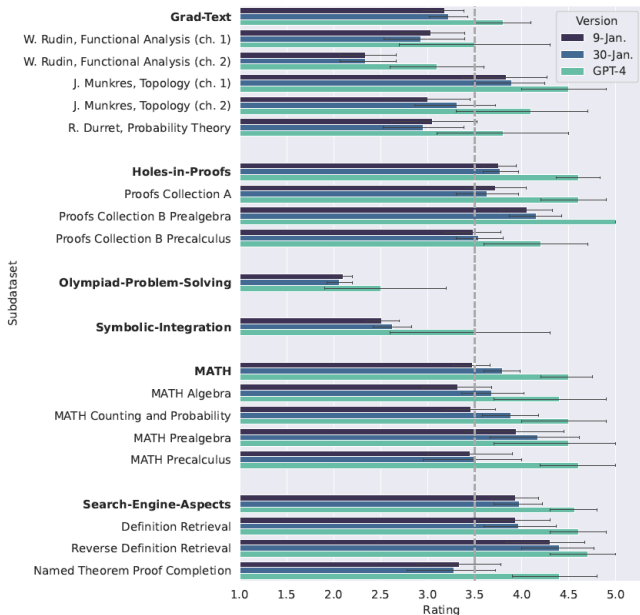
So, the complete factorization of the polynomial is:

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Indeed [2] shows that many of the answers fall behind the pass mark in most types of queries, although a notorious improvement in GPT-4, when compared with ChatGPT, is clearly noticeable in the next chart.



What next?



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As a tentative and not too risky (actually: almost trivial) **conclusion**:

Several recent studies show that, although presently somewhat lacking, we can expect a fast improvement of Chat-X mathematical capabilities in the near future, which will likely entail important impacts in the way we teach mathematics at university, and more acutely using distance learning models.

Mathematics and Online Education

Some references:

- 1 A. Costa, F.P. da Costa, M.A. Huertas: *e-Math Workshops: a Forum for Exchanging Experiences of Mathematics e-Learning at University Level*, *EMS Newsletter*, **101** (2016) 52–54.
- 2 S. Frieder et al: *Mathematical Capabilities of ChatGPT*, (20 July 2023) arXiv:2301.13867v2
- 3 O. Fidanian: *Historical Stages of the Distance Education Development*, *Pedagogy and Education Management Review*, **2** (2020)
- 4 B. Holmberg: *The Evolution, Principles and Practices of Distance Education*, Studien und Berichte der Arbeitsstelle Fernstudienforschung der Carl von Ossietzky Universität Oldenburg, Band 11, BIS Verlag, Oldenburg, 2008
- 5 Y. Wardat et al: *ChatGPT: A revolutionary tool for teaching and learning mathematics*, *EURASIA J. Math. Sci. Tech. Ed.*, **19** (7), (2023) em2286



Thank you very much!
Moltes gràcies!
Muchas gracias!
Muito obrigado!