Whitham Deformations of the Korteweg-de Vries Equation

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Zusammenfassung in deutscher Sprache

Die Beobachtung und Beschreibung von solitären Flachwasserwellen in einem Kanal hat im 19. Jahrhundert zur Formulierung der räumlich eindimensionalen Korteweg-de-Vries-Gleichung (KdV) geführt. Ihre Verallgemeinerung auf 2 räumliche Dimensionen ist die Kadomtsev-Petviashvili-Gleichung (KP). Beide Gleichungen sind *dispersive* nichtlineare partielle Differentialgleichungen. Besondere Aufmerksamkeit hat bei ihrem Studium die Tatsache erfahren, dass sie jeweils unendlich viele Erhaltungsgrößen und zugehörige Symmetrien besitzen, was es erlaubt, sie als Teil einer integrablen Hierarchie von kompatiblen Gleichungen zu verstehen. Mittels der inversen Streutheorie und der Finite-Gap-Methode kann eine weite Klasse von Lösungen beschrieben werden, die auch die anfänglich beobachteten Solitonenwellen beinhaltet. Es ist bemerkenswert, dass Lösungen mit der Finite-Gap-Methode durch Riemannsche Flächen parametrisiert werden.

Die vorliegende Arbeit beschäftigt sich mit *dispersionslosen* Versionen der integrablen KdV- und KP-Hierarchie, die auf dem Weg der Mittelung von dispersiven Lösungen erhalten werden. Dies liefert Hierarchien die wiederum integrabel sind, allerdings in einem allgemeineren Sinne. Während beispielsweise die Solitonenlösungen der dispersiven KdV-Gleichung stabil sind, tritt für die dispersionslose KdV-Gleichung das Phänomen brechender Wellen auf. Zu dem algebraisch-geometrischen Datum einer Riemannschen Fläche in der Finite-Gap-Methode kommen auf der dispersionslosen Seite algebraisch-geometrische und differential-geometrische Strukturen auf dem Modulraum Riemannscher Flächen. Diese Strukturen bilden den Hauptgegenstand dieser Arbeit. Dabei liegt das Augenmerk auf der einfacheren gemittelten Version der KdV-Hierarchie, der sogenannten *KdV-Whitham-Hierarchie*. Ihre Verallgemeinerung auf den KP-Fall wird als vereinheitlichender Rahmen benutzt. Die Mittelung, die den Übergang von der KdV-Hierarchie zur KdV-Whitham-Hierarchie ermöglicht, wird hier auf eine neue Weise durchgeführt. Dies liefert den einfachsten Fall in einer Klasse von Lösungen, die über die verallgemeinerte Hodograph-Methode zugänglich sind. Als Hauptresultat werden Lösungen dieser Klasse mittels Gleichungen beschrieben, die aus der klassischen Differentialgeometrie bekannt sind.

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Contents

CHAPTER 1

Introduction

The observation and description of solitary shallow-water waves led to the formulation of the famous *Korteweg-de Vries equation* (KdV) and its generalization, the *Kadomtsev-Petviashvili equation* (KP). They are *dispersive* and non-linear partial differential equations (PDEs) in $1+1$ dimensions and $2+1$ dimensions, respectively, where the first number refers to the spatial dimension and the second number to time. An aspect that has drawn a lot of attention is that the KdV and KP equations possess infinitely many conserved quantities and corresponding symmetries – giving rise to the structure of integrable hierarchies. The present thesis is about *dispersionless* versions of these integrable hierarchies which are obtained by averaging. They are again integrable, but in a more general way. While the dispersive equations admit stable solitary waves as solutions, on the dispersionless side breaking waves occur. The theoretical description of dispersionless hierarchies yields algebraic-geometric and differential-geometric structures on the spaces of the conserved quantities of the dispersive hierarchies. These structures are the main subject of the thesis. Its focus is on the more elementary averaged versions of the KdV hierarchy, but often the averaged versions of the KP hierarchy are considered since they provide a unifying framework. A new independent approach to obtain averaged versions of the KdV hierarchy is presented and used to describe a class of solutions that are accessible by the *generalized hodograph method*.

From Dispersive Equations to Dispersionless Analogues. On the side of dispersive equations, the $2+1$-dimensional KP equation can be expressed in the form

$$\partial_x\left(\partial_t u + u\partial_x u + \partial_x^3 u\right) + \lambda\partial_y^2 u = 0 \tag{1.0.1}$$

with $\lambda = \pm 1$. When asking for solutions that are constant in the spatial coordinate y, the (dispersive) KdV equation [**35**] appears as the $1+1$-dimensional reduction

$$\partial_t u + u\partial_x u + \partial_x^3 u = 0 \quad . \tag{1.0.2}$$

The KdV equation is an example of a *soliton equation* and admits a broad class of exact solutions by the *inverse scattering transform* [**49, 50, 26, 46**] and the *finite gap method* [**47, 18**]. Consecutively, the theory was generalized to other soliton equations, including the KP equation, which Krichever [**36, 37**] established as the natural framework for the algebraic-geometric finite gap method. The algebraic geometric data involved is a Riemann surface of finite genus, called *spectral curve*. It is a constant of motion for the dynamics that takes linear form on its Jacobian variety. For the KdV equation the finite gap method yields solutions described by ordinary differential equations (ODEs), i.e. $1+0$-dimensional equations. In more detail this important class of solutions will be explained in Chapter 3 in a classical, but not so well-known way.

Dispersionless versions of the KdV and KP equations can be obtained by several approaches. A simple method leading to the dispersionless KdV equation uses that by the reparameterization $x \to \epsilon x$, $t \to \epsilon t$ for $\epsilon > 0$, the KdV equation turns out to be equivalent to $\partial_t u + u\partial_x u + \epsilon^2 \partial_x^3 u = 0$. In the limit $\epsilon \to 0$ the *dispersionless KdV equation* (or *inviscid Burgers' equation*)

$$\partial_t u + u\partial_x u = 0 \tag{1.0.3}$$

appears. It is related to the averaging of finite gap solutions with spectral genus 0, as demonstrated in the introductory example of Chapter 2. Solutions of Burgers' equation share some features with those of the dispersive KdV equation, but also lead to new phenomena like breaking waves. Away from breaking points the simple method described at the beginning of Chapter 2 provides solutions of (1.0.3).

Geometric Structures on the Moduli Space of Finite Gap Solutions. Other approaches to derive dispersionless equations make use of the *Krylov-Bogoliubov averaging method* [**43, 44, 62**] and the *Wentzel–Kramers–Brillouin method* (WKB), which are established ways to approximate dynamics with oscillatory behavior. In [**69**] Whitham applied averaging to different types of non-linear dispersive waves. In particular, for finite gap KdV solutions with spectral genus 1 he obtained a *diagonal hydrodynamic system of PDEs* on the 3-dimensional parameter space of elliptic Riemann surfaces, now referred to as Whitham's equations. Analogous *Whitham equations* for KdV solutions from higher genus hyperelliptic spectral curves were described by Flaschka, Forest and McLaughlin [**25**] within an *algebraic-geometric* framework that uses meromorpic differential forms on varying spectral curves Γ. When for each Γ in the space of hyperelliptic Riemann surfaces, meromorphic differential forms $\mathrm{d}\Omega_0(\Gamma)$ and $\mathrm{d}\Omega_1(\Gamma)$ on Γ are uniquely chosen by a normalization, then the Whitham equations take the form of a conservation equation

$$\partial_T \mathrm{d}\Omega_0\left(\Gamma(X,T)\right) = \partial_X \mathrm{d}\Omega_1\left(\Gamma(X,T)\right) \ . \tag{1.0.4}$$

It is understood here that the space of Riemann surfaces Γ is parameterized in some way. Hence, the Whitam equation in conservation form is an implicit PDE for a map $(X,T) \mapsto \Gamma(X,T)$ with values in the parameter space. Moreover, in order to compare holomorphic objects on varying Riemann surfaces it will be essential throughout this thesis that all Riemann surfaces are equipped with a fixed chart at the point at infinity.

A *differential-geometric* framework for the KdV Whitham equations and similar equations was developed by Dubrovin and Novikov [**20**]. Their *Hamiltonian formalism* relates Whitham's hydrodynamic system of PDEs in $1+1$ dimensions to certain flat Riemannian metrics on the parameter space of elliptic Riemann surfaces. An analogous relation exists also for higher spectral genus. Generically, all solutions of diagonal Hamiltonian systems of hydrodynamic type can be described by Tsarev's *generalized hodograph method* [**65**]. In subsequent work crucial features of the differential-geometric approach were generalized to the wider class of *semi*-Hamiltonian systems of hydrodynamic type [**66**].

For the KP equation Krichever applied the averaging method to finite gap solutions [**38**] and interpreted the resulting equations in the algebraic-geometric framework of the *universal Whitham hierarchy* as a dynamics on the moduli space of Riemann surfaces with punctures and infinite jets of charts [**40**]. Due to the jets, this space is infinite dimensional and the PDEs appearing are integrable in the $2+1$-dimensional hydrodynamic sense. Considering finite dimensional subspaces described by *algebraic orbits* provides a reduction to $1+1$-dimensional hydrodynamic systems and allows again to construct solutions by the generalized hodograph method. The KdV Whitham equations are an example of an algebraic orbit. In Chapter 2 the universal Whitham hierarchy will be explained in more detail, with a focus on how a *hydrodynamic reduction* yields a transition to the differential-geometric side of algebraic orbits.

Averaging Methods for Finite Gap Solutions of the KdV Hierarchy. The averaging procedure used to derive Whitham equations in the context of finite gap theory usually involves the analysis of a multiscale system that results from coupling the "fast" KdV dynamics with a "slow" dynamics of the spectral curve. With the spectral curve as a constant of motion, also the integrability of the coupled system by the finite gap method is lost. Extracting the "slow" dynamics by averaging over the "fast" dynamics is rather complicated or often used as a mere heuristic, e.g. [**25**] addresses this issue by "its formal justification by 'two-timing' methods is too long to reproduce here, and we state the method as a prescription." On the other hand, Krichever's derivation in [**38**] is rigorous, but quite intricate.

One aim of the present thesis is to circumvent the explicit averaging procedure on the way from the KdV equation (1.0.2) to KdV Whitham equations (1.0.4) by using ideas from the theory of *adiabatic invariants*. This direction was already indicated by Whitham in [**69**] and was mentioned later on, e.g. in [**40**]. The basic idea, detailed in the following, is that when perturbing a classical Hamiltonian system "slowly," then its action variables are adiabatic invariants [**45, 4, 48**]. By definition, adiabatic invariants change very little under the perturbation. Demanding a perturbation to even preserve the action variables leads to the notion of *KdV Whitham deformations*. It turns out that KdV Whitham deformations induce solutions of the KdV Whitham equations. This relates to Krichver's "trivial but very useful" observation in [**39**] (see also Theorem 2 in [**38**] and [**40**]): the conservation form (1.0.4) of the Whitham equations is the compatibility condition for the system of differential equations

$$\begin{cases} \partial_X \mathrm{d}S = \mathrm{d}\Omega_0 \ , \\ \partial_T \mathrm{d}S = \mathrm{d}\Omega_1 \ . \end{cases} \tag{1.0.5}$$

The *generating differential form* $\mathrm{d}S$ is the complex derivative of a generating function S whose domain as subsets of the spectral curves and whose properties are yet to be determined. Here and in the following, the notation $\mathrm{d}(-)$ represents the complex derivatives on the spectral curves alone (derivatives in other directions are usually written as partial derivatives). It turns out that Whitham deformations yield particular generating differential forms that are defined on the underlying Riemann surface. In general, generating functions are a powerful tool for the analysis and construction of solutions of the Whitham equations.

Finite Gap Solutions of the KdV Hierarchy as a Classical Hamiltonian System. The first step, in order to apply adiabatic theory to finite gap solutions of KdV is to understand, how these solutions correspond to solutions of classical Hamiltonian systems. Since the work of Miura et al. [**49, 50**] the KdV equation is known to possess infinitely many constants of motion, corresponding to infinitely many symmetries and respective compatible higher time flows that form the *KdV hierarchy* [**46, 1**]. Compatibility means that a solution of the KdV hierarchy with times $t_1 = x, t_3 = t, t_5, \dots$ yields a solution $u = u(x, t, t_5, \dots)$ to the original KdV equation (1.0.2) for any choice of parameters $t_5, t_7, \dots$, which are called higher times. A $(2n+1)$-*stationary reduction* of the KdV hierarchy contains solutions constant in times t_{2n+1} and higher [**35, 55, 47**]. This reduction converts the KdV hierarchy into a system of ODEs whose constants of motion are encoded in the spectral curve from the finite gap soliton theory. For $(2n+1)$-stationary real solutions, the spectral curve is given in the form

$$\Gamma = \left\{(E, y) \in \mathbb{C}^2 \mid y^2 + g(E) = 0\right\} \cup \{\infty\} \tag{1.0.6}$$

for a polynomial $g(E) = \prod_{j=1}^{2n+1} (E - \gamma_j)$ with real roots $\gamma_1 < \cdots < \gamma_{2n+1}$. Moreover, the KdV dynamics can be interpreted as a finite dimensional completely integrable Hamiltonian system. Its Arnold-Liouville tori are given by real subsets of n-fold products of copies of the hyperelliptic spectral curves [**15, 51, 52, 2, 3, 53**]. Action-angle variables of this Hamiltonian system are available explicitly via Jacobi inversion. The actions I_j are the a-periods of the meromorphic differential form $y\mathrm{d}E$ on Γ

$$I_j(g) = 2\int_{\gamma_{2j}}^{\gamma_{2j+1}} y\mathrm{d}E \tag{1.0.7}$$

for $j = 1, \ldots, n$. In Chapter 3 the details of the Hamiltonian formulation of the stationary KdV hierarchy will be explained based mostly on work by S. I. Alber [**2**].

Application of Adiabatic Theory. In the second step, we consider perturbations of finite gap solutions in the Hamiltonian formulation of the stationary KdV hierarchy. The *adiabatic theorem* [**48**] can be applied to the actions of this completely integrable Hamiltonian system. Generally, for an integrable Hamiltonian system with parameters and a Hamiltonian flow in time t, this theorem guarantees that for a "slow" perturbation of the parameters (i.e. a *modulation* depending on the "slow" time $T = \epsilon t$), the actions $I = I(T, \epsilon)$ of the system remain uniformly close to a constant as the rate $\epsilon > 0$ of the modulation becomes smaller. Uniformly close here means that for a given ϵ the deviation is bound independently of $T = \epsilon t \in [0, 1]$ by some value $\rho(\epsilon)$. For the Hamiltonian formulation of the stationary KdV hierarchy, the parameters are the coefficients of the polynomial g_0 in

$$g(E) = g_0(E) + \sum_{k=0}^{n-1} h_k E^{n-1-k} \quad ,$$

while the coefficients h_k are the energies of the Hamiltonian system. Once the parameters are "slowly" perturbed in time, the dynamical system evolves in a non-integrable way. From its trajectories, the energies can be computed at each given point in time by the formulas of the integrable system at the same state. The uniform approximation of the actions I_j in (1.0.7) by constants implies that the energies $h_k = h_k(T, \epsilon)$ of the modulated system have to be uniformly close to a function in the modulation time T. Hence, the polynomial g itself and thus $y\mathrm{d}E = \sqrt{-g}\mathrm{d}E$ have to approximate a function in the modulation time. This suggests to consider the approximating objects which depend on T. We call a modulation *KdV Whitham deformation*, when the actions for $g = g(T)$ are constant in T. Equivalently, this means

$$0 = \partial_T I_j \tag{1.0.8}$$

for all $j = 1, \ldots, n$. Completely integrable Hamiltonian systems possess multiple symmetries and corresponding commuting flows in higher times. The perturbation of the system's parameters may also depend "slowly" on the higher times. For Whitham deformations the actions have to be constant in all those times. As an Ansatz for a Whitham deformation in the case $n = 1$ (i.e. 3-stationary KdV or 1-gap KdV) we take

$$g = E^3 + TE^2 + XE + h(X, T) = (E - \gamma_1)(E - \gamma_2)(E - \gamma_3) \quad . \tag{1.0.9}$$

The system of equations for Whitham deformations (1.0.8) then reads

$$\begin{aligned} 0 =& \partial_X I_1 = -\int_{\gamma_2}^{\gamma_3} (E + \partial_X h) \frac{\mathrm{d}E}{y} \ , \\ 0 =& \partial_T I_1 = -\int_{\gamma_2}^{\gamma_3} (E^2 + \partial_T h) \frac{\mathrm{d}E}{y} \ . \end{aligned}$$

These equations of elliptic integrals determine $\partial_X h$ and $\partial_T h$. From $\partial_T \partial_X I_1 = 0 = \partial_X \partial_T I_1$ we obtain the compatibility equations $\partial_T \partial_X h = \partial_X \partial_T h$ (see Example 4.2.4 for details). Hence, there is an energy function h such that the deformation of parameters (1.0.9) becomes a Whitham deformation. As a consequence $\mathrm{d}S := y\mathrm{d}E = \sqrt{-g(E)}\mathrm{d}E$ satisfies (1.0.5) for differential forms

$$\mathrm{d}\Omega_0 = (E + \partial_X h)\frac{\mathrm{d}E}{y} \quad \text{and} \quad \mathrm{d}\Omega_1 = (E^2 + \partial_T h)\frac{\mathrm{d}E}{y}$$

defined on the spectral curves $\Gamma = \Gamma(X, T)$ in (1.0.6) with $n = 1$. Note that these differential forms are not normalized yet, but a modification of the Ansatz (1.0.9) with polynomials in (X, T) as coefficients will allow this. In sum, the Whitham deformation of the spectral curve Γ provides the generating differential form $\mathrm{d}S = y\mathrm{d}E$ for the KdV Whitham hierarchy. When parameterizing the elliptic curves Γ by the roots of g, then (1.0.9) determines a solution $(X, T) \mapsto (\gamma_1, \gamma_2, \gamma_3)(X, T)$ of the Whitham equations in their form as a hydrodynamic system of PDEs. For the particular generating differential form $\mathrm{d}S = y\mathrm{d}E$ this corresponds to Krichever's version of the generalized Hodograph method in [**40**].

Applying the adiabatic theorem to the stationary KdV hierarchy in its Hamiltonian formulation is new. The necessary theory about adiabatic invariants is explained in Chapter 4 mostly by referring to work by Arnold, Kasuga and Neistadt as presented in [**48**]. Of course, multiscale analysis and averaging methods implicitly enter at this point again.

Generating Differential Forms and Euler-Poisson-Darboux Equations. *Euler-Poisson-Darboux equations* (EPDs) appear in the classical differential geometry of surfaces, see Chapter 3 in [**8**]. Solutions to particular EPDs, so called ϵ-systems [**59**], can be used to characterize general generating differential forms $\mathrm{d}S$ for the KdV Whitham hierarchy. Locally near a branch point $(E, y) = (\gamma_j, 0)$ of the spectral curve Γ from (1.0.6), there is a representation of generating differential forms by

$$\mathrm{d}S = \sum_{k \geq 0} a_k(E) y^{2k-1} \mathrm{d}E$$

with coefficient polynomials a_k of degree up to $2n$ in E. Under genericity assumptions, it will turn out that the differential equations (1.0.5) required for a generating differential form are satisfied if and only if $a_0 = 0$ and one coefficient polynomial a_j corresponds to a solution of an ϵ-systems with $\epsilon = (2j - 1)/2$. The other coefficient polynomials a_k are determined recursively. This will be shown in Chapter 5. A similar result was proved by Ferapontov and Pavlov for the quasi-classical limit of the coupled KdV hierarchy in [**24**]. In Sections 6 and 10 of [**59**] Pavlov seems to suggest this holds for the KdV Whitham hierarchy as well, but remains very vague. The more special case, when the generating differential form for the KdV Whitham hierarchy is defined on the spectral curve can be found in [**64**]. However, the general result and the proof given here are original.

The Soliton Limit. The spectral curve Γ from (1.0.6) degenerates when roots γ_j and γ_{j+1} of g coincide. For technical reasons this case is often excluded in the finite gap theory and Whitham theory. It is interesting nevertheless, since passing through such degeneracies allows to change the genus of Γ. Wave breaking in the dispersionless KdV equation (1.0.3) corresponds to changing genus, see [**64**]. The case when $\gamma_{2j-1} \to \gamma_{2j}$ for all $j = 1, \ldots, n$ leads to a genus 0 spectral curve. Corresponding finite gap solutions are called *solitons*. These describe solitary waves after which the KdV equation (1.0.2) was modeled initially. KdV solitons, their Whitham deformations and their Whitham equations are discussed in Section 3.3, Section 4.3 and Section 5.2, respectively.

CHAPTER 2

Hydrodynamic Integrability and the Universal Whitham Hierarchy

Whitham equations describe the modulation of integrable dispersive non-linear wave equations like the KdV equation [**69**]. The purpose of this chapter is to explain the basic differential-geometric structures and algebraic-geometric structures that emerged in Whitham theory [**40**].

Example 2.0.1. For an instructive example (resumed later on in the context of Section 4.3) let us start on the algebraic-geometric side. We are looking for a function $(X,T) \mapsto u(X,T)$ on a 2-dimensional real domain satisfying the conservation equation

$$\partial_T \mathrm{d}\Omega_0 = \partial_X \mathrm{d}\Omega_1 \tag{2.0.1}$$

for meromorphic differential forms

$$\mathrm{d}\Omega_0 = -\frac{1}{2}\frac{\mathrm{d}E}{\sqrt{u-E}}\ ,\ \mathrm{d}\Omega_1 = -\frac{1}{2}\frac{E-\frac{1}{2}u}{\sqrt{u-E}}\mathrm{d}E\ ,$$

on the double covering $\Gamma_0 = \{(E,y) \in \mathbb{C}^2 | y^2 = u - E\} \cup \{\infty\} \to \mathbb{C}\mathrm{P}^1,\ (E,y) \mapsto E$ whose finite branch value is u and thus depending on X and T. By using y as a chart at the branch point $(E,y) = (u,0)$, and using κ with $\kappa^2 = E$ as a chart at the point at infinity, the double covering yields a compact Riemann surface Γ_0 of genus zero. From $2y\mathrm{d}y = -\mathrm{d}E$ we see that $\mathrm{d}\Omega_0$ and $\mathrm{d}\Omega_1$ are holomorphic at the finite branch point. Note that the chart y depends on u and therefore also on (X,T). Hence, the conservation equation is equivalent to

$$0 = \left(\partial_T u - \frac{1}{2}u\partial_X u\right)\frac{\mathrm{d}y}{y^2}\ .$$

This identity for a differential form with a double pole at the branch point $(u,0)$ is equivalent to the quasi-linear equation

$$\partial_T u = \frac{1}{2}u\partial_X u\ , \tag{2.0.2}$$

which is called a hydrodynamic reduction of (2.0.1), since there is no more dependence on the parameter E. Moreover, (2.0.2) is a dispersionless KdV equation (or inviscid Burgers equation) and can be locally solved by the method of characteristics, see Appendix 2 to Chapter II in [**7**]. For some smooth function f consider the equation

$$0 = X + \frac{1}{2}uT + f(u)\ .$$

Let $(X_0, T_0, u_0) \in \mathbb{R}^3$ be a solution. If $0 \neq \frac{1}{2}T_0 + f'(u_0)$, then by the implicit function theorem, there is a solution $u = u(X,T)$ locally near $u_0 = u(X_0,T_0)$. For such a solution u, the derivatives can be found to by $0 = 1 + \left(\frac{1}{2}T + f'(u)\right) \partial_X u$ and $0 = \frac{1}{2}u + \left(\frac{1}{2}T + f'(u)\right) \partial_T u$. Hence,

$$0 = \left(\frac{1}{2}T + f'(u)\right) \left(\partial_T u - \frac{1}{2} u \partial_X u\right)$$

which implies (2.0.2), since by assumption and continuity $0 \neq \frac{1}{2}T + f'(u(X,T))$ near (X_0, T_0). Further away from (X_0, T_0) this inequality can become an equality. At such a "breaking point" the solution becomes discontinuous and a so called shock wave forms.

In this chapter the features of the previous example will be generalized to moduli spaces of Riemann surfaces of higher genus. As the central player we consider $1+1$-*dimensional systems of hydrodynamic type* (also called $1+1$-*dimensional hydrodynamic systems*)

$$\partial_T u_i = \sum_{j=1}^{m} v_{ij}(\boldsymbol{u}) \partial_X u_j \quad . \tag{2.0.3}$$

Like the dispersionless KdV equation in (2.0.2) these systems are quasi-linear systems of first order PDEs, but for a vector-valued function $(u_1, \dots, u_m) = \boldsymbol{u} = \boldsymbol{u}(X,T) \in \mathbb{R}^m$ instead of a scalar function. There is a rich theory of integrable quasi-linear first order PDEs related to classical geometry of conjugate and orthogonal nets. The associated theory involving linear PDEs goes back to Darboux [**9**], who adopted Riemann's method for the shock wave equation [**60**], see Chapter 4 in [**8**] and Chapter 22.3 in [**28**].

First, in Section 2.1 we discuss integrable and differential-geometric structures of $1+1$-dimensional hydrodynamic systems. Riemannian metrics in coordinates $\boldsymbol{u} \in \mathbb{R}^m$ appear to be associated to the hydrodynamic system and their curvature tensors mostly vanish. Then we address solutions obtained by Tsarev's generalized Hodograph method, which generalizes the method of characteristics in the example above. As a second part of this chapter the algebraic-geometric approach of Krichever's universal Whitham hierarchy is presented in Section 2.2. Here PDEs similar to the conservation equation (2.0.1) from the example above, are set on the algebraic-geometric data of Riemann surfaces with a marked point and a chart there. Initially, this configuration space is infinite dimensional. By considering an *algebraic orbit* in this data, the configuration space reduces to finitely many dimensions and a hydrodynamic reduction, similar to that in the example above, yields a $1+1$-dimensional hydrodynamic system. The associated Riemannian metric is set on the moduli space of Riemann surfaces with a marked point.

In the introductory example the moduli space is trivial, so the differential-geometric structure does not play a role there. The KdV Whitham hierarchy provides an example of the $1+1$-dimensional algebraic orbit setup with a flat Riemannian metric on the hydrodynamic side, see Example 2.2.4, Example 2.2.5 and for a more detailed treatment Chapter 5.

2.1. Semi-Hamiltonian Systems of Hydrodynamic Type

In the context of Whitham equations Dubrovin and Novikov established a Hamiltonian formalism for systems of hydrodynamic type [**20**]. In interesting examples like the KdV Whitham hierarchy, the tensor field v in (2.0.3) is diagonalizable with distinct eigenvalues (*hyperbolic case*), called *velocities*. Here, we take this as the starting point for the study of integrable hydrodynamic systems. A diagonal hyperbolic hydrodynamic system (2.0.3) is called *semi-Hamiltonian*, if the velocities satisfy

$$\partial_{u_k} \left(\frac{\partial_{u_i} v_j}{v_i - v_j} \right) = \partial_{u_i} \left(\frac{\partial_{u_k} v_j}{v_k - v_j} \right) \tag{2.1.1}$$

for $i \neq j \neq k \neq i$. The tensor field $v = (v_i(\boldsymbol{u})\delta_{ij})$ composed of the velocities is called *semi-Hamiltonian matrix*. In the case $m < 3$ the semi-Hamiltonian condition is empty. Note that the definition of semi-Hamiltonian hydrodynamic systems requires coordinates $\boldsymbol{u}$ which diagonalize the hydrodynamic systems and it also requires all the eigenvalues to be different. Let furthermore be assumed that no eigenvalue is zero, i.e. non-degeneracy. These are mostly technical assumptions and meant to simplify the arguments and statements below.

2.1.1. The Differential Geometry of Diagonal semi-Hamiltonian Systems. Geometry comes into play, since the velocities of a semi-Hamiltonian system give rise to a diagonal metric for which the Riemann curvature tensor (with respect to the induced Levi-Civita connection) vanishes except for R^j_{kkj}, compare to the first part of Theorem 3 in [**66**]. We see this as follows. The semi-Hamiltonian condition (2.1.1) is equivalent to the compatibility condition of the system

$$\partial_{u_i} v_j = \Gamma^j_{ji}(v_i - v_j) \tag{2.1.2}$$

for $i \neq j$. The coefficients Γ^j_{ji} then become Christoffel symbols of the diagonal metric $\mathfrak{g} = \sum_{j=1}^m \mathfrak{g}_{jj}\,(\mathrm{d}u_j)^2$ obtained by integrating

$$\Gamma^j_{ji} = \partial_{u_i} \log \sqrt{\mathfrak{g}_{jj}} \ . \tag{2.1.3}$$

For simplicity of notation we assume $\mathfrak{g}_{jj} > 0$, the case of pseudo-Riemannian metrics works in the same way, but the metric contains signs according to the signature. Since Γ^i_{ii} is not determined here, the integration is only unique up to multiplication with functions in one variable, that is $\mathfrak{g}_{ii}$ and $f_i(u_i)\mathfrak{g}_{ii}$ correspond to the same semi-Hamiltonian matrix. Applying the Koszul formula we see that the Christoffel symbols Γ^i_{jk} vanish for $i \neq j \neq k \neq i$ since the metric is diagonal. Hence, the metric property of the Levi-Civita connection is equivalent to $\mathfrak{g}_{kk}\Gamma^k_{ii} + \mathfrak{g}_{ii}\Gamma^i_{ik} = 0$ for $i \neq k$, so in this case Γ^k_{ii} is determined by (2.1.2) as well. The vanishing of torsion means $\Gamma^i_{ij} = \Gamma^i_{ji}$. Therefore, the Riemann curvature tensor corresponding to a diagonal metric $\mathfrak{g}$ is given by

$$R^j_{jki} = \partial_{u_k}\Gamma^j_{ji} - \partial_{u_i}\Gamma^j_{jk} \ , \tag{2.1.4}$$

$$R^j_{kij} = \partial_{u_i}\Gamma^j_{jk} - \Gamma^j_{jk}\Gamma^k_{ki} - \Gamma^j_{ji}\Gamma^i_{ik} + \Gamma^j_{ji}\Gamma^j_{jk} \ , \tag{2.1.5}$$

$$R^j_{kkj} = \partial_{u_k}\Gamma^j_{jk} - \partial_{u_j}\Gamma^j_{kk} + \Gamma^j_{jk}\Gamma^j_{jk} + \Gamma^k_{kj}\Gamma^j_{kk} - \sum_{r=1}^m \Gamma^r_{kk}\Gamma^j_{jr} \tag{2.1.6}$$

for $i \neq j \neq k \neq i$. Immediately, the semi-Hamiltonian property is seen as equivalent to the vanishing of (2.1.4). In the hyperbolic case, a short computation shows that this is equivalent to (2.1.5) vanishing as well, see page 403 f. in [**66**] for details. Only the expression (2.1.6) containing Γ^i_{ii} is not determined.

Conversely, if a diagonal metric on $\mathbb{R}^m$ (or equivalently an orthogonal coordinate system) is given such that the curvature tensor in (2.1.4) vanishes, then there is a family of semi-Hamiltonian matrices v, which satisfy (2.1.2) for the Christoffel symbols induced by the metric, see second part of Theorem 3 in [**66**]. The proof relies on a result by Darboux that the system (2.1.2) can be solved, if compatible, see pages 335-340 in [**9**]. Compatibility however, follows here from the semi-Hamiltonian property. That the semi-Hamiltonian matrices come in families originates from m functions in one variable used as "Cauchy data" for (2.1.2).

Remark 2.1.1. If the curvature in (2.1.6) vanishes as well, then the metric $\mathfrak{g}$ is flat and the system (2.0.3) is called a *Hamiltonian system of hydrodynamic type*. In this case there is a density h inducing a functional $H(\boldsymbol{u}) = \int h(\boldsymbol{u}(x))\mathrm{d}x$ as a *Hamiltonian of hydrodynamic type* and inducing the (semi-)Hamiltonian matrix by $v_{ij} = \mathfrak{g}^{ik}\nabla_k\nabla_j h = \nabla^i\nabla_j h$ (in coordinates, which are not necessarily diagonal). Assuming v in *general position* (see Definition 3 in [**66**]), the corresponding metric is unique up to a constant factor (see Theorem 2 there). The existence of a two-parameter family of flat metrics for the 1-phase KdV Whitham equations was shown in [**20**] in non-diagonal coordinates. For the n-phase KdV Whitham equations there is a flat metric on $\mathbb{R}^{2n+1}$ with signature $(+,-,+,\ldots,-,+)$, see [**16**] and Section 5.1 below.

For a diagonal metric the Christoffel symbols which are encoding the covariant derivative are particularly simple, only Γ^i_{ij} has to be known. Alternatively, the covariant derivative may be encoded in rotation coefficients. Let metric factors (*Lamé coefficients*) H_i be defined by $\mathfrak{g}_{ii} = H_i^2$. Then

$$\beta_{ij} = \frac{\partial_i H_j}{H_i} \tag{2.1.7}$$

for $i \neq j$ are called *rotation coefficients*. Here and in the following we often use $\partial_i = \partial_{u_i}$ for notational convenience. According to (2.1.3) the rotation coefficients are related to the Christoffel symbols by

$$\Gamma^i_{ij} = \frac{\partial_i H_j}{H_j} = \frac{H_i}{H_j}\beta_{ij} \ .$$

Inserting this into the Riemann curvature tensor then gives that the vanishing of (2.1.4) (and equivalently (2.1.5)) is equivalent to the *Darboux system*

$$\partial_k\beta_{ij} = \beta_{ik}\beta_{kj} \tag{2.1.8}$$

for $i \neq j \neq k \neq i$. This equation provides the compatibility of (2.1.7) seen as an equation for the metric factors $(H_i)_i$. The metric is flat if and only if additionally (2.1.6) vanishes, which becomes here

$$0 = \partial_i\beta_{ik} + \partial_k\beta_{ki} + \sum_{s\neq i,k}\beta_{si}\beta_{sk} \tag{2.1.9}$$

for $i \neq k$. If the rotation coefficients are symmetric, that is if $\beta_{ik} = \beta_{ki}$, then the condition for flatness (2.1.9) becomes simply

$$\sum_{s=1}^{m}\partial_s\beta_{ik} = 0 \tag{2.1.10}$$

for $i \neq k$. Such metrics whose rotation coefficients are symmetric are said to have the *Egorov property*. This is equivalent to $\partial_i\mathfrak{g}_{jj} = \partial_j\mathfrak{g}_{ii}$, which means that the metric is *potential* in the sense that there is a function $c = c(\boldsymbol{u})$ such that $\partial_i c = \mathfrak{g}_{ii}$. The KdV Whitham hierarchy provides an example of a flat Egorov metric, see [**16**] or Lemma 5.1.4 and Lemma 5.1.5 below. As a first example that can be treated directly, let us consider the following (for more of its context see Section 4.3).

Example 2.1.2 (1-soliton Whitham equations)**.** The Whitham equations of 1-solitons are given by

$$\partial_T\gamma_3 = \frac{1}{2}\gamma_3\partial_X\gamma_3 \ , \quad \partial_T\gamma_2 = \frac{1}{2}\left(2\gamma_2 - \gamma_3\right)\partial_X\gamma_2 \tag{2.1.11}$$

as a semi-Hamiltonian hydrodynamic system. We read off the velocities $v_2 = (2\gamma_2 - \gamma_3)/2$ and $v_3 = \gamma_3/2$ and find

$$\frac{\partial_{\gamma_3} v_2}{v_3 - v_2} = \frac{1}{2(\gamma_2 - \gamma_3)} \ , \quad \frac{\partial_{\gamma_2} v_3}{v_2 - v_3} = 0 \ . \tag{2.1.12}$$

Integrating (2.1.3) for $u_j = \gamma_j$, the corresponding metric is given by

$$\mathfrak{g} = \begin{pmatrix} e^{f_2(\gamma_2)}/(\gamma_2 - \gamma_3) & 0 \\ 0 & f_3(\gamma_3) \end{pmatrix} \tag{2.1.13}$$

with constants of integration f_2 and $f_3 \neq 0$. The resulting Gaussian curvature

$$\frac{-3f_3 + (\gamma_2 - \gamma_3)\partial_{\gamma_3} f_3}{4(\gamma_2 - \gamma_3)^2 f_3^2}$$

is non-zero. Hence, although the 1-soliton Whitham equations are a limiting case ($\gamma_1 \to \gamma_2$, see Section 4.3) of the 1-phase KdV Whitham hierarchy, the system of equations (2.1.11) cannot be Hamiltonian, unlike the KdV Whitham hierarchy. Analogously, the cylinder coordinates form an orthogonal coordinate system of the flat space $\mathbb{R}^3$, but when setting the angle coordinate and the height coordinate as equal, the resulting surface is the helicoid, which has non-zero Gaussian curvature.

2.1.2. Commuting Flows and Tsarev's Generalized Hodograph Method. An integrable feature of semi-Hamiltonian systems is that there are infinitely many flows commuting with (2.0.3). They originate from the infinitely many different solutions of (2.1.2) (see Theorem 6 in [**66**]). In more detail, two coupled semi-Hamiltonian systems

$$\partial_T u_i = v_i(\boldsymbol{u})\partial_X u_i \quad \text{and} \quad \partial_Y u_i = w_i(\boldsymbol{u})\partial_X u_i \tag{2.1.14}$$

commute, if and only if for all $i = 1, \dots, m$ we have

$$0 = \partial_Y \partial_T u_i - \partial_T \partial_Y u_i = \sum_{j=1}^{m} \left[\partial_{u_j} v_i (w_j - w_i) - \partial_{u_j} w_i (v_j - v_i)\right] \partial_X u_j \partial_X u_i \ .$$

Clearly, this equation is implied if $(v_i)_i$ and $(w_i)_i$ both satisfy (2.1.2) with the same Γ^j_{ji}, that is to say, if

$$\frac{\partial_{u_j} v_i}{v_j - v_i} = \frac{\partial_{u_j} w_i}{w_j - w_i} \tag{2.1.15}$$

holds for $i \neq j$. The solutions of (2.1.2) are parameterized by m functions in one variable (see Proposition 1 in [**66**] and [**9**]), so there are infinitely many independent commuting flows (2.1.14). They induce infinitely many solutions $\boldsymbol{u}$ for (2.1.14), justifying that (2.1.15) is also a necessary condition for the existence of commuting flows, see Section 2 in [**58**] and for a detailed variational argument see Sections 4 and 5 in [**66**].

Tsarev's remarkable result for semi-Hamiltonian diagonal systems is to establish a correspondence between solutions $\boldsymbol{u}$ of the system of PDEs

$$\partial_T u_j = v_j(\boldsymbol{u})\partial_X u_j \tag{2.1.16}$$

and solutions $\boldsymbol{u}$ of the transcendental (or algebraic) equations

$$w_i(\boldsymbol{u}) = v_i(\boldsymbol{u})T + X \tag{2.1.17}$$

for some coefficients (w_i) satisfying (2.1.15), see [**65**]. This is called the *generalized hodograph method.* Its precise formulations is given in the following.

Theorem 2.1.3 (Generalized Hodograph Method, see Theorem 10 in [**66**]). *A smooth solution of* (2.1.17) *is a solution of* (2.1.16). *Conversely, for any solution of* (2.1.16) *near a point* (X_0, T_0) *with* $\partial_X u_j(X_0, T_0) \neq 0$ *for all* $j = 1, \ldots, m$*, there is a commuting flow with coefficients* $(w_j)_j$ *such that* $\boldsymbol{u}$ *can be locally represented as a solution of* (2.1.17).

The genericity condition $\partial_X u_j \neq 0$ for all $j = 1, \ldots, m$ will be assumed throughout the following.

Remark 2.1.4. For the "classical" hodograph method Tsarev [**66**] refers to [**61**], pointing out that it allows to reduce any quasi-linear $1+1$-dimensional system of hydrodynamic type to a system of linear PDEs (2.1.15). The method is adopted from Riemann's work on shock waves, compare Equation (3) in [**60**] with Chap. 2, Sec. 9.1, Eq. (6) in [**61**]. However, the name "hodograph method" does not appear in [**60**].

In [**64**] Tian notes that Tsarev's hodograph method generalizes the method of characteristics used for solving the dispersionless KdV equation in the introductory Example 2.0.1.

Next, we apply the generalized hodograph method to Example 2.1.2.

Example 2.1.5 (Solutions for the Whitham hierarchy of the 1-soliton)**.** The velocities v_2, v_3 from Example 2.1.2 and

$$w_2 = \gamma_2^2 - \frac{1}{2}\gamma_2\gamma_3 - \frac{1}{8}\gamma_3^2 \ , \quad w_3 = \frac{3}{8}\gamma_3^2$$

satisfy (2.1.15). The generalized hodograph method then provides a solution for the 1-soliton Whitham hierarchy (2.1.11) by solving (2.1.17)

$$\gamma_2^2 - \frac{1}{2}\gamma_2\gamma_3 - \frac{1}{8}\gamma_3^2 = \frac{1}{2}(2\gamma_2 - \gamma_3)T + X \ , \quad \frac{3}{8}\gamma_3^2 = \frac{1}{2}\gamma_3 T + X \ .$$

With $\gamma_3^{\pm} = \frac{2}{3}\left(T \pm \sqrt{T^2 + 6X}\right)$, the solutions to this algebraic system are given by

$$(\gamma_2, \gamma_3) \in \{(\gamma_3^+, \gamma_3^+), (\gamma_3^-, \gamma_3^-), (\tfrac{1}{2}\gamma_3^+ + \tfrac{1}{3}T, \gamma_3^-), (\tfrac{1}{2}\gamma_3^- + \tfrac{1}{3}T, \gamma_3^+)\} \ .$$

In Example 4.3.4 we will arrive for the same equations at a solution which only differs by constants.

2.1.3. Conserved Densities and Conjugate Nets. Let us consider the conservation (or continuity) equation

$$\partial_T P(\boldsymbol{u}(X,T)) = \partial_X Q(\boldsymbol{u}(X,T)) \tag{2.1.18}$$

for a *conserved density* P and a *flux* Q. Integration over the spatial domain (and, e.g., assuming the primitve function Q to vanish at the boundary) turns this into the usual form of a conservation law $\partial_T \int P(\boldsymbol{u}(X,T))\mathrm{d}X = 0$.

Remark 2.1.6. In the setting of a Hamiltonian system of hydrodynamic type as in Remark 2.1.1, P induces a conservation law if and only if $w_{ij} := \nabla^i\nabla_j P$ induces a commuting flow for (2.1.16), see Lemma 3 in [**66**]. This amounts to a version of Noether's theorem for Hamiltonian systems of hydrodynamic type.

By using the diagonal hydrodynamic equation (2.1.16) and the chain rule, the continuity equation (2.1.18) is equivalent to

$$0 = \sum_{i=1}^{m} (v_i\partial_i P - \partial_i Q)\, \partial_X u_i$$

for all possible solutions $\boldsymbol{u}$ of the semi-Hamiltonian system. Certainly, this equation holds, if we have

$$v_i \partial_i P = \partial_i Q \tag{2.1.19}$$

for all $i = 1, \dots, m$ and considered for functions in $\boldsymbol{u} \in \mathbb{R}^m$ (or some subdomain of $\mathbb{R}^m$). On the other hand, the necessity of this condition can be justified by the infinite number of solutions for semi-Hamiltonian systems provided by commuting flows, see the similar argument in section 2.1.2. Here we content ourselves with considering the system of equations (2.1.19) in place of the continuity equation (2.1.18).

Taking into account how the semi-Hamiltonian matrix v is related to Christoffel symbols in (2.1.2), compatibility for the flux Q requires

$$\partial_i \partial_j P = \Gamma^j_{ji} \partial_j P + \Gamma^i_{ij} \partial_i P \tag{2.1.20}$$

for all $i \neq j$. This is a *Laplace equation*. The semi-Hamiltonian property (2.1.1) yields its compatibility. There are infinitely many linearly independent solutions of this equation, see Theorem 5 in [**66**] and pages 335-340 in [**9**]. Hence, each semi-Hamiltonian matrix v comes with infinitely many independent conserved densities and fluxes.

When considered for a vector P of $N \geq 1$ conserved densities, equation (2.1.20) describes a m-dimensional *conjugate net* in $\mathbb{R}^N$, i.e. $\partial_i \partial_j P$ is a linear combination of only $\partial_j P$ and $\partial_i P$. In case P defines an immersion, the compatibility condition $\partial_i(\partial_j \partial_k P) = \partial_j(\partial_i \partial_k P)$ is equivalent to the semi-Hamiltonian condition in the form that the curvature in (2.1.5) vanishes. The corresponding vector of fluxes Q, is a *Combescure transformation* of P by virtue of (2.1.19). In general, two conjugate nets $P, Q : \mathbb{R}^m \to \mathbb{R}^N$ (or defined on some subdomain of $\mathbb{R}^m$) are said to be related by a Combescure transformation, if $\partial_i P$ and $\partial_i Q$ are colinear for all $i = 1, \dots, m$, i.e. $\partial_i P = a_i \partial_i Q$ for some function $a_i : \mathbb{R}^m \to \mathbb{R}$. By the conjugate net equations

$$\partial_i \partial_j P = c_{ji} \partial_i P + c_{ij} \partial_j P \quad \text{and} \quad \partial_i \partial_j Q = \hat{c}_{ji} \partial_i Q + \hat{c}_{ij} \partial_j Q$$

with functions $c_{ij}, \hat{c}_{ij} : \mathbb{R}^m \to \mathbb{R}$ as linear coefficients, the factors a_i have to satisfy

$$\partial_i a_j = c_{ij}(a_i - a_j) \tag{2.1.21}$$

$$a_j \hat{c}_{ij} = a_i c_{ij} \ . \tag{2.1.22}$$

Conversely, let a conjugate net P with coefficients c_{ij} and, in addition, factors a_i be given such that (2.1.21) is satisfied. Then $\hat{c}_{ij}$ defined by (2.1.22) are compatible coefficients for a conjugate net Q which is related to P by a Combescure transformation, see Assertion 2 in [**66**].

In sum, choosing $c_{ij} = \Gamma^i_{ij}$ gives the conjugate net equation for conserved densities and turns (2.1.21) into (2.1.2). Hence, for a semi-Hamiltonian matrix v with conserved density P and flux Q each additional solution $w = (w_i)_i$ of (2.1.2) induces

- a commuting flow in time Y by (2.1.14) and
- a vector of fluxes Q^w related by a Combescure transformation to P.

In particular, P and Q^w satisfy the continuity equation $\partial_Y P = \partial_X Q^w$, that is P is a conserved density for all commuting flows.

A conjugate net is given uniquely by compatible coefficients $c_{ij} = \Gamma^i_{ij}$ and its values on the coordinate axes (Goursat problem, see page 3 in [**5**]). Explicitly the integration procedure for the conjugate net equation (2.1.20) is the following. Due to the semi-Hamiltonian property

$$\Gamma_{kj}^{k} = \partial_k \log H_j \tag{2.1.23}$$

can be integrated for $k \neq j$ and with initial values of H_j on the coordinate axes. The rotation coefficients for the metric factors H_j defined in (2.1.7) then turn the semi-Hamiltonian property into the Darboux system (2.1.8). This provides compatibility for integrating the systems $\partial_k X_j = \beta_{jk} X_k$ and $\partial_j P = H_j X_j$ successively. The resulting function P is a conjugate net satisfying (2.1.20). In particular, there is a solution $P : \mathbb{R}^m \to \mathbb{R}^m, \boldsymbol{u} \mapsto P(\boldsymbol{u})$ which is an immersion near the origin. Locally this is a coordinate transformation. Note, that in general, $\partial_j P$ and $\partial_i P$ will not be orthogonal in $\mathbb{R}^m$, that is the metric $\mathfrak{g}$ is not diagonal in coordinates induced by P.

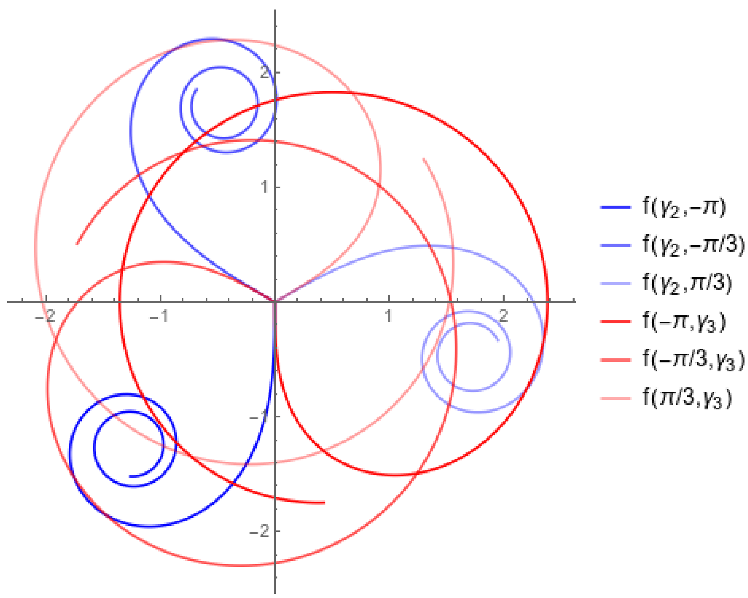

Figure 1. The conjugate net $f(\gamma_2, \gamma_3) = \int \frac{1}{\sqrt{\gamma_2 - \gamma_3}} X(\gamma_2) \mathrm{d}\gamma_2$ for $X(\gamma_2) = (\sin(\gamma_2), \cos(\gamma_2))$ drawn with Mathematica.

Example 2.1.7 (Conjugate nets for the 1-soliton Whitham equations)**.** For the semi-Hamiltonian system from Example 2.1.2 the coefficients of the related conjugate net equation are

$$c_{23} = \frac{1}{2(\gamma_2 - \gamma_3)} \quad \text{and} \quad c_{32} = 0 \ .$$

By the integration procedure just described the Whitham equations of 1-solitons yield a 2-dimensional conjugate net

$$f = \int \frac{1}{\sqrt{\gamma_2 - \gamma_3}} X(\gamma_2) \mathrm{d}\gamma_2$$

for some arbitrary function $X : \mathbb{R} \to \mathbb{R}^2$ and integration constants that are also constant in γ_3. The function f is real-valued only on $\{(\gamma_2, \gamma_3) \in \mathbb{R}^2 | \gamma_2 \geq \gamma_3\}$. We end the example with two illustrations of conjugate nets, see Figure 1 and Figure 2.

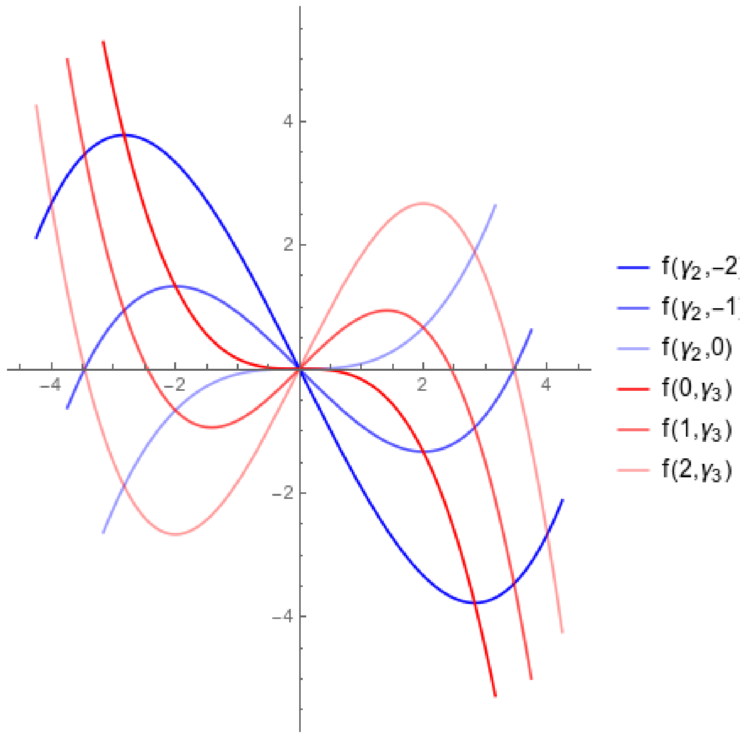

FIGURE 2. The conjugate net $f(\gamma_2,\gamma_3) = \int \frac{1}{\sqrt{\gamma_2-\gamma_3}} X(\gamma_2)\mathrm{d}\gamma_2$ for $X(\gamma_2) = (1,\gamma_2)$ drawn with Mathematica.

If the metric $\mathfrak{g}$ inducing a conjugate net equation (2.1.20) is flat, i.e. (2.1.9) holds, and functions are prescribed on the coordinate axes such that they intersect perpendicularly at the origin, then there is a conjugate net P whose coordinate lines are orthogonal to each other, that is $\partial_i P \perp \partial_j P$ for $i \neq j$. This is called an *orthogonal net*. Orthogonal nets yield flat coordinates [**41**]. Their coordinate lines are curvature lines. The metric $\mathfrak{g}$ is transformed into a constant pseudo-Euclidean metric, which is however not necessarily diagonal anymore, see [**14**] for an account on Frobenius structures.

Remark 2.1.8. As an outlook let us mention two occurrences of semi-Hamiltonian hydrodynamic systems with particular structures, characterized in terms of the metric $\mathfrak{g}$.

- For the multi-component KP hierarchy rotation coefficients appear in (and determine) the expansion of the Baker-Akhiezer function [**11**]. Generally they correspond to a conjugate net. Flatness of the corresponding metric $\mathfrak{g}$ yields BKP [**41**]. A metric which has the Egorov property gives CKP.
- The KdV Whitham hierarchy leads to a flat metric $\mathfrak{g}$ with the Egorov property, see [**16**] and Section 5.1. This allows for a Bihamiltonian structure [**17**].

Thereby, the differential geometry of conjugate nets, which was extensively studied in the late 19th and early 20th century by Darboux, Eisenhart and others, appears in the center of the more recent soliton theory.

2.2. Krichever's Universal Whitham Hierarchy

In [**40**] Krichever observes that "all the 'integrable' partial differential equations, that are considered in the framework of the 'soliton' theory, are equivalent to compatibility conditions of auxiliary linear problems." This is the starting point for his general approach to the *universal Whitham hierarchy* as "a certain 'shape' that has to be filled with a real content." In the following sections this shape will be explained with a focus on two particular cases. In Section 2.2.1 algebraic-geometric data given by the infinite dimensional moduli space of Riemann surfaces with one puncture and a chart centered there, will be discussed. The theory with multiple marked points and charts centered at them looks very similar, see [**40**]. However, the interesting case of the KP Whitham hierarchy only requires one marked point. For simplicity we only discuss this situation. In Section 2.2.3 the *algebraic orbit* will render the moduli space finite dimensional and $1+1$-dimensional hydrodynamic equations will appear. The Whitham hierarchy for stationary KdV is contained in this case.

2.2.1. Universal Whitham Equations on Moduli Spaces. As the domain of the universal Whitham hierarchy we consider the moduli space $\hat{M}_n$ of compact Riemann surfaces Γ_n of genus n with one marked point $P_0 \in \Gamma_n$ and a chart κ in its neighborhood such that $1/\kappa(P_0) = 0$, all up to conformal equivalence. The conformal equivalence classes of compact Riemann surfaces of genus $n > 1$ form a complex $(3n-3)$-dimensional moduli space. For $n = 1$ this moduli space has complex dimension 1. Choosing a marked point P_0 adds another dimension. Since any biholomorphic map κ on a neighborhood of P_0 serves as a chart, the moduli space $\hat{M}_n$ is always infinite dimensional. It is referred to as the "universal" set of algebraic-geometric data. "Forgetting" the chart κ gives a bundle projection $\hat{M}_n \to M_n$ to the finite dimensional moduli space M_n of compact Riemann surfaces Γ_n of genus n with one marked point $P_0 \in \Gamma_n$.

Dynamical systems may be used to investigate the structure of the moduli space $\hat{M}_n$ and its boundary. For a heuristics let us assume smooth local coordinates $I = [\Gamma_n, P_0, \kappa] \in \hat{M}_n$ of the infinite dimensional moduli space are given. Instead of vector fields to induce flows in times T_α, the dynamics we consider, are implicitly given by differential forms $\mathrm{d}\Omega_\alpha = \mathrm{d}\Omega_\alpha(P, I)$ on Γ_n, which are holomorphic for $P \in \Gamma_n \setminus \{P_0\}$ and normalized with respect to κ. Such differential forms will be chosen consistently on the moduli space in Section 2.2.2. Due to the possible pole at P_0 there will be infinitely many of them. Once they are given, still the compatibility of the dynamics in multiple times T_α has to be ensured. The resulting equations are known as Whitham equations and imply quasi-linear PDEs for the dynamics of the coordinate $I(\boldsymbol{T})$ of the moduli space. Here, the notation $\boldsymbol{T}$ represents a finite or infinite collection of compatible times T_α. Flaschka, Forest and McLaughlin [**25**] were the first to formulate Whitham equations using differential forms. For them the KdV Whitham equations take the form

$$\partial_{T_\alpha} \mathrm{d}\Omega_\beta\,(P, I(\boldsymbol{T})) = \partial_{T_\beta} \mathrm{d}\Omega_\alpha(P, I(\boldsymbol{T})) \tag{2.2.1}$$

with P_0 and κ determined by Γ_n. Hence, this is a system of 2-dimensional equations on a finite dimensional "slice" of $\hat{M}_n$ or, more precisely, on M_n. It will be shown in Section 5.1 how it reduces to a $1+1$-dimensional Hamiltonian hydrodynamic system for $I(\boldsymbol{T})$.

Provided that compatibility allows the dynamics to exhaust the entire moduli space $\hat{M}_n$, that is $\boldsymbol{T} \mapsto I(\boldsymbol{T})$ is a submersion, then some collection of times can be used as local coordinates. In this case it is equivalent to consider $\mathrm{d}\Omega_\alpha(P, \boldsymbol{T}) := \mathrm{d}\Omega_\alpha(P, I(\boldsymbol{T}))$ instead of $\mathrm{d}\Omega_\alpha(P, I)$. Likewise this can also be done on some "slice" of the moduli space $\hat{M}_n$. For example, when modulating a dynamical system with "fast" times $\boldsymbol{t}$, then one obtains "slow" times $\boldsymbol{T}$ as coordinates for some slice of the moduli space, rather than coordinates I of the entire moduli space, see Chapter 4.

In order to compare differential forms $\mathrm{d}\Omega_\alpha(P, I)$ on varying Riemann surfaces Γ_n we usually use the respective coordinates κ near $P_0 \in \Gamma_n$ and descend to $\mathrm{d}\Omega_\alpha(\kappa, I)$. This local version of the meromorphic differential form determines it globally. From now on we consider the differential forms $\mathrm{d}\Omega_\alpha(\kappa, \boldsymbol{T})$ given with $\boldsymbol{T}$ as coordinates of the moduli space and κ as the local variable on the Riemann surface.

In [**38**] Krichever generalized the KdV Whitham equations to the KP setting. There Theorem 1 presents the KP Whitham equations in the form

$$0 = \partial_\kappa \Omega_\alpha \left(\partial_{T_\gamma}\Omega_\beta - \partial_{T_\beta}\Omega_\gamma\right) - \partial_\kappa \Omega_\beta \left(\partial_{T_\gamma}\Omega_\alpha - \partial_{T_\alpha}\Omega_\gamma\right) + \partial_\kappa \Omega_\gamma \left(\partial_{T_\beta}\Omega_\alpha - \partial_{T_\alpha}\Omega_\beta\right) \quad . \tag{2.2.2}$$

Here $\Omega_\alpha = \Omega_\alpha(\kappa, \boldsymbol{T})$ denotes a primitive function of $\mathrm{d}\Omega_\alpha$. Locally, in the chart κ centered at P_0, each differential form $\mathrm{d}\Omega_\alpha$ that might have a pole at P_0 can be integrated on the universal covering of the punctured neighborhood. We assume all $\mathrm{d}\Omega_\alpha$ as residue-free such that their Abelian integrals are defined locally on Γ_n. The dependence on the chart κ is a choice corresponding to a section in the bundle $\hat{M}_n \to M_n$. Therefore each of the equations (2.2.2) is a 3-dimensional PDE set on the finite dimensional moduli space M_n with κ as a formal variable. That is to say κ is not coupled to the moduli space M_n. Coordinates of M_n can be determined by finitely many of these PDEs. In recent work Odesskii and Sokolov [**58, 57**] started to understand them as a hydrodynamically integrable system of $2+1$-dimensional quasi-linear PDEs for coordinates of the moduli space. The integrability of the system here means that it admits "sufficiently many" hydrodynamic reductions to $1+1$-dimensional hydrodynamic systems.

For the KP Whitham equations at hand, choosing the chart κ to depend on $\boldsymbol{T}$ in a certain way, allows to split each 3-dimensional equation into coupled 2-dimensional equations – with the trade-off that the dynamics take place on the infinite dimensional moduli space $\hat{M}_n$ instead of M_n and that there are infinitely many 2-dimensional equations to be considered. This is done in the framework of the Whitham hierarchy as explained in [**40, 21**] and in the following. Later on hydrodynamic reductions will occur in the context of algebraic orbits, see Section 2.2.4. First let us consider the chart κ as an independent variable and define the 1-form

$$\omega = \sum_\alpha \Omega_\alpha(\kappa, \boldsymbol{T}) \mathrm{d}T_\alpha \tag{2.2.3}$$

on the space of variables $(\kappa, \boldsymbol{T})$. The exterior derivative (denoted by δ) of $\omega(\kappa, \boldsymbol{T})$ is

$$\delta\omega = \sum_\alpha \partial_\kappa \Omega_\alpha \mathrm{d}\kappa \wedge \mathrm{d}T_\alpha + \sum_{\alpha,\beta} \partial_{T_\beta}\Omega_\alpha \mathrm{d}T_\beta \wedge \mathrm{d}T_\alpha \quad .$$

Here the notation $\mathrm{d}T_\alpha$ refers to differential forms on the domain of coordinates $\boldsymbol{T}$ of the moduli space.

Definition 2.2.1 ([**40**])**.** The family of Abelian integrals $(\Omega_\alpha)_\alpha$ forms a *Whitham hierarchy*, if

$$\delta\omega \wedge \delta\omega = 0 \quad . \tag{2.2.4}$$

In this general "shape," the Whitham hierarchy can be referred to as *universal*. As soon as it is "filled with real content" by concrete functions Ω_α the term "universal" is replaced by the context in which the Ω_α emerged, e.g. the KP Whitham hierarchy, see Section 2.2.2 below, arises by perturbation and averaging of the KP equation [**38**].

There are several equivalent versions of the Whitham hierarchy as we will see in the following. First note that the 2-form $\delta\omega$ can be written as the exterior product of two 1-forms exactly when (2.2.4) is satisfied. In tangential directions ∂_κ and $(\partial_{T_\alpha})_\alpha$ the Whitham hierarchy (2.2.4) is equivalent to

$$0 = \sum_{\sigma\in\mathrm{Sym}(\{\alpha,\beta,\gamma\})} \mathrm{sign}(\sigma)\partial_{T_{\sigma(\alpha)}}\Omega_{\sigma(\beta)}\partial_\kappa\Omega_{\sigma(\gamma)} \tag{2.2.5}$$

$$0 = \sum_{\sigma\in\mathrm{Sym}(\{\alpha,\beta,\gamma,\epsilon\})} \mathrm{sign}(\sigma)\partial_{T_{\sigma(\alpha)}}\Omega_{\sigma(\beta)}\partial_{T_{\sigma(\gamma)}}\Omega_{\sigma(\epsilon)} \ . \tag{2.2.6}$$

Writing the sum over the permutations out, the first equation is exactly (2.2.2), which is the form of the KP Whitham equations in [**38**]. So far the chart κ has been an independent variable. Let us now consider $\boldsymbol{T}$-dependent reparametrizations of this chart

$$\kappa = \kappa(p, \boldsymbol{T}) \tag{2.2.7}$$

where $\partial_p\kappa \neq 0$. This reparametrization changes how Ω_α depends on $\boldsymbol{T}$ in its second slot, that is $\Omega_\alpha(p,\boldsymbol{T}) := \Omega_\alpha(\kappa(p,\boldsymbol{T}),\boldsymbol{T})$. Hence, the derivative with respect to T_α changes in the following way

$$\partial_{T_\alpha}\Omega_\beta(p,\boldsymbol{T}) = \partial_\kappa\Omega_\beta(\kappa,\boldsymbol{T})\partial_{T_\alpha}\kappa + \partial_{T_\alpha}\Omega_\beta(\kappa,\boldsymbol{T}) \ . \tag{2.2.8}$$

An important aspect of the Whitham hierarchy (2.2.4) is its invariance under reparametrizations (or gauges) (2.2.7). The individual terms in the sums (2.2.5) and (2.2.6) depend on the parametrization, the sums do not. Here are two representations of the Whitham hierarchy that correspond to particular parametrizations (or gauge fixings).

The zero curvature form. Fixing an index α_0 induces a reparametrization by

$$p(\kappa,\boldsymbol{T}) = \Omega_{\alpha_0}(\kappa,\boldsymbol{T})$$

where $\partial_\kappa p \neq 0$. At the marked point P_0 that means p needs to have a simple pole there. As an Abelian integral p is defined on the universal covering of $\Gamma_n \setminus \{P_0\}$. The corresponding time T_{α_0} is usually renamed as X. Applying (2.2.8) for $\beta = \alpha_0$ gives $0 = \partial_\kappa p\partial_{T_\alpha}\kappa + \partial_{T_\alpha}p$. For functions $\Omega^p_\beta = \Omega_\beta(p,\boldsymbol{T})$ the first equation of the Whitham hierarchy (2.2.5) with indices α_0, α and β becomes a zero curvature equation

$$0 = \partial_{T_\alpha}\Omega^p_\beta - \partial_{T_\beta}\Omega^p_\alpha + \left\{\Omega^p_\alpha, \Omega^p_\beta\right\} \tag{2.2.9}$$

where the Poisson bracket is introduced by $\{f,g\} = \partial_X f\partial_p g - \partial_p f\partial_X g$. Conversely, if (2.2.9) is satisfied for all indices α and β, then this implies (2.2.5), so both equations are equivalent. By using the Jacobi identity, equation (2.2.9) can be seen as the compatibility equation for the system

$$\partial_{T_\alpha}E = \{E, \Omega^p_\alpha\} \ . \tag{2.2.10}$$

In [**58**] this system is called *pseudopotential representation* for the zero curvature representation [1]. Note, if Ω^p_α is *stationary* under some flow T_β, that is to say $\partial_{T_\beta}\Omega^p_\alpha = 0$, then the zero curvature equation (2.2.9) becomes (2.2.10) with $E = \Omega^p_\beta$. The KdV reduction of the KP Whitham hierarchy arises in this way, see Example 2.2.4.

[1]For $\beta = \alpha_0$ equation (2.2.9) becomes $0 = \partial_{T_\alpha}p - \partial_p\Omega_\alpha\partial_X p$, which is also a pseudopotential representation.

The conservation form. Let us consider a function $E = E(p, \boldsymbol{T})$ satisfying system (2.2.10) locally. If $\partial_p E \neq 0$, then this induces another reparametrization $(p, \boldsymbol{T}) \mapsto (E, \boldsymbol{T})$. For $\Omega_\alpha^E = \Omega_\alpha^p(E, \boldsymbol{T}) = \Omega_\alpha(E(p, \boldsymbol{T}), \boldsymbol{T})$ equation (2.2.10) takes the form

$$\partial_{T_\alpha} p^E = \partial_X \Omega_\alpha^E \tag{2.2.11}$$

and equations (2.2.9) from the Whitham hierarchy become its compatibility equations

$$\partial_{T_\alpha} \Omega_\beta^E = \partial_{T_\beta} \Omega_\alpha^E \ . \tag{2.2.12}$$

Moreover, the second set of equations (2.2.6) in the Whitham hierarchy is trivial when $(E, \boldsymbol{T})$ are used as coordinates. The Whitham equations (2.2.11) and (2.2.12) have the same form as the hydrodynamic conservation equation (2.1.18). Therefore they are said to be in *conservation form*. A difference is however, that (2.1.18) depends only on finitely many state variables $(u_1(\boldsymbol{T}), \ldots, u_m(\boldsymbol{T}))$, but solutions E of (2.2.10) come as infinite series in p with functions of $\boldsymbol{T}$ as coefficients. In other words, the moduli space $\hat{M}_n$ on which the Whitham hierarchy in conservation form is set, is infinite dimensional. In summary the Whitham hierarchy, which formed initially a system of 3-dimensional PDE on M_n with κ as a parameter can be reparametrized such that it becomes an infinite dimensional system of 2-dimensional PDE (2.2.11) and (2.2.12) on $\hat{M}_n$ with E as a parameter.

The Whitham hierarchy in $(E, \boldsymbol{T})$-coordinates provides the compatibility conditions for a *generating function* [**40**] (or *prepotential* [**21**]) $S = S(E, \boldsymbol{T})$ given by

$$\partial_{T_\alpha} S = \Omega_\alpha^E \ . \tag{2.2.13}$$

In other words, the 1-form ω in (2.2.3) is closed for coordinates $(E, \boldsymbol{T})$. Assuming that functions Ω_α^E are provided, the equations for the prepotential form a generally infinite system of ODEs (i.e. 1-dimensional equations) with E as a parameter. The question, how prepotentials can be used to get solutions of the Whitham hierarchy leads to *algebraic orbits*, which describe finite dimensional leafs in the moduli space $\hat{M}_n$ [**40**], or leads to *horizontal* families of Abelian differential forms [**17**], see Section 2.2.3. In both settings the Whitham hierarchy in 2-dimensional conservation form (2.2.11) will take the form (2.2.1) of the KdV Whitham equations given in [**25**]. A *hydrodynamic reduction* will allow then to understand each equation in (2.2.11) as a $1+1$-dimensional semi-Hamiltonian hydrodynamic system [**58**]. By (2.2.12) the other equations in (2.2.11) will correspond to commuting hydrodynamic flows (2.1.14) and, what is the same, to additional hydrodynamic conservation laws, see Section 2.1.3. This method for solving the Whitham hierarchy is also known as Krichever's version of Tsarev's generalized hodograph method, see Theorem 5 in [**17**].

Remark 2.2.2. Replacing the commutator in the isospectral theory of the KP equation (see Introduction in [**30**]) by a Poisson bracket produces the zero-curvature form and the prepotential form of the Whitham hierarchy for genus zero [**39**]. This is referred to as the *dispersionless limit* of the Lax hierarchy. In this limit the *Lax equation* becomes (2.2.10), the *Zakharov-Shabat equation* gives (2.2.9) and the *Sato-Wilson equation* gives (2.2.13).

2.2.2. The Whitham Hierarchy for KP. This section is meant to describe the KP and KdV Whitham hierarchy and thus providing "real content" for the "shape" of the Whitham hierarchy in Definition 2.2.1. In order to consistently provide differential forms on different Riemann surfaces, i.e. for different elements of the moduli space $\hat{M}_n$, let a basis of the first homology group $H_1(\Gamma_n, \mathbb{Z})$ be fixed on each Riemann surface Γ_n by

$$B = \{a_1, \ldots, a_n, b_1, \ldots, b_n | a_i, b_j \in H_1(\Gamma_n, \mathbb{Z}), a_i \cdot b_j = \delta_{ij}, a_i \cdot a_j = 0 = b_i \cdot b_j\} \ . \tag{2.2.14}$$

This adds an extra set of data to the moduli space $\hat{M}_n$ making it a manifold $\hat{M}_n^*$. For $[\Gamma_n, P_0, \kappa, B] \in \hat{M}_n^*$ Riemann's theorem (see Chapter 11, Theorem 3 in [**33**]) provides meromorphic differential forms $\mathrm{d}\Omega = \mathrm{d}\Omega(\kappa, P_0, \Gamma_n)$ on Γ_n characterized uniquely by

- $\mathrm{d}\Omega^h_\alpha$ for $\alpha = 1, \ldots, n$ is holomorphic on Γ_n and normalized by $\int_{a_j} \mathrm{d}\Omega^h_\alpha = \delta_{j\alpha}$, hence forming a basis of the holomorphic differential forms on Γ_n, and
- $\mathrm{d}\Omega_\alpha$ for $\alpha \geq 1$ is holomorphic on $\Gamma_n \setminus \{P_0\}$ with vanishing a-periods $\int_{a_j} \mathrm{d}\Omega_\alpha = 0$ and a pole at P_0 of the form

$$\mathrm{d}\Omega_\alpha = \left(\kappa^{\alpha-1} + \mathcal{O}(\kappa^{-2})\right) \mathrm{d}\kappa\ .^2 \tag{2.2.15}$$

  Note that $\mathrm{d}\kappa$ has a pole at P_0, since $1/\kappa(P_0) = 0$. The index $\alpha = 0$ is omitted, since the Abelian integral of (2.2.15) would not be single-valued near P_0.

The Whitham equations (2.2.5) and (2.2.6) consider κ as an independent variable and induce dynamics on the finite dimensional moduli space M_n of Riemann surfaces Γ_n with one marked point P_0. That means, P_0 and Γ_n become dependent on the times $\boldsymbol{T}$ and, by using some local parametrization of M_n, we may write

$$\mathrm{d}\Omega_\alpha(\kappa, \boldsymbol{T}) = \mathrm{d}\Omega_\alpha\left(\kappa, P_0(\boldsymbol{T}), \Gamma_n(\boldsymbol{T})\right)$$

and the same for $\mathrm{d}\Omega^h_\alpha$. The differential forms assumed for the Whitham hierarchy in Definition 2.2.1 are of this form. Note however, that Riemann's theorem as used above only provides that unique differential forms with the desired normalization do exist on Γ_n. Whether the differential forms depend smoothly on the underlying data $[\Gamma_n, P_0, \kappa, B] \in \hat{M}_n^*$ is an additional question which is usually omitted. In the KdV case a direct reasoning for the smooth dependence will be presented in Remark 5.0.2.

Since the differential forms $\mathrm{d}\Omega_\alpha$ correspond to isospectral flows of the KP hierarchy, the resulting KP Whitham hierarchy is called *basic*, see Chapter 7.1 in [**40**]. There are further extensions by particular holomorphic differential forms defined on the universal covering of $\Gamma_n \setminus \{P_0\}$, e.g. $\mathrm{d}\Omega^h_\alpha$ provide extensions. How to obtain such differential forms more generally will be explained in Section 5.3 in the setting of the KdV Whitham hierarchy.

The basic KP Whitham hierarchy may be represented in zero curvature form and prepotential form. Due to the normalization of the differential forms $\mathrm{d}\Omega_\alpha$ at the marked point P_0 the representation (2.2.12) is attained in coordinates $(\kappa, \boldsymbol{T})$ (instead of $(E, \boldsymbol{T})$) [3].

Lemma 2.2.3. *The basic KP Whitham hierarchy is equivalent to* $\partial_{T_\alpha}\Omega_\beta = \partial_{T_\beta}\Omega_\alpha$ *for all* $\alpha, \beta \geq 1$.

Proof. If the formula in the lemma holds, then clearly the equations of the Whitham hierarchy (2.2.5) (or equivalently (2.2.2)) and (2.2.6) also hold. Conversely, setting the index $\gamma = 1$ in (2.2.2), i.e. $\Omega_\gamma = p$ and $T_\gamma = X$, gives in coordinates $(\kappa, \boldsymbol{T})$

$$0 = \partial_\kappa\Omega_\alpha\left(\partial_X\Omega_\beta - \partial_{T_\beta}p\right) - \partial_\kappa\Omega_\beta\left(\partial_X\Omega_\alpha - \partial_{T_\alpha}p\right) + \partial_\kappa p\left(\partial_{T_\beta}\Omega_\alpha - \partial_{T_\alpha}\Omega_\beta\right)\ . \tag{2.2.16}$$

Asymptotically near P_0 the Abelian integral Ω_α is given by $\kappa^\alpha/\alpha + \mathcal{O}(\kappa^{-1})$ due to the normalization. Hence, we have $\partial_\kappa\Omega_\alpha = \kappa^{\alpha-1} + \mathcal{O}(\kappa^{-2})$ and $\partial_{T_\beta}\Omega_\alpha = \mathcal{O}(\kappa^{-1})$ near P_0.

[2] Alternatively, the real normalization $\Im(\int_c \mathrm{d}\Omega_\alpha) = 0$ for all $c \in H_1(\Gamma_n, \mathbb{Z})$ could be used here. But meromorphic differential forms normalized by their a-periods as done above, appear when modulating the KdV hierarchy (see [**25**] and Chapter 5 below) and KP hierarchy (see [**38**]). In [**40**] and related publications the principal parts are normalized by $\mathrm{d}\Omega_\alpha = \mathrm{d}\left(\kappa^\alpha + \mathcal{O}(\kappa^{-1})\right)$ instead.

[3] This is analogous to the zero genus case of the Whitham hierarchy described in Section 2 of [**40**].

Analogous formulas hold for $p = \Omega_1$ and Ω_β. In (2.2.16) this yields near P_0

$$0 = \kappa^{\alpha-1}\left(\partial_X\Omega_\beta - \partial_{T_\beta}p\right) - \kappa^{\beta-1}\left(\partial_X\Omega_\alpha - \partial_{T_\alpha}p\right) + \kappa^0\left(\partial_{T_\beta}\Omega_\alpha - \partial_{T_\alpha}\Omega_\beta\right) + \mathcal{O}(\kappa^{-3})$$

since the terms in brackets are in $\mathcal{O}(\kappa^{-1})$. For fixed α the order of the first and third term in the sum is bound from above independently of the value of β. The limit $\beta \to \infty$ then shows that all negative orders of $\partial_X\Omega_\alpha - \partial_{T_\alpha}p$ have to vanish, which implies that the expression has to be zero already. Using this in (2.2.16) shows the formula in the lemma. □

The argument in the proof is similar to that for showing the equivalence of the Zakharov-Shabat equation and the Lax equation in KP theory (see, e.g. Lemma 4.3 in [**30**]). In Theorem 2.1 and Theorem 7.7 in [**40**] Krichever derives the zero curvature form of the KP Whitham hierarchy in this way. The reparametrization $(\kappa, \boldsymbol{T}) \to (p, \boldsymbol{T})$ applied to

$$\partial_{T_\alpha}p = \partial_X\Omega_\alpha \tag{2.2.17}$$

works as follows.

- Where p is a coordinate, i.e. $\mathrm{d}p = \partial_\kappa p \mathrm{d}\kappa \neq 0$, we may write $\kappa = \kappa(p, \boldsymbol{T})$ with p considered as a parameter and $\Omega_\alpha(\kappa, \boldsymbol{T}) = \Omega_\alpha^p(p(\kappa, \boldsymbol{T}), \boldsymbol{T})$ as in Section 2.2.1. The coordinate change implies $0 = \partial_{T_\alpha}p(\kappa(p, \boldsymbol{T}), \boldsymbol{T}) = \partial_\kappa p(\kappa, \boldsymbol{T})\partial_{T_\alpha}\kappa + \partial_{T_\alpha}p(\kappa, \boldsymbol{T})$ and $\partial_X\Omega_\alpha(\kappa, \boldsymbol{T}) = \partial_p\Omega_\alpha^p\partial_X p + \partial_X\Omega_\alpha^p$. Hence, equation (2.2.17) gives
$$\partial_{T_\alpha}\kappa = -\partial_p\kappa\partial_X\Omega_\alpha = \partial_X\kappa\partial_p\Omega_\alpha^p - \partial_p\kappa\partial_X\Omega_\alpha^p = \{\kappa, \Omega_\alpha^p\} \quad , \tag{2.2.18}$$
which is already the pseudopotential equation (2.2.10).
- At points κ_i where $p = p(\kappa, \boldsymbol{T})$ is not a coordinate, i.e. $\mathrm{d}p|_{\kappa_i} = \partial_\kappa p|_{\kappa_i}\mathrm{d}\kappa = 0$ there is an expansion $p(\kappa, \boldsymbol{T}) = p_i + (\kappa - \kappa_i)^{l_i}\left(u_i + \mathcal{O}(\kappa - \kappa_i)\right)$ with $p_i := p(\kappa_i, \boldsymbol{T})$, $l_i \geq 2$ and $u_i = u_i(\boldsymbol{T}) \neq 0$. Using equation (2.2.17) gives
$$\partial_{T_\alpha}p_i = \partial_X\Omega_\alpha|_{\kappa_i} \quad . \tag{2.2.19}$$
- At last, equation (2.2.17) implies for all b-periods along b-cycles b_i in the basis of the first homology group (2.2.14)
$$\partial_{T_\alpha}\int_{b_i}\mathrm{d}p = \partial_X\int_{b_i}\mathrm{d}\Omega_\alpha \quad . \tag{2.2.20}$$
Since a-periods of the differential forms $\mathrm{d}p$ and $\mathrm{d}\Omega_\alpha$ vanish, stating (2.2.20) for b-cycles is equivalent to stating it for all elements of the first homology group.

Conversely, the equations (2.2.18), (2.2.19) and (2.2.20) also imply equation (2.2.17). In sum, this is the content of Theorem 7.7 in [**40**] in the case with one marked point P_0.

Let us point out again that (2.2.19) and (2.2.20) have the same form as a hydrodynamic conservation law in (2.1.18). However, the latter depends only on finitely many state variables $(u_1(\boldsymbol{T}), \ldots, u_m(\boldsymbol{T}))$, whereas solutions κ of (2.2.18) are set on the infinite dimensional moduli space $\hat{M}_n^*$. Choosing one chart κ suitably to reduce the moduli space to finitely many dimensions, will allow to reduce (2.2.17) to a diagonal semi-Hamiltonian hydrodynamic system in the sense of Section 2.1. How to get to such a reduction will be discussed in the following reduction and more generally in Section 2.2.3.

Example 2.2.4 (KdV reduction I)**.** The KdV equation and hierarchy are obtained from the KP equation and hierarchy by requiring the second spatial flow in $y = t_2$ (and thereby all even flows) to be trivial. Likewise, assuming the corresponding modulation in T_2 to be trivial, gives the KdV reduction of the basic KP Whitham hierarchy. In the representation of Lemma 2.2.3 this reduction means

$$\partial_{T_\alpha}\Omega_2 = \partial_{T_2}\Omega_\alpha = 0$$

for all $\alpha \geq 1$. That is to say, the Abelian integral Ω_2 is constant in all modulation times. In particular all b-periods of the differential form $\mathrm{d}\Omega_2$ are constant. So if they vanish at one point in time, they vanish at all times. By construction the a-periods of $\mathrm{d}\Omega_2$ are already zero. For a meromorphic differential form without residues, vanishing a-periods and b-periods imply that its Abelian integral defines a meromorphic function on the Riemann surface. If at some point in time this holds for Ω_2, then Ω_2 is a $2:1$-covering $\Gamma_n \to \mathbb{C}\mathrm{P}^1$ for all times. This means Γ_n has to be a hyperelliptic curve at all times. Altogether, in the KdV reduction Γ_n stays a hyperelliptic curve under Whitham flows, if it has been one initially.

On varying hyperelliptic curves Γ_n with $E := \Omega_2 : \Gamma_n \to \mathbb{C}\mathrm{P}^1$ as time-independent covering maps, each marked point P_0 corresponds to $E = \infty$ and the chart κ is given there by

$$\kappa^2 = 2E \ , \tag{2.2.21}$$

since $\mathrm{d}\Omega_2 = \mathrm{d}E = \kappa \mathrm{d}\kappa$ has the required normalization and is unique with that property. In particular, P_0 is a branch point of E with value infinity. Away form branch points of E, there is a reparametrization $(\kappa, \boldsymbol{T}) \mapsto (E, \boldsymbol{T})$ for the Whitham hierarchy in Lemma 2.2.3, giving

$$\partial_{T_\alpha}\mathrm{d}\Omega_\beta(E,\boldsymbol{T}) = \partial_{T_\beta}\mathrm{d}\Omega_\alpha(E,\boldsymbol{T}) \tag{2.2.22}$$

for all $\alpha, \beta \geq 1$. In [**25**] both parametrizations of the KdV Whitham hierarchy are used [4].

By their normalization all the even differential forms are determined as $\mathrm{d}\Omega_{2\alpha} = E^{\alpha-1}\mathrm{d}E$ for $\alpha \geq 1$ and thus time-independent. As a consequence, the flows in the corresponding times $T_{2\alpha}$ are trivial. This is the same for isospectral flows of KdV. Altogether, the KdV reduction of the KP Whitham hierarchy is constant in all times $T_{2\alpha}$ and set on the finite dimensional moduli space $M_n^{hyp} \subseteq M_n$ of hyperelliptic curves of genus n with marked point $E = \infty$ and fixed chart (2.2.21). A model for hyperelliptic curves of genus n is given by

$$\Gamma_n = \left\{(E,y) \in \mathbb{C}^2 \mid y^2 + g(E) = 0\right\} \cup \{\infty\} \tag{2.2.23}$$

with g a complex monic polynomial of degree $2n+1$ with distinct roots. By an affine transformation two of the roots of g can be mapped to 0 and 1, leaving the remaining roots to parametrize the moduli space of compact hyperelliptic Riemann surfaces of genus n. The point at infinity is a ramification point of the hyperelliptic covering $\Gamma_n \ni (E,y) \mapsto E \in \mathbb{C}\mathrm{P}^1$ and (2.2.21) gives a chart there. In the bundle $\hat{M}_n^{hyp} \to M_n^{hyp}$ this corresponds to a constant section

$$M_n^{hyp} \ni [\Gamma_n, \infty] \mapsto \left[\Gamma_n, \infty, \sqrt{2E}\right] \in \hat{M}_n^{hyp} \ .$$

For a detailed discussion of the KdV reduction of the KP Whitham hierarchy see Chapter 5. More generally, the case when the chart κ of the KP Whitham hierarchy is some root of an Abelian integral leads to algebraic orbits as discussed in the next section.

[4] More precisely, $\xi = 1/\kappa$ instead of κ is used as a variable in [**25**].

2.2.3. The KP Whitham Hierarchy on the Algebraic Orbit. The KP Whitham hierarchy in conservation form is an infinite system of PDEs set on the infinite dimensional moduli space $\hat{M}_n^*$, see Lemma 2.2.3. Trivially, it admits solutions $\boldsymbol{T} \mapsto I(\boldsymbol{T}) \in \hat{M}_n^*$ which are constant maps. A richer class of solutions comes from *algebraic orbits* [**40**]. As it will be defined below, an algebraic orbit is a finite dimensional subspace of the moduli space that is additionally invariant under the dynamics of the KP Whitham hierarchy. In Example 2.2.4 we have already seen that the KdV reduction of the KP Whitham hierarchy is set on the finite dimensional invariant subspace of hyperelliptic Riemann surfaces and thus forms an algebraic orbit.

Before developing the general framework of algebraic orbits, let us see in the following example, how the KdV reduction allows to make a connection between the KP Whitham hierarchy and the semi-Hamiltonian systems of hydrodynamic type from Section 2.1. As a consequence, solutions of the KP Whitham hierarchy which are constant in all even times $T_{2\alpha}$, are accessible by Tsarev's hodograph method.

Example 2.2.5 (KdV reduction II)**.** In Example 2.2.4 the chart κ at the marked point can be extended holomorphically to the universal covering of the hyperelliptic curve Γ_n by the algebraic relation $\kappa^2 = 2E$ with the Abelian integral E. Additionally, the Whitham hierarchy takes prepotential form (2.2.12) in coordinates $(\kappa, \boldsymbol{T})$ and due to the algebraic relation also in coordinates $(E, \boldsymbol{T})$, see (2.2.22). The Abelian integral E provides a $2:1$-covering $E : \Gamma_n \to \mathbb{C}\mathrm{P}^1$ with $2n+2$ branch points $P_0, P_1, \ldots, P_{2n+1}$ for which $\mathrm{d}E|_{P_j} = 0$ and with branch values $\infty, \gamma_1, \ldots, \gamma_{2n+1}$, respectively. The ramification index at all P_i is 2, so locally in a chart z_i of Γ_n near P_i we have

$$\begin{aligned} E - \gamma_i &= z_i^2 \ , \\ \mathrm{d}E &= 2 z_i \mathrm{d}z_i \ . \end{aligned}$$

Note that while E is time-independent, the charts z_i depend on time, since the branch values $\gamma_i = \gamma_i(\boldsymbol{T})$ do. In the chart z_i the differential forms of the KdV Whitham hierarchy take the form $\mathrm{d}\Omega_\alpha = C^i_\alpha(z_i, \boldsymbol{\gamma})\mathrm{d}z_i$ for some functions C^i_α that are holomorphic in the first argument and smooth in $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_{2n+1})$. Expressed in E this gives

$$\mathrm{d}\Omega_\alpha(E, \boldsymbol{\gamma}) = \frac{C^i_\alpha((E-\gamma_i)^{1/2}, \boldsymbol{\gamma})}{2(E-\gamma_i)^{1/2}} \mathrm{d}E \ .$$

Hence, near a branch value γ_i the KdV Whitham hierarchy (2.2.22) implies

$$\left[\frac{C^i_\beta(0, \boldsymbol{\gamma})}{4(E-\gamma_i)^{3/2}} \partial_{T_\alpha}\gamma_i - \frac{C^i_\alpha(0, \boldsymbol{\gamma})}{4(E-\gamma_i)^{3/2}} \partial_{T_\beta}\gamma_i \right] \mathrm{d}E = \mathcal{O}\left((E-\gamma_i)^{-1/2} \right) \mathrm{d}E$$

for all odd $\alpha, \beta \geq 1$. The equations for even indices are trivial. At branch values γ_i this means

$$\mathrm{d}\Omega_\beta|_{E=\gamma_i} \partial_{T_\alpha}\gamma_i = \mathrm{d}\Omega_\alpha|_{E=\gamma_i} \partial_{T_\beta}\gamma_i \ . \tag{2.2.24}$$

Assuming that generically $\mathrm{d}\Omega_\beta$ is not zero at branch values of E, we get that $v^{\alpha\beta} := \mathrm{d}\Omega_\alpha / \mathrm{d}\Omega_\beta$ can be evaluated at all these branch values γ_i. Therefore for each pair α and β

$$\partial_{T_\alpha}\gamma_i = v^{\alpha\beta}(\gamma_i) \partial_{T_\beta}\gamma_i \tag{2.2.25}$$

is a diagonal hydrodynamic system of the form (2.0.3). It is worth to note that the velocities $v^{\alpha\beta}(\gamma_i)$ all come by evaluation of the function $v^{\alpha\beta}$ at different branch values $\gamma_1, \ldots, \gamma_{2n+1}$. From Lemma 2.2.8 below it will follow that the hydrodynamic systems determine the KdV

Whitham hierarchy entirely. Furthermore, Corollary 2.2.10 will show that $v^{\alpha 1}$ induces a semi-Hamiltonian system and $v^{\beta 1}$ for $\alpha \neq \beta$ induces a commuting system. In Section 5.1 in the more concrete setup where the hyperelliptic curve is given by (2.2.23), the semi-Hamitonian system will appear as even a Hamiltonian system of hydrodynamic type.

More generally, algebraic orbits in the moduli space $\hat{M}_n^*$ of the universal KP hierarchy are described as follows (see Section 7.2 in [**40**] as well). For a point $[\Gamma_n, P_0, \kappa, B]$ in the moduli space let $\mathrm{d}E$ be a holomorphic differential form on $\Gamma_n \setminus \{P_0\}$, with vanishing a-periods and a pole of order n_0+1 at P_0 without residue. We assume $\mathrm{d}E$ to be a section in the bundle of meromorphic differential forms over the moduli space. This means that $\mathrm{d}E$ is a linear combination of $\mathrm{d}\Omega_1, \dots, \mathrm{d}\Omega_{n_0}$ with n_0 coefficients that may depend smoothly on times $\boldsymbol{T}$. The Abelian integral E of $\mathrm{d}E$ includes a constant of integration and is defined on the universal covering of Γ_n and has a pole of order n_0 at P_0.

Definition 2.2.6. The submanifold $\mathcal{N}_n(n_0) := \{[\Gamma_n, P_0, \kappa, B] \mid \kappa^{n_0} = E\} \subseteq \hat{M}_n^*$ is called *algebraic orbit*.

In other words, for a moduli space that is given with a fixed Abelian integral E (and without charts κ), the relation $\kappa^{n_0} = E$ induces an embedding into $\hat{M}_n^*$, thus turning the initial moduli space into an algebraic orbit. The complex dimension of $\mathcal{N}_n(n_0)$ is $3n-1+n_0$ if $n > 1$ and $3+n_0$ if $n = 1$, since E has n_0+1 parameters that depend on times $\boldsymbol{T}$. There are parametrizations of an algebraic orbit that correspond to the parameters in the zero curvature form and the conservation form of the Whitham hierarchy, see Section 2.2.1.

Parametrization by the Abelian integral p. At P_0 the Abelian integral Ω_1 has a simple pole, so $p = \Omega_1$ defines a chart there. Hence, we may write

$$E = p^{n_0} + u_{n_0-2}p^{n_0-2} + \dots + u_1 p + u_0 + \mathcal{O}(p^{-1})$$

for some coefficients u_i. It is assumed here that by a Möbius transformation $p(P_0)$ is normalized to ∞ and the leading coefficients u_{n_0} and u_{n_0-1} are 1 and 0, respectively. By the Riemann-Roch theorem $\mathrm{d}p$ has $2n$ zeros which give generically $2n$ critical values of p. Together with $u_0, \dots, u_{n_0-2}$, and the n b-periods of $\mathrm{d}p$ those critical values locally parametrize $\mathcal{N}_n(n_0)$. All this assumes that generically the chosen parameters are independent.

From equation (2.2.18) and $\kappa^{n_0} = E$ follows immediately the pseudopotential representation (2.2.10)

$$\partial_{T_\alpha} E = \{E, \Omega_\alpha^p\} \quad .$$

Conversely, this equation ensures that the algebraic orbit $\mathcal{N}_n(n_0)$ is invariant under flows of the KP Whitham hierarchy. It describes the KP Whitham hierarchy entirely, see Theorem 7.9 in [**40**]. Using $(E, \boldsymbol{T})$ as coordinates (that is E is considered as a parameter independent of $\boldsymbol{T}$) allows to represent the basic Whitham hierarchy from Lemma 2.2.3 by

$$\partial_{T_\alpha} \mathrm{d}\Omega_\beta(E, \boldsymbol{T}) = \partial_{T_\beta} \mathrm{d}\Omega_\alpha(E, \boldsymbol{T}) \quad .$$

Hence, like in Example 2.2.5, there is a hydrodynamic reduction for the KP Whitham hierarchy in terms of the branch values of E with additional equations for the b-periods of $\mathrm{d}E$.

Parametrization by the Abelian integral E. Let us parametrize $\mathcal{N}_n(n_0)$ directly by characterizing the differential form $\mathrm{d}E$ uniquely by its zeros and its b-periods. The parameters are the b-periods

$$U_{E,j} := U_{b_j}^E = \int_{b_j} \mathrm{d}E$$

for $j = 1, \ldots, n$ and the branch values E_j of E, with $j = 1, \ldots, 2n-1+n_0$ since again by Riemann-Roch $\mathrm{d}E$ has generically $2n-1+n_0$ zeros. The chosen parameters are generically independent. Then consider the normalized differential forms of the KP Whitham hierarchy on the algebraic orbit $\mathcal{N}_n(n_0)$ locally in these coordinates, that is

$$\mathrm{d}\Omega_\alpha = \mathrm{d}\Omega_\alpha(E; \boldsymbol{E}, U_E)$$

for $\boldsymbol{E} = (E_1, \ldots, E_{2n-1+n_0})$ and $U_E = (U_{E,1}, \ldots, U_{E,n})$. As a consequence the basic Whitham hierarchy from Lemma 2.2.3 turns into a system of PDEs for the function $\boldsymbol{T} \mapsto (\boldsymbol{E}(\boldsymbol{T}), U_E(\boldsymbol{T}))$ such that

$$\partial_{T_\alpha} \mathrm{d}\Omega_\beta \left(E; \boldsymbol{E}(\boldsymbol{T}), U_E(\boldsymbol{T})\right) = \partial_{T_\beta} \mathrm{d}\Omega_\alpha \left(E; \boldsymbol{E}(\boldsymbol{T}), U_E(\boldsymbol{T})\right) \quad . \tag{2.2.26}$$

For the KdV reduction of the KP Whitham hierarchy, a hydrodynamic reduction is given in (2.2.24). Similar to the argument there, in coordinates $(\boldsymbol{E}, U_E)$ of the algebraic orbit $\mathcal{N}_n(n_0)$, the following property allows a hydrodynamic reduction.

Proposition 2.2.7. *Let $\mathcal{N}_n(n_0)$ be an algebraic orbit with Abelian integral E that only has branch points of index 2. For the normalized differential forms of the basic KP Whitham hierarchy holds*

$$\mathrm{d}\Omega_\alpha|_{E_i} \partial_{E_i} \mathrm{d}\Omega_\beta = \mathrm{d}\Omega_\beta|_{E_i} \partial_{E_i} \mathrm{d}\Omega_\alpha \quad , \tag{2.2.27}$$

$$\partial_{U_{E,j}} \mathrm{d}\Omega_\alpha = 0 \tag{2.2.28}$$

for all $i = 1, \ldots, 2n-1+n_0$ and $j = 1, \ldots, n$

Proof. Aside from its branch points, E can be used as a chart and $\mathrm{d}\Omega_\alpha$ as well as its derivatives $\partial_{E_i} \mathrm{d}\Omega_\alpha$ and $\partial_{U_{E,j}} \mathrm{d}\Omega_\alpha$ are holomorphic away from branch points. Since all branch points P_i are assumed to be of index 2, there is a chart κ_i near P_i such that

$$\begin{aligned} E - E_i &= \kappa_i^2 \quad , \\ \mathrm{d}E &= 2\kappa_i \mathrm{d}\kappa_i \quad . \end{aligned}$$

Note that κ_i depends on the branch values $\boldsymbol{E}$ here. In this chart $\mathrm{d}\Omega_\alpha$ takes the form $C^i_\alpha(\kappa_i; \boldsymbol{E}, U_E)\mathrm{d}\kappa_i$ for some function C^i_α which is holomorphic in the first argument and smooth in the second and third argument. Therefore we have near E_i

$$\mathrm{d}\Omega_\alpha = \frac{C^i_\alpha((E-E_i)^{1/2}; \boldsymbol{E}, U_E)\mathrm{d}E}{2(E-E_i)^{1/2}} \quad . \tag{2.2.29}$$

Taking the derivative with respect to $U_{E,j}$ shows that $\partial_{U_{E,j}} \mathrm{d}\Omega_\alpha$ is holomorphic at all branch values E_i. Due to the normalization of the differential form $\mathrm{d}\Omega_\alpha$, both the principal part at P_0 and the a-periods of $\partial_{U_{E,j}} \mathrm{d}\Omega_\alpha$ vanish. Altogether $\partial_{U_{E,j}} \mathrm{d}\Omega_\alpha$ is a holomorphic differential form on a compact Riemann surface without a-periods and therefore zero. Taking the derivative with respect to E_j shows that $\partial_{E_j} \mathrm{d}\Omega_\alpha$ is holomorphic near E_i if $i \neq j$ and if $i = j$ we have

$$\partial_{E_i} \mathrm{d}\Omega_\alpha = \left(\frac{1}{2} - \frac{\partial_1 C^i_\alpha}{2C^i_\alpha}(E-E_i)^{1/2} + \mathcal{O}(E-E_i)\right) \frac{\mathrm{d}\Omega_\alpha}{E-E_i} \quad , \tag{2.2.30}$$

where $\partial_1 C^i_\alpha$ stands for the derivative with respect to the first argument of the function. Expressing $\partial_{E_i} \mathrm{d}\Omega_\alpha$ again in the chart κ_i shows a pole of order up to 2 at $\kappa_i = 0$. The normalization of $\mathrm{d}\Omega_\alpha$ implies that both, the a-periods of $\partial_{E_i} \mathrm{d}\Omega_\alpha$ and its principal part at P_0 vanish. By Riemann's theorem two differential forms on Γ_n with these properties coincide, if their principal parts at $E = E_i$ coincide. There is no residue present at $E = E_i$, since $\partial_{E_i}\mathrm{d}\Omega_\alpha$ is holomorphic everywhere else. From (2.2.30) it follows that both sides of (2.2.27) have the same principal part at $E = E_i$. □

The proof of Proposition 2.2.7 generalizes without any extra arguments to the case of normalized holomorphic differential forms $\mathrm{d}\Omega^h_\alpha$ for $\alpha = 1, \ldots, n$. More generally, in [**17**] *horizontal families of multivalued Abelian differential forms* contain differential forms $\mathrm{d}\Omega$ on the universal covering of Γ_n such that $\partial_{E_i}\mathrm{d}\Omega$ is a meromorphic differential form on Γ_n (i.e. not multivalued) without residues and holomorphic outside the branch points of E, where it has at most a pole of order 2. By (2.2.30), the differential forms $\mathrm{d}\Omega_\alpha$ of the KP Whitham hierarchy obviously have this property.

2.2.4. Hydrodynamic Reduction of the Basic KP Whitham Hierarchy on an Algebraic Orbit. As a consequence of Proposition 2.2.7 we obtain the following form of the KP Whitham hierarchy on algebraic orbits.

Lemma 2.2.8 (Hydrodynamic Reduction)**.** *Consider an algebraic orbit whose Abelian integral E has only branch points of index* 2*. Then the basic KP Whitham equations* (2.2.26) *are equivalent to the diagonal hydrodynamic system*

$$\mathrm{d}\Omega_\beta|_{E_i}\partial_{T_\alpha}E_i = \mathrm{d}\Omega_\alpha|_{E_i}\partial_{T_\beta}E_i \tag{2.2.31}$$

for all $i = 1, \ldots, 2n-1+n_0$.

Proof. Fron Proposition 2.2.7 we have that $\mathrm{d}\Omega_\alpha$ does not depend on the b-periods $U_{E,j}$ of $\mathrm{d}E$. Hence, the basic KP Whitham equations are equivalent to

$$0 = \sum_{l=1}^{2n-1+n_0} \partial_{E_l}\mathrm{d}\Omega_\beta\partial_{T_\alpha}E_l - \partial_{E_l}\mathrm{d}\Omega_\alpha\partial_{T_\beta}E_l \ . \tag{2.2.32}$$

Since E has only branch points of index 2, the principal part of this formula at E_i gives equation (2.2.31) by the help of (2.2.27). Conversely, the differential form on the right hand side of (2.2.32) has vanishing a-periods, is holomorphic on $\Gamma_n \setminus \{E_1, \ldots, E_{2n-1+n_0}\}$ and its principal parts at points $E = E_i$ vanish due to the hydrodynamic reduction (2.2.31). By Riemann's theorem this differential form has to be zero, which is equivalent to the basic KP Whitham equations. □

Note that if $\mathrm{d}\Omega_1|_{E_i} \neq 0$ for all $i = 1, \ldots, 2n-1+n_0$, then the equations (2.2.31) for $\alpha = 1$ and $\beta \geq 1$ imply the equations for all $\alpha, \beta \geq 1$. A convenient version of the compatibility equations (2.2.31) is their *Riemann invariant form*

$$\partial_{T_\beta}E_i = v^{(\beta)}(E_i)\partial_{T_1}E_i \tag{2.2.33}$$

with meromorphic functions $v^{(\beta)}(E) = \mathrm{d}\Omega_\beta/\mathrm{d}\Omega_1$ on $\mathbb{C}$. The ramification values E_i of the Abelian integral E are the *Riemann invariants* here. In particular, the system of equations (2.2.33) forms a diagonal system of hydrodynamic type.

Corollary 2.2.9. *Let $J \subseteq \mathbb{N}_{\geq 1}$ with $|J| = 2n-1+n_0$ and $1 \in J$ and assume there is a solution*

$$T_J := (T_j)_{j\in J} \mapsto \boldsymbol{E}(T_J) \tag{2.2.34}$$

of (2.2.31) *on some open domain in $\mathbb{R}^{|J|}$ such that at some point τ_J in the domain we have $\mathrm{d}\Omega_1|_{E_j(\tau_J)} \neq 0$ and $\partial_{T_1}E_j(\tau_J) \neq 0$ for all $j = 1, \ldots, 2n-1+n_0$. Then the map* (2.2.34) *is a submersion and an immersion at τ_J, i.e. it is invertible locally near τ_J.*

Proof. By assumption the equations (2.2.33) hold in a neighborhood of τ_J. Therefore the Jacobian matrix $J_{\boldsymbol{E}}$ given by $(J_{\boldsymbol{E}})_{\beta,i} = \partial_{T_\beta} E_i$ is invertible if $(v^{(\beta)}(E_i))_{\beta,i}$ is invertible and none of the $\partial_{T_1} E_i$ is zero. The latter holds at τ_J by assumption. Furthermore, the differential forms $(\mathrm{d}\Omega_\beta)_{\beta\in J}$ are linearly independent, due to their different pole order at the marked point P_0. If a linear combination $\mathrm{d}\Omega = \sum_{\beta\in J} k_\beta \mathrm{d}\Omega_\beta$ with $k_\beta \in \mathbb{C}$ vanishes at all branch values E_i, then Proposition 2.2.7 implies

$$\partial_{E_i}\mathrm{d}\Omega = \sum_{\beta\in J} k_\beta \left.\frac{\mathrm{d}\Omega_\beta}{\mathrm{d}\Omega_1}\right|_{E_i} \partial_{E_i}\mathrm{d}\Omega_1 = 0 \ .$$

Hence, $\mathrm{d}\Omega$ is constant in $\boldsymbol{E}$ and therefore also constant on Γ_n. An evaluation at E_i shows that this constant is zero, i.e. $\mathrm{d}\Omega = 0$. Altogether we get that $(v^{(\beta)}(E_i))_{\beta,i} = (\mathrm{d}\Omega_\beta/\mathrm{d}\Omega_1|_{E_i})_{\beta,i}$ is invertible. The inverse function theorem then yields the statement. □

The previous lemma and corollary are still valid, when times corresponding to the normalized holomorphic differential forms $\mathrm{d}\Omega^h_\beta$ are taken into consideration [5]. As a second consequence of Proposition 2.2.7 and the hydrodynamic reduction in Lemma 2.2.8, the basic KP Whitham hierarchy on the algebraic orbit appears as a semi-Hamiltonian hydrodynamic system.

Corollary 2.2.10. *For an algebraic orbit with Abelian integral E that only has branch values E_i of index 2, let $v_i^{(\alpha)} := v^{(\alpha)}(E_i)$ be defined as in* (2.2.33). *Then for $\alpha \geq 1$ and $i = 1, \ldots, 2n-1+n_0$ holds*

$$\partial_{E_i} v^{(\alpha)} = \frac{\partial_{E_i}\mathrm{d}\Omega_1}{\mathrm{d}\Omega_1}\left(v_i^{(\alpha)} - v^{(\alpha)}\right) \ . \tag{2.2.35}$$

In particular, by evaluation at E_j with $j \neq i$ follows the semi-Hamiltonian property (2.1.1) *(in state variables $\boldsymbol{E}$ instead of $\mathbf{u}$) for the hydrodynamic system in* (2.2.33).

After setting $\Gamma^j_{ji} = (\partial_{E_i}\mathrm{d}\Omega_1/\mathrm{d}\Omega_1)|_{E_j}$ for $i \neq j$ as Christoffel symbols, a comparison with (2.1.3) yields a candidate for a corresponding diagonal Riemann metric $\mathfrak{g} = \sum_{j=1}^{2n-1+n_0} \mathfrak{g}_{jj}(\boldsymbol{E})(\mathrm{d}E_j)^2$ by

$$\mathfrak{g}_{jj} = \left(\mathrm{d}\Omega_1(\partial_{\kappa_j})|_{E_j}\right)^2 = 2 \operatorname*{res}_{E=E_j} \frac{(\mathrm{d}\Omega_1)^2}{\mathrm{d}E} \ , \tag{2.2.36}$$

where $\kappa_j^2 = E - E_j$ induces a local coordinate κ_j of Γ_n near P_j and the second equality follows from $2\kappa_j \mathrm{d}\kappa_j = \mathrm{d}E$ and the representation (2.2.29) of $\mathrm{d}\Omega_1$ near branch points. In [**16**] Dubrovin showed for the KdV reduction that this metric has the Egorov property and is flat, see also Lemma 5.1.4 and Lemma 5.1.5 in Chapter 5. The more general case of algebraic orbits is treated in [**40**], see also Remark 2.2.15 below.

Example 2.2.11 (KdV Reduction III)**.** For the KdV Whitham hierarchy the velocities $v_{\alpha,i}(\boldsymbol{\gamma}) := v^{\alpha 1}(\gamma_i)$ (for $i = 1, \ldots, 2n+1$) corresponding to the flow in time T_α are given in (2.2.25), which is a special form of (2.2.33). Also velocities corresponding to normalized holomorphic differential forms $\mathrm{d}\Omega^h_\alpha$ may be allowed here. Due to (2.2.35) any two different velocities $(v_{\alpha,i})_i$ and $(v_{\beta,i})_i$ satisfy the equation for commuting flows (2.1.15). Therefore, these velocities can be used in Tsarev's generalized hodograph method for multiple commuting flows (see Theorem 2.1.3 and [**34**] for a version with more than two times). That is,

[5]In the proof of Corollary 2.2.9 the argument for the linear independence of the differential forms is provided by their different pole order at P_0 or their different a-periods.

in order to solve the KdV Whitham equations in the form of (2.2.25), we look for solutions $\boldsymbol{\gamma} = \boldsymbol{\gamma}(T_J)$ (with $T_J := (T_\alpha)_{\alpha\in J}$ and $J \subset 1 + 2\mathbb{N}$ finite) of

$$w_j(\boldsymbol{\gamma}) = \sum_{\alpha\in J} v_{\alpha,j}(\boldsymbol{\gamma})T_\alpha \tag{2.2.37}$$

for some w_j satisfying (2.1.15). Since (2.1.15) (with fixed $v_j = v_{1,j}$) is $\mathbb{C}$-linear in w_j, some w_j may be given by a convergent linear combination of velocities $v_{\alpha,j}$ with constant complex coefficients h_α

$$w_j(\boldsymbol{\gamma}) = \sum_{\alpha\geq 1} h_\alpha v_{\alpha,j}(\boldsymbol{\gamma}) \ .$$

Whether all possible solutions $(w_j)_j$ of (2.1.15) are of this form will be discussed in Section 5.3.

2.2.5. Krichever's Hodograph method. On algebraic orbits Krichever's hodograph method gives solutions of the KP Whitham hierarchy. Let us consider the Ansatz

$$\mathrm{d}S(E;\boldsymbol{E}(\boldsymbol{T}),U_E(\boldsymbol{T})) = \sum_{\beta\in J} T_\beta \mathrm{d}\Omega_\beta(E;\boldsymbol{E}(\boldsymbol{T}),U_E(\boldsymbol{T})) \tag{2.2.38}$$

for a generating function. Here also normalized holomorphic differential forms $\mathrm{d}\Omega^h_\alpha$ may be included, but for simplicity of notation they are not explicitly mentioned. Furthermore, let us first assume the sum to be finite, but large enough for $\boldsymbol{T} := (T_\alpha)_{\alpha\in J}$ to provide a submersion to the algebraic orbit, i.e. $|J| \geq \dim \mathcal{N}_n(n_0)$. Some subset of those times then yields a local parametrization. Inserting the Ansatz (2.2.38) into the characterizing equation for generating functions (2.2.13)

$$\partial_{T_\alpha}\mathrm{d}S = \mathrm{d}\Omega_\alpha \tag{2.2.39}$$

gives the following result which is a special case of Theorem 7.10 in [**40**].

Proposition 2.2.12. *Let $\mathcal{N}_n(n_0)$ be an algebraic orbit with Abelian integral E that only has branch points of index* 2. *If*

$$\mathrm{d}S|_{E=E_i} = 0 \tag{2.2.40}$$

for all branch values E_i of E, then $\boldsymbol{T} \mapsto (\boldsymbol{E}(\boldsymbol{T}), U_E(\boldsymbol{T}))$ with arbitrary smooth $U_E(\boldsymbol{T})$ is a solution for the basic KP Whitham hierarchy.

PROOF. Let us begin with the hodograph Ansatz (2.2.38) for finding a generating function in (2.2.39). For $\alpha \in J$ the characterizing equation for generating functions demands

$$\mathrm{d}\Omega_\alpha = \partial_{T_\alpha}\sum_{\beta\in J} T_\beta\mathrm{d}\Omega_\beta = \mathrm{d}\Omega_\alpha + \sum_{\beta\in J} T_\beta\partial_{T_\alpha}\mathrm{d}\Omega_\beta \ . \tag{2.2.41}$$

Applying the chain rule for $\mathrm{d}\Omega_\alpha = \mathrm{d}\Omega_\alpha(E;\boldsymbol{E},U_E)$ gives

$$\partial_{T_\alpha}\mathrm{d}\Omega_\beta = \sum_{i=1}^{2n-1+n_0} \partial_{E_i}\mathrm{d}\Omega_\beta\partial_{T_\alpha}E_i + \sum_{j=1}^{n}\partial_{U_{E,j}}\mathrm{d}\Omega_\beta\partial_{T_\alpha}U_{E,j} \ .$$

Therefore (2.2.41) is equivalent to

$$0 = \sum_{i=1}^{2n-1+n_0}\left(\sum_{\beta\in J} T_\beta\partial_{E_i}\mathrm{d}\Omega_\beta\right)\partial_{T_\alpha}E_i + \sum_{j=1}^{n}\left(\sum_{\beta\in J}T_\beta\partial_{U_{E,j}}\mathrm{d}\Omega_\beta\right)\partial_{T_\alpha}U_{E,j} \ . \tag{2.2.42}$$

Clearly, this equation is satisfied, if each term in the brackets vanishes, that is

$$0 = \sum_{\beta \in J} T_\beta \partial_{E_i} \mathrm{d}\Omega_\beta \quad \text{and} \quad 0 = \sum_{\beta \in J} T_\beta \partial_{U_{E,j}} \mathrm{d}\Omega_\beta \tag{2.2.43}$$

for all $i = 1, \ldots, 2n - 1 + n_0$ and $j = 1, \ldots, n$. In order to convert (2.2.43) into an equation for the zeros of the generating differential form $\mathrm{d}S$, we use Proposition 2.2.7. As a consequence, the second equation in (2.2.43) is automatically satisfied and the first equation becomes equivalent to

$$0 = \sum_{\beta \in J} T_\beta \left. \frac{\mathrm{d}\Omega_\beta}{\mathrm{d}\Omega_1} \right|_{E_i} , \tag{2.2.44}$$

where without loss of generality it is assumed that $1 \in J$ and $\mathrm{d}\Omega_1$ has the lowest vanishing order at E_i among all $\mathrm{d}\Omega_\beta$ with $\beta \in J$. Krichever's Hodograph formula (2.2.40) for the Ansatz (2.2.38) then implies (2.2.44) and the claim of the proposition. □

By a more general hodoraph Ansatz than (2.2.38), Theorem 7.10 in [**40**] provides all possible solutions of the KP Whitham hierarchy on an algebraic orbit $\mathcal{N}_n(n_0)$. Let us make a remark on extensions and assumptions that allow a generalization of Proposition 2.2.12 in this direction.

Remark 2.2.13. The Ansatz (2.2.38) only includes differential forms from the basic KP Whitham hierarchy for which we have $\partial_{U_{E,i}} \mathrm{d}\Omega_\beta = 0$. Therefore the b-periods U_E of $\mathrm{d}E$ are undetermined by the hodograph formula (2.2.40) in Proposition 2.2.12. As a consequence, the times of the basic KP Whitham hierarchy generally cannot even locally parametrize the algebraic orbit $\mathcal{N}_n(n_0)$. In order to mend this, the b-periods U_E can be taken as additional deformation times, corresponding to meromorphic differential forms $\mathrm{d}\Omega_i^E$ on the universal covering of Γ_n such that

$$\partial_{U_{E,i}} \mathrm{d}S = \mathrm{d}\Omega_i^E$$

for $i = 1, \ldots, n$, see Theorem 7.11 in [**40**]. In the KdV reduction all b-periods U_E are trivial. Horizontal families of multivalued Abelian differential forms provide a framework which allows to apply Krichever's hodograph Ansatz (2.2.38) more generally, see [**17**].

Conversely, let a generating differential form $\mathrm{d}S$ be given from some general hodograph Ansatz that satisfies (2.2.39). In order to show that $\mathrm{d}S$ satisfies the hodograph formula (2.2.40), we need in the proof of Proposition 2.2.12 that (2.2.42) implies (2.2.43). This implication holds under the assumption that $(\partial_{T_\alpha}(\boldsymbol{E}, U_E))_{\alpha \in J}$ has full rank $\dim \mathcal{N}_n(n_0) = (2n - 1 + n_0) + n$. A generalization of Corollary 2.2.9 to a map $T_J \mapsto (\boldsymbol{E}(T_J), U_E(T_J))$ provides this assumption. As before in Section 2.1.3 also consult Section 2 in [**58**] on this issue.

By the hodograph formula (2.2.40) all zeros of the differential form $\mathrm{d}E$ are also zeros of the generating differential form $\mathrm{d}S$. Therefore we conclude the following (see again Theorem 7.10 in [**40**]).

Corollary 2.2.14. *If Krichver's hodograph formula* (2.2.40) *holds, then there is a holomorphic function $Q = Q\,(E; \boldsymbol{E}(\boldsymbol{T}), U_E(\boldsymbol{T}))$ on $\Gamma_n \setminus \{P_0\}$ such that $\mathrm{d}S = Q\mathrm{d}E$.*

Conversely, when $Q = Q\,(E; \boldsymbol{E}, U_E)$ is considered as given on an algebraic orbit, then the differential equations (2.2.39) induce flows in times T_α. Krichever's hodograph Ansatz in the form (2.2.38) poses constraints on the possible functions Q. Each time T_α for $\alpha \in J$ can be extracted by

$$\operatorname*{res}_{1/\kappa=0} \kappa^{-\alpha} Q\mathrm{d}E = \sum_{\beta\in J} T_\beta \operatorname*{res}_{1/\kappa=0} \left(\kappa^{-\alpha+\beta-1} + \mathcal{O}(\kappa^0)\right)\mathrm{d}\kappa = T_\alpha \ . \tag{2.2.45}$$

This means that the times T_α become functions in the parameters $\boldsymbol{E}$ of the moduli space $\mathcal{N}_n(n_0)$, defining a map $(\boldsymbol{E}, U_E) \mapsto \boldsymbol{T}(\boldsymbol{E}, U_E)$. If this map is invertible, then its inverse gives a solution for the KP Whitham hierarchy. As a necessary condition for invertibility, there need to be more times than parameters of the moduli space, i.e. $|J| \geq \dim \mathcal{N}_n(n_0)$. Using more general functions Q than those originating form Ansatz (2.2.38) (with finite index set) as the starting point for finding solutions of the KP Whitham hierarchy, leads to the *universal configuration space* [**40, 42, 21**].

Remark 2.2.15. In the universal configuration space $Q = p = \Omega_1$ can be chosen. This function is defined on the universal covering of Γ_n. Flat coordinates for the metric (2.2.36) come from the times defined by (2.2.45) (and similar formulas), see Theorem 7.14 and the following page in [**40**].

Example 2.2.16 (KdV reduction IV)**.** Let us consider the hyperelliptic curve Γ_n represented by the algebraic equation $y^2 + g(E) = 0$ for some monic polynomial g of degree $2n+1$ with only simple zeros, see (2.2.23). Due to $2y\mathrm{d}y + \partial_E g(E)\mathrm{d}E = 0$, the roots $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_{2n+1})$ of g are the critical values of the hyperelliptic covering map $\Gamma_n \ni (E, y) \mapsto E \in \mathbb{C}\mathrm{P}^1$ here. According to Proposition 2.2.12, Krichever's hodograph Ansatz (2.2.38) provides a function $\boldsymbol{T} = (T_1, T_3, \dots, T_{2n+1}) \mapsto \boldsymbol{\gamma}(\boldsymbol{T})$ such that $\mathrm{d}S = \sum_{\beta=1}^{n+1} T_{2\beta-1}\mathrm{d}\Omega_{2\beta-1}(E, \boldsymbol{\gamma}(\boldsymbol{T}))$ satisfies (2.2.39). This means $\boldsymbol{\gamma}(\boldsymbol{T})$ is a solution for the KdV Whitham hierarchy.

Conversely, for $Q(E, \boldsymbol{\gamma}) = \sqrt{-g(E)}$ equation (2.2.45) defines a map $\boldsymbol{\gamma} \mapsto \boldsymbol{T}(\boldsymbol{\gamma})$. However, this map is not invertible and we cannot just add higher order times, since the pole order of $Q\mathrm{d}E$ at $1/\kappa = 0$ is $2n+4$ with leading coefficient 4, so in (2.2.45) $T_{2n+3} = 4$ and $T_{2j+1} = 0$ for all $j \geq n+2$. Considering the n normalized holomorphic differential forms $\mathrm{d}\Omega^h_\alpha$ and corresponding times $T^h_\alpha = \int_{a_\alpha} Q\mathrm{d}E$ in Chapter 5 will provide enough times for the invertibility, see Example 5.3.3.

In summary, given two out of the three objects: normalized differential forms $(\mathrm{d}\Omega_\alpha)_\alpha$, a generating differential form $\mathrm{d}S = Q\mathrm{d}E$ and times $\boldsymbol{T}$, the third one follows from Krichever's hodograph Ansatz.

- Given $(\mathrm{d}\Omega_\alpha)_\alpha$ and $\boldsymbol{T}$, then Q follows by Corollary 2.2.14.
- Given a function Q on Γ_n and $\boldsymbol{T}$, then differential forms $\mathrm{d}\Omega_j$ with the correct normalization are determined by (2.2.38).
- Given $(\mathrm{d}\Omega_\alpha)_\alpha$ and a function Q on Γ_n, then times $\boldsymbol{T}$ are found by

  $$T_\alpha = \operatorname*{res}_{1/\kappa=0} \kappa^{-\alpha} Q\mathrm{d}E \ , \text{ for } \alpha \geq 1 \ ,$$

  $$T^h_\alpha = \int_{a_\alpha} Q\mathrm{d}E \ , \text{ for } \alpha = 1, \dots, n \ .$$

  In (2.2.38) these times correspond to the coefficients of the normalized differential forms $\mathrm{d}\Omega_\alpha$ and $\mathrm{d}\Omega^h_\alpha$, respectively.

Either of the three cases allows for a generalization to the universal covering of Γ_n. In the first one this is quite implicit, though. The generating differential is given by a system of differential equations (2.2.39) depending holomorphically on the parameter E. Heuristically, holomorphic continuation yields $\mathrm{d}S = Q\mathrm{d}E$ on the universal covering of the Riemann surface Γ_n with possible singularities at the points covering $P_0 \in \Gamma_n$. In the third case times are redefined that have some meaning as a "slow" version of times in the KP hierarchy. Furthermore, times

$$T_i^E = U_{E,i} = \int_{b_\alpha} \mathrm{d}E \quad \text{and} \quad T_\alpha^Q = \int_{b_\alpha} \mathrm{d}Q$$

for $i, \alpha = 1, \ldots, n$ appear, when E and Q are multivalued on Γ_n, respectively. Similar to the second case, the general characterization of generating differential forms we aim at, will start with some power series expansion of Q and respect the originally given times. For the KdV Whitham equations this is going to be explained in Section 5.3.

CHAPTER 3

The KdV Hierarchy and Stationary Solutions

The Korteweg-de Vries equation is a non-linear PDE in one spatial variable x and one time t. It admits solutions u by the inverse scattering transform [**49, 50, 26**]. This works by interpreting the KdV equation $\partial_t u = -\partial_x^3 u + 6u\partial_x u$ as the compatibility condition of an auxiliary linear problem [1]. Here the initial data for $t = 0$ is described by the family of 2nd order linear ODEs

$$\partial_x^2 \phi = (u - E)\phi$$

with parameter E, see (3.1.1) below. Its solutions are deformed by the linear evolution in time t

$$\partial_t \phi = \left(-4\partial_x^3 + 6u\partial_x + 3\partial_x u\right)\phi$$

while preserving E, compare with (3.1.2) below. By using the framework of *Lax pairs*, compatible linear evolutions in higher times appear in a natural way at this point, leading to the *KdV hierarchy*.

One approach to studying solutions of the linear ODE in x with parameter E is to apply *Floquet theory*, another approach, employed by J. Drach in [**12**], uses *resolvents*, as will be explained in Section 3.1. On both ways a *spectral curve* covering the complex E-plane appears to parameterize solutions. Demanding the spectral curve to be a compact Riemann surface of finite genus provides *finite gap solutions* or *stationary solutions*, respectively. In the following both terms will be used synonymously and will be also applied to the corresponding time-dependent solutions of the KdV equation and the KdV hierarchy. This allows to study special solutions of the KdV hierarchy in the algebro-geometric setting [**46, 47, 18**], see Section 3.2. Extracting a solution of the KdV equation from a resolvent is straight forward, see Section 3.2.1. Explicit formulas for solutions are obtained by using separation of variables for the ODE in x and the Abel-Jacobi map on the spectral curve, see Section 3.2.3. This method was employed by Drach in [**13**] to solve his resolvent equation. From a limiting case of the spectral curve, soliton solutions of the KdV hierarchy are obtained in terms of trigonometric functions, see Section 3.3.

Following S. I. Alber [**1, 2, 3**] and J. Moser [**52**] there are interpretations of the stationary KdV hierarchy as completely integrable classical Hamiltonian systems, see Section 3.4 and Section 3.2.2, respectively. Real reductions of the Hamiltonian systems allow to identify their Arnold-Liouville torus with the real part of the Jacobi torus of the spectral curve.

3.1. Resolvent Formulation of the KdV Hierarchy

Hill's operator $L = -\partial_x^2 + u$ with a suitable $\mathcal{C}^\infty$-function u as the potential has a square root in the algebra of pseudo-differential operators. Let $P_j = [(-L)^{j/2}]_+$ denote the

[1]The transformation $u \mapsto -u/6$ converts solutions of this form of the KdV equation into solutions of its form in (1.0.2).

differential operator part of the $j/2$-th power of $-L$ [**30**]. For example we have $P_1 = \partial_x$, $P_2 = -L$ and $P_3 = \partial_x^3 - \frac{3}{2}u\partial_x - \frac{3}{4}\partial_x u$. Then the *Lax equation*

$$\partial_{t_j} L = [P_j, L]$$

gives rise to the KdV equation for $j = 3$ [2]. Note, for j an even number $P_j = (-L)^{j/2}$ implies the j-th flow to be trivial, i.e. $\partial_{t_j} L = 0$. The system of equations that is induced by the Lax equations (including higher times t_{2j+1} with $j \geq 1$) forms the *KdV hierarchy*. Equivalently to the Lax equations the *Zakharov-Shabat* equations hold

$$\partial_{t_i} P_j - \partial_{t_j} P_i + [P_i, P_j] = 0 \ .$$

The Lax equations are the compatibility equations for the (auxiliary) linear problem

$$L\phi = E\phi \ , \tag{3.1.1}$$

$$\partial_{t_j}\phi = P_j\phi \tag{3.1.2}$$

where the spectral parameter E is assumed independent of x (i.e. $\partial_x E = 0$) and a solution ϕ is called *wave function*. It follows immediately that E does not change under higher flows (i.e. $\partial_{t_j} E = 0$), in other words the deformations of ϕ by the higher flows are *isospectral*. Due to $\partial_{t_1}\phi = P_1\phi = \partial_x\phi$, we may consider ϕ as dependent on $x + t_1$ instead of x and t_1 separately and thus identify both times. In the following we always have $t_1 = x$.

The spectral equation (3.1.1) is a linear ODE of order two and has henceforth a solution space of dimension 2 spanned by linearly independent wave functions $\phi_\pm$. A *resolvent* (or *Green's function*) for (3.1.1) is given as

$$G = \frac{\phi_+\phi_-}{W(\phi_+, \phi_-)}$$

where dividing by the Wronskian $W(\phi_+, \phi_-) = \phi_+'\phi_- - \phi_+\phi_-'$ mods out the multiplicative gauge freedom of $\phi_\pm$, see Section 4 in [**52**]. Still the resolvent G depends on the choice of a basis $\{\phi_+, \phi_-\}$ for the solution space of the spectral equation. However, G satisfies a system of differential equations that does not depend on this choice.

Lemma 3.1.1 ([**12, 2, 52, 31**]). *For the resolvent G the KdV hierarchy becomes*

$$G''G - \frac{1}{2}(G')^2 - 2G^2(u - E) + \frac{1}{2} = 0 \ , \tag{3.1.3}$$

$$\partial_{t_{2i+1}} G^{-1} = (-1)^i \left(B_i G^{-1}\right)' \tag{3.1.4}$$

where $(-)' = \partial_x(-)$, $G^{-1} = 1/G$ and B_i is defined by $P_{2i+1}\phi_\pm = (-1)^i(-\frac{1}{2}B_i' + B_i\partial_x)\phi_\pm$ as a polynomial in E of degree i.

Proof. In order to derive equation (3.1.3) from the auxiliary linear problem we introduce $\tilde{G} = G \cdot W(\phi_+, \phi_-) = \phi_+\phi_-$. Its first and second derivatives are $\tilde{G}' = \phi_+'\phi_- + \phi_+\phi_-'$ and $\tilde{G}'' = 2(u - E)\tilde{G} + 2\phi_+'\phi_-'$. Therefore we have

$$\begin{aligned}\tilde{G}''\tilde{G} - \frac{1}{2}(\tilde{G}')^2 &= 2(u - E)\tilde{G}^2 + 2\tilde{G}\phi_+'\phi_-' - \frac{1}{2}\left((\phi_+'\phi_-)^2 + 2\tilde{G}\phi_+'\phi_-' + (\phi_+\phi_-')^2\right) \\ &= 2(u - E)\tilde{G}^2 - \frac{1}{2}W(\phi_+, \phi_-)^2 \ .\end{aligned}$$

[2] This form of the KdV equation differs in its constant coefficients from the form in the introduction. Both forms are equivalent by rescaling u, x and t.

Dividing by the square of the Wronskian which is constant in x yields (3.1.3). For obtaining equation (3.1.4) we represent the operator P_{2i+1} by a polynomial B_i as follows. Due to (3.1.1) we can substitute ∂_x^2 by the multiplication with $u-E$ (when applied to the wave function). This transforms $P_{2i+1}\phi_\pm$ into the above form with B_i a polynomial in E and coefficients depending on u and its derivatives, see [**1**]. As a consequence we get

$$\begin{aligned}\partial_{t_{2i+1}}\tilde{G} &= \left(\partial_{t_{2i+1}}\phi_+\right)\phi_- + \phi_+\partial_{t_{2i+1}}\phi_- \\ &= (-1)^i\left(-\frac{1}{2}B_i'\phi_+ + B_i\phi_+'\right)\phi_- + (-1)^i\phi_+\left(-\frac{1}{2}B_i'\phi_- + B_i\phi_-'\right) \\ &= (-1)^i\left(-B_i'\tilde{G} + B_i\tilde{G}'\right) \ .\end{aligned}$$

Since the Wronskian does not depend on t_{2i+1} (as follows form a straight forward computation) we arrive at (3.1.4) by using the quotient rule. □

Conversely, from a solution of the resolvent equation in (3.1.3) solutions of the auxiliary linear problem (3.1.1) are recovered by

$$\phi_\pm = \sqrt{G}e^{\pm\frac{1}{2}\int\frac{1}{G}\mathrm{d}x} \ .$$

Using the dynamics in the higher times (3.1.4) and the definition of B_i, these functions $\phi_\pm$ also satisfy (3.1.2) and thus they are wave functions. From now on we consider (3.1.3) and (3.1.4) as the equations of the KdV hierarchy.

Remark 3.1.2. The alternating sign in equation (3.1.4) originates from ∂_x^2 and E having opposite signs in the eigenvalue equation (3.1.1). This is due to the common definition of Hill's operator. For convenience of the following display, we reorient the higher flows of the KdV hieararchy by $t_{2i+1} \to (-1)^i t_{2i+1}$. Hence, (3.1.4) becomes $\partial_{t_{2i+1}}G^{-1} = \left(B_iG^{-1}\right)'$.

3.2. Stationary Solutions of the KdV Hierarchy

As an ODE with values in the formal Laurent series in E, the resolvent equation (3.1.3) is an infinite dimensional system of scalar ODEs. We are now looking for solutions which are already described by finitely many equations.

Definition 3.2.1 ([**55, 47**])**.** A resolvent G is called *m-stationary* if the m-th flow is a linear combination of the lower order flows. The corresponding system of equations in Lemma 3.1.1 forms the *m-stationary KdV hierarchy*.

This means a resolvent is $(2n+1)$-stationary if there are constants k_i such that

$$\partial_{t_{2n+1}}G + \sum_{i=0}^{n-1}k_i\partial_{t_{2i+1}}G = 0 \ . \tag{3.2.1}$$

Using equation (3.1.4) (with the modification from Remark 3.1.2) for the dynamics of G, this is equivalent to $\left(\sum_{i=0}^n k_iB_i\right)G' - \left(\sum_{i=0}^n k_iB_i'\right)G = 0$ where k_n is set to be 1. Here we see logarithmic derivatives, so integration yields for $\hat{B} = \sum_{i=0}^n k_iB_i$

$$G = \hat{B}e^P \tag{3.2.2}$$

where the constant of integration P is independent of x and $\hat{B}$ is a polynomial of order n since the B_i are polynomials. Substituting this form of G into (3.1.3) we obtain

$$\hat{B}''\hat{B} - \frac{1}{2}(\hat{B}')^2 - 2\hat{B}^2(u-E) + \frac{1}{2}e^{-2P} = 0 \ . \tag{3.2.3}$$

In particular, since $\hat{B}$ is a polynomial also e^{-2P} has to be polynomial in E. More precisely, $g = -\frac{1}{4}e^{-2P}$ is a monic polynomial of degree $2n+1$. For genericity we assume all the zeros $\gamma_1, \ldots, \gamma_{2n+1}$ of g to be simple.

Proposition 3.2.2. *The polynomial g is invariant under the higher KdV flows* (3.1.4).

PROOF. Substituting (3.2.2) into (3.1.4) (with the modification from Remark 3.1.2) and using that P does not depend on x gives after multiplying by $e^P \hat{B}^2$

$$B_i'\hat{B} - B_i\hat{B}' = -\partial_{t_{2i+1}}\hat{B} - \hat{B}\partial_{t_{2i+1}}P \quad . \tag{3.2.4}$$

About the first three terms in this equation we know that they are of order up to $n+i-1$ in E. The last term we rewrite by using $\partial_{t_{2i+1}}g = -2g\partial_{t_{2i+1}}P$. Multiplying (3.2.4) by g we arrive at

$$g(B_i'\hat{B} - B_i\hat{B}' + \partial_{t_{2i+1}}\hat{B}) = \frac{1}{2}\hat{B}\partial_{t_{2i+1}}g \quad .$$

At the zeros $\gamma_1, \ldots, \gamma_{2n+1}$ of g, the left hand side of this equation vanishes. Since $\hat{B}|_{E=\gamma_j} \neq 0$ generically (otherwise, following from (3.2.3) and $\partial_x\gamma_j = 0$ we would have γ_j as a double zero of g) we obtain $\partial_{t_{2i+1}}g|_{E=\gamma_j} = 0$. However, $\partial_{t_{2i+1}}g$ is a polynomial in E of degree at most $2n$ and therefore it must be zero, i.e. $\partial_{t_{2i+1}}g = 0$. □

Linear combinations of higher flows are again of the form in Lemma 3.1.1 for different polynomials B_i. For simplicity we assume now that the flow which becomes stationary is given such that the coefficients k_i of the linear combination in (3.2.1) are trivial, i.e. $\partial_{t_{2n+1}}G = 0$. However, this changes the polynomial g which will play a role later on in the study of non-isospectral deformations in Chapter 4. As a result Lemma 3.1.1 (with the modification from Remark 3.1.2) takes the following form.

Lemma 3.2.3. *For the $(2n+1)$-stationary KdV hierarchy the flows of the times $x = t_1, t_3, \ldots, t_{2n-1}$ are given by*

$$\hat{B}''\hat{B} - \frac{1}{2}(\hat{B}')^2 - 2\hat{B}^2(u-E) = 2g \quad , \tag{3.2.5}$$

$$\partial_{t_{2i+1}}\hat{B}^{-1} = \left(B_i\hat{B}^{-1}\right)' \tag{3.2.6}$$

where $(-)' = \partial_x(-)$ and $B_i = [E^{i-n}\hat{B}]_{\geq 0}$ is the polynomial part of $E^{i-n}\hat{B}$. The higher flows are trivial.

This is to say that finding $(2n+1)$-stationary solutions of the KdV hierarchy amounts to determining $\hat{B}$. The polynomial $\hat{B}$ and thereby the resolvent equation (3.2.5) can be parameterized in different ways:

- Parameterization by the monomial coefficients leads to a recursive system of second order non-linear ODEs as is presented in [**1**].
- A less obvious parameterization will allow to identify (3.2.5) with the finite dimensional integrable Hamiltonian system of the C. Neumann problem [**2**].
- Using the roots of $\hat{B}$ as parameters leads to the definition of the spectral curve and the Drach-Dubrovin equations [**13, 18, 6**] which may also be understood as a finite dimensional integrable Hamiltonian system [**3**].

The following sections will explain these three parameterizations of $\hat{B}$ in more detail.

3.2.1. The Recursive System. When sorted by powers of E the resolvent equation (3.2.5) forms a system of recursively related equations. Likewise, the dynamics in higher times can be determined recursively from (3.2.6) in its equivalent form $\partial_{t_{2i+1}}\hat{B} = B_i\hat{B}' - B_i'\hat{B}$. Let

$$\hat{B} = \sum_{j=0}^{n} \hat{b}_j E^{n-j} \quad \text{and} \quad g = \sum_{i=0}^{2n+1} c_{2n+1-i} E^i \tag{3.2.7}$$

where $\hat{b}_0 = 1$ and $c_0 = 1$. Then from (3.2.5) the potential u of Hill's operator can be read off as

$$u = 2\hat{b}_1 - c_1 \ . \tag{3.2.8}$$

Taking the derivative of the resolvent equation (3.2.5) with respect to x and dividing by $\hat{B}$ we obtain due to Proposition 3.2.2

$$\hat{B}''' - 4\hat{B}'(u - E) - 2\hat{B}u' = 0 \ . \tag{3.2.9}$$

This equation is linear in $\hat{B}$ and yields the recursion

$$4\hat{b}_{j+1}' = -\hat{b}_j''' + 4\hat{b}_j' u + 2\hat{b}_j u' \tag{3.2.10}$$

for $j = 0, \ldots, n$ and $\hat{b}_{n+1} = 0$ representing the $(2n+1)$-stationarity. Note however, that (3.2.9) determines $\hat{B}$ only up to a constant of integration which is included in (3.2.5). In [**1**] the recursion including the constants of integration is described.

The dynamics in higher times $\partial_{t_{2i+1}}\hat{B} = B_i\hat{B}' - B_i'\hat{B}$, i.e. (3.2.6), yields for the monomial E^{n-1} the coefficient equation

$$\partial_{t_{2i+1}}\hat{b}_1 = \hat{b}_{i+1}' \ .$$

In the case $i = 1$ we may express this equation in terms of Hill's potential u. By using (3.2.10) and substituting (3.2.8) we arrive at

$$\partial_{t_3} u = \frac{1}{4}u''' + \frac{3}{2}u'\left(u - \frac{c_1}{6}\right) \ . \tag{3.2.11}$$

After the transformations $u \mapsto \frac{1}{6}(u + c_1)$ and $t_3 \mapsto -4t$ we find the KdV equation in the form (1.0.2). So in particular, each solution of the stationary KdV hierarchy yields a solution of the KdV equation.

3.2.2. The C. Neumann Problem. In the stationary case the resolvent equation in (3.2.5) forms a finite dimensional (classical) completely integrable Hamiltonian system with phase space coordinates (q, p) defined (up to sign) by

$$\hat{B}(E) = \sum_{j=1}^{n+1}\left(\prod_{i\neq j}(E - e_i)\right) q_j^2 \ , \tag{3.2.12}$$

$$p = q' \tag{3.2.13}$$

and Hamiltonian $H = \frac{1}{2}\sum_{i=1}^{n+1} e_i q_i^2 + \frac{1}{4}\sum_{i,j=1}^{n+1}(q_i p_j - q_j p_i)^2$. Here the e_i are any $n+1$ of the $2n+1$ roots of g [**2**]. In order to get real solutions, the roots of g are chosen real and alternating in the following sense [**53**]: If $g(E) = \prod_{i=1}^{n+1}(E - e_i)\prod_{j=1}^{n}(E - \hat{e}_j)$, then

$$e_1 < \hat{e}_1 < e_2 < \hat{e}_2 < e_3 \cdots < \hat{e}_n < e_{n+1} \ . \tag{3.2.14}$$

Since $\hat{B}$ is also a monic polynomial we have $\sum_{j=1}^{n+1} q_j^2 = 1$, that is to say the Hamiltonian system describes a particle on the n-sphere (in a quadratic potential). This is the *C. Neumann Problem* [**52**]. Its integrals of motion

$$F_j = q_j^2 + \sum_{i \neq j} \frac{(q_i p_j - q_j p_i)^2}{e_j - e_i}$$

are obtained as the residues at e_j of the resolvent equation (3.2.5) divided by $\prod_{i=1}^{n+1}(E-e_i)^2$, see [**2**]. In particular we have $H = \frac{1}{2}\sum_{j=1}^{n+1} e_j F_j$. Since $F_1, \ldots, F_n$ are independent and Poisson commuting, we see that the C. Neumann Problem is completely integrable. A direct inspection shows

$$F_j = \frac{\prod_{k=1}^{n}(e_j - \hat{e}_k)}{\prod_{i \neq j}(e_j - e_i)} \quad . \tag{3.2.15}$$

Therefore the spectral curve Γ encodes the integrals of motion. The higher order flows (3.2.6) also become Hamiltonian with Hamiltonians that are linear combinations of $F_1, \ldots, F_n$.

In Section 3.4 a more direct way will be explained how to interpret the $(2n+1)$-stationary KdV hierarchy as a Hamiltonian system that can be integrated by separation of variables.

3.2.3. The Spectral Curve and Integration by Separation of Variables. Let $\eta_1, \ldots, \eta_n$ denote the roots of $\hat{B}$, that means $\hat{B}(E) = \prod_{l=1}^{n}(E-\eta_l)$. Evaluating the resolvent equation (3.2.5) at these roots gives

$$-(y_i)^2 = g(\eta_i) \tag{3.2.16}$$

for $y_i = \frac{1}{2}\hat{B}'(\eta_i)$. Therefore (η_i, y_i) can be interpreted as points on the complex curve $\Gamma = \{(E,y) \mid 0 = y^2 + g(E)\} \subseteq \mathbb{C}^2$ which is called *spectral curve*. The isospectral flows from Lemma 3.2.3 preserve Γ due to Proposition 3.2.2. Hence, the dynamics (3.2.6) takes place on the n-fold product of the spectral curve Γ. Evaluated at $E = \eta_i$ equation (3.2.6) has the form

$$\partial_{t_{2j+1}} \hat{B}|_{E=\eta_i} = \hat{B}'(\eta_i) B_j(\eta_i) = 2 y_i B_j(\eta_i) \ .$$

Then by writing $\partial_t \hat{B}|_{E=\eta_i} = -\prod_{k \neq i}(\eta_i - \eta_k)\partial_t \eta_i$ and using $0 = 2y_i \partial_t y_i + \partial_E g|_{E=\eta_i}\partial_t \eta_i$ on the spectral curve (3.2.16), an equivalent form of the equations in Lemma 3.2.3 are the *Drach-Dubrovin equations* [**13, 18**]

$$\begin{cases} \partial_{t_{2j+1}} \eta_i = \frac{-2y_i}{\prod_{k \neq i}(\eta_i - \eta_k)} B_j(\eta_i) \\ \partial_{t_{2j+1}} y_i = \frac{\partial_E g|_{E=\eta_i}}{\prod_{k \neq i}(\eta_i - \eta_k)} B_j(\eta_i) \ . \end{cases} \tag{3.2.17}$$

Note that introducing y_i as a coordinate avoids sign ambiguity when taking the square root of $-g(\eta_i)$. From a solution of (3.2.17) we can obtain by (3.2.8) a solution of the KdV equation $u = -2\sum_{l=1}^{n} \eta_l - c_1$.

Remark 3.2.4. The Drach-Dubrovin equations for any two times, say t_{2j+1} and t_{2k+1}, give rise to a $1+1$-dimensional system of hydrodynamic type by

$$\partial_{t_{2j+1}} \eta_i = \frac{B_j(\eta_i)}{B_k(\eta_i)} \partial_{t_{2k+1}} \eta_i \ .$$

Considering this system of PDEs amounts to “forgetting” the resolvent equation (3.2.5). The case $j = 1$ and $k = 0$ has been discussed in [**23**] as an example of a semi-Hamiltonian system of hydrodynamic type that is additionally *weakly non-linear*, i.e. $\partial_{\eta_i}(B_j(\eta_i)/B_k(\eta_i)) = 0$

for all $i = 1, \ldots, n$. Such systems allow to apply the methods explained in Section 2.1, notably Tsarev's generalized hodograph method. Moreover, the structure as a weakly non-linear system provides exactly $n-2$ linearly independent additional commuting flows that are weakly non-linear, see [**23**]. In the case of the KdV hierarchy these are the higher KdV flows in times t_{2l+1} for $l = 2, \ldots, n-1$. As a consequence, solutions of weakly non-linear systems of PDEs can be found by separation of variables and the inversion of integrals. Including the resolvent equation (3.2.5) again, turns this into the Jacobi inversion, as will be explained in the following.

Let us assume that the roots $\gamma_1, \ldots, \gamma_{2n+1}$ of g are simple, real and ordered increasingly. Consequently, $-g$ is non-negative on $[\gamma_{2i}, \gamma_{2i+1}]$, so if η_i is in this interval, then y_j is real. This gives:

Proposition 3.2.5. *For $\gamma_{2i} \leq \eta_i \leq \gamma_{2i+1}$ at the initial time, the Drach-Dubrovin equations* (3.2.17) *have a real solution. In particular, each η_i stays in its interval $[\gamma_{2i}, \gamma_{2i+1}]$.*

The combined Drach-Dubrovin equations read $\partial_{\boldsymbol{t}} \boldsymbol{\eta} = J_\eta$ for $\boldsymbol{t} = (t_{2i+1})_{i=0}^{n-1}$ and $\boldsymbol{\eta} = (\eta_1, \ldots, \eta_n)$ and

$$(J_\eta)_{i,j} = -2 \frac{\sqrt{-g(\eta_j)} B_i(\eta_j)}{\prod_{l \neq j} (\eta_j - \eta_l)}$$

for $i = 0, \ldots, n-1$. In other words

$$\mathrm{d}\boldsymbol{t} = \mathrm{d}\boldsymbol{\eta} \cdot J_\eta^{-1} \quad . \tag{3.2.18}$$

In this form, the Drach-Dubrovin equations can be integrated by separation of variables, and the integration necessary can be expressed by Abelian integrals of holomorphic differential forms on the spectral curve.

Lemma 3.2.6 ([**13**])**.** *The solution $\hat{B} = \prod_{r=1}^n (E - \eta_r)$ of the system in Lemma 3.2.3 is implicitly given by*

$$\boldsymbol{t} + b = -\frac{1}{2} \left(\sum_{r=1}^n \int^{\eta_r} \frac{\eta^{n-1-l} \mathrm{d}\eta}{\sqrt{-g(\eta)}} \right)_{l=0}^{n-1} \tag{3.2.19}$$

for some constant vector b.

Proof. Integrating (3.2.18) yields

$$(\boldsymbol{t} + b)^t = \int \left(J_\eta^{-1} \right)^t \mathrm{d}\boldsymbol{\eta}^t \tag{3.2.20}$$

for some constant b. We are left with finding the inverse matrix of J_η which is the product of $(B_i(\eta_j))_{i,j}$ with the diagonal matrix

$$\operatorname{diag} \left(\frac{-2\sqrt{-g(\eta_j)}}{\prod_{l \neq j} (\eta_j - \eta_l)} \right)_{j=1}^n \quad .$$

In order to invert the matrix $(B_i(\eta_j))_{i,j}$, note that by (3.2.7) and $B_{n-l} = [E^{-l} \hat{B}]_{\geq 0}$ we have $(B_{n-l}(E))_{l=1}^n = H \cdot \left(E^0, \ldots, E^{n-1}\right)^t$ for the Hankel matrix

$$H = \begin{pmatrix} \hat{b}_{n-1} & \ldots & \hat{b}_1 & 1 \\ \vdots & \iddots & \iddots & \\ \hat{b}_1 & 1 & & \\ 1 & & & 0 \end{pmatrix} \quad .$$

Therefore $(B_{n-l}(\eta_j))_{l,j} = H \cdot V^t$ for the Vandermonde matrix

$$V = \begin{pmatrix} 1 & \eta_1 & \dots & \eta_1^{n-1} \\ \vdots & \vdots & & \vdots \\ 1 & \eta_n & \dots & \eta_n^{n-1} \end{pmatrix} .$$

Since V diagonalizes H by $VHV^t = \operatorname{diag}(\partial_E \hat{B}|_{E=\eta_i})_{i=1}^n$ we arrive at

$$(B_{n-l}(\eta_j))_{l,j} = V^{-1} \operatorname{diag}\left(\partial_E \hat{B}|_{E=\eta_i}\right)_{i=1}^n .$$

The inverse of this matrix can be obtained explicitly by using $\partial_E \hat{B}|_{E=\eta_i} = \prod_{l\neq i}(\eta_i - \eta_l)$. Note furthermore that reversing the order of rows in $(B_{n-l}(\eta_j))_{l,j}$ yields $(B_i(\eta_j))_{i,j}$. Altogether we have now

$$J_\eta^{-1} = \operatorname{diag}\left(\frac{-1/2}{\sqrt{-g(\eta_j)}}\right)_{j=1}^n \cdot \begin{pmatrix} \eta_1^{n-1} & \dots & \eta_1 & 1 \\ \vdots & & \vdots & \vdots \\ \eta_n^{n-1} & \dots & \eta_n & 1 \end{pmatrix} . \tag{3.2.21}$$

Inserted into (3.2.20) this gives (3.2.19). $\square$

Inverting the primitive function in (3.2.19) such that we have a function $\boldsymbol{\eta} = \boldsymbol{\eta}(\boldsymbol{t})$, yields solutions of the $(2n+1)$-stationary KdV hierarchy. If the zeros of g are simple, the inversion appears to be the inverse *Abel map* (also called *Jacobi inversion*). In the case where all roots of g but one are double, the separation of variables leads to explicit solutions of the KdV hierarchy in terms of trigonometric functions. They are called *solitons*, see Section 3.3.

3.3. Soliton Solutions of Stationary KdV

The equations of the stationary KdV hierarchy in Lemma 3.2.3 with reality condition

$$\gamma_1 < \dots < \gamma_{2k-1} < \gamma_{2k} < \gamma_{2k+1} < \dots < \gamma_{2n+1}$$

(see Proposition 3.2.5) allow to consider limits $\gamma_{2k-1} \to \gamma_{2k}$ (and $\gamma_{2k+1} \to \gamma_{2k}$). When two branch points are joined, the hyperelliptic spectral curve degenerates and for its normalized version the genus drops by one. For a corresponding Hamiltonian system (see Section 3.2.2 or Section 3.4 below), the compact Arnold-Liouville torus has to be replaced by a cylinder (or a pinched torus) on which the Hamiltonian flows are linear again.

If the limit $\gamma_{2k-1} \to \gamma_{2k}$ is taken for all $k = 1, \dots, n$, the solution to the $(2n+1)$-stationary KdV equation is called a *n-soliton*. The corresponding normalized spectral curve has genus zero, i.e. it is equivalent to $\mathbb{C}\mathrm{P}^1$. Accordingly, when solving the Drach-Dubrovin equations by separation of variables (see Lemma 3.2.6) the hyperelliptic integrals become solvable by trigonometric functions as we will see in the following.

By Lemma 3.2.6 solving the $(2n+1)$-stationary KdV equation amounts to integrating

$$\mathrm{d}t_{2(n-j)+1} = -\frac{1}{2}\sum_{l=1}^n \frac{\eta_l^{j-1}\mathrm{d}\eta_l}{\sqrt{-g(\eta_l)}} \tag{3.3.1}$$

for $j = 1, \dots, n$. In the case of n-solitons we have $\sqrt{-g(E)} = \prod_{l=1}^n (E - \gamma_{2l})\sqrt{\gamma_{2n+1} - E}$. Using linear combinations of times, we can achieve some cancellation in the denominator and numerator of (3.3.1). There is a matrix $(\Omega_{j,r})_{j,r=1}^n$ such that for all $r = 1, \dots, n$

$$\sum_{j=1}^n \eta^{j-1}\Omega_{j,r} = \prod_{l=1, l\neq r}^n (\eta - \gamma_{2l}) =: \mathfrak{b}_r(\eta)$$

and [3] hence,

$$\begin{aligned} -2\sum_{j=1}^n \mathrm{d}t_{2(n-j)+1}\Omega_{j,r} &= \sum_{l=1}^n \left(\sum_{j=1}^n \eta_l^{j-1}\Omega_{j,r}\right) \frac{\mathrm{d}\eta_l}{\sqrt{-g(\eta_l)}} = \sum_{l=1}^n \frac{\mathfrak{b}_r(\eta_l)\mathrm{d}\eta_l}{\sqrt{-g(\eta_l)}} \\ &= \sum_{l=1}^n \frac{\mathrm{d}\eta_l}{(\eta_l - \gamma_{2r})\sqrt{\gamma_{2n+1} - \eta_l}} \ . \end{aligned}$$

In coordinates $u_l^2 = \gamma_{2n+1} - \eta_l$ each summand has a primitive function

$$\int \frac{\mathrm{d}\eta_l}{(\eta_l - \gamma_{2r})\sqrt{\gamma_{2n+1} - \eta_l}} = \frac{2}{v_r} \operatorname{arctanh}\left(\frac{u_l}{v_r}\right)$$

where $v_r^2 = \gamma_{2n+1} - \gamma_{2r}$. Altogether this gives

$$\Phi_r := -v_r \sum_{j=1}^n t_{2(n-j)+1}\Omega_{j,r} + \phi_{0,r} = \sum_{l=1}^n \operatorname{arctanh}\left(\frac{u_l}{v_r}\right) \tag{3.3.2}$$

where $\phi_{0,r} \in \mathbb{R}$ is the constant of integration. Therefore we are left with the task to solve for $u_1, \dots, u_n$ or some other expression that allows for extracting a solution $u = -2\sum_{l=1}^n \eta_l - c_1$ of the KdV equation in (3.2.11). Here the fact that all arctanh summands in (3.3.2) have the same factor becomes helpful, since we can now use the trigonometric identity

$$\tanh\left(\sum_{l=1}^n \operatorname{arctanh}(z_l)\right) = \frac{\prod_{l=1}^n (1+z_l) - \prod_{l=1}^n (1-z_l)}{\prod_{l=1}^n (1+z_l) + \prod_{l=1}^n (1-z_l)} = \frac{\sum_{S\subseteq\{1,\dots,n\}, |S| \ odd} z(S)}{\sum_{S\subseteq\{1,\dots,n\}, |S| \ even} z(S)}$$

where $z(S) = \prod_{j\in S} z_j$. Applied to (3.3.2) we obtain for all $r = 1, \dots, n$

$$a_r := \tanh(\Phi_r) = \frac{\sum_{S\subseteq\{1,\dots,n\}, |S| \ odd} u(S) v_r^{n-|S|}}{\sum_{S\subseteq\{1,\dots,n\}, |S| \ even} u(S) v_r^{n-|S|}} \ .$$

This in turn is equivalent to

$$0 = \Lambda \cdot \sigma$$

where $\sigma = (\sigma_0(\boldsymbol{u}), \dots, \sigma_n(\boldsymbol{u}))$ for $\sigma_m(\boldsymbol{u}) = \sum_{S\subseteq\{1,\dots,n\}, |S|=m} u(S)$ the elementary symmetric polynomial in $\boldsymbol{u} = (u_j)_{j=1}^n$ of degree m and

$$\Lambda_{r,m} = \begin{cases} a_r v_r^{n-m} & \text{if } m \text{ is even} \\ -v_r^{n-m} & \text{if } m \text{ is odd} \end{cases}$$

for $r = 1, \dots, n$ and $m = 0, \dots, n$. With that we have:

Lemma 3.3.1. *The n-soliton solutions of the Drach-Dubrovin equations* (3.2.18) *in parameters $\sigma_j(\boldsymbol{u})$ for $j = 0, \dots, n$ are*

$$\sigma_j(\boldsymbol{u}) = (-1)^j \det(\Lambda_j)/\det(\Lambda_0) \ , \tag{3.3.3}$$

[3] In more detail $\Omega = V^{-1} \cdot \mathrm{diag}(\mathfrak{b}_r(\gamma_{2r}))$ with $V = (\gamma_{2l}^{j-1})_{l,j=1}^n$ denoting a Vandermonde matrix.

where Λ_j is the $n \times n$-matrix obtained from Λ by deleting the j-th column. In particular, a solution of the KdV equation in (3.2.11) *is given by*

$$u = 2(\sigma_1(\boldsymbol{u})^2 - 2\sigma_2(\boldsymbol{u})) - (2n-1)\gamma_{2n+1} + 2\sum_{l=1}^{n} \gamma_{2l} \ . \tag{3.3.4}$$

Example 3.3.2 (1-solitons and 2-solitons)**.** In the case $n = 1$ we have $\Lambda = (a_1 v_1, -1)$ and thus from (3.3.3) $\sigma_1(\boldsymbol{u}) = -a_1 v_1$ and $\sigma_2(\boldsymbol{u}) = 0$. Therefore the KdV 1-soliton solution is

$$u = 2(a_1 v_1)^2 - \gamma_3 + 2\gamma_2 = 2(\gamma_3 - \gamma_2)\tanh^2 \Phi_1 - \gamma_3 + 2\gamma_2 = \gamma_3 - \frac{2(\gamma_3 - \gamma_2)}{\cosh^2 \Phi_1} \ .$$

In the case $n = 2$ we have

$$\Lambda = \begin{pmatrix} a_1 v_1^2 & -v_1 & a_1 \\ a_2 v_2^2 & -v_2 & a_2 \end{pmatrix}$$

with $a_j = \tanh(\Phi_j)$ and $v_j = \sqrt{\gamma_5 - \gamma_{2j}}$. By Lemma 3.3.1 we get $\sigma_1(\boldsymbol{u}) = (a_1 a_2 (v_2^2 - v_1^2))/(a_1 v_2 - a_2 v_1)$ and $\sigma_2(\boldsymbol{u}) = v_1 v_2 (a_2 v_2 - a_1 v_1)/(a_1 v_2 - a_2 v_1)$ and the KdV 2-soliton solution is

$$u = 2(\gamma_2 + \gamma_4) - 3\gamma_5 + 2\frac{\left(v_1^2 - v_2^2\right)^2 \tanh^2 \Phi_1 \tanh^2 \Phi_2}{\left(v_1 \tanh \Phi_2 - v_2 \tanh \Phi_1\right)^2} + 4\frac{v_1 v_2 \left(v_2 \tanh \Phi_2 - v_1 \tanh \Phi_1\right)}{v_1 \tanh \Phi_2 - v_2 \tanh \Phi_1} \ .$$

Choosing the phase $(\phi_{0,1}, \phi_{0,2}) = (0, i\pi/2)$ (and real times t_j) gives a bounded solution. Asymptotically for $\Phi_1, \Phi_2 \to -\infty$ we have $u \to \gamma_5$.

Which choice of phases ϕ_0 gives bounded n-soliton solutions involves the question where $\tanh \Phi_r$ has zeros and poles (which occur in case of complex arguments) and also the question of zeros and poles of $\det(\Lambda_0)$. This problem is not dealt with here, but in numerical experiments it looks like the phase $(\phi_{0,r})_{r=0}^n = (0, i\pi/2, 0, i\pi/2, 0, \dots)$ provides bounded n-soliton solutions u.

Asymptotically for large time $|t_{2(n-r)+1}| \to \infty$ a n-soliton decomposes into the sum of n 1-solitons as was shown in [**54**]. Here we have the following result.

Lemma 3.3.3. *For each $r \in \{1, \dots, n\}$ the limit of n-solitons for large time $|t_{2(n-r)+1}| \to \infty$ is*

$$\hat{B}(E) \to \prod_{k=1}^{n} (E - \gamma_{2k}) \quad \text{and} \quad u \to \gamma_{2n+1} \ . \tag{3.3.5}$$

This means in particular that $\eta_j \to \gamma_{2l}$ for some $l \in \{1, \dots, n\}$. The limit of η_j may differ for $t_{2(n-r)+1} \to \infty$ and $t_{2(n-r)+1} \to -\infty$.

Proof. For $t_{2(n-r)+1} \to \infty$ we have $a_r \to -1$ and therefore $\Lambda \to -(v_r^{n-m})_{r=1,m=0}^n$, which is the negative of a Vandermonde matrix with reverse column order and an extra column $(-v_r^n)_r$. For the determinant of Λ_j we obtain

$$\det \Lambda_j \to \sigma_j(\boldsymbol{v}) \prod_{i<j} (v_i - v_j)$$

where $\boldsymbol{v} = (v_1, \dots, v_n)$. Hence, in (3.3.3) we get

$$\sigma_j(\boldsymbol{u}) \to (-1)^j \sigma_j(\boldsymbol{v}) \ . \tag{3.3.6}$$

For $t_{2(n-r)+1} \to -\infty$ the same limit holds up to sign. The next step of the proof is to expand $\hat{B}$ in symmetric polynomials:

$$\hat{B}(E) = \prod_{i=1}^{n}(E-\eta_i) = \sum_{i=0}^{n}(-1)^i\sigma_i(\eta_1,\ldots,\eta_n)E^{n-i} \ .$$

By definition we have $\eta_l = \gamma_{2n+1} - u_l^2$ and thus $\sigma_i(\eta_1,\ldots,\eta_n) = \sigma_i((\gamma_{2n+1}-u_l^2)_{l=1}^n)$, which in turn is a symmetric polynomial in $\boldsymbol{u}$ and henceforth can be written as a polynomial in $(\sigma_1(\boldsymbol{u}),\ldots,\sigma_n(\boldsymbol{u}))$. Since the whole expression is even in $u_1,\ldots,u_n$ the sign of $\sigma_j(\boldsymbol{u})$ when j is odd does not matter. For $\sigma_j(\boldsymbol{u})$ we know the limit is $(-1)^j\sigma_j(\boldsymbol{v})$ with $\gamma_{2l} = \gamma_{2n+1} - v_l^2$ by definition. Putting everything together again, we obtain

$$\sigma_i(\eta_1,\ldots,\eta_n) \to \sigma_i((\gamma_{2l})_{l=1}^n)$$

which implies the first part of (3.3.5). Concluding the proof we get

$$u \to 2(\sigma_1(\boldsymbol{v})^2 - 2\sigma_2(\boldsymbol{v})) - (2n-1)\gamma_{2n+1} + 2\sum_{l=1}^{n}\gamma_{2l} = \gamma_{2n+1}$$

from (3.3.4) and (3.3.6). □

Remark 3.3.4. Alternatively, from the decomposition of an n-soliton into n 1-solitons it follows that the n-soliton KdV solution u is asymptotically constant and all its derivatives are asymptotically zero. By the recursion relation (3.2.10) then also $\hat{B}$ is asymptotically constant (in $t_1 = x$). From the resolvent equation (3.2.5) which determines $\hat{B}$ including constants of integration $g(E) = (E-\gamma_{2n+1})\prod_{k=1}^n(E-\gamma_{2k})^2$, we then get again (3.3.5).

3.4. Hamiltonian Formulation of the Stationary KdV Hierarchy

Following S. I. Alber [**2, 3**] and Novikov and Veselov [**67, 68**] we want to reinterpret the Drach-Dubrovin equations (3.2.17) as Hamiltonian flows. Generally, the Hamiltonian system will be complex-valued, but we are mainly interested in the real restriction that appeared for the Drach-Dubrovin equations in Proposition 3.2.5 before.

First let us consider $(\boldsymbol{y},\boldsymbol{\eta})$ as coordinates of the complex phase space $\mathbb{C}^n \times \mathbb{C}^n$ and Hamiltonians

$$H_l(\boldsymbol{y},\boldsymbol{\eta}) = -\sum_{j=1}^{n}\frac{y_j^2+g(\eta_j)}{\prod_{k\neq j}(\eta_j-\eta_k)}B_l(\eta_j)$$

for $l = 0,\ldots,n-1$ and g, B_l as defined in Section 3.2. The corresponding Hamiltonian flows are given by

$$\begin{cases}\partial_{t_{2l+1}}\boldsymbol{\eta} = \partial_{\boldsymbol{y}}H_l \\ \partial_{t_{2l+1}}\boldsymbol{y} = -\partial_{\boldsymbol{\eta}}H_l \ .\end{cases} \tag{3.4.1}$$

They are tangential to the n-fold product of the spectral curve Γ

$$M_g = \{(\boldsymbol{y},\boldsymbol{\eta})|y_i^2+g(\eta_i)=0,\ i=1,\ldots,n\} = \Gamma^n \subseteq \mathbb{C}^n\times\mathbb{C}^n$$

and therefore leave M_g invariant. Restricted to M_g the Hamilton equations become the Drach-Dubrovin equations (3.2.17). Hence, the Hamiltonian system can be integrated by separation of variables as explained in Lemma 3.2.6. In particular, it is completely integrable. On M_g the Hamiltonians H_l are constantly zero. We decompose the polynomial g

from the hyperelliptic spectral curve into scalar energies h_k and a fixed polynomial g_0 of constant parameters by

$$g(E) = g_0(E) + \sum_{k=0}^{n-1} h_k E^{n-1-k} \quad . \tag{3.4.2}$$

Here g_0 has to be a monic polynomial of degree $2n+1$. The decomposition is not unique, since g_0 might also contain powers of E less than n. With the vector of energies defined by $h = (h_0, \ldots, h_{n-1})$, then

$$S_{g_0}(\boldsymbol{\eta}, h) = \sum_{l=1}^{n} \int^{\eta_l} y \mathrm{d}\eta \tag{3.4.3}$$

is a *complete solution* [4] of each Hamilton-Jacobi equation

$$0 = H_l \left(\partial_{\boldsymbol{\eta}} S_{g_0}, \boldsymbol{\eta} \right) \quad .$$

The complete solution at hand is a sum in which the l-th term only depends on the l-th position η_l, so the mechanical description of the $(2n+1)$-stationary KdV hierarchy is completely separable in an additive way. This corresponds to solving the Drach-Dubrovin equations by separation of variables, see Section 3.2.3 above.

For the complex manifold M_g there is a real submanifold preserved under Hamiltonian flows [**68**]. Let $\alpha_j \subseteq \Gamma$ (for $j = 1, \ldots, n$) denote the cycle in Γ covering the *gap* $[\gamma_{2j}, \gamma_{2j+1}]$ and oriented clockwise. The polynomial $-g$ takes non-negative values on gaps, so $y = (-g(\eta))^{1/2}$ is real and so are the cycles α_j. For $(y_j, \eta_j) \in \alpha_j$ initially, we obtain from Proposition 3.2.5 that the Hamiltonian dynamics takes place on the compact and real phase torus $\alpha_1 \times \cdots \times \alpha_n \subseteq \Gamma^n$. The *actions* of the system are

$$I_j(h, g_0) = \int_{\alpha_j} y \mathrm{d}E \tag{3.4.4}$$

where $y^2 + g(E) = 0$. By construction the actions are constants of motion for the Hamiltonian dynamics. The real vectors $I = (I_1, \ldots, I_n)$ are called the *action variables* of the Hamiltonian system.

Proposition 3.4.1. *Let g_0 be as in* (3.4.2) *and fixed. Then the map $I : \mathbb{R}^n \to \mathbb{R}^n$, $h = (h_0, \ldots, h_{n-1}) \mapsto I(h, g_0)$ is locally a diffeomorphism.*

Proof. The claim follows by the inverse function theorem, if the Jacobian matrix of $h \mapsto I(h, g_0)$ is invertible at each point h and depends continuously on h. For the decomposition of g in (3.4.2) we compute

$$\partial_h I^t = \left(-\frac{1}{2} \int_{\alpha_j} \eta^{n-1-l} \frac{\mathrm{d}\eta}{y} \right)_{l=0,j=1}^{n-1,n} \quad . \tag{3.4.5}$$

To see that this matrix is invertible at h, we assume a row vector $v = (v_0, \ldots, v_{n-1})$ to be in the cokernel of the Jacobian matrix, that is $v \partial_h I^t = 0$. Equivalently, all a-periods of the

[4] Note that S_{g_0} is only locally a function, but globally multivalued. Related to the complete solution are Hamilton's principal function and Hamilton's characteristic function. Due to independence of time and $H_l = 0$ on M_g, the latter two coincide here.

holomorphic differential form

$$\sum_{l=0}^{n-1} v_l \eta^{n-1-l} \frac{\mathrm{d}\eta}{y}$$

vanish. This means that the differential form has to be zero. Hence, the vector v is zero and the Jacobian matrix is invertible. When taking higher derivatives, the integrand of (3.4.5) produces poles, without residues on Γ, however. Therefore higher derivatives exist and the map $h \mapsto I(h, g_0)$ is smooth. $\square$

With the local parametrization $h \mapsto I(h, g_0)$, the complete solution $S_{g_0}(\boldsymbol{\eta}, h(I))$ locally generates a canonical transformation to coordinates (I, Φ) with conjugate phases $\Phi = \partial_I S_{g_0}$, see [**4**]. In the following lemma the coordinates (I, Φ) turn out to be defined globally in the *angle variables* Φ. These coordinates are called *action-angle variables*.

Lemma 3.4.2. *The angle variables Φ are the real part of the Abel map on M_g thus they are defined on the real part of the Jacobi torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$. More explicitly, $\mathrm{d}\Phi_j(\boldsymbol{\eta}) = \sum_{k=1}^{n} \mathrm{d}\omega_j(\eta_k)$ for differential forms*

$$\mathrm{d}\omega_j(\eta) := \partial_{I_j} y \mathrm{d}\eta = -\frac{1}{2}\sum_{l=0}^{n-1} \partial_{I_j} h_l \frac{\eta^{n-1-l}\mathrm{d}\eta}{y} \tag{3.4.6}$$

which are holomorphic on Γ and satisfy $\int_{\alpha_k} \mathrm{d}\omega_j = \delta_{kj}$.

Proof. Using that the actions are constants of motion, we have for $j = 1, \dots, n$

$$\mathrm{d}\Phi_j = \mathrm{d}\left(\partial_{I_j} S_{g_0}\right) = \partial_{I_j} \mathrm{d}S_{g_0} = \sum_{k=1}^{n} \partial_{I_j} y_k \mathrm{d}\eta_k = \sum_{k=1}^{n} \mathrm{d}\omega_j(\eta_k) \ .$$

It is left to be shown that $\partial_{I_j} y \mathrm{d}\eta$ is the holomorphic differential form in (3.4.6). By construction $\partial_{I_j} g = \partial_{I_j} \sum_{l=0}^{n-1} h_l \eta^{n-1-l}$ and therefore we get

$$\partial_{I_j} y \mathrm{d}\eta = -\frac{1}{2}\partial_{I_j} g \frac{\mathrm{d}\eta}{y} = -\frac{1}{2}\sum_{l=0}^{n-1} \partial_{I_j} h_l \frac{\eta^{n-1-l}\mathrm{d}\eta}{y} \ .$$

The representation (3.4.4) of the actions and the definition of a-cycles yield

$$I_j(h, g_0) = 2\int_{\gamma_{2j}}^{\gamma_{2j+1}} y \mathrm{d}\eta \ .$$

Since each γ_l is a zero of y this implies

$$\delta_{jk} = \partial_{I_j} I_k = 2y|_{\gamma_{2k+1}} \partial_{I_j} \gamma_{2k+1} - 2y|_{\gamma_{2k}} \partial_{I_j} \gamma_{2k} + \int_{\alpha_k} \partial_{I_j} y \mathrm{d}\eta = \int_{\alpha_k} \mathrm{d}\omega_j$$

thus the a-periods of $\mathrm{d}\Phi_j$ are in $\mathbb{Z}$. $\square$

In action-angle variables the Hamiltonian flows become linear.

Corollary 3.4.3. *On $\mathbb{T}^n$ the Hamiltonian flows starting at Φ_j^0 are given by*

$$\Phi_j = \Phi_j^0 + \sum_{l=0}^{n-1} t_{2l+1} \partial_{I_j} h_l \mod \mathbb{Z} \ . \tag{3.4.7}$$

Note, $(\partial_{I_1} h_l, \dots, \partial_{I_n} h_l)$ are called the frequencies of the conditionally periodic motion in time t_{2l+1}.

Proof. Let us begin with the Hamiltonian dynamics in the form of solutions of the Drach-Dubrovin equations in Lemma 3.2.6

$$t_{2l+1} + b_{2l+1} = -\frac{1}{2}\sum_{k=1}^{n}\int_{\eta_k(0)}^{\eta_k(\boldsymbol{t})}\frac{\eta^{n-1-l}\mathrm{d}\eta}{y}$$

for $l = 0, \dots, n-1$. On the other hand, by integrating the differential form $\mathrm{d}\Phi$ in Lemma 3.4.2 we arrive at

$$\Phi_j = \sum_{k=1}^{n}\int_{\eta_k(0)}^{\eta_k(\boldsymbol{t})}\mathrm{d}\omega_j(\eta_k) = -\frac{1}{2}\sum_{l=0}^{n-1}\partial_{I_j}h_l\sum_{k=1}^{n}\int_{\eta_k(0)}^{\eta_k(\boldsymbol{t})}\frac{\eta^{n-1-l}\mathrm{d}\eta}{y} \ .$$

Combined, the previous two equations yield (3.4.7) on the phase torus $\mathbb{T}^n$. The starting point of the flow is given by $\Phi_j^0 = \sum_{l=0}^{n-1} b_{2l+1}\partial_{I_j}h_l$. □

As a consequence, $\boldsymbol{\eta} = (\eta_1, \dots, \eta_n)$ (or equivalently the coefficients of the resolvent $\hat{B}$) may be written as functions on the torus $\mathbb{T}^n$ composed with the angle variables Φ. This was first observed by Its and Matveev [**29**] for the potential of Hill's operator (see (3.2.8)) and later extended to the wave function in (3.1.1). The functions on the torus can be expressed in terms of the Riemann theta function.

Periodic functions composed with a linear function are the setting to apply the Krylov-Bogoliubov averaging method [**43, 44, 62**]. For the KdV hierarchy this was done in [**10**]. In [**38**] Krichever applied averaging techniques to the more general KP hierarchy. The equations obtained that way are called *Whitham equations*. However, in the following we are going to obtain the KdV Whitham equations in a different way by applying adiabatic theory to the stationary KdV hierarchy in its formulation as a Hamiltonian system.

CHAPTER 4

Whitham Deformations of Stationary KdV

A classical application of perturbation theory is the study of a non-integrable n-body system as the perturbation of several uncoupled integrable(!) two-body systems where the parameter of the perturbation controls the coupling. The perturbation parameter itself might also depend on time, e.g. imagine that the rope length of a simple gravity pendulum changes in time, see Section 52 in [**4**] and Example 4.1.1. This yields a time-dependent Hamiltonian system. An interesting question is, how the trajectories and integrals of motion of such a system behave.

Let us consider a completely integrable Hamiltonian system with dynamics in time t and parameters. Then assume these parameters become dependent on time by making them functions of $T := \epsilon t$, with a "small" rate $\epsilon > 0$. The trajectories of the resulting time-dependent Hamiltonian system then depend on t and ϵ or, what is the same, t and T. For an approximate description we use the representation of the time-independent completely integrable Hamiltonian system in action-angle variables. This provides a decoupling of the dynamics into a "slow" modulation of the actions and a "fast" dynamics of the angles [1]. In order to extract the slow modulation, usually the fast dynamics is averaged out, see for example [**38**]. This works generally for coordinates of the phase space that consist of integrals of motion and their conjugates. When taking the action variables as integrals of motion, the adiabatic theorem, explained in Section 4.1, allows to bypass explicit averaging and to obtain directly that the actions vary only "little" under time-dependent perturbation with rate ϵ. What is meant here by little, is quantified in the definition of *adiabatic invariance* formulated in Section 4.1.

In the following Section 4.2 the adiabatic theorem is applied to the Hamiltonian formulation of the stationary KdV hierarchy from Section 3.4, resulting in the observation that under the described perturbation the spectral curve $\Gamma = \{(E, y) \mid 0 = y^2 + g(E)\} \subseteq \mathbb{C}^2$ modulates "slowly" up to small "fast" oscillations.

We call a "slow" perturbation in time T a *Whitham deformation*, if it results in constant actions $I_j = \int_{\alpha_j} \sqrt{-g}\mathrm{d}E$ (for $j = 1, \ldots, n$). They can be found by demanding

$$\partial_T I_j = 0 \ . \tag{4.0.1}$$

It is worth to point out here that in the case $n = 1$ (1-gap) the action function I_1 was already used by Whitham in [**69**]. See also [**31**] for this observation.

When considering multiple integrable flows in times t_{2k+1} and their perturbed non-integrable counterparts, the previous equation has to hold for several "slow" times T_{2k+1}. Consequently, in the sense of Section 2.2.5 the differential form $\mathrm{d}S := \sqrt{-g(E)}\mathrm{d}E$ on the spectral curves appears as a generating differential form for the system

$$\partial_{T_{2j+1}} \mathrm{d}\Omega_i = \partial_{T_{2i+1}} \mathrm{d}\Omega_j \tag{4.0.2}$$

[1] Only variations on the "slow" time scale are called *modulations*, or redundantly *slow modulations* sometimes.

with $\mathrm{d}\Omega_k := \partial_{T_{2k+1}}\mathrm{d}S$. In order to meet the usual normalization of the differential forms in the Whitham hierarchy, particular Whitham deformations can be found. In [**25**] such differential forms are postulated as the starting point for the study of Whitham equations in the form (4.0.2). There are more general solutions to these equations than those given by the generating differential form here. In Section 5.3 an Ansatz will be presented that generalizes the above form of $\mathrm{d}S$.

Finally, in Section 4.3 the Whitham deformations are customized to modulate soliton solutions of the KdV hierarchy.

4.1. Perturbations and Adiabatic Invariants

Averaging of perturbed integrable Hamiltonian systems is a key method in perturbation theory. It allows to identify adiabatic invariants. For systems with one phase (i.e. one degree of freedom) this was explained by Arnold in [**4**]. In the higher dimensional multiphase case not all initial conditions are admissible any more, but the measure of the exceptional set can be controlled. This and the question of optimal results was developed by Neistadt, based upon ideas by Kasuga, see [**48**]. Without giving details, the present section follows Arnold's description in order to motivate Neistadt's more general result on adiabatic invariants which is applicable to the Hamiltonian system of the stationary KdV hierarchy from Section 3.4.

Given a completely integrable Hamiltonian $H(p,q)$ with action variables $I(p,q)$, then locally there is a canonical transformation $(p,q) \mapsto (I,\Phi)$ such that Hamilton's equations are transformed in the following way

$$\begin{cases} p' = -\partial_q H \\ q' = \partial_p H \end{cases} \mapsto \begin{cases} I' = 0 \\ \Phi' = \omega(I) \end{cases}$$

where the frequencies are given as $\omega = \partial_I H_0$ for the transformed Hamiltonian $H_0(I)$. In coordinates (p,q) of the $2n$-dimensional phase space $\mathbb{R}^n \times \mathbb{R}^n$ Lagrangian submanifolds are described by n constant energies. Expressed in local coordinates (I,Φ) a Lagrangian submanifold becomes a subset of $\{I\} \times \mathbb{R}^n$ and is called a *phase torus*. The moduli space of phase tori is parameterized by I. If angle variables Φ exist globally and the phase torus is compact, then they are defined on the torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$.

Provided a completely integrable Hamiltonian system is already given globally in action-angle variables, then perturbations of the following form can be considered

$$\begin{cases} I' = \epsilon f_1(I,\Phi) \\ \Phi' = \omega(I) + \epsilon f_2(I,\Phi) \end{cases}$$

with smooth functions f_1 and f_2 and small $\epsilon \geq 0$. By the *averaging principle* (see Section 52B in [**4**]) we are led to the differential equation

$$J' = \langle f_1 \rangle(J) \tag{4.1.1}$$

with $\langle f_1 \rangle(-) = \int_{\mathbb{T}^n} f_1(-,\Phi)\,\mathrm{d}vol_\Phi$. An *averaging-type theorem* is a theorem stating that for all initial conditions outside a (small) set, the solution I for the perturbed system stays close to the solution J for the averaged system for all times in the interval $[0,1/\epsilon]$. Section 52C in [**4**] is about an averaging theorem for systems with one phase. Under which assumptions such a theorem holds more generally is presented in detail in Chapter 6 in [**48**].

Given a family of completely integrable Hamiltonians $H(p,q|g_0)$ with parameter g_0, then a canonical transformation to action-angle variables will also depend on the parameter. Such a transformation is generated by a (multivalued) function $S = S(I,q|g_0)$. Here we assume the map $(I, g_0) \mapsto \omega(I, g_0)$ to the frequencies to have full rank, which is Kolmogorov's non-degeneracy condition. When the parameter changes "slowly" over time, i.e. $g_0 = g_0(\epsilon t)$, then the transformed Hamiltonian is $K(I, \Phi | g_0) = H(p, q | g_0) + \partial_t S$ with H independent of Φ and we arrive at the perturbed system [2]

$$\begin{cases} I' = -\partial_\Phi K = -\epsilon g_0' \partial_\Phi \partial_{g_0} S \\ \Phi' = \partial_I K = \omega + \epsilon g_0' \partial_I \partial_{g_0} S \end{cases} .$$

Applying an averaging-type theorem to this system (and Stokes' theorem on $\mathbb{T}^n$) gives

$$J' = -g_0' \langle \partial_\Phi \partial_{g_0} S \rangle (J) = 0$$

and results in the following *adiabatic theorem* (see Chapter 9.2 in [**48**]): Let V be the set of non-degenerate phase tori (parameterized by I) and respective initial values Φ_0 for the phase flow; let furthermore ρ be a continuous function satisfying $\sqrt{\epsilon} \leq \rho(\epsilon)$. Then for all $\epsilon > 0$ sufficiently small, there is a partition $V = V' \cup V''$ such that (a) for each initial condition in V' we have for the action $I_\epsilon(t)$ along trajectories of the time-dependent Hamiltonian system

$$\sup_{t \in [0, 1/\epsilon]} \| I_\epsilon(t) - I_\epsilon(0) \| \leq \rho(\epsilon) \tag{4.1.2}$$

and (b) the measure of V'' is of order $\sqrt{\epsilon}/\rho(\epsilon)$. A quantity with property (a) is called *adiabatic invariant*.

Typically the function ρ is chosen such that $\lim_{\epsilon \to 0} \rho(\epsilon) = 0$ and $\lim_{\epsilon \to 0} \sqrt{\epsilon}/\rho(\epsilon) = 0$, e.g. $\rho(\epsilon) = \epsilon^{1/3}$. Hence, the smaller the perturbation rate ϵ, (a) the less an adiabatic invariant varies along a trajectory of the time-dependent Hamiltonian, and (b) the smaller the exceptional set V'' becomes. Roughly speaking, the perturbed system is a mix between the "fast" oscillations on the phase torus (Arnold-Liouville torus) of the integrable system and the "slow" modulation/deformation of this torus. The "slow" modulations are obtained by averaging over the "fast" oscillations. In other contexts this operation can appear as a projection, e.g. the semi-classical limit or the dispersionless limit in [**63**].

We end this section by discussing the aforementioned pendulum with slowly changing rope length.

Example 4.1.1 (see Section 52 in [**4**])**.** We describe the motion of a simple gravity pendulum by the angle between the rope and the vertical axis as position variable q and use the usual small angle approximation. So the Hamiltonian is given by

$$H = \frac{1}{l^2} \frac{p^2}{2} + lg \frac{q^2}{2} ,$$

where l is the rope length and g is the gravitational constant. At energy $h > 0$, the action I of this system is the area of the ellipse $E : \{H \leq h\}$ in the phase space, that is

$$I = \int_{\partial E} p \mathrm{d}q = \int_E \mathrm{d}p \wedge \mathrm{d}q = area(E) = 2\pi h \sqrt{\frac{l}{g}} .$$

Clearly, the frequency $\partial_I h = \frac{1}{2\pi}\sqrt{g/l}$ has full rank as a map in either l or g. Now, let the length of rope double over time, i.e. $g_0 := l(\epsilon t) = l_0(1 + \epsilon t)$ with $\epsilon > 0$ and $t \in [0, 1/\epsilon]$. When we start the time-dependent Hamiltonian system with an initial action I_0, then as an adiabatic invariant the action varies only little while we change the rope length at small speed ϵl_0 over the time interval $t \in [0, 1/\epsilon]$. Asymptotically for $\epsilon \to 0$ the action is constant.

[2] It has to be checked that the derivative $\partial_{g_0} S$ is single-valued.

However, what happens to the energy h meanwhile? From its relation to the action we get asymptotically for $\epsilon \to 0$ and $T := \epsilon t \in [0,1]$ fixed

$$h_\epsilon(t) = \sqrt{\frac{g}{l(T)}}\frac{I_\epsilon(t)}{2\pi} \to \sqrt{\frac{g}{l(T)}}\frac{I_0}{2\pi} \ . \tag{4.1.3}$$

Hence, while l doubles (i.e. $l(1) = 2l(0)$), asymptotically the energy decreases slowly by a factor of $\sqrt{2}$. By putting the pendulum in an elevator slowly accelerating downwards (that is g declines slowly) the energy decreases as well.

4.2. Adiabatic Invariants and Whitham Deformations of Stationary KdV

In this section we consider "slow" perturbations of the stationary KdV hierarchy understood as the completely integrable Hamiltonian system (3.4.1) with energies in the vector $h = (h_0, \dots, h_{n-1})$ and parameter g_0 (see (3.4.2))

$$g(E) = g_0(E) + \sum_{k=0}^{n-1} h_k E^{n-1-k} \ . \tag{4.2.1}$$

The monic polynomial g_0 of degree $2n+1$ is given by $2n+1$ coefficients which we consider as parameters. Out of the n commuting Hamiltonian flows of the system, at first we consider only perturbations in a single one. Let t denote the time of this flow. Modulating the parameters g_0 in time t with rate $\epsilon \geq 0$, that is $t \mapsto g_0(\epsilon t)$, means to modulate the spectral curve $\Gamma : \{y^2 + g(E) = 0\}$. As a result of the perturbation, the dynamical system becomes non-integrable. However, each state of the system in space and time can be seen as the initial state of an integrable system with the current parameters. Hence, along a trajectory of the non-integrable system its current energies and actions can be obtained by the formulas for the integrable system at the corresponding point in phase space and parameter space. Thereby the energies and actions depend on ϵ and t, i.e. $h = h_\epsilon(t)$ and $I = I_\epsilon(t)$.

In Corollary 3.4.3 the frequencies of the l-th Hamiltonian flow are given by $\partial_I h_l$. For each $l \in \{0, \dots, n-1\}$ Kolmogorov's non-degeneracy condition requires the frequencies to have full rank as a function in (g_0, I). In Proposition 3.4.1 however, the actions I are rather described as a function in (g_0, h), which makes it complicated to compute the Hessian $\partial_I^2 h_l$. The order 3 tensor $\partial_h^2 I$ is more accessible. It can be shown by counting the dimension of the space of certain meromorpic differential forms on Γ (as done in the proof of Proposition 3.4.1) that its hyperdeterminant does not vanish. Equivalently, the hyperdeterminant of $\partial_I^2 h$ does not vanish either (see Chapter 14 in [**27**] [3]). This implies for $\partial_I^2 h_l$ to have full matrix-rank, yielding Kolmogorov's non-degeneracy. Hence, the action along trajectories of the time-dependent Hamiltonian system is an adiabatic invariant: for $\rho(\epsilon) \geq \sqrt{\epsilon}$ with $\rho(\epsilon) \to 0$ and $\sqrt{\epsilon}/\rho(\epsilon) \to 0$ as $\epsilon \to 0$ we have

$$\sup_{T \in [0,1]} \| I_\epsilon(T/\epsilon) - I_\epsilon(0) \| \leq \rho(\epsilon) \tag{4.2.2}$$

for initial conditions outside a set with measure of order $\sqrt{\epsilon}/\rho(\epsilon)$, see (4.1.2). Here, the adiabatic invariance is expressed in the "slow" version $T = \epsilon t \in [0,1]$ of the time t. As a consequence we have for initial conditions with action I_0 independent of ϵ that generically the ϵ-family of functions $T \mapsto I_\epsilon(T/\epsilon)$ approximates the constant I_0 uniformly on $[0,1]$ as ϵ goes to zero.

[3] See also https://math.stackexchange.com/questions/4401956/on-hessians-of-inverse-vector-functions

For the modulated pendulum in Example 4.1.1 we found in (4.1.3) that asymptotically the energy $h_\epsilon(t)$ changes at the rate of the modulation, that is, asymptotically a function $h = h(T)$ approximates $h_\epsilon(t)$. In order to get an analogous result for the modulations of the stationary KdV hierarchy, we study the relation between the actions

$$I_j = \int_{\alpha_j} \sqrt{-g}\mathrm{d}E$$

(for $j = 1, \dots, n$) and the energies h in more detail.

Proposition 4.2.1. *Let g_0 be a monic polynomial of degree $2n+1$ and $h \in \mathbb{R}^n$ (see (4.2.1)). Then the map*

$$\Xi : (g_0, h) \mapsto (g_0, I)$$

is smooth and a local diffeomorphism outside the set of lower dimension where $g_0(E) + \sum_{k=0}^{n-1} h_k E^{n-1-k}$ has multiple zeros. In particular, for a smooth modulation $T \mapsto g_0(T)$, (generically) the map

$$(T, I) \mapsto h(T, I) = \mathrm{proj}_h \left[\Xi^{-1}(g_0(T), I)\right]$$

is locally defined and smooth.

Proof. The Jacobian matrix of Ξ is given by

$$\partial_{g_0,h}\Xi = \left(\begin{array}{c|c} \mathbb{1} & 0 \\ \hline \star & \partial_h I \end{array}\right) \ .$$

By Proposition 3.4.1 the matrix $\partial_h I$ is invertible, under the condition that $g(E) = g_0(E) + \sum_{k=0}^{n-1} h_k E^{n-1-k}$ has no multiple zeros and therefore defines a hyperelliptic surface. Hence, $\partial_{g_0,h}\Xi$ is invertible and the inverse function theorem yields the statement. □

Now we can compare the energies $h(T) := h(T, I_0)$ and $h_\epsilon(T/\epsilon) = h(T, I_\epsilon(T/\epsilon))$ by using the mean value inequality.

Lemma 4.2.2. *Let $r > 0$ and $T \mapsto g_0(T)$ be a modulation such that Ξ^{-1} is defined on an open set including $g_0([0,1]) \times \overline{B}_r(I_0)$. Then we have $\sup_{T\in[0,1]}\|h_\epsilon(T/\epsilon) - h(T)\| \in \mathcal{O}(\rho(\epsilon))$ outside the exception sets of (4.2.2).*

Proof. By hypothesis, the energies h in Proposition 4.2.1 depend smoothly on $(T, I) \in [0,1] \times B_r(I_0)$. Hence, for $T \in [0,1]$ and $I_1, I_2 \in B_r(I_0)$ we have

$$\|h(T, I_2)) - h(T, I_1)\| \leq \sup_{(T,I)\in\overline{(T,I_2)(T,I_1)}} \|D_{(T,I)}h|_{(T,I)}\| \|I_2 - I_1\| \ ,$$

where the matrix norm is induced by the norm on $\mathbb{R}^{n+1}$ and $\overline{(T, I_2)(T, I_1)}$ denotes the straight line between the two points. Since $[0,1] \times \overline{B}_r(I_0)$ is compact and the derivative $D_{(T,I)}h$ is continuous, we have

$$M := \sup_{(T,I)\in[0,1]\times\overline{B}_r(I_0)} \|D_{(T,I)}h|_{(T,I)}\| < \infty \ .$$

Using the adiabatic invariance of the action, we find $\epsilon > 0$ small enough such that $I_1 = I_\epsilon(T/\epsilon)$ and $I_2 = I_0$ have distance less than r for all $T \in [0,1]$. Then for all $T \in [0,1]$ we have

$$\|h_\epsilon(T/\epsilon) - h(T)\| \leq M \sup_{T\in[0,1]} \|I_\epsilon\,(T/\epsilon) - I_0\| \leq M\rho(\epsilon) \ ,$$

where the second estimate follows again from the adiabatic invariance of the action. □

This lemma implies that $h(T)$ approximates $h_\epsilon(T/\epsilon)$ uniformly (outside some small exception sets of initial values), which is what we found before for the modulated pendulum in Example 4.1.1.

In summary, we consider the Hamiltonian system (3.4.1) that becomes time-dependent by modulating its parameters g_0 by $[0,1] \ni T = \epsilon t \mapsto g_0(T)$. For initial values of this system that lie outside small exception sets, we have that the actions and energies remain close to a constant vector I_0 and a function $h(T)$, respectively. This approximation is uniform on $[0,1]$. The approximate values $h_k(T)$ correspond to constant a-periods I_0 of $\sqrt{-g}\mathrm{d}E$ where

$$g(T) = g_0(T) + \sum_{k=0}^{n-1} h_k(T)E^{n-1-k} \quad ,$$

see Proposition 4.2.1. Conversely, given a function $h(T)$ and a modulation of g_0 such that the resulting a-periods I are constant in T, then for any generic initial value of the Hamiltonian system with energies $h(0)$, we get that its actions and energies converge uniformly to I and $h(T)$, respectively.

In order to avoid the implicit genericity assumption on g_0 and h to yield a polynomial g with only simple zeros, it is convenient to directly consider deformations of g which avoid multiple zeros of g as a polynomial in E.

Definition 4.2.3. For the completely integrable Hamiltonian system of stationary KdV a deformation $T := \epsilon t \mapsto g(T)$ is called *Whitham deformation*, if the resulting variation of the action $T \mapsto I(T)$ is constant.

Equivalently, the condition when a smooth deformation of g yields a Whitham deformation is given by the *Whitham deformation equation*

$$0 = \partial_T I \quad .$$

Whitham deformations can also be considered when modulating g in all times $(t_{2l+1})_{l=0}^{n-1}$ corresponding to Hamiltonian flows. Note here that the time t_{2n+1} and thereby all higher times were assumed trivial in the $(2n+1)$-stationary case, however, the way in which $\partial_{t_{2n+1}} G$ in (3.2.1) is a linear combination of lower times can be modulated over time. Hence, we include perturbations in time t_{2n+1}, that is altogether

$$\boldsymbol{T} := \epsilon(t_{2l+1})_{l=0}^{n} \mapsto g(\boldsymbol{T}) \quad .$$

Now Whitham deformations are obtained, if the a-periods of $y\mathrm{d}E$ (with $y^2 = -g(E)$) are constant, or equivalently if the deformation equations

$$0 = \partial_{T_{2l+1}} I_j = \partial_{T_{2l+1}} \int_{\alpha_j} \sqrt{-g(E)}\mathrm{d}E \tag{4.2.3}$$

are satisfied for all $l = 0, \dots, n$ and $j = 1, \dots, n$. In the following example (mentioned before in the introduction in Chapter 1) we discuss how to find Whitham deformations by using Whitham deformation equations.

Example 4.2.4 (Whitham deformation of 1-gap KdV solutions)**.** In the case $n = 1$ we take as an Ansatz for the deformation

$$g = E^3 + TE^2 + XE + h(X,T) \tag{4.2.4}$$

where we write $T_1 = X$ and $T_3 = T$ for convenience. The system of equations for Whitham deformations (4.2.3) then reads

$$\begin{aligned} 0 =& \partial_X I_1 = -\frac{1}{2}\int_{\alpha_1} (E + \partial_X h)\frac{\mathrm{d}E}{y} \ , \\ 0 =& \partial_T I_1 = -\frac{1}{2}\int_{\alpha_1} (E^2 + \partial_T h)\frac{\mathrm{d}E}{y} \ . \end{aligned}$$

These equations of elliptic integrals determine $\partial_X h$ and $\partial_T h$. In order to see that a function h with these derivatives exists at least locally, we have to show $\partial_T \partial_X h = \partial_X \partial_T h$. From $\partial_T \partial_X I_1 = 0 = \partial_X \partial_T I_1$ and

$$0 = \partial_X \partial_T I_1 = -\frac{1}{2}\int_{\alpha_1} \partial_X \partial_T h \frac{\mathrm{d}E}{y} + \frac{1}{4}\int_{\alpha_1} \frac{\partial_X g \partial_T g}{g}\frac{\mathrm{d}E}{y}$$

we get (since the second term of the sum is symmetric in X and T)

$$\int_{\alpha_1} \partial_T \partial_X h \frac{\mathrm{d}E}{y} = \int_{\alpha_1} \partial_X \partial_T h \frac{\mathrm{d}E}{y} \ .$$

This implies compatibility of the equations for h. Hence, there is an energy function h such that the deformation of parameters (4.2.4) becomes a Whitham deformation.

Given a modulation of g_0, the general system of Whitham deformation equations (4.2.3) for the Ansatz

$$\boldsymbol{T} \mapsto g = g_0(E, \boldsymbol{T}) + \sum_{k=0}^{n-1} h_k(\boldsymbol{T}) E^{n-1-k} \tag{4.2.5}$$

consists of the $n(n+1)$ equations (4.2.3) for the n unknown energies $h_0, \ldots, h_{n-1}$. Their solvability follows in the same way as in Example 4.2.4. By the Whitham deformation equations (4.2.3) we are naturally led to consider meromorphic differential forms on the spectral curve

$$\mathrm{d}\Omega_i := \partial_{T_{2i+1}} (y \mathrm{d}E) = -\frac{1}{2}\partial_{T_{2i+1}} g \frac{\mathrm{d}E}{y} \tag{4.2.6}$$

with vanishing a-periods. Flaschka, Forest and McLaughlin [**25**] formulated the Whitham equations in the framework of differential forms with vanishing a-periods and normalized principal parts at $E = \infty$ as

$$\partial_{T_{2j+1}} \mathrm{d}\Omega_i = \partial_{T_{2i+1}} \mathrm{d}\Omega_j \ . \tag{4.2.7}$$

for all $i, j = 0, \ldots n$. Due to the generating differential form $\sqrt{-g}\mathrm{d}E$ that originates from Whitham deformations, the differential forms $\mathrm{d}\Omega_i$ in (4.2.6) are compatible, so (4.2.7) is satisfied automatically. On the other hand, for the equations (4.2.7) there are more general generating differential forms $\mathrm{d}S$ (i.e. $\partial_{T_{2i+1}} \mathrm{d}S = \mathrm{d}\Omega_i$) that may have higher order poles at infinity and do not have to be defined on the spectral curve. This will be discussed in Section 5.3 for the KdV Whitham hierarchy.

Remark 4.2.5. Let us sketch a more common approach to the KdV Whitham equations that uses averaging more directly and point out some differences to the approach using adiabatic theory. A more general comparison of averaging methods can be found in Paragraph 5 in [**19**].

For the direct approach to averaging, two properties of the stationary KdV hierarchy as formulated in Lemma 3.2.3 are crucial.

- Separation of variables allows to linearize the KdV dynamics, so after Jacobi inversion the resolvent becomes a function $\hat{B} = \hat{B}(\Phi;\Gamma)$ with $\Phi = \omega \boldsymbol{t} + \Phi_0$ on the real part of the Jacobi torus $\mathbb{T}^n$ of the spectral curve Γ, see Lemma 3.2.6. This matches how action-angle variables of the Hamiltonian formulation of the stationary KdV hierarchy are used to obtain Whitham deformations. Considering the spectral curve as dependent on time in a "slow" way, that is $\Gamma = \Gamma(\boldsymbol{T})$ for $\boldsymbol{T} := \epsilon \boldsymbol{t}$, leads to a dynamics on two time scales. Averaging out the "fast" dynamics in $\boldsymbol{t}$ can be replaced by an average over the real part of the Jacobi torus, that is
$$\langle f(\Phi;\Gamma)\rangle := \int_{\mathbb{T}^n} f(\Phi;\Gamma)\,\mathrm{d}vol_\Phi \quad .$$
However, the arguments to justify this are rather involved, see Lemma 3 in [**38**].
- The equations for the higher KdV flows (3.2.6) are in conservation form. Hence, exchanging time derivatives with the average over the real part of the Jacobi torus gives
$$\partial_{T_{2i+1}} \left\langle \frac{\sqrt{-g}}{\hat{B}} \right\rangle = \partial_X \left\langle \frac{\sqrt{-g}B_i}{\hat{B}} \right\rangle \quad , \tag{4.2.8}$$
where $X = T_1$, $\hat{B}$ is a monic polynomial of degree n in E and $B_i = [E^{i-n}\hat{B}]_{\geq 0}$ (in particular $B_0 = 1$). It can be shown that $\Omega_i := \langle \sqrt{-g}B_i/\hat{B}\rangle$ is multivalued on the spectral curve Γ, but taking the derivative on Γ yields $\mathrm{d}\Omega_i$ as a meromorphic differential form on Γ with constant a-periods and a single pole at $E = \infty$. Therefore, (4.2.8) implies that $\mathrm{d}\Omega_i$ satisfies the KdV Whitham equations (4.2.7) [4].

The ODE for the resolvent (3.2.5) is not in the form of a conservation equation, thus averaging does not directly apply as in (4.2.8). In contrast, adiabatic theory applies to the entire stationary KdV hierarchy in its Hamiltonian description in Section 3.4. As a result we rather arrive at the more particular equations (4.2.6) for a generating differential form $\sqrt{-g(E)}\mathrm{d}E$ than at their compatibility equations (4.2.7).

To conclude this section, we prove that Whitham deformations in the form (4.2.5) exist such that the principal parts of the differential forms $\mathrm{d}\Omega_i$ in (4.2.6) are normalized in a way similar to the normalization in Section 2.2.2 (and exactly as in Chapter 5 below) by
$$\mathrm{d}\Omega_i = (E^{i+1} + \mathcal{O}(E^0))\mathrm{d}\xi$$
for the chart $\xi^2 = -1/E$ at the point at infinity $E = \infty$. If the principal part of $\sqrt{-g}\mathrm{d}E$ is given by
$$\left(2E^{n+2} + \sum_{l=0}^{n} T_{2l+1}E^{l+1}\right)\mathrm{d}\xi \quad , \tag{4.2.9}$$
then by the definition in (4.2.6) the differential form $\mathrm{d}\Omega_i = \partial_{T_{2i+1}}(\sqrt{-g}\mathrm{d}E)$ has the desired normalized principal part. That (4.2.9) can be realized as a principal part of $\sqrt{-g}\mathrm{d}E$ by a modulation of g_0 in the Ansatz (4.2.5) is a consequence of the following result.

Proposition 4.2.6. *The principal part of $\sqrt{-g}\mathrm{d}E$ at $E = \infty$ is determined by the parameter g_0 in a polynomial way. Conversely, if $P(E)\mathrm{d}\xi$ denotes the principal part we have*
$$4E^3 g_0(E) = P(E)^2 + \mathcal{O}(E^{n+2}) \quad . \tag{4.2.10}$$

[4] We are going to see in Section 5.1 that the system of Whitham equations (4.2.7) with one time fixed to be X (i.e. $j = 0$), implies the equations for all combinations of times.

That is, P determines the parameter g_0 up to terms of order $n-1$.

PROOF. The differential form $\mathrm{d}E$ becomes $2\xi^{-3}\mathrm{d}\xi$ in the coordinate ξ. Hence, for ξ near 0 (that is for large E), we have

$$\sqrt{-g(E)}\mathrm{d}E = 2E^{n+2}\sqrt{E^{-2n-1}g(E)}\mathrm{d}\xi$$

with the term under the second square root of the form $E^{-2n-1}g(E) = 1 + \sum_{k=1}^{2n+1} c_k E^{-k}$. Using the binomial formula then gives

$$\begin{aligned}\sqrt{-g(E)}\mathrm{d}E &= 2E^{n+2}\sum_{l\geq 0}\binom{1/2}{l}\left(\sum_{k=1}^{2n+1} c_k E^{-k}\right)^l \mathrm{d}\xi \\ &= \left(2E^{n+2}\sum_{l=0}^{n+1}\binom{1/2}{l}\left(\sum_{k=1}^{n+1} c_k E^{-k}\right)^l + \mathcal{O}(E^0)\right)\mathrm{d}\xi\end{aligned}$$

for large E. Therefore the principal part of this meromorphic form is a polynomial in E of degree $n+2$, whose coefficients depend on $c_1, \dots, c_{n+1}$ in a polynomial way. Since $g(E) - g_0(E) \in \mathcal{O}(E^{n-1})$, the coefficients $c_1, \dots, c_{n+1}$ and therefore the principal part depend only on g_0 up to order $n-1$.

Conversely, if P is given such that $\sqrt{-g(E)}\mathrm{d}E = \big(P(E) + \mathcal{O}(E^0)\big)\mathrm{d}\xi$, then taking squares and using $(\mathrm{d}E)^2 = -4E^3(\mathrm{d}\xi)^2$ gives (4.2.10). □

4.3. Whitham Deformations of Solitons

The soliton limit of $(2n+1)$-stationary KdV from Section 3.3 turns the spectral curve into a genus zero surface and therefore the action variables

$$I_j = \int_{\alpha_j} y\mathrm{d}E$$

of the Hamiltonian system in Section 3.4 are trivial. Keeping them trivial under a deformation $\boldsymbol{T} \mapsto g(\boldsymbol{T})$ of the spectral curve $\Gamma = \{(E,y)|0 = y^2 + g(E)\}$ means that the double order of zeros of g has to be preserved. Hence, *Whitham deformations of solitons* are those deformations that preserve the multiplicity of roots of g. Among the corresponding Whitham equations we are going to find the dispersionless KdV equation (1.0.3) from the introduction and the introductory example of Chapter 2.

Remark 4.3.1. The stationary KdV hierarchy is a dynamical system on polynomials $\hat{B}$, see Lemma 3.2.3. Its constants of motion g can be used to form the spectral curve. In the case of solitons, the roots of the polynomial g are directly encoded in the asymptotic behavior of $\hat{B}$ and u by Lemma 3.3.3. Hence, the Whitham deformations of solitons may be understood as deformations of their asymptotic behavior (including $\hat{B}$).

In (4.2.5) we prescribed the higher terms of g and then the preservation of the actions determined the lower order terms. Here, in the soliton case, preservation of the multiplicity of roots can be used in an analogous way.

Proposition 4.3.2. *Let g be a polynomial of the form $g(E) = R(E)^2(E-\gamma)$ for some monic polynomial R of degree n and some $\gamma \in \mathbb{C}$. Then the map*

$$(R,\gamma) \mapsto P = \left[E^{-n}g\right]_{\geq 0} \tag{4.3.1}$$

to the monic polynomials of degree $n+1$ is bijective. Furthermore, the coefficients of P are real if and only if those of R and γ are real.

Proof. Let $R = \sum_{j=0}^{n} r_j E^{n-j}$ with $r_0 = 1$. Then writing g ordered by the powers of E gives

$$(E-\gamma)R^2 = c_0 E^{2n+1} - \gamma c_{2n} E^0 + \sum_{l=1}^{2n} (c_l - \gamma c_{l-1})\, E^{2n+1-l}$$

for $c_l := \sum_{i+j=l} r_i r_j$ and $r_m := 0$ if $m \notin \{0,\ldots,n\}$. Given $P = \sum_{l=0}^{n+1} p_l E^{n+1-l}$ with $p_0 = 1$ we then have

$$P = \left[E^{-n} g\right]_{\geq 0}$$

if and only if $p_l = c_l - \gamma c_{l-1}$ for $l = 0,\ldots,n+1$. For $l = 0$ this is trivial and for $l = 1,\ldots,n$ the equation means

$$p_l = 2r_l - \gamma r_{l-1} + \sum_{i=1}^{l-1} r_i \left(r_{l-i} - \gamma r_{l-1-i}\right) \quad . \tag{4.3.2}$$

Hence, r_l is determined uniquely by p_l and $r_0,\ldots,r_{l-1}$. The equation for $l = n+1$, i.e. $p_{n+1} = c_{n+1} - \gamma c_n$ then determines γ uniquely, if $c_n \neq 0$. Otherwise, the equation for $l = n$ reads $p_n = -\gamma c_{n-1}$ which determines γ, if $c_{n-1} \neq 0$. Otherwise, we consider $p_{n-1} = -\gamma c_{n-2}$. Going through the equations like that, we will arrive at some $c_m \neq 0$ at least when $m = 0$. This determines $\gamma \in \mathbb{C}$ uniquely. Altogether, given P there is a (γ, R) that is mapped to P, which proves the surjectivity of (4.3.1). Since (γ, R) is unique we also get injectivity of this map. From (4.3.2) we see that reality of P implies reality of (γ, R) and vice versa. □

This proposition can be understood in the following way: Given a real and monic polynomial P of degree $n+1$, then there exists a generically unique polynomial Q of degree up to $n-1$ such that $g := E^n P(E) + Q(E)$ has n double zeros and one simple zero. When prescribing a deformation $\boldsymbol{T} \mapsto P(\boldsymbol{T})$ Proposition 4.3.2 can be applied pointwise. In analogy to the deformation (4.2.5), we prescribe in a neighborhood of some initial time $\boldsymbol{\tau}$ the deformation

$$\boldsymbol{T} = (T_{2j+1})_{j=0}^{n} \mapsto P(\boldsymbol{T}) = E^{n+1} + \sum_{j=0}^{n} T_{2j+1} E^j \quad .$$

The initial time $\boldsymbol{\tau}$ is chosen such that locally near $\boldsymbol{\tau}$ the real polynomial Q, which exists due to Proposition 4.3.2, induces a polynomial $g = E^n P + Q$ with n real double roots and one real simple root which is the largest one [5]. This means that the deformation $\boldsymbol{T} \mapsto g(\boldsymbol{T})$ preserves the multiplicity of the roots of g, hence, it is a Whitham deformation of solitons. In this sense Proposition 4.3.2 provides for solitons an analogous version of Proposition 4.2.1. The prescribed modulation P for Whitham deformations of solitons can be normalized, as it was done in (4.2.9) for the generic case.

Lemma 4.3.3. *The deformation $\boldsymbol{T} \mapsto P(\boldsymbol{T})$ given by*

$$P = \left[E^{-n-3}\left(E^{n+2} + \frac{1}{2}\sum_{j=0}^{n} T_{2j+1} E^{j+1}\right)^2\right]_{\geq 0} \tag{4.3.3}$$

[5] A suitable initial time τ exists, since $\boldsymbol{T} \mapsto P(\boldsymbol{T})$ is a surjective map to the monic polynomials of degree $n+1$.

induces a Whitham deformation of n-solitons with asymptotic behavior

$$y\mathrm{d}E = \left(2E^{n+2} + \sum_{j=0}^{n} T_{2j+1}E^{j+1} + \mathcal{O}(E^0)\right)\mathrm{d}\xi \ . \tag{4.3.4}$$

PROOF. Using $y^2 = -g$ and $(\mathrm{d}\xi)^2 = -\frac{1}{4}E^{-3}(\mathrm{d}E)^2$ the square of (4.3.4) gives

$$-g = -\frac{1}{4}E^{-3}\left(\left(2E^{n+2} + \sum_{j=0}^{n} T_{2j+1}E^{j+1}\right)^2 + \mathcal{O}(E^{n+2})\right) \ . \tag{4.3.5}$$

If we write $g = E^nP + Q$ with P and Q polynomials of degree $n+1$ and $n-1$, respectively, then (4.3.5) determines P completely to be (4.3.3). According to Proposition 4.3.2 the polynomial Q is uniquely determined by P and the condition that g can be written as $g = R^2(E-\gamma)$, i.e. that g has generically only double zeros except for a single simple zero. □

Example 4.3.4 (Whitham deformation of 1-solitons)**.** In the case $n = 1$ the deformation $\boldsymbol{T} \mapsto P(\boldsymbol{T})$ from (4.3.3) is given as

$$P = E^2 + TE + \tfrac{1}{4}T^2 + X$$

where we use the notation $X = T_1$ and $T = T_3$. In order to find $Q(E) = q_0 \in \mathbb{R}$ such that $g = EP + Q$ has one real double zero and one real simple zero, we search for zeros of $\partial_E g$ and then choose Q such that g is zero there, too. We have $0 = \partial_E g = 3E^2 + 2TE + \frac{1}{4}T^2 + X$ if and only if

$$E_\pm = -\frac{T}{3} \pm \frac{1}{6}\sqrt{T^2 - 12X} \ .$$

Note here that these roots are real and simple if and only if $T^2 - 12X > 0$. Since we want the smaller root of g to be double, we choose $\gamma_1 := \gamma_2 := E_-$. Then Q is determined by $g(\gamma_1) = 0$, i.e. $q_0 = -\gamma_1 P(\gamma_1)$. The simple root γ_3 of g can be obtained from $q_0 = g(0) = -\gamma_1^2\gamma_3$ as

$$\gamma_3 = -\frac{T}{3} + \frac{1}{3}\sqrt{T^2 - 12X} \ .$$

Altogether, $(X,T) \mapsto (\gamma_1, \gamma_3)$ is a Whitham deformation of 1-solitons with normalization (4.3.4).

As in the case with higher genus, Whitham deformations of solitons yield differential forms

$$\mathrm{d}\Omega_i := \partial_{T_{2i+1}}(y\mathrm{d}E)$$

for $i = 0, \dots, n-1$. They are defined on the compactification Γ_0 of the double covering $\{(E,\upsilon) | \upsilon^2 = \gamma_{2n+1} - E\} \to \mathbb{CP}^1$, $(E,\upsilon) \mapsto E$ and holomorphic except for the point at infinity $0 = \xi^2 = -1/E$. By construction

$$\partial_{T_{2j+1}}\mathrm{d}\Omega_i = \partial_{T_{2i+1}}\mathrm{d}\Omega_j \tag{4.3.6}$$

holds for all $i, j = 0, \dots, n-1$. If the deformation $\boldsymbol{T} \mapsto P(\boldsymbol{T})$ is normalized according to Lemma 4.3.3, then $\mathrm{d}\Omega_i$ has asymptotic behavior

$$\mathrm{d}\Omega_i = \left(E^{i+1} + \mathcal{O}(E^0)\right)\mathrm{d}\xi \ . \tag{4.3.7}$$

Conversely, $y\mathrm{d}E$ in (4.3.4) is one primitive of (4.3.7) among others. Only the differential forms $\mathrm{d}\Omega_i$ on Γ_0 are uniquely determined by the asymptotic behavior and solely depend on γ_{2n+1}, the only parameter of Γ_0.

Proposition 4.3.5. *The normalized differential forms for the Whitham deformation of solitons are given by*

$$\mathrm{d}\Omega_i = -\frac{1}{2}\sum_{k=0}^{i}\binom{1/2}{k}(-\gamma_{2n+1})^k \frac{E^{i-k}\mathrm{d}E}{\sqrt{-E+\gamma_{2n+1}}} \ . \tag{4.3.8}$$

Proof. Meromorphic differentials on Γ_0 with asymptotic behavior (4.3.7) can be written in the form

$$\mathrm{d}\Omega_i = \left(\sum_{l=0}^{i} c_l E^l\right)\frac{\mathrm{d}E}{\upsilon} \tag{4.3.9}$$

with $c_l \in \mathbb{C}$. By using $\mathrm{d}\xi = -\xi/(2E)\mathrm{d}E$ and the expansion $\upsilon\xi = \sqrt{1-\gamma_{2n+1}/E} = \sum_{k\geq 0}\binom{1/2}{k}(-\gamma_{2n+1}/E)^k$ this is equivalent to

$$(E^i + \mathcal{O}(E^{-1}))\sum_{k\geq 0}\binom{1/2}{k}\left(-\frac{\gamma_{2n+1}}{E}\right)^k = -2\left(\sum_{l=0}^{i} c_{i-l}E^{i-l}\right) \ .$$

By comparing the coefficients of E^{i-k} for $k = 0, \ldots, i$ we obtain $\binom{1/2}{k}(-\gamma_{2n+1})^k = -2c_{i-k}$. Substituting this into (4.3.9) then gives (4.3.8). ☐

Note that differential forms $\mathrm{d}\Omega_i$ with asymptotic behavior (4.3.7) exist for all $i \geq 0$, although Whitham deformations only provide them for $i = 0, \ldots, n-1$. The most simple non-trivial case $n = 1$ gives equations for the modulation of 1-solitons.

Example 4.3.6. The differential forms for the Whitham deformations of 1-solitons are given by

$$\mathrm{d}\Omega_0 = -\frac{1}{2}\frac{\mathrm{d}E}{\sqrt{-E+\gamma_3}} \ , \ \mathrm{d}\Omega_1 = -\frac{1}{2}\frac{E-\frac{1}{2}\gamma_3}{\sqrt{-E+\gamma_3}}\mathrm{d}E \ .$$

Their compatibility equation (4.3.6) (in times $X = T_1$ and $T = T_3$) is equivalent to $\partial_T\gamma_3 = \frac{1}{2}\gamma_3\partial_X\gamma_3$, which is just the dispersionless KdV equation (2.0.2). General generic solutions for this equation were described in the introductory example of Chapter 2. One particular solution is given by the second component of $(X, T) \mapsto (\gamma_1, \gamma_3)$ in Example 4.3.4. The first component is not involved. By a limit of higher genus Whitham deformation equations, the equation $\partial_T\gamma_1 = \frac{1}{2}(2\gamma_1 - \gamma_3)\partial_X\gamma_1$ can be found, see Equation (2.90) in [**22**]. The Whitham deformation from Example 4.3.4 satisfies this equation.

In more generality, the limiting case of KdV Whitham equations for solitons is going to be discussed in Section 5.2.

CHAPTER 5

The KdV Whitham Hierarchy

The KdV Whitham hierarchy in its algebraic-geometric form takes values in the parameter space of hyperelliptic curves. In the present chapter, by applying a hydrodynamic reduction we arrive at differential-geometric structures on this parameter space – a flat diagonal Riemannian metric with corresponding flat coordinates and, furthermore, particular Euler-Poisson-Darboux equations (EPDs) that allow to characterize generating differential forms for the KdV Whitham hierarchy.

We start by revising some basic constructions and definitions from the framework of Krichever's KP Whitham hierarchy that contains the KdV Whitham hierarchy as a special type of algebraic orbits, see Section 2.2. Some minor changes are applied here. Let $g(E) = \prod_{j=1}^{2n+1} (E - \gamma_j)$ be a polynomial with real roots $\gamma_1 < \cdots < \gamma_{2n+1}$ and consider the hyperelliptic curve (2.2.23)

$$\Gamma_n = \big\{(E, y) \in \mathbb{C}^2 \mid y^2 + g(E) = 0\big\} \cup \{\infty\}$$

with coordinate $\xi^2 = -1/E$ at the point at infinity and fixed real a-cycles $\alpha_j \subseteq \Gamma_n$ covering the gaps $[\gamma_{2j}, \gamma_{2j+1}]$ and oriented clockwise.

- The corresponding parameter space M_n^{hyp} of hyperelliptic curves has real coordinates $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_{2n+1})$. By an affine transformation γ_1 and γ_2 could be normalized to 0 and 1, respectively. However, unlike in the moduli spaces in Section 2.2 before, conformally equivalent curves are not identified with each other here.
- The coordinate ξ at the point at infinity is used as a reference that allows to compare differential forms on varying curves Γ_n. It is related to the coordinate κ in Example 2.2.4 by $\kappa = \sqrt{-2}/\xi$.
- On a hyperelliptic curve Γ_n of genus n there are meromorphic differential forms $(\mathrm{d}\Omega_i)_{i \geq -n}$ characterized uniquely by
  - $(\mathrm{d}\Omega_i)_{i=-n,\ldots,-1}$ is a basis of the holomorphic differential forms on Γ_n normalized by $\int_{\alpha_j} \mathrm{d}\Omega_i = \delta_{-i,j}$, and
  - $\mathrm{d}\Omega_i$ for $i \geq 0$ is holomorphic except at the point at infinity where it is given by $(E^{i+1} + \mathcal{O}(E^0))\mathrm{d}\xi$ and all a-periods of $\mathrm{d}\Omega_i$ vanish [1].

  For solely practical reasons their normalization and indexing differ from that of the more general KP case. To avoid confusion about the normalization, Greek indizes are used for KP and Latin indices are used for KdV.

As a specification of the universal Whitham hierarchy in Definition 2.2.1 to the KdV setting we have the following.

Definition 5.0.1. The *Whitham hierarchy for KdV* is a system of PDEs for functions $\boldsymbol{T} := (T_{2i+1})_{i \geq -n} \mapsto \boldsymbol{\gamma}(\boldsymbol{T})$ such that for all $i, j \geq -n$ we have

$$\partial_{T_{2j+1}} \mathrm{d}\Omega_i = \partial_{T_{2i+1}} \mathrm{d}\Omega_j \quad . \tag{5.0.1}$$

[1] Oddness with respect to the hyperelliptic involution is implied by the expansion at the point at infinity.

Remark 5.0.2. Recall from Section 2.2.2 that the sections $\Gamma_n \mapsto \mathrm{d}\Omega_i(\Gamma_n)$ in the bundle of meromorphic differential forms over the parameter space M_n^{hyp} exist uniquely as a consequence of Riemann's theorem. Since these sections are also smooth and Γ_n can be represented by $\boldsymbol{\gamma}$, the Whitham equations (5.0.1) implicitly induce PDEs for $\boldsymbol{T} \mapsto \boldsymbol{\gamma}(\boldsymbol{T})$.

We outline a proof that $\boldsymbol{\gamma} \mapsto \mathrm{d}\Omega_i(\boldsymbol{\gamma})$ is smooth for $i \geq -n$. First note that meromorphic differential forms on Γ_n, which are odd with respect to the hyperelliptic involution $\sigma : (E, y) \mapsto (E, -y)$ and holomorphic outside $\xi = 0$, are of the form

$$\mathrm{d}\Omega = C(E)\frac{\mathrm{d}E}{y} \tag{5.0.2}$$

for some polynomial C and $y = \sqrt{-g}$. The pole order of $\mathrm{d}\Omega$ at $\xi = 0$ is $2[\deg_E(C) - (n-1)]$. Let us consider the map Ξ that assigns to (g, C) the principal part of $\mathrm{d}\Omega$ and the a-periods of $\mathrm{d}\Omega$. This is a smooth surjective map (even for fixed g) and $\mathbb{C}$-linear in C. Moreover, if restricted to polynomials C with real coefficients and degree up to a fixed number k, where $k \geq i + n$ if $i \geq 0$ and $k \geq n - 1$ if $i < 0$, then Ξ is a submersion to $\mathbb{R}^{k-(n-1)} \times \mathbb{R}^n$ (by identifying polynomials with their coefficient vector). Hence, if $i \geq 0$, then $\Xi^{-1}(E^{i+1}, 0)$ (and if $i < 0$ then $\Xi^{-1}(0, e_{-i})$) is a smooth submanifold. It can be parametrized as a graph of $g \mapsto C = C(g)$, since $(g, C) \mapsto \Xi(g, C)$ is a submersion already, if g is kept constant.

Starting from the algebraic-geometric description using differential forms $\mathrm{d}\Omega_i$ on hyperelliptic curves, we will study in Section 5.1 the differential-geometric aspects of the KdV Whitham hierarchy that originate from its reduction to a Hamiltonian hydrodynamic system. The geometry appearing here is that of flat and diagonal Riemannian metrics on the real parameter space M_n^{hyp}. Corresponding *orthogonal nets* will be given. In Section 5.2 the KdV Whitham hierarchy of solitons will be introduced and reduced to a semi-Hamiltonian system. Finally, Section 5.3 is dedicated to characterizing generating differential forms for the KdV Whitham hierarchy by solutions of EPDs.

5.1. A Flat Metric on the Parameter Space of Hyperelliptic Curves

The KdV Whitham hierarchy is a special case of an algebraic orbit, hence the hydrodynamic reduction works in the same way as in Section 2.2.4. For the sake of completeness, the statements and proofs are revisited here, using the more explicit representation of meromorphic differential forms $\mathrm{d}\Omega_i$ on hyperelliptic curves by (5.0.2). Corresponding to the hydrodynamic system there are Riemannian metrics on the parameter space M_n^{hyp}. The goal of this section is to show flatness for one of these metrics, i.e. the hydrodynamic reduction of the KdV Whitham hierarchy is a Hamiltonian system of hydrodynamic type.

As an intermediate step (parallel to Proposition 2.2.7 above), the following result explains how taking the derivative of some $\mathrm{d}\Omega_i$ with respect to γ_k is related to evaluating at γ_k.

Lemma 5.1.1. *For the normalized differential forms* $\mathrm{d}\Omega_i$ *(*$i \geq -n$*) and all* $\gamma_1, \ldots, \gamma_{2n+1}$ *holds that*

$$\partial_{\gamma_k}\mathrm{d}\Omega_i \tag{5.1.1}$$

has vanishing a-periods, is holomorphic on $\Gamma_n \backslash \{\gamma_k\}$ *and has principal part* $\mathrm{d}\Omega_i|_{E=\gamma_k}/(2(\gamma_k - E))$*. Conversely, these properties determine* (5.1.1) *uniquely. In particular we have*

$$\mathrm{d}\Omega_i|_{E=\gamma_k}\partial_{\gamma_k}\mathrm{d}\Omega_j = \mathrm{d}\Omega_j|_{E=\gamma_k}\partial_{\gamma_k}\mathrm{d}\Omega_i \ . \tag{5.1.2}$$

PROOF. By construction of $\mathrm{d}\Omega_i$ its a-periods vanish and its principal part at the point at infinity is constant. Therefore $\partial_{\gamma_k}\mathrm{d}\Omega_i$ has vanishing a-periods and vanishing principal part at infinity. For describing the principal part at $E=\gamma_k$ consider

$$\mathrm{d}\Omega_i = C_i(E)\frac{\mathrm{d}E}{y} \tag{5.1.3}$$

for some uniquely given polynomial C_i of degree $n+i$ if $i \geq 0$ and degree up to $n-1$ if $i<0$. The derivative of $1/y$ with respect to γ_k is

$$\partial_{\gamma_k}\left(\frac{1}{y}\right) = \partial_{\gamma_k}\left(\prod_{j=1}^{2n+1}(E-\gamma_j)^{-1/2}\right) = \frac{1}{2(E-\gamma_k)}\frac{1}{y} \tag{5.1.4}$$

and therefore we have

$$\partial_{\gamma_k}\mathrm{d}\Omega_i = \left(\frac{C_i}{2(E-\gamma_k)} + \partial_{\gamma_k}C_i\right)\frac{\mathrm{d}E}{y} = \left(\frac{1}{2(E-\gamma_k)} + \frac{\partial_{\gamma_k}C_i}{C_i}\right)\mathrm{d}\Omega_i \ . \tag{5.1.5}$$

Now, the principal part at $E=\gamma_k$ can be read off as $\mathrm{d}\Omega_i|_{E=\gamma_k}/(2(E-\gamma_k))$. By Riemann's theorem the a-periods and principal parts determine (5.1.1) uniquely. Furthermore, the factors in (5.1.2) are chosen such that the principal parts on both sides coincide (while all a-periods still vanish). Hence, the differential forms are the same. □

As a corollary (parallel to Lemma 2.2.8 with identical proof) the Whitham equations (5.0.1) are equivalent to a system of first order quasi-linear PDE in diagonal form.

Corollary 5.1.2 (Hydrodynamic Reduction)**.** *The Whitham equation* (5.0.1) *is equivalent to*

$$\mathrm{d}\Omega_i|_{E=\gamma_l}\partial_{T_{2j+1}}\gamma_l = \mathrm{d}\Omega_j|_{E=\gamma_l}\partial_{T_{2i+1}}\gamma_l \tag{5.1.6}$$

for all $l=1,\ldots,2n+1$.

Under the genericity assumption $\mathrm{d}\Omega_0|_{E=\gamma_l} \neq 0$ for all $l=1,\ldots,2n+1$, the equations (5.1.6) for $i=0$ and $j \geq -n$ imply the equations for all $i,j \geq -n$. Hence, the KdV Whitham hierarchy takes the form of a diagonal $1+1$-dimensional system of hydrodynamic type

$$\partial_{T_{2k+1}}\gamma_l = v_l^{(k)}\partial_{T_1}\gamma_l \tag{5.1.7}$$

for velocities $v_l^{(k)} := v^{(k)}(\gamma_l)$ given by the meromorphic functions $v^{(k)}(E) = \mathrm{d}\Omega_k/\mathrm{d}\Omega_0$ on $\mathbb{C}$. The ramification points $\gamma_1,\ldots,\gamma_{2n+1}$ of the spectral curve are the *Riemann invariants* here. They were found in [**69**] in the case of one phase (i.e. $n=1$) and generalized to the multiphase case in [**25**].

As a second consequence (partly parallel to Corollary 2.2.10) we get that the normalized differential forms of the KdV Whitham hierarchy satisfy two types of Laplace equations.

Lemma 5.1.3. *For each differential form* $\mathrm{d}\Omega_k$ *with* $k \geq -n$ *and* $i \neq j$ *we have the Laplace equation*

$$\partial_{\gamma_i}\partial_{\gamma_j}\mathrm{d}\Omega_k = c_j^k(\gamma_i)\partial_{\gamma_i}\mathrm{d}\Omega_k + c_i^k(\gamma_j)\partial_{\gamma_j}\mathrm{d}\Omega_k \quad \textit{with} \quad c_j^k(E) := \frac{\partial_{\gamma_j}\mathrm{d}\Omega_k}{\mathrm{d}\Omega_k} \ . \tag{5.1.8}$$

Furthermore, the velocities $v^{(k)}(E) = \mathrm{d}\Omega_k/\mathrm{d}\Omega_0$ *satisfy*

$$\partial_{\gamma_i}v^{(k)} = c_i^0(E)\left(v_i^{(k)} - v^{(k)}\right) \ , \tag{5.1.9}$$

so as a consequence, $\partial_{\gamma_i}\mathrm{d}\Omega_k = v_i^{(k)}\partial_{\gamma_i}\mathrm{d}\Omega_0$ *induces a Combescure transformation (as defined in Section 2.1.3) from* $\mathrm{d}\Omega_0$ *to* $\mathrm{d}\Omega_k$*. The Laplace equation*

$$\partial_{\gamma_i}\partial_{\gamma_j}\mathrm{d}\Omega_k = c_{ji}(E)\partial_{\gamma_i}\mathrm{d}\Omega_k + c_{ij}(E)\partial_{\gamma_i}\mathrm{d}\Omega_k \quad \textit{with} \quad c_{ji}(E) := c_i^0(\gamma_j)\frac{\partial_{\gamma_j}\mathrm{d}\Omega_0}{\partial_{\gamma_i}\mathrm{d}\Omega_0} \tag{5.1.10}$$

has coefficients independent of the index k*, but depending on the parameter* E*.*

Proof. Under the genericity assumption $\mathrm{d}\Omega_0|_{\gamma_l} \neq 0$ for all $l = 1, \dots, 2n+1$ Lemma 5.1.1 gives for all $k \geq -n$ that $\mathrm{d}\Omega_k|_{\gamma_l} \neq 0$ and

$$v_i^{(k)} = \frac{\partial_{\gamma_i}\mathrm{d}\Omega_k}{\partial_{\gamma_i}\mathrm{d}\Omega_0} \quad . \tag{5.1.11}$$

Hence, a direct computation shows (5.1.9). In order to show the Laplace equation (5.1.8), we want to show that the differential forms on the left and the right hand side have identical a-periods and principal parts. The equality then follows by Riemann's theorem. By construction $\mathrm{d}\Omega_k$ has constant a-periods and a constant principal part at the point at infinity, thus its derivatives with respect to branch values γ_i have vanishing a-periods and at $E = \infty$ they are holomorphic. It is left to be shown that the principal parts coincide. From the representation $\mathrm{d}\Omega_k = C_k(E)(\mathrm{d}E)/y$ we have as in (5.1.5)

$$\partial_{\gamma_j}\mathrm{d}\Omega_k = \left(\frac{C_k}{2(E-\gamma_j)} + \partial_{\gamma_j}C_k\right)\frac{\mathrm{d}E}{y} = \left(\frac{1}{2(E-\gamma_j)} + \frac{\partial_{\gamma_j}C_k}{C_k}\right)\mathrm{d}\Omega_k \quad .$$

Taking the derivative of this expression with respect to γ_i and using (5.1.4) for the derivative of $1/y$ gives

$$\partial_{\gamma_i}\partial_{\gamma_j}\mathrm{d}\Omega_k = \left[\frac{C_k}{4(E-\gamma_i)(E-\gamma_j)} + \frac{\partial_{\gamma_i}C_k}{2(E-\gamma_j)} + \frac{\partial_{\gamma_j}C_k}{2(E-\gamma_i)} + \partial_{\gamma_i}\partial_{\gamma_j}C_k\right]\frac{\mathrm{d}E}{y} \quad .$$

By the help of partial fraction decomposition for $\frac{1}{4(E-\gamma_i)(E-\gamma_j)}$ this is

$$\partial_{\gamma_i}\partial_{\gamma_j}\mathrm{d}\Omega_k = \left[\frac{1}{2(E-\gamma_i)}\left(\frac{1}{2(\gamma_i-\gamma_j)} + \frac{\partial_{\gamma_j}C_k}{C_k}\right) + \frac{1}{2(E-\gamma_j)}\left(\frac{1}{2(\gamma_j-\gamma_i)} + \frac{\partial_{\gamma_i}C_k}{C_k}\right)\right]\mathrm{d}\Omega_k \quad .$$

The differential form $\partial_{\gamma_i}\partial_{\gamma_j}\mathrm{d}\Omega_k$ is therefore holomorphic on $\Gamma_n \setminus \{\gamma_i, \gamma_j\}$ and its principal part at $E = \gamma_i$ is

$$\frac{1}{2(E-\gamma_i)}\left(\frac{1}{2(\gamma_i-\gamma_j)} + \left.\frac{\partial_{\gamma_j}C_k}{C_k}\right|_{\gamma_i}\right)\mathrm{d}\Omega_k|_{\gamma_i} = \frac{1}{2(E-\gamma_i)}\left(\partial_{\gamma_j}\mathrm{d}\Omega_k|_{\gamma_i}\right) \quad ,$$

which is the same as the principal part of the differential form

$$\left.\frac{\partial_{\gamma_j}\mathrm{d}\Omega_k}{\mathrm{d}\Omega_k}\right|_{\gamma_i}\partial_{\gamma_i}\mathrm{d}\Omega_k$$

at $E = \gamma_i$. In the the same way the principal parts at $E = \gamma_j$ can be treated. For (5.1.8) this gives that the left and the right hand side have identical principal parts. Hence, the formula is shown. Finally, equation (5.1.10) follows by combining (5.1.9) and (5.1.8)

$$\partial_{\gamma_i}\partial_{\gamma_j}\mathrm{d}\Omega_k = \partial_{\gamma_i}\left(v_j^{(k)}\partial_{\gamma_j}\mathrm{d}\Omega_0\right) = c_i^0(\gamma_j)v_i^{(k)}\partial_{\gamma_j}\mathrm{d}\Omega_0 + c_j^0(\gamma_i)v_j^{(k)}\partial_{\gamma_i}\mathrm{d}\Omega_0 \quad .$$

Equation (5.1.11) allows to identify this expression with (5.1.10). ◻

As a consequence of (5.1.9) the hydrodynamic system of the KdV Whitham hierarchy (5.1.7) is semi-Hamiltonian. Comparing (2.1.2) with (5.1.9) we find $c_i^0(\gamma_j)$ as candidates for Christoffel symbols. A possible corresponding Riemannian metric in (2.1.3) is

$$\mathfrak{g}_{ii} = (\mathrm{d}\Omega_0(\partial_{\xi_i})|_{\gamma_i})^2 = 2 \operatorname*{res}_{E=\gamma_i} \frac{(\mathrm{d}\Omega_0)^2}{\mathrm{d}E} \tag{5.1.12}$$

with a chart ξ_i induced by $\xi_i^2 = E - \gamma_i$, see (2.2.36). For Riemannian metrics associated with a semi-Hamiltonian system the Riemann curvature tensor is conveniently expressed in terms of rotation coefficients, see Section 2.1.1. Given Lamé coefficients H_i by $\mathfrak{g}_{ii} = (H_i)^2$ the rotation coefficients are

$$\beta_{ij} = \frac{\partial_{\gamma_i} H_j}{H_i}$$

for $i \neq j$. In (5.1.12) the Lamé coefficients are $H_i = \mathrm{d}\Omega_0(\partial_{\xi_i})|_{\gamma_i}$. The metric is said to have the Egorov property, if the rotation coefficients are symmetric or, equivalently, if $\partial_{\gamma_i}\mathfrak{g}_{jj} = \partial_{\gamma_j}\mathfrak{g}_{ii}$. In this case there is a potential $c_0 = c_0(\boldsymbol{\gamma})$ for the metric such that $\partial_{\gamma_i} u_0 = \mathfrak{g}_{ii}$ for all $i = 1, \ldots, 2n+1$.

Lemma 5.1.4 (Assertion 4 in [**16**])**.** *The Riemannian metric* (5.1.12) *has the Egorov property.*

Proof. By construction $\mathrm{d}\Omega_0$ has an expansion at $\xi^2 = -1/E = 0$ of the form

$$\mathrm{d}\Omega_0 = \left(E + u_0 E^0 + \mathcal{O}(E^{-1})\right) \mathrm{d}\xi \tag{5.1.13}$$

for some function $u_0 = u_0(\boldsymbol{\gamma})$. Hence, there is a primitive function of $\mathrm{d}\Omega_0$ with the asymptotic $\Omega_0 = \xi^{-1} + u_0\xi + \mathcal{O}(\xi^3)$ near $\xi = 0$. Riemann's period relations (see Chapter 11.3 in [**33**]) imply

$$\sum_{j=1}^{n} \int_{a_j} \partial_{\gamma_i}\mathrm{d}\Omega_0 \int_{a_j} \partial_{\gamma_i}\mathrm{d}\Omega_0 - \int_{a_j} \partial_{\gamma_i}\mathrm{d}\Omega_0 \int_{a_j} \partial_{\gamma_i}\mathrm{d}\Omega_0 = \mathrm{Res}\,(\partial_{\gamma_i}\Omega_0)\,\mathrm{d}\Omega_0 \ . \tag{5.1.14}$$

The left hand side of the relation is zero, since by construction $\mathrm{d}\Omega_0$ has vanishing a-periods and thus also the a-periods of $\partial_{\gamma_i}\mathrm{d}\Omega_0$ vanish. From Lemma 5.1.1 we get that $\partial_{\gamma_i}\mathrm{d}\Omega_0$ is holomorphic on $\Gamma_n \setminus \{\gamma_i\}$, so $(\partial_{\gamma_i}\Omega_0)\mathrm{d}\Omega_0$ can have poles only at $E = \gamma_i$ and $\xi = 0$. The expansions given above yield

$$\operatorname*{res}_{\xi=0}(\partial_{\gamma_i}\Omega_0)\mathrm{d}\Omega_0 = -\partial_{\gamma_i} u_0 \ . \tag{5.1.15}$$

Near $E = \gamma_i$ a chart ξ_i is defined by $\xi_i^2 = E - \gamma_i$, which makes ξ_i depend on γ_i. Derivation gives $2\xi_i \mathrm{d}\xi_i = \mathrm{d}E$. Therefore the differential forms $\mathrm{d}\Omega_0$ and $\partial_{\gamma_i}\mathrm{d}\Omega_0$ as represented in (5.1.3) and (5.1.5), respectively, read

$$\mathrm{d}\Omega_0 = \left(\frac{2C_0(\gamma_i)}{\sqrt{-\prod_{j\neq i}(\gamma_i - \gamma_j)}} + \mathcal{O}\left(\xi_i^1\right)\right) \mathrm{d}\xi_i \ ,$$

$$\partial_{\gamma_i}\mathrm{d}\Omega_0 = \left(\frac{C_0(\gamma_i)}{\xi_i^2\sqrt{-\prod_{j\neq i}(\gamma_i - \gamma_j)}} + \mathcal{O}\left(\xi_i^0\right)\right) \mathrm{d}\xi_i \ .$$

Hence, the Abelian integral of $\partial_{\gamma_i}\mathrm{d}\Omega_0$ has a first order pole at $E=\gamma_i$

$$\partial_{\gamma_i}\Omega_0 = -\frac{C_0(\gamma_i)}{\xi_i\sqrt{-\prod_{j\neq i}(\gamma_i-\gamma_j)}} + \mathcal{O}\left(\xi_i^0\right)$$

and we obtain the following result

$$\operatorname*{res}_{\xi_i=0}(\partial_{\gamma_i}\Omega_0)\mathrm{d}\Omega_0 = \frac{2C_0(\gamma_i)^2}{\prod_{j\neq i}(\gamma_i-\gamma_j)} = -\mathfrak{g}_{ii} \ .$$

Together with (5.1.15) and the vanishing of Riemann's period relation (5.1.14), this gives $-\partial_{\gamma_i}u_0 = \mathfrak{g}_{ii}$. Hence, $-u_0$ is a potential for the metric $\mathfrak{g}$ which has thereby the Egorov property. □

A metric associated to a semi-Hamiltonian system and with the Egorov property is flat if (2.1.10) holds, that is if $\sum_{s=1}^{2n+1}\partial_{\gamma_s}\beta_{ik}=0$ for $i\neq k$.

Lemma 5.1.5 (Example 2 in [**20**])**.** *The Riemannian metric* (5.1.12) *is flat.*

Proof. It suffices to check equation (2.1.10) since $\mathfrak{g}$ is already known to come from a semi-Hamiltonian system and have the Egorov property. If $s\neq i,k$ then $\partial_{\gamma_s}\beta_{ik} = \sum_{s=1}^{2n+1}\beta_{is}\beta_{sk}$ by (2.1.8). For $s=k$ we compute

$$\partial_{\gamma_k}\beta_{ik} = \frac{1}{H_i^2}\left(H_i\partial_{\gamma_i}\partial_{\gamma_k}H_k - \partial_{\gamma_i}H_k\partial_{\gamma_k}H_i\right) = \frac{1}{H_i}\partial_{\gamma_i}\partial_{\gamma_k}H_k - \frac{H_k}{H_i}\beta_{ik}\beta_{ki} \ . \tag{5.1.16}$$

Now we want to express $\partial_{\gamma_k}H_k$ in terms of rotation coefficients as well. In order to do this, we use that $-u_0$ in the expansion (5.1.13) of $\mathrm{d}\Omega_0$ is a potential for the metric $\mathfrak{g}$, i.e. $-\partial_{\gamma_k}u_0 = H_k^2$ and thus $-\partial_{\gamma_k}^2u_0 = 2H_k\partial_{\gamma_k}H_k$. On the other hand $\mathrm{d}\Omega_0$ satisfies

$$\partial_E\mathrm{d}\Omega_0 = -\sum_{s=1}^{2n+1}\partial_{\gamma_s}\mathrm{d}\Omega_0 \tag{5.1.17}$$

due to Riemann's theorem, since both sides have vanishing a-periods and identical principal parts. We see this as follows. For $\mathrm{d}\Omega_0 = (C_0\mathrm{d}E)/y$ the derivative with respect to E is

$$\partial_E\mathrm{d}\Omega_0 = \left(-\frac{1}{2}\sum_{s=1}^{2n+1}\frac{1}{E-\gamma_s} + \frac{\partial_E C_0}{C_0}\right)\mathrm{d}\Omega_0 \ .$$

Comparing this expansion to $\partial_{\gamma_k}\mathrm{d}\Omega_0$ in (5.1.5) we see that the principal parts at $E=\gamma_k$ are the negative of each other. By the fundamental theorem of calculus and since there are no residues, $\partial_E\mathrm{d}\Omega_0$ has only trivial periods. Hence, (5.1.17) holds. Together with the expansion of $\mathrm{d}\Omega_0$ in (5.1.13) the relation $1=-\sum_{s=1}^{2n+1}\partial_{\gamma_s}u_0$ follows. Therefore we get

$$2H_k\partial_{\gamma_k}H_k = -\partial_{\gamma_k}^2u_0 = \sum_{s\neq k}\partial_{\gamma_k}\partial_{\gamma_s}u_0 = -\sum_{s\neq k}\partial_{\gamma_k}H_s^2 = -2\sum_{s\neq k}H_sH_k\beta_{ks}$$

and more simply $\partial_{\gamma_k} H_k = \sum_{s\neq k} H_s \beta_{ks}$. Inserted into (5.1.16) this yields

$$\begin{aligned}\partial_{\gamma_k}\beta_{ik} &= -\frac{1}{H_i}\partial_{\gamma_i}\left(\sum_{s\neq k} H_s\beta_{ks}\right) - \frac{H_k}{H_i}\beta_{ik}^2 \\ &= -\frac{1}{H_i}\sum_{s\neq k}\left(\partial_{\gamma_i}H_s\beta_{ks} + H_s\partial_{\gamma_i}\beta_{ks}\right) - \frac{H_k}{H_i}\beta_{ik}^2 \\ &= -\sum_{s\neq i,k}\left(\beta_{is}\beta_{ks} + \frac{H_s}{H_i}\beta_{ki}\beta_{is}\right) - \frac{1}{H_i}\partial_{\gamma_i}H_i\beta_{ki} - \partial_{\gamma_i}\beta_{ki} - \frac{H_k}{H_i}\beta_{ik}^2 \ .\end{aligned}$$

The term $\partial_{\gamma_i} H_i$ can be replaced by $-\sum_{s\neq i,k} H_s\beta_{is} - H_k\beta_{ik}$. After cancellation and rearrangement we find

$$\partial_{\gamma_k}\beta_{ik} + \partial_{\gamma_i}\beta_{ik} + \sum_{s\neq i,k}\beta_{is}\beta_{sk} = 0 \ .$$

hence, the metric $\mathfrak{g}$ is flat. □

Remark 5.1.6. Another candidate for a Riemannian metric corresponding to Christoffel symbols $c_i^0(\gamma_j)$ is given by

$$\mathfrak{h}_{ii} = \frac{1}{\gamma_i}\left(\mathrm{d}\Omega_0(\partial_{\xi_i})|_{\gamma_i}\right)^2 = 2\operatorname*{res}_{E=\gamma_i}\frac{(\mathrm{d}\Omega_0)^2}{E\mathrm{d}E} \ .$$

It can be shown that this metric has the Egorov property and is flat as well. Additionally, Theorem 1 in [**17**] provides flat coordinates for the metric $\mathfrak{g}$ on the parameter space M_n^{hyp} of hyperelliptic curves by

$$\begin{aligned}T_0(\boldsymbol{\gamma}) &= \operatorname*{res}_{\xi=0}\xi\mathrm{d}\Omega_0 \ , \\ T_\alpha^S(\boldsymbol{\gamma}) &= \int_{a_\alpha}\Omega_0\mathrm{d}E \ \text{, for } \alpha = 1,\ldots,n \\ T_\alpha^Q(\boldsymbol{\gamma}) &= \int_{b_\alpha}\mathrm{d}\Omega_0 \ \text{, for } \alpha = 1,\ldots,n \ .\end{aligned}$$

Derivatives with respect to the branch values γ_i can be permuted with integration with respect to E and also with taking the residue at $\xi = 0$. Hence, by Lemma 5.1.3 the coordinates individually satisfy the Laplace equation (5.1.8) for $k = 0$, so the map $\boldsymbol{\gamma} \mapsto \left(T_0, T_\alpha^S, T_\alpha^Q\right)(\boldsymbol{\gamma}) \in \mathbb{R}^{2n+1}$ forms an orthogonal net.

5.2. The Whitham Hierarchy for Solitons

For spectral curves Γ_0 of genus zero, the Whitham hierarchy from Definition 5.0.1 is set on the 1-dimensional parameter space M_0^{hyp}. However, when considering the hydrodynamic reduction of the Whitham equations on M_n^{hyp}, that is (5.1.7), then in the "soliton limit" [**22**] a more particular system of equations appears that additionally to $\gamma_{2n+1} \in M_0^{hyp}$ includes n double points of the spectral curve.

Let us start by recalling the soliton limit $\gamma_{2j-1} \to \gamma_{2j}$ (for $j = 1,\ldots,n$) of the $(2n+1)$-stationary KdV hierarchy with a time-independent spectral curve Γ_n. It yields a degenerate spectral curve that has double points $\gamma_1 = \gamma_2, \ldots, \gamma_{2n-1} = \gamma_{2n}$, see Section 3.3. By normalizing the double points, we arrive at the genus zero curve $\Gamma_0 = \{(E, y)|y^2 = \gamma_{2n+1} - E\} \cup \{\infty\}$.

The space of such curves M_0^{hyp} is parametrized by $\gamma_{2n+1} \in \mathbb{R}$. On M_0^{hyp} the Whitham equations (5.0.1) involve differential forms $(\mathrm{d}\Omega_i)_{i\geq 0}$ which are determined entirely by their asymptotic behavior at $E = \infty$. By a hydrodynamic reduction the compatibility equations (5.0.1) are equivalent to

$$\partial_{T_{2i+1}}\gamma_{2n+1} = \frac{\mathrm{d}\Omega_i}{\mathrm{d}\Omega_0}\bigg|_{\gamma_{2n+1}} \partial_X \gamma_{2n+1} \quad .$$

This is a generalization of the dispersionless KdV equation in the introductory example of Chapter 2 and Example 4.3.6. More specifically, the hydrodynamic reduction allows to include equations for the double points, that is for all $r = 2, 4, \ldots, 2n, 2n+1$ we consider

$$\partial_{T_{2i+1}}\gamma_r = v_r^{(i)}\partial_X\gamma_r \quad \text{with} \quad v_r^{(i)} := \frac{\mathrm{d}\Omega_i}{\mathrm{d}\Omega_0}\bigg|_{\gamma_r} = \sum_{k=0}^{i}\binom{1/2}{k}(-\gamma_{2n+1})^k\gamma_r^{i-k} \tag{5.2.1}$$

and $X = T_1$. The formula for $v_r^{(i)}$ follows from Proposition 4.3.5. Ignoring that the parameters γ_j are dependent variables in the KdV Whitham equations, heuristically, the system of equations (5.2.1) is the soliton limit $\gamma_{2j-1} \to \gamma_{2j}$ of the hydrodynamic reduction (5.1.7) of the Whitham equations on M_n^{hyp}. In [**32**] and [**22**] this type of "soliton limit" is presented in detail for the case $n = 1$. We do not give a formal derivation here and rather postulate the system of equations (5.2.1) as the *Whitham equations of n-solitons*. In Lemma 4.3.3 we have seen how some solutions of (5.2.1) can be obtained from the Whitham deformations in Proposition 4.3.2.

Unlike the generic KdV Whitham equations, the Whitham equations of 1-solitons are not Hamiltonian systems of hydrodynamic type, see Example 2.1.2. However, for $n \leq 2$ the semi-Hamiltonian condition (2.1.1) is empty, so trivially the Whitham equations of 1-solitons (2.1.11) are semi-Hamiltonian. For the KdV Whitham equations of n-solitons this is still true as we will see now.

Proposition 5.2.1. *For $j, r = 2, 4, \ldots 2n, 2n+1$, $i \geq 1$ and $r \neq j$ we have*

$$\frac{\partial_{\gamma_j} v_r^{(i)}}{v_j^{(i)} - v_r^{(i)}} = \begin{cases} \frac{1}{2(\gamma_r - \gamma_{2n+1})} & \text{if } j = 2n+1 \\ 0 & \text{if } j \neq 2n+1 \end{cases} \quad . \tag{5.2.2}$$

Proof. The case $j \neq 2n+1$ is clear, since $v_r^{(i)}$ only depends on γ_{2n+1} and γ_r, but $j \neq r$. In the case $j = 2n+1$ we use $a^k - b^k = (a-b)\sum_{m=0}^{k-1} a^m b^{k-1-m}$ to obtain

$$\begin{aligned} v_j^{(i)} - v_r^{(i)} &= (\gamma_j - \gamma_r)\sum_{l=0}^{i-1}\sum_{m=0}^{i-l-1}\binom{1/2}{l}(-1)^l\gamma_{2n+1}^{l+m}\gamma_r^{i-1-(l+m)} \\ &= (\gamma_j - \gamma_r)\sum_{l=0}^{i-1}\sum_{k=l+1}^{i}\binom{1/2}{l}(-1)^l\gamma_{2n+1}^{k-1}\gamma_r^{i-k} \quad , \end{aligned}$$

where $k = l + m + 1$ has been substituted. Swapping the order of summation then gives

$$v_j^{(i)} - v_r^{(i)} = (\gamma_j - \gamma_r)\sum_{k=1}^{i}\sum_{l=0}^{k-1}\binom{1/2}{l}(-1)^l\gamma_{2n+1}^{k-1}\gamma_r^{i-k} \quad .$$

The inner summation simplifies to $\sum_{l=0}^{k-1}\binom{1/2}{l}(-1)^l = (-1)^{k-1}\binom{-1/2}{k-1} = -(-1)^k 2\binom{1/2}{k}k$. Hence,

$$v_j^{(i)} - v_r^{(i)} = -2(\gamma_j - \gamma_r)\sum_{k=1}^{i}\binom{1/2}{k}(-k)(-\gamma_{2n+1})^{k-1}\gamma_r^{i-k} = 2(\gamma_r - \gamma_j)\partial_{\gamma_j}v_r^{(i)} \ ,$$

which gives (5.2.2). □

As a consequence, the semi-Hamiltonian condition (2.1.1) is satisfied for the velocities (5.2.2) of the KdV Whitham hierarchy of solitons.

Corollary 5.2.2. *The KdV Whitham equations of n-solitons* (5.2.1) *are semi-Hamiltonian.*

Semi-Hamiltonian systems possess infinitely many commuting flows, see Section 2.1.2. For the KdV Whitham hierarchy of n-solitons they are characterized as follows.

Lemma 5.2.3. *All commuting flows of* (5.2.1) *are induced by $w_r = f(\gamma_r)$ for $r = 2n+1$ and some function $f : \mathbb{R} \to \mathbb{R}$; and by*

$$w_r = -\sqrt{\gamma_{2n+1} - \gamma_r}\left(\frac{1}{2}\int \frac{f(\gamma_{2n+1})}{\sqrt{\gamma_{2n+1} - \gamma_r}^3}\mathrm{d}\gamma_{2n+1} + f_r(\gamma_r)\right) \tag{5.2.3}$$

for $r \in \{2, 4, \ldots, 2n\}$ and some functions $f_r : \mathbb{R} \to \mathbb{R}$.

Proof. A commuting flow is induced by w_r satisfying the system of differential equations

$$\partial_{\gamma_j} w_r = c_{jr}(w_j - w_r)$$

with $j \neq r$ and c_{jr} defined by the right hand side of (5.2.2)

$$c_{jr} := \begin{cases} \frac{1}{2(\gamma_r - \gamma_{2n+1})} & \text{if } j = 2n+1 \\ 0 & \text{if } j \neq 2n+1 \end{cases} \ . \tag{5.2.4}$$

For $r = 2n+1$ this system simply reads $\partial_{\gamma_j} w_r = 0$ for $j = 2, 4, \ldots, 2n$ and henceforth w_r may only depend on γ_{2n+1}. Similarly, if $r, j \neq 2n+1$ we have $\partial_{\gamma_j} w_r = 0$ and therefore w_r may only depend on γ_r and γ_{2n+1}. We are left with the case $r \neq 2n+1$ and $j = 2n+1$:

$$\partial_{\gamma_j} w_r = -\frac{1}{2}\frac{f(\gamma_j) - w_r}{\gamma_j - \gamma_r} \ ,$$

where we have set $w_{2n+1} = f$ for some function $f : \mathbb{R} \to \mathbb{R}$. Nearby a starting point $\gamma_{2n+1,0} \neq \gamma_r$ this ODE has a unique solution with starting value $w_r(\gamma_r, \gamma_{2n+1,0})$. Direct inspection shows that (5.2.3) (which can be found by variation of constants) satisfies the ODE. □

As a consequence, Tsarev's hodograph method for multiple commuting flows (see Example 2.2.11) provides a solution of the n-soliton Whitham hierarchy by solving a transcendental equation.

Corollary 5.2.4. *All solutions $(\gamma_r)_r$ of the Whitham hierarchy of n-solitons* (5.2.1) *are (locally) given as roots of*

$$w_r = X + \sum_{i=1}^{n} v_r^{(i)} T_{2i+1} \ ,$$

for some w_r as in Lemma 5.2.3.

In Example 2.1.5 hodograph solutions for the 1-soliton Whitham equation were constructed from velocities $w_2 = v_2^{(2)}$ and $w_3 = v_3^{(2)}$.

Example 5.2.5 (Another solution for the Whitham hierarchy of 1-solitons)**.** Let $w_3 = f(\gamma_3) = \gamma_3$ and $f_2(\gamma_2) = 1$ in Lemma 5.2.3. Then $w_2 = 2\gamma_2 - \gamma_3 + \sqrt{\gamma_3 - \gamma_2}$. According to Corollary 5.2.4 a solution (γ_2, γ_3) of

$$2\gamma_2 - \gamma_3 + \sqrt{\gamma_3 - \gamma_2} = \frac{1}{2}\left(2\gamma_2 - \gamma_3\right) T + X \ , \quad \gamma_3 = \frac{1}{2}\gamma_3 T + X$$

gives a solution for the 1-soliton Whitham hierarchy (2.1.11). Explicitly, the solutions are here $\gamma_3 = 2X/(2-T)$ and $\gamma_2 = \gamma_3$ or $\gamma_2 = \gamma_3 - 1/(2-T)^2$.

A further aspect of the semi-Hamiltonian property of the coefficients c_{jr} defined in (5.2.4) is that they serve as compatible coefficients of a conjugate net. How to obtain the conjugate net by an integration procedure was described in Section 2.1.3 right in front of Example 2.1.7. In Example 2.1.7 conjugate nets for the 1-soliton Whitham equation were given.

Example 5.2.6 (Conjugate nets for the n-soliton Whitham equations)**.** The Whitham equations of n-solitons yield a $(n+1)$-dimensional conjugate net by

$$f = \sum_{l \in \{2,4,\dots 2n\}} \int \frac{1}{\sqrt{\gamma_l - \gamma_{2n+1}}} X_l(\gamma_l) \mathrm{d}\gamma_l \tag{5.2.5}$$

for some arbitrary pointwise linear independent functions $X_l : \mathbb{R} \to \mathbb{R}^{n+1}$ and integration constants that are constant in all γ_j. When we assume $\gamma_l < \gamma_{2n+1}$ as usual, then the root in (5.2.5) will be purely imaginary and so will be the resulting conjugate net. This issue originates from the chosen normalization (4.3.7) of $\mathrm{d}\Omega_i$ in the Whitham hierarchy and may be resolved by replacing E^{i+1} by $(-E)^{i+1}$ there.

5.3. Generating Differential Forms and EPDs

In the present section, generating differential forms $\mathrm{d}S$ of the KdV Whitham hierarchy are investigated more closely. They are given by the system of differential equations

$$\partial_{T_{2j+1}} \mathrm{d}S = \mathrm{d}\Omega_j \ . \tag{5.3.1}$$

It turns out that generating differential forms can be represented by solutions of particular EPDs on the parameter space M_n^{hyp}. Once a generating differential form on M_n^{hyp} is given, finding a solution of the KdV Whitham hierarchy reduces to solving a system of ODEs.

In Chapter 2 we have seen two versions of the generalized hodograph method that provide solutions of the KP Whitham hierarchy. Let us revisit them in the context of the KdV Whitham hierarchy.

- Tsarev's generalized hodograph method in Theorem 2.1.3 describes all generic solutions of the KdV Whitham hierarchy by commuting flows. By a result about linear PDEs that goes back to Darboux, commuting flows are parametrized by $2n+1$ (smooth) functions in one variable.
- Krichever's generalized hodograph method in the special case formulated in Proposition 2.2.12 uses
$$\mathrm{d}S = \sum_{l \geq -n} T_{2l+1} \mathrm{d}\Omega_l$$

as an Ansatz for generating differential forms. This Ansatz is parametrized by a sequence of coefficients $(T_{2l+1})_{l\geq -n}$ such that the series converges. The coefficients also serve as times.

Considering the parameter sets, Tsarev's version appears vastly more general. In more detail, the velocities $v_j^{(l)} = (\mathrm{d}\Omega_l/\mathrm{d}\Omega_0)|_{E=\gamma_j}$ induce commuting flows for the KdV Whitham hierarchy (see (5.1.9)), so Tsarev's version of the hodograph method contains Krichever's version as a special case. On the other hand, the differential equations (5.3.1) for a generating differential form provide a compact expression for the system of equations for commuting flows (2.1.15). This is due to their algebraic-geometric structure that originates from the spectral curves.

The aim of the present section is to obtain a version of the generalized hodograph method for the KdV Whitham hierarchy that is as general as Tsarev's version and admits the structure of a generating differential form. As the main result, Theorem 5.3.10 describes generating differential forms of the KdV Whitham hierarchy by a series expansion

$$\mathrm{d}S = \sum_{l\geq 0} a_l(E) g(E)^{l-1/2} \mathrm{d}E$$

with coefficient polynomials a_l of degree up to $2n$ whose coefficients are functions in $\boldsymbol{\gamma}$. The lowest order term of this expansion is determined as $a_0 = 0$ by the hodograph method. The higher order terms that describe $\mathrm{d}S$ are determined by a particular *Euler–Poisson–Darboux equation* (EPD) – a so called ϵ-system with ϵ in the half-integers. An EPD is the special case of a Laplace equation (2.1.20) given by

$$\partial_{\gamma_i}\partial_{\gamma_j}\mathfrak{a} = \frac{1}{\gamma_i - \gamma_j}\left(\epsilon_j \partial_{\gamma_i}\mathfrak{a} - \epsilon_i \partial_{\gamma_j}\mathfrak{a}\right)$$

for $i \neq j$ and constants $\epsilon_i, \epsilon_j \in \mathbb{C}$. If all constants are the same $\epsilon_j = \epsilon$, then the EPD is called an ϵ-system. Solutions of ϵ-systems are described by generalized hypergeometric functions (or, more specifically, by Lauricella functions). For a description of these solutions and the claim that KdV Whitham equations correspond to an ϵ-system, see [**59**].

Remark 5.3.1. For his more general construction of integrable Whitham hierarchies in [**56**], Odesskii uses hypergeometric functions and points out that they "can be constructed and studied in two dual ways: as solutions of holonomic linear systems of PDEs and/or as periods of some multiple-valued differential forms." The present section rather follows the first way.

The structure of this section is as follows: Section 5.3.1 shows how EPDs appear in Krichever's hodograph method from Proposition 2.2.12. In Section 5.3.2 the factor representation of generating differential forms will be introduced and then used as an Ansatz in Section 5.3.3. This yields the main theorem. The final section then contains the proof for the main theorem.

For genericity in all this, the conditions of Corollary 2.2.9 are assumed as fulfilled, i.e. already $2n+1$ different times suffice to make a map

$$T_J \mapsto \boldsymbol{\gamma}(T_J) \tag{5.3.2}$$

satisfying the equations of the KdV Whitham hierarchy in Corollary 5.1.2 a submersion. Here $J \subseteq \{l \mid l \geq -n\}$ denotes the index set of active times $T_J := (T_{2j+1})_{j\in J}$ and $|J| \geq 2n+1$ is assumed. If not stated otherwise, T_J represents finitely many times such that $|J| = 2n+1$ and (5.3.2) becomes a local diffeomorphism.

5.3.1. The Hodograph Method and EPDs. Krichever's hodograph method (see Section 2.2.5) relies on a Laurent expansion of the generating differential form at $E=\infty$

$$\mathrm{d}S = \left(\sum_{j\geq 0} T_{2j+1}E^{j+1} + \sum_{k=1}^{n} H_k E^{k-n} + \mathcal{O}\left(E^{-n}\right)\right)\mathrm{d}\xi \ . \tag{5.3.3}$$

Assuming only finitely many times T_{2j+1} are non-zero, then there is only a pole at infinity. The positive powers E^{j+1} are realized by $\mathrm{d}\Omega_j = (E^{j+1}+\mathcal{O}(E^0))\mathrm{d}\xi$ with $j\geq 0$. Linear combinations of the normalized holomorphic differential forms $(\mathrm{d}\Omega_j)_{j=-n,\ldots,-1}$, allow to realize the powers E^{k-n} for $k=1,\ldots,n$. Accordingly, the coefficients H_k depend on times that correspond to the holomorphic differential forms. If only terms related to $\mathrm{d}\Omega_j$ with $j\geq -n$ appear in the sum, then Krichever's Ansatz for a solution of (5.3.1) reads

$$\mathrm{d}S\left(E;\boldsymbol{\gamma}(T_J)\right) = \sum_{l\geq -n} T_{2l+1}\mathrm{d}\Omega_l\left(E;\boldsymbol{\gamma}(T_J)\right) \tag{5.3.4}$$

which is defined everywhere on Γ_n. In this case, the hodograph method tells that the Ansatz (5.3.4) solves (5.3.1), if and and only if $T_J\mapsto\boldsymbol{\gamma}(T_J)$ satisfies

$$\mathrm{d}S\left(\gamma_j(T_J);\boldsymbol{\gamma}(T_J)\right) = 0 \ , \tag{5.3.5}$$

for all $j=1,\ldots,2n+1$, see Proposition 2.2.12. For a generating differential form $\mathrm{d}S$ that is not defined on Γ_n, but rather on its universal covering, terms of order $\mathcal{O}(E^{-n})$ appear in (5.3.3) that do not belong to the Laurent expansion of (5.3.4). In other words, generating differential forms $\mathrm{d}S$ which have a pole at infinity, but are not defined on Γ_n can not be constructed by only using the normalized differential forms $\mathrm{d}\Omega_l$ with $l\geq -n$ in the hodograph Ansatz.

In order to see an ϵ-system appear, let us consider a hodograph Ansatz of the most simple form

$$\mathrm{d}S_m = \mathrm{d}\Omega_{n+m} + \sum_{l=-n}^{n} T_{2l+1}\mathrm{d}\Omega_l \tag{5.3.6}$$

for some $m\geq 1$. The following result is a refined version of Corollary 2.2.14 and the discussion after it.

Proposition 5.3.2. *Given* $\mathrm{d}S_m$ *in* (5.3.6)*, then there are some uniquely determined polynomials* $a_j=a_j(E)$ *of degree up to* $2n$*, whose coefficients depend on* $\boldsymbol{\gamma}$ *and* T_J *(with* $J=\{-n,\ldots,n\}$*) such that*

$$\mathrm{d}S_m = -i\sum_{j=0}^{k} a_j(E)g(E)^{j-1/2}\mathrm{d}E \tag{5.3.7}$$

for k *the least integer with* $k(2n+1)\geq m$*. Conversely, given* (5.3.7) *with polynomials* $a_j=a_j(E)$ *of degree up to* $2n$*, whose coefficients depend on* $\boldsymbol{\gamma}$*, then*

$$T_{2l+1}(\boldsymbol{\gamma}) = \begin{cases} \int_{\alpha_{-l}}\mathrm{d}S_m & \text{for } l=-n,\ldots,-1 \\ (-1)^{l+1}\operatorname{res}_{\xi=0}\xi^{2l+1}\mathrm{d}S_m & \text{for } l=0,\ldots,n \end{cases}$$

gives $\mathrm{d}S_m$ *in the form* (5.3.6)*. The condition* (5.3.5) *for the hodograph Ansatz to provide a solution of the KdV Whitham hierarchy is simply* $a_0=0$*.*

PROOF. Transforming (5.3.6) into (5.3.7) uses mainly polynomial long division. To start with the proof, let each normalized differential form be represented by $\mathrm{d}\Omega_l = (C_l(E)\mathrm{d}E)/y$ for some polynomial C_l of degree $\max(n+l, n-1)$. The normalization determines the polynomial uniquely. Hence, the generating differential form is given by $\mathrm{d}S_m = (P_m(E)\mathrm{d}E)/y$ for the polynomial $P_m = C_{n+m} + \sum_{l=-n}^{n} T_{2l+1}C_l$. Since the hodograph method requires evaluation at the branch values γ_j, which are the roots of g, we are going to factor P_m by g. Polynomial long division gives

$$P_m(E) = \sum_{j=0}^{k} a_j(E)g(E)^j$$

for some polynomials a_j and k as characterized in the statement of the proposition. The generating differential form then takes the form

$$\mathrm{d}S_m = \sum_{j=0}^{k} a_j(E)g(E)^j \frac{\mathrm{d}E}{y} = -i\sum_{j=0}^{k} a_j(E)g(E)^{j-1/2}\mathrm{d}E$$

since $y^2 = -g(E)$. Represented like this, the condition (5.3.5) for the hodograph Ansatz to provide a solution of the KdV Whitham hierarchy, simply becomes $a_0 = 0$.

Conversely, let a differential form $\mathrm{d}S_m$ be given by (5.3.7) with polynomials $a_j = a_j(E)$ of degree up to $2n$ whose coefficients depend on $\boldsymbol{\gamma}$. Assuming (5.3.6) as an Ansatz allows to determine $T_{2l+1} = T_{2l+1}(\boldsymbol{\gamma})$ by using the normalization of the differential forms $\mathrm{d}\Omega_l$

$$\int_{\alpha_{-l}} \mathrm{d}S_m = \sum_{j=-n}^{n} T_{2j+1}\int_{\alpha_{-l}} \mathrm{d}\Omega_j = T_{2l+1}$$

for $l = -n, \dots, -1$ and

$$\begin{aligned}\operatorname*{res}_{\xi=0} \xi^{2l+1}\mathrm{d}S_m &= \sum_{j=-n}^{n} T_{2j+1} \operatorname*{res}_{\xi=0} \xi^{2l+1}\mathrm{d}\Omega_j = \sum_{j=-n}^{n} T_{2j+1} \operatorname*{res}_{\xi=0} \xi^{2l+1}E^{j+1}\mathrm{d}\xi \\ &= \sum_{j=-n}^{n} T_{2j+1} \operatorname*{res}_{\xi=0} \xi^{2l+1}(-1)^{j+1}\xi^{-2(j+1)}\mathrm{d}\xi = (-1)^{l+1}T_{2l+1}\end{aligned}$$

for $l = 0, \dots, n$. The inverse function $T_J \mapsto \boldsymbol{\gamma}(T_J)$ is a solution of the KdV Whitham hierarchy if $a_0 = 0$. ☐

Example 5.3.3 ($m = 1$)**.** Studying adiabatic invariants of the stationary KdV hierarchy led to the generating differential form $\mathrm{d}S = y\mathrm{d}E$ in Chapter 4. Due to $y^2 = -g(E)$ we find

$$\mathrm{d}S = y\mathrm{d}E = ig(E)^{1/2}\mathrm{d}E$$

which is (5.3.7) for $m = 1$, $k = 1$, $a_0 = 0$ and $a_1 = -1$. This example can be seen as a continuation of Example 2.2.16.

In the following example we see, how a_1 induces an ϵ-system with $\epsilon = 1/2$.

Example 5.3.4 ($m = 3$)**.** In the case $m = 3$ the generating differential form (5.3.7) is given by

$$\mathrm{d}S_3 = (a_1 g + a_0)\,\frac{\mathrm{d}E}{y}$$

for $a_1 = \alpha_1 E^2 + \alpha_2 E + \alpha_3$ with scalar coefficients α_j. When we express $\mathrm{d}S_3$ by $\mathrm{d}\Omega_{n+3} + \sum_{l=-n}^{n} T_{2l+1}\mathrm{d}\Omega_l$ it turns out that $\mathrm{d}\Omega_{n+3}$ alone determines the coefficients of a_1 as functions in $\boldsymbol{\gamma}$. In contrast, the coefficients of a_0 depend on $\boldsymbol{\gamma}$ and T_J. The condition $a_0 = 0$ imposes a dependence between $\boldsymbol{\gamma}$ and T_J that makes $\boldsymbol{\gamma}(T_J)$ a solution of the KdV Whitham hierarchy. We want to show now that $h_i := a_1|_{\gamma_i}$ satisfies the equation

$$h_l - h_k = 2(\gamma_l - \gamma_k)\partial_{\gamma_k} h_l \quad . \tag{5.3.8}$$

In particular this means $\partial_{\gamma_k} h_l = \partial_{\gamma_l} h_k$, thus there is a function $\mathfrak{a} = \mathfrak{a}(\boldsymbol{\gamma})$ with $\partial_{\gamma_i}\mathfrak{a} = h_i$. Inserted into (5.3.8) we then find that $\mathfrak{a}$ satisfies an ϵ-system with $\epsilon = 1/2$.

We start by comparing principal parts of $\mathrm{d}S_3$ and $\mathrm{d}\Omega_{n+3} + \sum_{l=-n}^{n} T_{2l+1}\mathrm{d}\Omega_l$ at $E = \infty$. When the differential forms $\mathrm{d}\Omega_j$ are represented by $\mathrm{d}\Omega_j = (C_j \mathrm{d}E)/y$ for some polynomial C_j of degree $n+j$, then $\mathrm{d}S_3 = \mathrm{d}\Omega_{n+3} + \sum_{l=-n}^{n} T_{2l+1}\mathrm{d}\Omega_l$ implies $C_{n+3} = a_1 g + \mathcal{O}(E^{2n})$. Using the expansions

$$C_{n+3} = c_1 E^{2n+3} + c_2 E^{2n+2} + c_3 E^{2n+1} + \mathcal{O}(E^{2n}) \quad \text{and}$$
$$g = E^{2n+1} - \sum_{j=1}^{2n+1} \gamma_j E^{2n} + \sum_{1\le i<j\le 2n+1} \gamma_i\gamma_j E^{2n-1} + \mathcal{O}(E^{2n-2})$$

this becomes equivalent to

$$\begin{aligned} &c_1 E^{2n+3} + c_2 E^{2n+2} + c_3 E^{2n+1} + \mathcal{O}(E^{2n}) \\ &\quad = \alpha_1 E^{2n+3} + \left(\alpha_2 - \alpha_1 \sum_j \gamma_j\right) E^{2n+2} \\ &\qquad + \left(\alpha_1 \sum_{i<j} \gamma_i\gamma_j - \alpha_2 \sum_j \gamma_j + \alpha_3\right) E^{2n+1} + \mathcal{O}(E^{2n}) \quad . \end{aligned}$$

Hence, in order to express α_1, α_2 and α_3 in terms of $\boldsymbol{\gamma}$, we are left to determine the coefficients c_1, c_2 and c_3. They follow from the normalization $\mathrm{d}\Omega_{n+3} = \left(E^{n+4} + \mathcal{O}(E^0)\right)\mathrm{d}\xi$ which by squaring both sides implies

$$C_{n+3}^2 \frac{(\mathrm{d}E)^2}{-g} = \left(E^{2n+8} + \mathcal{O}(E^{n+4})\right)(\mathrm{d}\xi)^2 \quad .$$

From $\xi^{-2} = -E$ it follows that $-2\xi^{-3}\mathrm{d}\xi = -\mathrm{d}E$ and $(\mathrm{d}E)^2 = -4E^3(\mathrm{d}\xi)^2$, so we arrive at

$$4C_{n+3}^2 = g\left(E^{2n+5} + \mathcal{O}(E^{n+1})\right)$$

and, by using the expansions for C_{n+3} and g, equivalently

$$\begin{aligned} &4\left(c_1^2 E^{4n+6} + 2c_1c_2 E^{4n+5} + (c_1c_3 + c_2^2)E^{4n+4} + \mathcal{O}(E^{4n+3})\right) \\ &\quad = E^{4n+6} - \sum_j \gamma_j E^{4n+5} + \sum_{i<j} \gamma_i\gamma_j E^{4n+4} + \mathcal{O}(E^{4n+3}) \quad . \end{aligned}$$

Hence, the principal part of $\mathrm{d}\Omega_{n+3}$ determines the coefficients c_1, c_2 and c_3 of C_{n+3} by

$$c_1 = \pm\frac{1}{2} \ , \quad c_2 = \mp\frac{1}{4}\sum_j \gamma_j \quad \text{and} \quad c_3 = \pm\frac{1}{8}\left(\sum_{i<j}\gamma_i\gamma_j - \frac{1}{2}\sum_j \gamma_j^2\right) \quad .$$

Now the coefficients α_j can be expressed in terms of $\gamma_1, \ldots, \gamma_{2n+1}$

$$\alpha_1 = c_1 = \pm\frac{1}{2} \ , \quad \alpha_2 = c_2 + \alpha_1 \sum_j \gamma_j = \pm\frac{1}{4}\sum_j \gamma_j \quad \text{and}$$

$$\alpha_3 = c_3 + \alpha_2 \sum_j \gamma_j - \alpha_1 \sum_{i<j} \gamma_i\gamma_j = \pm\frac{1}{8}\sum_{i<j}\gamma_i\gamma_j \pm \frac{3}{16}\sum_j \gamma_j^2 \ .$$

Therefore the functions $h_l = a_1|_{\gamma_l}$ defined above are given as

$$h_l = \pm\frac{1}{2}\left(\gamma_l^2 + \frac{1}{2}\gamma_l \sum_j \gamma_j + \frac{1}{4}\sum_{i<j}\gamma_i\gamma_j + \frac{3}{8}\sum_j \gamma_j^2\right) \ .$$

Finally, direct inspection shows that the functions h_l (with $l = 1, \ldots, 2n+1$) satisfy (5.3.8).

More generally, we find an ϵ-system with $\epsilon = 1/2$ when we consider in (5.3.6) instead of $\mathrm{d}\Omega_{n+m}$ a differential form $\mathrm{d}\Omega = \sum_{j\geq -n} k_j \mathrm{d}\Omega_j$ with constants k_j such that $\mathrm{d}\Omega$ is defined on Γ_n, see Theorem 4.2. in [**64**]. However, this still excludes generating differential forms $\mathrm{d}S$ that are not defined on Γ_n (but rather on the universal covering of Γ_n). Therefore we look for a representation of generating differential forms that does not require them to be defined on Γ_n.

5.3.2. The Factor Representation of the Generating Differential Form. An alternative to the Laurent expansion (5.3.3) of the generating differential form at $E = \infty$ is to consider a series expansion at the branch values γ_j given as the roots of g. For the hyperelliptic curve Γ_n defined by $y^2 + g(E) = 0$, at all branch points $(E, y) = (\gamma_j, 0) \in \Gamma_n$ simultaneously y is a chart. Hence, for any generating differential form there is an expansion near $(\gamma_1, 0)$ of the form

$$\mathrm{d}S = \sum_{k\geq 0} c_k y^k \mathrm{d}y \tag{5.3.9}$$

for scalar coefficients $c_k = c_k(\boldsymbol{\gamma})$. This expansion can be considered formally, but usually at least local convergence is assumed for generating differential forms. When $\mathrm{d}S$ is multivalued on Γ_n, the expansion converges only locally. Due to $y = 0$ at all branch points $(\gamma_j, 0)$, the series converges there as well, but does not necessarily represent the same generating differential form. The expansion at $(\gamma_1, 0)$ determines $\mathrm{d}S$ entirely and by holomorphic continuation the expansion coefficients at other branch points can be found.

Since $\mathrm{d}S$ is supposed to be a generating differential form, the equations (5.3.1) have to hold. That is to say, the time derivatives of $\mathrm{d}S$ are related to the normalized differential forms, which are represented as $\mathrm{d}\Omega_l = (C_l(E)\mathrm{d}E)/y$. Accordingly, we want to express (5.3.9) in the chart E again. Using $2y\mathrm{d}y = -g'(E)\mathrm{d}E$ gives

$$\mathrm{d}S = -\frac{1}{2}\sum_{k\geq 0} c_k g'(E) y^{k-1} \mathrm{d}E \ .$$

All normalized differential forms $\mathrm{d}\Omega_l$ are odd with respect to the hyperelliptic involution σ, i.e. $\sigma^*\mathrm{d}\Omega_i = -\mathrm{d}\Omega_i$. Decomposing $\mathrm{d}S$ into an even and an odd part gives

$$\mathrm{d}S = (f_1(E)y + f_2(E))\,\mathrm{d}E$$

with $f_1(E) = -\frac{1}{2}\sum_{k\geq 0} c_{2k} g'(E)(-g(E))^{k-1}$ and $f_2(E) = -\frac{1}{2}\sum_{k\geq 0} c_{2k+1} g'(E)(-g(E))^{k}$. Due to the oddness of $\mathrm{d}\Omega_j$ equation (5.3.1) implies for the even part of $\mathrm{d}S$

$$\partial_{T_{2j+1}} f_2 = 0$$

for all $j \in J$. Hence, f_2 is constant in T_J and also as a function in $\boldsymbol{\gamma}$, since the map in (5.3.2) is a local diffeomorphism. What this constant is, is not described by (5.3.1). Without loss of generality we may assume $f_2 = 0$ that is $c_{2k+1} = 0$ for all $k \geq 0$ and arrive at the following *factor representation* of generating differential forms.

Proposition 5.3.5. *Any generating differential form* $\mathrm{d}S$ *for the KdV Whitham hierarchy can be represented locally near* $(E, y) = (\gamma_1, 0)$ *by*

$$\mathrm{d}S(E;\boldsymbol{\gamma}) = \sum_{k\geq 0} a_k(E) g(E)^{k-1/2} \mathrm{d}E \tag{5.3.10}$$

with $2a_k(E) = i(-1)^{k+1} c_{2k} g'(E)$ *a polynomial in* E *of degree* $2n$ *and coefficients that depend on* $\boldsymbol{\gamma}$.

The system of differential equations (5.3.1) has to determine the coefficient polynomials a_k [2]. Let $\mathrm{d}\Omega_\alpha = (C_\alpha(E)\mathrm{d}E)/y$ for a polynomial C_α of degree $\alpha+n$ in E that is determined by the normalization of the differential $\mathrm{d}\Omega_\alpha$. (To avoid confusion with the index of summation, Greek letters are used to index the normalized differential forms of the KdV Whitham hierarchy.)

Proposition 5.3.6. *For the factor representation* (5.3.10) *of a generating differential form* $\mathrm{d}S$ *the equations* (5.3.1) *take the form* $a_0 = 0$ *and for all* $\alpha \in J$ *holds*

$$-iC_\alpha = \frac{1}{2} a_1 \partial_{T_{2\alpha+1}} g + \sum_{l\geq 1} \left[\partial_{T_{2\alpha+1}} a_l + \left(l + \frac{1}{2}\right) a_{l+1} \partial_{T_{2\alpha+1}} g\right] g^l \ . \tag{5.3.11}$$

PROOF. Inserting the factor representation into the differential equation (5.3.1) gives

$$C_\alpha \frac{\mathrm{d}E}{y} = \partial_{T_{2\alpha+1}} \mathrm{d}S = \sum_{k\geq 0} \left[(\partial_{T_{2\alpha+1}} a_k) g^{k-1/2} + a_k \left(k - \frac{1}{2}\right) g^{k-3/2} \partial_{T_{2\alpha+1}} g\right] \mathrm{d}E \ .$$

This is equivalent to

$$-iC_\alpha = \sum_{k\geq 0} \left[(\partial_{T_{2\alpha+1}} a_k) g^k + a_k \left(k - \frac{1}{2}\right) g^{k-1} \partial_{T_{2\alpha+1}} g\right] \ . \tag{5.3.12}$$

At all branch values γ_j ($j = 1, \ldots, 2n+1$) only the term for $k = 0$ on the right hand side would have a pole, so its coefficient a_0 has to vanish there. However, since a_0 is a polynomial of degree up to $2n$, it is already 0. A shift in the index of summation for the second term in the square brackets of (5.3.12) gives the formula in the statement of the proposition. □

In other words, for the factor representation the differential equation (5.3.1) implies $a_0 = 0$, which is the hodograph condition $\mathrm{d}S|_{\gamma_j} = 0$. By $-2iC_\alpha|_{\gamma_j} = a_1|_{\gamma_j} \left(\partial_{T_{2\alpha+1}} g\right)|_{\gamma_j}$ the next polynomial coefficient a_1 turns out to provide an hydrodynamically reduced version of the equations (5.3.1), i.e. it is free of the spectral parameter E (compare this to the hydrodynamically reduced KdV Whitham equations in Corollary 5.1.2).

[2] The series expansions in Proposition 5.3.5 only contains polynomials a_k of degree $2n$ and generally, does not converge at $E = \infty$. In contrast, the Laurent expansion in Proposition 5.3.2 allows coefficient polynomials of degree up to $2n$ and converges at $E = \infty$. Therefore the two expansions are not the same in general.

Lemma 5.3.7. *Any solution $T_J \mapsto \boldsymbol{\gamma}(T_J)$ of the KdV Whitham hierarchy which is given by a generating differential form* $\mathrm{d}S$ *that converges near* $(E, y) = (\gamma_1, 0)$*, is determined by the system of ODEs*

$$\mathrm{d}\Omega_\alpha|_{\gamma_j} = -\frac{1}{2}\left.\frac{\mathrm{d}S}{E-\gamma_j}\right|_{\gamma_j} \partial_{T_{2\alpha+1}}\gamma_j \tag{5.3.13}$$

for all $\alpha \in J$ *and* $j = 1, \dots, 2n+1$*. The initial values* $X = T_1 \mapsto \boldsymbol{\gamma}(X)$ *for the PDEs of the hydrodynamic reduction* (5.1.6) *correspond to a solution of the ODE* (5.3.13) *for* $\alpha = 0$*.*

Proof. Evaluating the differential equations (5.3.11) that describe $\mathrm{d}S$ at $E = \gamma_j$ gives $-2iC_\alpha|_{\gamma_j} = a_1|_{\gamma_j}\partial_{T_{2\alpha+1}} g|_{\gamma_j}$. By

$$\partial_{T_{2\alpha+1}} g = -\sum_{k=1}^{2n+1}\left(\prod_{l\neq k}(E-\gamma_l)\right)\partial_{T_{2\alpha+1}}\gamma_l$$

then follows

$$C_\alpha|_{\gamma_j} = -\frac{i}{2}a_1|_{\gamma_j}\prod_{l\neq j}(\gamma_j-\gamma_l)\partial_{T_{2\alpha+1}}\gamma_j \quad . \tag{5.3.14}$$

On the other hand the factor representation (5.3.10) gives

$$\begin{aligned}\left.\left(\mathrm{d}\Omega_\alpha + \frac{1}{2}\frac{\mathrm{d}S}{E-\gamma_j}\partial_{T_{2\alpha+1}}\gamma_j\right)\right|_{\gamma_j} &= \left.\left(C_\alpha + \frac{i}{2}a_1\frac{g}{E-\gamma_j}\partial_{T_{2\alpha+1}}\gamma_j\right)\frac{\mathrm{d}E}{y}\right|_{\gamma_j} \\ &= \left.\left(C_\alpha|_{\gamma_j} + \frac{i}{2}a_1|_{\gamma_j}\prod_{l\neq j}(\gamma_j-\gamma_l)\partial_{T_{2\alpha+1}}\gamma_j\right)\frac{\mathrm{d}E}{y}\right|_{\gamma_j} \quad .\end{aligned}$$

The term in the bracket vanishes due to (5.3.14), thus we arrive at (5.3.13). From (5.3.13) the KdV Whitham equations in their hydrodynamically reduced form (5.1.6) follow. □

Note that the factor representation (5.3.10) allows here to trade the derivation of $\mathrm{d}S$ for an evaluation. For the differential forms $\mathrm{d}\Omega_\alpha$ this was shown before in Lemma 5.1.1.

Remark 5.3.8. When we compare the generating differential form $\mathrm{d}S$ of the KdV Whitham hierarchy and the resolvent $\hat{B}$ of the stationary KdV hierarchy, we notice that the system of ODEs (5.3.13) plays the same role as the Drach-Dubrovin equations (3.2.17). However, the resolvent is given by an ODE, while some generating differential form has to be assumed in Lemma 5.3.7 in order to get an ODE.

Next we are going to see, how the higher coefficient polynomials a_i with $i \geq 2$ are determined. The recursion scheme works similar to polynomial long division. Let the polynomials C_α be represented by

$$C_\alpha = \sum_{l\geq 0} C_{\alpha l} g^l$$

for polynomials $C_{\alpha l} = C_{\alpha l}(E)$ of degree up to $2n$. If $J = \{-n, \dots, n\}$, then $\deg_E C_\alpha \leq 2n$ for all $\alpha \in J$ and therefore $C_{\alpha l} = 0$ for all $l \geq 1$. Given a solution $T_J \mapsto \boldsymbol{\gamma}(T_J)$ for the KdV Whitham hierarchy, then the evaluation of (5.3.11) at $E = \gamma_j$ for $j = 1, \dots, 2n+1$

determines the coefficient polynomial a_1 by (5.3.14), since a_1 has degree up to $2n$ in E. Hence, there is a unique polynomial f_α^1 in E of degree up to $2n-1$ such that

$$-iC_{\alpha 0} = \frac{1}{2} a_1 \partial_{T_{2\alpha+1}} g + f_\alpha^1 g \ .$$

This turns (5.3.11) into

$$0 = g \left(-f_\alpha^1 + \sum_{l \geq 1} \left[\partial_{T_{2\alpha+1}} a_l + \left(l + \frac{1}{2} \right) a_{l+1} \partial_{T_{2\alpha+1}} g + iC_{\alpha l} \right] g^{l-1} \right) \ .$$

The expression in the bracket determines a_2 by evaluation at all $E = \gamma_j$

$$0 = \left(-f_\alpha^1 + \partial_{T_{2\alpha+1}} a_1 + \frac{3}{2} a_2 \partial_{T_{2\alpha+1}} g + iC_{\alpha 1} \right) \bigg|_{\gamma_j} \ .$$

Again, there is a unique polynomial f_α^2 in E of degree up to $2n-1$ such that

$$0 = -f_\alpha^1 + \partial_{T_{2\alpha+1}} a_1 + \frac{3}{2} a_2 \partial_{T_{2\alpha+1}} g + iC_{\alpha 1} + f_\alpha^2 g \ .$$

Iterating this process of evaluation and factoring g gives the following recursion.

Proposition 5.3.9. *There are uniquely determined auxiliary polynomials f_α^l of degree up to $2n-1$ ($l \geq 1$) such that* (5.3.11) *takes the form*

$$0 = -f_\alpha^l + iC_{\alpha l} + \partial_{T_{2\alpha+1}} a_l + \left(l + \frac{1}{2} \right) a_{l+1} \partial_{T_{2\alpha+1}} g + f_\alpha^{l+1} g \ . \tag{5.3.15}$$

By setting $f_\alpha^0 = 0$, this equation also holds for $l = 0$.

An alternative representation of the infinite system of equations (5.3.15) is

$$0 = \partial_{T_{2\alpha+1}} \boldsymbol{a} + \partial_{T_{2\alpha+1}} g A \boldsymbol{a} + (gF - \mathrm{I}) \boldsymbol{f}_\alpha + i\boldsymbol{C}_\alpha$$

for $\boldsymbol{a} = (a_0, a_1, \dots)^t$ with $a_0 = 0$, $\boldsymbol{f_\alpha} = (f_\alpha^0, f_\alpha^1, \dots)^t$, $\boldsymbol{C_\alpha} = (C_{\alpha 0}, C_{\alpha 1}, \dots)^t$, $F = (\delta_{i+1,j})_{i,j \geq 0}$, $\mathrm{I} = (\delta_{i,j})_{i,j \geq 0}$ and $A = \frac{1}{2} \operatorname{diag}(2l+1)_{l \geq 0} F$, i.e.

$$F = \begin{pmatrix} 0 & 1 & 0 & \dots \\ 0 & 0 & 1 & \\ \vdots & & \ddots & \ddots \end{pmatrix} \quad \text{and} \quad A = \frac{1}{2} \begin{pmatrix} 0 & 1 & 0 & \dots \\ 0 & 0 & 3 & \\ \vdots & & \ddots & \ddots \end{pmatrix} \ . \tag{5.3.16}$$

5.3.3. The Factor Ansatz for the Generating Differential Form. So far a generating differential form $\mathrm{d}S$ providing a solution $T_J \mapsto \boldsymbol{\gamma}(T_J)$ for the KdV Whitham hierarchy was assumed as given. For the factor representation of $\mathrm{d}S$ we have seen that the coefficient polynomials a_k have to satisfy (5.3.11) or, equivalently, the recursion scheme (5.3.15). Conversely, we now want to use the factor representation

$$\mathrm{d}S = \sum_{k \geq 0} a_k(E) g(E)^{k-1/2} \mathrm{d}E \tag{5.3.17}$$

as an Ansatz for finding solutions of the KdV Whitham hierarchy. This means we start with an admissible coefficient polynomial a_l. The recursion scheme in Proposition 5.3.9 then determines coefficient polynomials a_k for $k \neq l$ such that the *factor Ansatz* (5.3.17) (formally) gives a generating differential form for the KdV Whitham hierarchy. Only such polynomials a_l are considered admissible that yield coefficient polynomials $2a_k(E) = i(-1)^{k+1} c_{2k} g'(E)$ such that the series in (5.3.9) converges near $(E, y) = (\gamma_1, 0)$.

The system of differential equations for the generating differential form and the equivalent recursion scheme for the factor representation have a version in coordinates $\boldsymbol{\gamma}$ instead of coordinates T_J. In order to obtain this, let differential forms $\mathrm{d}\Omega_l^\gamma = \mathrm{d}\Omega_l^\gamma(\boldsymbol{\gamma}, \partial_{T_J}\boldsymbol{\gamma})$ for $l = 1, \dots 2n+1$ be defined by

$$\mathrm{d}\Omega_\alpha = \sum_{l=1}^{2n+1} (\partial_{T_{2\alpha+1}}\gamma_l)\mathrm{d}\Omega_l^\gamma \ . \tag{5.3.18}$$

For the polynomials C_l^γ with $\mathrm{d}\Omega_l^\gamma = (C_l^\gamma(E)\mathrm{d}E)/y$ we have $C_\alpha = \sum_{l=1}^{2n+1}(\partial_{T_{2\alpha+1}}\gamma_l)C_l^\gamma$. The differential equations (5.3.1) for the generating differential form then yield

$$0 = \mathrm{d}\Omega_\alpha - \partial_{T_{2\alpha+1}}\mathrm{d}S = \sum_{l=1}^{2n+1} (\partial_{T_{2\alpha+1}}\gamma_l)\left[\mathrm{d}\Omega_l^\gamma - \partial_{\gamma_l}\mathrm{d}S\right] \ .$$

Since the Jacobian matrix $\partial_{T_J}\boldsymbol{\gamma}$ is invertible, the term in the square brackets has to vanish that is

$$\partial_{\gamma_l}\mathrm{d}S = \mathrm{d}\Omega_l^\gamma \tag{5.3.19}$$

for all $l = 1, \dots 2n+1$. Here $\mathrm{d}S$ and also the coefficient polynomials a_k of its factor representation depend on $\boldsymbol{\gamma}$ and $\partial_{T_J}\boldsymbol{\gamma}$. Conversely, given some coefficient polynomials a_k (depending only on $\boldsymbol{\gamma}$) such that the factor Ansatz solves (5.3.19), then (5.3.18) induces a system of ODEs whose solution is a solution $T_J \mapsto \boldsymbol{\gamma}(T_J)$ for the KdV Whitham hierarchy. For example, assuming $0 \in J$, the ODE for $X = T_1 \mapsto \boldsymbol{\gamma}(X)$ reads

$$\mathrm{d}\Omega_0(E;\boldsymbol{\gamma}) = \sum_{l=1}^{2n+1} (\partial_X\gamma_l)\mathrm{d}\Omega_l^\gamma(E;\boldsymbol{\gamma}) \ .$$

A version of this equation without the spectral parameter E can be obtained by evaluation of E at $\gamma_1, \dots \gamma_{2n+1}$ or by considering the coefficients of the Laurent expansion at $E = \infty$.

The main result of the present section is about how to characterize those coefficient polynomials a_k that correspond to a generating differential form $\mathrm{d}S$ in (5.3.19). For simplicity of the description it is assumed from now on that $J = \{-n, \dots, n\}$.

Theorem 5.3.10. *Given a generating differential form* $\mathrm{d}S$*, then coefficient polynomials* a_k *in Ansatz* (5.3.17) *are recursively determined such that*

- $a_0 = 0$ *and*
- *there is a solution* $\mathfrak{a}$ *for an EPD with* $\epsilon = 3/2$*, which is related to* a_2 *by*

$$\frac{3}{2}a_2|_{E=\gamma_j} = \partial_{\gamma_j}\mathfrak{a} \tag{5.3.20}$$

and all a_k *are determined by the recursion uniquely.*

Conversely, let a solution $\mathfrak{a}$ *for an EPD with* $\epsilon = 3/2$ *be given such that*

- (5.3.20) *induces a coefficient polynomial* a_2 *of degree* $2n$ *and*
- *the recursively determined coefficient polynomials* a_1 *and* a_k *for* $k \geq 3$ *yield a power series in the factor Ansatz* (5.3.17) *converging locally near* $(E, y) = (\gamma_1, 0)$*,*

then this series defines a generating differential form in (5.3.19).

A proof of Theorem 5.3.10 will be given in Section 5.3.4. It will be mostly about the compatibility of a recursion scheme related to the one in Proposition 5.3.9. We end this section with the following remarks.

Remark 5.3.11. For the recursively determined coefficient polynomials a_k with $k \geq 3$ as well, there are solutions $\mathfrak{a}_k$ for an ϵ-system with $\epsilon = (2k-1)/2$ such that

$$\frac{1}{2}(2k-1)a_k|_{E=\gamma_j} = \partial_{\gamma_j}\mathfrak{a}_k \ .$$

Once indices $\alpha > n$ are considered as part of the index set J, then the corresponding normalized differential forms $\mathrm{d}\Omega_\alpha$ will show up as $C_{\alpha l}$ in the recursion (5.3.15) beyond the step $l = 0$. In this case, one of the ϵ-systems with $\epsilon > 3/2$ can be used to formulate the theorem.

Remark 5.3.12. Given a solution $\mathfrak{a}$ for an EPD with $\epsilon = 3/2$, it is difficult to see, whether the induced sequence of coefficient polynomials $2a_k(E) = i(-1)^{k+1}c_{2k}g'(E)$ leads to a generating function in (5.3.17) that converges at least locally near $(E, y) = (\gamma_1, 0)$. This question will not be addressed here. However, note that the following proof does not rely on the convergence of $\mathrm{d}S$. Without convergence Theorem 5.3.10 can be seen as an alternative version of Tsarev's generalized hodograph method in Theorem 2.1.3 with an ϵ-system taking the place of equation (2.1.15) for commuting flows.

5.3.4. Proof of the Theorem. The proof of Theorem 5.3.10 relies on the use of coordinates $\boldsymbol{\gamma}$ instead of T_J. In this section it is always assumed that $J = \{-n, \dots, n\}$, hence all normalized differential forms $\mathrm{d}\Omega_\alpha = (C_\alpha \mathrm{d}E)/y$ are represented by polynomials C_α of degree up to $2n$. It follows that also the polynomials C_l^γ defined in (5.3.18) are of degree up to $2n$ only.

For the recursion scheme of the factor representation there is a version in coordinates $\boldsymbol{\gamma}$. With the same proofs as for Proposition 5.3.6 and Proposition 5.3.9 (or as a corollary) we get the following.

Proposition 5.3.13. *For the factor representation* (5.3.17) *of a generating differential form* $\mathrm{d}S$ *the equations* (5.3.19) *take the form* $a_0 = 0$ *and for all* $k = 1, \dots, 2n+1$ *holds*

$$-iC_k^\gamma = \frac{1}{2}a_1\partial_{\gamma_k}g + \sum_{l\geq 1}\left[\partial_{\gamma_k}a_l + \left(l + \frac{1}{2}\right)a_{l+1}\partial_{\gamma_k}g\right]g^l \ . \tag{5.3.21}$$

Equivalently, there are uniquely determined auxiliary polynomials f_k^l *of degree up to* $2n-1$ *(*$l \geq 1$*) such that* (5.3.21) *takes the form*

$$-iC_k^\gamma = \frac{1}{2}\partial_{\gamma_k}g a_1 + g f_k^1 \ , \tag{5.3.22}$$

$$0 = -f_k^l + \partial_{\gamma_k}a_l + \left(l + \frac{1}{2}\right)a_{l+1}\partial_{\gamma_k}g + f_k^{l+1}g \ . \tag{5.3.23}$$

The first equation (5.3.22) can always be solved for some polynomials a_1 and f_k^1. For the recursion in (5.3.23) there is the more compact form

$$0 = \partial_{\gamma_k}\boldsymbol{a} + \partial_{\gamma_k}gA\boldsymbol{a} + (gF - \mathrm{I})\boldsymbol{f}_k \tag{5.3.24}$$

with $\boldsymbol{a} = (a_1, a_2, \dots)^t$, $\boldsymbol{f}_k = (f_k^1, f_k^2, \dots)^t$, and $F = (\delta_{i+1,j})_{i,j\geq 1}$, $\mathrm{I} = (\delta_{i,j})_{i,j\geq 1}$ and $A = \frac{1}{2}\operatorname{diag}(2l+1)_{l\geq 1}F$. Note that the matrices F, A, I here are those defined in (5.3.16), but with the first column and row deleted. The proof of Theorem 5.3.10 will be mainly about the compatibility of equation (5.3.24), that is

$$\partial_{\gamma_k}\partial_{\gamma_l}\boldsymbol{a} = \partial_{\gamma_l}\partial_{\gamma_k}\boldsymbol{a} \tag{5.3.25}$$

for all $k, l = 1, \dots, 2n+1$. We begin by expressing the compatibility in terms of the vectors of polynomials $\boldsymbol{f}_k$.

Proposition 5.3.14. *The compatibility conditions* (5.3.25) *are equivalent to the existence of some auxiliary coefficients* $\boldsymbol{f}_{lk} = (f_{lk}^1, f_{lk}^2, \dots)^t$ *such that for all* $l, k = 1, \dots, 2n+1$ *holds*

$$0 = \partial_{\gamma_k} \boldsymbol{f}_l - \partial_{\gamma_l} \boldsymbol{f}_k + \partial_{\gamma_l} \partial_{\gamma_k} g \boldsymbol{f}_{lk} \ , \tag{5.3.26}$$

$$0 = (F - A) \left[(E - \gamma_l) \boldsymbol{f}_l - (E - \gamma_k) \boldsymbol{f}_k \right] - \boldsymbol{f}_{lk} \ . \tag{5.3.27}$$

Proof. We consider the recursion scheme in the form (5.3.24) and express its derivative with respect to γ_l by

$$\begin{aligned} 0 &= \partial_{\gamma_l} \partial_{\gamma_k} \boldsymbol{a} + \partial_{\gamma_l} \partial_{\gamma_k} g A \boldsymbol{a} + \partial_{\gamma_k} g A \partial_{\gamma_l} \boldsymbol{a} + \partial_{\gamma_l} g F \boldsymbol{f}_k + (gF - \mathrm{I}) \partial_{\gamma_l} \boldsymbol{f}_k \\ &= \partial_{\gamma_l} \partial_{\gamma_k} \boldsymbol{a} + \partial_{\gamma_l} \partial_{\gamma_k} g A \boldsymbol{a} - \partial_{\gamma_k} g A \left(\partial_{\gamma_l} g A \boldsymbol{a} + (gF - \mathrm{I}) \boldsymbol{f}_l \right) + \partial_{\gamma_l} g F \boldsymbol{f}_k + (gF - \mathrm{I}) \partial_{\gamma_l} \boldsymbol{f}_k \ . \end{aligned}$$

The second equation is obtained by substituting $\partial_{\gamma_l} \boldsymbol{a}$ from the recursion (5.3.24). Now the compatibility condition $\partial_{\gamma_k} \partial_{\gamma_l} \boldsymbol{a} = \partial_{\gamma_l} \partial_{\gamma_k} \boldsymbol{a}$ of the recursion becomes equivalent to

$$0 = \left(-A(gF - \mathrm{I}) - F \right) \left(\partial_{\gamma_k} g \boldsymbol{f}_l - \partial_{\gamma_l} g \boldsymbol{f}_k \right) + (gF - \mathrm{I}) \left(\partial_{\gamma_l} \boldsymbol{f}_k - \partial_{\gamma_k} \boldsymbol{f}_l \right) \ . \tag{5.3.28}$$

For $r \neq k, l$ evaluating the previous equation at $E = \gamma_r$ gives $0 = -\left(\partial_{\gamma_l} \boldsymbol{f}_k - \partial_{\gamma_k} \boldsymbol{f}_l \right) |_{\gamma_r}$. The polynomials in each $\boldsymbol{f}_l$ are of degree strictly less than $2n$, thus there are auxiliary scalar coefficients f_{lk}^j such that for $\boldsymbol{f}_{lk} = (f_{lk}^1, f_{lk}^2, \dots)^t$ it holds

$$0 = \partial_{\gamma_k} \boldsymbol{f}_l - \partial_{\gamma_l} \boldsymbol{f}_k + \partial_{\gamma_l} \partial_{\gamma_k} g \boldsymbol{f}_{lk} \ .$$

This is the equation stated in (5.3.26). Inserting (5.3.26) into the compatibility equation (5.3.28) and using $\partial_{\gamma_k} g = -(E - \gamma_l) \partial_{\gamma_l} \partial_{\gamma_k} g$ yields

$$0 = \left(-A(gF - \mathrm{I}) - F \right) \left(-(E - \gamma_l) \boldsymbol{f}_l + (E - \gamma_k) g \boldsymbol{f}_k \right) + (gF - \mathrm{I}) \boldsymbol{f}_{lk} \ .$$

By evaluation at $E = \gamma_r$ for $r = 1, \dots, 2n+1$ this equation turns into

$$0 = (-A + F) \left[(\gamma_r - \gamma_l) \boldsymbol{f}_l |_{\gamma_r} - (\gamma_r - \gamma_k) \boldsymbol{f}_k |_{\gamma_r} \right] - \boldsymbol{f}_{lk} \ .$$

The polynomials in each coefficient vector $\boldsymbol{f}_j$ are of degree at most $2n - 1$, so the previous equation has to hold even without the evaluation at γ_r. Hence, the equation in (5.3.27) has been shown under the assumption of the compatibility (5.3.25).

Conversely, when (5.3.26) and (5.3.27) are given, then (5.3.28) reduces to

$$0 = AF \left[(E - \gamma_l) \boldsymbol{f}_l - (E - \gamma_k) \boldsymbol{f}_k \right] + F \boldsymbol{f}_{lk} \ .$$

This equation holds due to (5.3.27). □

There is another way to express the recursion (5.3.24) and its compatibility (5.3.25), as we are going to see now in Proposition 5.3.15 and Proposition 5.3.16, respectively. Combining both descriptions will then allow to eliminate the auxiliary coefficients $\boldsymbol{f}_l$ and $\boldsymbol{f}_{lk}$, see Lemma 5.3.17 below.

Since $g = \prod_{l=1}^{2n+1} (E - \gamma_l)$ has only simple zeros its derivatives $\partial_{\gamma_j} g$ $(j = 1, \dots, 2n+1)$ can be used as a basis for the polynomials of degree up to $2n$. That is, for a polynomial P with $\deg P \leq 2n$ there are linear coefficients $p_i \in \mathbb{C}$ such that $P(E) = \sum_{j=1}^{2n+1} p_j \partial_{\gamma_j} g(E)$. Likewise, each polynomial f_k^l of degree up to $2n - 1$ can be expressed by

$$f_k^l = \sum_{r \neq k} \phi_{kr}^l \partial_{\gamma_k} \partial_{\gamma_r} g = -\sum_{r \neq k} \phi_{kr}^l \frac{\partial_{\gamma_k} g}{E - \gamma_r} \tag{5.3.29}$$

for some auxiliary scalar coefficients ϕ_{kr}^l. Define $\boldsymbol{\phi}_{kr} = (\phi_{kr}^1, \phi_{kr}^2, \dots)^t$. As a consequence, the recursion scheme (5.3.24) can be rewritten as

$$0 = \partial_{\gamma_k} \boldsymbol{a} + \partial_{\gamma_k} g \left(A\boldsymbol{a} - (gF - \mathrm{I}) \sum_{r \neq k} \boldsymbol{\phi}_{kr} \frac{1}{E - \gamma_r} \right) \quad . \tag{5.3.30}$$

Proposition 5.3.15. *There are auxiliary coefficients* $\boldsymbol{h}_k = (h_k^1, h_k^2, \dots)^t$ *such that the recursion* (5.3.30) *becomes*

$$0 = \partial_{\gamma_k} \boldsymbol{a} + \partial_{\gamma_k} g \left(\boldsymbol{h}_k + \sum_{r \neq k} \boldsymbol{\phi}_{kr} \frac{1}{E - \gamma_r} \right) \quad . \tag{5.3.31}$$

These coefficients are determined by $\boldsymbol{h}_k = A\boldsymbol{a}|_{\gamma_k}$ *and satisfy*

$$0 = \boldsymbol{h}_k - \boldsymbol{h}_r - \partial_{\gamma_r} g|_{\gamma_r} F \boldsymbol{\phi}_{kr} \quad . \tag{5.3.32}$$

Proof. For $j \neq k$ the polynomial $\partial_{\gamma_k} g$ has a zero of order one at $E = \gamma_j$, so evaluating (5.3.30) there gives

$$0 = \left(\partial_{\gamma_k} \boldsymbol{a} + \partial_{\gamma_k} g \sum_{r \neq k} \boldsymbol{\phi}_{kr} \frac{1}{E - \gamma_r} \right) \Bigg|_{\gamma_j} \quad .$$

The degree of the term in the bracket is strictly less than $2n+1$. Hence, there are auxiliary scalar coefficients h_k^j such that for $\boldsymbol{h}_k = (h_k^1, h_k^2, \dots)^t$ it holds

$$0 = \partial_{\gamma_k} \boldsymbol{a} + \partial_{\gamma_k} g \sum_{r \neq k} \boldsymbol{\phi}_{kr} \frac{1}{E - \gamma_r} + \boldsymbol{h}_k \partial_{\gamma_k} g \quad .$$

This is the recursion scheme in the form (5.3.31). The difference with (5.3.30) then eliminates $\partial_{\gamma_k} \boldsymbol{a}$ and yields

$$0 = \boldsymbol{h}_k - A\boldsymbol{a} + gF \sum_{r \neq k} \boldsymbol{\phi}_{kr} \frac{1}{E - \gamma_r} \quad .$$

By evaluation at $E = \gamma_k$ follows the first identity for the coefficients $\boldsymbol{h}_k$. Evaluation at $E = \gamma_r$ with $r \neq k$ and $\partial_{\gamma_r} g = -g/(E - \gamma_r)$ imply the second identity. □

Now for the compatibility of the recursion in the form of (5.3.31) we have.

Proposition 5.3.16. *The compatibility conditions* (5.3.25) *are equivalent to*

$$0 = \partial_{\gamma_l} \boldsymbol{h}_k - \partial_{\gamma_k} \boldsymbol{h}_l \quad , \tag{5.3.33}$$

$$0 = \boldsymbol{h}_l - \boldsymbol{h}_k + (\gamma_k - \gamma_l) \partial_{\gamma_k} \boldsymbol{h}_l - \sum_{r \neq l} \partial_{\gamma_k} \boldsymbol{\phi}_{lr} + \sum_{r \neq k} \partial_{\gamma_l} \boldsymbol{\phi}_{kr} \quad , \tag{5.3.34}$$

$$0 = \boldsymbol{\phi}_{lr} - \boldsymbol{\phi}_{kr} - (\gamma_r - \gamma_k) \partial_{\gamma_k} \boldsymbol{\phi}_{lr} + (\gamma_r - \gamma_l) \partial_{\gamma_l} \boldsymbol{\phi}_{kr} \quad \textit{for} \quad r \neq l, k \quad . \tag{5.3.35}$$

Proof. Let us start by differentiating the recursion scheme in the form (5.3.31) with respect to γ_l

$$0 = \partial_{\gamma_l}\partial_{\gamma_k}\boldsymbol{a} + \partial_{\gamma_l}\partial_{\gamma_k} g \left(\boldsymbol{h}_k + \sum_{r\neq k} \boldsymbol{\phi}_{kr} \frac{1}{E-\gamma_r} \right) + \partial_{\gamma_k} g \left(\partial_{\gamma_l}\boldsymbol{h}_k + \partial_{\gamma_l}\left(\boldsymbol{\phi}_{kl}\frac{1}{E-\gamma_l} \right) + \sum_{r\neq k,l} \partial_{\gamma_l}\boldsymbol{\phi}_{kr}\frac{1}{E-\gamma_r} \right) .$$

Then, by the help of $\partial_{\gamma_k} g = -(E-\gamma_l)\partial_{\gamma_l}\partial_{\gamma_k} g$ the compatibility condition $\partial_{\gamma_k}\partial_{\gamma_l}\boldsymbol{a} = \partial_{\gamma_l}\partial_{\gamma_k}\boldsymbol{a}$ is equivalent to

$$\begin{aligned} 0 = {} & \boldsymbol{h}_l - \boldsymbol{h}_k + \boldsymbol{\phi}_{lk}\frac{1}{E-\gamma_k} - \boldsymbol{\phi}_{kl}\frac{1}{E-\gamma_l} + \sum_{r\neq k,l} (\boldsymbol{\phi}_{lr} - \boldsymbol{\phi}_{kr}) \frac{1}{E-\gamma_r} \\ & - (E-\gamma_k)\left(\partial_{\gamma_k}\boldsymbol{h}_l + \partial_{\gamma_k}\left(\boldsymbol{\phi}_{lk}\frac{1}{E-\gamma_k} \right) + \sum_{r\neq k,l} \partial_{\gamma_k}\boldsymbol{\phi}_{lr}\frac{1}{E-\gamma_r} \right) \\ & + (E-\gamma_l)\left(\partial_{\gamma_l}\boldsymbol{h}_k + \partial_{\gamma_l}\left(\boldsymbol{\phi}_{kl}\frac{1}{E-\gamma_l} \right) + \sum_{r\neq k,l} \partial_{\gamma_l}\boldsymbol{\phi}_{kr}\frac{1}{E-\gamma_r} \right) . \end{aligned}$$

At $E = \gamma_l$ and $E = \gamma_k$ the poles in this equation cancel out, so we arrive at the simpler version of the compatibility conditions

$$\begin{aligned} 0 = {} & \boldsymbol{h}_l - \boldsymbol{h}_k + \sum_{r\neq k,l} (\boldsymbol{\phi}_{lr} - \boldsymbol{\phi}_{kr}) \frac{1}{E-\gamma_r} \\ & - (E-\gamma_k)\left(\partial_{\gamma_k}\boldsymbol{h}_l + \sum_{r\neq l} \partial_{\gamma_k}\boldsymbol{\phi}_{lr}\frac{1}{E-\gamma_r} \right) \\ & + (E-\gamma_l)\left(\partial_{\gamma_l}\boldsymbol{h}_k + \sum_{r\neq k} \partial_{\gamma_l}\boldsymbol{\phi}_{kr}\frac{1}{E-\gamma_r} \right) . \end{aligned} \tag{5.3.36}$$

The asymptotic behavior for $E \to \infty$ and the poles at $E = \gamma_r$ determine this equation up to a constant part in E.

- For $E \to \infty$ the compatibility equation (5.3.36) implies (5.3.33).
- Looking at the principal part of (5.3.36) at $E = \gamma_r$ implies (5.3.35).
- The constant part of (5.3.36) is

  $$0 = \boldsymbol{h}_l - \boldsymbol{h}_k + \gamma_k\partial_{\gamma_k}\boldsymbol{h}_l - \sum_{r\neq l}\partial_{\gamma_k}\boldsymbol{\phi}_{lr} - \gamma_l\partial_{\gamma_l}\boldsymbol{h}_k + \sum_{r\neq k}\partial_{\gamma_l}\boldsymbol{\phi}_{kr}$$

  and implies together with (5.3.33) the equation (5.3.34).

Conversely, the compatibility conditions for the recursion scheme in the form (5.3.36) follow from (5.3.33), (5.3.34) and (5.3.35). □

The equations (5.3.33) provide compatibility for the system of equations

$$\partial_{\gamma_k}\mathfrak{a} = \boldsymbol{h}_k \tag{5.3.37}$$

with $\mathfrak{a} = (\mathfrak{a}^1, \mathfrak{a}^2, \mathfrak{a}^3, \dots)^t$ a vector of scalar-valued functions. We aim now at proving that

- the equations (5.3.37) for $\mathfrak{a}^1$ are equivalent to the compatibility (5.3.25) of the recursion scheme, and
- that each $\mathfrak{a}^j$ satisfies an ϵ-system with $\epsilon = (2j-1)/2$, respectively.

By eliminating the auxiliary coefficients $\boldsymbol{f}_k, \boldsymbol{f}_{lk}, \boldsymbol{\phi}_{kr}$ and $\boldsymbol{h}_k$ again, the compatibility of the recursion scheme takes the following form.

Lemma 5.3.17. *The compatibility conditions* (5.3.25) *are equivalent to*

$$(\gamma_k - \gamma_l)\partial_{\gamma_l}\partial_{\gamma_k}\mathfrak{a} = \frac{3}{2}\left(\partial_{\gamma_k}\mathfrak{a} - \partial_{\gamma_l}\mathfrak{a}\right) \tag{5.3.38}$$

for $\mathfrak{a} := \mathfrak{a}^1$ *and all* $l, k = 1, \dots, 2n+1$ *with* $l \neq k$. *Given compatibility, then*

$$(\gamma_k - \gamma_l)\partial_{\gamma_l}\partial_{\gamma_k}\mathfrak{a}^j = \frac{1}{2}(2j+1)\left(\partial_{\gamma_k}\mathfrak{a}^j - \partial_{\gamma_l}\mathfrak{a}^j\right) \tag{5.3.39}$$

holds for all $j \geq 2$.

Proof. We start by comparing the two versions of the compatibility conditions in Proposition 5.3.14 and Proposition 5.3.16 and conclude that compatibility implies the equations in the statement. Equation (5.3.26) from Proposition 5.3.14 gives $\partial_{\gamma_l}\partial_{\gamma_k} g \boldsymbol{f}_{lk} = \partial_{\gamma_l}\boldsymbol{f}_k - \partial_{\gamma_k}\boldsymbol{f}_l$. From the definition of the coefficients $\boldsymbol{\phi}_{kl}$ in (5.3.29) we have

$$\begin{aligned}
\partial_{\gamma_l}\boldsymbol{f}_k &= \sum_{r\neq k}\partial_{\gamma_l}\boldsymbol{\phi}_{kr}\partial_{\gamma_k}\partial_{\gamma_r}g + \sum_{r\neq l,k}\boldsymbol{\phi}_{kr}\partial_{\gamma_l}\partial_{\gamma_k}\partial_{\gamma_r}g \\
&= \left(\sum_{r\neq k}\partial_{\gamma_l}\boldsymbol{\phi}_{kr}\frac{E-\gamma_l}{E-\gamma_r} - \sum_{r\neq l,k}\boldsymbol{\phi}_{kr}\frac{1}{E-\gamma_r}\right)\partial_{\gamma_l}\partial_{\gamma_k}g \ .
\end{aligned}$$

Together this implies

$$\boldsymbol{f}_{lk} = \sum_{r\neq k}\partial_{\gamma_l}\boldsymbol{\phi}_{kr}\frac{E-\gamma_l}{E-\gamma_r} - \sum_{r\neq l}\partial_{\gamma_k}\boldsymbol{\phi}_{lr}\frac{E-\gamma_k}{E-\gamma_r} - \sum_{r\neq l,k}\left(\boldsymbol{\phi}_{kr} - \boldsymbol{\phi}_{lr}\right)\frac{1}{E-\gamma_r} \ .$$

In the limit $E \to \infty$ we see that the auxiliary coefficients $\boldsymbol{f}_{lk}$ and $\boldsymbol{\phi}_{jr}$ are related by

$$\boldsymbol{f}_{lk} = \sum_{r\neq k}\partial_{\gamma_l}\boldsymbol{\phi}_{kr} - \sum_{r\neq l}\partial_{\gamma_k}\boldsymbol{\phi}_{lr} \ .$$

This turns (5.3.34) from Proposition 5.3.16 into

$$0 = \boldsymbol{h}_l - \boldsymbol{h}_k + (\gamma_k - \gamma_l)\partial_{\gamma_k}\boldsymbol{h}_l + \boldsymbol{f}_{lk} \ . \tag{5.3.40}$$

Furthermore, evaluating the second relation in Proposition 5.3.14 at $E = \gamma_k$ yields

$$\boldsymbol{f}_{lk} = (F-A)(\gamma_k - \gamma_l)\boldsymbol{f}_l|_{\gamma_k} = (F-A)(\gamma_k-\gamma_l)\boldsymbol{\phi}_{lk}\partial_{\gamma_l}\partial_{\gamma_k}g|_{\gamma_k} \ .$$

For the second equality the definition of $\boldsymbol{\phi}_{lr}$ in (5.3.29) has been used again. With $(E - \gamma_l)\partial_{\gamma_l}\partial_{\gamma_k}g = -\partial_{\gamma_k}g$ it follows that

$$\boldsymbol{f}_{lk} = -(F-A)\boldsymbol{\phi}_{lk}\partial_{\gamma_k}g|_{\gamma_k} \ .$$

Due to $A - F = \frac{1}{2}\operatorname{diag}(2j-1)_{j\geq 1}F$ and (5.3.32), the previous equation simplifies to

$$\boldsymbol{f}_{lk} = \frac{1}{2}\operatorname{diag}(2l-1)_{l\geq 1}\left(\boldsymbol{h}_l - \boldsymbol{h}_k\right) \ .$$

Together with (5.3.40) we then arrive at

$$0 = (\gamma_k - \gamma_l)\partial_{\gamma_k}\boldsymbol{h}_l + \frac{1}{2}\operatorname{diag}(2j+1)_{j\geq 1}(\boldsymbol{h}_l - \boldsymbol{h}_k) \quad .$$

Substituting $\partial_{\gamma_l}\mathfrak{a} = \boldsymbol{h}_l$ from (5.3.37) yields finally

$$(\gamma_k - \gamma_l)\partial_{\gamma_l}\partial_{\gamma_k}\mathfrak{a} = \frac{1}{2}\operatorname{diag}(2j+1)_{j\geq 1}(\partial_{\gamma_k}\mathfrak{a} - \partial_{\gamma_l}\mathfrak{a}) \quad .$$

This equation implies the equations (5.3.38) and (5.3.39) from the statement of the lemma.

Conversely, let a solution $\mathfrak{a}$ of (5.3.38) be given. Then $\partial_{\gamma_k}\mathfrak{a}$ determines h_k^1 in (5.3.37). Since we have $\boldsymbol{h}_k = A\boldsymbol{a}|_{\gamma_k}$ in Proposition 5.3.15, we set $h_k^1 =: \frac{3}{2}a_2|_{\gamma_k}$, so a polynomial a_2 of degree up to $2n$ is determined. Inserting a_2 into (5.3.23) for $l = 1$ gives the system (for unknowns f_k^1, f_k^2 and a_1)

$$f_k^1 - \partial_{\gamma_k}a_1 = \frac{3}{2}a_2\partial_{\gamma_k}g + f_k^2 g$$

which is compatible by construction. Hence, evaluation at $E = \gamma_j$ for $j = 1, \dots, 2n+1$ determines polynomials $P_k := f_k^1 - \partial_{\gamma_k}a_1$ of degree up to $2n$. It follows that

- there are polynomials f_k^2 of degree up to $2n-1$ such that $(5.3.23)_{l=1}$ holds, and
- there are a scalar coefficient c_2 and polynomials f_k^1 of degree up to $2n-1$ such that

$$f_k^1 - \partial_{\gamma_k}\left(\frac{i}{2}c_2 g'\right) = P_k \quad .$$

  The polynomial $2a_1 = ic_2g'$ is chosen such that it is of the form required by the factor representation (5.3.10).

Inserting a_2 and f_k^2 into (5.3.23) for $l = 2$ then gives the system (for unknowns f_k^3 and a_3)

$$f_k^2 - \partial_{\gamma_k}a_2 = \frac{5}{2}a_3\partial_{\gamma_k}g + f_k^3 g$$

which is compatible by construction. Hence, evaluation at $E = \gamma_j$ for $j = 1, \dots, 2n+1$ determines a_3. Then there are polynomials f_k^3 of degree up to $2n-1$ such that $(5.3.23)_{l=2}$ holds.

Following the recursion scheme (5.3.23) for $l \geq 3$ in the same way, determines suitable polynomials a_l and f_k^l. The resulting data $\boldsymbol{a} := (a_1, a_2, \dots)^t$, $\boldsymbol{f}_k := (f_k^1, f_k^2, \dots)^t$ satisfies the recursion equation (5.3.24) and is in particular compatible. $\square$

Remark 5.3.18. When considering a general index set $J \subseteq \{-n, \dots, 0, 1, \dots\}$ with $|J| = 2n+1$ instead of $J = \{-n, \dots, n\}$, then (5.3.39) will be still valid starting from a higher index than 1. This is due to the appearance of polynomials C_{kl}^{γ} for $l \geq 1$ on the left hand side of (5.3.23).

Now, the proof of Theorem 5.3.10 can be concluded as follows. Given a generating differential form $\mathrm{d}S$ satisfying (5.3.19), then the recursion scheme for the coefficient polynomials a_k of the factor Ansatz (5.3.17) is compatible. By Lemma 5.3.17 there is a function $\mathfrak{a} = \mathfrak{a}^1$ satisfying an ϵ-system with $\epsilon = 3/2$ such that (5.3.20) holds, i.e. $\frac{3}{2}a_2|_{E=\gamma_j} = \partial_{\gamma_j}\mathfrak{a}$. This determines the polynomial a_2 uniquely. The other coefficient polynomials are uniquely determined by the recursion scheme (5.3.24).

Conversely, let a solution $\mathfrak{a}$ satisfying an ϵ-system with $\epsilon = 3/2$ be given such that the polynomial a_2 determined by $\frac{3}{2}a_2|_{E=\gamma_j} = \partial_{\gamma_j}\mathfrak{a}$ has degree $2n$. Then the recursion scheme (5.3.24) provides further polynomials a_k for $k = 1$ and $k \geq 3$. Inserted into the factor

Ansatz (5.3.17), the a_k give a formal differential form $\mathrm{d}S$. If this power series converges locally near $(E, y) = (\gamma_1, 0)$, then $\mathrm{d}S$ defines a generating differential form in (5.3.19). Altogether, the Theorem 5.3.10 has been shown.

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