# On Pairwise <br> Graph Connectivity 

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#### Abstract

A graph on at least $k+1$ vertices is said to have global connectivity $k$ if any two of its vertices are connected by $k$ independent paths. The local connectivity of two vertices is the number of independent paths between those specific vertices. This dissertation is concerned with pairwise connectivity notions, meaning that the focus is on local connectivity relations that are required for a number of or all pairs of vertices. We give a detailed overview about how uniformly $k$-connected and uniformly $k$-edge-connected graphs are related and provide a complete constructive characterization of uniformly 3 -connected graphs, complementing classical characterizations by Tutte. Besides a tight bound on the number of vertices of degree three in uniformly 3 -connected graphs, we give results on how the crossing number and treewidth behaves under the constructions at hand. The second central concern is to introduce and study cut sequences of graphs. Such a sequence is the multiset of edge weights of a corresponding Gomory-Hu tree. The main result in that context is a constructive scheme that allows to generate graphs with prescribed cut sequence if that sequence satisfies a shifted variant of the classical Erdős-Gallai inequalities. A complete characterization of realizable cut sequences remains open. The third central goal is to investigate the spectral properties of matrices whose entries represent a graph's local connectivities. We explore how the spectral parameters of these matrices are related to the structure of the corresponding graphs, prove bounds on eigenvalues and related energies, which are sums of absolute values of all eigenvalues, and determine the attaining graphs. Furthermore, we show how these results translate to ultrametric distance matrices and touch on a Laplace analogue for connectivity matrices and a related isoperimetric inequality.


Keywords. Uniform connectivity, cut sequences, connectivity matrices, ultrametric matrices, graph constructions

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## Introduction

Connectivity properties belong to the fundamental concepts studied in graph theory. We say that a graph on at least $k+1$ vertices is $k$-connected if there are $k$ independent, meaning internally vertex disjoint, paths between any two of its vertices. Likewise, a graph on at least two vertices is called $k$-edgeconnected if any two vertices are connected by $k$ edge-disjoint paths. Such concepts are often referred to as global connectivity measures, as they require to be satisfied by all vertex pairs. While these concepts certainly provide important structural information about the strength of connectivity in a graph, they often provide only a rather rough picture of a graph's connectivity properties. For example, the graph in Figure 1 is connected, but not 2-connected, because it contains a vertex of degree one. So the graph at hand has the same connectivity as a tree, although, intuitively, we may say that both graphs are far from being connected in a comparable way. From the perspective of network reliability, such a global connectivity measure corresponds to a worst-case analysis. Only one pair of poorly connected vertices causes a low global connectivity value. On the other hand, the local connectivity of two vertices is defined as the number of independent paths between those specific vertices. Likewise, the local edge-connectivity is the number of edge-disjoint paths between two vertices. In Figure 1, the local connectivity of $v$ and $w$ is one, the local edge-connectivity is two. Those are quite precise measures of connectivity for the respective vertices. But, in general, it tells little about the rest of the graph.


Figure 1: A graph and its Gomory-Hu tree

This dissertation is concerned with what we call pairwise connectivity relations. This means that we are interested in local connectivities, but within the questions we tackle, we typically require local connectivity relations for a number of or all pairs of vertices. This is why the connectivity notions we are concerned with have both local and global flavors. The main interest of this work is to contribute to the theory of such pairwise connectivity measures.

## The structure of this work

Following the introduction, Chapter 2 introduces basic concepts and connectivity notions that we make use of throughout our investigation. A particular interest is in Menger's Theorem [81] and several of its versions, presented by Diestel [35] or Göring [55]. Furthermore, we discuss the rich properties of Gomory-Hu trees [50], for which there exist now algorithms of nearly quadratic running time due to the remarkable progress by Chen, Kyng, Liu, Peng, Gutenberg, and Sachdeva [21], Abboud, Krauthgamer, Li, Panigrahi, Saranurak, and Trabelsi [1] or Zhang [109]. The subsequent topics of this dissertation are organized according to how restrictive the pairwise connectivity relations are that we are concerned with.

We begin with uniform connectivity in Chapter 3. A uniformly $k$-connected graph is a graph on at least $k+1$ vertices where each pair of vertices is connected by $k$ independent paths, and no pair is connected by more than $k$ independent paths. A graph on at least two vertices is called uniformly $k$-edge-connected if any two of its vertices are connected by $k$ and not more than $k$ edge-disjoint paths. The graph in Figure 1 is far from being uniformly connected, but the block containing $v$ is uniformly 2 -connected as well as uniformly 2 -edge-connected. The vertex version of this concept is studied by Beineke, Oellermann, and Pippert [5], the edge version is studied independently by Kingsford and Marçais [70]. A purpose of this chapter is to show how both concepts are related [Example 20, Theorem 22]. A key contribution is a constructive characterization of uniformly 3-connected graphs [Theorem 35], which complements classical results by Tutte [102, 103].

This has several applications. For example, it allows to answer questions about the minimum number of vertices of minimum degree in uniformly 3 -connected graphs [Theorem 37]. For minimally $k$-connected graphs, this parameter is studied extensively by Halin [60], Mader [77], or Schmidt [90]. We also discuss how crossing numbers and treewidths behave under the constructions at hand [Theorem 39, Theorem 51]. The latter is of importance as many hard combinatorial problems admit polynomial time algorithms if the input graph has bounded treewidth, as discussed by Arnborg and Proskurowski [3] or Kleinberg and Tardos [72, Chapter 10].

In Chapter 4, we do not require the same connectivity between each pair of vertices, but we investigate graphs for which we prescribe, possibly different, cut values. For this purpose, we introduce a graph's cut sequence. This is the multiset of edge weights of a corresponding Gomory-Hu tree. The cut sequence of the graph in Figure 1 is $4,3,3,2,2,2,1$. Although, in general, a graph allows for different Gomory-Hu trees, which can be listed using an approach of Yamada, Kataoka, and Watanabe [106], we check that a graph's cut sequence is uniquely determined [Corollary 59]. Furthermore, we discuss the classical characterization of degree sequences by Erdős and Gallai [40] and tackle the question whether there is an analogous criterion for a graph's cut sequence. For this purpose, we review conditions under which graphs are maximally local-edge-connected. These graphs, surveyed by Hellwig and Volkmann [65], satisfy that for any vertex pair the local edge-connectivity equals the minimum of the respective degrees. Building on the constructive characterization of graphic sequences by Tripathi, Venugopalan, and West [101], this chapter's central result is a constructive scheme that allows to generate graphs for a prescribed cut sequence if such a sequence satisfies a shifted variant of the classical Erdős-Gallai inequalities [Theorem 67, Theorem 73].

In Chapter 5, we do not strictly prescribe connectivity values, but change our perspective to that of spectral graph theory. We investigate a graph's connectivity matrix, which is the matrix whose off-diagonal $v$ - $w$ entry is the maximum number of independent $v-w$ paths and whose diagonal entries are set to zero. We study the relations between the spectral parameters of such matrices and how they are linked to the structural properties of the underlying graph. We aim for eigenvalue bounds, ask for which graphs they are attained, and investigate certain energies. For a given matrix, its energy is the sum of the absolute values of its eigenvalues. The interest in these invariants stems from applications in chemical graph theory, as described by Li, Shi, and Gutman [76, Chapter 2]. Many more energy variants and ap-
plications are surveyed by Gutman and Furtula [54]. A central element of Chapter 5 is a conjecture raised by Shikare, Malavadkar, Patekar, and Gutman [93]. The authors ask, whether the energy of the connectivity matrix is bounded by $2(n-1)^{2}$, where $n$ is the order of the respective graph. While we present mostly negative results with respect to this conjecture [Theorem 81, Figures 39 and 40], the situation is more tractable for edge-connectivity matrices, whose off-diagonal entries represent the number of edge-disjoint paths. The entries of such a matrix $C=\left[c_{i j}\right]$ satisfy the so-called ultrametric inequality, meaning that $c_{i k} \geq \min \left\{c_{i j}, c_{j k}\right\}$ for all $i, j, k \in\{1, \ldots, n\}$. This relation turns out to be useful for proving key results of this chapter. For edge-connectivity matrices, we obtain the above bound [Theorem 91], discuss several facts about the eigenvalues and eigenvector structure, and give a refined estimate for the energy [Theorem 94]. Furthermore, we discuss some links to matrices whose structure is similar to that of edge-connectivity matrices [Corollary 99]. For example, the inverse local connectivities satisfy the conditions of an ultrametric, that is a metric where addition in the triangle inequality is replaced by taking a maximum. Such distances are natural measures when data exhibits some sort of hierarchical structure, as discussed by Murtagh, Downs, and Contreras [82] or Chehreghani [59]. Finally, we introduce a Laplace analogue for connectivity matrices and adapt a proof of Chung [24] to verify a version of Cheeger's inequality [20] for this specific Laplace matrix [Theorem 100]. This may open the way to new research questions concerning respective spectral graph partitioning properties.

## Publications related to this work

Parts of this dissertation are already published in the following articles.
[58] Frank Göring, Tobias Hofmann, and Manuel Streicher
Uniformly connected graphs
Journal of Graph Theory, 101(2):210-225, 2022.
[66] Tobias Hofmann and Uwe Schwerdtfeger
Edge-connectivity matrices and their spectra
Linear Algebra and its Applications, 640:34-47, 2022.
[57] Frank Göring and Tobias Hofmann
Properties of uniformly 3-connected graphs arXiv:2211.16966, 2023.

Since one of the purposes of Chapter 2 is to provide basic terminology and connectivity concepts, there is some natural overlap with respective passages of [58], [66], and [57]. That being said, the vast majority of Chapter 2 is compiled specifically for this dissertation.

In Chapter 3, Sections 3.1, 3.2, and 3.3, though completely revised and supplemented by several new illustrations, essentially contain the material of [58]. Section 3.4 is based on [57], extended by some proofs of preliminary results and additional illustrations.

The results of Chapter 4 appear in this dissertation for the first time.
Sections 5.1 and 5.2 overlap with [66] in some basic terms and concepts as well as the discussion around Figure 39, but largely contain new material. Section 5.3 is a revised version of [66]. A few passages of Section 5.4, in particular, Corollary 99 can also be found in [66].


## Graph Connectivity

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One of the most basic questions one can ask about a graph is whether it is connected. Furthermore, we may take a closer look and be interested in how strong it is tied together or how the strength of the connectivity varies over the graph. Section 2.1 is intended to provide a solid foundation for such questions. An aspect that we shall think about particularly is how to measure connectivity. One possibility is to count the minimum number of vertices whose deletion disconnects a given graph. Another approach is to determine the minimum number of independent paths between each pair of its vertices. The fact that these perspectives are actually two sides of the same coin is the substance of Menger's Theorem [81]. Section 2.2 is devoted to this duality statement, which is perhaps the most fundamental tool for proving connectivity properties. We discuss a short proof and focus on some variants of Menger's Theorem that we make use of throughout this investigation. For answering our questions about edge-connectivity, we intensively work with Gomory-Hu trees of graphs. In section 2.3, we therefore treat some contents of the work of Gomory and $\mathrm{Hu}[50]$.

### 2.1 Basic concepts

We define a graph to be an ordered pair $G=(V, E)$ consisting of a finite vertex set $V$ and an edge set $E \subseteq\{\{v, w\}: v, w \in V$ with $v \neq w\}$ with $V \cap E=\emptyset$. For an edge $\{v, w\}$, we often make use of the shorthand $v w$. The occurring $v$ and $w$ are called endvertices of $v w$. All graphs in this work neither contain directed edges nor contain loops, which means edges of the form $v v$. Our definition of a graph also does not permit multiple edges. In some places, however, allowing multiple edges will be a useful practice. There we use the term multigraph for an ordered pair $(V, E)$ of a nonempty finite set $V$ and a finite set $E$ together with a function $E \rightarrow\{v w: v, w \in V$ with $v \neq w\}$, which specifies for each edge its two endvertices. In this context, an edge is not determined uniquely by its endvertices. We nevertheless use here too $v w$ to address some particular edge whose endvertices are $v$ and $w$.

Graph theoretical terminology that we do not explicitly define in this work follows the standard introduced by Diestel [35]. We focus in this section on those concepts that are of particular interest to our considerations. So let us take a graph $G$. Its vertex set is addressed by $V(G)$ and its edge set by $E(G)$. A subgraph of $G$ is a graph $(X, F)$ where $X \subseteq V(G)$ and $F \subseteq E(G)$. We call a subgraph $(X, F)$ induced if it contains all edges $v w \in E(G)$ with $v, w \in X$ and refer to it by $G[X]$. A clique is a vertex set $C \subseteq V(G)$ where $G[C]$ is a complete subgraph of $G$, where a complete graph is a graph in which every pair of vertices is joined by an edge. For another graph $H$, we denote by $G \uplus H$ the graph whose vertex set is $V(G) \uplus V(H)$ and whose edge set is $E(G) \cup E(H)$. For a vertex set $X \subseteq V(G)$ we define $G-X$ to be the graph on vertex set $V(G) \backslash X$ and edge set $\{v w \in E(G): v \notin X$ and $w \notin X\}$. When subtracting only one vertex $v \in V(G)$, we may write $G-v$ instead of $G-\{v\}$. For an edge set $F \subseteq\{v w: v, w \in V(G)$ with $v \neq w\}$ we address the graph $(V(G), E(G) \backslash F)$ by $G-F$ and the graph $(V(G), E(G) \cup F)$ by $G+F$. For an edge $e \in E(G)$ we write $G-e$ instead of $G-\{e\}$ and $G+e$ instead of $G+\{e\}$. Contracting $e=v w$ yields the graph $G / e$ on vertex set $(V(G) \backslash\{v, w\}) \cup\left\{v_{e}\right\}$ and edge set

$$
\{x y \in E(G): x y \cap v w=\emptyset\} \cup\left\{v_{e} x: v x \in E(G) \backslash\{e\} \text { or } w x \in E(G) \backslash\{e\}\right\} .
$$

Similarly, contracting a vertex set $X \subseteq V(G)$ yields the graph $G / X$ by identifying all vertices in $X$ to one vertex $x$, omitting all edges in $G[X]$, and replacing each edge $v w$ where $v \in X$ and $w \notin X$ by $x w$. We call a graph $H$ minor of the graph $G$ if it can be obtained from $G$ by contracting edges and deleting edges or vertices.

We denote $E_{G}(X, Y):=\{v w \in E(G): v \in X$ and $w \in Y\}$ for $X, Y \subseteq V(G)$. In particular, we use the shorthand $E_{G}(X):=E(X, V(G) \backslash X)$. For a vertex $v \in V(G)$ we write $E_{G}(v):=E_{G}(v, V(G) \backslash\{v\})$, denote the neighborhood of $v$ by $N_{G}(v):=\{w \in V(G): v w \in E(G)\}$, and the degree of $v$ by $\operatorname{deg}_{G}(v):=\left|N_{G}(v)\right|$. In these notations, we omit to address the respective graph in the index if there is no need for a reference.

A path is a graph $P=(V, E)$ whose vertex set $V=\left\{v_{1}, \ldots, v_{k}\right\}$ is nonempty and whose edge set is of the form $E=\left\{v_{i} v_{i+1}: i \in\{1, \ldots, k-1\}\right\}$. This definition allows for paths consisting of a single vertex. Often, we refer to a path by its sequence of vertices $v_{1} \ldots v_{k}$. As for edges, we may call $v_{1}$ and $v_{k}$ the endvertices of $P$ and say that $P$ connects $v_{1}$ and $v_{k}$ or leads from $v_{1}$ to $v_{k}$, which means it also leads from $v_{k}$ to $v_{1}$. Note that all paths in this work are undirected objects, although our notation and use of language may occasionally suggest something else, as it can be useful to mentally focus on one specific direction. For a path $Q=w_{1} \ldots w_{\ell}$ with $V(P) \cap V(Q)=\emptyset$, we address the path formed via $V(P) \cup V(Q)$ and $E(P) \cup E(Q) \cup\left\{v_{k} w_{1}\right\}$ by $P Q$ or by $P w_{1} \ldots w_{\ell}$. Similar path concatenations are denoted analogously. A cycle is a graph $P+v_{k} v_{1}$ where $P=v_{1} \ldots v_{k}$ is a path and $k \geq 3$. For vertex sets $X$ and $Y$, a path $P=v_{1} \ldots v_{k}$ is called $X-Y$ path if $V(P) \cap X=\left\{v_{1}\right\}$ and $V(P) \cap Y=\left\{v_{k}\right\}$. By $W_{n}$ we address the wheel graph on $n \geq 4$ vertices that results from a cycle on $n-1$ vertices by adding a new vertex adjacent to all other vertices.

A set $S \subseteq V(G) \cup E(G)$ separates two vertex sets $X, Y \subseteq V(G)$ if each $X-Y$ path contains some element of $S$. We say the set $S \subseteq(V(G) \backslash\{v, w\}) \cup E(G)$ separates two vertices $v, w \in V(G)$ if $S$ separates $\{v\}$ and $\{w\}$. A subset of vertices $S \subseteq V(G)$ is a separator if it separates two vertices. A subset of vertices $S \subseteq V(G) \backslash\{v, w\}$ is a $v$-w separator, if it separates the vertices $v, w \in V(G)$. A graph $G$ is called connected if any two vertices of $G$ are connected by a path. We refer to a connected induced subgraph $C$ of $G$ as a component of $G$ if it is not contained in any connected subgraph $H$ of $G$ with $|V(H)|>|V(C)|$. For $k \in \mathbb{N}:=\{1,2, \ldots\}$ the graph $G$ is $k$-connected if $|V(G)| \geq k+1$ and $G-S$ is connected for any set $S \subseteq V(G)$ with $|S| \leq k-1$. A cut in $G$ is an edge set $E(S, V(G) \backslash S)$ where $S$ is a nonempty proper subset of $V(G)$. We refer to $S$ and $V(G) \backslash S$ as the sides of the cut. For two vertices $v, w \in V(G)$, a $v$ - $w$ cut is a cut in $G$ such that $v$ and $w$ are in different sides. A bridge is a cut containing exactly one edge. A vertex whose deletion increases the number of components is called cutvertex. A maximal connected subgraph without a cutvertex is called block. So a block is either an isolated vertex, a bridge with its incident vertices, or a
maximal 2-connected subgraph. We refer to the latter as nontrivial block. For $k \in \mathbb{N}$ a graph $G$ is called $k$-edge-connected if $G-F$ is connected for any set $F \subseteq E(G)$ with $|F| \leq k-1$. To shorten notation, we use the terms $k$-cut or $k$-separator to indicate that they contain $k \in \mathbb{N}$ elements. We call a vertex set $X \subseteq V(G)$ independent if none of its vertices are adjacent. We say two or more paths are independent if every vertex that is contained in more than one path is an endpoint of all paths it is contained in. Two or more paths are edge-disjoint if the edge sets of any pair of these paths are disjoint. For a set $X \subseteq V(G)$ and a vertex $v \in V(G) \backslash X$ we call a set of $v$ - $X$ paths $v$ - $X$ fan if any two of the paths only have the vertex $v$ in common.

### 2.2 Menger's Theorem

Menger's Theorem [81] is one of the corner stones of graph theory and plays an important role in this investigation. An overview about several versions and proof strategies is given by Diestel [35, Secion 3.3], from where we take Theorems 2 to 5 . The subsequent proof of Menger's Theorem is based on an edge contraction argument, given by Göring [56]. When in the following we contract an edge $e$ of a graph $G$ that is incident to a vertex from some vertex set $V \subseteq V(G)$, we denote the vertex appearing by contraction by $v_{e}$ and regard it as a vertex of $V(G)$. Also note that Menger's Theorem holds for graphs and multigraphs, as is presented by Göring [56]. We make use of the respective references when treating uniformly edge-connected graphs in Chapter 3, as their definition allows for multiple edges. This is, however, the only occasion in this work where we allow for multiple edges.

Theorem 1. Let $G$ be a graph or multigraph and consider two vertex sets $V, W \subseteq V(G)$. Then the minimum cardinality of a vertex set $S$ separating $V$ and $W$ equals the maximum cardinality of a set of disjoint $V-W$ paths.

Proof. Clearly, the number of disjoint $V-W$ paths cannot be larger than the minimum cardinality of a vertex set separating $V$ and $W$. So we have to show that if a minimum vertex set separating $V$ and $W$ is of cardinality $k$, then there are $k$ disjoint $V$ - $W$ paths. If $E(G)=\emptyset$, then $|V \cap W|=k$ and we find $k$ disjoint $V-W$ paths, consisting each of a single vertex in $V \cap W$. Let now $G$ be a graph that is a counterexample to our claim with minimal $|E(G)|$. Consider an edge $e=x y \in E(G)$. Because there are less than $k$ disjoint $V$ - $W$ paths in $G$, also $G / e$ contains less than $k$ such paths. So in $G / e$ there is a vertex set $S$ separating $V$ and $W$ with $|S|<k$. We know that $v_{e} \in S$, as otherwise $S$ separates $V$ and $W$ also in $G$. Then $T:=\left(S \backslash\left\{v_{e}\right\}\right) \cup\{x, y\}$ is a
vertex set separating $V$ and $W$ in $G$ and thus $|T|=|S|+1=k$. Furthermore, each vertex set separating $V$ and $T$ in $G-e$, and likewise each vertex set separating $T$ and $W$ in $G-e$, also separates $V$ and $W$ in $G$ and thus contains at least $k$ vertices. So we find $k$ disjoint $V-T$ paths and $k$ disjoint $T-W$ paths, meeting only in $T$, because $T$ separates $V$ and $W$. This provides $k$ disjoint $V-W$ paths in $G$, contradicting that $G$ is a counterexample to our claim.

We also use Menger's Theorem in the following fan version.
Theorem 2. Given a graph $G$, a vertex set $W \subseteq V(G)$ and a vertex $v \in V(G) \backslash W$, the minimum cardinality of a set of vertices separating $v$ and $W$ in $G$ equals the maximum number of paths building a $v-W$ fan in $G$.

Proof. The claim follows by Theorem 1 for the vertex sets $N(v)$ and $W$.
Theorem 3 is a local vertex version and Theorem 4 a local edge version of Menger's Theorem. Global vertex and edge versions follow with Theorems 5 and 6 , which we use in particular when characterizing uniformly connected and uniformly edge-connected graphs in Chapter 3.

Theorem 3. Given a graph $G$ and two nonadjacent vertices $v, w \in V(G)$, the minimum cardinality of a vertex set $S \subseteq V(G) \backslash\{v, w\}$ separating $v$ and $w$ in $G$ equals the maximum number of independent $v-w$ paths in $G$.

Proof. The claim follows by Theorem 1 for the vertex sets $N(v)$ and $N(w)$.
Theorem 4. Consider a graph or multigraph $G$ and two vertices $v, w \in V(G)$. Then the minimum cardinality of an edge set $S \subseteq E(G)$ separating $v$ and $w$ in $G$ equals the maximum number of edge-disjoint $v-w$ paths in $G$.

Proof. Define the line graph of $G$ on vertex set $E(G)$ in which two vertices $e, f \in E(G)$ are adjacent if $e$ and $f$ share an endvertex in $G$. The sets $E(v)$ and $E(w)$ are edge sets in $G$ and vertex sets in the corresponding line graph. Applying Theorem 1 for them proves our claim.

Theorem 5. For a number $k \in \mathbb{N}$, a graph $G$ on at least $k+1$ vertices is $k$-connected if and only if there are $k$ independent paths connecting any two vertices of $G$.

Proof. If there are $k$ independent paths connecting any two vertices of $G$, then $G$ contains at least $k+1$ vertices and cannot be separated by less than $k$ vertices. So we only have to show that in a $k$-connected graph each pair of vertices is connected by $k$ independent paths. In view of Theorem 3, assume, for contradiction, that there are two adjacent vertices $v, w \in V(G)$
connected by not more than $k-1$ independent paths in $G$ and hence not more than $k-2$ independent paths in $G-v w$. Again Theorem 3 says that there is a set $S$ with $|S| \leq k-2$ separating $v$ and $w$ in $G-v w$. Because $G$ contains at least $k+1$ vertices, there is some $x \in V(G) \backslash(S \cup\{v, w\})$. The set $S$ separates $x$ from $v$ or $w$ in $G-v w$. But then $S \cup\{v\}$ separates $x$ from $w$ or $S \cup\{w\}$ separates $x$ from $v$ in $G$. This is a contradiction to $G$ being $k$-connected, because $|S \cup\{v\}|=|S \cup\{w\}| \leq k-1$.

Theorem 6. For a number $k \in \mathbb{N}$, a graph or multigraph $G$ is $k$-edgeconnected if there are $k$ edge-disjoint paths connecting any two vertices of $G$.

Proof. The claim is a consequence of Theorem 4.

### 2.3 Gomory-Hu trees

A graph's minimum cut capacities carry rich structural properties. In what follows, we take a look at some of the relationships, on which also Gomory and Hu build on in their seminal article [50].

Lemma 7. Consider a graph $G$ and three distinct vertices $v, w, x \in V(G)$. Then the minimum cut capacities $c_{v w}, c_{v x}$, and $c_{x w}$ satisfy the so-called $u l$ trametric inequality

$$
c_{v w} \geq \min \left\{c_{v x}, c_{x w}\right\} .
$$

Proof. Suppose the assertion is false. Then there are vertices $v, w, x \in V(G)$ satisfying $c_{v w}<\min \left\{c_{v x}, c_{x w}\right\}$. This means there exist nonempty vertex sets $S, T \subseteq V(G)$ such that $S \cup T=V(G), S \cap T=\emptyset, v \in S, w \in T$, and $|E(S, T)|=c_{v w}$. The vertex $x$ has to be contained either in $S$ or in $T$. If $x \in S$, then we obtain the contradiction $c_{x w} \leq c_{v w}$, because then the edge set $E(S, T)$ is of cardinality $c_{v w}$ and separates $x$ and $w$. If $x \in T$, we obtain the contradiction $c_{v x} \leq c_{v w}$, because then $E(S, T)$ separates $x$ and $v$.

Lemma 8. Consider numbers $c_{i j} \geq 0$, for $i, j \in\{1, \ldots, n\}$, that satisfy $c_{i \ell} \geq \min \left\{c_{i j}, c_{j \ell}\right\}$ for all $i, j, \ell \in\{1, \ldots, n\}$. Then for $k \in\{1, \ldots, n\}$

$$
c_{1 k} \geq \min _{i=2}^{k} c_{i-1, i} .
$$

Proof. Lemma 7 serves as an induction base. Assuming our statement to be true for $k-1$ instead of $k$, we conclude by induction that

$$
\min _{i=2}^{k} c_{i-1, i}=\min \left\{\min _{i=2}^{k-1} c_{i-1, i}, c_{k-1, k}\right\} \leq \min \left\{c_{1, k-1}, c_{k-1, k}\right\} \leq c_{1 k} .
$$

Lemma 9. Consider numbers $c_{i j} \geq 0$, for $i, j \in\{1, \ldots, n\}$, satisfying $c_{i j}=c_{j i}$ and $c_{i \ell} \geq \min \left\{c_{i j}, c_{j \ell}\right\}$ for all $i, j, \ell \in\{1, \ldots, n\}$. Then for $c_{v w} \leq c_{v x} \leq c_{w x}$ there holds $c_{v w}=c_{v x}$.
Proof. We are given that $c_{v w} \leq c_{v x}$. So we just have to show that $c_{v w} \geq c_{v x}$. Applying the ultrametric inequality, we obtain

$$
c_{v w} \geq \min \left\{c_{v x}, c_{x w}\right\}=\min \left\{c_{v x}, c_{w x}\right\}=c_{v x} .
$$

To determine a minimum cut between two vertices, we may use methods based on network flows, which is justified by the max-flow min-cut theorem of Ford and Fulkerson [46]. If we are interested in the minimum cuts between all pairs of a set of vertices $V$, we might do this naively by performing $\binom{|V|}{2}$ maximum flow computations. In [50], however, Gomory and Hu showed that there is a tree structure, called Gomory-Hu tree, that neatly encodes all the minimum cuts of a graph and that indeed $|V|-1$ maximum flow computations suffice to generate that tree.

Definition 10. Let $G=(V, E)$ be a graph. A Gomory-Hu tree for $G$ is a tree $T=(V, F)$ where for each edge $e=v w \in F$ the graph $T-e$ contains a component on vertex set $W$ such that $E(W)$ is a $v-w$ cut of minimum capacity in $G$.

Having Lemma 8 at hand, we restate in the following lemma how to obtain a minimum $s$ - $t$ cut for some pair of vertices $s$ and $t$ from a graph's GomoryHu tree. Our goal for the remainder of this section is to review that there indeed exists a Gomory-Hu tree for any given graph. We largely follow the presentation by Schrijver [91, Section 15.4].

Lemma 11. Let $(V, E)$ be a graph and let $(V, F)$ be a corresponding Gomory-Hu tree. Consider also two vertices $s, t \in V$ and the unique path $P$ connecting $s$ and $t$ in $T$. Furthermore, let $v w$ be an edge of minimum weight $c_{v w}$ in $E(P)$. Then $c_{s t}=c_{v w}$ and denoting one of the two components of $T-v w$ by $W$ the edges $E(W)$ form a minimum s-t cut in $G$.
Proof. Lemma 8 says that $c_{s t} \geq c_{v w}$. On the other hand, the edges $E(W)$ form an $s$ - $t$ cut, which implies $c_{s t} \leq|E(W)| \leq c_{v w}$.

Lemma 12. For a graph $G$ and all vertex sets $U, W \subseteq V(G)$ there holds
(i) $|E(U)|+|E(W)|=|E(U \backslash W)|+|E(W \backslash U)|$

$$
+2|E(U \cap W, V(G) \backslash(U \cup W))| \quad \text { and }
$$

(ii) $|E(U)|+|E(W)|=|E(U \cup W)|+|E(U \cap W)|+2|E(U \backslash W, W \backslash U)|$.

Proof. Both statements follow directly by double counting the edges of $G$.

In particular, the equations from Lemma 12 provide the estimates

$$
\begin{aligned}
& |E(U)|+|E(W)| \geq|E(U \backslash W)|+|E(W \backslash U)| \quad \text { and } \\
& |E(U)|+|E(W)| \geq|E(U \cup W)|+|E(U \cap W)|
\end{aligned}
$$

for vertex sets $U, W \subseteq V(G)$ for a graph $G$. A set function satisfying the second inequality is called submodular, which we may interpret as a diminishing returns property. This is why such functions have not only technical value when proving the following graph-theoretical statement, but occur quite naturally in a variety of applications.

Lemma 13. Consider a graph $G$ and for two vertices $s, t \in V(G)$ a minimum $s$ - $t$ cut $E(U)$ with $s \in U$. Then for two vertices $v, w \in U$, there is a minimum $v$ - $w$ cut $E(W)$ such that $W \subseteq U$.

Proof. Let $E(X)$ be a minimum $v$ - $w$ cut. If necessary, rename $w$ into $v$, and $v$ into $w$, so that $v \in X$. Furthermore, we assume $s \in X$, as we may consider $V \backslash X$ instead of $X$. Then $E(U \cap X)$ and $E(U \backslash X)$ are $v$-w cuts. If $t \notin X$, we obtain that $E(U \cup X)$ is an $s$ - $t$ cut. So $|E(U \cup X)| \geq|E(U)|$. By statement (ii) of Lemma 12,

$$
|E(X)| \geq|E(U \cap X)|+|E(U \cup X)|-|E(U)| \geq|E(U \cap X)| .
$$

Thus $E(U \cap X)$ is a minimum $v$-w cut. If $t \in X$, we obtain that $E(X \backslash U)$ is an $s$ - $t$ cut, because $s \in U$. This implies $|E(X \backslash U)| \geq|E(U)|$. By statement (i) of Lemma 12,

$$
|E(X)| \geq|E(U \backslash X)|+|E(X \backslash U)|-|E(U)| \geq|E(U \backslash X)|
$$

So $E(U \backslash X)$ is a minimum $v$ - $w$ cut, which concludes our proof.
Theorem 14. Each graph has a corresponding Gomory-Hu tree.
Proof. Consider a graph $G$ and define a partial Gomory-Hu tree on a subset of vertices $X \subseteq V(G)$ as a pair consisting of a tree $(X, F)$ and a partition $\left\{C_{x}: x \in X\right\}$ of $V(G)$ satisfying the following conditions.
(i) For each $x \in X$ holds $x \in C_{x}$.
(ii) For each edge $e=v w \in F$ the graph $(X, F \backslash\{e\})$ contains a component on vertex set $W$ such that $E\left(\cup_{x \in W} C_{x}\right)$ is a minimum $v-w$ cut in $G$.

Our goal is to show that there is a partial Gomory-Hu tree for each $X \subseteq V(G)$. This in particular implies the existence of a Gomory-Hu tree for the graph $G$. For $|X|=1$, it is trivial to find a corresponding partial Gomory-Hu tree.

Let us proceed inductively for $|X|>1$ and consider a minimum cut $E(W)$ that separates at least two vertices of $X$. Contracting $V(G) \backslash W$ to one vertex $v^{1}$ provides a graph $G^{1}$. The graph $G^{1}$, by induction, has a partial Gomory-Hu tree on $X^{1}:=X \cap W$ that we denote by $\left(X^{1}, F^{1}\right),\left\{C_{x}^{1}: x \in X^{1}\right\}$. On the other hand, contracting $W$ to one vertex $v^{2}$ provides a graph $G^{2}$ and a respective partial Gomory-Hu tree $\left(X^{2}, F^{2}\right),\left\{C_{x}^{2}: x \in X^{2}\right\}$ for $X^{2}:=X \backslash W$.

Denoting a vertex $x^{1} \in X^{1}$ such that $v^{1} \in C_{x^{1}}^{1}$ and a vertex $x^{2} \in X^{2}$ such that $v^{2} \in C_{x^{2}}^{2}$, we define $F:=F^{1} \cup F^{2} \cup\left\{x^{1} x^{2}\right\}, C_{x^{1}}:=C_{x^{1}}^{1} \backslash\left\{v^{1}\right\}$, as well as $C_{x}:=C_{x}^{1}$ for all $x \in X^{1} \backslash\left\{x^{1}\right\}$. Furthermore, we set $C_{x^{2}}:=C_{x^{2}}^{2} \backslash\left\{v^{2}\right\}$, and $C_{x}:=C_{x}^{2}$ for all $x \in X^{2} \backslash\left\{x^{2}\right\}$. It remains to be shown that $(X, F)$, $\left\{C_{x}: x \in X\right\}$ is a partial Gomory-Hu tree on $X$. Property (i) is satisfied by definition. Property (ii) follows for all $e \in F \backslash\left\{x^{1} x^{2}\right\}$ by Lemma 13. For $e=x^{1} x^{2}$, we have $\cup_{x \in W} C_{x}=W$. Recalling that $E(W)$ is chosen to be a minimum cut that separates at least two vertices from $X$, we obtain that $E(W)$ is a minimum $x^{1}-x^{2}$ cut, which is what remained to be shown.

Along the way, the previous proof also describes an algorithm to construct a Gomory-Hu tree, which requires $|V|-1$ minimum cut computations. In practical implementations, often a variant by Gusfield [52] is used, which involves the same amount of minimum cut computations, but is easier to implement. Ford and Fulkerson's max-flow min-cut theorem [46] says that the necessary minimum cut values can be obtained by maximum flow computations. Building on these insights, the algorithm of Edmonds and Karp [38] successively looks for shortest augmenting paths in an appropriately constructed auxiliary network to compute a maximum flow in running time $\mathcal{O}\left(|V||E|^{2}\right)$. Dinitz' algorithm [36] improves this to $\mathcal{O}\left(|V|^{2}|E|\right)$, by determining blocking flows in a so-called level graph. Another competitive approach is the pushrelabel algorithm by Goldberg and Tarjan [49], also involving $\mathcal{O}\left(|V|^{2}|E|\right)$ running time. The recent breakthrough by Chen, Kyng, Liu, Peng, Gutenberg, and Sachdeva [21] provides the first algorithm that allows maximum flow computations in almost linear time. The introduced techniques, combined with the insights from Abboud, Krauthgamer, Li, Panigrahi, Saranurak, and Trabelsi [1] or Zhang [109], also allow for an $|E|^{1+o(1)}$ time algorithm to compute Gomory-Hu trees. Notwithstanding this theoretical milestone, there remains a lot of work to be done, as the constants hidden in the above Landau notation are currently too large for practical use.


## Uniform Connectivity

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This chapter deals with the concept of uniform graph connectivity. It is in a sense the sharpest connectivity measure that we study in this work. Beginning with the central definitions, we investigate how they relate to other connectivity and regularity notions in Section 3.1. As it is the case with other connectivity concepts, uniform connectivity too can be related to vertex or edge separation. Section 3.2 is concerned with the relations between these two types, which in earlier studies have been introduced independently. A key result of this section is that for $k \leq 3$ every uniformly $k$-connected graph is uniformly $k$-edge-connected, whereas for $k>3$ we find an infinite family of examples for which this inclusion does not hold. In Section 3.3, we proceed with constructions for both uniformly $k$-connected and uniformly $k$-edge-connected graphs in cases where $k \leq 3$ and describe how these ideas are related to classical characterizations by Tutte. Section 3.4 concludes this chapter with applications of our constructive results. In particular, we give a tight bound on the number of vertices of minimum degree in uniformly 3 -connected graphs and address questions about their treewidths and crossing numbers.

### 3.1 Basic terms and relations

At the heart of this chapter are the following two connectivity concepts.
Definition 15. For a number $k \in \mathbb{N}$, a graph $G$ on at least $k+1$ vertices is called uniformly $k$-connected if any two vertices of $G$ can be connected by $k$ and not more than $k$ independent paths.

Definition 16. For a number $k \in \mathbb{N}$, a multigraph $G$ on at least two vertices is called uniformly $k$-edge-connected if any two vertices of $G$ can be connected by $k$ and not more than $k$ edge-disjoint paths.

When suppressing the parameter $k$ in these terms, we mean graphs that are uniformly $k$-connected or multigraphs that are uniformly $k$-edge-connected for some $k \in \mathbb{N}$. Also note that Definition 16 is one of the few places in this work where we explicitly refer to multigraphs. For those we may use terminology that we introduced for graphs. All the terms we need naturally extend to this setting. One reason to allow multiple edges here is to be in line with the definition of uniformly edge-connected graphs given by Kingsford and Marçais [70], who contributed considerably to understanding the structure of uniformly 3 -edge-connected graphs. Another reason is that this reveals an interesting difference between uniform vertex- and edge-connectivity. Whereas allowing multiple edges in the latter case adds structural aspects, using the word multigraph in Definition 15 actually would not make any difference, which is discussed in more detail at the beginning of Section 3.2.

Now let us turn to some examples. Figure 2 displays a tree on the left and indeed trees on at least two vertices are exactly the uniformly 1-connected graphs which are exactly the graphs that are uniformly 1-edge-connected. Each of these three classes is defined to contain precisely the graphs on at least two vertices in which each pair of vertices is connected by a unique path. All further examples in Figure 2 are uniformly edge-connected and all but the hourglass graph on the right are also uniformly connected. This may immediately invoke the question of whether there are graphs that are uniformly connected but not uniformly edge-connected, which is one of the


Figure 2: Small uniformly edge-connected graphs
major topics that we discuss in Section 3.2. Note also that all graphs in Figure 2 are connected, as one is the smallest value that we allow for the parameter $k$ in our definitions. To let them also comprise uniformly 0 -connected or uniformly 0-edge-connected graphs is a matter of taste. But this would include merely those graphs without any edges at all. Beyond small examples, checking a graph to be uniformly connected or uniformly edge-connected requires to know the number of independent or edge-disjoint paths for all pairs of vertices. Those can be calculated efficiently by the classical approach of Even and Tarjan [43], which is based on network flows.

A very useful fact about our classes is that Menger's Theorem provides a concise dual characterization for them. It says that a graph $G$ on at least $k+1$ vertices is uniformly $k$-connected if and only if for each two nonadjacent vertices $v, w \in V(G)$ there is a minimum $v-w$ separator of cardinality $k$ in $V(G) \backslash\{v, w\}$ and for each two adjacent vertices $v, w \in V(G)$ there is a minimum $v$ - $w$ separator of cardinality $k-1$ in $V(G) \backslash\{v, w\}$. Similarly, a multigraph $G$ on at least two vertices is uniformly $k$-edge-connected if and only if for each two vertices $v, w \in V(G)$ there is a minimum $v$ - $w$ cut of cardinality $k$. Let us also record here that a uniformly $k$-connected graph is $k$-connected and a uniformly $k$-edge-connected multigraph is $k$-edge-connected.

Though the definitions of uniformly connected and uniformly edge-connected graphs are very similar, both classes have initially been studied independently. Uniformly connected graphs are introduced by Beineke, Oellermann, and Pippert in their work about the average connectivity of a graph [5]. Motivated by the fact that a graph's connectivity is a lower bound for its average connectivity, they are concerned with uniformly connected graphs as those for which this bound is attained. As it is our interest in Section 3.3, their focus is on constructions that preserve uniform connectivity and produce infinite families of uniformly connected graphs. Uniformly edge-connected graphs, on the other hand, have first been studied by Kingsford and Marçais in [70], calling them exactly edge-connected. They too are concerned with constructive characterizations for uniformly edge-connected graphs.

A link between uniform connectivity and the topics discussed in Chapter 5 arises from Corollary 92. There we prove a tight bound on the energy, this means the sum of the absolute values of the eigenvalues, of a graph's edgeconnectivity matrix. It turns out that the bound that we study there, is sharp for uniformly edge-connected graphs. From the perspective of spectral graph theory, it is therefore natural to ask about the structure of these graphs. This is a typical procedure, as it helps to understand how certain graph structures are related to the spectral parameter in mind.


Figure 3: A minimally and critically connected graph that is neither uniformly connected nor uniformly edge-connected

To gain a clearer picture of the graph classes introduced, we shall next discuss relationships to other common connectivity concepts. For this purpose, we recall that a graph is called minimally $k$-connected if it is $k$-connected and deleting any of its edges results in a graph that is no longer $k$-connected. A graph is called critically $k$-connected if it is $k$-connected and deleting any of its vertices leaves a graph that is no longer $k$-connected. Minimally $k$-edge-connected and critically $k$-edge-connected multigraphs are defined analogously and we may omit the parameter $k$ if it is not relevant.

The following Theorem is a slight generalization of a result of Beineke, Oellermann, and Pippert [5]. They show how uniform connectivity is related to minimal and critical connectivity. And in fact, their reasoning can easily be adopted to the case of uniformly edge-connected graphs. They also provide the example in Figure 3, which demonstrates that a graph can be both minimally and critically connected without having to be uniformly connected.

Theorem 17. A uniformly $k$-edge-connected multigraph is
(i) minimally $k$-edge-connected for $k \geq 1$ and
(ii) critically $k$-edge-connected for $k \geq 2$.

A uniformly $k$-connected graph is
(iii) minimally $k$-connected for $k \geq 1$ and
(iv) critically $k$-connected for $k \geq 2$.

Proof. We prove only statement (i), because the others can be shown analogously. Suppose, for the sake of contradiction, that there is some uniformly $k$-edge-connected multigraph $G$ that is not minimally $k$-connected. So it contains an edge $e=v w \in E(G)$ such that $G-e$ remains $k$-edge-connected. Thus Theorem 4 says that $v$ and $w$ are connected by $k$ edge-disjoint paths in $G-e$. But then $G$ contains $k+1$ edge-disjoint paths between $v$ and $w$, because the edge $v w$ itself forms another such path. This contradicts that $G$ is uniformly $k$-connected and thus proves our claim.

Note that it is indeed necessary to formulate the statements about critical connectivity for $k \geq 2$ only. This is because trees are uniformly 1 -connected and uniformly 1-edge-connected, but neither critically connected nor critically edge-connected, as they remain connected when deleting a leaf.

As uniform connectivity requires the same strength of connectivity between all vertex pairs of a graph, it can also be understood as a regularity measure. However, regularity by degree is not implied by uniform connectivity, for which Figure 2 shows a few examples. But if a graph is regular, then we have a very accurate picture.

Lemma 18. Let $G$ be a $k$-regular graph. Then
(i) $G$ is uniformly $k$-connected if and only if $G$ is $k$-connected and
(ii) $G$ is uniformly $k$-edge-connected if and only if $G$ is $k$-edge-connected.

Proof. As $G$ is $k$-regular, it can contain at most $k$ independent and at most $k$ edge-disjoint paths between every pair of its vertices. Conversely, Theorem 6 says that there are at least $k$ edge-disjoint paths between every pair of vertices if and only if $G$ is $k$-edge-connected, which proves statement (ii). Furthermore, a $k$-regular graph has at least $k+1$ vertices. So Theorem 5 says that there are at least $k$ independent paths between any pair of vertices if and only if $G$ is $k$-connected, which proves statement (i).

This very concise description encourages us to focus essentially on irregular graphs when characterizing uniformly edge-connected and uniformly connected graphs in Section 3.3. It also sheds light on relations to some other graph classes. For example, the edge graphs of simple $k$-dimensional polytopes are $k$-connected, and hence $k$-edge-connected, by Balinski's Theorem [4] and the word simple just means that their edge graphs are regular. So Lemma 18 says that those are uniformly $k$-connected and uniformly $k$-edgeconnected. Also distance regular graphs, studied for example by Brouwer, Cohen, and Neumaier [15] are in particular regular. Furthermore, Brouwer and Koolen [17] prove distance regular graphs of degree $k$ to be $k$-connected, and thus also $k$-edge-connected. So Lemma 18 says that those graphs are uniformly $k$-connected and uniformly $k$-edge-connected. But note that the converse inclusions do not hold because uniformly connected or uniformly edge-connected graphs do not have to be regular by degree.

A term that might be easily confused with our notion is that of $k$-uniform connectivity. Notionally, the only difference to uniform $k$-connectivity is that Chartrand and Zhang [19] attach the parameter $k$ at another place. But they call a graph $G$ to be $k$-uniformly connected if it is of order $n \geq 2$ and if for


\left.| Relevant shortest paths |  |  |
| :--- | :--- | :--- |
| 132 | 142 | 152 |
| 314 | 315 | 415 |
| 324 | 325 | 425 |$\right] \times 2$

Figure 4: A path-regular graph that is not uniformly connected


Figure 5: A uniformly connected graph that is not path-regular
some integer $k$ with $1 \leq k \leq n-1$ for every two distinct vertices $v, w \in V(G)$ there is a $v-w$ path of length $k$ in $G$. The diamond, which is the graph resulting from the complete graph on four vertices by deleting some edge, is 2-uniformly connected, but certainly not uniformly $k$-connected or uniformly $k$-edge-connected, for any $k$. Conversely, a path on three vertices is uniformly connected and uniformly edge-connected, but there is no $k$ for which it is $k$-uniformly connected.

Another regularity concept whose name and definition suggests some overlap with uniform connectivity is a graph's path-regularity introduced by Matula and Dolev [78]. Their definition involves shortest path lists that may contain multiple copies of certain paths. Furthermore, only those lists are of relevance for which each pair of vertices are the endvertices of the same number of paths. Then a graph is said to be edge-path-regular if there exists such a list in which each edge occurs in the same number of paths. The table in Figure 4 contains a list of shortest paths of length two that contains six times each pair of endvertices. Note that for such lists we can always specify paths of lengths zero and one that meet the criteria of the preceding definition. This is why these paths are typically omitted in this context. As the list given in Figure 4 contains each edge the same number of times, we can identify the corresponding graph as edge-path-regular. However, the graph obviously is neither uniformly edge-connected nor uniformly connected. Figure 5, on the other hand, shows a tree and thus a uniformly edge-connected and uniformly connected graph for which we cannot provide a list of shortest paths that certifies path-regularity. To see that, we first observe that the shortest paths
in the list in Figure 5 are uniquely determined, because the corresponding graph is a tree. To meet the requirement that each pair of vertices are the endvertices of the same number of paths, we may only choose multiple copies of the entire list. Thus we always find that the edge 34 appears more often than the other edges. In addition to edge-path-regular graphs, the authors in [78] analogously define vertex-path-regular graphs. Those too do not include our classes, because a path on three vertices is not vertex-pathregular. On the other hand, it is not too difficult to see that the Cartesian product of the wheel graph on five vertices and the path on two vertices is neither uniformly edge-connected nor uniformly connected, whereas it is vertex-path-regular by [78, Figure 2 and Theorem 7].

### 3.2 Uniform vertex- versus edge-connectivity

In this section, we work out the relations between uniformly connected and uniformly edge-connected graphs. A first difference between Definition 15 and 16 is that the latter allows multiple edges, in line with the definition given by Kingsford and Marçais [70]. Remarkably, the following Lemma says that we may replace the word graph by multigraph in Definition 15 and yet shall never see multiple edges in uniformly connected graphs.

Lemma 19. If a $k$-connected multigraph contains two vertices $v$ and $w$ that are joined by parallel edges, then $v$ and $w$ are connected by at least $k+1$ independent paths.

Proof. To obtain a contradiction, suppose there is a graph $G$ containing two vertices $v$ and $w$ that are joined by parallel edges, but not connected by more than $k$ independent paths. Deleting all edges between $v$ and $w$ in $G$ leaves not more than $k-2$ independent paths between $v$ and $w$ in the resulting graph that we denote by $H$. By Theorem $1, H$ contains a $v$-w separator $S \subseteq V(H) \backslash\{v, w\}=V(G) \backslash\{v, w\}$ with $|S| \leq k-2$. The definition of $k$-connectedness requires $G$ to have at least $k+1$ vertices. So $H$ too contains at least $k+1$ vertices and $H-S$ contains more than two components or one of its components contains more than one vertex. So $S \cup\{v\}$ or $S \cup\{w\}$ is a separator in $G$ with $|S \cup\{v\}|=|S \cup\{w\}| \leq k-1$, which contradicts our prerequisite that $G$ is $k$-connected.

This section's key fact is that for $k \leq 3$ any uniformly $k$-connected graph is uniformly $k$-edge-connected. Before we turn to its proof, let us investigate a family of counterexamples in case $k>3$.


Figure 6: Constructing uniformly connected but not uniformly edgeconnected graphs

Example 20. Let us consider for $k \geq 4$ graphs $G_{k}$ on a vertex set of the form $S \cup T$ where $|S|=k-1$ and $|T|=k$ and let the induced subgraph $G_{k}[T]$ be a tree with at least two inner vertices. This means we require $G_{k}$ to contain two vertices whose degree in $G_{k}[T]$ is larger than one. All the remaining edges of $G_{k}$ shall result from joining each vertex in $S$ with all vertices in $T$. In other words, we define $N(s)=T$ for each vertex $s \in S$. Now let us take a look at Figure 6, where this example is illustrated. Depicted there solid is the result for $k=4$, which is the only case where the constructed graph is uniquely determined, because the only tree on four vertices with two inner vertices is a path and for $k>4$ such a tree is not uniquely determined. Note that, as expected, our example does not work for $k<4$, because we cannot find a corresponding tree containing two inner vertices.

The claim to be proven is that for all $k \geq 4$ a graph $G_{k}$ is not uniformly $k$-edge-connected, but uniformly $k$-connected.

Proof. Denoting two inner vertices of $G_{k}[T]$ by $v$ and $w$, we find $k-1$ edgedisjoint paths of the form $v s w$ for $s \in S$ and further paths of the form vxsyw for vertices $x \in N(v), y \in N(w)$, and $s \in S$. In total, we count a number of $k-1+\min \{|N(v)|,|N(w)|\} \geq k+1$ edge-disjoint paths. So $G_{k}$ is not uniformly $k$-edge-connected.

Only the inner vertices of $G_{k}[T]$ are possible candidates to be connected by more than $k$ independent paths, because all the other vertices are of degree $k$. But two vertices $v, w \in T$ can be separated by deleting $S$ and some element of the unique path connecting $v$ and $w$ in $G_{k}[T]$. So no two vertices in $G_{k}$ are connected by more than $k$ independent paths.

To conclude that $G_{k}$ is uniformly $k$-connected, it remains to be shown that $G_{k}$ is $k$-connected. Because each vertex in $S$ is adjacent to all vertices in $T$, a minimum separator of $G_{k}$ contains either $S$ or $T$. So a minimum separator of $G_{k}$ contains at least $k$ vertices, because $|S|=k-1$ and $S$ itself does not separate $G$ and $|T|=k$. Thus $G_{k}$ is $k$-connected.

In fact, Figure 6 shows us the smallest uniformly $k$-connected graph that is not uniformly $k$-edge-connected. This is because such an example has to contain at least two vertices that are connected by $k+1$ edge-disjoint paths and thus have to be of degree at least $k+1$. So for $k \geq 4$ suitable graphs clearly cannot have five or fewer vertices. If there are two vertices of degree five on six vertices, then those vertices are connected by five independent paths. So the degree sequence $5,5,4,4,4,4$, belonging to the graph drawn solid in Figure 6, contains smallest possible values and this example therefore contains fewest possible edges. To prove that for $k \leq 3$ there is no uniformly $k$-connected graph that is not uniformly $k$-edge-connected takes a bit more effort. We begin with the following fact about fans in $k$-connected graphs.

Lemma 21. Consider a $k$-connected graph $G$ where $k \in \mathbb{N}$. For distinct vertices $v, w \in V(G)$, consider subgraphs $G_{v} \subseteq G-w$ and $G_{w} \subseteq G-v$ such that $V\left(G_{v}\right) \cup V\left(G_{w}\right)=V(G)$ and $E\left(G_{v}\right) \cup E\left(G_{w}\right)=E(G)$. Furthermore, let $S:=V\left(G_{v}\right) \cap V\left(G_{w}\right)$ be a separator of cardinality $|S|=k$. Then $G_{w}$ contains for each $x \in S$ an $x-S \backslash\{x\}$ fan consisting of $\min \left\{\left|N_{G_{w}}(x)\right|, k-1\right\}$ paths.

Proof. Let us consider a vertex set $T \subseteq V\left(G_{w}\right) \backslash\{x\}$ separating $x$ and $S \backslash\{x\}$ in $G_{w}$. Our statement is implied by Theorem 2 if we manage to prove that $|T| \geq \min \left\{\left|N_{G_{w}}(x)\right|, k-1\right\}$. This is certainly true when $|T| \geq\left|N_{G_{w}}(x)\right|$. So let us suppose that $|T|<\left|N_{G_{w}}(x)\right|$. Then there is a vertex $y \in N_{G_{w}}(x) \backslash T$ separated by $T$ from $S \backslash\{x\}$. Theorem 2 then says there is a $y$ - $S$ fan in $G$ that consists of $k$ paths, because $G$ is $k$-connected. These paths end in pairwise distinct vertices of $S$ and hence cannot contain vertices from $G_{v}$. because $T$ separates $y$ and $S \backslash\{x\}$, it has to be of cardinality $|T| \geq k-1$, what remained to be shown.

Theorem 22. Consider a $k$-connected graph $G$ where $k \in\{0,1,2,3\}$ that contains two vertices connected by $k+1$ edge-disjoint paths. Then $G$ contains vertices which are connected by $k+1$ independent paths.

Proof. Let us first record that we can assume $G$ to be a simple graph. This is because $G$ is $k$-connected and thus Lemma 19 says that parallel edges already imply the existence of $k+1$ independent paths. Furthermore, the statement to be shown is certainly true for $k=0$. So let $k \in\{1,2,3\}$, suppose that our


Figure 7: A 3-connected graph containing two vertices which are connected by four edge-disjoint paths
claim is true for $k-1$ and take two vertices $v, w \in V(G)$ which are connected by $k+1$ edge-disjoint paths.

In the case where $v w \in E(G)$, the graph $G-v w$ is still $(k-1)$-connected and there remain $k$ edge-disjoint paths connecting $v$ and $w$. We obtain by induction that there exist $k$ independent paths in $G-v w$. In $G$ the edge $v w$ is another such path. So we find $v$ and $w$ to be connected by $k+1$ independent paths in $G$.

We now turn to the case where $v w \notin E(G)$. Suppose, for the sake of contradiction, that $v$ and $w$ are not connected by $k+1$ independent paths. In view of Theorem 1, this means that $G$ contains a $v-w$ separator $S$ of cardinality $|S| \leq k$, which we may choose as close as possible to $v$. More precisely, we consider $S \subseteq V(G) \backslash\{v, w\}$ to be the only $\{v\}-S$ separator containing $k$ or fewer vertices. Referring to the component of $G-S$ containing $v$ by $H$, we denote the subgraphs

$$
\begin{aligned}
G_{v} & :=(V(H) \cup S, E(H) \cup E(H, S)) \quad \text { and } \\
G_{w} & :=G-V(H)
\end{aligned}
$$

With this setup, illustrated for $k=3$ in Figure 7, we have $G_{v} \subseteq G-w$, $G_{w} \subseteq G-v$ as well as $V\left(G_{v}\right) \cup V\left(G_{w}\right)=V(G), E\left(G_{v}\right) \cup E\left(G_{w}\right)=E(G)$, $V\left(G_{v}\right) \cap V\left(G_{w}\right)=S$, and $|S|=k$, which allows to employ Lemma 21 in what follows. We also observe that the $k+1$ edge-disjoint $v-w$ paths given in $G$ contain $k+1$ edge-disjoint $v$ - $S$ subpaths. Because $|S|=k$, there is one vertex $x \in S$ which is contained in two of these subpaths. Because $G$ does not contain parallel edges, the vertex $x$ has two distinct neighbors in $G_{v}$ and two distinct neighbors in $G_{w}$. Denoting a neighbor of $x$ in $G_{v}$ that is not $v$ by $y$, we observe that $y \notin S$, because we defined $G_{v}$ not to contain edges between
vertices of $S$. So the vertex set $Y:=S \cup\{y\}$ is of cardinality $|Y|=k+1$ and at least $k+1$ vertices are required to separate $Y$ from $v$, as we have chosen $S$ to be a separator of cardinality $|S| \leq k$ closest possible to $v$. So Theorem 2 says that there is a $v-Y$ fan in $G_{v}$ consisting of $k+1$ independent paths, of which none can contain the edge $x y$. Consequently, there are $k+1$ independent $v-S$ paths and two of them have $v$ and $x$ as endpoints.

For $\left|N_{G_{w}}(x)\right| \geq 2$ and $k \leq 3$, we evaluate $\min \left\{\left|N_{G_{w}}(x)\right|, k-1\right\}=k-1$. So we find an $x-S \backslash\{x\}$ fan in $G_{w}$ consisting of $k-1$ paths, by Lemma 21. As is illustrated in Figure 7 for $k=3$, these paths can be concatenated with the $k+1$ independent $v$ - $S$ paths which exist in $G_{v}$ to obtain $k+1$ independent $v-x$ paths in $G$.

The proof cannot work for $k \geq 4$, because Figure 6 displays examples that show that our statement does not hold in this case. And indeed, the step in which we evaluate $\min \left\{\left|N_{G_{w}}(x)\right|, k-1\right\}=k-1$ is not valid for $\left|N_{G_{w}}(x)\right| \geq 2$ and $k \geq 4$, which is why we cannot rely on Lemma 21 to find enough independent paths. To conclude this section with an extra corollary, we employ the following consequence of Menger's Theorem.

Lemma 23. A graph which is $k$-connected and uniformly $k$-edge-connected is uniformly $k$-connected.

Proof. Theorem 5 says that we find at least $k$ independent paths between each pair of vertices of a $k$-connected graph. On the other hand, there are at most $k$ independent paths between each pair of vertices, because those are also edge-disjoint, of which we cannot have more than $k$ in a uniformly $k$-edge-connected graph.

Corollary 24. For $k \leq 3$ a $k$-connected graph is uniformly $k$-connected if and only if it is uniformly $k$-edge-connected.

Proof. This follows from Lemma 23 and Theorem 22.

### 3.3 Constructing uniformly connected graphs

In Section 3.1, we already observed that uniformly 1-connected graphs are exactly the uniformly 1 -edge-connected graphs which comprise exactly all trees. The goal of this section is to continue with such descriptions. Whereas uniformly $k$-connected and uniformly $k$-edge-connected graphs still have a pretty neat structure for $k=2$, their variety increases considerably for $k \geq 3$. In this case, constructive descriptions become all the more interesting.


Figure 8: Illustration that uniformly 2-connected graphs are cycles only

Lemma 25. A graph is uniformly 2-connected if and only if it is a cycle.
Proof. A cycle certainly is uniformly 2-connected.
Conversely, a uniformly 2-connected graph $G$ has at least three vertices and is 2 -connected. So it contains a cycle $C$ as a subgraph. Suppose for the sake of contradiction that $G \neq C$. If there is a chord in $C$, then we immediately find three independent paths in $G$. So there must be some vertex $x$ not contained in $V(C)$. Because $G$ is 2 -connected, we know from Theorem 2 that there exists an $x-\{v, w\}$ fan $F$ for any two distinct vertices $v, w \in V(C)$. We find again three independent paths, which is illustrated in Figure 8 and concludes our proof.

Let us continue with the following statements by Kingsford and Marçais [70], which allow to characterize the class of uniformly 2-edge-connected graphs.

Theorem 26. A connected graph is uniformly $k$-edge-connected if and only if each of its blocks is uniformly $k$-edge-connected.

Proof. We observe first that each path connecting any two vertices that are contained in the same block of a graph must be completely contained in this block. This shows already that a uniformly $k$-edge-connected graph contains only uniformly $k$-edge-connected blocks.

Now let $G$ be a connected graph whose blocks are all uniformly $k$-edge-connected. As paths between two vertices $v$ and $w$ that belong to the same block have to be completely contained in this block, we are sure to find exactly $k$ edge-disjoint paths connecting $v$ and $w$. So let us consider the remaining case where $v$ is in a block $B_{v}$ other than the block $B_{w}$ in which $w$ is contained. For this purpose, let $P$ be the unique path connecting $B_{v}$ and $B_{w}$ in the blockcutpoint tree of $G$. We denote $c_{0}=v$ and $c_{\ell}=w$ as well as the cut-vertices of $G$ on $P$ that are not in $\{v, w\}$ by $c_{1}, \ldots, c_{\ell-1}$, where these vertices shall be sorted in the order in which they appear when traversing $P$ from $v$ to $w$.


Figure 9: The structure of uniformly 2-edge-connected graphs

For $i \in\{1, \ldots, \ell\}$ each path that connects $c_{i-1}$ with $c_{i}$ in $G$ only contains edges of the unique block in $G$ that contains both $c_{i-1}$ and $c_{i}$. Because each block of $G$ is uniformly $k$-edge-connected, we find $k$ edge-disjoint paths between $c_{i-1}$ and $c_{i}$ that we denote by $P_{i}^{1}, \ldots, P_{i}^{k}$. Those can be concatenated to $k$ edge-disjoint paths

$$
\bigcup_{i=1}^{\ell} P_{i}^{1}, \ldots, \bigcup_{i=1}^{\ell} P_{i}^{k}
$$

that connect $v$ and $w$. There also cannot be more than $k$ edge-disjoint paths between $v$ and $w$, as this already requires at least $k+1$ edge-disjoint paths between $c_{0}$ and $c_{1}$, which contradicts the fact that the corresponding block is uniformly $k$-edge-connected. So we conclude that $G$ is uniformly $k$-edgeconnected.

Corollary 27. A connected graph is uniformly 2-edge-connected if and only if each of its blocks is a cycle.

Proof. Certainly, any connected graph whose blocks are cycles is uniformly 2-edge-connected.

Let now $G$ be a uniformly 2-edge-connected graph. By Theorem 26, each block $B$ of $G$ is uniformly 2-edge-connected. If $B$ is 2 -connected, then $B$ is a cycle by Lemma 23 and Lemma 25 . Otherwise, $B$ is a graph with two vertices that are joined by two edges, which is also a cycle.

So we can get an impression of the structure of uniformly 2-edge-connected graphs by taking a look at Figure 9. Note too that although the formulations of Lemma 25 and Corollary 27 do not stress that point, they also describe how to construct uniformly 2 -connected and uniformly 2 -edge-connected graphs. As such constructive descriptions often allow further insight into a graph


Figure 10: A vertex split that does not preserve uniform connectivity
class, we shall proceed in that direction. A famous characterization is Tutte's Wheel Theorem [102]. It says that any 3-connected graph is a wheel or can be obtained from a wheel by successively adding edges between distinct nonadjacent vertices and splitting vertices. This splitting construction is allowed for any vertex $x$ of degree at least four. All neighbors of $x$ shall be collected in two disjoint sets $V$ and $W$ such that $|V| \geq 2$ and $|W| \geq 2$. Then splitting $x$ means that it is replaced by two vertices $v$ and $w$ and incident edges such that $N(v)=V \cup\{w\}$ and $N(w)=W \cup\{v\}$, whereas all further incidence relations remain unchanged. This construction though preserving 3 -connectivity does not preserve uniform connectivity. This can be seen from Figure 10, which displays a vertex split that produces two vertices $v$ and $w$ that are connected by four independent paths. Another attractive characterization by Tutte describes how to build all 3-regular 3-connected graphs. It is based on the following construction.

Definition 28. Joining two edges $k \ell, v w \in E(G)$ of a 3-regular 3-connected graph $G$ means to take two new vertices $x, y \notin V(G)$ and to form the graph

$$
G+x+y-k \ell-v w+k x+x \ell+v y+y w+x y .
$$



Figure 11: Joining the edges $v w$ and $k \ell$


Figure 12: Joining two graphs by the bridge construction

This construction is also illustrated in Figure 11. Also note that $k \ell$ and $v w$ are two distinct edges, but they are allowed to share one endvertex. Tutte [103, Chapter 12] characterized 3-regular 3-connected graphs exactly as those that can be obtained from the complete graph on four vertices by successively joining edges. This construction cannot produce all uniformly 3 -connected graphs, as they comprise also non-regular ones, but it preserves uniform connectivity, because 3-regular 3-connected graphs are uniformly 3 -connected by statement (i) of Lemma 18 . So the class of uniformly 3 -connected graphs sits between those characterized by Tutte and we require further constructions to produce them all.

Definition 29. For graphs $G_{1}$ and $G_{2}$ which contain vertices $v_{1} \in V\left(G_{1}\right)$ and $v_{2} \in V\left(G_{2}\right)$ having both three neighbors, denoted by $N\left(v_{1}\right)=\left\{x_{1}, x_{2}, x_{3}\right\}$ and $N\left(v_{2}\right)=\left\{y_{1}, y_{2}, y_{3}\right\}$, by bridge construction we refer to forming

$$
\left(\left(G_{1}-v_{1}\right) \cup\left(G_{2}-v_{2}\right)\right)+x_{1} y_{1}+x_{2} y_{2}+x_{3} y_{3}
$$

The set of all graphs obtained this way from $G_{1}$ and $G_{2}$, for any such $v_{1}$ and $v_{2}$, is denoted by $G_{1} \oplus G_{2}$.

Definition 30. Let $G$ be a graph containing distinct vertices $v, w, x \in V(G)$ and an edge $v w \in E(G)$, that satisfies $\operatorname{deg}(u)=3$ for all $u \in V(G) \backslash\{x\}$ and $\operatorname{deg}(x) \geq 3$. For a vertex $y \notin V(G)$, by spoke construction we refer to forming

$$
(G+y)-v w+x y+v y+w y .
$$

The set of all graphs obtained this way from $G$, for any such $v, w$, and $x$, is denoted by $\theta(G)$. Furthermore, we speak of a primary spoke construction if $\operatorname{deg}(x)=3$ holds above, and call it a secondary spoke construction if $\operatorname{deg}(x)>3$.


Figure 13: Expanding a graph by the spoke construction

At some points, we just say bridge for bridge construction or spoke for spoke construction. Our goal now is to prove that these constructions preserve uniform 3 -connectivity and that they indeed suffice to construct all uniformly 3 -connected graphs. Before entering into this discussion, let us recall the following notion.

Definition 31. A degenerate cut is a cut having one side that contains a single vertex.

Lemma 32. Let $G_{1}$ and $G_{2}$ be graphs and $H$ be a graph in $G_{1} \oplus G_{2}$. Then $H$ is uniformly 3 -connected if and only if $G_{1}$ and $G_{2}$ are both uniformly 3-connected.

Proof. We are given $\left.H=\left(G_{1}-v_{1}\right) \cup\left(G_{2}-v_{2}\right)\right)+x_{1} y_{1}+x_{2} y_{2}+x_{3} y_{3}$. for distinct vertices $v_{1}, x_{1}, x_{2}, x_{3} \in V\left(G_{1}\right)$ with $N\left(v_{1}\right)=\left\{x_{1}, x_{2}, x_{3}\right\}$ and distinct vertices $v_{2}, y_{1}, y_{2}, y_{3} \in V\left(G_{2}\right)$ with $N\left(v_{2}\right)=\left\{y_{1}, y_{2}, y_{3}\right\}$.

We begin be assuming that $G_{1}$ and $G_{2}$ are both uniformly 3-connected and showing that this implies $H$ to be 3 -connected. Consider first two vertices $v \in V\left(G_{1}\right)$ and $w \in V\left(G_{2}\right)$. Having three independent $v$ - $v_{1}$ paths in $G_{1}$, we also have three independent $v-\left\{x_{1}, x_{2}, x_{3}\right\}$ paths in $G_{1}-v_{1}$. Likewise, there are three independent $\left\{y_{1}, y_{2}, y_{3}\right\}-w$ paths in $G_{2}-v_{2}$. Combining these paths with the edges $x_{1} y_{1}, x_{2} y_{2}$, and $x_{3} y_{3}$ gives three independent $v-w$ paths in $H$. So to prove that $H$ is 3 -connected, it remains to be shown that there is no 2 -separator for vertices $v, w \in V\left(G_{i}\right)$, for $i \in\{1,2\}$. We concentrate, without loss of generality, on the case where $v, w \in V\left(G_{1}\right)$ and assume, for a contradiction, that there is a $v-w$ separator $S \subseteq V(H) \backslash\{v, w\}$ of cardinality $|S|=2$ in $H$. In $G_{1}-S$ there is a $v$-w path $P$, because $G_{1}$ is 3-connected. If $v_{1} \notin V(P)$, then $P$ also exists in $H-S$, against our assumption that $S$ separates $v$ and $w$ in $H$. So all $v$-w paths in $G_{1}-S$ have to contain $v_{1}$. Certainly, this requires $S \subseteq V\left(G_{1}\right)$. But such a path $P$ contains a subpath $x_{i} v_{1} x_{j}$
for certain $i, j \in\{1,2,3\}$ with $i \neq j$. We define $Q$ as the path that results from $P$ by replacing the subpath $x_{i} v_{1} x_{j}$ by $x_{i} y_{i} R y_{j} x_{j}$ for some $y_{i}-y_{j}$ path $R$ in $G_{2}-v_{2}$, which is well-defined because $S \subseteq V\left(G_{1}\right)$. The path $Q$ exists in $H-S$, which is a contradiction. So $H$ is 3 -connected.

To obtain that $H$ is uniformly 3 -connected, assume, for a contradiction, there are vertices $v, w \in V(H)$ that are connected by four independent paths. Figure 12 , which illustrates the construction by which $H$ arises, shows that there cannot be four independent paths connecting a vertex in $V\left(G_{1}\right)$ with a vertex in $V\left(G_{2}\right)$. So, without loss of generality, let $v, w \in V\left(G_{1}\right)$. Again, in view of Figure 12, just one of four independent paths between $v$ and $w$, can contain vertices of $G_{2}$. We denote this path by $P$ and observe that it contains a subpath of the form $x_{i} y_{i} Q y_{j} x_{j}$ for certain $i, j \in\{1,2,3\}$ with $i \neq j$ and some $y_{i}-y_{j}$ path $Q$ in $G_{2}-v_{2}$. The path that is obtained by replacing the subpath $x_{i} y_{i} Q y_{j} x_{j}$ in $P$ by $x_{i} v_{1} x_{j}$ remains independent to the other three paths between $v$ and $w$ and is contained in $G_{1}$. This contradicts that $G_{1}$ is uniformly 3 -connected and thus proves that $H$ is uniformly 3 -connected.

Now we assume that $H$ is uniformly 3 -connected and aim to prove that then $G_{1}$ and $G_{2}$ have to be uniformly 3 -connected as well. We focus only on $G_{1}$ as the following arguments apply to $G_{2}$ analogously. To show that $G_{1}$ is 3 -connected, consider two arbitrary vertices $v, w \in V\left(G_{1}\right)$. For the case where $v_{1} \in\{v, w\}$, we may consider $v=v_{1}$ and observe that there are three independent $w-y_{1}$ paths in $H$. These paths induce a $w$ - $\left\{x_{1}, x_{2}, x_{3}\right\}$ fan $F$ in $G_{1}$. Including the edges $x_{1} v_{1}, x_{2} v_{1}, x_{3} v_{1} \in E\left(G_{1}\right)$ to $F$ shows that there are three independent $w$ - $v_{1}$ paths in $G_{1}$. This leaves us with the case $v_{1} \notin\{v, w\}$. In $H$ the vertices $v$ and $w$ are connected by three independent paths. Only one of them, say $P$, can contain vertices of $G_{2}$. In $P$ we can replace the subpath $x_{i} y_{i} Q y_{j} x_{j}$, where $i, j \in\{1,2,3\}$ with $i \neq j$ and some $y_{i}-y_{j}$ path $Q$ in $G_{2}-v_{2}$, by $x_{i} v_{1} x_{j}$ to obtain a path contained in $G_{1}$ that remains independent of the other two paths between $v$ and $w$. So $G_{1}$ is 3-connected.

To show that $G_{1}$ is indeed uniformly 3-connected, suppose there are vertices $v, w \in V\left(G_{1}\right)$ connected by four independent paths in $G_{1}$. As the degree of $v_{1}$ is three, we know that $v_{1} \notin\{v, w\}$. If $v_{1}$ is contained in none of the four paths, then they exist in $H$ as well, contradicting that $H$ is uniformly 3 -connected. So exactly one of the three independent paths contains a subpath of the form $x_{i} v_{1} x_{j}$ for some $i, j \in\{1,2,3\}$ with $i \neq j$. Replacing this subpath by $x_{i} y_{i} Q y_{j} x_{j}$ for some $y_{i}-y_{j}$ path $Q$ in $G_{2}-v_{2}$ shows that there are four independent paths in $H$, which is again a contradiction and concludes our proof.

Lemma 33. Each graph in $Q(G)$ is uniformly 3-connected if $G$ is a uniformly 3 -connected graph.

Proof. We consider a graph $H=G+y-v w+x y+v y+w y$ for distinct vertices $v, w, x \in V(G)$, an edge $v w \in E(G)$ and a vertex $y \notin V(G)$ such that $\operatorname{deg}(u)=3$ for all $u \in V(G) \backslash\{x\}$. We are given that $G$ is a uniformly 3 -connected graph. To show that also $H$ is uniformly 3 -connected, we just need to prove that $H$ is 3 -connected, because only one of its vertices is of degree larger then three, which is why two vertices in $V(H)$ cannot be connected by more than three independent paths.

Now let us consider two arbitrary vertices $s, t \in V(G)=V(H) \backslash\{y\}$. For sure, we find three independent $s$ - $t$ paths in $G$, as $G$ is 3 -connected. If none of these paths contains the edge $v w$, these three paths are present in $H$ as well. If one of these paths, say $P$, contains $v w$, we obtain a path $Q$ in $H$ that results from $P$ when replacing the edge $v w$ by the subpath $v y w$. As neither the vertex $y$ nor the edges $v y$ and $y w$ exist in $G$, we find that $Q$ is independent of the other two paths. What remains to be checked is whether $y$ is connected to any other vertex $t \in V(H)$ by three independent paths. Let us assume, for contradiction, that there is a $y$ - $t$ separator $S \subseteq V(H) \backslash\{y, t\}$ of cardinality $|S|=2$. In $H-S$ the component of $y$ contains besides $y$ at least one further vertex, because $y$ has three neighbors in $V(G)$. But this shows that $S$ separates two vertices from $V(G)$ in $H$, which contradicts what we have shown already. So $H$ is 3 -connected and the proof is complete.

Lemma 34. Let $H$ be a graph in $\otimes(G)$ for some graph $G$. If $H$ is uniformly 3 -connected and if each 3 -cut in $H$ is degenerate, then also $G$ is uniformly 3 -connected.

Proof. Let $H=G+y-v w+x y+v y+w y$ be a uniformly 3-connected graph for distinct $v, w, x \in V(G), v w \in E(G)$ and $y \notin V(G)$ with $\operatorname{deg}(u)=3$ for all $u \in V(G) \backslash\{x\}$. We also know that each 3-cut in $H$ is degenerate. As in the proof of Lemma 33, we only have to show that $G$ is 3 -connected, because at most one of its vertices is of degree larger than three.

Our first goal is to show that $x$ is connected to all other vertices in $V(G)$ by three independent paths. To obtain a contradiction, suppose that some vertex $z \in V(G)$ can be separated from $x$ by a set $S=\left\{s_{1}, s_{2}\right\} \in V(G) \backslash\{x, z\}$. Let the component of $G-S$ containing $x$ be denoted by $G_{x}$ and the component containing $z$ by $G_{z}$. This situation is illustrated in Figure 14. Both vertices $s_{1}$ and $s_{2}$ are of degree three which implies

$$
\min \left\{E\left(\left\{s_{i}\right\}, V\left(G_{x}\right)\right), E\left(\left\{s_{i}\right\}, V\left(G_{z}\right)\right)\right\}=1 \quad \text { for } i \in\{1,2\}
$$



Figure 14: A situation supposed in the proof of Lemma 34

So we may denote the single edge joining $s_{i}$ with the respective component $G_{x}$ or $G_{z}$ by $e_{i}$, which is also depicted in Figure 14. This shows that there is a cut $\left\{e_{1}, e_{2}\right\}$ in $G$. In case $v w \notin\left\{e_{1}, e_{2}\right\}$, then $\left\{e_{1}, e_{2}, x y\right\}$ is a cut separating $x$ and $z$ in $H$. We know that all edges in this cut are incident to one vertex of degree three, because we are given that $H$ only contains degenerate 3-cuts. But the degree of $x$ in $H$ is larger than three and the edges $e_{1}$ and $e_{2}$ are not incident to $y$, because $y \notin V(G)$. This is a contradiction. In case $v w \in\left\{e_{1}, e_{2}\right\}$, the cut $\left\{e_{1}, e_{2}, x y\right\} \backslash\{v w\} \cup\{v y\}$ separates $x$ and $z$ in $H$. Against our assumption, this 3 -cut is again not degenerate, because one of its edges is not incident to $y$ while the other two are.
To conclude that $G$ is 3 -connected, suppose, for contradiction, that there is some separator $S \subseteq V(G)$ of cardinality $|S|=2$. We know that $S$ has to contain $x$, as otherwise $S$ also separates $x$ from some other vertex in $V(G)$, which is not possible according to our previous reasoning. However, if $S$ contains $x$, then $S$ also is a separator in $H$, which contradicts the fact that $H$ is uniformly 3 -connected.
The condition in Lemma 34 requiring only degenerate cuts may seem somewhat strange. Yet it is a crucial one. The graph $H$ depicted on the left in Figure 15 is uniformly 3 -connected and is in $\Theta(G)$ for the graph $G$ depicted on the right in Figure 15. But $H$ contains a non-degenerate 3 -cut, which is drawn in dashed lines there and indeed the graph $G$ is not 3 -connected, as its highlighted vertices form a 2 -separator. Now that we have investigated in which sense the $\oplus$ and $\theta$ constructions preserve uniform connectivity, we may proceed to the main result of this section.
Theorem 35. Consider the following recursively defined inclusionwise minimal graph class $\mathcal{C}$. Let each 3 -connected 3 -regular graph be in $\mathcal{C}$. For $G \in \mathcal{C}$ the class $\mathcal{C}$ has to contain each graph in $\otimes(G)$ and for $G_{1}, G_{2} \in \mathcal{C}$ the class $\mathcal{C}$ has to contain each graph in $G_{1} \oplus G_{2}$. Then a graph is uniformly 3-connected if and only if it is contained in $\mathcal{C}$.


Figure 15: An example showing that it is necessary for $H$ to only contain non-degenerate 3 -cuts in Lemma 34

Proof. We know that each 3-connected 3-regular graph is uniformly 3-connected by Lemma 18. Moreover, the $\oplus$ and $\otimes$ constructions preserve uniform 3 -connectivity by Lemmas 32 and 33, respectively. This proves that each graph in $\mathcal{C}$ is uniformly 3 -connected.

To show that the class $\mathcal{C}$ contains indeed all uniformly 3 -connected graphs, we proceed by induction on the number of vertices. The smallest uniformly 3 -connected graph is the complete graph on four vertices. It is contained in $\mathcal{C}$, as it is 3 -connected and 3 -regular, and it is the only uniformly 3 -connected graph on four vertices. So let $G$ be an arbitrary uniformly 3 -connected graph with $|V(G)| \geq 5$ and let us suppose that any graph on less than $|V(G)|$ vertices is contained in $\mathcal{C}$.

We begin with the case in which $G$ contains only degenerate 3 -cuts. We also consider $G$ to be irregular, because otherwise $G$ belongs to $\mathcal{C}$ already by definition. Furthermore, we assume, for contradiction, that there are two vertices $s, t \in V(G)$ of degree larger than three. From Corollary 24, we know that $G$ is 3 -edge-connected, and thus contains a 3-cut that separates $s$ and $t$. As we assume each 3 -cut in $G$ to be degenerate and because the degrees of $s$ and $t$ are larger than three, we find that all edges of this 3 -cut are incident to one vertex in $V(G) \backslash\{s, t\}$. But then this vertex is an $s$ - $t$ separator, contradicting that $G$ is 3 -connected. This shows that $G$ contains exactly one vertex of degree larger than three, which we call $x$. For a neighbor $y$ of $x$ we also denote $N(y)=\{v, w, x\}$. Now suppose, for contradiction, that $v w \in E(G)$. Then $G$ contains the triangle on $\{v, w, y\}$ as a subgraph. Because all vertices except $x$ have degree three, we conclude that $E(\{v, w, y\}, V(G) \backslash\{v, w, y\})$ contains exactly three edges, which separate $x$ and $\{v, w, y\}$. Recalling that $\operatorname{deg}(x)>3$, we found a 3 -cut that is not degenerate, against our assumption. Therefore, $v w \notin E(G)$ and hence $H:=G-y+v w$ is well-defined. In consequence, $G$ is a graph in $Q(H)$. Because all 3-cuts in $G$ are de-
generate, we obtain by Lemma 34 that $H$ is uniformly 3-connected. Because $|V(H)|<|V(G)|$, our induction hypothesis says that $H$ is contained in $\mathcal{C}$. This in turn implies that $G$ is in $\mathcal{C}$, as $G$ is in $\otimes(H)$.

It remains the case in which $G$ contains a 3 -cut $F=\left\{e_{1}, e_{2}, e_{3}\right\}$ that is nondegenerate. We begin by showing that the edges in $F$ have no common endvertices. Now assume, for contradiction, that at least two edges in $F$ have a common endvertex $x$. Let those edges be $e_{1}$ and $e_{2}$. In $G-F$ the edge $e_{3}$ has an endvertex $y$ that is not contained in the component of $x$. Since $y$ cannot be incident to both edges $e_{1}$ and $e_{2}$ and $x$ is not incident to $e_{3}$, we find that $\{x, y\}$ is a separator in $G$, which is a contradiction to $G$ being 3 -connected. So the edges in $F$ have all distinct endvertices. We denote the two components of $G-F$ by $X$ and $Y$ and address for $i \in\{1,2,3\}$ the endvertex of $e_{i}$ that is contained in $X$ by $x_{i}$ and the endvertex that is contained in $Y$ by $y_{i}$. Defining for two vertices $v_{1}, v_{2} \notin V(G)$ the graphs $G_{1}:=X+v_{1}+x_{1} v_{1}+x_{2} v_{1}+x_{3} v_{1}$ and $G_{2}:=Y+v_{2}+y_{1} v_{2}+y_{2} v_{2}+y_{3} v_{2}$, we obtain that $G$ is in $G_{1} \oplus G_{2}$. So Lemma 32 implies that both graphs $G_{1}$ and $G_{2}$ are uniformly 3-connected. By their construction they satisfy $\left|V\left(G_{1}\right)\right|<|V(G)|$ and $\left|V\left(G_{2}\right)\right|<|V(G)|$. Consequently, our induction hypothesis says that $G_{1}$ and $G_{2}$ are in $\mathcal{C}$ and because $G$ is in $G_{1} \oplus G_{2}$, this shows that $G$ is contained in $\mathcal{C}$.

With this we achieved a concise description of the class of uniformly 3connected graphs. The strength of such a constructive characterization is that it precisely describes how more complex graphs can be constructed from smaller building blocks, which in turn comes in very useful when proving further structural properties. This is what we focus on in the next section.

Before embarking on that, let us compare our constructive results for uniformly 3 -connected graphs with the characterization of uniformly 3-edge-connected graphs by Kingsford and Marçais [70]. Their construction builds on the dumbbell graph, which consists of two vertices connected by three parallel edges. They prove that each uniformly 3 -edge-connected graph can be obtained from dumbbell graphs by successive block gluings and cycle expansions. By gluing two graphs $G_{1}$ and $G_{2}$ we simply mean to build their union and to replace two vertices $v_{1} \in V\left(G_{1}\right)$ and $v_{2} \in V\left(G_{2}\right)$ by a vertex $v$ adjacent to all neighbors of $v_{1}$ and $v_{2}$. By a cycle expansion we mean to replace a vertex $v$ of a graph by a cycle $C$ containing $c \leq \operatorname{deg}(v)$ vertices. One vertex of $C$ shall be adjacent to $\operatorname{deg}(v)-c+1$ neighbors of $v$ and the remaining vertices of $C$ shall be adjacent to exactly one neighbor of $v$. Figure 16 illustrates how to construct the wheel graph $W_{5}$ from a smaller uniformly edge-connected graph. As in this example, and beginning already with the dumbbell graph, these constructions heavily rely on the existence of multiple edges. They also


Figure 16: A cycle expansion producing a wheel graph
produce separators of cardinality one and two. So they do not preserve uniform vertex-connectivity. Conversely, the constructions from Definition 29 and 30 can certainly not produce all uniformly 3 -edge-connected graphs, but they preserve uniform edge-connectivity, because of Corollary 24 uniformly 3 -connected graphs are also uniformly 3 -edge-connected.

### 3.4 Properties of uniformly connected graphs

The complete graph on four vertices, which can also be regarded as the wheel graph $W_{4}$, is the smallest building block when constructing uniformly 3 -connected graphs. It is the smallest uniformly 3 -connected graph and, in view of Theorem 35, belongs to the base class of 3-regular 3-connected graphs. All further wheel graphs $W_{n}$ on $n \geq 5$ vertices can be obtained from $\otimes\left(W_{n-1}\right)$. In this recursion, each $\theta\left(W_{n}\right)$ gives a unique graph. But let us recall at this point that the result of both $\theta$ and $\oplus$ are sets of graphs. So to make the nested use of these symbols precise, we introduce for a graph $G$ and a set of graphs $\mathcal{H}$ the conventions

$$
\mathcal{H} \oplus G:=\bigcup_{H \in \mathcal{H}}(H \oplus G) \quad \text { and } \quad \otimes(\mathcal{H}):=\bigcup_{H \in \mathcal{H}} \otimes(H) .
$$

In this section, we are concerned with structural properties of certain uniformly connected graphs. Let us begin to study which vertex degrees are possible. Uniformly $k$-connected graphs clearly have minimum degree $k$ and the wheel graph on $n$ vertices is an example showing that uniformly connected graphs may contain a vertex of degree $n-1$. For $k<n-1$, however, no uniformly $k$-connected graph contains more than one vertex of degree $n-1$, because otherwise we find the corresponding vertices to be connected by $n-1$ independent paths. We may further ask how many vertices of minimum degree can possibly exist. Formally, for a graph $G$ we can ask for the parameter

$$
\nu(G):=\left|\left\{v \in V(G): \operatorname{deg}(v)=\min _{w \in V(G)} \operatorname{deg}(w)\right\}\right| .
$$

This question attracted wide interest in extremal graph theory. For example, Kingsford and Marçais [71] showed for $k \in \mathbb{N}$ that a uniformly $k$-edge-connected graph $G$ satisfies $\nu(G) \geq 2$. A multigraph resulting from a path graph on $n$ vertices in which we replace each edge by exactly $k$ parallel edges is an example that attains this bound for general $n$. Minimally connected graphs have also been studied extensively and all the results obtained in this context also hold for uniformly $k$-connected as well as uniformly $k$-edge-connected graphs. This is implied by Theorem 17, which states that graphs from both classes are minimally $k$-connected. The starting point of these investigations is a result by Halin [60], who proved that each minimally $k$-connected graph contains a vertex of degree $k$. Dirac [37] showed $\nu(G) \geq(n+4) / 3$ for minimally 2 -connected graphs $G$ on $n$ vertices and Halin [61] proved that $\nu(G) \geq(2 n+6) / 5$ for minimally 3 -connected graphs $G$ on $n$ vertices. Both bounds are tight and are included as special cases of the inequality $\nu(G) \geq((k-1) n+2 k) /(2 k-1)$ for minimally $k$-connected graphs $G$ on $n$ vertices, given by Mader [77]. This result cannot be improved if we are interested in formulas that only depend on a graph's number of vertices. However, Oxley [83] achieved stronger bounds depending on a graph's number of vertices $n$ and number of edges $m$. The question of whether Oxley's result is best possible has recently been answered by Schmidt [90], by providing the tight bound $\nu(G) \geq \max \{(k+1) n-2 m,\lceil(m-n+k) /(k-1)\rceil\}$, which deviates from Oxley's result only for small $m$.

All mentioned results are valid for uniformly connected as well as uniformly edge-connected graphs. We may see even stronger bounds for them, since Figure 3 shows that minimally $k$-connected graphs neither have to be uniformly $k$-connected nor have to be uniformly $k$-edge-connected. Indeed, all vertices contained in a uniformly 2 -connected graph have to be of minimum degree two, as we characterized those graphs exactly as cycles in Lemma 25. In the following, our goal is to show a sharp bound on the number of vertices of minimum degree for uniformly 3 -connected graphs.

Recall that Definition 30 for forming $Q(G)$ requires that $G$ has at most one distinguished vertex $x$ whose degree is allowed to exceed three. We introduced the terms primary spoke if $\operatorname{deg}(x)=3$ in Definition 30 and secondary spoke if $\operatorname{deg}(x)>3$. Also recall that Tutte's edge joining construction, given in Definition 28 , is only defined for 3 -regular 3 -connected graphs. Tutte [103, Chapter 12] characterized these graphs as exactly those that can be built from the complete graph on four vertices by successively joining edges. So the base class in Theorem 35 is built by edge joins, which is why their number appears in the following counting statement.

Theorem 36. A uniformly 3 -connected graph on $n$ vertices satisfies

$$
n=4+2 t+2 b+p+s
$$

if it is built from complete graphs on four vertices by a sequence of $t$ edge joins, $b$ bridges, $p$ primary spokes, and $s$ secondary spokes.

Proof. The complete graph on four vertices is the smallest uniformly 3-connected graph. It satisfies our statement, as in this case $t=b=p=s=0$. So let now $G$ be a graph on $n$ vertices and suppose our statement holds for all graphs on less than $n>4$ vertices.

At first, let us consider the case where $G$ is constructed by a sequence that ends with an edge join. This means $G$ is built from a uniformly 3 -connected graph $G^{\prime}$ with $n=|V(G)|=\left|V\left(G^{\prime}\right)\right|+2$, because joining two edges adds two new vertices. Denoting by $t^{\prime}$ the number of edge joins that went into form$\operatorname{ing} G^{\prime}$, we have $t=t^{\prime}+1$. We conclude by induction that

$$
\begin{aligned}
n & =|V(G)|=\left|V\left(G^{\prime}\right)\right|+2 \\
& =4+2 t^{\prime}+2 b+2+p+s \\
& =4+2 t+2 b+p+s .
\end{aligned}
$$

Analogously, we can treat the cases where the sequence from which $G$ is constructed ends with a primary or secondary spoke, as in both cases we simply add one vertex, as is illustrated in Figure 13.

Finally, let $G$ be constructed by a sequence that ends by employing a bridge. This means that $G$ is built out of two uniformly 3-connected graphs $G_{1}$ and $G_{2}$ satisfying $n=|V(G)|=\left|V\left(G_{1}\right)\right|+\left|V\left(G_{2}\right)\right|-2$ as well as $t=t_{1}+t_{2}$, $b=b_{1}+b_{2}+1, p=p_{1}+p_{2}$, and $s=s_{1}+s_{2}$, where $t_{i}, b_{i}, p_{i}, s_{i}$ denote the respective numbers of edge joins, bridges, primary spokes, and secondary spokes used when constructing $G_{i}$, where $i \in\{1,2\}$. This is also illustrated in Figure 12. We conclude by induction that

$$
\begin{aligned}
n & =|V(G)|=\left|V\left(G_{1}\right)\right|+\left|V\left(G_{2}\right)\right|-2 \\
& =4+2 t_{1}+2 b_{1}+p_{1}+s_{1}+4+2 t_{2}+2 b_{2}+p_{2}+s_{2}-2 \\
& =4+2\left(t_{1}+t_{2}\right)+2\left(b_{1}+b_{2}+1\right)+\left(p_{1}+p_{2}\right)+\left(s_{1}+s_{2}\right) \\
& =4+2 t+2 b+p+s .
\end{aligned}
$$

This allows to prove a bound on the number of vertices of minimum degree in uniformly 3 -connected graphs. Along the way, we obtain additional conditions on the numbers of constructions required.

Theorem 37. A uniformly 3 -connected graph $G$ on $n$ vertices satisfies

$$
\nu(G) \geq\lceil(2 n+2) / 3\rceil .
$$

Proof. Theorem 36 establishes the equation

$$
\begin{equation*}
n=4+2 t+2 b+p+s \tag{1}
\end{equation*}
$$

By definition, a primary spoke can only be used on 3-regular graphs and it increases one of the degrees to four. Thus it can be used only once per graph going into a bridge. This implies

$$
\begin{equation*}
b+1 \geq p \Rightarrow 2 b \geq 2 p-2 \tag{2}
\end{equation*}
$$

By combining Equations (1) and (2), we obtain

$$
\begin{equation*}
n \geq 2+2 t+3 p+s \geq 2+3 p \Rightarrow p \leq\lfloor(n-2) / 3\rfloor . \tag{3}
\end{equation*}
$$

We also observe that primary spokes are the only constructions through which degrees can increase. In fact, per spoke, one degree increases by exactly one. Thus,

$$
\begin{equation*}
\nu(G) \geq n-p \geq\lceil(2 n+2) / 3\rceil, \tag{4}
\end{equation*}
$$

which is the statement to be shown.

The result we obtained is in fact best possible in terms of the number of vertices. This is easily seen for $n=4$. For $n \in \mathbb{N}, n \geq 5$, graphs of the form

$$
\left(\bigoplus_{i=1}^{\lfloor(n-5) / 3\rfloor} W_{5}\right) \oplus W_{5+(n+1 \bmod 3)}
$$

attain the bound stated in Theorem 37. Consider a graph $G$ built according to this scheme. Then its construction involves $p=\lfloor(n-5) / 3\rfloor+1$ primary spokes, required to form $W_{5}$ 's from $W_{4}$ 's. In view of Condition (4), we obtain $\nu(G) \geq n-p=n-(\lfloor(n-5) / 3\rfloor+1)=\lceil(2 n+2) / 3\rceil$. Thus $G$ indeed attains the given bound. For an illustration, an example on $n=15$ vertices, where the above formula reads $W_{5} \oplus W_{5} \oplus W_{5} \oplus W_{6}$, is displayed in Figure 17. Naturally, we can move on asking for a complete description of the extremal graphs, meaning those uniformly 3 -connected graphs for which the bound of Theorem 37 is attained. Also, we may ask for special properties this subclass possesses. As it turns out, following these questions leads to further insights into the constructions involved in forming uniformly 3 -connected graphs. The first property we focus on concerns the crossing number $\operatorname{cro}(G)$ of a graph $G$.


Figure 17: A uniformly 3-connected graph on 15 vertices with minimum number of vertices of minimum degree

This is the smallest possible number of edge crossings when drawing $G$ in a plane. To prove that the bridge construction preserves this parameter, we build on the following fact about graph embeddings, presented for example by West [104, Chapter 6].

Lemma 38. For any edge set $E$ of a face of some planar embedding of a graph $G$, there is an embedding of $G$ having $E$ as edge set of the outer face.

Proof. Projecting a drawing of a graph stereographically onto the sphere lets the edge sets of any face remain the same. By choosing the center of projection inside the face bounded by $E$ and projecting back onto the plane, we obtain an embedding of $G$ with $E$ as the edge set of the outer face.

Theorem 39. If $G=G_{1} \oplus G_{2}$ for some graphs $G_{1}$ and $G_{2}$, then

$$
\operatorname{cro}(G) \leq \operatorname{cro}\left(G_{1}\right)+\operatorname{cro}\left(G_{2}\right) .
$$

Proof. Let $G_{1}$ and $G_{2}$ be two graphs with vertices $v_{1} \in V\left(G_{1}\right)$ and $v_{2} \in V\left(G_{2}\right)$ whose neighborhoods are $N\left(v_{1}\right)=\left\{x_{1}, y_{1}, z_{1}\right\}$ and $N\left(v_{2}\right)=\left\{x_{2}, y_{2}, z_{2}\right\}$. Furthermore, let $G$ be any graph of the form

$$
G:=\left(\left(G_{1}-v_{1}\right) \cup\left(G_{2}-v_{2}\right)\right)+x_{1} x_{2}+y_{1} y_{2}+z_{1} z_{2}
$$

We begin by investigating some drawing of the graph $G_{1}$ in the plane. Such a drawing may have crossings. But replacing each existing crossing by a new vertex gives us a planarization $P$. When forming $P$, some of the edges in $\left\{x_{1} v_{1}, y_{1} v_{1}, z_{1} v_{1}\right\}$ may have to be subdivided. By $x_{1}^{\prime}$ we denote the vertex on the former edge $x_{1} v_{1}$ including $x_{1}$ but excluding $v_{1}$ that is closest to $v_{1}$. The vertices $y_{1}^{\prime}$ and $z_{1}^{\prime}$ are defined analogously. Because $\operatorname{deg}\left(v_{1}\right)=3$, two of the three edges $x_{1}^{\prime} v_{1}, y_{1}^{\prime} v_{1}$, and $z_{1}^{\prime} v_{1}$ have to be contained in the same edge set of a face of $P$. Let those two edges be denoted by $x_{1}^{\prime} v_{1}$ and $y_{1}^{\prime} v_{1}$. Lemma 38


Figure 18: The bridge construction's effect on crossings
then says that there is an embedding $P^{\prime}$ of $P$ in which $x_{1}^{\prime} v_{1}$ and $y_{1}^{\prime} v_{1}$ are contained in the edge set of the outer face. When replacing in $P^{\prime}$ those vertices we had to introduce when planarizing $G$ back to crossings, we obtain a drawing of $G_{1}$ where both edges $x_{1} v_{1}$ and $y_{1} v_{1}$ are incident to the outer face. Moreover, by reflecting the resulting embedding of $G_{1}$ across a line through $v_{1}$, we can choose the orientation of the edges $x_{1} v_{1}$ and $y_{1} v_{1}$. The exact same reasoning holds for the graph $G_{2}$. In short, there is no loss of generality in assuming that $G_{1}$ and $G_{2}$ are drawn as in Figure 18.

Because the graphs we embedded are finite, there exist radii $\varepsilon, \delta>0$ for which the discs $U_{\varepsilon}\left(v_{1}\right)=\left\{x \in \mathbb{R}^{2}:\left\|x-v_{1}\right\|_{2}<\varepsilon\right\}$ and $U_{\delta}\left(v_{2}\right)$ do not contain any $x_{1}^{\prime}, y_{1}^{\prime}, z_{1}^{\prime}, x_{2}^{\prime}, y_{2}^{\prime}$, and $z_{2}^{\prime}$. Let $x_{1}^{\prime \prime}$ denote the intersection of the edge $x_{1} v_{1}$ with the boundary of the disc $U_{\varepsilon}\left(v_{1}\right)$. Likewise, denote by $x_{2}^{\prime \prime}$ the intersection of the edge $x_{2} v_{1}$ with the boundary of $U_{\delta}\left(v_{2}\right)$. The arising points form a polygonal arc, leading from $x_{1}$ to $x_{1}^{\prime \prime}$ to $x_{2}^{\prime \prime}$ to $x_{2}$. Analogously, there are polygonal arcs connecting $y_{1}$ with $y_{2}$ and $z_{1}$ with $z_{2}$. Since we are given a drawing of $G_{1}$ and $G_{2}$ as in Figure 18, all three polygonal arcs can be drawn without additional crossings. In other words, forming $G=G_{1} \oplus G_{2}$ can be done without adding additional crossings. Equivalently, $\operatorname{cro}(G) \leq \operatorname{cro}\left(G_{1}\right)+\operatorname{cro}\left(G_{2}\right)$.

Our next goal is to study how the treewidths of input graphs behave under the bridge construction. Let us therefore recall the following concepts.

Definition 40. A pair $\left(\left\{X_{i}: i \in I\right\}, T=(I, F)\right)$ where $T$ is a tree is a tree decomposition of a graph $G$ if for each node $i \in I$ there is a bag $X_{i} \subseteq V(G)$ such that the following conditions are satisfied.
(i) Each vertex in $V(G)$ is contained in some bag. Formally, $\cup_{i \in I} X_{i}=V(G)$.
(ii) For each $v w \in E(G)$ there exists a node $i \in I$ with $v, w \in X_{i}$.
(iii) For each $v \in V(G)$ the nodes in $\left\{i \in I: v \in X_{i}\right\}$ induce a subtree of $T$.

By the width of a tree decomposition $\left(\left\{X_{i}: i \in I\right\}, T=(I, F)\right)$, we mean the number $\max _{i \in I}\left|X_{i}\right|-1$. The treewidth $\operatorname{tw}(G)$ of a graph $G$ is the minimum width taken over all tree decompositions of $G$.

Note that we address the elements of $V(G)$ as vertices and those of $I$ as nodes. We also build on the following fact, presented for example by Diestel [35, Chapter 12].

Lemma 41. For a minor $H$ of a graph $G$ there holds $\operatorname{tw}(H) \leq \operatorname{tw}(G)$.
Proof. Consider a tree decomposition $D:=\left(\left\{X_{i}: i \in I\right\}, T=(I, F)\right)$ of the graph $G$. By Definition 40, deleting any edge or vertex of $G$ leaves $D$ to be a tree decomposition for the resulting minor $H$ and hence $\operatorname{tw}(H) \leq \operatorname{tw}(G)$.

Consider now a minor $H$ that is obtained from $G$ by contracting an arbitrary edge $v w \in E(G)$. Denoting the vertex to which $v w$ is contracted by $x$, we define $X_{i}^{\prime}:=\left(X_{i} \backslash\{v, w\}\right) \cup\{x\}$ for $i \in I$ with $\{v, w\} \cap X_{i} \neq \emptyset$ and $X_{i}^{\prime}:=X_{i}$ for $i \in I$ with $\{v, w\} \cap X_{i}=\emptyset$. Then $\left(\left\{X_{i}^{\prime}: i \in I\right\}, T=(I, F)\right)$ is a tree decomposition for $H$ of width at most $\operatorname{tw}(G)$, so $\operatorname{tw}(H) \leq \operatorname{tw}(G)$.

Another important fact we make use of is the following clique containment lemma, presented by Scheffler [89].

Lemma 42. Let $\left(\left\{X_{i}: i \in I\right\}, T=(I, F)\right)$ be a tree decomposition of a graph $G$. Then for each clique $W \subseteq V(G)$ there is a node $i \in I$ with $W \subseteq X_{i}$.

Proof. For cliques containing one or two vertices our claim holds by Conditions (i) and (ii) of Definition 40. We proceed by induction on the number of vertices contained in the respective cliques. So consider a clique $W \subseteq V(G)$ with $|W|=: k \geq 3$ and let $v \in W$. By induction, for $W^{\prime}:=W \backslash\{v\}$ there is a node $j \in I$ such that $W^{\prime} \subseteq X_{j}$. If $v \in X_{j}$, then there is nothing left to show. So suppose $v \notin X_{j}$. By Condition (iii) of Definition 40, the subgraph $T_{v}$ of $T$ that contains those nodes $i \in I$ where $v \in X_{i}$ forms a subtree of $T$. So there is a unique $\{j\}-V\left(T_{v}\right)$ path $P$ in $T$. Denoting by $\ell$ the endvertex of $P$ that lies in $V\left(T_{v}\right)$, we observe that $W \subseteq X_{\ell}$. This is because Condition (ii) of Definition 40 requires that all edges of $G[W]$ have to be covered by some bag and Condition (iii) of Definition 40 requires that that for each $w \in W$ the nodes $\left\{i \in I: w \in X_{i}\right\}$ induce a subtree of $T$. So $W^{\prime} \subseteq X_{i}$ for each $i \in V(P)$ and $W \subseteq X_{\ell}$, as claimed.

In what follows, we show that the bridge construction preserves the treewidth of the input graphs only under a certain condition. To this end, for a graph $G$, we say a vertex $v \in V(G)$ of $\operatorname{deg}(v)=3$ is safe if $G$ admits a tree decomposition of $\operatorname{tw}(G)$ in which there is a bag containing $v$ together with two of


Figure 19: Attaching a wheel graph at an unsafe vertex
its neighbors. With Lemma 42 at hand, we may rephrase this as follows. A vertex of degree three is safe if two of its neighbors are joined by an edge or if two of its neighbors can be joined by an edge without increasing the treewidth. If a vertex of degree three is not safe, we call it unsafe. Suppose the vertex $v$, with neighborhood $N(v)=\{x, y, z\}$, in Figure 19 is unsafe in the indicated graph $G$. If we now take the bridge construction to join $G$ at $v$ with a wheel graph on four vertices, then the resulting graph has $G+x y$ as minor. In Figure 19, this can be checked by contracting the vertex pairs shaded in gray. Consequently, the bridge construction can increase the treewidth. The following theorem justifies the notion of a safe vertex.

Theorem 43. Let $G_{1}$ and $G_{2}$ be two graphs with vertices $v_{1} \in V\left(G_{1}\right)$ and $v_{2} \in V\left(G_{2}\right)$ with $N\left(v_{1}\right)=\left\{x_{1}, y_{1}, z_{1}\right\}$ and $N\left(v_{2}\right)=\left\{x_{2}, y_{2}, z_{2}\right\}$. Furthermore, take any graph $G$ of the form

$$
G:=\left(\left(G_{1}-v_{1}\right) \cup\left(G_{2}-v_{2}\right)\right)+x_{1} x_{2}+y_{1} y_{2}+z_{1} z_{2} .
$$

If $\max \left\{\operatorname{tw}\left(G_{1}\right), \operatorname{tw}\left(G_{2}\right)\right\} \geq 3$ and $v_{1}$ as well as $v_{2}$ are safe vertices, then

$$
\operatorname{tw}(G)=\max \left\{\operatorname{tw}\left(G_{1}\right), \operatorname{tw}\left(G_{2}\right)\right\} .
$$

Proof. To verify $\operatorname{tw}(G) \geq \max \left\{\operatorname{tw}\left(G_{1}\right), \operatorname{tw}\left(G_{2}\right)\right\}$, it is enough to check that both $G_{1}$ and $G_{2}$ are minors of $G$, because then Lemma 41 applies. Contracting the vertices of $G$ that originate from $G_{2}$ to a single vertex gives $G_{1}$ and the same argument can be made for $G_{2}$.

For the converse inequality, let $\left(\left\{X_{i}: i \in I_{1}\right\}, T_{1}=\left(I_{1}, F_{1}\right)\right)$ be a tree decomposition of minimum width of $G_{1}$ and $\left(\left\{Y_{j}: j \in I_{2}\right\}, T_{2}=\left(I_{2}, F_{2}\right)\right)$ be a tree decomposition of minimum width of $G_{2}$. Because $v_{1}$ and $v_{2}$ are safe, we can further assume that there is a node $s \in I_{1}$ whose corresponding bag $X_{s}$ contains $v_{1}$ and two of its neighbors. We may relabel them as $x_{1}$ and $y_{1}$ if necessary. Likewise, we can assume that there is a node $t \in I_{2}$ whose


Figure 20: Joining tree decompositions at bags of safe vertices if $|F|=1$
corresponding bag $Y_{t}$ contains $v_{2}$ and two of its neighbors. To verify the inequality $\operatorname{tw}(G) \leq \max \left\{\operatorname{tw}\left(G_{1}\right), \operatorname{tw}\left(G_{2}\right)\right\}$, we provide a tree decomposition for $G$ whose width does not exceed $\max \left\{\operatorname{tw}\left(G_{1}\right), \operatorname{tw}\left(G_{2}\right)\right\}$. Whereas we were free to label the neighbors of $v_{1}$ in bag $X_{s}$ by $x_{1}$ and $y_{1}$, we have to distinguish two cases according to how the vertices in $X_{s}$ and $Y_{t}$ are joined by edges in $G$. For the edge set $F:=E\left(G\left[X_{s} \cup Y_{t}\right]\right) \cap\left\{x_{1} x_{2}, y_{1} y_{2}, z_{1} z_{2}\right\}$, there holds either $|F|=1$ or $|F| \geq 2$.

First, we focus on the case $|F|=1$, where we address the neighbors of $v_{2}$ in $G_{2}$ that are contained in $Y_{t}$ by $x_{2}$ and $z_{2}$. Recalling that $v_{1}, v_{2} \notin V(G)$, we can safely replace these vertices, when defining

$$
\begin{array}{rll}
X_{i}^{\prime} & :=X_{i} \backslash\left\{v_{1}\right\} \cup\left\{z_{2}\right\} & \text { for each } i \in I_{1} \text { with } v_{1} \in X_{i}, \\
X_{i}^{\prime} & :=X_{i} & \text { for each } i \in I_{1} \text { with } v_{1} \notin X_{i}, \\
Y_{j}^{\prime} & :=Y_{j} \backslash\left\{v_{2}\right\} \cup\left\{y_{1}\right\} & \text { for each } j \in I_{2} \text { with } v_{2} \in Y_{j}, \\
Y_{j}^{\prime} & :=Y_{j} & \text { for each } j \in I_{2} \text { with } v_{2} \notin Y_{j} .
\end{array}
$$

With this redefinition, we have not changed any bag's cardinality. Taking a new node $v \notin I_{1} \cup I_{2}$, let us define the bag $X_{v}:=\left\{x_{1}, x_{2}, y_{1}, z_{2}\right\}$ as well as the tree $T:=T_{1} \uplus T_{2}+v+s v+v t$. Because $\left|X_{v}\right|=4$, we conclude that

$$
\max \left\{\max _{i \in I_{1}}\left|X_{i}\right|, \max _{j \in I_{2}}\left|Y_{j}\right|\right\}=\max \left\{\max _{i \in I_{1}}\left|X_{i}^{\prime}\right|, \max _{j \in I_{2}}\left|Y_{j}^{\prime}\right|,\left|X_{v}\right|\right\} .
$$

Herein, we used our assumption that $\max \left\{\operatorname{tw}\left(G_{1}\right), \operatorname{tw}\left(G_{2}\right)\right\} \geq 3$. What remains to be shown is that $D:=\left(\left\{X_{i}^{\prime}: i \in I_{1}\right\} \cup\left\{Y_{j}^{\prime}: j \in I_{2}\right\} \cup\left\{X_{v}\right\}, T\right)$ is a tree decomposition of the graph $G$. Certainly, $D$ satisfies Condition (i) of Definition 40, because the only vertices we deleted when defining the bags of $D$ were $v_{1}$ and $v_{2}$, which are not contained in $V(G)$. This is also the reason why for each edge $v w \in E\left(G_{1}\right) \cup E\left(G_{2}\right)$ there exists a bag in $D$ containing $v$ and $w$. Moreover, Condition (ii) of Definition 40 ensures that there is some $k \in I_{1}$ with $v_{1}, z_{1} \in X_{k}$. This implies that $z_{1}, z_{2} \in X_{k}^{\prime}$. Analogously,


Figure 21: Joining tree decompositions at bags of safe vertices if $|F|=2$
there exists an $\ell \in I_{2}$ with $y_{1}, y_{2} \in Y_{\ell}^{\prime}$. Because the endvertices of $x_{1} x_{2}$ are both contained in $X_{v}$, Condition (ii) of Definition 40 holds. Condition (iii) of Definition 40 certainly holds for vertices in $V(G) \backslash X_{v}$. This is because $T$, by construction, is a tree that contains $T_{1}$ and $T_{2}$ as subtrees. Furthermore, the only vertices that were deleted when forming $D$ were $v_{1}$ and $v_{2}$, which are not present in $G$, and the only vertices that were included in some bag of $D$ were those of $X_{v}$. This is also illustrated by Figure 20. We included $z_{2}$ in each bag $X_{i}$ that contained $v_{1}$, represented by $z_{2}$ inside a gray box with subscript $v_{1}$ in Figure 20. Consequently, $\left\{i \in I_{1}: z_{2} \in X_{i}^{\prime}\right\}$ induces a subtree of $T_{1}$. Because $\left\{j \in I_{2}: z_{2} \in Y_{j}^{\prime}\right\}=\left\{j \in I_{2}: z_{2} \in Y_{j}\right\}$ induces a subtree of $T_{2}$ and the fact that $z_{2} \in X_{v}$, we see that the set of nodes whose bags contain $z_{2}$ induces a subtree of $T$. With Figure 20 at hand, it is easy to argue analogously for the remaining vertices in $X_{v}$.

Let us now turn to the case $|F| \geq 2$, in which we address the neighbors of $v_{2}$ in $G_{2}$ that are contained in $Y_{t}$ by $x_{2}$ and $y_{2}$. We set

$$
\begin{array}{ll}
X_{i}^{\prime}:=X_{i} \backslash\left\{v_{1}\right\} \cup\left\{z_{1}\right\} & \text { for each } i \in I_{1} \text { with } v_{1} \in X_{i}, \\
X_{i}^{\prime}:=X_{i} & \text { for each } i \in I_{1} \text { with } v_{1} \notin X_{i}, \\
Y_{j}^{\prime}:=Y_{j} \backslash\left\{v_{2}\right\} \cup\left\{z_{1}\right\} & \text { for each } j \in I_{2} \text { with } v_{2} \in Y_{j}, \\
Y_{j}^{\prime}:=Y_{j} & \text { for each } j \in I_{2} \text { with } v_{2} \notin Y_{j} .
\end{array}
$$

Taking two new nodes $v, w \notin I_{1} \cup I_{2}$, we define the bags $X_{v}:=\left\{x_{1}, y_{1}, y_{2}, z_{1}\right\}$, $X_{w}:=\left\{x_{1}, x_{2}, y_{2}, z_{1}\right\}$ and the tree $T:=T_{1} \uplus T_{2}+v+w+s v+v w+w t$. Investigating Figure 21, we find that this defines a tree decomposition of $G$ whose width does not exceed $\max \left\{\operatorname{tw}\left(G_{1}\right), \operatorname{tw}\left(G_{2}\right)\right\}$.

Note that although Theorem 43 shows cases in which the bridge construction preserves the treewidth of the input graphs, in what follows, we face situations where the respective vertices $v_{1}$ and $v_{2}$ at which we wish to join two graphs are not safe. Moreover, general uniformly 3-connected graphs can be seen to have large treewidth. In fact, this is already the case for 3 -regular 3 -connected


Figure 22: Forming uniformly 3-connected graphs of arbitrary treewidth
graphs. The example in Figure 22, taken from Meeks [80], illustrates that for any $k \in \mathbb{N}$ there are 3-regular 3 -connected graphs containing a $k \times k$ grid as minor. This is a graph on vertex set $\{1, \ldots, k\} \times\{1, \ldots, k\}$ and edge set $\left\{\left(x_{1}, y_{1}\right)\left(x_{2}, y_{2}\right):\left|x_{1}-x_{2}\right|+\left|y_{1}-y_{2}\right|=1\right\}$. Such a graph has treewidth $k$, as is shown by Seymour and Thomas [92] or Bellenbaum and Diestel [6]. Together with Lemma 41, this implies that there are 3-regular 3-connected graphs of treewidth $k$ for $k \geq 3$. Having worked out some facts about our bridge construction, we now come back to our question about how extremal uniformly 3 -connected graphs look like. Let us have a look at the following example to recall Conditions (1) to (4), which we obtained as a byproduct of the proof of Theorem 37 .

Example 44. Consider an extremal uniformly 3-connected graph on $n=10$ vertices. According to Condition (4), when constructing such a graph, we have to use as many primary spokes as possible. In view of Condition (3), this means $p=2$. Condition (1) then reads $4=2 t+2 b+s$ and Condition (2) requires $b \geq 1$. So we obtain exactly three options, in which $p=2$ and

$$
t=1, b=1, s=0 \quad \text { or } \quad t=0, b=2, s=0 \quad \text { or } \quad t=0, b=1, s=2 .
$$

A graph for the setting $t=1, b=1, p=2, s=0$ is illustrated in Figure 23.


Figure 23: An extremal uniformly 3-connected graph on ten vertices

Our goal for the remainder of this section is to generalize the findings from this example and so to obtain a precise picture of extremal uniformly 3-connected graphs and some of their properties.

Lemma 45. Consider an extremal uniformly 3-connected graph containing $n=3 k+\ell \geq 5$ vertices, for some $k \in \mathbb{N} \backslash\{1\}$ and $\ell \in\{-1,0,1\}$, formed by a number of $t$ edge joins, $b$ bridge constructions, $p$ primary and $s$ secondary spokes. Then these quantities are related as follows.
(i) It holds that $p=k-1$.
(ii) If $\ell=-1$, then $b=k-2, t=0, s=0$.
(iii) If $\ell=0$, then $b=k-2, t=0, s=1$.
(iv) If $\ell=1$, then $b=k-1, t=0, s=0$
or $b=k-2, t=1, s=0$
or $b=k-2, t=0, s=2$.
Proof. From (3) and (4), we know that forming an extremal graph involves

$$
p=\lfloor(n-2) / 3\rfloor=\lfloor(3 k+\ell-2) / 3\rfloor=k+\lfloor(\ell-2) / 3\rfloor=k-1
$$

primary spoke constructions, which verifies Claim (i). Condition (2) requires that $b \geq p-1=k-2$ and so by Condition (1) it follows that

$$
\begin{aligned}
n & =4+2 t+2 b+p+s \\
\Rightarrow 3 k+\ell & \geq 4+2 t+2(k-2)+k-1+s \\
\Rightarrow \quad 1+\ell & \geq 2 t+s .
\end{aligned}
$$

Also note that $b \leq k-1$, as otherwise the right hand side of Equation (1) exceeds its left hand side. So $b \in\{k-2, k-1\}$. If $\ell=-1$, we see from above that $b=k-2, t=s=0$, which verifies Claim (ii). If $\ell=0$, it follows $b=k-2, t=0, s=1$, which proves Claim (iii). It remains the case where $\ell=1$. If $b=k-2$, we obtain $t=0$ and $s=2$ or $t=1$ and $s=0$, which are the last two alternatives in Claim (iv). If $b=k-1$, then Condition (1) requires $t=s=0$, which is the remaining alternative in Claim (iv).

Example 46. Consider an extremal graph in whose construction an edge join is involved. Recall that in Tutte's characterization [103], and so in Theorem 35, edge joins are only allowed to be used on 3-regular 3-connected graphs. From Condition (i) of Lemma 45, we see that the construction of any extremal graph on at least five vertices has to involve a primary spoke. So all extremal graphs except $W_{4}$ are nonregular. Consequently, when forming extremal graphs, edge joins take only $W_{4}$ as input. Up to graph isomorphism,


Figure 24: Small extremal uniformly 3 -connected graphs
the only possible outcomes of an edge join on $W_{4}$ are the complete bipartite graph $K_{3,3}$ and the envelope graph, illustrated in the middle of Figure 24. Those graphs, by a primary spoke construction, can give rise to the graphs on the right in Figure 24. The dashed green edges drawn in the bottom right graph are to be understood as alternatives, indicating the three nonisomorphic graphs that can be obtained from the envelope graph. The alternative where edge $f$ is added to the envelope graph is isomorphic to the graph in the top right corner of Figure 24. This can be checked by investigating the gray vertex labels. Furthermore, the alternative where edge $e$ is added to the envelope graph is isomorphic to the graph resulting from $W_{4} \oplus W_{5}$. Similarly, the envelope graph can be obtained via $W_{4} \oplus W_{4}$, which is why nonplanar extremal graphs can arise even without using edge joins.

Combining what we have learned about extremal uniformly 3-connected graphs with our knowledge on how crossing numbers behave under the bridge construction, we come to the following conclusion.

Theorem 47. An extremal graph $G$ on $n=3 k+\ell \geq 4$ vertices, with $k \in \mathbb{N}$ and $\ell \in\{-1,0,1\}$ satisfies $\operatorname{cro}(G) \leq 1$ and $G$ is planar if $n=4$ or $\ell \in\{-1,0\}$.

Proof. For $n=4$, the only uniformly 3-connected graph is the complete graph on four vertices, which is extremal and planar. So let $G$ be an extremal graph on $n=3 k+\ell \geq 5$ vertices, for some $k \in \mathbb{N} \backslash\{1\}$.

If $\ell \in\{-1,0\}$, then Conditions (i) to (iii) of Lemma 45 say that $G$ can be formed by $k-1$ primary spoke and $k-2$ bridge constructions. If $\ell=0$, an additional secondary spoke has to be used. Otherwise, no secondary
spoke is involved. This means that $G$ can be formed by using the bridge construction recursively to combine wheels $W_{5}$, and one $W_{6}$, in any order, in case $\ell=0$. By Theorem 39, we conclude that $G$ is planar.

Let now $\ell=1$. From (i) and (iv) of Lemma 45, we know that forming $G$ requires $k-1$ primary spoke constructions. If $b=k-1$, then $t=s=0$. This means that $G$ can be obtained by recursively using the bridge construction to combine $k-1$ wheels $W_{5}$ with one $W_{4}$ or, as we have seen in Example 46, by combining $k-2$ wheels $W_{5}$ with one of the graphs in the bottom right corner of Figure 24. This implies $\operatorname{cro}(G) \leq 1$, by Theorem 39.

Finally, let $\ell=1$ and $b=k-2$. If $t=1$, then $s=0$ and $G$ is obtained by using the bridge construction recursively to combine wheels $W_{5}$ with one of the graphs on the right of Figure 24. If $t=0$, then $s=2$ and $G$ can be obtained by using the bridge construction recursively to combine wheels $W_{5}$ with two $W_{6}$ or one $W_{7}$. In all cases, Theorem 39 implies $\operatorname{cro}(G) \leq 1$.

One may also notice a certain similarity between uniformly 3 -connected graphs and Halin graphs, of which Brandstadt, Le, and Spinrad [12] give an overview. They are defined as those graphs that can be obtained by embedding a tree in the plane that has no vertices of degree two and connecting its leafs by a cycle without crossing any of the tree's edges. We have already met such a graph in Figure 17. Halin graphs are uniformly 3 -connected, because any two of their vertices are connected by exactly three independent paths, one along the inner tree and the other two along the enclosing cycle. To check whether the converse inclusion holds, we may take nonplanar uniformly 3 -connected graphs as counterexamples. But even planar extremal ones may not be Halin graphs. For $\ell=-1$, an example is given by Figure 25. Furthermore, from what we have seen in the previous proof, the extremal graphs include those Halin graphs whose inner vertices are all of degree four. In addition, if $\ell=0$, we can have one further inner vertex of degree five. If $\ell=1$, we may have two additional vertices of degree five or one of degree six.

Halin graphs are classical examples for graphs of low treewidth and so for a graph class on which many hard combinatorial problems become tractable. For example, Cornuéjols, Naddef, and Pulleyblank [28] discuss this in the context of the travelling salesperson problem. Since extremal uniformly 3-connected graphs show a certain similarity to Halin graphs, we may ask if they also possess a certain tree-like structure. Bodlaender [9] establishes that the treewidth of Halin graphs is bounded by three. On the other hand, Figure 22 illustrates that the treewidth of general uniformly 3 -connected


Figure 25: An extremal uniformly 3-connected graph that is not a Halin graph
graphs is unbounded. But since we observed in Example 46 and the proof of Theorem 39 that the extremal graphs are built by successively joining a relatively small set of building blocks by the bridge construction, this raises the question of whether a small treewidth bound can be shown. However, Theorem 43 only ensures that the bridge construction preserves treewidths if the input graphs are joined at safe vertices. Figure 26 demonstrates that unsafe vertices may appear in an extremal graph. The example shown there is the graph from the bottom right corner of Figure 24, obtained when edge $f$ is inserted. In Figure 26, the vertex $v$ has an independent neighborhood and joining any of its neighbors gives rise to a $K_{5}$ minor. The three possibilities are indicated by dashed green lines. The respective $K_{5}$ minor can be obtained by contracting the vertex pairs shaded in gray. The graph without any of the dashed green edges has treewidth three. To see this, delete in the left example the vertices highlighted by gray circles. What remains is a tree, for which we easily find a tree decomposition of width one. Putting the highlighted vertices in all bags, gives rise to a tree decomposition of width three. Furthermore, attaching a wheel graph on four vertices at $v$ would in fact increase the treewidth from three to four. This is what we discussed in the context of Figure 19. But note that Figure 26 shows the only example for a graph containing an unsafe vertex that we identified so far. Indeed, in the remainder of this section, we verify that the treewidth of extremal graphs


Figure 26: An extremal uniformly 3-connected graph containing an unsafe vertex $v$
is bounded. For this purpose, let us recall the notion of a line graph of a graph $G$. This is the graph $L(G)$ on vertex set $E(G)$ whose vertices are adjacent exactly when they are incident in $G$. Harvey and Wood [63] investigate treewidths of line graphs, for which they present the following fact.

Lemma 48. Any graph $G$ satisfies

$$
\operatorname{tw}(G) \leq 2 \operatorname{tw}(L(G))+1
$$

Proof. If we replace each edge by both its endpoints in a tree decomposition of $L(G)$, then we obtain a tree decomposition of $G$. Recalling that a tree decomposition's width is the largest bag size minus one, this implies

$$
\operatorname{tw}(G) \leq 2(\operatorname{tw}(L(G))+1)-1=2 \operatorname{tw}(L(G))+1 .
$$

Bodlaender, Van Leeuwen, Tan, and Thilikos [11] show another useful fact about a clique sum of two graphs $G_{1}$ and $G_{2}$. For two cliques $S \subseteq V\left(G_{1}\right)$ and $T \subseteq V\left(G_{2}\right)$ with $|S|=|T|$, this is a graph that arises by forming $G_{1} \cup G_{2}$ and then identifying $S$ and $T$.

Lemma 49. For graphs $G_{1}$ and $G_{2}$ with cliques $S \subseteq V\left(G_{1}\right)$ and $T \subseteq V\left(G_{2}\right)$ such that $|S|=|T|$, a clique sum $G$ of $G_{1} \cup G_{2}$ obtained by identifying $S$ and $T$ satisfies

$$
\operatorname{tw}(G)=\max \left\{\operatorname{tw}\left(G_{1}\right), \operatorname{tw}\left(G_{2}\right)\right\} .
$$

Proof. Certainly, the graphs $G_{1}$ and $G_{2}$ are both subgraphs of $G$. This implies $\operatorname{tw}(G) \geq \max \left\{\operatorname{tw}\left(G_{1}\right), \operatorname{tw}\left(G_{2}\right)\right\}$, by Lemma 41 .

For the converse relation, consider two minimum width tree decompositions $\left(\left\{X_{i}: i \in I_{1}\right\}, T_{1}=\left(I_{1}, F_{1}\right)\right)$ and $\left(\left\{Y_{j}: j \in I_{2}\right\}, T_{2}=\left(I_{2}, F_{2}\right)\right)$ of $G_{1}$ and $G_{2}$, respectively. By Lemma 42, there are nodes $s \in I_{1}$ and $t \in I_{2}$ with $S \subseteq X_{s}$ and $T \subseteq Y_{t}$. So $\left(\left\{X_{i}: i \in I_{1}\right\} \cup\left\{Y_{j}: j \in I_{2}\right\}, T_{1} \cup T_{2}+s t\right)$ defines a tree decomposition of $G$ whose width does not exceed $\max \left\{\operatorname{tw}\left(G_{1}\right), \operatorname{tw}\left(G_{2}\right)\right\}$.

Lemma 50. Let $G_{1}$ and $G_{2}$ be two graphs with vertices $v_{1} \in V\left(G_{1}\right)$ and $v_{2} \in V\left(G_{2}\right)$ with $N\left(v_{1}\right)=\left\{x_{1}, y_{1}, z_{1}\right\}$ and $N\left(v_{2}\right)=\left\{x_{2}, y_{2}, z_{2}\right\}$. Furthermore, take any graph $G$ of the form

$$
G:=\left(\left(G_{1}-v_{1}\right) \cup\left(G_{2}-v_{2}\right)\right)+x_{1} x_{2}+y_{1} y_{2}+z_{1} z_{2} .
$$

Let further $H$ a be a clique sum of $L\left(G_{1}\right)$ and $L\left(G_{2}\right)$ obtained by identifying $v_{1} x_{1}$ with $v_{2} x_{2}, v_{1} y_{1}$ with $v_{2} y_{2}$, and $v_{1} z_{1}$ with $v_{2} z_{2}$. Then $L(G)$ is a proper subgraph of $H$.


Figure 27: The bridge construction's effect on the respective line graphs

Proof. Figure 27 shows the bridge construction and, alongside, how it acts on the corresponding line graphs. We observe that joining $G_{1}$ and $G_{2}$ by adding the edges $x_{1} x_{2}, y_{1} y_{2}$, and $z_{1} z_{2}$ translates to identifying $v_{1} x_{1}$ with $v_{2} x_{2}, v_{1} y_{1}$ with $v_{2} y_{2}$, and $v_{1} z_{1}$ with $v_{2} z_{2}$ in the corresponding line graphs. This is highlighted by the dashed green lines in Figure 27. Removing $v_{1}$ in $G_{1}$ and $v_{2}$ in $G_{2}$ has the effect of removing the edges $\left\{v_{1} x_{1}, v_{1} y_{1}\right\},\left\{v_{1} x_{1}, v_{1} z_{1}\right\}$, and $\left\{v_{1} y_{1}, v_{1} z_{1}\right\}$ in $L\left(G_{1}\right)$ and $\left\{v_{2} x_{2}, v_{2} y_{2}\right\},\left\{v_{2} x_{2}, v_{2} z_{2}\right\}$, and $\left\{v_{2} y_{2}, v_{2} z_{2}\right\}$ in $L\left(G_{2}\right)$. So $H$ can be obtained by building the clique sum of $L\left(G_{1}\right)$ and $L\left(G_{2}\right)$ and then removing all edges of the subgraph induced by the clique at which the clique sum is formed.

Theorem 51. Consider a class of graphs $\mathcal{C}$ that includes a base class containing only graphs whose line graph's treewidth is bounded by $w$. If all other graphs in $\mathcal{C}$ can be obtained by employing the bridge construction iteratively, then any $G \in \mathcal{C}$ satisfies

$$
\operatorname{tw}(G) \leq 2 w+1
$$

Proof. This is a consequence of Lemmas 48, 49, and 50.
Corollary 52. Any extremal graph $G$ satisfies $\operatorname{tw}(G) \leq 13$.
Proof. By Example 46 and our proof of Theorem 47, we find that the extremal graphs belong to a class that is generated by successively using the bridge construction to join wheels on at most six vertices and the graphs

$W_{6}$

$L\left(W_{6}\right)$

$L\left(W_{6}\right)$

Figure 28: A wheel graph, its line graph, and a bramble certifying that the line graph's treewidth is bounded by six
illustrated in Figure 24. In view of Theorem 51, our claim follows if we manage to prove that those graph's line graphs have treewidth at most six. At first, investigate Figure 28. Deleting the vertices highlighted by gray circles in the middle graph, we are left with a tree for which there is a tree decomposition of width one. Putting the highlighted vertices in all bags, yields a tree decomposition of width six. Line graphs of smaller wheels are minors of the example we discussed. Whithin Figure 24, we essentially have to check those graphs depicted in Figure 29. To see this, recall that the graph in the bottom right corner of Figure 24 where edge $e$ is included can be obtained from $W_{5} \oplus W_{4}$. Also recall that the graph in the top right corner is isomorphic to the graph in the bottom right corner where edge $f$ is included. All remaining graphs of Figure 24 are minors of those in Figure 29. As before, for the depicted line graphs, the vertices highlighted by gray circles indicate how to obtain a tree decomposition of width five.

Note that the bound $\operatorname{tw}\left(L\left(W_{6}\right)\right) \leq 6$ is indeed best possible. To check this, let us recall the notion of a bramble of a graph $G$, given by Seymour and Thomas [92]. This is a set of connected mutually touching subgraphs of $G$. Hereby, we say two subgraphs $G_{1}$ and $G_{2}$ touch each other if they have a common vertex or if there is an edge $v w \in E(G)$ with $v \in V\left(G_{1}\right)$ and $w \in V\left(G_{2}\right)$. The order of a bramble is the smallest size of a hitting set. This is a set of vertices of $G$ having a nonempty intersection with each of the bramble's subgraphs. The bramble illustrated on the right in Figure 28 has five subgraphs containing a single vertex, for each of the inner vertices, and five subgraphs containing all but one outer vertex. Certainly, those subgraphs can only be hit by at least seven vertices, which certifies the claimed treewidth bound because a graph has treewidth at least $k-1$ if and only if it has a bramble


Figure 29: Extremal graphs and corresponding line graphs
of order $k$. The latter result is known as treewidth duality theorem, which is shown by Seymour and Thomas [92]. Statements as those in Theorem 43, Theorem 51, or Corollary 52 have their value because a lot of computationally hard combinatorial problems on graphs become efficiently solvable by dynamic programming methods if the treewidth of the given graph is bounded. Bodlaender and Koster [10] provide an overview of the relevant algorithmic techniques. In fact, our treewidth results imply that it is possible to find an optimal coloring for an extremal graph in $\mathcal{O}(n)$ time. This is a consequence of the work of Arnborg and Proskurowski [3] for general graphs of bounded treewidth. Furthermore, uniformly 3-connected graphs, except wheels on an even number of vertices, are indeed 3 -colorable, which is shown by Aboulker, Brettell, Havet, Marx, and Trotignon [2].
A problem that remains open is to determine the best upper bound $C$ such that $\operatorname{tw}(G) \leq C$ holds for any extremal uniformly 3-connected graph $G$. By Figures 26 and 19, we know that $C \geq 4$. By Corollary 52, we know that $C \leq 13$.


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In this chapter, we ask which values can possibly occur as edge weights of Gomory-Hu trees and we are interested in how to construct graphs for which we prescribe such weights. While basic facts about Gomory-Hu trees can be found in Section 2.3, we introduce the concept of a graph's cut sequence in Section 4.1. There, we verify that the cut sequence of a graph is determined uniquely. In Section 4.2, we review basic facts about maximally local-edge-connected graphs and discuss why this graph class is of interest when investigating cut sequences. Revisiting the classical characterization of degree sequences by Erdős and Gallai [40] at the beginning of Section 4.3, we work towards a similar criterion for sequences of integer numbers to be the cut sequence of a graph. Unlike for degrees, the sum of cut values does not have to be even. However, the latter satisfy a related parity condition, which we discuss in Section 4.3. Building on the constructive characterization of graphic sequences by Tripathi, Venugopalan, and West [101], we demonstrate how to form graphs with prescribed cut sequence if they satisfy a shifted variant of the classical Erdős-Gallai inequalities in Section 4.4.

### 4.1 Cut sequences

The following definition is the central concept that we study in this chapter.
Definition 53. A finite sequence of nonnegative numbers $c_{1} \geq \ldots \geq c_{n-1}$ is called cut sequence if there is a Gomory-Hu tree for some graph on $n$ vertices that has the multiset of numbers $c_{1}, \ldots, c_{n-1}$ as edge weights.

We refer to the numbers in a graph's cut sequence as cut values. The word multiset in this definition means that we have to regard the numbers $c_{1} \geq \ldots \geq c_{n-1}$ with their exact multiplicities. Note also that we consider unweighted graphs without loops or multiple edges here. Many of the investigations about Gomory-Hu trees straightforwardly generalize to the case of weighted graphs. However, the questions that we discuss in this chapter are only meaningful in the case of unweighted graphs. This is because each finite sequence of $n-1$ numbers can be the edge weights of a Gomory-Hu tree of a weighted graph. We may just take some tree with $n-1$ edges and write exactly the prescribed weights on its edges. In contrast, all attempts to find an unweighted graph that has a Gomory-Hu tree with weights $2,1,1$ are in vain. Natural constraints for the values in a cut sequence are $n-1 \geq c_{1}$ and $c_{n-1} \geq 0$. If there are some cut values equal to zero, this just means that the graph in question has several components, which can be considered independently. So formulating some statements in this chapter for the case $c_{n-1} \geq 1$ is not a loss of generality, but may help to simplify notation.

Figure 30 shows a graph on the left side and two corresponding GomoryHu trees in the middle and on the right side. The figure illustrates that a Gomory-Hu tree is not necessarily a subgraph of the graph from which it originates. We also see that a graph's Gomory-Hu tree is in general not unique. Yet this does not carry over to cut sequences. Remarkably, they are uniquely determined for a given graph. To prove this fact is the goal of the remainder of this section. For this purpose, we first recall the cycle property of maximum spanning trees, presented by Jungnickel [69], for example.


Figure 30: A graph and two corresponding Gomory-Hu trees

Theorem 54. Consider a connected graph $G$ with edge weights $c \in \mathbb{R}^{E(G)}$. A spanning tree $T$ has maximum weight if and only if for all $e \in E(G) \backslash E(T)$ it holds that

$$
c_{e} \leq c_{f} \quad \text { for all } f \in C_{T}(e)
$$

where $C_{T}(e)$ denotes the unique cycle in $T+e$.
Proof. Let $T$ be a spanning tree of maximum weight and suppose for a contradiction that there is an edge $e \in E(G) \backslash E(T)$ such that $c_{e}>c_{f}$ for some $f \in C_{T}(e)$. Then $T-e+f$ is a spanning tree whose weight is larger than the weight of $T$, against our assumption.

For the reverse statement, let us write $E:=E(T)$ and choose among all spanning trees of maximum weight the tree $T^{\prime}=\left(V(G), E^{\prime}\right)$ for which $\left|E^{\prime} \backslash E\right|$ is smallest possible. If $\left|E^{\prime} \backslash E\right|=0$, then $E=E^{\prime}$ and $T$ is indeed a spanning tree of maximum weight of $G$. We shall argue that the remaining case $\left|E^{\prime} \backslash E\right|>0$ contradicts the extremal choice of $T^{\prime}$. For this purpose, take an edge $e=v w \in E^{\prime} \backslash E$. In $T^{\prime}-e$ we find two components whose vertex sets shall be denoted by $V$ and $W$ such that $v \in V$ and $w \in W$. We observe that $C_{T}(e) \backslash\{e\}$ forms a path in $T$ with endpoints $v$ and $w$. This path contains an edge $f \neq e$ joining a vertex from $V$ with a vertex from $W$. As we are given that $c_{e} \leq c_{f}$, we find that the graph $\left(V(G), E^{\prime \prime}\right)$ where $E^{\prime \prime}:=E^{\prime} \backslash\{e\} \cup\{f\}$ is again a spanning tree of maximum weight. However,

$$
\left|E^{\prime \prime} \backslash E^{\prime}\right|=\left|E^{\prime} \backslash E\right|-1,
$$

which contradicts the choice of $T$ and concludes the proof.
Theorem 54 can be used to prove the following observation by Gomory and Hu [50], which connects Gomory-Hu trees with certain maximum spanning trees. This link in turn shall be useful to rephrase our question about the uniqueness of a graph's cut sequence as a question about the uniqueness of a maximum spanning tree's multiset of edge weights.

Theorem 55. Consider a graph $G$ and a weighted complete graph $K$ on the same vertex set $V(G)$ in which each edge $v w$ is assigned the minimum capacity of a $v-w$ cut $c_{v w}$ in $G$. Then each Gomory-Hu tree of $G$ is a spanning tree of maximum weight of $K$.

Proof. A Gomory-Hu tree $T$ of a graph $G$ is defined on the vertex set $V(G)$. So it is certainly a spanning tree of $K$. It remains to be shown that $T$ is of maximum weight. So let us take an arbitrary edge $e=v w \in E(G) \backslash E(T)$ and verify the condition of Theorem 54. For each $f \in C_{T}(e) \backslash\{e\}$ we obtain in $T-f$ two components whose vertex sets shall be denoted by $V$ and $W$
such that $v \in V$ and $w \in W$. By the definition of a Gomory-Hu tree, we have $c_{f}=|E(V, W)|$ and because $c_{e}=c_{v w}$ is the capacity of a minimum $v$ - $w$ cut, we obtain that

$$
c_{e}=c_{v w} \leq|E(V, W)|=c_{f},
$$

which was to be shown.
Theorem 55 says that our question of whether the cut sequence of a graph is uniquely determined rests on proving that all maximum spanning trees of a graph have exactly the same multiset of edge weights. Another consequence of Theorem 54 is that it implies the correctness of Algorithm 1. This is a variant of Kruskal's [75] algorithm, which is a greedy algorithm, originally formulated for the minimum spanning tree problem, whose principle idea can be adapted for the generation of maximum spanning trees as well.

```
Algorithm 1
Input: graph \(G=(V, E)\), edge weights \(c \in \mathbb{R}^{E(G)}\)
Output: maximum spanning tree \((V, F)\) of \(G\)
    \(F:=\emptyset\)
    while \(E \neq \emptyset\) do
        Choose \(e \in E\) of largest weight \(c(e)\)
        \(E \leftarrow E \backslash\{e\}\)
        if \((V, F \cup\{e\})\) is a forest then
                \(F \leftarrow F \cup\{e\}\)
```

Theorem 56. For a graph $G=(V, E)$ and edge weights $c \in \mathbb{R}^{E(G)}$, Algorithm 1 correctly outputs a maximum spanning tree $T=(V, F)$.

Proof. Our goal is to show that Algorithm 1 produces a tree that satisfies the conditions of Theorem 54. Suppose, for contradiction, that there is an edge $e \in E \backslash F$ for which $c_{e}>c_{f}$ for some $f \in C_{T}(e)$. Then $e$ is processed before $f$ in the while loop of Algorithm 1. Denote by $F^{\prime} \subseteq F$ the set of edges that Algorithm 1 added to $T$ up to the state where $e$ is processed. Because then $f \notin F^{\prime}$, there is no cycle in $\left(V, F^{\prime} \cup\{e\}\right)$. But then $e$ is to be included in the set of edges of $T$, contrary to our assumption that $e \in E \backslash F$.

Kruskal's algorithm possesses a number of remarkable properties, about which Cormen, Leiserson, Rivest, and Stein [27, Section 21] give an overview. Among them are Theorems 57 and 58, which enable to verify the uniqueness of a graph's cut sequence.

Theorem 57. For a graph $G=(V, E)$ and edge weights $c \in \mathbb{R}^{E(G)}$, Algorithm 1 is able to find each maximum spanning tree of $G$.

Proof. Let us suppose, for contradiction, that there is a maximum spanning tree $T=(V, F)$ of $G$ that cannot be the output of Algorithm 1. It is within the scope of Algorithm 1 to choose among those edges of largest weight whenever possible an edge belonging to $F$ in State 3 . So let us suppose this behavior and let $T^{\prime}=\left(V, F^{\prime}\right)$ be a resulting output of Algorithm 1. By our initial assumption, there is an edge $f \in F \backslash F^{\prime}$. Denote by $F^{\prime \prime} \subseteq F^{\prime}$ the set of edges that Algorithm 1 added to $T$ up to the state where $f$ is processed. We know that $\left(V, F^{\prime \prime} \cup\{f\}\right)$ contains a cycle $C$, because $f \notin F^{\prime}$. Furthermore, all edges in $C$ have larger weight than $f$, as we assumed that Algorithm 1 prefers to choose $f$ whenever possible. So we observe for an arbitrary edge $e \in E(C) \backslash E(T)$ that $c_{e}>c_{f}$ where $f \in C_{T}(e)$. By Lemma 54, this contradicts that $T$ is a maximum spanning tree, concluding the proof.

For how to enumerate all maximum spanning trees of a graph algorithmically, see Eppstein [39] or Yamada, Kataoka, and Watanabe [106]. To our discussion, Theorem 57 contributes the fact that we can assume a maximum spanning tree to be generated by Algorithm 1 when aiming to verify some of its properties. This is the strategy in the proof of the following statement.

Theorem 58. For a graph $G=(V, E)$ and a vector of weights from $\mathbb{R}^{E}$, all maximum spanning trees have the same edge weights with exactly the same multiplicities.

Proof. Denoting by $c_{1}, \ldots, c_{k}$ the distinct weights appearing on the edges of $G$, we proceed by induction on their number $k$. In the base case $k=1$, all spanning trees of $G$ have exactly the same weight and thus our claim is true.

Now let $k>1$ and suppose that our statement is true for $k-1$ distinct weights. Furthermore, let $T=(V, F)$ be an arbitrary maximum spanning tree of $G$. By Theorem 57, we can assume that $T$ is produced by Algorithm 1. The graph which is obtained by deleting those edges from $G$ whose weight is at most $c_{k}$ shall be denoted by $G^{\prime}=\left(V, E^{\prime}\right)$ where $E^{\prime}:=\left\{e \in E: c_{e}>c_{k}\right\}$. Similarly, denote the tree $T^{\prime}=\left(V, F^{\prime}\right)$ with $F^{\prime}:=\left\{e \in F: c_{e}>c_{k}\right\}$. When generating $T$, Algorithm 1 processes the edges from $E^{\prime}$ earlier than those of weight $c_{k}$. Consequently, Algorithm 1 builds first the components of $T^{\prime}$ until they span the components of $G^{\prime}$ and the components of $T^{\prime}$ are of maximum weight, as otherwise $T$ cannot be of maximum weight. By induction, we know that $T$ has the same multiset of edge weights on the components of $G^{\prime}$ as all other maximum spanning trees of $G$. After Algorithm 1 has processed
all edges from $E^{\prime}$, there remain only edges of weight $c_{k}$ to choose from in State 3. So $T$ has the same multiset of edge weights as all other maximum spanning trees of $G$, because they all have to have the same total weight.

Corollary 59. A graph's cut sequence is uniquely determined.
Proof. By Theorem 55, each Gomory-Hu tree is a maximum spanning tree of a certain weighted complete graph and Theorem 58 says that all maximum spanning trees of a graph have exactly the same multiset of edge weights.

### 4.2 Maximally local-edge-connected graphs

We already pointed out that the values in a cut sequence $c_{1} \geq \ldots \geq c_{n-1}$ naturally satisfy $n-1 \geq c_{1}$ and $c_{n-1} \geq 0$. Having these bounds at hand, we may ask whether they can be attained simultaneously.

Lemma 60. Consider a graph $G$ with cut sequence $c_{1} \geq \ldots \geq c_{n-1}$ and let $c_{k}=n-1$ for some $k \in\{1, \ldots, n-1\}$. Then $c_{n-1} \geq k$.

Proof. If $c_{1}=\ldots=c_{k}=n-1$, then a Gomory-Hu tree of $G$ contains $k$ pairwise distinct edges $v_{i} w_{i}$, where $i \in\{1, \ldots, k\}$, of weight $n-1$. So for each $i \in\{1, \ldots, k\}$ there have to be $n-1$ edge-disjoint $v_{i}-w_{i}$ paths in $G$. This implies that all the vertices $v_{i}$ and $w_{i}$ are adjacent to all vertices in $V(G)$. Because the edges $v_{i} w_{i}$ are pairwise distinct, we find at least $k+1$ vertices in $G$ which are adjacent to all vertices in $V(G)$. This shows that $G$ is $k$-connected, which is the same as $c_{1} \geq k$.

If in particular $c_{1}=n-1$, then the respective graph is 2 -connected or $c_{1} \geq 2$. So two values of the $n-1$ numbers in a cut sequence have to have the same value. We shall also see that there are graphs on any number of vertices that have only two repeated cut values.

Let us proceed with a few concepts that play a crucial role in the subsequent observations. A graph $G$ is called maximally edge-connected if its minimum degree $\delta(G)$ equals its edge-connectivity $\lambda(G)$ and it is called maximally local-edge-connected if $c_{v w}=\min \left\{d_{v}, d_{w}\right\}$ for all vertices $v, w \in V(G)$. Such graphs are surveyed by Hellwig and Volkmann [65]. A dominating vertex is a vertex that is adjacent to all other vertices in $V(G)$. Threshold graphs, first introduced by Chvátal and Hammer [26], are graphs that can be generated from an empty graph by recursively adding isolated vertices or dominating vertices. In other words, a threshold graph on $n$ vertices can be encoded by a sequence $b_{1}, \ldots, b_{n}$ with $b_{i} \in\{0,1\}$ requiring that in iteration $i$ we shall add
an isolated vertex if $b_{i}=0$ and we shall add a dominating vertex if $b_{i}=1$. Starting from the empty graph, the value of $b_{1}$ clearly does not matter, which is why we may start such a sequence with some placeholder symbol $\epsilon$. This is also illustrated in Figure 31. Defining the distance $d(v, w)$ between two vertices $v$ and $w$ as the number of edges in a shortest path from $v$ to $w$, the diameter of a $\operatorname{graph} G$ is $\operatorname{diam}(G):=\max \{d(v, w): v, w \in V(G)\}$. The following statement relates a graph's diameter with its local-edge-connectivity.
Lemma 61. A graph $G$ with $\operatorname{diam}(G) \leq 2$ is maximally local-edge-connected.
Proof. Taking two arbitrary vertices $v, w \in V(G)$, we only have to verify that $c_{v w} \geq d_{v}$ or $c_{v w} \geq d_{w}$, because $c_{v w} \leq \min \left\{d_{v}, d_{w}\right\}$ is certainly fulfilled. Denoting a minimum $v$-w cut by $S$, we find in $G-S$ two components whose vertex sets shall be denoted by $V$ and $W$ such that $v \in V$ and $w \in W$. Our first goal is to prove that each vertex in $V$ or each vertex in $W$ is incident to at least one edge in $S$. Suppose that not all vertices in $W$ are incident to some edge in $S$. If there is also a vertex $x \in V$ not incident to some edge in $S$, then $d(x, y) \geq 3$, contrary to our prerequisite that $\operatorname{diam}(G) \leq 2$.

We may assume, without loss of generality, that $V$ is the set containing only vertices that are incident to at least one edge in $S$. Setting

$$
\begin{aligned}
& S_{1}:=\{x y \in E(G): x=v \text { and } y \in W\} \quad \text { and } \\
& S_{2}:=\{x y \in E(G): x \in V \backslash\{v\} \text { and } y \in W\},
\end{aligned}
$$

we obtain $S=S_{1} \cup S_{2}, S_{1} \cap S_{2}=\emptyset$, and $\left|S_{2}\right| \geq|V \backslash\{v\}|$, because we know that each vertex in $V \backslash\{v\}$ is incident to at least one edge in $S_{2}$. Consequently,

$$
\begin{aligned}
d_{v}= & |\{x y \in E(G): x=v\}| \\
= & \mid\{x y \in E(G): x=v \text { and } y \in W\} \mid \\
& +\mid\{x y \in E(G): x=v \text { and } y \in V \backslash\{v\}\} \mid \\
\leq & \left|S_{1}\right|+|V \backslash\{v\}| \\
\leq & \left|S_{1}\right|+\left|S_{2}\right|=|S|=c_{v w},
\end{aligned}
$$

which concludes our proof.
The previous statement and further conditions under which graphs are maximally local-edge-connected can be found in the survey by Hellwig and Volkmann [65], although referring to an unpublished manuscript for a proof of the above statement. The beginning of our proof is inspired by arguments of Chartrand, Lesniak, and Zhang [18, Theorem 4.7]. Their reasoning, however, only demonstrates that the edge-connectivity $\lambda(G)$ of a graph $G$ is equal to
its minimum degree $\delta(G)$ if $\operatorname{diam}(G) \leq 2$. This is implied by Lemma 61, because then

$$
\lambda(G)=\min _{v, w \in V(G)} c_{v w}=\min _{v, w \in V(G)} \min \left\{d_{v}, d_{w}\right\}=\delta(G)
$$

However, the statement that a graph is maximally local-edge-connected is stronger than the statement that it is maximally edge-connected. For an example, consider some tree $T$ containing two vertices $v$ and $w$ whose degrees are at least two. Then $c_{v w}=1 \neq 2 \leq \min \left\{d_{v}, d_{w}\right\}$ and so $T$ is not maximally local-edge-connected, but certainly $\lambda(T)=1=\delta(T)$ and so $T$ is maximally edge-connected. Let us proceed with further consequences of Lemma 61, which appear to be quite useful in what follows.

Corollary 62. Graphs containing a dominating vertex are maximally local-edge-connected.

Proof. A dominating vertex in a graph $G$ provides a path of length at most two between each two vertices in $V(G)$. This implies $\operatorname{diam}(G) \leq 2$ and our claim follows by Lemma 61.

Corollary 63. Threshold graphs are maximally local-edge-connected.
Proof. Let us observe first that the condition $c_{v w}=\min \left\{d_{v}, d_{w}\right\}$ is always fulfilled if one of the vertices $v$ or $w$ is an isolated vertex. So when verifying a graph to be maximally local-edge-connected, we may omit isolated vertices. Now consider a threshold graph $G$ encoded by the binary sequence $b_{1}, \ldots, b_{n}$. We may assume that $b_{n}=1$, because otherwise $G$ contains isolated vertices. From $b_{n}=1$, we know that $G$ contains a dominating vertex. So our claim follows by Corollary 62.

Our interest in maximally local-edge-connected graphs when discussing cut sequences originates from the following lemma. In its proof, we use the term star graph. This is a tree containing one dominating vertex.

Lemma 64. Consider a maximally local-edge-connected graph $G$ with degree sequence $d_{1} \geq \ldots \geq d_{n}$ and cut sequence $c_{1} \geq \ldots \geq c_{n-1}$. Then $d_{i}=c_{i-1}$ for $i \in\{2, \ldots, n\}$.

Proof. Let us address the vertices of $G$ whose degrees are $d_{1} \geq \ldots \geq d_{n}$ by $v_{1}, \ldots, v_{n}$, respectively. Our claim is that the star graph $T$ on vertex set $\left\{v_{1}, \ldots, v_{n}\right\}$ and edge set $\left\{v_{1} v_{2}, v_{1} v_{3}, \ldots, v_{1} v_{n}\right\}$ with weight $d_{i}$ for edge $v_{1} v_{i}$ for $i \in\{2, \ldots, n\}$ is a Gomory-Hu tree for $G$. Because $G$ is maximally local-edge-connected, we obtain for an arbitrary edge $v_{1} w \in E(T)$


Figure 31: A threshold graph encoded by $\epsilon, 1,0,1,0,1,0,1$ and a corresponding Gomory-Hu tree having cut values $1,2,3,4,4,5,6$
that $c_{v_{1} w}=\min \left\{d_{v_{1}}, d_{w}\right\}=d_{w}$. Deleting $v_{1} w$ from $T$ produces two components on vertex sets $\{w\}$ and $V \backslash\{w\}$. The edges $E(\{w\}, V \backslash\{w\})$ form a $v_{1}-w$ cut of minimum capacity in $G$. So $T$ is a Gomory-Hu tree for $G$. This immediately implies that $d_{i}=c_{i-1}$ for $i \in\{2, \ldots, n\}$.

This lemma is our key tool to prove a parity condition for cut sequences in Section 4.3. It also allows to provide graphs, on any number of vertices, that have only two repeated cut values. For $n \in \mathbb{N}$ consider the threshold graph encoded by the sequence $b_{1}, \ldots, b_{n}$ where $b_{i}:=1$ if $i \equiv n \bmod 2$ and $b_{i}:=0$ otherwise. We obtain a threshold graph whose degree sequence is of the form

$$
n-1, n-2, \ldots,\lfloor n / 2\rfloor+1,\lfloor n / 2\rfloor,\lfloor n / 2\rfloor,\lfloor n / 2\rfloor-1, \ldots, 2,1
$$

which is also illustrated in Figure 31. By Corollary 63, the constructed graph is maximally local-edge-connected. So Lemma 64 says that we just have to omit the value $n-1$ in the above sequence in order to obtain the cut sequence of the given threshold graph. Apart from their role in this example, threshold sequences, meaning degree sequences of threshold graphs, possess a variety of further interesting properties. First studied by Chvátal and Hammer [26] in the context of integer programming, Hammer, Ibaraki, and Simeone [62] showed that threshold sequences are in a sense the least graphic ones. This means, they are exactly those that satisfy the classical Erdős-Gallai inequalities [40] as equalities. Lemma 64 additionally says that in this case we can easily read off the corresponding cut sequence. In the following sections, we shall go on asking about the connections between degree sequences, cut sequences and the Erdős-Gallai inequalities.

### 4.3 Erdős-Gallai characterizations

In order to recognize cut sequences and eventually to construct graphs with prescribed cut values, we aim for conditions that characterize cut sequences. As our results show parallels to the Erdős-Gallai theorem [40], we shall begin by reviewing their landmark result on the structure of degree sequences. They precisely describe when a sequence of integer numbers is the degree sequence of a graph. We call such sequences graphic and say that a graph having a specific sequence of numbers as its degrees realizes that sequence. Since the original proof of Erdős and Gallai [40], various techniques have been used to reproduce their result. Berge [8, Chapter 6] uses arguments based on network theory. Other short proofs are given by Choudum [22] or Tripathi, Venugopalan, and West [101]. We recall the latter, because its algorithmic nature fits perfectly with our subsequent discussion.

Theorem 65. A sequence of integer numbers $d_{1} \geq \ldots \geq d_{n} \geq 0$ is graphic if and only if their sum $\sum_{i=1}^{n} d_{i}$ is even and the Erdős and Gallai inequalities

$$
\sum_{i=1}^{k} d_{i} \leq k(k-1)+\sum_{i=k+1}^{n} \min \left\{k, d_{i}\right\}
$$

hold for all $k \in\{1, \ldots, n\}$.
Proof. For the necessity, let us consider a graph $G$ and denote its vertices by $v_{1}, \ldots, v_{n}$ such that $d_{1} \geq \ldots \geq d_{n}$. The fact that the sum $\sum_{i=1}^{n} d_{i}$ is even is known as Handshaking Lemma and follows by counting the edges of $G$ twice, which was first shown by Euler in his seminal article [42] on the Seven Bridges of Königsberg. The adjacency matrix of $G$ has the structure

Summing over the first $k$ columns yields the left hand side of the $k$-th Erdős-Gallai inequality. As indicated in the matrix, this sum is bounded by $k(k-1)+\sum_{i=k+1}^{n} \min \left\{k, d_{i}\right\}$, which was to be shown.

```
Algorithm 2
Input: graphic sequence \(d_{1} \geq \ldots \geq d_{n} \geq 1\)
Output: realizing graph \((V, E)\)
    \(V \leftarrow\left\{v_{1}, \ldots, v_{n}\right\}\)
    \(E \leftarrow \emptyset\)
    for \(t=1, \ldots, n\) do
        while \(\operatorname{deg}\left(v_{t}\right)<d_{t}\) do
            if \(v_{t} v_{i} \notin E\) for some \(v_{i}\) with \(\operatorname{deg}\left(v_{i}\right)<d_{i}\) then \(\quad\) Case 1
                \(E \leftarrow E \cup\left\{v_{t} v_{i}\right\}\)
        else if \(v_{t} v_{i} \notin E\) for some \(i<t\) then \(\triangleright\) Case 2
            Choose \(w \in N\left(v_{i}\right) \backslash\left(N\left(v_{t}\right) \cup\left\{v_{t}\right\}\right)\)
                if \(d_{t}-\operatorname{deg}\left(v_{t}\right) \geq 2\) then
                    \(E \leftarrow\left(E \backslash\left\{v_{i} w\right\}\right) \cup\left\{v_{i} v_{t}, v_{t} w\right\}\)
                else \(\quad \triangleright\) Case 2.2
                    Choose \(k>t\) such that \(d\left(v_{k}\right)<d_{k}\)
                    \(E \leftarrow\left(E \backslash\left\{v_{k} v_{t}, v_{i} w\right\}\right) \cup\left\{v_{i} v_{t}, v_{t} w\right\}\)
        else if \(\operatorname{deg}\left(v_{k}\right) \neq \min \left\{t, d_{k}\right\}\) for some \(k>t\) then \(\quad \triangleright\) Case 3
                Choose \(i<t\) such that \(v_{i} v_{k} \notin E\)
                Choose \(w \in N\left(v_{i}\right) \backslash\left(N\left(v_{t}\right) \cup\left\{v_{t}\right\}\right)\)
                \(E \leftarrow\left(E \backslash\left\{v_{i} w\right\}\right) \cup\left\{v_{i} v_{k}, v_{t} w\right\}\)
        else \(\quad \triangleright\) Case 4
            Choose \(w \in N\left(v_{i}\right) \backslash\left(N\left(v_{t}\right) \cup\left\{v_{t}\right\}\right)\)
                Choose \(x \in N\left(v_{j}\right) \backslash\left(N\left(v_{t}\right) \cup\left\{v_{t}\right\}\right) \quad \triangleright x=w\) is allowed
                \(E \leftarrow\left(E \backslash\left\{v_{i} w, v_{j} x\right\}\right) \cup\left\{v_{i} v_{j}, v_{t} w\right\}\)
```

For the converse case, suppose that we are given a sequence of integer numbers $d_{1} \geq \ldots \geq d_{n} \geq 0$ that satisfies the Erdős-Gallai inequalities and whose sum is even. Also assume that $d_{n} \geq 1$, because zeros in such a sequence are realized just by adding isolated vertices. Our claim is that Algorithm 2, initializing a graph $(V, E)$ on $n$ isolated vertices, successively adapts $(V, E)$ to realize the given sequence. The parameter $t$, ranging from 1 to $n$, denotes the position in the sequence that the algorithm is working on. Our goal is to show that for any position $t$, in any case of the algorithm's subordinate while loop, the gap $d_{t}-\operatorname{deg}\left(v_{t}\right)$ is reduced, whereas we maintain that $\operatorname{deg}\left(v_{i}\right)=d_{i}$ for $i<t$. So Algorithm 2 only terminates with a realization of $d_{1} \geq \ldots \geq d_{n}$.

We observe first that, while the algorithm is working at position $t$, the vertex set $S_{t}:=\left\{v_{t+1}, \ldots, v_{n}\right\}$ remains independent. To check this, note that when updating $E$, in lines $6,10,13,17$, or 21 , only two of the new edges do not have $v_{t}$ as an endvertex. The first exception is $v_{i} v_{k}$ in Case 3 , where $i<t$,
the second is $v_{i} v_{j}$ in Case 4 , where $i<j<t$. So all new edges have at least one endvertex that is not in $S_{t}$, as claimed.

To see that Algorithm 2 maintains $\operatorname{deg}\left(v_{i}\right)=d_{i}$ for $i<t$, we consider again the updates of $E$ in the respective cases. Case 1 certainly behaves as claimed. In Case 2.1 all degrees, except that of $v_{t}$, are preserved. In Case 2.2, the update preserves the degrees of $v_{i}$ and $w$, while it increases the degree of $v_{t}$ and decreases the degree of $v_{k}$, which is allowed because $k>t$. In Case 3 , the degrees of $v_{i}$ and $w$ are preserved, and the degrees of $v_{t}$ and $v_{k}$ are increased, which is allowed because $k>t$. In Case 4 , the degrees of $v_{i}$ and $v_{j}$ are preserved, the degree of $v_{t}$ is increased, and the degree of $x$ is decreased. The latter is in accordance with our claim, because $x \notin N\left(v_{t}\right) \cup\left\{v_{t}\right\}$ and so it is in $S_{t}$, as otherwise Case 2 applies.

Finally, let us check that while $\operatorname{deg}\left(v_{t}\right)<d_{t}$, there is some case that applies to reduce the respective gap $d_{t}-\operatorname{deg}\left(v_{t}\right)$. Case 1 certainly works as intended.

Case 2 applies if we are not in Case 1 and if $v_{t} v_{i} \notin E$ for some $i$ with $i<t$. By the fact that $\operatorname{deg}\left(v_{i}\right)=d_{i} \geq d_{t}>\operatorname{deg}\left(v_{t}\right)$, a vertex $w \in N\left(v_{i}\right) \backslash\left(N\left(v_{t}\right) \cup\left\{v_{t}\right\}\right)$ can be chosen. If $d_{t}-\operatorname{deg}\left(v_{t}\right) \geq 2$, then Subcase 2.1 applies and $d_{t}-\operatorname{deg}\left(v_{t}\right)$ can be reduced as described in line 10. Otherwise, $d_{t}-\operatorname{deg}\left(v_{t}\right)=1$ and because the sum

$$
\sum_{i=1}^{n} d_{i}-\sum_{i=1}^{n} \operatorname{deg}\left(v_{i}\right)=\sum_{i=t}^{n} d_{i}-\sum_{i=t}^{n} \operatorname{deg}\left(v_{i}\right)
$$

is even, there is a $k>t$ with $\operatorname{deg}\left(v_{k}\right)<d_{k}$. Furthermore, $v_{k}$ is adjacent to $v_{t}$, because otherwise Case 1 applies. So we can reduce the gap $d_{t}-\operatorname{deg}\left(v_{t}\right)$ as described in line 17.

If neither Case 1 nor Case 2 applies, then $v_{1}, \ldots, v_{t-1} \in N\left(v_{t}\right)$. We are in Case 3 if, in addition, there is some $k>t$ such that $\operatorname{deg}\left(v_{k}\right) \neq \min \left\{t, d_{k}\right\}$. In fact, we have $\operatorname{deg}\left(v_{k}\right)<\min \left\{t, d_{k}\right\}$, because $\operatorname{deg}\left(v_{k}\right) \leq d_{k}$ and $\operatorname{deg}\left(v_{k}\right) \leq t$, as $S_{t}$ is independent. We also know that $v_{t} v_{k} \in E$, because otherwise Case 1 applies. Furthermore, there is an $i$ with $i<t$ such that $v_{i} v_{k} \notin E$, because $\operatorname{deg}\left(v_{k}\right)<t$. By $\operatorname{deg}\left(v_{i}\right)>\operatorname{deg}\left(v_{t}\right)$, there is some $w \in N\left(v_{i}\right) \backslash\left(N\left(v_{t}\right) \cup\left\{v_{t}\right\}\right)$ to reduce $d_{t}-\operatorname{deg}\left(v_{t}\right)$ as described in line 17 .

Case 4 applies if none of the Cases 1 to 3 applies. Suppose first that there are $i<j<t$ with $v_{i} v_{j} \notin E$. Also, $v_{i}, v_{j} \in N\left(v_{t}\right)$, as otherwise Case 1 applies. By $\operatorname{deg}\left(v_{i}\right) \geq \operatorname{deg}\left(v_{j}\right)>\operatorname{deg}\left(v_{t}\right)$, we find $w \in N\left(v_{i}\right) \backslash\left(N\left(v_{t}\right) \cup\left\{v_{t}\right\}\right)$ as well as $x \in N\left(v_{j}\right) \backslash\left(N\left(v_{t}\right) \cup\left\{v_{t}\right\}\right)$. Note that $w, x \in S_{t}$, as otherwise Case 1 applies. The gap $d_{t}-\operatorname{deg}\left(v_{t}\right)$ can be reduced as specified in line 21.

Now consider the situation where none of the Cases 1 to 3 applies and where $v_{i} v_{j} \in E$ for all $i<j<t$. Then the vertices $v_{1}, \ldots, v_{t}$ are pairwise adjacent and $\operatorname{deg}\left(v_{k}\right)=\min \left\{t, d_{k}\right\}$ for $k>t$. Because $S_{t}$ is independent, we obtain

$$
\sum_{i=1}^{n} \operatorname{deg}\left(v_{i}\right)=t(t-1)+\sum_{k=t+1}^{n} \min \left\{t, d_{k}\right\},
$$

which, by the first part of this proof, says that $\operatorname{deg}\left(v_{t}\right)$ is already at maximum and so $\operatorname{deg}\left(v_{t}\right)=d_{t}$.

So the proof of Tripathi, Venugopalan, and West [101] actually yields a bit more than is claimed in Theorem 65. Algorithm 2 not only establishes the existence of a realizing graph but also shows how to construct it. Our next goal is to discuss related conditions for cut sequences. Probably the simplest requirement in Theorem 65 is that degree sequences have to add up to an even number. However, this is not the case for cut values. This can already be observed from the graph consisting of two vertices connected by an edge, whose cut sequence is just 1 . Nevertheless, cut values satisfy a somewhat similar parity condition, for whose proof we build on the following fact.
Lemma 66. Consider a graph $G$ with cut sequence $c_{1} \geq \ldots \geq c_{n-1}$. An edge $e \in E(G)$ is a bridge in $G$ if and only if $e$ is an edge of weight one in any corresponding Gomory-Hu tree of $G$. In particular, the maximal $\ell \in \mathbb{N} \cup\{0\}$ with $c_{n-\ell}=\ldots=c_{n-1}=1$ gives the number of bridges in $G$.

Proof. We consider some Gomory-Hu tree $T$ of $G$. For any bridge $v w \in E(G)$ there is a unique cut $E(V, W)=\{v w\}$ in $G$ that we denote such that $v \in V$ and $w \in W$. Assuming, for contradiction, that $v w \notin E(T)$, there must be another edge $x y \in E(T)$ representing the cut $E(V, W)$. So $x y$ has at least one endvertex not in common with $v w$, say $x \notin\{v, w\}$, and $T-x y$ contains two components on vertex sets $V$ and $W$, denoted such that $x \in V$ and $y \in W$. There is a unique path $P$ in $T$ that leads from $v$ to $x$. By Lemma 11, each edge with smallest weight on $P$ induces a minimum $v-x$ cut $C$ in $G$. Because $P$ is a subpath of the unique path leading from $v$ to $x$ to $y$ to $w$, in which possibly $y=w$, we conclude that $C$ also separates $v$ and $w$. This implies that $v w \in C$. But $C \backslash\{v w\}$ is a $v-x$ cut, as otherwise $v$ or $x$ must be contained in $W$, contradicting the minimality of $C$. This proves that each bridge in $G$ appears as an edge in $T$. Clearly, its weight is one.

Suppose conversely that there is an edge $v w \in E(T)$ whose weight is one, but with $v w \notin E(G)$. Then there exists a cut $E(V, W)$ with $|E(V, W)|=1$, which we denote such that $v \in V$ and $w \in W$. So $E(V, W)$ contains a single edge $x y \neq v w$ with $x \in V$ and $y \in W$ whose deletion disconnects $G$. In other
words, the edge $x y$ is a bridge in $G$ and so $x y \in E(T)$, by our reasoning from above. As there is a unique, possibly empty, path $P$ that connects $v$ and $x$ in $T[V]$ as well as a unique, possibly empty, path $Q$ that connects $w$ and $y$ in $T[W]$, we find a cycle $P+v w+Q+x y$ in $T$, contradicting that $T$ is a tree. Consequently, $v w$ is contained in $E(G)$ and so it is a bridge.

Theorem 67. Consider a graph $G$ with cut sequence $c_{1} \geq \ldots \geq c_{n-1}$ satisfying $c_{1}=n-1-\ell>1$ where $\ell:=\left|\left\{c_{i}: c_{i}=1\right\}\right|$. Then the following statements hold.
(i) The graph $G$ has only one nontrivial block that contains $n-\ell$ vertices.
(ii) The sum $\sum_{i=2}^{n-1-\ell} c_{i}$ is even.

Proof. Condition $c_{1}=n-1-\ell>1$ says that there are vertices $v, w \in V(G)$ that are connected by $n-1-\ell>1$ edge-disjoint paths. These paths can only exist when $v$ has at least $n-1-\ell$ neighbors. This requires that there is a nontrivial block in $G$ containing $|N(v) \cup\{v\}| \geq n-\ell$ vertices. To conclude statement (i), denote by $n_{1}$ the number of vertices of the largest block of $G$ and suppose, for contradiction, that there is another nontrivial block on $n_{2} \geq 3$ vertices. As the parameter $\ell$, by Lemma 66, counts the number of bridges in $G$ and because two different blocks in $G$ can share at most one vertex, we obtain $n_{1} \leq n-\ell-\left(n_{2}-1\right) \leq n-\ell-2$, which contradicts the fact that there is a block containing $n-\ell$ vertices.

Let us turn to statement (ii). By statement (i), there is only one nontrivial block $B$ in $G$ that contains $n-\ell$ vertices and because $c_{1}=n-1-\ell>1$, there are two vertices $v$ and $w$ in $B$ which are connected by $n-1-\ell$ edgedisjoint paths. In other words, both $v$ and $w$ are dominating vertices in $B$. Also note that no pair of vertices of $B$ is connected by any path that uses any of the bridges of $G$. So the cut sequence of $B$ is $c_{1} \geq \ldots \geq c_{n-1-\ell}$. Denoting the degree sequence of $B$ by $d_{1} \geq \ldots \geq d_{n-\ell}$, Corollary 62 and Lemma 64 imply that $c_{i-1}=d_{i}$ for $i \in\{2, \ldots, n-\ell\}$. We also have $d_{1}=d_{2}=n-1-\ell$, because there are $n-1-\ell$ edge-disjoint paths between $v$ and $w$. Consequently,

$$
\sum_{i=2}^{n-1-\ell} c_{i}=\sum_{i=3}^{n-\ell} d_{i}=\sum_{i=1}^{n-\ell} d_{i}-\left(d_{1}+d_{2}\right)=2|E(B)|-2(n-1-\ell)
$$

which is an even sum and thus proves statement (ii).
We now turn to sufficient conditions on integer sequences to be cut sequences, beginning with the following simple but quite useful observation.

Lemma 68. Consider for a sequence of natural numbers $c_{1} \geq \ldots \geq c_{n-1}$ the shifted sequence defined by $d_{i}:=c_{i}-1$ for $i \in\{1, \ldots, n-1\}$. Then the sequence $c_{1} \geq \ldots \geq c_{n-1}$ satisfies the shifted Erdős-Gallai inequalities

$$
\sum_{i=1}^{k} c_{i} \leq k^{2}+\sum_{i=k+1}^{n-1} \min \left\{k, c_{i}-1\right\}
$$

for all $k \in\{1, \ldots, n-1\}$ if and only if the sequence $d_{1} \geq \ldots \geq d_{n-1}$ satisfies the Erdős-Gallai inequalities for all $k \in\{1, \ldots, n-1\}$.

Proof. If the sequence $c_{1} \geq \ldots \geq c_{n-1}$ satisfies the shifted Erdős-Gallai inequalities, then we obtain for all $k \in\{1, \ldots, n-1\}$ that

$$
\begin{aligned}
\sum_{i=1}^{k} d_{i}=\left(\sum_{i=1}^{k} c_{i}\right)-k & \leq k^{2}-k+\sum_{i=k+1}^{n-1} \min \left\{k, c_{i}-1\right\} \\
& =k(k-1)+\sum_{i=k+1}^{n-1} \min \left\{k, d_{i}\right\}
\end{aligned}
$$

The other direction follows analogously.
This relation is a first key tool that we use when constructing graphs with prescribed cut values in the following section.

### 4.4 Constructing graphs with prescribed cut values

Let us begin our constructive attempts with an example. It is easy to verify that the sequence $4,4,3,3,3$ satisfies the shifted Erdős-Gallai inequalities. So the sequence 3, 3, 2, 2, 2 satisfies the Erdős-Gallai inequalities. Note that we do not have to check the latter directly, it suffices to apply Lemma 68. Even more, the numbers $3,3,2,2,2$ add up to an even sum. So the sequence $3,3,2,2,2$ is graphic. For those sequences, Algorithm 2 constructs a realizing graph $G$. Adding a dominating vertex to $G$, provides a graph $G^{\prime}$ whose degree sequence is $5,4,4,3,3,3$. The resulting graph $G^{\prime}$, by Corollary 62 , is maximally local-edge-connected. Therefore, Lemma 64 ensures that $4,4,3,3,3$ is the cut sequence of $G^{\prime}$. So we have found that $4,4,3,3,3$ is a cut sequence and we know how to construct a realizing graph. But this procedure, which is illustrated in Figure 32, has one crucial requirement.


Figure 32: Constructing a graph whose cut sequence is $4,4,3,3,3$

It works only if the shifted sequence, $3,3,2,2,2$ in this case, adds up to an even sum. If this is not the case, then we can perform the following modification to still obtain a graphic sequence.

Lemma 69. Consider an integer sequence $d_{1} \geq \ldots \geq d_{n} \geq 0$ with $d_{1} \geq 1$ that satisfies the Erdős-Gallai inequalities. Furthermore, define the numbers $d_{t}^{\prime}:=d_{t}-1$ for $t:=\max \left\{i: d_{i}=d_{1}\right\}$ and $d_{i}^{\prime}:=d_{i}$ for $i \in\{1, \ldots, n\} \backslash\{t\}$. If the sum $\sum_{i=1}^{n} d_{i}$ is odd, then the sequence $d_{1}^{\prime} \geq \ldots \geq d_{n}^{\prime}$ is graphic.

Proof. If $\sum_{i=1}^{n} d_{i}$ is odd, then $\sum_{i=1}^{n} d_{i}^{\prime}=\left(\sum_{i=1}^{n} d_{i}\right)-1$ is even. To verify that $d_{1}^{\prime} \geq \ldots \geq d_{n}^{\prime}$ is graphic, it hence remains to be checked if this sequence satisfies the $k$ Erdős-Gallai inequalities. If $k \in\{t, \ldots, n\}$, then $d_{t}$ occurs only on their left side and we obtain that

$$
\begin{aligned}
\sum_{i=1}^{k} d_{i}^{\prime}=\left(\sum_{i=1}^{k} d_{i}\right)-1 & \leq k(k-1)-1+\sum_{i=k+1}^{n} \min \left\{k, d_{i}\right\} \\
& <k(k-1)+\sum_{i=k+1}^{n} \min \left\{k, d_{i}^{\prime}\right\}
\end{aligned}
$$

So let $k \in\{1, \ldots, t-1\}$ for the remainder of our proof. In the case $d_{t}>k$, we obtain $d_{t}^{\prime} \geq k$ and thus $\min \left\{k, d_{i}\right\}=\min \left\{k, d_{i}^{\prime}\right\}$ for all $i$. Consequently,

$$
\begin{aligned}
\sum_{i=1}^{k} d_{i}^{\prime}=\sum_{i=1}^{k} d_{i} & \leq k(k-1)+\sum_{i=k+1}^{n} \min \left\{k, d_{i}\right\} \\
& =k(k-1)+\sum_{i=k+1}^{n} \min \left\{k, d_{i}^{\prime}\right\}
\end{aligned}
$$

If $d_{t}<k$, we have $d_{1}^{\prime}=\ldots=d_{t-1}^{\prime}<k$, because $t=\max \left\{i: d_{i}=d_{1}\right\}$. Thus,

$$
\sum_{i=1}^{k} d_{i}^{\prime} \leq \sum_{i=1}^{k}(k-1)=k(k-1) \leq k(k-1)+\sum_{i=k+1}^{n} \min \left\{k, d_{i}^{\prime}\right\} .
$$

So only the case $d_{t}=k$ remains. Then the left hand side of the $k$-th ErdősGallai inequality reads

$$
\sum_{i=1}^{k} d_{i}^{\prime}=\sum_{i=1}^{k} k=k(k-1)+k
$$

and we are left to verify that $\sum_{i=k+1}^{n} \min \left\{k, d_{i}^{\prime}\right\} \geq k$. For this purpose, we use the relation $d_{k+1}^{\prime} \geq d_{t}^{\prime}=k-1$, which holds because $k<t$. Furthermore, we know that $\sum_{i=k+2}^{n} \min \left\{k, d_{i}^{\prime}\right\} \geq 1$, as otherwise $d_{1}=\ldots=d_{t}=d_{k+1}=k$ and $d_{k+2}=\ldots=d_{n}=0$. But then

$$
\sum_{i=1}^{n} d_{i}=\sum_{i=1}^{k+1} k=k(k+1)
$$

is an even number, contrary to our assumption. This implies

$$
\begin{aligned}
\sum_{i=k+1}^{n} \min \left\{k, d_{i}^{\prime}\right\} & =\min \left\{k, d_{k+1}^{\prime}\right\}+\sum_{i=k+2}^{n} \min \left\{k, d_{i}^{\prime}\right\} \\
& \geq k-1+\sum_{i=k+2}^{n} \min \left\{k, d_{i}^{\prime}\right\} \geq k
\end{aligned}
$$

which is the last relation to be shown.

Let us summarize what we obtained so far. If we are given a sequence of natural numbers $c_{1} \geq \ldots \geq c_{n-1}$ which satisfy the shifted Erdős-Gallai inequalities, then either the shifted sequence $d_{i}:=c_{i}-1, i \in\{1, \ldots, n-1\}$,


Figure 33: A graph containing a dominating vertex $s$ whose cut sequence $2,2,1,1$ is changed to $2,2,2,1$ by replacing edges $v w$ and $s x$ by $v x$ and $w x$
is already graphic or we can adjust it as described in Lemma 70 to obtain a graphic sequence. When proceeding in the latter case as illustrated in Figure 32 , then we obtain a graph whose cut sequence is close to the sequence $c_{1} \geq \ldots \geq c_{n-1}$ we are aiming for. The modification from Lemma 69, however, causes some cut value to be too low by one. In what follows, we discuss constructions to correct this deficiency.

Lemma 70. Consider a graph $G$ that contains a dominating vertex $s$ and whose degree sequence is $n-1=d_{1}>d_{2} \geq \ldots \geq d_{n}$. If there is a vertex $x$ with $\operatorname{deg}_{G}(x) \geq d_{2}-1$ so that for some edge $v w \in E(G)$ there is $v x \notin E(G)$ and $w x \notin E(G)$, then the graph

$$
G^{\prime}:=(V(G),(E(G) \backslash\{v w, s x\}) \cup\{v x, w x\})
$$

realizes the cut sequence $\operatorname{deg}_{G^{\prime}}(x) \geq d_{2} \geq \ldots \geq d_{n}$ where $d_{2} \geq \ldots \geq d_{n}$ is to be read without one number for $\operatorname{deg}_{G}(x)$.

Proof. From $v x \notin E(G)$ and $w x \notin E(G)$, we know that $v \neq s$ and $w \neq s$ as well as $\operatorname{deg}_{G}(x) \leq n-3$. So the graph $G^{\prime}$ is well-defined. The edge exchange that transforms $G$ into $G^{\prime}$ is illustrated in Figure 33. It preserves the degrees of $v$ and $w$, increases the degree of $x$ by one, and decreases the degree of $s$ by one. So $n-2 \geq \operatorname{deg}_{G^{\prime}}(x) \geq d_{2} \geq \ldots \geq d_{n}$ is the degree sequence of $G^{\prime}$, where $d_{2} \geq \ldots \geq d_{n}$ is to be read without one number for $\operatorname{deg}_{G}(x)$. We now aim to show that $G^{\prime}$ is maximally local-edge-connected, because then Lemma 64 implies our claim.

We consider two arbitrary vertices $i, j \in V\left(G^{\prime}\right)$ and denote them such that $\operatorname{deg}_{G^{\prime}}(i) \leq \operatorname{deg}_{G^{\prime}}(j)$. Our goal is to show that there exists a set of $\operatorname{deg}_{G^{\prime}}(i)$ edge-disjoint $i-j$ paths in $G^{\prime}$. Note that we cannot use Corollary 62 for that purpose, because $G^{\prime}$ does not contain the edge $x s$ and so there is no dominating vertex in $G^{\prime}$. Consider the set of paths

$$
\begin{array}{ll}
i j & \text { if } i j \in E\left(G^{\prime}\right), \\
i y j & \text { for } y \in N_{G^{\prime}}(i) \cap N_{G^{\prime}}(j), \\
\text { iyszj } & \text { for } y \in N_{G^{\prime}}(i) \backslash\left(N_{G^{\prime}}(j) \cup\{j, s\}\right) \text { and suitable } \\
& z \in N_{G^{\prime}}(j) \backslash\left(N_{G^{\prime}}(i) \cup\{i, s\}\right) .
\end{array}
$$

The first two groups are well-defined for all $i, j \in V\left(G^{\prime}\right)$. The third group establishes the required local edge-connectivity unless $x \in N_{G^{\prime}}(i) \backslash\left(N_{G^{\prime}}(j) \cup\{j\}\right)$ or $\left[x \in N_{G^{\prime}}(j) \backslash\left(N_{G^{\prime}}(i) \cup\{i\}\right)\right.$ and $\left.\operatorname{deg}_{G^{\prime}}(i)=\operatorname{deg}_{G^{\prime}}(j)\right]$. Indeed, all vertices except $x$ are adjacent to $s$ and $N_{G^{\prime}}(s)=V\left(G^{\prime}\right) \backslash\{s, x\}$. Even if $j=s$, the set $N_{G^{\prime}}(i) \backslash\left(N_{G^{\prime}}(s) \cup\{s\}\right)$ is empty whenever $x \notin N_{G^{\prime}}(i)$ and if $i=s$, we
have $\operatorname{deg}_{G^{\prime}}(s)=n-2=\operatorname{deg}_{G^{\prime}}(j)$ and $N_{G^{\prime}}(s)=N_{G^{\prime}}(j)$ unless $x \in N_{G^{\prime}}(j)$. If $i=s$ and $x \in N_{G^{\prime}}(j)$, then swap $i$ and $j$ and consider the remaining case.

We are left with the situation where $i \in V\left(G^{\prime}\right) \backslash\{s, x\}$ and $j \in V\left(G^{\prime}\right) \backslash\{x\}$ with $\operatorname{deg}_{G^{\prime}}(i) \leq \operatorname{deg}_{G^{\prime}}(j)$ and $x \in N_{G^{\prime}}(i) \backslash\left(N_{G^{\prime}}(j) \cup\{j\}\right)$. In this case, all paths iyszj of the third group can be constructed except for $y=x$. For that path, there also remains some $z=u \in N_{G^{\prime}}(j) \backslash\left(N_{G^{\prime}}(i) \cup\{i, s\}\right)$ that we can choose suitably. If $N_{G^{\prime}}(x) \cap\left(N_{G^{\prime}}(j) \backslash\left(N_{G^{\prime}}(i) \cup\{i\}\right)\right) \neq \emptyset$, we may assume $u$ to be within this set and then $i x u j$ is a feasible path. Otherwise, by $\operatorname{deg}_{G^{\prime}}(x) \geq \operatorname{deg}_{G^{\prime}}(j) \geq \operatorname{deg}_{G^{\prime}}(i)$, there is some $t \in N_{G^{\prime}}(x) \backslash\left(N_{G^{\prime}}(i) \cup N_{G^{\prime}}(j)\right)$ and with it we obtain the feasible path ixtsuj. Indeed, if $x$ has no neighbors in $\left(N_{G^{\prime}}(j) \backslash\left(N_{G^{\prime}}(i) \cup\{i\}\right)\right)$ there is such a $t$, because $i$ is adjacent to $s$, whereas $x$ is not. This shows that $G^{\prime}$ is maximally local-edge-connected, which is what remained to be shown.

With Lemma 70 at hand, we have a tool to adapt the cut sequence of certain graphs while retaining their local edge-connectivity. However, it is not always possible to perform the edge exchange described in Lemma 70, because for the vertex $x \in V(G)$ with $\operatorname{deg}_{G}(x) \geq d_{2}-1$ there might be no suitable edge $v w \in E(G)$ with $v x \notin E(G)$ and $w x \notin E(G)$. An example for this situation is depicted in Figure 34. So we have to look for exchange operations that also work in such a situation.

Lemma 71. Consider a graph $G$ containing a dominating vertex $s$ with degree sequence $n-1=d_{1}>d_{2} \geq \ldots \geq d_{n} \geq 2$ and suppose that there is a vertex $x$ with $n-3 \geq \operatorname{deg}_{G}(x) \geq d_{2}-1$ that is adjacent to at least one endvertex of each edge of $G$. Furthermore, let $v, w \in V(G) \backslash\{x, s\}$ be two vertices such that $v x \notin E(G)$ and $w x \notin E(G)$. If there are vertices $v^{\prime} \in N_{G}(v) \backslash\{s\}$ and $w^{\prime} \in N_{G}(w) \backslash\left(N_{G}\left(v^{\prime}\right) \cup\left\{v^{\prime}, s\right\}\right)$, then the graph

$$
G^{\prime}:=\left(V(G),\left(E(G) \backslash\left\{s x, v v^{\prime}, w w^{\prime}\right\}\right) \cup\left\{v x, w x, v^{\prime} w^{\prime}\right\}\right)
$$

realizes the cut sequence $\operatorname{deg}_{G^{\prime}}(x) \geq d_{2} \geq \ldots \geq d_{n}$, where $d_{2} \geq \ldots \geq d_{n}$ is to be read without one number for $\operatorname{deg}_{G}(x)$.

Proof. First of all, note that there are suitable vertices $v, w \in V(G) \backslash\{x, s\}$ with $v x \notin E(G)$ and $w x \notin E(G)$, because $\operatorname{deg}_{G}(x) \leq n-3$. If there are vertices $v^{\prime} \in N_{G}(v)$ and $w^{\prime} \in N_{G}(w) \backslash\left(N_{G}\left(v^{\prime}\right) \cup\left\{v^{\prime}\right\}\right)$, then $G^{\prime}$ is well-defined. The edge exchange that transforms $G$ into $G^{\prime}$, which is illustrated in Figure 34, decreases the degree of $s$ by one, increases the degree of $x$ by one and preserves all other degrees. So $n-2 \geq \operatorname{deg}_{G^{\prime}}(x) \geq d_{2} \geq \ldots \geq d_{n}$, where one number for $\operatorname{deg}_{G}(x)$ is deleted in $d_{2} \geq \ldots \geq d_{n}$, is the degree sequence of $G^{\prime}$.


Figure 34: A graph containing a dominating vertex $s$ whose cut sequence $3,3,3,2,2$ is changed to $4,3,3,2,2$ by replacing $s x, v v^{\prime}, w w^{\prime}$ by $v x, w x, v^{\prime} w^{\prime}$

It remains to be shown that $G^{\prime}$ is maximally local-edge-connected, because then Lemma 64 implies our claim.

Let us verify that $\operatorname{diam}(G) \leq 2$ to show that $G^{\prime}$ is maximally local-edgeconnected. First, observe that $d(y, z) \leq 2$ for all $y, z \in V\left(G^{\prime}\right) \backslash\{s, x\}$, because the paths $y s z$ are present in $G^{\prime}$. Furthermore, we have $d(s, y)=1$ for all $y \in V\left(G^{\prime}\right) \backslash\{s, x\}$ and $d(s, x)=2$, because there is the path xvs. Now consider an arbitrary vertex $y \in V\left(G^{\prime}\right) \backslash\{s\}$. As $\operatorname{deg}_{G^{\prime}}(y) \geq d_{n} \geq 2$, there is an edge $e$ incident to $y$ but not to $s$. Because $x$ is adjacent to at least one endvertex of $e$ in $G$, this is also true for $e$ in $G^{\prime}$, because $x$ only loses the adjacency to $s$ when building $G^{\prime}$ and the newly added edges $v x, w x$, and $v^{\prime} w^{\prime}$ do not harm, either. This provides an $x-y$ path of length at most two. So $\operatorname{diam}\left(G^{\prime}\right) \leq 2$ and Lemma 61 can be applied.

Once again, Lemma 71 might not always be applicable. Fortunately, there is a third exchange idea that can be employed if the previous ones fail.

Lemma 72. Consider a graph $G$ containing a dominating vertex $s$ with degree sequence $n-1=d_{1}>d_{2} \geq \ldots \geq d_{n} \geq 2$ and suppose that there is a vertex $x$ with $n-3 \geq \operatorname{deg}_{G}(x) \geq d_{2}-1$ that is adjacent to at least one endvertex of each edge of $G$. Let $W:=V(G) \backslash\left(N_{G}(x) \cup\{x\}\right)$ and let $v \in W$. If $N_{G}(W)$ forms a clique, then the graph

$$
G^{\prime}:=(V(G),(E(G) \backslash\{s v\}) \cup\{x v\})
$$

realizes the cut sequence $\operatorname{deg}_{G^{\prime}}(x) \geq d_{2} \geq \ldots \geq d_{n}$, where $d_{2} \geq \ldots \geq d_{n}$ is to be read without one number for $\operatorname{deg}_{G}(x)$.


Figure 35: A graph containing a dominating vertex $s$ whose cut sequence $4,4,3,3,2,2$ is changed to $5,4,3,3,2,2$ by replacing $s v$ by $x v$

Proof. First, note that $|W| \geq 2$, because $\operatorname{deg}_{G}(x) \leq n-3$. Also note that $W$ is independent, as otherwise $x$ is adjacent to some vertex in $W$. The set $C:=N_{G^{\prime}}(W) \backslash\{s, x\} \subseteq N_{G^{\prime}}(x)$ is nonempty, as $v$ is not adjacent to $s$ in $G^{\prime}$ and $\operatorname{deg}_{G^{\prime}}(v) \geq d_{n} \geq 2$. Furthermore, $C$ forms a clique in $G^{\prime}$, because $N_{G}(W)$ is a clique in $G$. We also know that $x$ is adjacent to all neighbors of $v$, in $G$ and $G^{\prime}$, because $x$ is adjacent to at least one endvertex of each edge of $G$ and so of $G^{\prime}$. This implies $\operatorname{deg}_{G^{\prime}}(x)>\operatorname{deg}_{G^{\prime}}(v)$, as $x$ is adjacent to $s$ but $v$ is not. Also, denote a neighbor of $v$ in $C$ by $v^{\prime} \in C$ and recall that $x v^{\prime} \in E\left(G^{\prime}\right)$.

The edge exchange that transforms $G$ into $G^{\prime}$, which is illustrated in Figure 35 , decreases the degree of $s$ by one, increases the degree of $x$ by one, and preserves all other degrees. So $n-2 \geq \operatorname{deg}_{G^{\prime}}(x) \geq d_{2} \geq \ldots \geq d_{n}$, where one number for $\operatorname{deg}_{G}(x)$ is deleted in $d_{2} \geq \ldots \geq d_{n}$, is the degree sequence of $G^{\prime}$. It remains to be shown that $G^{\prime}$ is maximally local-edge-connected, because then Lemma 64 implies our claim.

For two arbitrary vertices $i, j \in V\left(G^{\prime}\right)$, denoted such that $\operatorname{deg}_{G^{\prime}}(i) \leq \operatorname{deg}_{G^{\prime}}(j)$, our goal is to provide a set of $\operatorname{deg}_{G^{\prime}}(i)$ edge-disjoint $i-j$ paths. Consider

$$
\begin{array}{ll}
\text { ij } & \text { if } i j \in E\left(G^{\prime}\right), \\
\text { iyj } & \text { for } y \in N_{G^{\prime}}(i) \cap N_{G^{\prime}}(j), \\
\text { iyszj } & \text { for } y \in N_{G^{\prime}}(i) \backslash\left(N_{G^{\prime}}(j) \cup\{j, s\}\right) \text { and suitable } \\
& z \in N_{G^{\prime}}(j) \backslash\left(N_{G^{\prime}}(i) \cup\{i, s\}\right) .
\end{array}
$$

The first two groups are well-defined for all $i, j \in V\left(G^{\prime}\right)$. The third group establishes the required local edge-connectivity unless $v \in N_{G^{\prime}}(i) \backslash\left(N_{G^{\prime}}(j) \cup\{j\}\right)$ or $\left[v \in N_{G^{\prime}}(j) \backslash\left(N_{G^{\prime}}(i) \cup\{i\}\right)\right.$ and $\left.\operatorname{deg}_{G^{\prime}}(i)=\operatorname{deg}_{G^{\prime}}(j)\right]$. Indeed, all vertices except $v$ are adjacent to $s$ and $N_{G^{\prime}}(s)=V(G) \backslash\{s, v\}$. Even if $j=s$, the
set $N_{G^{\prime}}(i) \backslash\left(N_{G^{\prime}}(s) \cup\{s\}\right)$ is empty whenever $v \notin N_{G^{\prime}}(i)$ and if $i=s$, we have $\operatorname{deg}_{G^{\prime}}(s)=n-2=\operatorname{deg}_{G^{\prime}}(j)$ and $N_{G^{\prime}}(s)=N_{G^{\prime}}(j)$ unless $v \in N_{G^{\prime}}(j)$.

So for the remaining proof, let $i \in V\left(G^{\prime}\right) \backslash\{s, v\}$ and $j \in V\left(G^{\prime}\right) \backslash\{v\}$ with $\operatorname{deg}_{G^{\prime}}(i) \leq \operatorname{deg}_{G^{\prime}}(j)$, and $v \in N_{G^{\prime}}(i) \backslash\left(N_{G^{\prime}}(j) \cup\{j\}\right)$. Because of the degree condition and the fact that $i$ is adjacent to $v$ but $j$ is not, we obtain that $N_{G^{\prime}}(j) \backslash\left(N_{G^{\prime}}(i) \cup\{i\}\right) \neq \emptyset$, in all following cases.

Case 1: $i=x$. Consider some $w \in N_{G^{\prime}}(j) \backslash\left(N_{G^{\prime}}(x) \cup\{x\}\right) \subseteq W$. We have $w \notin C$. Recall that $W$ is independent, $\operatorname{deg}_{G^{\prime}}(w) \geq 2$, and $w$ is not adjacent to $x$. This implies that $w$ has a neighbor $w^{\prime} \in C$, possibly $w^{\prime}=v^{\prime}$.

Subcase 1.1: $j=s$. We find the feasible paths
$\begin{array}{ll}x v v^{\prime} w s & \text { if } w^{\prime}=v^{\prime}, \\ x v v^{\prime} w^{\prime} w s & \text { if } w^{\prime} \neq v^{\prime},\end{array}$
xs,
xys for $y \in N_{G^{\prime}}(x) \backslash\{v, s\}$.
Subcase 1.2: $j \neq s$. This case is only relevant when $\operatorname{deg}_{G^{\prime}}(x)=\operatorname{deg}_{G^{\prime}}(j)$. Furthermore, $j \neq v^{\prime}$, because $j \notin N_{G^{\prime}}(v)$. We provide the feasible paths

$$
\begin{array}{ll}
x v v^{\prime} s w j & \text { if } v^{\prime} j \in E\left(G^{\prime}\right), \\
x v v^{\prime} w j & \text { if } v^{\prime} j \notin E\left(G^{\prime}\right) \text { and } v^{\prime}=w^{\prime}, \\
x v v^{\prime} w^{\prime} w j & \text { if } v^{\prime} j \notin E\left(G^{\prime}\right) \text { and } v^{\prime} \neq w^{\prime}, \\
x j & \text { if } x j \in E\left(G^{\prime}\right), \\
x y j & \text { for } y \in N_{G^{\prime}}(x) \cap N_{G^{\prime}}(j), \\
x y s z j & \text { for } y \in N_{G^{\prime}}(x) \backslash\left(N_{G^{\prime}}(j) \cup\{j, v, s\}\right) \text { and suitable } \\
& z \in N_{G^{\prime}}(j) \backslash\left(N_{G^{\prime}}(x) \cup\{x, w, s\}\right) .
\end{array}
$$

Note that we carefully select exactly one of the first three paths so that it does not share an edge with any path of the last two groups.

Case 2: $i \neq x$. Note that $i \in C$, because $v \in N_{G^{\prime}}(i)$ and $v \in W$. This implies that $i x \in E\left(G^{\prime}\right)$

Subcase 2.1: $j=s$. Consider the following paths

```
ivxzs, if }xz\inE(\mp@subsup{G}{}{\prime})\mathrm{ for some z }z\in\mp@subsup{N}{\mp@subsup{G}{}{\prime}}{}(s)\(\mp@subsup{N}{\mp@subsup{G}{}{\prime}}{}(i)\cup{i}
is if is }\inE(\mp@subsup{G}{}{\prime})
iys for }y\in\mp@subsup{N}{\mp@subsup{G}{}{\prime}}{}(i)\{v,s}
```

These paths verify the required local-edge-connectivity unless $x$ is not adjacent to some vertex in $N_{G^{\prime}}(s) \backslash\left(N_{G^{\prime}}(i) \cup\{i\}\right)$. But then $x$ is adjacent to all vertices in $N_{G^{\prime}}(i)$, because $\operatorname{deg}_{G^{\prime}}(x) \geq \operatorname{deg}_{G^{\prime}}(i)$. Consider a vertex $z \in N_{G^{\prime}}(s) \backslash\left(N_{G^{\prime}}(i) \cup\{i\}\right)$ and recall that $\operatorname{deg}_{G^{\prime}}(z) \geq d_{n} \geq 2$. Because $x$ is adjacent to at least one endvertex of each edge of $G^{\prime}$, the vertex $z$ must be adjacent to some $u \in N_{G^{\prime}}(i)=N_{G^{\prime}}(x)$, and we provide the paths

```
ivxuzs,
is if is }\inE(\mp@subsup{G}{}{\prime})\mathrm{ ,
iys for }y\in\mp@subsup{N}{\mp@subsup{G}{}{\prime}}{}(i)\{v,s}
```

Subcase 2.2: $j \neq s$. Recall that if $x j \notin E\left(G^{\prime}\right)$, then $x$ is adjacent to all neighbors of $j$. Let us specify the feasible paths

```
\(i v x s z j\) for a suitable \(z \in N_{G^{\prime}}(j) \backslash\left(N_{G^{\prime}}(i) \cup\{i, v, x, s\}\right)\) if \(x j \in E\left(G^{\prime}\right)\),
\(i v x z j \quad\) for a suitable \(z \in N_{G^{\prime}}(j) \backslash\left(N_{G^{\prime}}(i) \cup\{i, v, s\}\right) \quad\) if \(x j \notin E\left(G^{\prime}\right)\),
ij if \(i j \in E\left(G^{\prime}\right)\),
iyj \(\quad\) for \(y \in\left(N_{G^{\prime}}(i) \cap N_{G^{\prime}}(j)\right)\),
iyszj \(\quad\) for \(y \in N_{G^{\prime}}(i) \backslash\left(N_{G^{\prime}}(j) \cup\{j, v, s\}\right)\) and suitable \(z \in N_{G^{\prime}}(j)\)
```

In all cases, we obtain that $G^{\prime}$ is maximally local-edge-connected.
Now we are ready to summarize the findings of this section.
Theorem 73. Consider natural numbers $c_{1} \geq \ldots \geq c_{n-1}$ that satisfy the shifted Erdős-Gallai inequalities for all $k \in\{1, \ldots, n-1\}$. Moreover, in case $c_{1}=n-1-\ell$ for $\ell:=\left|\left\{c_{i}: c_{i}=1\right\}\right|$, let the sum $\sum_{i=2}^{n-1-\ell} c_{i}$ be even. Then there is a graph that realizes the cut sequence $c_{1} \geq \ldots \geq c_{n-1}$.

Proof. First of all, suppose a graph $G$ is found that realizes the cut sequence $c_{1} \geq \ldots \geq c_{n-1-\ell}$. Then we can easily account for the remaining cut values $c_{n-\ell} \geq \ldots \geq c_{n-1}$, which are equal to one. We can simply append $\ell$ leafs to $G$. So we may assume that $\ell=0$, or equivalently that $c_{n-1} \geq 2$.

Let us consider the sequence given by $d_{i}:=c_{i}-1$ for $i \in\{1, \ldots, n-1\}$. Because $c_{1} \geq \ldots \geq c_{n-1}$ satisfies the shifted Erdős-Gallai inequalities, we know by Lemma 68 that the sequence $d_{1} \geq \ldots \geq d_{n-1}$ satisfies the Erdős-Gallai inequalities. In case their sum $\sum_{i=1}^{n-1} d_{i}$ is even, we may follow the process outlined in Figure 32. We can employ Algorithm 2 to generate a graph $G$ that realizes the degree sequence $d_{1} \geq \ldots \geq d_{n-1}$. If we add a dominating vertex to $G$, we obtain a graph that realizes the cut sequence $c_{1} \geq \ldots \geq c_{n-1}$. This is because of Corollary 62 and Lemma 64.

In case the sum $\sum_{i=1}^{n-1} d_{i}$ is odd, Algorithm 2 is not applicable directly. But Lemma 69 ensures that the numbers $d_{t}^{\prime}:=d_{t}-1$ for $t:=\max \left\{i: d_{i}=d_{1}\right\}$ and $d_{i}^{\prime}:=d_{i}$ for $i \in\{1, \ldots, n\} \backslash\{t\}$ constitute a graphic sequence. For the adapted sequence $d_{1}^{\prime} \geq \ldots \geq d_{n-1}^{\prime}$, Algorithm 2 outputs a realizing graph to which we append a dominating vertex $s$ to obtain the graph $H$. The degree sequence of $H$ is $n-1>d_{1}^{\prime}+1 \geq \ldots \geq d_{n-1}^{\prime}+1$. This sequence is almost the same as $n-1>c_{1} \geq \ldots \geq c_{n-1}$. The only differing number is $d_{t}^{\prime}+1=c_{t}-1$. Also note the assertion $n-1>d_{1}^{\prime}+1$ built in the previous lines. Indeed, otherwise $n-1=d_{1}^{\prime}+1=c_{1}$. Then $\sum_{i=2}^{n-1} c_{i}$ is even and

$$
\sum_{i=1}^{n-1} d_{i}=\sum_{i=1}^{n-1}\left(c_{i}-1\right)=\sum_{i=2}^{n-1} c_{i}+c_{1}-(n-1)=\sum_{i=2}^{n-1} c_{i}
$$

is even as well. But then Algorithm 2 is applicable and we are not in the case that we discuss in this paragraph. The key question remaining is whether we can adapt the graph $H$ to a graph whose cut sequence is $c_{1} \geq \ldots \geq c_{n-1}$. There is a vertex $x \in V(H)$ with $\operatorname{deg}_{H}(x)=d_{t}^{\prime}+1=c_{t}-1 \leq c_{1}-1 \leq n-3$ and $\operatorname{deg}_{H}(x)=d_{t}^{\prime}+1 \geq d_{1}^{\prime}$, by the definition of $t$. In case there exists an edge $v w \in E(H)$ such that $v, w \notin N_{H}(x)$, then Lemma 70 is applicable and provides a graph realizing the cut sequence $c_{1} \geq \ldots \geq c_{n-1}$. In case there is no such edge $v w$, then $x$ is adjacent to at least one endvertex of each edge of $H$. Denoting the vertex set $W:=V(H) \backslash\left(N_{H}(x) \cup\{x\}\right)$, we observe that $|W| \geq 2$, because $\operatorname{deg}_{H}(x) \leq n-3$. If for some $i, j \in W$ there are vertices $i^{\prime} \in N_{H}(i)$ and $j^{\prime} \in N_{H}(j) \backslash\left(N_{H}\left(i^{\prime}\right) \cup\left\{i^{\prime}\right\}\right)$, then Lemma 71 applies. Otherwise, $N_{H}(W)$ forms a clique and Lemma 72 applies. In all cases, we know how to construct a graph realizing the cut sequence $c_{1} \geq \ldots \geq c_{n-1}$.

Let us recall that the constructions addressed in Lemmas 70, 71, and 72 and consequently the constructions summarized in Theorem 73 all yield maximally local-edge-connected graphs. Such graphs are of particular interest in the article [34] of Dankelmann and Oellermann. Calling a graphic sequence edge-optimal if some realizing graph exists which is maximally local-edgeconnected, they raise the intriguing conjecture that all graphic sequences with smallest term at least two are edge-optimal. As a byproduct of this chapter, we obtain techniques for generating certain maximally local-edgeconnected graphs that possess additional connectivity properties. Although the proposed ideas do not resolve the mentioned conjecture, they might serve as tools for constructive approaches.

The program of this chapter is concluded by Theorem 73. It provides a class of cut sequences for which we can generate a realizing graph. The question


Figure 36: The structure of complete sun graphs
whether the conditions of Theorem 73 characterize all cut sequences has a negative answer. Figure 36 provides examples that do not satisfy the shifted Erdős-Gallai inequalities. The structure displayed there is that of complete sun graphs, surveyed by Brandstadt, Le, and Spinrad [12, Chapter 7]. For a formal definition, take for $k \in\{3,4, \ldots\}$ a graph $G$ whose vertex set $V(G):=V \cup W$ is partitioned into $V=\left\{v_{1}, \ldots, v_{k}\right\}$ and $W=\left\{w_{1}, \ldots, w_{k}\right\}$, and form the edge set as

$$
E(G):=\left\{v_{i} w_{i}, w_{i} v_{(i \bmod k)+1}: i \in\{1, \ldots, k\}\right\} \cup\left\{v_{i} v_{j}: i, j \in\{1, \ldots, k\}\right\} .
$$

The cut sequences belonging to that class are of the form

$$
\underbrace{k+1, \ldots, k+1}_{k-1 \text { times }}, \underbrace{2, \ldots, 2}_{k \text { times }} .
$$

For those, the left-hand side of the $(k-1)$ th shifted Erdős-Gallai inequality exceeds the respective right-hand side by

$$
\sum_{i=1}^{k-1}(k+1)-(k-1)^{2}-\sum_{i=k}^{2 k-1} \min \{1, k-1\}=k-2 .
$$

An interesting open problem is to characterize those graphs that violate the shifted Erdős-Gallai inequalities. One can observe that quite a few counterexamples are split graphs, or are somewhat similar to them. A split graph is a graph whose vertex set can be partitioned into a clique and an independent set, which is the case for the example in Figure 36. It might be fruitful to make precise how the counterexamples are related to that class.

| $n$ | connected graphs | counterexamples | fraction in $\%$ |
| :---: | ---: | ---: | ---: |
| $1-5$ | 31 | 0 | 0 |
| 6 | 112 | 2 | 1.79 |
| 7 | 853 | 13 | 1.52 |
| 8 | 261080 | 109 | 0.98 |
| 9 | 11716571 | 831 | 0.32 |
| 10 | 11989764 | 6790 | 0.06 |
| $1-10$ | 7745 | 0.06 |  |
| $n$ | tested graphs | edge probability | fraction in $\%$ |
| 20 | 1000000 | 0.2 | $\ll 0.00$ |
| 20 | 1000000 | 0.4 | $\ll 0.00$ |
| 20 | 1000000 | 0.6 | $\ll 0.00$ |
| 20 | 1000000 | 0.8 | $\ll 0.00$ |

Table 4.1: The fraction among connected graphs having a cut sequence violating the shifted Erdős-Gallai inequalities

Furthermore, the shifted Erdős-Gallai inequalities seem to characterize a large proportion of graphs. When checking all connected graphs on up to ten vertices, listed by McKay and Piperno [79], there appear 7745 out of 11989764 graphs that violate the shifted Erdős-Gallai inequalities. This is a proportion of merely $0.065 \%$ and the numbers in Table 4.1 indicate that this percentage decreases for increasing order $n$. There are exact results for $n \in\{1, \ldots, 10\}$ as well as results of some randomized tests for $n=20$ and varying probabilities for edges being present or not, as in the Erdős-Rényi model [41]. Encouraged by these empirical results, we may ask whether asymptotically all graphs satisfy the shifted Erdős-Gallai inequalities.

For another outlook, let us mention the field of complex network modeling, where constructive approaches as that of this section are of value. When realworld graphs are too large, hard to capture empirically, or simply unknown a priori, as in molecule or drug design, a key idea is to extract desired features and to generate representative network structures artificially. Such works are conducted by Chung and Lu [25], or in a refined version by Brissette, Liu, and Slota [14], where graphs with expected degree sequences are generated. Moreover, Heath and Parikh [64] focus on graphs with tunable clustering coefficients.


## Connectivity Matrices

## Contents

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In contrast to the previous chapters, our perspective on pairwise connectivity here changes to that of spectral graph theory. We ask structural questions about matrices whose entries reflect the pairwise connectivity relations. We thoroughly introduce the relevant matrices in Section 5.1. As spectral graph theory often relies on machinery from linear algebra, we collect the tools we make use of in the Appendix. We apply them in Section 5.2 to learn about the spectral properties of vertex-connectivity matrices. We focus on eigenvalue bounds, the attaining graphs, and a conjecture about the energy of such matrices. Remarkably, analogues to certain questions that remain open for vertex-connectivity matrices can all be resolved for edge-connectivity matrices. This is what we investigate in Section 5.3. Finally, in Section 5.4, we discuss how our results about connectivity matrices can be transferred to specific distance matrices, whose entries satisfy similar conditions as those of connectivity matrices. For an outlook, we touch on a version of Šoltés problem [96] and a variant of Cheeger's inequality [20], which suggests applications in spectral graph partitioning.

### 5.1 Graph matrices

Spectral graph theory is an intriguing subject concerned with the relationships between the structure of graphs and the spectral properties of matrices associated with graphs, which we assume to be nonempty throughout this chapter. The simplest relevant matrix is the adjacency matrix, defined for a graph $G$ on vertex set $\left\{v_{1}, \ldots, v_{n}\right\}$ as $A=\left[a_{i j}\right] \in\{0,1\}^{n \times n}$ having entries $a_{i j}=1$ if $v_{i} v_{j} \in E(G)$ and $a_{i j}=0$ if $v_{i} v_{j} \notin E(G)$. Denoting the diagonal matrix of vertex degrees by $D:=\operatorname{diag}\left(\operatorname{deg}\left(v_{1}\right), \ldots, \operatorname{deg}\left(v_{n}\right)\right)$, the Laplace matrix of $G$ is $L:=D-A$. In applications, this matrix sometimes is used in a normalized form $\mathcal{L}:=D^{-1 / 2} L D^{-1 / 2}$ and also the signless Laplace ma$\operatorname{trix} Q:=D+A$ is investigated intensively, of which Cvetković and Simić provide a comprehensive overview [29, 30, 31]. A general survey about spectral graph theory is given by Spielman [97]. In-depth works on the theory of graph spectra are the monographs of Chung [23], Brouwer and Haemers [16], or Cvetković, Rowlinson, and Simić [32]. To mention just a few of the various related applications, see Brin and Page [13] on how google ranks pages according to the Perron-Frobenius eigenvector of the world wide web's link matrix. We refer to Spielman and Teng [98] for spectral clustering methods and refer to Shuman, Narang, Frossard, Ortega, and Vandergheynst [94] for an overview about the emerging field of signal processing on graphs. Note that although the matrices from above are natural candidates for representing a graph's structure, the fundamental ideas of spectral graph theory are not strictly bound to the given definitions. Different graph matrices emphasize different structural properties, which we may aim to understand by spectral analysis. Our focus here shall be on connectivity matrices, introduced by Shikare, Malavadkar, Patekar, and Gutman [93].

Definition 74. For a graph on vertex set $\left\{v_{1}, \ldots, v_{n}\right\}$ its connectivity matrix is $P=\left[p_{i j}\right] \in\{0, \ldots, n-1\}^{n \times n}$ where $p_{i i}=0$ for $i \in\{1, \ldots, n\}$ and $p_{i j}$ is the maximum number of independent paths between vertices $v_{i}$ and $v_{j}$ for $i \neq j$. Analogously, the edge-connectivity matrix is $C=\left[c_{i j}\right] \in\{0, \ldots, n-1\}^{n \times n}$ where $c_{i i}=0$ for $i \in\{1, \ldots, n\}$ and $c_{i j}$ is the maximum number of edgedisjoint paths between vertices $v_{i}$ and $v_{j}$ for $i \neq j$.

Before we go on, let us think about how the matrices we introduced in Definition 74 can be determined. The edge-connectivity matrix of a graph can be read off directly from its Gomory-Hu tree, which always exists due to Theorem 14. According to Lemma 11, for two vertices we just have to follow the unique path connecting them in the Gomory-Hu tree and take the smallest edge weight on that path. For the actual flow computations, there


$$
P=\begin{aligned}
& 1 \\
& 1 \\
& 2 \\
& 3 \\
& 4
\end{aligned}\left[\begin{array}{lllll}
0 & 2 & 3 & 4 & 1 \\
2 & 0 & 2 & 1 & 1 \\
5 & 2 & 0 & 2 & 2 \\
1 & 1 & 2 & 0 & 2 \\
1 & 1 & 2 & 2 & 0
\end{array}\right]
$$

$$
C=\begin{gathered}
1 \\
1 \\
2 \\
2
\end{gathered}\left[\begin{array}{llll}
0 & 2 & 3 & 4 \\
3 \\
2 & 0 & 2 & 2 \\
2 \\
4 \\
2 & 2 & 0 & 2 \\
2 \\
2 & 2 & 2 & 0 \\
2 & 2 & 2 & 2 \\
0
\end{array}\right]
$$

Figure 37: The hourglass graph with its associated connectivity matrix $P$ and edge-connectivity matrix $C$
are several algorithms available, as we reviewed in the final paragraph of Section 2.3. For the connectivity matrix we need another approach, because there is no analogue to Gomory-Hu trees for vertex separators, as is pointed out by Benczúr [7]. To determine the connectivity matrix of a graph $G$, one can employ the following procedure, proposed by Ford and Fulkerson [47, Chapter 1]. We transform $G$ into a directed graph $D$ by replacing each edge $v w$ by two directed edges $(v, w)$ and $(w, v)$. In a second step, each vertex $v \in V(D)$ is replaced by two copies $v_{\text {in }}$ and $v_{\text {out }}$, connected by an edge ( $v_{\text {in }}, v_{\text {out }}$ ). All other edges $(v, w)$ are replaced by $\left(v_{\text {out }}, w_{\text {in }}\right)$. This construction is illustrated in Figure 38. Assigning capacity one to each edge of $D$, we compute the number of independent paths between two vertices $v, w \in V(G)$ as the maximum flow value between $v_{\text {out }}$ and $w_{\text {in }}$ in $D$. This works as intended as a flow in $D$ can pass a vertex $v_{\text {in }}$ only over the edge ( $v_{\text {in }}, v_{\text {out }}$ ). So each vertex can only be passed once.

In what follows, we rely on several facts from linear algebra, which are summarized in the Appendix. For basic concepts and the spectral theorem for symmetric matrices, we refer to Horn and Johnson [67, Chapters 0, 1, and 4]. To characterize extremal eigenvalues, we frequently use the principles by Rayleigh [100] and Ritz [88]. Further tools originate from the work of Perron [85] and Frobenius [48], of which Horn and Johnson [67, Chapter 8] give


Figure 38: An auxiliary network for computing a graph's connectivity matrix
an overview. The latter theory in particular refers to irreducible matrices. So let us recall that a matrix $A$ is called reducible if it is of the form

$$
P^{\top} A P=\left[\begin{array}{ll}
B & 0 \\
0 & C
\end{array}\right]
$$

where $B$ and $C$ are square matrices of size at least one, 0 is a matrix containing only zeros and $P$ is a permutation matrix. If a matrix is not reducible, it is called irreducible. So matrices of size one are irreducible and the adjacency, connectivity, or edge-connectivity matrix of a graph on at least two vertices is irreducible if and only if the graph is connected. Furthermore, we call a matrix $A=\left[a_{i j}\right] \in \mathbb{R}^{n \times n}$ nonnegative and write $A \geq 0$ if $a_{i j} \geq 0$ for all $i, j \in\{1, \ldots, n\}$ as well as $A>0$ if $a_{i j}>0$ for all $i, j \in\{1, \ldots, n\}$. For another matrix $B \in \mathbb{R}^{n \times n}$ we denote $A \geq B$ if $A-B \geq 0$. Our next goal is to see how the introduced matrix types are related.

Lemma 75. Let $P=\left[p_{i j}\right]$ be the connectivity matrix and $C=\left[c_{i j}\right]$ be the edge-connectivity matrix of a graph with given vertex labeling. Then $C \geq P$.

Proof. An independent $i-j$ path is automatically edge-disjoint. So $c_{i j} \geq p_{i j}$ for all $i, j \in\{1, \ldots, n\}$, which confirms our claim.

We refer to the eigenvalues of a matrix $A \in \mathbb{R}^{n \times n}$ by $\lambda_{1}(A) \geq \ldots \geq \lambda_{n}(A)$. The eigenvalues of the connectivity matrix of a graph $G$ on $n$ vertices are addressed by $\rho_{1}(G) \geq \ldots \geq \rho_{n}(G)$ and the eigenvalues of the corresponding edge-connectivity matrix by $\gamma_{1}(G) \geq \ldots \geq \gamma_{n}(G)$. Herein, we omit to specify the respective matrices or graphs if there is no need for a reference. Another spectral parameter that we investigate is the energy of a graph. This concept is introduced by Gutman [53] as the sum of the absolute values of the eigenvalues of a graph's adjacency matrix. Since his seminal work, a variety of applications and mathematical links have been discovered, of which Li, Shi, and Gutman [76] or Gutman and Furtula [54] give an overview. The roots of this subject stem from a correspondence between the orbital energy of $\pi$-electrons in conjugated hydrocarbon molecules and the eigenvalues of the adjacency matrix of a suitably constructed graph. Intuitively, one may see the energy as a measure of complexity of the underlying graph structure, potentially depending on specific edge weights, as in the case of connectivity matrices. This is discussed by Sinha, and de Weck [95] and has various applications. Those range from measurements in space engineering, as described by Pugliese and Nilchiani [87], to understanding network breakdowns in Alzheimer's disease, as discussed by Daianu, Mezher, Jahanshad, Hibar, Nir, Jack, Weiner, Bernstein, and Thompsonas [33]. As the definition of a
graph's energy, and so the investigations in that field, are not specifically limited to the spectrum of the adjacency matrix, we here denote

$$
\mathcal{E}(A):=\sum_{i=1}^{n}\left|\lambda_{i}(A)\right|
$$

for a general symmetric matrix $A \in \mathbb{R}^{n \times n}$. To address the connectivity or path energy of a graph $G$, we use the notion $\mathcal{E}_{P}(G):=\mathcal{E}(P(G))$ and to address its edge-connectivity or edge-path energy, we write $\mathcal{E}_{C}(G):=\mathcal{E}(C(G))$. For certain graph classes, it is not too hard to determine the corresponding connectivity or edge-connectivity matrix spectra entirely.

Example 76. The connectivity matrix of a uniformly $k$-connected graph $G$ on $n$ vertices is $k\left(\mathbb{1} \mathbb{1}^{\top}-I\right)$ for appropriate matrix sizes. A basis of eigenvectors for such a matrix is given by

$$
\left[\begin{array}{c}
1 \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
1
\end{array}\right],\left[\begin{array}{c}
1 \\
-1 \\
0 \\
\vdots \\
\vdots \\
0
\end{array}\right],\left[\begin{array}{c}
1 \\
0 \\
-1 \\
0 \\
\vdots \\
0
\end{array}\right], \ldots,\left[\begin{array}{c}
1 \\
0 \\
\vdots \\
\vdots \\
0 \\
-1
\end{array}\right] .
$$

The corresponding eigenvalues are $\rho_{1}=k(n-1)$ and $\rho_{2}=\ldots=\rho_{n}=-k$. Analogously, the eigenvalues of a uniformly $k$-edge-connected graph's edgeconnectivity matrix are $\gamma_{1}=k(n-1)$ and $\gamma_{2}=\ldots=\gamma_{n}=-k$.

### 5.2 Spectra of vertex-connectivity matrices

Let us now turn to the spectral properties of connectivity matrices. We focus on eigenvalue bounds and discuss a conjecture about connectivity energy. For our first eigenvalue property, we define for a graph $G$ and corresponding connectivity matrix $P=\left[p_{i j}\right] \in\{0, \ldots, n-1\}^{n \times n}$ the potential of vertex $i \in V(G)$ by

$$
p_{i}:=\sum_{j=1}^{n} p_{i j} .
$$

Lemma 77. Let $\rho$ be an eigenvalue of a connectivity matrix $P$ with corresponding eigenvector $x=\left[x_{1}, \ldots, x_{n}\right]^{\top}$. Then

$$
\text { either } \quad \sum_{i=1}^{n} x_{i}=\sum_{i=1}^{n} p_{i} x_{i}=0 \quad \text { or } \quad \rho=\left(\sum_{i=1}^{n} p_{i} x_{i}\right)\left(\sum_{i=1}^{n} x_{i}\right)^{-1} .
$$

Proof. By summing over the entries of the vector $P x$, we get

$$
\mathbb{1}^{\top} P x=\sum_{i=1}^{n}\left(\sum_{j=1}^{n} p_{i j}\right) x_{i}=\sum_{i=1}^{n} p_{i} x_{i}
$$

We are also given that $x$ is an eigenvector to $\rho$. Thus $\mathbb{1}^{\top} P x=\mathbb{1}^{\top} \rho x=\rho \mathbb{1}^{\top} x$ and therefore

$$
\rho \sum_{i=1}^{n} x_{i}=\sum_{i=1}^{n} p_{i} x_{i},
$$

which concludes the proof.
Shikare, Malavadkar, Patekar, and Gutman [93] provide for a connectivity matrix' spectral radius $\rho_{1}$ the bound $(n-1) \leq \rho_{1} \leq(n-1)^{2}$. The lower bound is attained if and only if the respective graph is a tree. The upper bound is attained if and only if the respective graph is complete. An overview about how the adjacency matrix' spectral radius is linked with the underlying graph's structure is given by Stevanović [99]. The Rayleigh principle and Lemma 77 will help us to obtain a slight improvement of the result from above. The attaining graphs are old friends from Chapter 3.

Theorem 78. Let $G$ be a $k$-connected, $\ell$-edge-connected graph on $n$ vertices whose minimum degree is $\delta$, whose connectivity matrix has spectral radius $\rho_{1}$, and whose edge-connectivity matrix has spectral radius $\gamma_{1}$. Then

$$
k(n-1) \leq \rho_{1} \leq 2|E(G)|-\delta .
$$

The lower bound is attained if and only if $G$ is uniformly $k$-connected, the upper bound is attained if and only if $G$ is $k$-regular and $k$-connected. Furthermore,

$$
\ell(n-1) \leq \gamma_{1} \leq 2|E(G)|-\delta .
$$

The lower bound is attained if and only if $G$ is uniformly $\ell$-edge-connected, the upper bound is attained if and only if $G$ is $\ell$-regular and $\ell$-edge-connected.

Proof. We only prove the assertion concerning a graph's connectivity matrix $\left[p_{i j}\right]$, because the statement about the edge-connectivity matrix follows completely analogously. Let us recall that $p_{i j} \leq \min \{\operatorname{deg}(i), \operatorname{deg}(j)\}$ for any $i, j \in\{1, \ldots, n\}$ and $p_{i i}=0$ for any $i \in\{1, \ldots, n\}$. This implies

$$
p_{i}=\sum_{j=1}^{n} p_{i j} \leq \sum_{\substack{j=1 \\ j \neq i}}^{n} \min \{\operatorname{deg}(i), \operatorname{deg}(j)\} \leq \sum_{\substack{j=1 \\ j \neq i}}^{n} \operatorname{deg}(j) \leq 2|E(G)|-\delta .
$$

Let now $x$ be an eigenvector to $\rho_{1}$ and observe that the connectivity matrix of a connected graph is nonnegative and irreducible. Statement (ii) of Theorem 106 says that $x$ has only positive or only negative entries. So Lemma 77 implies

$$
\begin{aligned}
\rho_{1} & =\left(\sum_{i=1}^{n} p_{i} x_{i}\right)\left(\sum_{i=1}^{n} x_{i}\right)^{-1} \\
& \leq\left(\sum_{i=1}^{n}\left(\sum_{j=1}^{n} \operatorname{deg}(j)-\delta\right) x_{i}\right)\left(\sum_{i=1}^{n} x_{i}\right)^{-1} \\
& =\sum_{j=1}^{n} \operatorname{deg}(j)-\delta=2|E(G)|-\delta .
\end{aligned}
$$

This inequality is sharp if and only if

$$
p_{i}=\sum_{j=1}^{n} p_{i j}=\sum_{j=1}^{n} \operatorname{deg}(j)-\delta=2|E(G)|-\delta \quad \text { for each } i \in\{1, \ldots, n\}
$$

These equations are certainly satisfied for $k$-regular $k$-connected graphs. On the other hand, suppose that $(*)$ holds but we are given a connectivity matrix where $p_{i j}<\operatorname{deg}(j)$ for some $i$ and $j$. Therefore, because $p_{i h} \leq \operatorname{deg}(h)$ for all $h \in\{1, \ldots, n\}$, relation $(\star)$ implies that $\operatorname{deg}(i)<\delta$, which is a contradiction. So $p_{i j}=\operatorname{deg}(j)$ for all $i, j \in\{1, \ldots, n\}$. Again $(\star)$ implies $\operatorname{deg}(i)=\delta$ for all $i \in\{1, \ldots, n\}$ and in turn $p_{i j}=\delta$ for all $i, j \in\{1, \ldots, n\}$. So graphs that attain the upper bound on $\rho_{1}$ have to be $k$-regular and $k$-connected. For the lower bound on $\rho_{1}$, let us set $y:=\frac{1}{\sqrt{n}} \mathbb{1}$ and apply Theorem 104 to obtain

$$
\begin{aligned}
\rho_{1} & =\max _{x \neq 0} \frac{x^{\top} P x}{x^{\top} x} \geq \frac{y^{\top} P y}{y^{\top} y}=\frac{1}{n} \mathbb{1}^{\top} P \mathbb{1} \\
& =\frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} p_{i j} \geq \frac{1}{n} \sum_{\substack{i=1}}^{n} \sum_{\substack{j=1 \\
j \neq i}}^{n} k=k(n-1) .
\end{aligned}
$$

Uniformly $k$-connected graphs attain this bound, as we have seen in Example 76. On the other hand, if we are given a $k$-connected graph that is not uniformly $k$-connected, then there exists some $p_{i j}>k$. But then the last inequality in the above calculation is strict and the bound is not attained.

We proceed by investigating the energy of the connectivity matrix. Shikare, Malavadkar, Patekar, and Gutman [93] first studied the connectivity energy, established basic properties, and raised the following conjecture which motivated larger parts of this section's contents.

Conjecture 79. The energy of a graph $G$ on $n$ vertices satisfies

$$
\mathcal{E}_{P}(G) \leq 2(n-1)^{2}
$$

The bound is attained if and only if $G$ is a complete graph.
The following lower bound is a slight refinement of a result by Ilić and Bašić [68] for connected graphs. We formulate it for $k$-connected and $\ell$-edgeconnected graphs. The proof essentially uses Perron-Frobenius arguments.

Lemma 80. For a $k$ connected and $\ell$-edge-connected graph $G$ on $n$ vertices there holds

$$
2 k(n-1) \leq \mathcal{E}_{P}(G)
$$

The bound is attained if and only if $G$ is uniformly $k$-connected. Furthermore,

$$
2 \ell(n-1) \leq \mathcal{E}_{C}(G)
$$

The bound is attained if and only if $G$ is uniformly $\ell$-edge-connected.
Proof. We only show the bound concerning the connectivity matrix $P=\left[p_{i j}\right]$, because the other bound follows analogously. The connectivity matrix of a uniformly $k$-connected graph $G$ on $n$ vertices is $k\left(\mathbb{1} \mathbb{1}^{\top}-I\right)$ for appropriate matrix sizes. The eigenvalues of this matrix are $k(n-1)$ with multiplicity one and $-k$ with multiplicity $n-1$, as we already observed in Example 76. This adds up to $\mathcal{E}_{P}(G)=2 k(n-1)$. The connectivity matrix $P$ of any $k$-connected graph $H$ satisfies $P \geq k\left(1 \mathbb{1}^{\top}-I\right)$. If $H$ is not uniformly $k$-connected, then there is some $p_{i j}>k$. By Corollary 107, we know that that then $\rho_{1}>k(n-1)$ and because of $\sum_{i=1}^{n} \rho_{i}=\operatorname{tr}(P)=0$, we conclude that $\mathcal{E}_{P}(H) \geq 2 \rho_{1}>2 k(n-1)$.

A typical approach to prove upper bounds on graph energies is that of Koolen and Moulten [74]. Ilić and Bašić [68] followed such an idea with a small error in their proof. If we correct the argument, we obtain the following bound.

Theorem 81. For a graph $G$ on $n$ vertices with degree sequence $d_{1} \geq \ldots \geq d_{n}$ holds

$$
\mathcal{E}_{P}(G) \leq \sqrt{2 n \sum_{i=1}^{n}(n-i) d_{i}^{2}}
$$

Proof. By the Cauchy-Schwarz inequality, we obtain

$$
\mathcal{E}_{P}(G)=\rho_{1}+\sum_{i=2}^{n} \rho_{i} \leq \rho_{1}+\sqrt{(n-1) \sum_{i=2}^{n} \rho_{i}^{2}}=\rho_{1}+\sqrt{(n-1)\left(\operatorname{tr}\left(P^{2}\right)-\rho_{1}^{2}\right)} .
$$

The trace of $P^{2}$ satisfies

$$
\begin{aligned}
\operatorname{tr}\left(P^{2}\right) & =\sum_{i=1}^{n} \sum_{j=1}^{n} p_{i j}^{2}=2 \sum_{i=1}^{n} \sum_{j=i+1}^{n} p_{i j}^{2} \leq 2 \sum_{i=1}^{n} \sum_{j=i+1}^{n} \min \left\{d_{i}, d_{j}\right\}^{2} \\
& \leq 2 \sum_{i=1}^{n} \sum_{j=i+1}^{n} d_{i}^{2}=2 \sum_{i=1}^{n}(n-i) d_{i}^{2}=: t
\end{aligned}
$$

Consequently, $\mathcal{E}_{P}(G) \leq f\left(\rho_{1}\right)$ where $f(x):=x+\sqrt{(n-1)\left(t-x^{2}\right)}$. Because of $0 \leq \rho_{1}$ and $\rho_{1}^{2} \leq \operatorname{tr}\left(P^{2}\right) \leq t$, we analyze $f$ for $x \in[0, \sqrt{t}]$. Let us determine

$$
\frac{\mathrm{d} f(x)}{\mathrm{d} x}=1-\frac{(n-1) x}{\sqrt{(n-1)\left(t-x^{2}\right)}}=1-\frac{\sqrt{n-1} x}{\sqrt{t-x^{2}}}
$$

So $\frac{\mathrm{d} f(x)}{\mathrm{d} x}=0$ if and only if $\sqrt{t-x^{2}}=\sqrt{n-1} x$, which is equivalent to $t=n x^{2}$. The solutions to this equation are $x_{ \pm}= \pm \sqrt{t / n}$. The root $x_{-}$is not contained in $[0, \sqrt{t}]$. At $x_{+}$, we evaluate

$$
\begin{aligned}
f\left(x_{+}\right) & =\sqrt{t / n}+\sqrt{(n-1)\left(t-\sqrt{t / n}^{2}\right)} \\
& =\sqrt{t / n}+\sqrt{(n-1) \frac{(n-1) t}{n}} \\
& =\sqrt{t / n}+(n-1) \sqrt{t / n}=n \sqrt{t / n}=\sqrt{t n}
\end{aligned}
$$

Because $f(0)=\sqrt{(n-1) t} \leq \sqrt{n t}=f\left(x_{+}\right)$and $f(\sqrt{t})=\sqrt{t} \leq \sqrt{n t}=f\left(x_{+}\right)$, we conclude that $\max _{x \in[0, \sqrt{t}]} f(x)=f\left(x_{+}\right)$and therefore

$$
\mathcal{E}_{P}(G) \leq f\left(x_{+}\right)=\sqrt{t n}=\sqrt{2 n \sum_{i=1}^{n}(n-i) d_{i}^{2}}
$$

However, the appearing vertex degrees $d_{i}$ may well equal $n-1$ and therefore the preceding result allows in general only for the estimate

$$
\begin{aligned}
\sqrt{2 n \sum_{i=1}^{n}(n-i) d_{i}^{2}} & \leq \sqrt{2 n \sum_{i=1}^{n}(n-i)(n-1)^{2}} \\
& =\sqrt{2 n(n-1)^{2} \frac{n(n-1)}{2}}=n(n-1)^{3 / 2}
\end{aligned}
$$

which exceeds the conjectured bound of $2(n-1)^{2}$ for all $n \in \mathbb{N}$. Or, to put


Figure 39: A graph for which $D+P$ is not positive semidefinite
it positively, denoting the maximum degree by $\Delta$ and setting

$$
\sqrt{2 n \sum_{i=1}^{n}(n-i) \Delta^{2}}=\sqrt{2 n \Delta^{2} \frac{n(n-1)}{2}}=n \Delta \sqrt{n-1}
$$

equal to $2(n-1)^{2}$, we find that Conjecture 79 holds for any graph whose maximum degree satisfies $\Delta \leq 2(n-1)^{3 / 2} n^{-1}$. Another attempt, eventually ineffective, to approach Conjecture 79 can be summarized as follows. Patekar and Shikare [84] suggest the matrix $D+P$ as a connectivity analogue for the signless Laplacian. Its classical variant $D+A$ is positive semidefinite, implying $\lambda_{n}(A) \geq \lambda_{n}(-\Delta I+D+A)=-\Delta+\lambda_{n}(D+A) \geq-(n-1)$, where $\Delta$ denotes the largest degree of the respective graph. If we had such a bound on the smallest connectivity matrix eigenvalue, by $\sum_{i=1}^{n} \rho_{i}=\operatorname{tr}(P)=0$, we would obtain $\mathcal{E}_{P}(G) \leq 2 \sum_{i: \rho_{i}<0}^{n}\left|\rho_{i}\right| \leq \sum_{i=2}^{n}\left|\rho_{n}\right| \leq 2(n-1)^{2}$. But $D+P$ is not guaranteed to be positive semidefinite. This can be seen from the graph in Figure 39 for which

$$
\left.P+D=\begin{array}{c}
1 \\
2
\end{array} \begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
3 & 4 & 4 & 4 & 3 & 3 \\
4 & 4 & 4 & 4 & 3 & 3 \\
4 & 4 & 4 & 3 & 3 & 3 \\
4 & 4 & 3 & 4 & 3 & 3 \\
3 & 3 & 3 & 3 & 3 & 3 \\
3 & 3 & 3 & 3 & 3 & 3
\end{array}\right] .
$$

Because this matrix has a negative principal minor

$$
\operatorname{det}\left[\begin{array}{lll}
4 & 4 & 4 \\
4 & 4 & 3 \\
4 & 3 & 4
\end{array}\right]=\operatorname{det}\left[\begin{array}{rrr}
4 & 0 & 0 \\
4 & 0 & -1 \\
4 & -1 & 0
\end{array}\right]=4 \operatorname{det}\left[\begin{array}{rr}
0 & -1 \\
-1 & 0
\end{array}\right]=-4
$$

it cannot be positive semidefinite. Also the slightly weaker hypothesis that the minimum eigenvalue of $P$ is bounded by at least $-(n-1)$ turns out


Figure 40: A graph whose corresponding connectivity matrix has an eigenvalue smaller than $-(n-1)$
to be false, for which Figure 40 provides an example. Recalling Lemma 102, this can be seen by investigating the matrix $P+(n-1) I$, which has the minor

$$
\operatorname{det}\left[\begin{array}{cccccc}
8 & 6 & 6 & 7 & 7 & 7 \\
6 & 8 & 6 & 7 & 7 & 7 \\
6 & 6 & 8 & 7 & 7 & 7 \\
7 & 7 & 7 & 8 & 7 & 7 \\
7 & 7 & 7 & 7 & 8 & 7 \\
7 & 7 & 7 & 7 & 7 & 8
\end{array}\right]=-4
$$

But note that, up to now, we identifed only examples where the smallest eigenvalue is just slightly below the value $-(n-1)$. For the example above, the smallest eigenvalue is about -8.06 . So there seems to be some leeway to the general lower bound known for the smallest eigenvalue of a real symmetric matrix $A$ whose entries are in a given interval $[a, b]$. Such a bound is shown by Zhan [108]. For $n \geq 2, a<b$, and $|a|<b$ the smallest eigenvalue satisfies

$$
\lambda_{n}(A) \geq \begin{cases}n(a-b) / 2 & \text { if } n \text { is even } \\ \left(n a-\sqrt{a^{2}+\left(n^{2}-1\right) b^{2}}\right) / 2 & \text { if } n \text { is odd }\end{cases}
$$

Up to simultaneous permutations of rows and columns, the attaining matrices are

$$
\begin{array}{ll}
{\left[\begin{array}{ll}
a\left[11^{\top}\right]_{\frac{n}{}} & b\left[\mathbb{1} 1^{\top}\right]_{\frac{n}{2}} \\
b\left[\mathbb{1} 1^{\top}\right]_{\frac{n}{2}} & a\left[\mathbb{1} 1^{\top}\right]_{\frac{n}{2}}
\end{array}\right] \text { if } n \text { is even, }} \\
{\left[\begin{array}{ll}
a\left[11^{\top}\right]_{\frac{n-1}{}} & b\left[11^{\top}\right]_{\frac{n-1}{2}}, \frac{n+1}{2} \\
b\left[\mathbb{1} 1^{\top}\right]_{\frac{n+1}{2}, \frac{n-1}{2}} & a\left[\mathbb{1} 1^{\top}\right]_{\frac{n+1}{2}}
\end{array}\right] \quad \text { if } n \text { is odd, }}
\end{array}
$$

where the subscripts denote the respective matrix sizes. In the case of connectivity matrices, where $a=0$ and $b=n-1$, we obtain $-(n-1) n / 2$ if $n$ is even and $-(n-1) \sqrt{\left(n^{2}-1\right)} / 2$ if $n$ is odd. This bound is quadratic in $n$, in contrast to the nearly linear behavior that we observed for graphs on up to ten vertices. Also, the matrices attaining Zhan's bound, for $a=0$ and $b=n-1$, cannot be connectivity matrices, because an entry equal to $n-1$ means that there are two vertices connected by $n-1$ independent paths, contradicting $a=0$. So there is room for improvement.

### 5.3 Spectra of edge-connectivity matrices

In contrast to connectivity matrices, edge-connectivity matrices possess certain structural properties which allow for stronger spectral estimates. This allows to answer questions about the energy, whose analogues for connectivity matrices remained open in the preceding section. Those properties stem from the ultrametric inequality, established by Gomory and Hu [50], which we discussed in Section 2.3. Our first goal in this section is to develop an equivalent formulation for the ultrametric inequality that is more amenable to our spectral considerations. In what follows, for a set $X$, we denote by $X^{2}:=X \times X$ the Cartesian product. For $X \subseteq\{1, \ldots, n\}$, we denote by $\mathbb{1}_{X}$ the vector $\left[x_{1}, \ldots, x_{n}\right]^{\top}$ where $x_{i}=1$ if $i \in X$ and $x_{i}=0$ if $i \notin X$. For a matrix $A \in \mathbb{R}^{n \times n}$ and index sets $X, Y \subseteq\{1, \ldots, n\}$, we denote by $A_{X Y}:=\left[a_{i j}\right]_{i \in X, j \in Y}$ the submatrix that results from $A$ by deleting the rows of the index set $\{1, \ldots, n\} \backslash X$ and columns of the index set $\{1, \ldots, n\} \backslash Y$. If $X=Y$, we write $A_{X^{2}}$ for $A_{X Y}$.
Definition 82. For a symmetric matrix $C=\left[c_{v w}\right] \in \mathbb{R}^{n \times n}$ and some number $\ell \in \mathbb{R}$ we define the superlevel set of $C$ for the level $\ell$ by

$$
S_{\ell}(C):=\left\{(i, j) \in\{1, \ldots, n\}^{2}: c_{i j} \geq \ell\right\}
$$

The matrix $C$ is called block diagonally layered, or layered for short, if for each level $\ell \in \mathbb{R}$ there is a set $\mathcal{T}_{\ell}(C)$ of pairwise disjoint subsets of $\{1, \ldots, n\}$ satisfying

$$
S_{\ell}(C)=\bigcup_{X \in \mathcal{T}_{\ell}(C)} X^{2} .
$$

Remark 83. By Definition 82, a layered matrix with distinct values of entries $\ell_{0}<\ldots<\ell_{k}$ permits the decomposition

$$
C=\ell_{0} \mathbb{1}_{\{1, \ldots, n\}} \mathbb{1}_{\{1, \ldots, n\}}^{\top}+\sum_{i=1}^{k} \sum_{X \in \mathcal{T}_{i}(C)}\left(\ell_{i}-\ell_{i-1}\right) \mathbb{1}_{X} \mathbb{1}_{X}^{\top} .
$$

When setting $s_{i}:=\ell_{i}-\ell_{i-1}$ for $i \in\{1, \ldots, n\}$ as well as $s_{1}:=\ell_{0}$, we may also write more compactly

$$
C=\sum_{i=0}^{k} \sum_{X \in \mathcal{T}_{e_{i}}(C)} s_{i} \mathbb{1}_{X} \mathbb{1}_{X}^{\top} .
$$

In fact, this decomposition into a sum of block diagonal matrices is the reason for the term block diagonally layered.

Example 84. To become acquainted with the previous definition, let us check for

$$
C={ }_{3}^{1}\left[\begin{array}{llll}
3 & 2 & 3 & 4 \\
3 & 3 & 0 & 0 \\
3 & 4 & 0 & 0 \\
0 & 0 & 1 & 1 \\
0 & 0 & 1 & 3
\end{array}\right] \quad \text { that } \quad S_{\ell}(C)= \begin{cases}\{1,2,3,4\}^{2} & \text { for } \quad \ell \leq 0 \\
\{1,2\}^{2} \cup\{3,4\}^{2} & \text { for } 0<\ell \leq 1 \\
\{1,2\}^{2} \cup\{4\}^{2} & \text { for } 1<\ell \leq 3 \\
\{2\}^{2} & \text { for } 3<\ell \leq 4 \\
\emptyset & \text { for } 4<\ell\end{cases}
$$

The corresponding decomposition into block diagonal matrices is

$$
C=\left[\begin{array}{llll}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{array}\right]+\left[\begin{array}{llll}
1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 \\
0 & 0 & 1 & 1
\end{array}\right]+\left[\begin{array}{llll}
2 & 2 & 0 & 0 \\
2 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2
\end{array}\right]+\left[\begin{array}{llll}
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{array}\right] .
$$

Theorem 85. A nonnegative layered matrix $C=\left[c_{i j}\right] \in \mathbb{R}^{n \times n}$ is positive semidefinite.

Proof. Let us denote all distinct values of entries of $C$ by $\ell_{0}<\ldots<\ell_{k}$. Because $C$ is nonnegative, also the numbers $s_{i}:=\ell_{i}-\ell_{i-1}$ for $i \in\{1, \ldots, n\}$ and $s_{1}:=\ell_{0}$ are nonnegative. Furthermore, the matrix $\mathbb{1}_{X} \mathbb{1}_{X}^{\top}$ is positive semidefinite, because $y^{\top} \mathbb{1}_{X} \mathbb{1}_{X}^{\top} y=\left(\mathbb{1}_{X}^{\top} y\right)^{\top}\left(\mathbb{1}_{X}^{\top} y\right)=\left\|\mathbb{1}_{X}^{\top} y\right\|_{2}^{2} \geq 0$ for $y \in \mathbb{R}^{n}$. In view of Remark 83,

$$
C=\sum_{i=0}^{k} \sum_{X \in \mathcal{T}_{i}(C)} s_{i} \mathbb{1}_{X} \mathbb{1}_{X}^{\top}
$$

is positive semidefinite, because it is a nonnegative linear combination of positive semidefinite matrices.

The key to our spectral questions is the following link between layered matrices and the ultrametric inequality, established by Gomory and Hu. Note that in the subsequent statement we neither require the respective matrices to stem from graphs nor that the diagonal entries have to be zero.

Theorem 86. A symmetric matrix $C \in \mathbb{R}^{n \times n}$ satisfies $c_{i k} \geq \min \left\{c_{i j}, c_{j k}\right\}$ for all $i, j, k \in\{1, \ldots, n\}$ if and only if it is layered.

Proof. We begin by showing that a layered matrix $C$ satisfies the ultrametric inequality. So take arbitrary $i, j, k \in\{1, \ldots, n\}$ and set $\min \left\{c_{i j}, c_{j k}\right\}=: \ell$. Defining $\mathcal{T}_{\ell}(C)$ as in Definition 82 , some $X \in \mathcal{T}_{\ell}(C)$ with $(i, j) \in X^{2}$ has to exist. This is because $c_{i j} \geq \ell$. Also, there is a set $Y \in \mathcal{T}_{\ell}(C)$ that contains $j$ and $k$, as $c_{j k} \geq \ell$. Thus $j \in X \cap Y$, which implies that $X=Y$, because the sets in $\mathcal{T}_{\ell}(C)$ are pairwise disjoint. Consequently, the set $X$ contains all three indices $i, j$, and $k$. This implies that $c_{i k} \geq \ell$, which was to be shown.

Suppose now that $C$ satisfies $c_{i k} \geq \min \left\{c_{i j}, c_{j k}\right\}$ for all $i, j, k \in\{1, \ldots, n\}$. We show that $C$ is layered by induction on $n$, which is certainly true for $n=1$. For $n \geq 2$ we set $m:=\min \left\{c_{i j}: i, j \in\{1, \ldots, n\}\right\}$. Let $k \in\{1, \ldots, n\}$ be a column of $C$ that contains an entry equal to $m$. For it, we define

$$
X:=\left\{i \in\{1, \ldots, n\}: i \neq k \text { and } c_{i k}=m\right\} \text { and } Y:=\{1, \ldots, n\} \backslash X .
$$

We are given that $c_{k k} \geq \min \left\{c_{k j}, c_{j k}\right\}=c_{j k}$ for all $j, k \in\{1, \ldots, n\}$. As $n \geq 2$, this shows that $X \neq \emptyset$. Furthermore, $Y \neq \emptyset$, as $k \in Y$. We now take arbitrary $i \in X$ and $j \in Y$ and aim to show that $c_{i j}=m$. By the choice of $i$ and $j$, we have $c_{i k} \leq c_{j k}$. Herein, by the definition of $X$ and $Y$, equality is only possible if $c_{k k}=m$ and $Y=\{k\}$. So if $j \neq k$, then $c_{i k}<c_{j k}$ and thus $c_{i k} \geq \min \left\{c_{i j}, c_{j k}\right\}=c_{i j}$. Altogether, we obtain $c_{i j} \leq c_{i k} \leq c_{j k}$, which is trivially fulfilled in the remaining case where $j=k$. By Lemma 9 , we conclude that $c_{i j}=c_{i k}=m$. We have shown that, by a suitable simultaneous permutation of rows and columns, $C$ is of the form

$$
\left.C=\begin{array}{cc}
X & X \\
Y & Y \mathbb{1} \mathbb{1}^{\top} \\
m \mathbb{1} \mathbb{1}^{\top} & B
\end{array}\right]
$$

This implies that for $\ell>m$ the superlevel set $S_{\ell}(C)$ is contained in $X^{2} \cup Y^{2}$. The submatrices $A=C_{X^{2}}$ and $B=C_{Y^{2}}$ satisfy the ultrametric inequality. So they are layered, by induction. Consequently,

$$
S_{\ell}(C)=S_{\ell}\left(C_{X^{2}}\right) \cup S_{\ell}\left(C_{Y^{2}}\right) \quad \text { for } \ell>m
$$

and because $S_{m}(C)=\{1, \ldots, n\}$, we obtain that $C$ is layered.
The preceding proof also allows to verify that a layered matrix can have at most $2 n-1$ different entries, which is certainly true for $n=1$. Denoting the number of different entries a matrix can have by $\#(\cdot)$, we see
from the preceding proof that $\#(C) \leq \#(A)+\#(B)+1$. Because $A \in \mathbb{R}^{k \times k}$ and $B \in \mathbb{R}^{(n-k) \times(n-k)}$ for some $k \in\{1, \ldots, n-1\}$, we conclude by induction that $\#(C) \leq 2 k-1+2(n-k)-1+1=2 n-1$. Lemma 7 states that the off-diagonal entries of an edge-connectivity matrix satisfy the ultrametric inequality. Thus all entries except the diagonal ones exhibit a layered structure and at most $n-1$ different numbers can occur as matrix entries. This resembles a classical result by Gomory and Hu [50], presented in connection with Theorem 14, saying that $n-1$ minimum cut computations suffice to obtain the minimum cut values for all pairs of vertices of a graph.

Theorem 87. Consider a graph $G$ on $n$ vertices, its edge-connectivity matrix $C=\left[c_{i j}\right]$, and set $c_{i \text { max }}:=\max \left\{c_{i j}: j \in\{1, \ldots, n\} \backslash\{i\}\right\}$. Then the matrix $\bar{C}:=C+\operatorname{diag}\left(c_{1 \max }, \ldots, c_{n \max }\right)$ is positive semidefinite.

Proof. All off-diagonal entries of the matrix $C$, and so of $\bar{C}$, satisfy the ultrametric inequality by Theorem 7 . By the definition of $\bar{C}=\left[\overline{c_{i j}}\right]$, also $\overline{c_{i i}} \geq \overline{c_{i j}}$ for all $i, j \in\{1, \ldots, n\}$. Thus $\bar{C}$ is layered by Theorem 86. Because it is also real and nonnegative, it is positive semidefinite by Theorem 85 .

Theorem 87 also includes the fact that taking a graph's edge-connectivity matrix $C=\left[c_{i j}\right]$ and adding its degree matrix $D=\operatorname{diag}\left(d_{1}, \ldots, d_{n}\right)$ gives a positive semidefinite matrix $C+D$. This is because $c_{i j} \leq \min \left\{d_{i}, d_{j}\right\}$ for all $i, j \in\{1, \ldots, n\}$ and therefore $c_{i \max } \leq d_{i}$ for all $i \in\{1, \ldots, n\}$. Note that this is different from the situation with vertex-connectivity matrices, for which Figure 39 shows a graph with indefinite matrix $P+D$. Furthermore, layered matrices, and so edge connectivity matrices, make it easier to retrieve some spectral information directly from the matrix entries.

Theorem 88. Consider a symmetric matrix $C=\left[c_{i j}\right] \in \mathbb{R}^{n \times n}$ whose offdiagonal entries satisfy the ultrametric inequality. The matrix $C$ has an eigenvalue $c_{i i}-c_{i j}$, where $i, j \in\{1, \ldots, n\}$ with $i \neq j$, with corresponding eigenvector $e_{i}-e_{j}$ if and only if $c_{i i}=c_{j j}$ and $c_{i j}$ is the maximum entry among the off-diagonal elements of row $i$ and column $j$.

Proof. Suppose $C$ has an eigenvalue $c_{i i}-c_{i j}$ with corresponding eigenvector $e_{i}-e_{j}$ for some $i, j \in\{1, \ldots, n\}$. Considering row $k \in\{1, \ldots, n\} \backslash\{i, j\}$ of the eigenequation $C\left(e_{i}-e_{j}\right)=\left(c_{i i}-c_{i j}\right)\left(e_{i}-e_{j}\right)$, we find $c_{k i}-c_{k j}=0$. We conclude that $c_{i k}=c_{k j}=\min \left\{c_{i k}, c_{k j}\right\} \leq c_{i j}$, by the symmetry of $C$ and the ultrametric inequality. So $c_{i j}$ is the largest off-diagonal entry of row $i$ and column $j$. Considering row $j$ of the equation $C\left(e_{i}-e_{j}\right)=\left(c_{i i}-c_{i j}\right)\left(e_{i}-e_{j}\right)$, we find $c_{j i}-c_{j j}=-\left(c_{i i}-c_{i j}\right)$ and thus $c_{i i}=c_{j j}$, what remained to be shown.

For the other direction, let $i, j \in\{1, \ldots, n\}$ with $i \neq j$ and $c_{i i}=c_{j j}$ such that $c_{i j}$ is the maximum off-diagonal of row $i$ and column $j$. Employing the ultrametric inequality, we find for each $k \in\{1, \ldots, n\}$ with $k \neq i$ and $k \neq j$ that

$$
c_{i k} \geq \min \left\{c_{i j}, c_{j k}\right\}=c_{j k} \geq \min \left\{c_{j i}, c_{i k}\right\}=c_{i k}
$$

From this chain, we obtain $c_{i k}=c_{j k}$ and therefore

$$
\left(C\left(e_{i}-e_{j}\right)\right)_{k}= \begin{cases}0 & \text { if } k \neq i \text { and } k \neq j, \\ c_{i i}-c_{i j} & \text { if } k=i, \\ c_{j i}-c_{j j}=c_{i j}-c_{i i} & \text { if } k=j\end{cases}
$$

So $C$ has an eigenvalue $c_{i i}-c_{i j}$ with corresponding eigenvector $e_{i}-e_{j}$.
Theorem 89. Consider a nonnegative symmetric matrix $C=\left[c_{i j}\right] \in \mathbb{R}^{n \times n}$ with only zeros on its diagonal whose off-diagonal entries satisfy the ultrametric inequality. Then the smallest eigenvalue of $C$ satisfies

$$
\lambda_{n}=-\max \left\{c_{i j}: i, j \in\{1, \ldots, n\}\right\} .
$$

Proof. Let us consider indices $k, \ell \in\{1, \ldots, n\}$ with $k \neq \ell$ chosen such that $c_{k, \ell}=\max \left\{c_{i j}: i, j \in\{1, \ldots, n\}\right\}=: c_{\text {max }}$. Theorem 88 implies that $C$ has an eigenvalue $c_{k k}-c_{k \ell}=-c_{\max }$. Furthermore, the matrix $c_{\max } I+C$ is nonnegative and it is layered, by Theorem 86. So $c_{\max } I+C$ is positive semidefinite, by Theorem 85, and we obtain for the smallest eigenvalue of $C$ that

$$
\lambda_{n}(C)=\lambda_{n}\left(-c_{k \ell} I+c_{k \ell} I+C\right)=-c_{k \ell}+\lambda_{n}\left(c_{k \ell} I+C\right) \geq-c_{k \ell} .
$$

This implies that $\lambda_{n}=-c_{k \ell}$.
Corollary 90. Consider a graph on $n$ vertices and its corresponding edgeconnectivity matrix $C=\left[c_{i j}\right]$. Then the smallest eigenvalue of $C$ satisfies

$$
\gamma_{n}=-\max \left\{c_{i j}: i, j \in\{1, \ldots, n\}\right\} .
$$

Proof. The matrix $C$ is clearly nonnegative and has only zeros on its diagonal. Furthermore, Lemma 7 says that the off-diagonal entries of edge-connectivity matrices satisfy the ultrametric inequality. So Theorem 89 applies.

Note that this result implies the bound $\gamma_{n} \geq-(n-1)$, whereas such a bound does not hold for $\rho_{n}$. We can now proceed with a tight bound on the energy of a graph's edge-connectivity matrix.

Theorem 91. Consider a nonnegative symmetric matrix $C=\left[c_{i j}\right] \in \mathbb{R}^{n \times n}$ with only zeros on its diagonal whose off-diagonal entries satisfy the ultrametric inequality and denote $c_{\max }:=\max \left\{c_{i j}: i, j \in\{1, \ldots, n\}\right\}$. Then

$$
\mathcal{E}(C) \leq 2(n-1) c_{\max } .
$$

Herein, equality holds if and only if $C=c_{\max }\left(11^{\top}-I\right)$.
Proof. We are given that $\operatorname{tr}(C)=0$ and we assume that $C$ has at least one positive eigenvalue, as otherwise all eigenvalues must be zero, in which case our statement is certainly true. Then, by Theorem 89,

$$
\begin{aligned}
\mathcal{E}(C) & =\sum_{i=1}^{n}\left|\lambda_{i}\right|=2 \sum_{\substack{i=1 \\
\lambda_{i}<0}}^{n}\left|\lambda_{i}\right| \leq 2 \sum_{\substack{i=1 \\
\lambda_{i}<0}}^{n}\left|\lambda_{n}\right| \\
& =2\left|\left\{i \in\{1, \ldots, n\}: \lambda_{i}<0\right\}\right|\left|\lambda_{n}\right| \\
& \leq 2(n-1)\left|\lambda_{n}\right| \leq 2(n-1) c_{\max },
\end{aligned}
$$

which is the bound we stated. In Example 76, we determined the spectrum for matrices of the form $c_{\max }\left(11^{\top}-I\right)$ and obtained $\lambda_{1}=(n-1) c_{\max }$ as well as $\lambda_{2}=\ldots=\lambda_{n}=-c_{\max }$, which shows that our bound is attained for matrices of the form $c_{\max }\left(\mathbb{1} \mathbb{1}^{\top}-I\right)$. To see that those are the only attaining matrices, suppose first that $C$ has two or more positive eigenvalues. Then $\mathcal{E}(C)=2\left|\left\{i \in\{1, \ldots, n\}: \lambda_{i}<0\right\}\right|\left|\lambda_{n}\right| \leq 2(n-2) c_{\text {max }}$. In other words, only matrices with exactly one positive eigenvalue can attain the upper bound on $\mathcal{E}(C)$. So consider a matrix $A \neq c_{\max }\left(\mathbb{1} \mathbb{1}^{\top}-I\right)$ with only zeros on its diagonal and maximum entry $c_{\text {max }}$ that has exactly one positive eigenvalue. Because of $\operatorname{tr}(A)=0$, we have $\mathcal{E}(A)=2 \lambda_{1}$. The largest eigenvalue of such a matrix satisfies $\lambda_{1}(A)<(n-1) c_{\max }$, by Corollary 107. Consequently, $\mathcal{E}(A)=2 \lambda_{1}<2(n-1) c_{\text {max }}$. This shows that equality in the stated bound holds only for matrices of the form $c_{\max }\left(\mathbb{1} \mathbb{1}^{\top}-I\right)$.

Corollary 92. Let $G$ be a graph on $n$ vertices with maximum local edgeconnectivity $k$, that is, containing no pair of vertices connected by more than $k$ edge-disjoint paths. Then its edge-connectivity energy satisfies

$$
\mathcal{E}_{C}(G) \leq 2 k(n-1)
$$

Herein, equality holds if and only if $G$ is uniformly $k$-edge-connected.
In the previous Theorem, the parameter $k$ can be at most $n-1$, providing us with a general upper bound of $2(n-1)^{2}$ on the energy of edge-connectivity
matrices, which is the value from Conjecture 79 about the energy of vertexconnectivity matrices. This bound is attained only for the complete graph on $n$ vertices. In fact, a complete graph's edge-connectivity matrix also attains the upper bound $(n-1)^{2}$ we found for the largest eigenvalue and simultaneously the lower bound $-(n-1)$ we found for the smallest eigenvalue. This provides a tight upper bound on the spread of edge-connectivity matrices, which is the largest distance between any two eigenvalues of a matrix, specializing to $s(A):=\lambda_{1}(A)-\lambda_{n}(A)$ when $A$ is a symmetric matrix.

Corollary 93. The spread of an edge-connectivity matrix $C$ satisfies

$$
s(C) \leq(n-1)(n-2)
$$

This answers a special case of an open problem formulated by Zhan [108, Problem 2] for which Fallat and Xing [44] provide a more detailed conjecture. There, the authors ask for the spread of a general symmetric matrix with entries in a given interval. Note that our result is owed to the fact that our bounds on the smallest and largest eigenvalue are attained by the same matrix, which is a behavior one cannot expect in general.

Our next goal is to refine our energy results. For this purpose, let us recall the notion of an equitable matrix partition, as presented by Brouwer and Haemers [16, Chapter 2]. This is a partition of a matrix $A$ of the form

$$
A=\left[\begin{array}{ccc}
A_{11} & \cdots & A_{1 k} \\
\vdots & \ddots & \ddots \\
A_{k 1} & \cdots & A_{k k}
\end{array}\right]
$$

where $A_{i j}$ are blocks with constant row sums $q_{i j}$. The matrix defined by $Q:=\left[q_{i j}\right]_{i, j=1}^{k}$ is the corresponding equitable quotient matrix of $A$. You, Yang, So, and $\mathrm{Xi}[107]$ show that the spectrum of $Q$ is a subset of the spectrum of $A$ and that their spectral radii are the same if $A$ is nonnegative.

Theorem 94. Consider a symmetric matrix $C=\left[c_{i j}\right] \in \mathbb{R}^{n \times n}$ and denote by $c_{i \text { max }}:=\max \left\{c_{i j}: j \in\{1, \ldots, n\} \backslash\{i\}\right\}$ the largest off-diagonal entry for each row $i \in\{1, \ldots, n\}$. If the off-diagonal entries of $C$ satisfy the ultrametric inequality, then the following statements hold.
(i) With $C$ we are given an equivalence relation on $\{1, \ldots, n\}$ via

$$
i \sim j: \Leftrightarrow i=j \text { or } c_{i j}=c_{i \max }=c_{j \max }
$$

(ii) If $i \sim j$ and $k \sim \ell$, then $c_{i k}=c_{j \ell}$. In other words, an entry $c_{i j}$ depends only on the equivalence classes of $i$ and $j$.
(iii) Denote by $X_{1}, \ldots, X_{m}$ the equivalence classes induced by the relation $\sim$. If $c_{i i}=c_{j j}$ for all elements $i, j \in\{1, \ldots, n\}$ with $i \sim j$, then the submatrices $C_{X_{i} X_{j}}$ form an equitable partition of $C$. The corresponding equitable quotient matrix is $Q=\left[q_{k}\right]_{k, \ell=1}^{m}$ where

$$
q_{k \ell}= \begin{cases}c_{i j}\left|X_{\ell}\right| & \text { if } k \neq \ell, \text { for some } i \in X_{k} \text { and } j \in X_{\ell}, \\ c_{i i}+c_{i \max }\left(\left|X_{\ell}\right|-1\right) & \text { if } k=\ell, \text { for some } i \in X_{\ell} .\end{cases}
$$

(iv) Assume, in addition, that $C$ is nonnegative and has only zeros on its diagonal. Using $c\left(X_{k}\right)$ to denote the common value $c_{i \max }$ for $i \in X_{k}$, it holds

$$
\mathcal{E}(C)=\mathcal{E}(Q)+\operatorname{tr}(Q) \geq 2 \sum_{k=1}^{m}\left(\left|X_{k}\right|-1\right) c\left(X_{k}\right) .
$$

This inequality is attained if and only if $Q$ has no negative eigenvalues.

Proof. Let us first prove Item (i). By definition, the relation $\sim$ is reflexive. The symmetry follows directly from the symmetry of $C$. To verify transitivity, let $i \sim j$ and $j \sim k$. We assume that $i, j, k$ are pairwise distinct, as otherwise there is nothing to show. So we have $c_{i \text { max }}=c_{i j}=c_{j \text { max }}=c_{j k}=c_{k \text { max }}$ and by the ultrametric inequality, $c_{i j}=\min \left\{c_{i j}, c_{j k}\right\} \leq c_{i k} \leq c_{i \max }=c_{i j}$. This shows that $\sim$ is transitive, because it implies $c_{i k}=c_{i j}=c_{i \max }=c_{k \text { max }}$.

For statement (ii), let $i \sim j$ and $k \sim \ell$. So we have $c_{i j}=c_{i \max }=c_{j \max }$ as well as $c_{k \ell}=c_{k \max }=c_{\ell \max }$. The ultrametric inequality and symmetry of $C$ invoke

$$
\begin{aligned}
c_{i k} & \geq \min \left\{c_{i \ell}, c_{\ell k}\right\}=c_{i \ell} \geq \min \left\{c_{i j}, c_{j \ell}\right\}=c_{j \ell} \\
& \geq \min \left\{c_{j k}, c_{k \ell}\right\}=c_{j k} \geq \min \left\{c_{j i}, c_{i k}\right\}=c_{i k} .
\end{aligned}
$$

We find that this is actually a chain of equalities and that therefore $c_{i k}=c_{j \ell}$.
For statement (iii), note that for $k, \ell \in\{1, \ldots, m\}$ with $k \neq \ell$ the submatrices $C_{X_{k} X_{\ell}}$, by Item (ii), have identical entries and therefore identical row sums. In view of the relation $\sim$, for $k \in\{1, \ldots, m\}$ all off-diagonal entries of $C_{X^{2}}$ have the same value. The additional assumption in (iii) thus requires all diagonal entries to have the same value. This implies the stated formula for the entries $q_{i j}$ of the corresponding equitable quotient matrix $Q$.

Finally, to verify statement (iv), consider an equivalence class $X=\left\{i_{1}, \ldots, i_{s}\right\}$ induced by the relation $\sim$. Then Theorem 88 says that $C$ has $s-1$ linearly independent eigenvectors $e_{i_{1}}-e_{i t}$, for $t \in\{2, \ldots, s\}$, which all correspond to
the eigenvalue $-c(X) \leq 0$. In total, this contributes $-(s-1) c(X)$ to the sum of nonpositive eigenvalues. Summation over all equivalence classes adds

$$
\sum_{k=1}^{m}\left(\left(\left|X_{k}\right|-1\right) c\left(X_{i}\right)\right)=\operatorname{tr}(Q)
$$

to the energy $\mathcal{E}(C)$. From the equivalence classes, we obtain, when counting multiplicities, $n-m$ nonpositive eigenvalues. The corresponding linearly independent eigenvectors are of the form $e_{i}-e_{j}$ with $i \sim j$. The remaining $m$ eigenvectors can be chosen orthogonal to them. In other words, they can be chosen constant on the classes $X_{k}$ or, which is the same, of the form

$$
x=\sum_{k=1}^{m} y_{k} \mathbb{1}_{X_{k}} \quad \text { for appropriate } y_{k} \in \mathbb{R}
$$

Setting $y=\left[y_{1}, \ldots, y_{m}\right]^{\top}$, the corresponding eigenequation $C x=\lambda x$ is equivalent to $Q y=\lambda y$. So the remaining $m$ eigenvalues of $C$, in particular the positive ones, are those of $Q$. Consequently, $\mathcal{E}(C)=\mathcal{E}(Q)+\operatorname{tr}(Q)$. This implies the statement to be shown, because $\mathcal{E}(Q) \geq \operatorname{tr}(Q)$, wherein equality holds if and only if all eigenvalues of $Q$ are nonnegative.

Corollary 95. The equitable quotient matrix $Q$ of Theorem 94 is similar to the symmetric matrix $S=\left[s_{k k}\right]$ with

$$
s_{k \ell}= \begin{cases}c_{i j} \sqrt{\left|X_{k}\right|\left|X_{\ell}\right|} & \text { if } k \neq \ell, \text { for some } i \in X_{k} \text { and } j \in X_{\ell}, \\ c_{i i}+c_{i \max }\left(\left|X_{k}\right|-1\right) & \text { if } k=\ell, \text { for some } i \in X_{k} .\end{cases}
$$

If $S$ is positive semidefinite, the inequality in Theorem 94 is an equality.
Proof. For $D:=\operatorname{diag}\left(\sqrt{\left|X_{k}\right|}: k \in\{1, \ldots, m\}\right)$, we obtain $S=D Q D^{-1}$.
Example 96. We consider the graph from Figure 39. Its edge-connectivity matrix is

$$
C=\begin{gathered}
1 \\
1 \\
2 \\
2
\end{gathered}\left[\begin{array}{llllll}
0 & 3 & 4 & 5 & 6 \\
3 & 4 & 4 & 4 & 3 & 3 \\
4 & 0 & 4 & 4 & 3 & 3 \\
4 & 4 & 0 & 4 & 3 & 3 \\
4 & 4 & 4 & 0 & 3 & 3 \\
3 & 3 & 3 & 3 & 0 & 3 \\
3 & 3 & 3 & 3 & 3 & 0
\end{array}\right] .
$$

Partitioning the index set $\{1, \ldots, n\}$ into $X_{1}=\{1,2,3,4\}$ and $X_{2}=\{5,6\}$
yields the equitable matrix partition and quotient matrix

$$
C=\left[\begin{array}{cc}
C_{X_{1}^{2}} & C_{X_{1} X_{2}} \\
C_{X_{2} X_{1}} & C_{X_{2}^{2}}
\end{array}\right], \quad Q=\left[\begin{array}{ll}
12 & 6 \\
12 & 3
\end{array}\right] .
$$

In view of Corollary 95, the matrix $Q$ is similar to the symmetric matrix

$$
S=D Q D^{-1}=\left[\begin{array}{rr}
2 & 0 \\
0 & \sqrt{2}
\end{array}\right]\left[\begin{array}{ll}
12 & 6 \\
12 & 3
\end{array}\right]\left[\begin{array}{cc}
1 / 2 & 0 \\
0 & 1 / \sqrt{2}
\end{array}\right]=\left[\begin{array}{rr}
12 & 6 \sqrt{2} \\
6 \sqrt{2} & 3
\end{array}\right] .
$$

The block to $X_{1}$ gives us the eigenvalue -4 and three linearly independent eigenvectors $e_{1}-e_{2}, e_{2}-e_{3}$, and $e_{3}-e_{4}$. From the block to $X_{2}$, we read off the eigenvalue -3 and eigenvector $e_{5}-e_{6}$. This leads to the energy bound

$$
\mathcal{E}(C) \geq 2 \sum_{k=1}^{m}\left(\left|X_{k}\right|-1\right) c\left(X_{k}\right)=2((4-1) 4+(2-1) 3)=30 .
$$

We also know that this bound is not tight, because $\operatorname{det}(Q)=-36$, and thus $Q$ has a negative eigenvalue. Nevertheless, the value we achieved is not all too far from the actual energy $\mathcal{E}(C)=15+3 \sqrt{41} \approx 34.21$.

### 5.4 Ultrametric distance matrices

Another perspective on the results of the previous section reveals itself when replacing the local edge-connectivities by an ultrametric distance measure. Consider a graph $G$ on vertex set $\{1, \ldots, n\}=V(G)$ and let $C=\left[c_{i j}\right]$ be its edge-connectivity matrix. For two vertices $i, j \in V(G)$ we define the inverse connectivity distance, or flow distance, by $d(i, j):=1 / c_{i j}$ if $i \neq j$ and $d(i, j):=0$ if $i=j$. To see how this distance is related to other resistance distance notions, see Gurvich [51]. This measure satisfies a strong triangle inequality, meaning that

$$
d(i, k)=\frac{1}{c_{i k}} \leq \frac{1}{\min \left\{c_{i j}, c_{j k}\right\}}=\max \left\{\frac{1}{c_{i j}}, \frac{1}{c_{j k}}\right\}=\max \{d(i, j), d(j, k)\}
$$

for all vertices $i, j, k \in V(G)$. Indeed, the flow distance satisfies all the properties required to fit into the following distance concept.

Definition 97. An ultrametric distance $d: V \times V \rightarrow \mathbb{R}$ is a map satisfying
(i) $d(i, j) \geq 0$ for all $i, j \in V$,
(ii) $d(i, j)=d(j, i)$ for all $i, j \in V$,
(iii) $d(i, j)=0$ if and only if $i=j$, where $i, j \in V$, and
(iv) $d(i, k) \leq \max \{d(i, j), d(j, k)\}$ for all $i, j, k \in V$.

Remark 98. Many notions used when studying shortest path distances in graphs, or general metric spaces, naturally translate to the flow distance concept. For example, for a flow distance $d: V(G) \times V(G) \rightarrow \mathbb{R}$ on a graph $G$ the transmission of a vertex $i \in V(G)$ can be defined as $t(i):=\sum_{j \in V(G) \backslash\{i\}} d(i, j)$. The transmission of the graph $G$ is then $\sigma(G):=\frac{1}{2} \sum_{i \in V(G)} t(i)$. For the shortest path distance, the latter object is called Wiener index [105], denoted by $W(G)$, which is used in chemistry to describe the topological structure of molecules. A challenging graph theoretical problem about transmission is that of Šoltés [96]. He raised the problem to find all those graphs whose transmission remains stable upon vertex removal. Formally, Šoltés problem is to find all graphs $G$ satisfying $W(G)=W(G-v)$ for all vertices $v \in V(G)$. Up to now, this problem remains unsolved. Attempts towards its solution focus, for example, on relaxing Šoltés condition to be satisfied for some, but not all, vertices in $V(G)$, as is done by Knor, Majstorović, and Škrekovski [73]. For the original problem, only the cycle on eleven vertices is known to satisfy Soltés condition for all its vertices. For the flow distance, we observe a remarkably similar behavior. Denoting by $T_{n}$ a tree on $n$ vertices, we determine

$$
\sigma\left(T_{n}\right)=\frac{1}{2} \sum_{i \in V\left(T_{n}\right)} t(i)=\frac{1}{2} \sum_{i \in V\left(T_{n}\right)}(n-1)=\frac{n(n-1)}{2}
$$

and for a cycle $C_{n}$ on $n$ vertices, we find

$$
\sigma\left(C_{n}\right)=\frac{1}{2} \sum_{i \in V\left(C_{n}\right)} t(i)=\frac{1}{2} \sum_{i \in V\left(C_{n}\right)} \frac{n-1}{2}=\frac{n(n-1)}{4} .
$$

For a given cycle, on $n \geq 3$ vertices, Šoltés condition $\sigma(G)=\sigma(G-v)$ reads $\sigma\left(C_{n}\right)=\sigma\left(T_{n-1}\right)$, which is equivalent to

$$
\frac{n(n-1)}{4}=\frac{(n-1)(n-2)}{2} .
$$

Consequently, the only cycle that satisfies Šoltés condition for the flow distance is the cycle on $n=4$ vertices. In fact, it is the only graph we know so far, having investigated all graphs on at most ten vertices.

Now let us see how our results from section 5.3 carry over to the ultrametric distance setting. We derive counterparts to Theorems 88 and 94.

Corollary 99. Consider an ultrametric distance $d: V \times V \rightarrow \mathbb{R}$ with distance matrix $D=[d(i, j)]_{i, j \in V}$. Denoting the distance of $i \in V$ to a nearest point by $d_{i \min }:=\min \{d(i, j): j \in V \backslash\{i\}\}$, the following statements hold.
(i) Two points $i, j \in V, i \neq j$, are mutually nearest points, meaning that $d(i, j) \leq d(i, k)$ and $d(i, j) \leq d(k, j)$ for any $k \in V \backslash\{i, j\}$, if and only if $D$ has an eigenvalue $-d(i, j)$ with corresponding eigenvector $e_{i}-e_{j}$.
(ii) With $D$ we are given an equivalence relation on $V$ via

$$
i \sim j: \Leftrightarrow i=j \text { or } d(i, j)=d_{i \min }=d_{j \min } .
$$

(iii) If $i \sim j$ and $k \sim \ell$, then $d(i, k)=d(j, \ell)$. In other words, an entry $d(i, j)$ depends only on the equivalence classes of $i$ and $j$.
(iv) Denoting by $X_{1}, \ldots, X_{m}$ the equivalence classes induced by the relation $\sim$, the submatrices $D_{X_{i} X_{j}}$ form an equitable partition of $D$. The corresponding equitable quotient matrix is $Q=\left[q_{k \ell}\right]_{k, \ell=1}^{m}$ where

$$
q_{k \ell}= \begin{cases}d(i, j)\left|X_{\ell}\right| & \text { if } k \neq \ell, \text { for some } i \in X_{k} \text { and } j \in X_{\ell}, \\ d_{i \min }\left(\left|X_{\ell}\right|-1\right) & \text { if } k=\ell, \text { for some } i \in X_{\ell} .\end{cases}
$$

(v) Using $d_{\min }\left(X_{k}\right)$ to denote the common value $d_{i \text { min }}$ for $i \in X_{k}$, the energies of $D$ and its equitable quotient matrix $Q$ are related by

$$
\mathcal{E}(D)=\mathcal{E}(Q)+\operatorname{tr}(Q) \geq 2 \sum_{k=1}^{m}\left(\left|X_{k}\right|-1\right) d_{\min }\left(X_{k}\right) .
$$

This inequality is attained if and only if $Q$ has no negative eigenvalues.
Proof. At first, observe that a matrix $D=[d(i, j)]_{i, j \in V}$ whose entries $d(i, j)$ originate from an ultrametric distance has only zeros on its diagonal. Furthermore, because the entries $d(i, j)$ satisfy the strong triangle inequality, we obtain for all $i, j, k \in V$ that

$$
\begin{aligned}
d(i, k) & \leq \max \{d(i, j), d(j, k)\} \\
\Rightarrow-d(i, k) & \geq-\max \{d(i, j), d(j, k)\}=\min \{-d(i, j),-d(j, k)\} .
\end{aligned}
$$

So the entries of $-D$ satisfy the ultrametric inequality and hence Theorems 88 and 94 apply. Furthermore, observe that for mutually nearest points $i, j \in V, i \neq j$, the entry $-d(i, j)$ is maximal among the off-diagonal elements of row $i$ and column $j$ of $-D=[-d(i, j)]$. This directly implies statements (ii) to (iv) of Corollary 99. To check statement (i), observe that, by Theorem 88 , an entry $-d(i, j)$ is maximal among the off-diagonal elements of row $i$ and column $j$ of $-D$ if and only if $-D$ has an eigenvalue $-d(i, i)-(-d(i, j))=d(i, j)$ with corresponding eigenvector $e_{i}-e_{j}$. This
holds exactly if $D$ has an eigenvalue $-d(i, j)$ with corresponding eigenvector $e_{i}-e_{j}$. Statement (v) of Corollary 99 can be shown completely analogous to statement (iv) of Theorem 94. To see this, recall that $D$ is nonnegative and has only zeros on its diagonal. Also note that in the proof of statement (iv) of Theorem 94 we do not work with the ultrametric inequality directly, but only refer to Theorem 88 and the other statements of Theorem 94, whose counterparts in Corollary 99 we have already verified.

Another example, where ultrametric distances arise comes from data analysis. Consider a complete graph $G$ with nonnegative edge weights $\omega(e)$ for $e \in E(G)$. We may think about the vertices of $G$ as data points and about the edge weights as some pairwise distance measure. Furthermore, for a minimum spanning tree $T$ of $G$ and two vertices $i, j \in V(T)$ we denote by $P(i, j)$ the edge set of the unique path between $i$ and $j$ in $T$. Then $d: V \times V \rightarrow \mathbb{R}$ where $d(i, j):=\max _{e \in P(i, j)} \omega(e)$ for $i \neq j$ and $d(i, j):=0$ for $i=j$ is an ultrametric distance. Properties (i), (ii), and (iii) of Definition 97 are certainly fulfilled and Property (iv) holds because

$$
\begin{aligned}
d(i, k) & =\max _{e \in P(i, k)} \omega(e) \\
& \leq \max _{e \in P(i, j) \cup P(j, k)} \omega(e)=\max \left\{\max _{e \in P(i, j)} \omega(e), \max _{e \in P(j, k)} \omega(e)\right\}=\max \{d(i, j), d(j, k)\}
\end{aligned}
$$

Distance measures of such kind are called min-max distances and find their application naturally when data inhabits some kind of hierarchical structure. This is demonstrated, for example, by Murtagh, Downs, and Contreras [82], who show how to find hierarchical clusters in very large, high dimensional data sets, or by Chehreghani [59], who present an unsupervised representation learning approach, which is based on min-max distances.

Let us conclude this chapter with a glance at a Laplace analogue for connectivity matrices and how it can be utilized for spectral graph partitioning. We consider a graph $G$ on vertex set $V(G)=\{1, \ldots, n\}$. For its connectivity or edge-connectivity matrix $C=\left[c_{i j}\right] \in\{0, \ldots, n-1\}^{n \times n}$, we define for a vertex $i \in\{1, \ldots, n\}$ its potential by $c_{i}:=\sum_{j=1, j \neq i}^{n} c_{i j}$, as in the beginning of Section 5.2. Writing those potentials in the diagonal matrix $T=\operatorname{diag}\left(c_{1}, \ldots, c_{n}\right)$, we define a graph's Laplace connectivity matrix by $L:=T-C$ and denote by $\mathcal{L}:=T^{-1 / 2} L T^{-1 / 2}=I-T^{-1 / 2} C T^{-1 / 2}$ a normalized version. Those matrices are positive semidefinite, because for any $x \in \mathbb{R}^{n}$ they satisfy

$$
x^{\top} \mathcal{L} x=x^{\top} T^{-1 / 2} L T^{-1 / 2} x=y^{\top} L y=\sum_{\substack{i, j=1 \\ i<j}}^{n} c_{i j}\left(y_{i}-y_{j}\right)^{2} \geq 0,
$$

where $y:=T^{-1 / 2} x$. Furthermore, considering $\mathcal{L} T^{1 / 2} \mathbb{1}=T^{-1 / 2} L \mathbb{1}=0$, we obtain that $\lambda_{1}=0$ is an eigenvalue with corresponding eigenvector $T^{-1 / 2} \mathbb{1}$. Note that, in contrast to the rest of this work, we here follow the convention $\lambda_{1} \leq \lambda_{2} \leq \ldots \leq \lambda_{n}$, which is common practice for Laplace matrices. By Lemma 101 and Theorem 105, we obtain

$$
\begin{align*}
\lambda_{2} & =\inf _{\substack{x \in \mathbb{R}^{n} \\
x \perp T^{1 / 2} \mathbb{1}}} \frac{x^{\top} \mathcal{L} x}{x^{\top} x} \\
& =\inf _{\substack{y \mathbb{R}^{n} \\
y \perp T \mathbb{1}}} \frac{y^{\top} L y}{\left(T^{1 / 2} y\right)^{\top} T^{1 / 2} y} \\
& =\inf _{\substack{y \in \mathbb{R}^{n} \\
y \perp T \mathbb{1}}} \frac{\sum_{i, j=1: i<j}^{n} c_{i j}\left(y_{i}-y_{j}\right)^{2}}{\sum_{i=1}^{n} c_{i} y_{i}^{2}} \tag{R}
\end{align*}
$$

For a graph $G$ and a subset of vertices $S \subseteq V(G)$, we define the flow volume by $v(S):=\sum_{i \in S} c_{i}$ or $v(G):=\sum_{i \in V(G)} c_{i}$, respectively. The parameter

$$
\alpha(S):=\sum_{i \in S, j \in V(G) \backslash S} c_{i j}
$$

adds up the number of edge-disjoint paths from $i \in S$ to $j \in V(G) \backslash S$ over all such vertex pairs and can be seen as a measure of connectivity between $S$ and the rest of the graph. The flow Cheeger ratio of a set $S \subseteq V(G)$ balances this by the flow volume of $S$ and $V(G) \backslash S$, respectively. Formally, we define it by

$$
h(S):=\frac{\alpha(S)}{\min \{v(S), v(G)-v(S)\}}
$$

The flow Cheeger constant of a graph $G$ is the parameter

$$
h(G):=\min _{S \subseteq V(G)} h(S)
$$

Adapting a proof of Chung [24] to the situation of connectivity matrices, we obtain the following version of Cheeger's inequality [20].

Theorem 100. For a connected graph $G$ holds

$$
\frac{h(G)^{2}}{2} \leq \lambda_{2} \leq 2 h(G) .
$$

Proof. We begin with the upper bound. Consider any $S \subseteq V(G)$ and substitute $y=\mathbb{1}_{S}-\frac{v(S)}{v(G)} \mathbb{1}$ in (R). This yields

$$
\begin{aligned}
\lambda_{2} & \leq \frac{\sum_{i \in V, j \in V(G) \backslash S} c_{i j}}{\sum_{i \in S} c_{i}\left[1-\frac{v(S)}{v(G)}\right]^{2}+\sum_{i \in V(G) \backslash S} c_{i}\left[-\frac{v(S)}{v(G)}\right]^{2}} \\
& =\frac{\alpha(S)}{v(S)\left[1-\frac{v(S)}{v(G)}\right]^{2}+[v(G)-v(S)]\left[\frac{v(S)}{v(G)}\right]^{2}} \\
& =\frac{\alpha(S)}{v(S)-2 \frac{v(S)^{2}}{v(G)}+\frac{v(S)^{3}}{v(G)^{2}}+\frac{v(S)^{2}}{v(G)}-\frac{v(S)^{3}}{v(G)^{2}}} \\
& =\frac{\alpha(S) v(G)}{v(S)(v(G)-v(S))}=\frac{\alpha(S)}{v(S)}+\frac{\alpha(S)}{v(G)-v(S)} \\
& \leq \frac{2 \alpha(S)}{\min \{v(S), v(G)-v(S)\}}=2 h(S) .
\end{aligned}
$$

For the lower bound, consider an eigenvector $y$ to $\lambda_{2}$ and assume, without loss of generality, that rows and columns of $\mathcal{L}$ are simultaneously permuted such that $y_{1} \geq \ldots \geq y_{n}$ and corresponding vertices are denoted by $v_{1}, \ldots, v_{n}$. Setting $S_{k}:=\{1, \ldots, k\}$ and $\ell:=\max \left\{k \in\{1, \ldots, n\}: v\left(S_{k}\right) \leq v(G) / 2\right\}$, we obtain

$$
\sum_{i=1}^{n} c_{i} y_{i}^{2}=\min _{\xi \in \mathbb{R}} \sum_{i=1}^{n} c_{i}\left(y_{i}-\xi\right)^{2} \leq \sum_{i=1}^{n} c_{i}\left(y_{i}-y_{\ell}\right)^{2} .
$$

Herein, we rely on the fact that $\sum_{i=1}^{n} y_{i} c_{i}=0$. For $i \in\{1, \ldots, n\}$ let us denote

$$
\stackrel{+}{y}_{i}:=\left\{\begin{array}{ll}
y_{i}-y_{\ell} & \text { if } y_{i} \geq y_{\ell}, \\
0 & \text { if } y_{i}<y_{\ell}
\end{array} \quad \text { and } \quad \bar{y}_{i}:= \begin{cases}\left|y_{i}-y_{\ell}\right| & \text { if } y_{i} \leq y_{\ell} \\
0 & \text { if } y_{i}>y_{\ell}\end{cases}\right.
$$

and assume $\left({ }_{y}{ }^{\top} \mathcal{L}{ }^{+}\right) /\left({ }_{y}{ }^{\top}+{ }_{y}^{+}\right) \leq\left(\bar{y}^{\top} \mathcal{L} \bar{y}\right) /\left(\bar{y}^{\top} \bar{y}\right)$, without loss of generality. We obtain

$$
\begin{aligned}
& \lambda_{2}=\frac{\sum_{i, j=1: i<j}^{n} c_{i j}\left(y_{i}-y_{j}\right)^{2}}{\sum_{i=1}^{n} c_{i} y_{i}^{2}} \geq \frac{\sum_{i, j=1: i<j}^{n} c_{i j}\left(y_{i}-y_{j}\right)^{2}}{\sum_{i=1}^{n} c_{i}\left(y_{i}-y_{\ell}\right)^{2}} \\
& \geq \frac{\sum_{i, j=1: i<j}^{n} c_{i j}\left[\left({ }_{y}^{\prime}\right.\right.}{\left.\left.y_{i}-\stackrel{+}{y}_{j}\right)^{2}+\left(\bar{y}_{i}-\bar{y}_{j}\right)^{2}\right]} \\
& \sum_{i=1}^{n} c_{i}\left[{ }_{y}^{+}{ }_{i}+\bar{y}_{i}^{2}\right] \\
& \geq \frac{\sum_{i, j=1: i<j}^{n} c_{i j}\left(\stackrel{+}{y}_{i}-\stackrel{+}{y}_{j}\right)^{2}}{\sum_{i=1}^{n} c_{i} \stackrel{+}{y}_{i}^{2}} .
\end{aligned}
$$

In the preceding line, we used the inequality $\frac{a+b}{c+d} \geq \min \left\{\frac{a}{c}, \frac{b}{d}\right\}$, which holds for
any $a, b \in \mathbb{R}$ and $c, d \in \mathbb{R} \backslash\{0\}$. Denoting $\eta\left(S_{i}\right):=\min \left\{v\left(S_{i}\right), v(G)-v\left(S_{i}\right)\right\}$ for $i \in\{1, \ldots, n\}$, recall that

$$
h(G)=\min _{S \subseteq V(G)} h(S) \leq h\left(S_{i}\right)=\frac{\alpha\left(S_{i}\right)}{\min \left\{v\left(S_{i}\right), v(G)-v\left(S_{i}\right)\right\}}=\frac{\alpha\left(S_{i}\right)}{\eta\left(S_{i}\right)}
$$

and thus $\alpha\left(S_{i}\right) \geq h(G) \eta\left(S_{i}\right)$. We conclude that

$$
\begin{aligned}
& \lambda_{2} \geq \frac{\sum_{i, j=1: i<j}^{n} c_{i j}\left(\stackrel{+}{y}_{i}-\stackrel{+}{y}_{j}\right)^{2}}{\sum_{i=1}^{n} c_{i} \dot{y}_{i}^{2}} \\
& \left.\left.\left.=\frac{\left[\sum_{i, j=1: i<j}^{n} c_{i j}\left(\stackrel{+}{y}_{i}-\stackrel{+}{y_{j}}\right)^{2}\right]\left[\sum _ { i , j = 1 : i < j } ^ { n } c _ { i j } \left({ }_{( }^{y}\right.\right.}{i}++_{y_{j}}\right)^{2}\right]\right) \\
& \geq \frac{\left[\sum_{i, j=1: i<j}^{n} c_{i j}\left(\stackrel{+}{y}_{i}^{2}-\stackrel{+}{y}_{j}^{2}\right)\right]^{2}}{2\left[\sum_{i=1}^{n} c_{i} \dot{y}_{i}\right]\left[\sum_{i, j=1: i<j}^{n} c_{i j}\left({ }_{y}^{2}{ }_{i}^{2}+\stackrel{+}{y}_{j}^{2}\right)\right]} \quad\left[\frac{\text { Cauchy-Schwarz }}{(a+b)^{2} \leq 2\left(a^{2}+b^{2}\right)}\right] \\
& =\frac{\left[\sum_{i, j=1: i<j}^{n} c_{i j}\left(\stackrel{+}{y}_{i}-\stackrel{+}{y}_{i+1}^{2}+\stackrel{+}{y}_{i+1}^{2} \mp \ldots-\stackrel{+}{y}_{j-1}^{2}+\stackrel{+}{y}_{j-1}^{2}-\stackrel{+}{y}_{j}^{2}\right)\right]^{2}}{2\left[\sum_{i=1}^{n} c_{i} \stackrel{+}{y}_{i}\right]^{2}} \\
& =\frac{\left[\sum_{i=1}^{n-1} \alpha\left(S_{i}\right)\left({ }_{y}^{+}{ }_{i}-\stackrel{+}{y}_{i+1}{ }^{2}\right)\right]^{2}}{2\left[\sum_{i=1}^{n} c_{i} \stackrel{+}{y}_{i}^{2}\right]^{2}} \\
& \geq \frac{\left[\sum_{i=1}^{n-1} h(G) \eta\left(S_{i}\right)\left(\stackrel{+}{y}_{i}-\stackrel{+}{y}_{i+1}\right)\right]^{2}}{2\left[\sum_{i=1}^{n} c_{i} \dot{y}_{i}^{2}\right]^{2}} \\
& =\frac{h(G)^{2}}{2} \frac{\left[\eta\left(S_{1}\right) \dot{y}_{1}^{2}+\sum_{i=2}^{n}\left(\eta\left(S_{i}\right)-\eta\left(S_{i-1}\right)\right) y_{i}\right]^{2}}{\left[\sum_{i=1}^{n} c_{i} \dot{y}_{i}{ }^{2}\right]^{2}} \\
& =\frac{h(G)^{2}}{2} \frac{\left[v\left(S_{1}\right) \dot{y}_{1}^{2}+\sum_{i=2}^{n}\left(v\left(S_{i}\right)-v\left(S_{i-1}\right)\right) \dot{y}_{i}\right]^{2}}{\left[\sum_{i=1}^{n} c_{i} \dot{y}_{i}^{2}\right]^{2}} \\
& =\frac{h(G)^{2}}{2} \frac{\left[\sum_{i=1}^{n} c_{i} \dot{y}_{i}^{2}\right]^{2}}{\left[\sum_{i=1}^{n} c_{i} \dot{y}_{i}^{2}\right]^{2}}=\frac{h(G)^{2}}{2} \text {. }
\end{aligned}
$$

For the second to last equality, recall that $\stackrel{+}{y}_{i}=0$ whenever $y_{i}<y_{\ell}$. By the definition of $\ell$, this is exactly the case when the respective minimum in $\eta\left(S_{i}\right)=\min \left\{v\left(S_{i}\right), v(G)-v\left(S_{i}\right)\right\}$ is realized at the second argument.

The idea behind this proof can be utilized algorithmically. Pothen, Simon, and Liou [86] present a framework that works for general weighted Laplace matrices. We also refer to the seminal work of Fiedler [45], which framed the term algebraic connectivity for the second smallest eigenvalue of a graph's Laplace matrix, and attracted great research interest in this parameter. A question that arises here is whether the second smallest eigenvalue of the connectivity Laplace matrix shows some specific properties. Although determining the latter involves additional computational effort, for determining the given graph's cut values, as discussed in Section 5.1, this effort could be worthwhile if one could demonstrate theoretical or empirical benefits for spectral graph partitioning.

## 16

## Conclusion and Outlook

This dissertation focuses on pairwise connectivity relations in graphs. To say that a graph is $k$-connected gives information about its global structure. On the other hand, the number of independent or edge-disjoint paths between two vertices measures how strong those specific vertices are connected. This thesis contributes to exploring the interplay of these two perspectives.

## Main results

Whereas Chapters 1 and 2 lay the foundations of this work, Chapter 3 presents the current state of research on uniform graph connectivity. The central results of this chapter are the following.

- Section 3.2 gives a detailed account of how the classes of uniformly $k$-connected and uniformly $k$-edge-connected graphs are related.
- Section 3.3 provides a complete constructive characterization of uniformly 3 -connected graphs.
- Section 3.4 demonstrates how to utilize the constructive results of Section 3.3. It contains a tight bound on the number of vertices of degree three in uniformly 3 -connected graphs as well as results on how crossing numbers and treewidths behave under the constructions at hand.

In Chapter 4, cut sequences are suggested as graph invariants. Many structural questions that have been studied intensively for degree sequences translate naturally to this concept. The following are the key results.

- Section 4.1 conceptualizes the notion of a graph's cut sequence.
- Section 4.4 shows how to construct graphs having a given cut sequence in case that sequence satisfies a shifted variant of the Erdős-Gallai inequalities.

Chapter 5 presents the current state of research on the spectral properties of connectivity and edge-connectivity matrices, whose entries represent a graph's local connectivities. This chapter's main results are the following.

- In Section 5.2, we present our knowledge about the spectral properties of connectivity matrices. In particular, we report on the status of a conjecture that the energy of a connectivity matrix is bounded from above by $2(n-1)^{2}$.
- Section 5.3 provides spectral properties of edge-connectivity matrices. In particular, this includes a tight lower bound on the smallest eigenvalue of an edge-connectivity matrix and a tight upper bound on the energy of such a matrix.


## A selection of open problems

For many questions we answered in this work, several more arise. Let us conclude this investigation by compiling a few of them.

- How can we characterize uniformly $k$-connected and uniformly $k$-edgeconnected graphs for $k \geq 4$ ?
- We know that the treewidth of uniformly 3 -connected graphs with minimum number of vertices of degree three is greater than or equal to 4 and less than or equal to 13 . What is the best bound that can be given?
- Find a concise characterization of graphs whose cut sequences violate the shifted Erdős-Gallai inequalities and show how to construct them.
- Prove or disprove that asymptotically all graphs satisfy the shifted Erdős-Gallai inequalities.
- Prove or disprove that the energy of a connectivity matrix is bounded from above by $2(n-1)^{2}$.
- Provide a lower bound on the smallest eigenvalue of a connectivity matrix that is better than the general bound by Zhan [108].
- Prove or disprove that the cycle on four vertices is the only graph that satisfies Šoltés condition for the inverse flow distance.
- What are specific characteristics of the connectivity Laplace matrix and can they be utilized for spectral graph partitioning?


## 13

## Symmetric Matrices

In this appendix, we summarize the necessary tools from linear algebra which we employ in particular in Chapter 5. For the following results and further terminology, we refer to Horn and Johnson [67, Chapters 0, 1, and 4]. Our focus here is specifically on matrices that are symmetric and nonnegative, which are typical properties of graph matrices, as discussed in Section 5.1.

Lemma 101. A symmetric matrix $A \in \mathbb{R}^{n \times n}$ has only real eigenvalues and eigenvectors belonging to different eigenvalues are orthogonal.

Lemma 102. Let $A=\left[a_{i j}\right] \in \mathbb{R}^{n \times n}$ be a matrix with eigenvalues $\lambda_{1} \geq \ldots \geq \lambda_{n}$ and let $\mu \in \mathbb{R}$. Then $A+\mu I$ has eigenvalues $\lambda_{1}+\mu \geq \ldots \geq \lambda_{n}+\mu$.

Theorem 103. For each symmetric matrix $A \in \mathbb{R}^{n \times n}$ there exists an orthogonal matrix $U$ containing the eigenvectors of $A$ as columns such that $\Lambda=U^{\top} A U$ is a real diagonal matrix.

Since the matrix $U$ in Theorem 103 is orthogonal, we can write the matrix $A$ as

$$
A=U \Lambda U^{\top}=\sum_{i=1}^{n} \lambda_{i} u_{i} u_{i}^{\top}
$$

where $\lambda_{1}, \ldots, \lambda_{n}$ are the eigenvalues of $A$ that correspond to the eigenvectors $u_{1}, \ldots, u_{n}$. This expression is also known as the spectral decomposition of $A$. When bounding the eigenvalues of a matrix, we often rely on the following principle by Rayleigh [100] and Ritz [88].

Theorem 104. Let $\lambda_{1} \geq \ldots \geq \lambda_{n}$ be the eigenvalues of a real symmetric matrix $A \in \mathbb{R}^{n \times n}$. Then for all $x \in \mathbb{R}^{n \times n}$ there holds

$$
\lambda_{n} x^{\top} x \leq x^{\top} A x \leq \lambda_{1} x^{\top} x
$$

Equality in the lower bound holds if $x$ is an eigenvector to $\lambda_{n}$, equality in the upper bound holds if $x$ is an eigenvector to $\lambda_{1}$.

Alternatively, we may state the contents of Theorem 104 via

$$
\lambda_{1}=\max _{\substack{x \in \mathbb{R}^{n} \\\|x\|_{2}=1}} x^{\top} A x=\max _{\substack{x \in \mathbb{R}^{n} \\ x \neq 0}} \frac{x^{\top} A x}{x^{\top} x} \quad \text { and } \quad \lambda_{n}=\min _{\substack{x \in \mathbb{R}^{n} \\\|x\|_{2}=1}} x^{\top} A x=\min _{\substack{x \in \mathbb{R}^{n} \\ x \neq 0}} \frac{x^{\top} A x}{x^{\top} x} .
$$

The terms $x^{\top} A x /\left(x^{\top} x\right)$ appearing herein are called Rayleigh-quotients. This principle can be generalized to the $k$-th largest or $k$-th smallest eigenvalue.

Theorem 105. Let $\lambda_{1} \geq \ldots \geq \lambda_{n}$ be the eigenvalues of a real symmetric ma$\operatorname{trix} A \in \mathbb{R}^{n \times n}$ and let $\left\{u_{1}, \ldots, u_{n}\right\}$ be a corresponding basis of orthonormal eigenvectors. Then for $k \in\{1, \ldots, n\}$ there holds

$$
\lambda_{n-k}=\min _{\substack{x \neq 0 \\ x \perp u_{n}, \ldots, u_{n-k+1}}} \frac{x^{\top} A x}{x^{\top} x} \quad \text { and } \quad \lambda_{k}=\max _{\substack{x \neq 0 \\ x \perp u_{1}, \ldots, u_{k-1}}} \frac{x^{\top} A x}{x^{\top} x} .
$$

Another key tool in spectral graph theory goes back to Perron [85] and Frobenius [48] and comes in many variants, of which Horn and Johnson [67, Chapter 8] give an overview. We make use of the following version.

Theorem 106. Let $A=\left[a_{i j}\right] \in \mathbb{R}^{n \times n}$ be a nonnegative irreducible matrix. Then the following statements apply.
(i) It holds $\lambda_{1}>0$.
(ii) An eigenvector corresponding to $\lambda_{1}$ contains only positive or only negative entries.
(iii) The eigenvalue $\lambda_{1}$ is simple.
(iv) For all eigenvalues $\lambda$ of $A$ there holds $|\lambda| \leq \lambda_{1}$.

Corollary 107. Let $A=\left[a_{i j}\right]$ and $B=\left[b_{i j}\right]$ be nonnegative irreducible matrices in $\mathbb{R}^{n \times n}$. If $A \leq B$, then $\lambda_{1}(A) \leq \lambda_{1}(B)$. If $A \leq B$ and if $a_{i j}<b_{i j}$ for some $i, j \in\{1, \ldots, n\}$, then $\lambda_{1}(A)<\lambda_{1}(B)$.

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## Declaration of Authorship

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Chemnitz, Mai 30, 2023

