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THE SHARPER VERSION FOR GENERALIZED POWER MEAN INEQUALITIES WITH NEGATIVE EXPONENT

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Abstract. In this study, the generalized power mean inequalities with a negative parameter are refined using an optimality theorem on the generator function. The optimality theorem requires the study of different cases for the exponents and yields a refinement of the inequality in a neighbourhood of the vectors for which the equality occurs. Then, these local inequalities are generalized to all positive vectors by an appropriate selection of parameters. Also, some of the results are exemplified by numerical calculations.

1. Introduction

The mean inequalities have an important place in the realm of inequalities. The well-known means such as the arithmetic, quadratic, geometric and harmonic means are the special cases of the generalized power mean function

$$M_t(\boldsymbol{\alpha};x) = \left(\sum_{i=1}^{n} \alpha_i x_i^t\right)^{\frac{1}{t}}$$

where $t \in \mathbb{R}$, $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ with $\sum_{i=1}^{n} \alpha_i = 1$, $\alpha_i \geqslant 0$ and $x = (x_1, x_2, \ldots, x_n)$, $x_i > 0$.

Note that $M_{-1}(\boldsymbol{\alpha};x)$, $M_1(\boldsymbol{\alpha};x)$ and $M_2(\boldsymbol{\alpha};x)$ give weighted harmonic, arithmetic and quadratic means, respectively.

$M_0(\boldsymbol{\alpha};x)$, $M_{-\infty}(\boldsymbol{\alpha};x)$ and $M_{\infty}(\boldsymbol{\alpha};x)$ are defined as the limit of $M_t(\boldsymbol{\alpha};x)$ as $t \to 0$, $t \to -\infty$ and $t \to \infty$, respectively. They correspond to the following functions:

$$M_0(\boldsymbol{\alpha};x) = \prod_{i=1}^{n} x_i^{\alpha_i}, \quad M_{-\infty}(\boldsymbol{\alpha};x) = \min_{i\in\{1,..,n\}} x_i, \quad M_{\infty}(\boldsymbol{\alpha};x) = \max_{i\in\{1,..,n\}} x_i.$$

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The inequalities between these means are based on the fact in [17] that $M_t(\boldsymbol{\alpha};x)$ is an increasing function with respect to t, i.e., if $p,q \in \mathbb{R}$ with $p < q$ then

$$M_p(\boldsymbol{\alpha};x) \leqslant M_q(\boldsymbol{\alpha};x). \tag{1}$$

Equality occurs if and only if $x_1 = x_2 = \cdots = x_n$.

There are numerous refinements or sharper versions of the mean inequalities. [3, 15, 19, 4, 6, 8, 18, 5, 2, 7] can be referred as instances for further reading.

One way to obtain new or refined inequalities is to use the notions from abstract convexity and its applications in optimization. Rubinov in [12] proposed the utilization of the optimality properties of the abstract convex functions to sharpen inequalities with an example on the arithmetic-geometric mean. Then Adilov and Tınaztepe in [1] applied the results to geometric-harmonic, aritmethic-quadratic mean inequalities. Same approach is used to sharpen the classical inequalities such as Hölder, Aczel, Bergström and Brunn-Minkowski inequalities [21, 22, 23, 20]. We refer to [24, 25, 26, 9, 16, 13, 14] for other types of convexities and their applications to inequalities.

In this paper, the generalized power mean inequality (1) involving a negative exponent is sharpened by means of the results that are based on abstract convexity.

The paper is outlined as follows: In the second section, some definitions and theorems related to abstract convexity given in [12] are presented. In the third section, sharper version of generalized power mean inequality is derived and some numerical examples are given.

2. Preliminaries

The sharpening of an inequality or sharper version of an inequality can be described as follows: Let $f : \mathbb{R}^n \to \mathbb{R}_+$. If one can find $u(x) > 0$ such that $f(x) \geqslant u(x)$, then $f(x) \geqslant u(x)$ is sharper than the inequality $f(x) \geqslant 0$.

Let H be the class of functions on $\Omega \subseteq \mathbb{R}^n$. If there exists a function $h \in H$ such that $h \geqslant f$ then $f : \Omega \to \mathbb{R} \cup \{-\infty\}$ is majorized by H.

In [11], A. M. Rubinov gives the definition of abstract concave function as follows: Let H be the set of functions $h : \Omega \to \mathbb{R} \cup \{-\infty\}$. A function $f : \Omega \to \mathbb{R} \cup \{-\infty\}$ is called abstract concave with respect to H (or H-concave) if there exists a set $U \subseteq H$ such that $f(x) = \inf\limits_{h \in U} h(x)$ for all $x \in \Omega$.

Let X be a Banach space and H be the set of all quadratic functions h in the form of

$$h(x) = a\|x\|^2 + \langle l, x\rangle + c, \quad x \in X \tag{2}$$

where $a > 0$, $l \in X$ and $c \in \mathbb{R}$. Let $\Omega \subset X$. Then a function $f : \Omega \to \mathbb{R} \cup \{-\infty\}$ is abstract concave with respect to H if and only if f is majorized by H and f is upper semicontinuous (see [11]).

The following theorem asserts that the function f satisfying certain conditions can be abstract concave with respect to the functions in the form of (2).

THEOREM 1. [12] *Let $\Omega \subset X$ be a convex set and let f be a differentiable function defined on an open set containing Ω. Assume that the mapping $x \mapsto \nabla f(x)$ is continuous on Ω with the Lipschitz constant L, i.e., L is the least real number satisfying*

$$\|\nabla f(x) - \nabla f(y)\| \leqslant L\|x-y\|$$

for $x, y \in \Omega$. Let $a \geqslant L$. For each $t \in \Omega$, consider the function

$$f_t(x) = f(t) + \langle \nabla f(x), x-t \rangle + a\|x-t\|^2, \ \ x \in X.$$

Then $f(x) = \min_{t\in\Omega} f_t(x), \ x \in \Omega$.

The optimality conditions for a function f that can be represented as the infimum of the family of the functions $(f_t)_{t\in\Omega}$ over a convex set are elaborated in [12]. In this context, for a function $f : X \to \mathbb{R}$ satisfying $\|\nabla f(x) - \nabla f(y)\| \leqslant a\|x-y\|$ for all $x, y \in X$, the following result is obtained in [12]:

If x^* is a global minimum point of f over X, then

$$\frac{1}{4a}\|\nabla f(x)\|^2 \leqslant f(x) - f(x^*) \tag{3}$$

for all $x \in X$.

The following theorem which suggests the general case of the inequality (3) is proved in [12].

THEOREM 2. [12] *Let $\|.\|$ be Euclidean norm and $\|.\|_\circ$ be any norm on $\mathbb{R}^n$. Let $\Omega \subset \mathbb{R}^n$ have nonempty interior and let f be differentiable on Ω and let $\nabla f(x)$ be continuous with Lipschitz constant L on Ω. Suppose f has a global minimum at $x^* \in$ int Ω over Ω. Accept that*

$$B_\circ(x^*, r) = \{x : \|x - x^*\|_\circ \leqslant r\} \subset int\Omega$$

and

$$T = \max\{\|\nabla f(x)\|_\circ : x \in B_\circ(x^*, r)\}.$$

Consider $s > 0$ such that $B_\circ(x^*, r+s) \subset \Omega$ and let $a \geqslant \max\left(L, \dfrac{T}{2s}\right)$. Then

$$\frac{1}{4a}\|\nabla f(x)\|^2 \leqslant f(x) - f(x^*), \ x \in B_\circ(x^*, r).$$

3. Main results

Using Theorem 2, we obtain a sharper version of (1) for $p \leqslant 0$ and for any $q \in \mathbb{R}$. In doing this, we analyze the cases $p < 0$ and $p = 0$ for any $q \in \mathbb{R}$ separately. While all cases are managed using the same sharpening scheme, there are some different technical details in the proofs. Therefore, we provide separate theorems for three cases $p < 0$ and $q \neq 0$, $p < 0$ and $q = 0$, and $p = 0$. Before stating the theorems, we introduce some notations for ease of use. In the statements of the theorems and the proofs, we will use the following notations: $\mathbb{R}_+^n = \{(x_1,x_2,\dots,x_n)\in\mathbb{R}^n : x_i \geqslant 0\}$, $\mathbb{R}_{++}^n = \{(x_1,x_2,\dots,x_n)\in\mathbb{R}^n : x_i > 0\}$, $\mathbf{1} = (1,\dots,1)\in\mathbb{R}^n$, $\|\cdot\|_\infty$ denotes maximum norm for $\mathbb{R}^n$, the indices for all $\sum$'s are from 1 to n unless stated otherwise.

First we state the theorem for $p<0$ and $q\neq 0$.

THEOREM 3. *Let $\lambda > r > 0$, $p<0$, $q\in\mathbb{R}\backslash\{0\}$ with $p<q$ and $\boldsymbol{\alpha}\in\mathbb{R}_+^n$ such that $\sum_i \alpha_i = 1$. Define $Z_{u,v} = \frac{u+v}{u-v}$ for $u,v\in\mathbb{R}$ with $u\neq v$ and suppose*

$$C = \left[Z_{\lambda,r}^{1-p} + Z_{\lambda,r}^{1-q}\right]\|\boldsymbol{\alpha}\|_\infty,\quad B = \min_{r<d<\lambda}\max\left\{\Lambda(d), \frac{C}{2(d-r)}\right\}$$

where $\Lambda(d)$ is defined for the following cases of q:

Case 1: For $q<1$ and $q\neq 0$,

$$\Lambda(d) = \frac{2(1-p)}{\lambda-d} Z_{\lambda,d}^{1-2p}\left[\sum_i \alpha_i^2\left(1-2\alpha_i+\sum_k \alpha_k^2\right)\right]^{\frac{1}{2}},$$

Case 2: For $1\leqslant q<2$,

$$\Lambda(d) = \frac{\left[\sum_i \alpha_i^2\left\{(1-\alpha_i)^2\left[(q-1)Z_{\lambda,d}^{q}+(1-p)Z_{\lambda,d}^{1-2p}\right]^2+\sum_{k\neq i}\alpha_k^2\left[(q-1)Z_{\lambda,d}^{2q-2}+(1-p)Z_{\lambda,d}^{1-2p}\right]^2\right\}\right]^{1/2}}{\lambda-d}$$

Case 3: For $2\leqslant q$,

$$\Lambda(d) = \frac{1}{\lambda-d}\left[(q-1)Z_{\lambda,d}^{2q-2}+(1-p)Z_{\lambda,d}^{2p-1}\right]\left[\sum_i(\alpha_i^2-2\alpha_i^3)+\left(\sum_i\alpha_i^2\right)^2\right]^{\frac{1}{2}}.$$

Then for $x\in\mathbb{R}_{++}^n$ satisfying $\|x-\lambda\mathbf{1}\|_\infty\leqslant r$, the following inequality holds:

$$M_q(\boldsymbol{\alpha};x)\geqslant M_p(\boldsymbol{\alpha};x)+\frac{1}{4B}\sum_k \alpha_k^2\left[x_k^{q-1}M_q(\boldsymbol{\alpha};x)^{1-q}-x_k^{p-1}M_p(\boldsymbol{\alpha};x)^{1-p}\right]^2. \quad (4)$$

Proof. Let us consider the function $f:\mathbb{R}_{++}^n\to\mathbb{R}$ defined by

$$f(x) = M_q(\boldsymbol{\alpha};x) - M_p(\boldsymbol{\alpha};x).$$

It is known that $f(x)$ attains its minimum value 0 on any point x such that $x_1 = x_2 = \cdots = x_n$. Let us denote such minimum point by $\lambda\mathbf{1}$. We will apply Theorem 2 to $f(x)$. Let $\|\ \|_\circ$ in Theorem 2 be maximum norm on $\mathbb{R}_{++}^n$. For ease of use we define

$$\rho_i(x) := \frac{df}{dx_i} = \alpha_i x_i^{q-1}M_q(\boldsymbol{\alpha};x)^{1-q} - \alpha_i x_i^{p-1}M_p(\boldsymbol{\alpha};x)^{1-p}.$$

In this case $\|\nabla f(x)\|^2 = \sum_i \rho_i^2(x)$.

First, we will show the Lipschitz continuity of $\|\nabla f\|$ on the set

$$V_{\lambda,d} := \{x \in \mathbb{R}^n : \|x - \lambda \mathbf{1}\|_\infty \leqslant d\} \tag{5}$$

where $\lambda > d > 0$. It is clear that $V_{\lambda,d} \subset \mathbb{R}^n_{++}$. Let us estimate $\|\nabla \rho_i(x)\|$ for $x \in V_{\lambda,d}$ to show the Lipschitz continuity of ∇f to ensure the conditions of Theorem 2. We have

$$\begin{aligned}
\frac{\partial \rho_i}{\partial x_i}(x) = \alpha_i \Bigg\{ &(q-1)x_i^{q-2} M_q(\boldsymbol{\alpha};x)^{1-2q} \left(\sum_{k\neq i} \alpha_k x_k^q\right) \\
&-(p-1)x_i^{p-2} M_p(\boldsymbol{\alpha};x)^{1-2p} \left(\sum_{k\neq i} \alpha_k x_k^p\right) \Bigg\}, \\
\frac{\partial \rho_i}{\partial x_j}(x) = \alpha_i \alpha_j \{ &(1-q)(x_i x_j)^{q-1} M_q(\boldsymbol{\alpha};x)^{1-2q} \\
&-(1-p)(x_i x_j)^{p-1} M_p(\boldsymbol{\alpha};x)^{1-2p} \}, \ (i \neq j).
\end{aligned}$$

The estimates for $\left|\frac{\partial \rho_i}{\partial x_i}(x)\right|$ and $\left|\frac{\partial \rho_i}{\partial x_j}(x)\right|$ on $V_{\lambda,d}$ using (5) depend on the powers of x_i, thus requires the investigation of different cases of q.

Case 1. $q < 1$ and $q \neq 0$. We will prove it in three parts:
i) $q < 0$, ii) $0 < q < \frac{1}{2}$ and iii) $\frac{1}{2} \leqslant q < 1$.

Proof of i). For a better tracking of the calculations, we present the following auxiliary inequalities, which follows from (5):

$$\begin{array}{ll}
\text{a)} & (\lambda+d)^{q-2} \leqslant x_i^{q-2} \leqslant (\lambda-d)^{q-2}, \\
\text{b)} & (\lambda-d)^{1-2q} \leqslant M_q(\boldsymbol{\alpha};x)^{1-2q} \leqslant (\lambda+d)^{1-2q}, \\
\text{c)} & (1-\alpha_i)(\lambda+d)^q \leqslant \sum\limits_{k\neq i}^{n} \alpha_k x_k^q \leqslant (1-\alpha_i)(\lambda-d)^q, \\
\text{d)} & (\lambda+d)^{2q-2} \leqslant (x_i x_j)^{q-1} \leqslant (\lambda-d)^{2q-2}.
\end{array} \tag{6}$$

One can replace q with p in (6) since $p < 0$. Using the inequalities in (6), triangle inequality and

$$(1-q)Z_{\lambda,d}^{1-2q} < (1-p)Z_{\lambda,d}^{1-2p}, \tag{7}$$

we have

$$\begin{aligned}
\left|\frac{\partial \rho_i}{\partial x_i}(x)\right| &\leqslant \frac{\alpha_i(1-\alpha_i)}{\lambda-d}\left[(1-q)Z_{\lambda,d}^{1-2q} + (1-p)Z_{\lambda,d}^{1-2p}\right] \\
&\leqslant \frac{2\alpha_i(1-\alpha_i)(1-p)Z_{\lambda,d}^{1-2p}}{\lambda-d}, \\
\left|\frac{\partial \rho_i}{\partial x_j}(x)\right| &\leqslant \frac{\alpha_i\alpha_j}{\lambda-d}\left[(1-q)Z_{\lambda,d}^{1-2q} + (1-p)Z_{\lambda,d}^{1-2p}\right] \qquad (j \neq i) \\
&< \frac{2\alpha_i\alpha_j}{\lambda-d}(1-p)Z_{\lambda,d}^{1-2p}
\end{aligned} \tag{8}$$

so

$$\|\nabla \rho_i(x)\| \leqslant \frac{2\alpha_i(1-p)}{\lambda-d}\left(1-2\alpha_i+\sum_k \alpha_k^2\right)^{\frac{1}{2}} Z_{\lambda,d}^{1-2p}. \tag{9}$$

Let $x,z \in V_{\lambda,d}$. By means of Cauchy-Schwarz inequality and the mean value theorem, it can be deduced that there exist numbers $\theta_i \in (0,1)$, $i=1,\dots,n$ such that $x+\theta_i(z-x) \in V_{\lambda,d}$ for all i

$$\rho_i(x)-\rho_i(z) = \nabla\rho_i(x+\theta_i(z-x))(x-z).$$

Using the inequality (9)

$$\begin{aligned}
\|\nabla f(x)-\nabla f(z)\| &= \left[\sum_{i=1}^{n}[\nabla\rho_i(x+\theta_i(z-x))(x-z)]^2\right]^{\frac{1}{2}} \\
&= \left[\sum_{i=1}^{n}\|\nabla\rho_i(x+\theta_i(z-x))\|^2\right]^{\frac{1}{2}} \|x-z\| \\
&\leqslant \frac{2(1-p)Z_{\lambda,d}^{1-2p}}{\lambda-d}\left[\sum_{i}^{n}\alpha_i^2\left(1-2\alpha_i+\sum_{k}^{n}\alpha_k^2\right)\right]^{\frac{1}{2}}\|x-z\|
\end{aligned}$$

we conclude that

$$\|\nabla f(x)-\nabla f(z)\| \leqslant \Lambda(d)\,\|x-z\|, \quad x,z \in V_{\lambda,d}$$

where

$$\Lambda(d) = \frac{2(1-p)Z_{\lambda,d}^{1-2p}}{\lambda-d}\left[\sum_i \alpha_i^2\left(1-2\alpha_i+\sum_k \alpha_k^2\right)\right]^{\frac{1}{2}}.$$

Hence ∇f is continuous on $V_{\lambda,d}$ with the Lipschitz constant $L \leqslant \Lambda(d)$ for $q<0$.

Now, let us determine the number T to apply Theorem 2 to the set $\Omega = V_{\lambda,d}$. Let $r \in (0,d)$. For the case i), $T = \max\left\{\|\nabla f(x)\|_\infty : x \in V_{\lambda,r}\right\}$ can be estimated by the triangle inequality, (5) and (6) as follows:

$$\begin{aligned}
T &= \max_{x\in V_{\lambda,r}}\{\|\nabla f(x)\|_\infty\} = \max_{x\in V_{\lambda,r}}\left\{\max_{1\leqslant i\leqslant n}\{|\rho_i(x)|\}\right\} \\
&\leqslant \max_{1\leqslant i\leqslant n}\left\{\alpha_i(\lambda-r)^{q-1}(r+\lambda)^{1-q}+\alpha_i(\lambda-r)^{p-1}(r+\lambda)^{1-p}\right\} \\
&= \max_{1\leqslant i\leqslant n}\{\alpha_i\}\left[Z_{\lambda,r}^{1-q}+Z_{\lambda,r}^{1-p}\right].
\end{aligned}$$

Let us define

$$C = \max_{1\leqslant i\leqslant n}\{\alpha_i\}\left[Z_{\lambda,r}^{1-q}+Z_{\lambda,r}^{1-p}\right].$$

and

$$\beta(d) = \max\left\{\Lambda(d), \frac{C}{2(d-r)}\right\}.$$

Due to the fact that Λ and $d \mapsto \dfrac{C}{2(d-r)}$ are continuous on (r,λ) and $\lim\limits_{d\to\lambda^-}\beta(d) = \lim\limits_{d\to r^+}\beta(d) = +\infty$, $\beta(d)$ attains its minimum on the interval (r,λ). Let $B = \min\limits_{r<d<\lambda}\beta(d)$. Applying Theorem 2 we conclude that

$$M_q(\boldsymbol{\alpha};x) \geqslant M_p(\boldsymbol{\alpha};x) + \frac{1}{4B}\sum_i \alpha_i^2\left[x_i^{q-1}M_q(\boldsymbol{\alpha};x)^{1-q} - x_i^{p-1}M_p(\boldsymbol{\alpha};x)^{1-p}\right]^2 \tag{10}$$

for $x \in V_{\lambda,r}$. Thus the proof is completed for the part i).

Proof of ii) and iii). In both parts, (6c) transforms into

$$(1-\alpha_i)(\lambda-d)^q \leqslant \sum_{k\neq i}^n \alpha_k x_k^q \leqslant (1-\alpha_i)(\lambda+d)^q.$$

In iii) (6b) changes into

$$(\lambda+d)^{1-2q} \leqslant M_q(\boldsymbol{\alpha};x)^{1-2q} \leqslant (\lambda-d)^{1-2q}$$

while the others remain unchanged. Also the following additional inequalities are used instead of (7) in both ii) and iii), respectively,

$$(1-q)Z_{\lambda,d}^{2-q} < (1-p)Z_{\lambda,d}^{2-2p} \quad \text{and} \quad Z_{\lambda,d}^{q} < Z_{\lambda,d}^{1-2p}.$$

Consequently, $\Lambda(d)$ for the part i) is found to be the same $\Lambda(d)$ as for ii) and iii).

On the other hand, the determination of T follows exactly the same path as the case i). So (10) is obtained. Thus the proof has been completed for Case 1.

Case 2: $1 \leqslant q < 2$. The proof proceeds the same as the proof of Case 1 except for a different $\Lambda(d)$. The inequalities (6b), (6c) and (6d) transform into the following corresponding inequalities

$$(\lambda+d)^{1-2q} \leqslant M_q(\boldsymbol{\alpha};x)^{1-2q} \leqslant (\lambda-d)^{1-2q},$$
$$(1-\alpha_i)(\lambda-d)^q \leqslant \sum_{k\neq i} \alpha_k x_k^q \leqslant (1-\alpha_i)(\lambda+d)^q,$$
$$(\lambda-d)^{2q-2} \leqslant (x_i x_j)^{q-1} \leqslant (\lambda+d)^{2q-2}$$

which leads to the $\Lambda(d)$ for Case 2.

Case 3: $2 \leqslant q$. The proof proceeds the same as the proof of Case 1 except for a different $\Lambda(d)$. The inequalities (6a), (6b), (6c), (6d) change into the following corresponding inequalities

$$(\lambda-d)^{q-2} \leqslant x_i^{q-2} \leqslant (\lambda+d)^{q-2},$$
$$(\lambda+d)^{1-2q} \leqslant M_q(\boldsymbol{\alpha};x)^{1-2q} \leqslant (\lambda-d)^{1-2q},$$
$$(1-\alpha_i)(\lambda-d)^q \leqslant \sum_{k\neq i} \alpha_k x_k^q \leqslant (1-\alpha_i)(\lambda+d)^q,$$
$$(\lambda-d)^{2q-2} \leqslant (x_i x_j)^{q-1} \leqslant (\lambda+d)^{2q-2}$$

which leads to the $\Lambda(d)$ for Case 2.

The determination of T for Case 2 and Case 3 follows exactly the same path as Case 1 and necessary computations yield to the same T value in Case 1. □

REMARK 1. Although Theorem 3 appears to give a local refinement of the power mean inequality, it is possible to extend it to any $x \in \mathbb{R}^n_{++}$ by taking $\lambda = \max\{x\}$ and $r = \max\{x\} - \min\{x\}$. Note that by this choice of λ and r, $\|x - \lambda \mathbf{1}\|_\infty \leqslant r$ is satisfied for any $x \in \mathbb{R}^n_{++}$ and this applies to the other theorems.

REMARK 2. In the inequality (10), we note that $\Lambda(d) \to \infty$ as $d \to \lambda$ and $\frac{C}{2(d-r)} \to \infty$ as $d \to r$ implying that $B \leqslant \Lambda(d_0)$ for some d_0 very close to λ and $B \leqslant \frac{C}{2(d_1 - r)}$ for some d_1 very close to r. Note that one can choose $d_1 - r = \varepsilon$ sufficiently small and replace B with $\frac{C}{2\varepsilon}$.

In the light of Remark 1, we present a sharper version of arithmetic-harmonic mean (for $p = -1$ and $q = 1$) as an example.

COROLLARY 1. *The inequality*

$$M_1\left(\frac{\mathbf{1}}{n};x\right) \geqslant M_{-1}\left(\frac{\mathbf{1}}{n};x\right) + \frac{1}{4Bn^2}\left[1 - nM_{-1}^2\left(\frac{\mathbf{1}}{n};x\right)\left(\sum_k x_k^{-2}\right)\right]^2.$$

holds for any $x \in \mathbb{R}^n_{++}$ *where* $B = \frac{1}{n}\min_{r<d<\lambda}\max\left\{2\sqrt{n-1}\frac{(\lambda+d)^3}{(\lambda-d)^4}, \frac{(\lambda+r)^2+\lambda-r}{2(\lambda-r)^2(d-r)}\right\}$, $\lambda = \max\{x\}$ *and* $r = \max\{x\} - \min\{x\}$.

We provide a numerical illustration of Theorem 3

EXAMPLE 1. Let $p=-2$, $q=-1$, $n=5$, $x=(6,7,8,9,10)$, $\boldsymbol{\alpha}=(0.2,0.1,0.15,0.3,0.25)$, $\lambda = 10$ and $r = 4$. Then $M_p(\boldsymbol{\alpha};x) \approx 7.870$, $M_q(\boldsymbol{\alpha};x) \approx 8.019$ and the rightmost term in (4) equals to 9×10^{-5} approximately.

In the following theorem, we give the inequality for $p = 0$. Note that the same proof scheme as Theorem 3 is applied . However, some details in the proof differ from the proof of Theorem 3. Hence, we present it as a separate theorem.

THEOREM 4. *Let* $q > 0$, $\lambda > r > 0$, $\boldsymbol{\alpha} \in \mathbb{R}^n_+$ *such that* $\sum_{i=1}^n \alpha_i = 1$. *Define* $Z_{u,v} = \frac{u+v}{u-v}$ *for* $u, v \in \mathbb{R}$ *with* $u \neq v$ *and*

$$C = 2\|\boldsymbol{\alpha}\|_\infty Z_{\lambda,r} \text{ and } B = \min_{r<d<\lambda}\max\left\{\Lambda(d), \frac{C}{2(d-r)}\right\}$$

where $\Lambda(d)$ *is given with respect to* q *in the following cases:*

Case 1: For $0 < q \leqslant 1$,

$$\Lambda(d) = \frac{2(\lambda+d)}{(\lambda-d)^2}\left[\sum_i \alpha_i^2 \left\{\left(1-2\alpha_i+\sum_k \alpha_k^2\right)\right\}\right]^{\frac{1}{2}},$$

Case 2: For $1 < q \leqslant 2$,

$$\Lambda(d) = \frac{(\lambda+d)}{(\lambda-d)^2}\left[\sum_i \alpha_i^2(1-\alpha_i)^2\left[(q-1)Z_{\lambda,d}+1\right]^2 + 4\left(-\sum_i \alpha_i^4 + \left(\sum_i \alpha_i^2\right)^2\right)\right]^{\frac{1}{2}},$$

Case 3: For $2 < q$,

$$\Lambda(d) = \frac{2(\lambda+d)}{(\lambda-d)^2}\left[\sum_i^n \alpha_i^2(1-\alpha_i)^2\left[(q-1)Z_{\lambda,d}^{q-1}+1\right]^2 + q^2\left(-\sum_i \alpha_i^4 + \left(\sum_i \alpha_i^2\right)^2\right)\right]^{\frac{1}{2}}.$$

Then for all $x \in \mathbb{R}_{++}^n$ *satisfying* $\|x-\lambda\mathbf{1}\|_\infty \leqslant r$, *the following inequality holds:*

$$M_q(\boldsymbol{\alpha};x) \geqslant M_0(\boldsymbol{\alpha};x) + \frac{1}{4B}\sum_k^n \alpha_k^2\left[x_k^{q-1}M_q(\boldsymbol{\alpha};x)^{1-q} - x_k^{-1}M_0(\boldsymbol{\alpha};x)\right]^2. \tag{11}$$

Proof. Since the proof follows the same path as the proof of Theorem 2, we will give the important highlights of the proof. First we define

$$f(x) = M_q(\boldsymbol{\alpha};x) - M_0(\boldsymbol{\alpha};x).$$

Then we define

$$\rho_i(x) := \frac{df}{dx_i} = \alpha_i x_i^{q-1} M_q(\boldsymbol{\alpha};x)^{1-q} - \alpha_i x_i^{-1} M_0(\boldsymbol{\alpha};x).$$

Next we show the Lipschitz continuity of ∇f on the set $V_{\lambda,d} = \{x \in \mathbb{R}^n : \|x-\lambda\mathbf{1}\|_\infty \leqslant d\}$ where $\lambda > d > 0$. To do this, we find the estimates for $\left|\frac{\partial\rho_i}{\partial x_i}(x)\right|$ and $\left|\frac{\partial\rho_i}{\partial x_j}(x)\right|$ on $V_{\lambda,d}$, which depends on the cases q.

We analyze Case 1, $0 < q < 1$, by dividing it into two cases:
i) $0 < q < \frac{1}{2}$ and ii) $\frac{1}{2} \leqslant q < 1$.

For i) and ii), we derive the auxiliary equations similar to (6) and get the following estimation

$$\|\nabla\rho_i(x)\| \leqslant 2\alpha_i\left(1-2\alpha_i+\sum_k \alpha_k^2\right)^{\frac{1}{2}}\frac{\lambda+d}{(\lambda-d)^2}. \tag{12}$$

By using a similar mean value argument in the previous theorem and using (12), we get

$$\|\nabla f(x) - \nabla f(z)\| \leqslant \frac{2(\lambda+d)}{(\lambda-d)^2}\left[\sum_i \alpha_i^2\left(1-2\alpha_i+\sum_k \alpha_k^2\right)\right]^{\frac{1}{2}}\|x-z\|$$

for $x,z \in V_{\lambda,d}$, which yields

$$\Lambda(d) = \frac{2(\lambda+d)}{(\lambda-d)^2}\left[\sum_i^n \alpha_i^2\left(1-2\alpha_i+\sum_k^n \alpha_k^2\right)\right]^{\frac{1}{2}}. \tag{13}$$

for both subcases. Hence ∇f is continuous on $V_{\lambda,d}$ with a Lipschitz constant less than $\Lambda(d)$.

Theorem 2 is applied for the set $\Omega = V_{\lambda,d}$. For $r \in (0,d)$ we obtain

$$\max_{x\in V_{\lambda,r}}\{\|\nabla f(x)\|_\infty\} \leqslant 2\max_{1\leqslant i\leqslant n}\{\alpha_i\}\frac{\lambda+r}{\lambda-r}$$

then let us denote

$$C = 2\max_{1\leqslant i\leqslant n}\{\alpha_i\}\frac{\lambda+r}{\lambda-r} \quad \text{and} \quad \beta(d) = \max\left\{\Lambda(d), \frac{C}{2(d-r)}\right\}.$$

Using the same argument in the proof of the previous theorem, we deduce the existence of the minimum of $\min\limits_{r<d<\lambda}\beta(d)$. We let $B = \min\limits_{r<d<\lambda}\beta(d)$ and conclude that

$$M_q(\boldsymbol{\alpha};x) \geqslant M_0(\boldsymbol{\alpha};x) + \frac{1}{4B}\sum_k^n \alpha_k^2\left[x_k^{q-1}M_q(\boldsymbol{\alpha};x)^{1-q} - x_k^{-1}M_0(\boldsymbol{\alpha};x)\right]^2$$

for $x \in V_{\lambda,r}$.

For Case 2 and Case 3, new auxiliary inequalities are derived and some additional inqualities are used to obtain $\Lambda(d)$. Then $\Lambda(d)$ is obtained as indicated in the statement of theorem. The existence of B is similarly seen. So the indicated result in the expression of theorem is obtained. □

We present a numerical illlustration of Theorem 4

EXAMPLE 2. Let $p=0$, $q=2$, $n=5$, $x=(3,5,8,11,14)$, $\boldsymbol{\alpha}=(0.15,0.2,0.25,0.1,0.30)$, $\lambda=14$ and $r=8$. Then $M_p(\boldsymbol{\alpha};x)\approx 7.676$, $M_q(\boldsymbol{\alpha};x)\approx 9.657$ and the rightmost term in (11) equals to 0.005 approximately.

Now we deal with the case $p<0$ and $q=0$. Note that $q=0$ yields $M_0(\boldsymbol{\alpha};x)$ which corresponds to the generalized geometric mean.

THEOREM 5. *Let $p<0$, $\lambda>r>0$, $\boldsymbol{\alpha}\in\mathbb{R}_+^n$ such that $\sum\limits_{i=1}^n \alpha_i = 1$ and*

$$C = 2\|\boldsymbol{\alpha}\|_\infty\frac{\lambda+r}{\lambda-r} \quad \textit{and} \quad B = \min_{r<d<\lambda}\max\left\{\Lambda(d), \frac{C}{2(d-r)}\right\}$$

where

$$\Lambda(d) = \frac{2(1-p)}{(\lambda-d)}\left(\frac{\lambda+d}{\lambda-d}\right)^{1-2p}\left[\sum_i \alpha_i^2\left(1-2\alpha_i+\sum_k \alpha_k^2\right)\right]^{\frac{1}{2}}.$$

Then for all $x \in \mathbb{R}^n_{++}$ *satisfying* $\|x - \lambda\mathbf{1}\| \leqslant r$, *the following inequality holds*

$$M_0(\boldsymbol{\alpha};x) \geqslant M_p(\boldsymbol{\alpha};x) + \frac{1}{4B}\sum_k \alpha_k^2 \left[x_k^{-1}M_0(\boldsymbol{\alpha};x) - x_k^{p-1}M_p^{1-p}(\boldsymbol{\alpha};x)\right]^2. \tag{14}$$

Proof. The sketch of the proof is same as Theorem 2 and 3. However, different auxiliary inequalities are derived and used in the intermediate steps. □

COROLLARY 2. *Let* $\lambda > r > 0$ *and* $p < q$. *Define* $\Lambda_i(d)$ *and* C_i *for* $\Lambda(d)$ *and* C *for each case* i $(i \in \{1,2,3\})$ *in Theorem* 3 *respectively. Let* $\Lambda_{\min}(d) = \min_{1\leqslant i\leqslant 3} \Lambda_i(d)$ *for* $r < d < \lambda$ *and* $C_{\max} = \max_{1\leqslant i\leqslant 3} C_i$.

$$B = \min_{r<d<\lambda} \max\left\{\Lambda_{\min}(d), \frac{C_{\max}}{2(d-r)}\right\}.$$

Then for $x \in \mathbb{R}^n_{++}$ *satisfying* $\|x - \lambda\mathbf{1}\|_\infty \leqslant r$, *the following inequality holds:*

$$M_q(\boldsymbol{\alpha};x) \geqslant M_p(\boldsymbol{\alpha};x) + \frac{1}{4B}\sum_k \alpha_k^2 \left[x_k^{q-1}M_q^{1-q}(\boldsymbol{\alpha};x) - x_k^{p-1}M_p^{1-p}(\boldsymbol{\alpha};x)\right]^2$$

where $\sum_{i=1}^n \alpha_i = 1$ *and* $\boldsymbol{\alpha} \in \mathbb{R}^n_+$.

In the light of Remark 1, we present the following two examples for $p = 0$, which establish the sharper versions of arithmetic-geometric mean inequality and geometric-harmonic mean inequality.

COROLLARY 3. *The inequality*

$$M_1\left(\frac{\mathbf{1}}{n};x\right) \geqslant M_0\left(\frac{\mathbf{1}}{n};x\right) + \frac{1}{4Bn^2}\sum_k \left[1 - x_k^{-1}M_0\left(\frac{\mathbf{1}}{n};x\right)\right]^2$$

holds for any $x \in \mathbb{R}^n_{++}$ *where* $B = \frac{1}{n}\min_{r<d<\lambda} \max\left\{\sqrt{n-1}\frac{\lambda+d}{(\lambda-d)^2}, \frac{r}{(\lambda-r)(d-r)}\right\}$, $\lambda = \max\{x\}$ *and* $r = \max\{x\} - \min\{x\}$.

COROLLARY 4. *The inequality*

$$M_0\left(\frac{\mathbf{1}}{n};x\right) \geqslant M_{-1}\left(\frac{\mathbf{1}}{n};x\right) + \frac{1}{4B}\sum_i \left[\frac{1}{n}x_i^{-1}M_0\left(\frac{\mathbf{1}}{n};x\right) - nx_i^{-2}M_{-1}\left(\frac{\mathbf{1}}{n};x\right)^2\right]^2.$$

holds for any $x \in \mathbb{R}^n_{++}$ *where* $B = \frac{1}{n}\min_{r<d<\lambda} \max\left\{2\sqrt{n-1}\frac{(\lambda+d)^3}{(\lambda-d)^4}, \frac{(\lambda+r)^2+\lambda-r}{2(\lambda-r)^2(d-r)}\right\}$, $\lambda =$ $\max\{x\}$ *and* $r = \max\{x\} - \min\{x\}$.

We give a numerical example for Theorem 5.

EXAMPLE 3. Let $p = -2$, $q = 0$, $n = 5$, $x = (13,15,18,10,14)$, $\boldsymbol{\alpha} = (0.32,0.23,$ $0.13,0.2,0.12)$, $\lambda = 18$ and $r = 8$. Then $M_p(\boldsymbol{\alpha};x) \approx 12.987$, $M_q(\boldsymbol{\alpha};x) \approx 13.418$ and the rightmost term in (14) equals to 0.0003 approximately.

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