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On Gaussian Leonardo Hybrid Polynomials

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Abstract: In the present paper, we first study the Gaussian Leonardo numbers and Gaussian Leonardo hybrid numbers. We give some new results for the Gaussian Leonardo numbers, including relations with the Gaussian Fibonacci and Gaussian Lucas numbers, and also give some new results for the Gaussian Leonardo hybrid numbers, including relations with the Gaussian Fibonacci and Gaussian Lucas hybrid numbers. For the proofs, we use the symmetric and antisymmetric properties of the Fibonacci and Lucas numbers. Then, we introduce the Gaussian Leonardo polynomials, which can be considered as a generalization of the Gaussian Leonardo numbers. After that, we introduce the Gaussian Leonardo hybrid polynomials, using the Gaussian Leonardo polynomials as coefficients instead of real numbers in hybrid numbers. Moreover, we obtain the recurrence relations, generating functions, Binet-like formulas, Vajda-like identities, Catalan-like identities, Cassini-like identities, and d'Ocagne-like identities for the Gaussian Leonardo polynomials and hybrid polynomials, respectively.

Keywords: Fibonacci number; Leonardo number; Gaussian number; hybrid number; hybrid polynomial

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1. Introduction

The significance of special integer sequences extends beyond the confines of pure and applied mathematics, transcending into various scientific domains such as physics and engineering. The most famous integer sequence is the Fibonacci sequence, named after the Italian mathematician Leonardo Pisano, more commonly known as Fibonacci. The Fibonacci sequence starts with 0 and 1, and each subsequent number is generated by the sum of the two preceding ones. The Fibonacci numbers of the Fibonacci sequence are given by 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, and so on. The Fibonacci sequence finds extensive application in various scientific fields such as mathematics, physics, and engineering. On the other hand, one of the most important reasons why the Fibonacci sequence is so interesting and mysterious, and of interest to many researchers is that Fibonacci numbers are widely found in nature and appear in various biological phenomena, including the arrangement of leaves on plants and the proportions of the human body. Fibonacci numbers are also observed in living organisms. Many flowers exhibit a petal arrangement that follows the Fibonacci numbers. Flowers, such as irises and lilies, frequently have a total number of petals that corresponds to a number in the Fibonacci sequence. Sunflowers often have a number of leaves that corresponds to a Fibonacci number, such as 55 or 89. The arrangement of the seed heads also adheres to the Fibonacci spiral. Pineapples commonly exhibit spiral patterns with a count of either 5, 8, 13, or 21, which are also Fibonacci numbers. Moreover, the Fibonacci numbers are found in the family tree of a male honeybee. Male bees, also known as drones, are the result of parthenogenesis, as they are produced from an unfertilized egg laid by the queen. Therefore, male bees only have a mother and no father. On the other hand, female worker bees have both a male (drone) and a female (queen) as their parents. This reproductive pattern can be effectively illustrated by the Fibonacci sequence. Furthermore, the majority of body parts adhere to the numerical patterns of one, two, three, and five. For instance, humans possess a singular nose, a pair of eyes, three segments in each limb, and five fingers on each hand. Additionally, the proportions and measurements of the human body can be further categorized using the concept of the golden ratio. For more information, one can see Refs. [1,2] (see also the studies cited within).

The golden ratio, often represented by the Greek letter ϕ (phi), is one of the most famous and important ratios in mathematics and some other areas such as art and design. The golden ratio, also known as the golden number, golden mean, golden section, golden proportion, or divine proportion, is an algorithm of mathematical symmetry. The golden ratio, which appears frequently in nature, is an irrational number $\frac{1+\sqrt{5}}{2}$ that approximately equals to 1.618.

The sequence of Fibonacci numbers is defined recursively by the relation

$$F_n = F_{n-1} + F_{n-2}, \quad n \geq 2 \tag{1}$$

with initial conditions $F_0 = 0$ and $F_1 = 1$.

The Lucas numbers are closely related to the Fibonacci numbers. In a similar way, the sequence of Lucas numbers is defined recursively by the relation

$$L_n = L_{n-1} + L_{n-2}, \quad n \geq 2 \tag{2}$$

with initial conditions $L_0 = 2$ and $L_1 = 1$.

The Binet formulas of the Fibonacci and Lucas numbers are given by

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \tag{3}$$

and

$$L_n = \alpha^n + \beta^n, \tag{4}$$

respectively, where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$ are the roots of the characteristic equation $x^2 - x - 1 = 0$ of the recurrences (1) and (2). For details on Fibonacci and Lucas numbers, we refer to Ref. [3].

When we look at the ratios of consecutive Fibonacci numbers, these ratios are strongly related to the golden ratio. We have seen this number $\frac{1+\sqrt{5}}{2}$ in the Binet formula (3). It must be noted that the Binet formula is used to find the nth term of the sequence.

The Fibonacci and Lucas numbers are generalized in a variety of ways by different researchers. One of the generalizations of these numbers are Fibonacci polynomials, introduced by Catalan in 1883, and Lucas polynomials, introduced by Bicknell in 1970. These polynomials are defined by the recurrence relations

$$F_0(x) = 0, F_1(x) = 1;\ F_n(x) = xF_{n-1}(x) + F_{n-2}(x), \quad n \geq 2$$

and

$$L_0(x) = 2, L_1(x) = x; L_n(x) = xL_{n-1}(x) + L_{n-2}(x), \quad n \geq 2,$$

respectively.

There are some studies in the literature associated with the Fibonacci and Lucas polynomials, for example, see Refs. [4–7], among others.

In this paper, we will consider the Leonardo numbers, which are closely related to the Fibonacci numbers. The sequence of Leonardo numbers, denoted as A001595 in the On-Line Encyclopedia of Integer Sequences (or OEIS) [8] (available at https://oeis.org/A001595 (accessed on 30 April 1991)) is defined by the recurrence relation

$$Le_n = Le_{n-1} + Le_{n-2} + 1, \quad n \geq 2 \tag{5}$$

with initial conditions $Le_0 = Le_1 = 1$ and $Le_2 = 3$.

The sequence of Leonardo numbers is also defined by the relation

$$Le_n = 2Le_{n-1} - Le_{n-3}, \quad n \geq 3. \tag{6}$$

It must be noted that Catarino and Borges [9] used Le_n to denote the nth Leonardo number instead of L_n (denoted the nth Lucas number) to avoid confusion. So, throughout the paper we use the notation Le_n for the nth Leonardo number.

The Leonardo numbers are

$$1, 1, 3, 5, 9, 15, 25, 41, 67, 109, 177, 287, \ldots$$

The properties of Leonardo numbers are similar to those of Fibonacci numbers, and are connected to the Fibonacci numbers. The following relation between Fibonacci and Leonardo numbers holds:

$$Le_n = 2F_{n+1} - 1, \tag{7}$$

where F_{n+1} is the $(n+1)$th Fibonacci number and Le_n is the nth Leonardo number.

The Binet formula of the Leonardo numbers are given by

$$Le_n = \frac{2\alpha^{n+1} - 2\beta^{n+1} - \alpha + \beta}{\alpha - \beta}, \tag{8}$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$.

In recent times, there has been a huge amount of interest in the Leonardo sequence. In Ref. [9], Catarino and Borges gave some properties for the Leonardo sequence such as generating function, summation formulas, Catalan's identity, Cassini's identity, and d'Ocagne's identity. Then, in Ref. [10], Alp and Koçer obtained some new identities for this sequence, and they gave some relations among the Leonardo, Fibonacci, and Lucas numbers.

Kürüz et al. [11] introduced a generalization of the Leonardo numbers called Leonardo Pisano polynomials. These polynomials are defined by

$$Le_n(x) = 2xLe_{n-1}(x) - Le_{n-3}(x), \quad n \geq 3 \tag{9}$$

with initial conditions $Le_0(x) = Le_1(x) = 1$ and $Le_2(x) = x + 2$.

The first few Leonardo polynomias are: $1, 1, x+2, 2x^2+4x-1, 4x^3+8x^2-2x-1, 8x^4+16x^3-4x^2-3x-2, 16x^5+32x^4-8x^3-8x^2-8x+1$.

For some studies involving the Leonardo sequence, one can see, for example, Refs. [12–16], among others.

Two-dimensional number systems, such as complex, hyperbolic, and dual numbers, have found numerous applications in the fields of science and engineering. Let a and b be two real numbers. A complex number is in the form $a + b\mathbf{i}$, where $\mathbf{i}$ is the imaginary unit satisfying $\mathbf{i}^2 = -1$. Hyperbolic numbers and dual numbers are similar to complex numbers, but both hyperbolic and dual numbers differ from complex numbers because of their hyperbolic and dual units, respectively. More clearly, a hyperbolic number is in the form $a + b\mathbf{h}$, where $\mathbf{h}$ is the hyperbolic unit satisfying $\mathbf{h}^2 = 1$ for $\mathbf{h} \neq \pm 1$, and a dual number is in the form $a + b\varepsilon$, where ε is the dual unit satisfying $\varepsilon^2 = 0$ for $\varepsilon \neq 0$. The hybrid number system, which can be considered as a generalization of the complex, hyperbolic, and dual number systems, is defined by Özdemir in Ref. [17]. The set of the hybrid numbers is defined as

$$\mathbb{K} = \{z = a + b\mathbf{i} + c\varepsilon + d\mathbf{h} : a, b, c, d \in \mathbb{R}, \mathbf{i}^2 = -1, \varepsilon^2 = 0, \mathbf{h}^2 = 1, \mathbf{ih} = -\mathbf{hi} = \varepsilon + \mathbf{i}.\}.$$

Let $z = a + b\mathbf{i} + c\varepsilon + d\mathbf{h}$ be a hybrid number. Here, a is called the scalar part, and $b\mathbf{i} + c\varepsilon + d\mathbf{h}$ is called the vector part.

From the relation $\mathbf{ih} = -\mathbf{hi} = \varepsilon + \mathbf{i}$, the multiplication rules for the hybrid units $\mathbf{i}$, ε, $\mathbf{h}$ can be obtained as follows:

$$\mathbf{i}\varepsilon = 1 - \mathbf{h}, \quad \varepsilon\mathbf{i} = \mathbf{h} + 1, \quad \mathbf{ih} = \varepsilon + \mathbf{i}, \quad \mathbf{hi} = -\varepsilon - \mathbf{i}, \quad \varepsilon\mathbf{h} = -\varepsilon, \quad \mathbf{h}\varepsilon = \varepsilon. \tag{10}$$

Let $z_1 = a_1 + b_1\mathbf{i} + c_1\varepsilon + d_1\mathbf{h}$ and $z_2 = a_2 + b_2\mathbf{i} + c_2\varepsilon + d_2\mathbf{h}$ be two hybrid numbers. Then the addition of these two hybrid numbers is given by

$$z_1 + z_2 = (a_1 + a_2) + (b_1 + b_2)\mathbf{i} + (c_1 + c_2)\varepsilon + (d_1 + d_2)\mathbf{h},$$

and the multiplication of these two hybrid numbers is given by

$$\begin{aligned} z_1 \times z_2 = \; & (a_1a_2 - b_1b_2 + b_1c_2 + c_1b_2 + d_1d_2) + (a_1b_2 + b_1a_2 + b_1d_2 - d_1b_2)\mathbf{i} \\ & + (a_1c_2 + b_1d_2 - d_1b_2 + c_1a_2 - c_1d_2 + d_1c_2)\varepsilon \\ & + (a_1d_2 + d_1a_2 - b_1c_2 + c_1b_2)\mathbf{h}. \end{aligned}$$

Note that the operation of addition in the hybrid numbers is commutative and associative, and the operation of multiplication in the hybrid numbers is associative but not commutative. The set of the hybrid numbers form a non-commutative ring with respect to the addition and multiplication operations. By defining the map $\varphi : \mathbb{K} \to \mathbb{M}_{2\times2}$, where

$$\varphi(a + b\mathbf{i} + c\varepsilon + d\mathbf{h}) = \begin{pmatrix} a + c & b - c + d \\ c - b + d & a - c \end{pmatrix}$$

for $a + b\mathbf{i} + c\varepsilon + d\mathbf{h} \in \mathbb{K}$, Özdemir [17] showed that the ring of the hybrid numbers $\mathbb{K}$ is isomorphic to the ring of the real 2×2 matrices $\mathbb{M}_{2\times2}$. He also obtained several properties of the hybrid numbers. For further information, we refer to Ref. [17].

In the literature, hybrid numbers and their generalizations with different integer sequence coefficients have been studied by many researchers. For example, in Ref. [18], Szynal-Liana and Wloch defined the Fibonacci hybrid numbers, using the Fibonacci numbers as coefficients instead of real numbers in hybrid numbers. Kızılateş [19] defined the q-Fibonacci hybrid numbers and q-Lucas hybrid numbers, which are defined by means of the q-integer. Moreover, in Ref. [20], Tan and Ait-Amrane gave a generalization of Fibonacci and Lucas hybrid numbers and investigated some of their properties. Szynal-Liana and Wloch [21] introduced a new notion called the hybrinomials (alias hybrid polynomials). The Fibonacci hybinomials that generalize the Fibonacci hybrid numbers are obtained by using the Fibonacci polynomials as components of hybrid numbers. The authors studied the Fibonacci and Lucas hybrinomials and investigated some properties of them. In Ref. [22], Ait-Amrane et al. introduced a new generalization of the Fibonacci and Lucas hybrinomials.

In Ref. [23], Leonardo hybrid numbers are introduced and studied by Alp and Koçer. The nth Leonardo hybrid number is defined by

$$HLe_n = Le_n + Le_{n+1}\mathbf{i} + Le_{n+2}\varepsilon + Le_{n+3}\mathbf{h}, \tag{11}$$

where Le_n is the nth Leonardo number, and $\mathbf{i}, \varepsilon, \mathbf{h}$ are the hybrid units that satisfy the rules (10). Alp and Koçer also investigated some algebraic properties of these numbers in their studies.

Furthermore, as a generalization of the Equation (11), Kürüz et al. introduced the Leonardo hybrid polynomials, called the Leonardo Pisano hybrinomials, in Ref. [11] by the following:

$$HLe_n(x) = Le_n(x) + Le_{n+1}(x)\mathbf{i} + Le_{n+2}(x)\varepsilon + Le_{n+3}(x)\mathbf{h},$$

where $Le_n(x)$ is the nth Leonardo polynomial, and $\mathbf{i}, \varepsilon, \mathbf{h}$ are the hybrid units which satisfy the rules (10). The authors obtained some basic properties, including the generating function and Binet-like formula, of the Lenardo Pisano hybrinomials.

Several studies related to hybrid numbers with different integer sequence coefficients can be found in Refs. [24–32], among others. See also the studies cited by these papers.

Kara and Yılmaz [33], Taşçı [34], as well as Prasad et al. [35] studied the Gaussian Leonardo numbers. Some basic properties related to Gaussian Leonardo numbers are investigated separately by the authors in Refs. [33–35]. Furthermore, in Ref. [33], Kara and Yılmaz obtained the $n \times n$ Hessenberg matrices whose permanents give the Leonardo and Gaussian Leonardo numbers. The nth Gaussian Leonardo number is defined as

$$GLe_n = Le_n + Le_{n-1}\mathbf{i}, \tag{12}$$

where Le_n is the nth Leonardo number. The Gaussian Leonardo number sequence is defined recursively by the relation

$$GLe_n = GLe_{n-1} + GLe_{n-2} + (1 + \mathbf{i}), \quad n \geq 2 \tag{13}$$

or

$$GLe_n = 2GLe_{n-1} - GLe_{n-3}, \quad n \geq 3 \tag{14}$$

with initial conditions $GLe_0 = 1 - \mathbf{i}$, $GLe_1 = 1 + \mathbf{i}$, and $GLe_2 = 3 + \mathbf{i}$.

Furthermore, the following identities are true [34,36]:

$$GF_n = F_n + F_{n-1}\mathbf{i}, \tag{15}$$
$$GL_n = L_n + L_{n-1}\mathbf{i}, \tag{16}$$
$$GLe_n = 2GF_{n+1} - (1 + \mathbf{i}), \tag{17}$$

where F_n is the nth Fibonacci, L_n is the nth Lucas, GF_n is the nth Gaussian Fibonacci, GL_n is the nth Gaussian Lucas, and GLe_n is the nth Gaussian Leonardo numbers.

Moreover, the Gaussian Leonardo hybrid numbers are studied by Kara and Yılmaz [33]. The recurerence relation, generating function, and Binet formula of the Gaussian Leonardo hybrid numbers are obtained by the authors. The nth Gaussian Leonardo hybrid number is defined by

$$HGLe_n = GLe_n + GLe_{n+1}\mathbf{i} + GLe_{n+2}\varepsilon + GLe_{n+3}\mathbf{h}, \tag{18}$$

where GLe_n is the nth Gaussian Leonardo number, and $\mathbf{i}, \varepsilon, \mathbf{h}$ are the hybrid units that satisfy the rules (10).

In this study, we first obtain some new results for the Gaussian Leonardo numbers [33–35] and Gaussian Leonardo hybrid numbers [33]. After that, motivated by the above mentioned papers, we introduce a new notion called Gaussian Leonardo polynomials. Furthermore, by the aid of the Gaussian Leonardo polynomials, we introduce the Gaussian Leonardo hybrid polynomials. We also present and prove some results that relate the Gaussian Leonardo polynomials and hybrid polynomials.

2. Some New Results for Gaussian Leonardo Numbers and Related Hybrid Numbers

In this section, we first obtain some identities for the Gaussian Leonardo numbers, including relations with the Gaussian Fibonacci and Lucas numbers. Then, we give some results for the Gaussian Leonardo hybrid numbers, including relations with the Gaussian Fibonacci and Lucas hybrid numbers.

Theorem 1. *Let GLe_n be the nth Gaussian Leonardo number. Then the following identities hold:*

$$\begin{aligned} GLe_{n-1} + GLe_{n+1} &= 2GL_{n+1} - 2(1+\mathbf{i}), \quad (19)\\ GLe_{n+2} - GLe_{n-2} &= 2GL_{n+1}, \quad (20)\\ GLe_n + 2GF_n &= GLe_{n+1}, \quad (21)\\ GLe_n + GF_n + GL_n &= 2GLe_n + (1+\mathbf{i}), \quad (22)\\ GLe_n^2 + GLe_{n-1}^2 &= 2Le_{2n-1} - 2Le_{n-1}(1 - 2\mathbf{i}(L_n - 1)). \quad (23) \end{aligned}$$

Proof. (19): By virtue of the Equation (12), we have

$$\begin{aligned} GLe_{n-1} + GLe_{n+1} &= (Le_{n-1} + Le_{n-2}\mathbf{i}) + (Le_{n+1} + Le_n\mathbf{i})\\ &= (Le_{n-1} + Le_{n+1}) + (Le_{n-2} + Le_n)\mathbf{i}\\ &= (2L_{n+1} - 2) + (2L_n - 2)\mathbf{i}\\ &= 2(L_{n+1} + L_n\mathbf{i}) - 2(1+\mathbf{i})\\ &= 2GL_{n+1} - 2(1+\mathbf{i}). \end{aligned}$$

Here, we use the equation $Le_{n-1} + Le_{n+1} = 2L_{n+1} - 2$ (see Ref. [10]).

(20): Using the equation $Le_{n+2} - Le_{n-2} = 2L_{n+1}$ (see Ref. [10]), the proof is similar to Equation (19).

(21): From Equations (12) and (15), we have

$$\begin{aligned} GLe_n + 2GF_n &= (Le_n + Le_{n-1}\mathbf{i}) + 2(F_n + F_{n-1}\mathbf{i})\\ &= 2(F_{n+1} + F_n) - 1 + 2(F_n + F_{n-1})\mathbf{i} - \mathbf{i}\\ &= 2(F_{n+2} + F_{n+1}\mathbf{i}) - (1+\mathbf{i})\\ &= 2GF_{n+2} - (1+\mathbf{i})\\ &= GLe_{n+1}. \end{aligned}$$

Here, we use Equations (7) and (17).

(22): Using the following equation, $F_n + L_n = 2F_{n+1}$ (see Ref. [3]), the proof can be done in a similar manner.

(23): Using Equation (12) and considering Equations $Le_n^2 + Le_{n+1}^2 = 2(Le_{2n+2} - Le_{n+2} + 1)$ and $Le_n + Le_{n-2} = 2L_n - 2$ (see Ref. [10]), we have

$$\begin{aligned} GLe_n^2 + GLe_{n-1}^2 &= (Le_n + Le_{n-1}\mathbf{i})^2 + (Le_{n-1} + Le_{n-2}\mathbf{i})^2\\ &= (Le_n^2 + Le_{n-1}^2) - (Le_{n-1}^2 + Le_{n-2}^2) + 4\mathbf{i}Le_{n-1}(L_n - 1)\\ &= 2Le_{2n-1} - 2Le_{n-1}(1 - 2\mathbf{i}(L_n - 1)). \end{aligned}$$

□

Theorem 2. *Let $HGLe_n$ be the nth Gaussian Leonardo hybrid number. Then the following identities hold:*

$$HGLe_{n-1} + HGLe_{n+1} = 2HGL_{n+1} - 2(1 + 3\mathbf{i} + 2\varepsilon), \quad (24)$$
$$HGLe_n + HGLe_{n+1} = HGLe_{n+2} - (1 + 3\mathbf{i} + 2\varepsilon), \quad (25)$$
$$HGLe_{n+2} - HGLe_{n-2} = 2HGL_{n+1}, \quad (26)$$
$$HGLe_n + 2HGF_n = HGLe_{n+1}, \quad (27)$$
$$HGLe_n + HGF_n + HGL_n = 2HGLe_n + (1 + 3\mathbf{i} + 2\varepsilon), \quad (28)$$

where $HGF_n = GF_n + GF_{n+1}\mathbf{i} + GF_{n+2}\varepsilon + GF_{n+3}\mathbf{h}$ is the nth Gaussian Fibonacci hybrid number and $HGL_n = GL_n + GL_{n+1}\mathbf{i} + GL_{n+2}\varepsilon + GL_{n+3}\mathbf{h}$ is the nth Gaussian Lucas hybrid number [37].

Proof. (24): By virtue of the Equations (18) and (19), we have

$$\begin{aligned} HGLe_{n-1} + HGLe_{n+1} &= (GLe_{n-1} + GLe_{n+1}) + (GLe_n + GLe_{n+2})\mathbf{i} \\ &\quad + (GLe_{n+1} + GLe_{n+3})\varepsilon + (GLe_{n+2} + GLe_{n+4})\mathbf{h} \\ &= (2GL_{n+1} - 2(1+\mathbf{i})) + (2GL_{n+2} - 2(1+\mathbf{i}))\mathbf{i} \\ &\quad + (2GL_{n+3} - 2(1+\mathbf{i}))\varepsilon + (2GL_{n+4} - 2(1+\mathbf{i}))\mathbf{h} \\ &= 2(GL_{n+1} + GL_{n+2}\mathbf{i} + GL_{n+3}\varepsilon + GL_{n+4}\mathbf{h}) - 2(1+\mathbf{i})(1 + \mathbf{i} + \varepsilon + \mathbf{h}) \\ &= 2HGL_{n+1} - 2(1 + 3\mathbf{i} + 2\varepsilon). \end{aligned}$$

(25): By virtue of Equation (18) and using the relation $GLe_n + GLe_{n+1} = GLe_{n+2} - (1 + \mathbf{i})$ (see Ref. [34]), we have

$$\begin{aligned} HGLe_n + HGLe_{n+1} &= (GLe_n + GLe_{n+1}) + (GLe_{n+1} + GLe_{n+2})\mathbf{i} \\ &\quad + (GLe_{n+2} + GLe_{n+3})\varepsilon + (GLe_{n+3} + GLe_{n+4})\mathbf{h} \\ &= GLe_{n+2} + GLe_{n+3}\mathbf{i} + GLe_{n+4}\varepsilon + GLe_{n+5}\mathbf{h} - (1+\mathbf{i})(1 + \mathbf{i} + \varepsilon + \mathbf{h}) \\ &= HGLe_{n+2} - (1 + 3\mathbf{i} + 2\varepsilon). \end{aligned}$$

Thus, the proof is completed.

Equations (26)–(28) can be obtained in a similar manner. □

Example 1. *If $n = 1$ for Equation (19), $n = 2$ for Equation (20), and $n = 0$ for Equation (22) in Theorem 1, then we obtain*

$$\begin{aligned} GLe_0 + GLe_2 &= (1 - \mathbf{i}) + (3 + \mathbf{i}) = 4 \\ 2GL_2 - 2(1 + \mathbf{i}) &= 2(3 + \mathbf{i}) - 2(1 + \mathbf{i}) = 4, \end{aligned}$$

$$\begin{aligned} GLe_4 - GLe_0 &= (9 + 5\mathbf{i}) - (1 - \mathbf{i}) = 8 + 6\mathbf{i} \\ 2GL_3 &= 2(4 + 3\mathbf{i}) = 8 + 6\mathbf{i}, \end{aligned}$$

and

$$\begin{aligned} GLe_0 + GF_0 + GL_0 &= (1 - \mathbf{i}) + \mathbf{i} + (2 - \mathbf{i}) = 3 - \mathbf{i} \\ 2GLe_0 + (1 + \mathbf{i}) &= 2(1 - \mathbf{i}) + (1 + \mathbf{i}) = 3 - \mathbf{i}, \end{aligned}$$

respectively.

Example 2. *If $n = 1$ for Equation (24), $n = 0$ for Equations (25), (27), and (28) in Theorem 2, then we obtain*

$$\begin{aligned} HGLe_0 + HGLe_2 &= (1 + 3\mathbf{i} + 6\varepsilon + 4\mathbf{h}) + (5 + 15\mathbf{i} + 18\varepsilon + 10\mathbf{h}) = 6 + 18\mathbf{i} + 24\varepsilon + 14\mathbf{h} \\ 2HGL_2 - 2(1 + 3\mathbf{i} + 2\varepsilon) &= 2(4 + 12\mathbf{i} + 14\varepsilon + 7\mathbf{h}) - 2(1 + 3\mathbf{i} + 2\varepsilon) = 6 + 18\mathbf{i} + 24\varepsilon + 14\mathbf{h}, \end{aligned}$$

$$\begin{aligned} HGLe_0 + HGLe_1 &= (1 + 3\mathbf{i} + 6\varepsilon + 4\mathbf{h}) + (3 + 9\mathbf{i} + 10\varepsilon + 6\mathbf{h}) = 4 + 12\mathbf{i} + 16\varepsilon + 10\mathbf{h} \\ HGLe_2 - (1 + 3\mathbf{i} + 2\varepsilon) &= (5 + 15\mathbf{i} + 18\varepsilon + 10\mathbf{h}) - (1 + 3\mathbf{i} + 2\varepsilon) = 4 + 12\mathbf{i} + 16\varepsilon + 10\mathbf{h}, \end{aligned}$$

$$\begin{aligned} HGLe_0 + 2HGF_0 &= (1 + 3\mathbf{i} + 6\varepsilon + 4\mathbf{h}) + 2(1 + 3\mathbf{i} + 2\varepsilon + \mathbf{h}) = 3 + 9\mathbf{i} + 10\varepsilon + 6\mathbf{h} \\ HGLe_1 &= = 3 + 9\mathbf{i} + 10\varepsilon + 6\mathbf{h}, \end{aligned}$$

and

$$\begin{aligned} HGLe_0 + HGF_0 + HGL_0 &= (1 + 3\mathbf{i} + 6\varepsilon + 4\mathbf{h}) + (1 + 3\mathbf{i} + 2\varepsilon + \mathbf{h}) + (1 + 3\mathbf{i} + 6\varepsilon + 3\mathbf{h}) \\ &= 3 + 9\mathbf{i} + 14\varepsilon + 8\mathbf{h} \\ 2HGLe_0 + (1 + 3\mathbf{i} + 2\varepsilon) &= 2(1 + 3\mathbf{i} + 6\varepsilon + 4\mathbf{h}) + (1 + 3\mathbf{i} + 2\varepsilon) = 3 + 9\mathbf{i} + 14\varepsilon + 8\mathbf{h}, \end{aligned}$$

respectively.

3. The Gaussian Leonardo Polynomials

In this section, we first define the Gaussian Leonardo polynomials. Gaussian Leonardo polynomials are a generalization of the Gaussian Leonardo hybrid numbers. Then we give some properties of these polynomials.

Definition 1. *The nth Gaussian Leonardo polynomial $GLe_n(x)$ is defined by*

$$GLe_n(x) = Le_n(x) + Le_{n-1}(x)\mathbf{i}, \tag{29}$$

where $Le_n(x)$ is the nth Leonardo polynomial.

The first few Gaussian Leonardo polynomials are: $1 + (x-2)\mathbf{i}$, $1 + \mathbf{i}$, $x + 2 + \mathbf{i}$, $2x^2 + 4x - 1 + (x+2)\mathbf{i}$, $4x^3 + 8x^2 - 2x - 1 + (2x^2 + 4x - 1)\mathbf{i}$, $8x^4 + 16x^3 - 4x^2 - 3x - 2 + (4x^3 + 8x^2 - 2x - 1)\mathbf{i}$, $16x^5 + 32x^4 - 8x^3 - 8x^2 - 8x + 1 + (8x^4 + 16x^3 - 4x^2 - 3x - 2)\mathbf{i}$.

Remark 1. *If we put $x = 1$ in (29), then we obtain the nth Gaussian Leonardo number in Refs. [33–35].*

Theorem 3. *Let $n \geq 3$ be an integer. Then the recurrence relation of the Gaussian Leonardo polynomial sequence is*

$$GLe_n(x) = 2xGLe_{n-1}(x) - GLe_{n-3}(x) \tag{30}$$

with initial conditions $GLe_0(x) = 1 + (x-2)\mathbf{i}$, $GLe_1(x) = 1 + \mathbf{i}$, and $GLe_2(x) = x + 2 + \mathbf{i}$.

Proof. If $n = 3$, then using the Equation (29) we get

$$\begin{aligned} 2xGLe_2(x) - GLe_0(x) &= 2x(Le_2(x) + Le_1(x)\mathbf{i}) - (Le_0(x) + Le_{-1}(x)\mathbf{i}) \\ &= 2x(x + 2 + \mathbf{i}) - (1 + (x-2)\mathbf{i}) \\ &= 2x^2 + 4x - 1 + (x+2)\mathbf{i} \\ &= Le_3(x) + Le_2(x)\mathbf{i} \\ &= GLe_3(x). \end{aligned}$$

If $n > 3$, then using Equations (9) and (29) we get

$$\begin{aligned} GLe_n(x) &= Le_n(x) + Le_{n-1}(x)\mathbf{i} \\ &= 2xLe_{n-1}(x) - Le_{n-3}(x) + (2xLe_{n-2}(x) - Le_{n-4}(x))\mathbf{i} \\ &= 2x(Le_{n-1}(x) + Le_{n-2}(x)\mathbf{i}) - (Le_{n-3}(x) + Le_{n-4}(x)\mathbf{i}) \\ &= 2xGLe_{n-1}(x) - GLe_{n-3}(x). \end{aligned}$$

This completes the proof. □

Theorem 4. *The generating function for the Gaussian Leonardo polynomial sequence is*

$$g(x,t) = \frac{1 + (x-2)\mathbf{i} + (1 - 2x + (1 + 4x - 2x^2)\mathbf{i})t + (2 - x + (1-2x)\mathbf{i})t^2}{1 - 2xt + t^3}. \tag{31}$$

Proof. In order to find the generating function of the Gaussian Leonardo polynomial sequence, we have to write the sequence as a power series in which each term of the sequence corresponds to the coefficients of the series. Let $g(x,t)$ be the generating function of the Gaussian Leonardo polynomial sequence. Then we can write the following:

$$\begin{aligned} g(x,t) &= \sum_{n=0}^{\infty} GLe_n(x)t^n, \\ -2xtg(x,t) &= -2x\sum_{n=0}^{\infty} GLe_n(x)t^{n+1}, \\ t^3 g(x,t) &= \sum_{n=0}^{\infty} GLe_n(x)t^{n+3}. \end{aligned}$$

Using the recurrence relation (30) of the Gaussian Leonardo polynomials, we obtain

$$(1 - 2xt + t^3)g(x,t) = GLe_0(x) + (GLe_1(x) - 2xGLe_0(x))t + (GLe_2(x) - 2xGLe_1(x))t^2.$$

Considering $GLe_0(x) = 1 + (x-2)\mathbf{i}$, $GLe_1(x) = 1 + \mathbf{i}$, and $GLe_2(x) = x + 2 + \mathbf{i}$, the desired result can be obtained. □

Remark 2. *If we put $x = 1$ in (31), then we obtain the generating function of the Gaussian Leonardo numbers [33–35] as*

$$g(t) = \frac{1 - \mathbf{i} + (-1 + 3\mathbf{i})t + (1 - \mathbf{i})t^2}{1 - 2t + t^3}.$$

Theorem 5. *The Binet-like formula for the Gaussian Leonardo polynomial sequence is given by*

$$GLe_n(x) = A{r_1}^n + B{r_2}^n + C{r_3}^n, \tag{32}$$

where r_1, r_2, and r_3 are the roots of the characteristic equation $t^3 - 2xt^2 + 1 = 0$, and

$$\begin{aligned} A &= \frac{{r_1}^2(1 + (x-2)\mathbf{i}) + r_1(1 - 2x + (1 + 4x - 2x^2)\mathbf{i}) + 2 - x + (1-2x)\mathbf{i}}{(r_1 - r_2)(r_1 - r_3)}, \\ B &= \frac{{r_2}^2(1 + (x-2)\mathbf{i}) + r_2(1 - 2x + (1 + 4x - 2x^2)\mathbf{i}) + 2 - x + (1-2x)\mathbf{i}}{(r_2 - r_1)(r_2 - r_3)}, \\ C &= \frac{{r_3}^2(1 + (x-2)\mathbf{i}) + r_3(1 - 2x + (1 + 4x - 2x^2)\mathbf{i}) + 2 - x + (1-2x)\mathbf{i}}{(r_3 - r_1)(r_3 - r_2)}. \end{aligned}$$

Proof. The characteristic equation $f(t) = t^3 - 2xt^2 + 1 = 0$ of the recurrence relation (30) should have three distinct roots: r_1, r_2, and r_3. Then $\frac{1}{r_1}$, $\frac{1}{r_2}$, and $\frac{1}{r_3}$ are the roots of the equation $f(\frac{1}{t}) = \frac{1}{t^3} - \frac{2x}{t^2} + 1 = 0$. Here, since $t^3 \neq 0$, we have $1 - 2xt + t^3 = 0$.

By virtue of the generating function of the Gaussian Leonardo polynomial sequence, we can write

$$\frac{1 + (x-2)\mathbf{i} + (1 - 2x + (1 + 4x - 2x^2)\mathbf{i})t + (2 - x + (1-2x)\mathbf{i})t^2}{1 - 2xt + t^3} = \frac{A}{1 - r_1 t} + \frac{B}{1 - r_2 t} + \frac{C}{1 - r_3 t}.$$

Then we have

$$\begin{aligned} &1 + (x-2)\mathbf{i} + (1 - 2x + (1 + 4x - 2x^2)\mathbf{i})t + (2 - x + (1-2x)\mathbf{i})t^2 \\ &\quad = A(1 - r_2 t)(1 - r_3 t) + B(1 - r_1 t)(1 - r_3 t) + C(1 - r_1 t)(1 - r_2 t). \end{aligned}$$

If we take $t = \frac{1}{r_1}$ then we get

$$1 + (x-2)\mathbf{i} + (1 - 2x + (1 + 4x - 2x^2)\mathbf{i})t + (2 - x + (1-2x)\mathbf{i})t^2 = A(1 - \frac{r_2}{r_1})(1 - \frac{r_3}{r_1})$$

and so

$$r_1{}^2(1 + (x-2)\mathbf{i}) + r_1(1 - 2x + (1 + 4x - 2x^2)\mathbf{i}) + 2 - x + (1-2x)\mathbf{i} = A(r_1 - r_2)(r_1 - r_3).$$

Then we obtain

$$A = \frac{r_1{}^2(1 + (x-2)\mathbf{i}) + r_1(1 - 2x + (1 + 4x - 2x^2)\mathbf{i}) + 2 - x + (1-2x)\mathbf{i}}{(r_1 - r_2)(r_1 - r_3)}.$$

In a similar manner, if we take $t = \frac{1}{r_2}$ and $t = \frac{1}{r_3}$ then we obtain

$$B = \frac{r_2{}^2(1 + (x-2)\mathbf{i}) + r_2(1 - 2x + (1 + 4x - 2x^2)\mathbf{i}) + 2 - x + (1-2x)\mathbf{i}}{(r_2 - r_1)(r_2 - r_3)}$$

and

$$C = \frac{r_3{}^2(1 + (x-2)\mathbf{i}) + r_3(1 - 2x + (1 + 4x - 2x^2)\mathbf{i}) + 2 - x + (1-2x)\mathbf{i}}{(r_3 - r_1)(r_3 - r_2)},$$

respectively.

Thus, we have

$$\begin{aligned} g(x,t) = \sum_{n=0}^{\infty} GLe_n(x)t^n &= \frac{A}{1 - r_1 t} + \frac{B}{1 - r_2 t} + \frac{C}{1 - r_3 t} \\ &= \sum_{n=0}^{\infty} A r_1{}^n t^n + \sum_{n=0}^{\infty} B r_2{}^n t^n + \sum_{n=0}^{\infty} C r_3{}^n t^n \\ &= \sum_{n=0}^{\infty} (A r_1{}^n + B r_2{}^n + C r_3{}^n) t^n. \end{aligned}$$

Hence, we get $GLe_n(x) = A r_1{}^n + B r_2{}^n + C r_3{}^n$, which completes the proof. □

Theorem 6. *(Vajda-like Identity) For any non-negative integers m, n, and r, we have*

$$\begin{aligned} GLe_{n+m}(x)GLe_{n+r}(x) - GLe_n(x)GLe_{n+m+r}(x) &= AB(r_1 r_2)^n(r_1{}^m - r_2{}^m)(r_2{}^r - r_1{}^r) \\ &\quad + AC(r_1 r_3)^n(r_1{}^m - r_3{}^m)(r_3{}^r - r_1{}^r) \\ &\quad + BC(r_2 r_3)^n(r_2{}^m - r_3{}^m)(r_3{}^r - r_2{}^r). \end{aligned} \tag{33}$$

Proof. By virtue of Equation (32), we have

$$
\begin{aligned}
GLe_{n+m}(x)GLe_{n+r}(x) - GLe_n(x)GLe_{n+m+r}(x) \\
&= (Ar_1{}^{n+m} + Br_2{}^{n+m} + Cr_3{}^{n+m})(Ar_1{}^{n+r} + Br_2{}^{n+r} + Cr_3{}^{n+r}) \\
&\quad -(Ar_1{}^{n} + Br_2{}^{n} + Cr_3{}^{n})(Ar_1{}^{n+m+r} + Br_2{}^{n+m+r} + Cr_3{}^{n+m+r}) \\
&= AB(r_1{}^{n+m}r_2{}^{n+r} + r_1{}^{n+r}r_2{}^{n+m} - r_1{}^{n}r_2{}^{n+m+r} - r_1{}^{n+m+r}r_2{}^{n}) \\
&\quad +AC(r_1{}^{n+m}r_3{}^{n+r} + r_1{}^{n+r}r_3{}^{n+m} - r_1{}^{n}r_3{}^{n+m+r} - r_1{}^{n+m+r}r_3{}^{n}) \\
&\quad +BC(r_2{}^{n+m}r_3{}^{n+r} + r_2{}^{n+r}r_3{}^{n+m} - r_2{}^{n}r_3{}^{n+m+r} - r_2{}^{n+m+r}r_3{}^{n}) \\
&= AB(r_1r_2)^n(r_2{}^r(r_1{}^m - r_2{}^m) + r_1{}^r(r_2{}^m - r_1{}^m)) \\
&\quad +AC(r_1r_3)^n(r_3{}^r(r_1{}^m - r_3{}^m) + r_1{}^r(r_3{}^m - r_1{}^m)) \\
&\quad +BC(r_2r_3)^n(r_3{}^r(r_2{}^m - r_3{}^m) + r_2{}^r(r_3{}^m - r_2{}^m)) \\
&= AB(r_1r_2)^n(r_1{}^m - r_2{}^m)(r_2{}^r - r_1{}^r) \\
&\quad +AC(r_1r_3)^n(r_1{}^m - r_3{}^m)(r_3{}^r - r_1{}^r) \\
&\quad +BC(r_2r_3)^n(r_2{}^m - r_3{}^m)(r_3{}^r - r_2{}^r).
\end{aligned}
$$

□

The following particular cases are obtained from the Vajda-like identity (33).

Corollary 1. *(Catalan-like Identity) If we put $m = -r$ in Equation (33), then we have*

$$
\begin{aligned}
GLe_{n-r}(x)GLe_{n+r}(x) - (GLe_n(x))^2 &= AB(r_1r_2)^{n-r}(r_2{}^r - r_1{}^r)^2 \\
&\quad +AC(r_1r_3)^{n-r}(r_3{}^r - r_1{}^r)^2 \\
&\quad +BC(r_2r_3)^{n-r}(r_3{}^r - r_2{}^r)^2.
\end{aligned}
$$

Corollary 2. *(Cassini-like Identity) If we put $r = -m = 1$ in Equation (33), then we have*

$$
\begin{aligned}
GLe_{n-1}(x)GLe_{n+1}(x) - (GLe_n(x))^2 &= AB(r_1r_2)^{n-1}(r_2 - r_1)^2 \\
&\quad +AC(r_1r_3)^{n-1}(r_3 - r_1)^2 \\
&\quad +BC(r_2r_3)^{n-1}(r_3 - r_2)^2.
\end{aligned}
$$

Corollary 3. *(d'Ocagne-like Identity) If we put $r = k - n$ and $m = 1$ in Equation (33), then we have*

$$
\begin{aligned}
GLe_{n+1}(x)GLe_k(x) - GLe_n(x)GLe_{k+1}(x) &= AB(r_1r_2)^n(r_1 - r_2)(r_2{}^{k-n} - r_1{}^{k-n}) \\
&\quad +AC(r_1r_3)^n(r_1 - r_3)(r_3{}^{k-n} - r_1{}^{k-n}) \\
&\quad +BC(r_2r_3)^n(r_2 - r_3)(r_3{}^{k-n} - r_2{}^{k-n}).
\end{aligned}
$$

4. The Gaussian Leonardo Hybrid Polynomials

In this section, we first define the Gaussian Leonardo hybrid polynomials. Then we give some properties of these polynomials.

Definition 2. *The nth Gaussian Leonardo hybrid polynomial $HGLe_n(x)$ is defined by*

$$HGLe_n(x) = GLe_n(x) + GLe_{n+1}(x)\mathbf{i} + GLe_{n+2}(x)\varepsilon + GLe_{n+3}(x)\mathbf{h}, \quad n \geq 0 \tag{34}$$

where $GLe_n(x)$ is the nth Gaussian Leonardo polynomial, and $\mathbf{i}, \varepsilon, \mathbf{h}$ are the hybrid units.

Remark 3. *If we put $x = 1$ in the Equation (34) then we obtain the nth Gaussian Leonardo hybrid number in [33].*

Theorem 7. *Let $n \geq 3$ be an integer. Then the recurrence relation of the Gaussian Leonardo hybrid polynomial sequence is*

$$HGLe_n(x) = 2xHGLe_{n-1}(x) - HGLe_{n-3}(x) \tag{35}$$

with initial conditions

$$\begin{aligned} HGLe_0(x) &= 1 + (2x+1)\mathbf{i} + (2x+4)\varepsilon + (2x^2+4x-2)\mathbf{h}, \\ HGLe_1(x) &= x + 2 + (2x^2+5x+2)\mathbf{i} + (4x^2+8x-2)\varepsilon + (4x^3+8x^2-3x-3)\mathbf{h}, \\ HGLe_2(x) &= 2x^2+4x-1 + (4x^3+10x^2+2x-1)\mathbf{i} + (8x^3+16x^2-4x-2)\varepsilon \\ &\quad +(8x^4+16x^3-6x^2-7x-1)\mathbf{h}. \end{aligned}$$

Proof. If $n = 3$, then using the multiplication rules of the hybrid units we get

$$\begin{aligned} & 2xHGLe_2(x) - HGLe_0(x) \\ = \; & 2x[2x^2+4x-1+(4x^3+10x^2+2x-1)\mathbf{i} + (8x^3+16x^2-4x-2)\varepsilon \\ & \quad +(8x^4+16x^3-6x^2-7x-1)\mathbf{h}] \\ & -[1+(2x+1)\mathbf{i} + (2x+4)\varepsilon + (2x^2+4x-2)\mathbf{h}] \\ = \; & 4x^3+8x^2-2x-1+(8x^4+20x^3+4x^2-4x-1)\mathbf{i} + (16x^4+32x^3-8x^2-6x-4)\varepsilon \\ & +(16x^5+32x^4-12x^3-16x^2-6x+2)\mathbf{h} \\ = \; & [2x^2+4x-1+(x+2)\mathbf{i}] + [4x^3+8x^2-2x-1+(2x^2+4x-1)\mathbf{i}]\mathbf{i} \\ & +[8x^4+16x^3-4x^2-3x-2+(4x^3+8x^2-2x-1)\mathbf{i}]\varepsilon \\ & +[16x^5+32x^4-8x^3-8x^2-8x+1+(8x^4+16x^3-4x^2-3x-2)\mathbf{i}]\mathbf{h} \\ = \; & GLe_3(x) + GLe_4(x)\mathbf{i} + GLe_5(x)\varepsilon + GLe_6(x)\mathbf{h} \\ = \; & HGLe_3(x). \end{aligned}$$

If $n > 3$, then using Equations (30) and (34) we get

$$\begin{aligned} HGLe_n(x) &= GLe_n(x) + GLe_{n+1}(x)\mathbf{i} + GLe_{n+2}(x)\varepsilon + GLe_{n+3}(x)\mathbf{h} \\ &= (2xGLe_{n-1}(x) - GLe_{n-3}(x)) + (2xGLe_n(x) - GLe_{n-2}(x))\mathbf{i} \\ &\quad +(2xGLe_{n+1}(x) - GLe_{n-1}(x))\varepsilon + (2xGLe_{n+2}(x) - GLe_n(x))\mathbf{h} \\ &= 2x(GLe_{n-1}(x) + GLe_n(x)\mathbf{i} + GLe_{n+1}(x)\varepsilon + GLe_{n+2}(x)\mathbf{h}) \\ &\quad -(GLe_{n-3}(x) + GLe_{n-2}(x)\mathbf{i} + GLe_{n-1}(x)\varepsilon + GLe_n(x)\mathbf{h}) \\ &= 2xHGLe_{n-1}(x) - HGLe_{n-3}(x). \end{aligned}$$

Thus, the proof is completed. □

Remark 4. *If we take $x = 1$ in the Equations (34) and (35), then we obtain the nth Gaussian Leonardo hybrid number as*

$$HGLe_n = GLe_n + GLe_{n+1}\mathbf{i} + GLe_{n+2}\varepsilon + GLe_{n+3}\mathbf{h},$$

and the recurrence relation of the Gaussian Leonardo hybrid number sequence as

$$HGLe_n = 2HGLe_{n-1} - HGLe_{n-3}$$

with initial conditions $HGLe_0 = 1 + 3\mathbf{i} + 6\varepsilon + 4\mathbf{h}$, $HGLe_1 = 3 + 9\mathbf{i} + 10\varepsilon + 6\mathbf{h}$, *and* $HGLe_2 = 5 + 15\mathbf{i} + 18\varepsilon + 10\mathbf{h}$, *respectively (see Ref. [33]).*

Theorem 8. *The generating function for the Gaussian Leonardo hybrid polynomial sequence is*

$$G(x,t)=\frac{HGLe_0(x)+(HGLe_1(x)-2xHGLe_0(x))t+(HGLe_2(x)-2xHGLe_1(x))t^2}{1-2xt+t^3}. \quad (36)$$

Proof. Since the proof is very similar to Theorem 4, we omit it. □

Remark 5. *If we put $x=1$ in the Equation (36) then we obtain the generating function of the Gaussian Leonardo hybrid numbers [33] as*

$$\begin{aligned} g(t) &= \frac{HGLe_0+(HGLe_1-2HGLe_0)t+(HGLe_2(x)-2HGLe_1)t^2}{1-2t+t^3} \\ &= \frac{1+3\mathbf{i}+6\varepsilon+4\mathbf{h}+(1+3\mathbf{i}-2\varepsilon-2\mathbf{h})t+(-1-3\mathbf{i}-2\varepsilon-2\mathbf{h})t^2}{1-2t+t^3}. \end{aligned}$$

Theorem 9. *The Binet-like formula for the Gaussian Leonardo hybrid polynomial sequence is given by*

$$HGLe_n(x)=Ar_1{}^*r_1{}^n+Br_2{}^*r_2{}^n+Cr_3{}^*r_3{}^n, \quad (37)$$

where $r_1{}^*=1+r_1\mathbf{i}+r_1{}^2\varepsilon+r_1{}^3\mathbf{h}$, $r_2{}^*=1+r_2\mathbf{i}+r_2{}^2\varepsilon+r_2{}^3\mathbf{h}$, $r_3{}^*=1+r_3\mathbf{i}+r_3{}^2\varepsilon+r_3{}^3\mathbf{h}$, and A,B,C are defined as in Theorem 5.

Proof. By virtue of the Equations (32) and (34), we have

$$\begin{aligned} HGLe_n(x) &= GLe_n(x)+GLe_{n+1}(x)\mathbf{i}+GLe_{n+2}(x)\varepsilon+GLe_{n+3}(x)\mathbf{h} \\ &= (Ar_1{}^n+Br_2{}^n+Cr_3{}^n)+(Ar_1{}^{n+1}+Br_2{}^{n+1}+Cr_3{}^{n+1})\mathbf{i} \\ &\quad +(Ar_1{}^{n+2}+Br_2{}^{n+2}+Cr_3{}^{n+2})\varepsilon+(Ar_1{}^{n+3}+Br_2{}^{n+3}+Cr_3{}^{n+3})\mathbf{h} \\ &= Ar_1{}^n(1+r_1\mathbf{i}+r_1{}^2\varepsilon+r_1{}^3\mathbf{h})+Br_2{}^n(1+r_2\mathbf{i}+r_2{}^2\varepsilon+r_2{}^3\mathbf{h}) \\ &\quad +Cr_3{}^n(1+r_3\mathbf{i}+r_3{}^2\varepsilon+r_3{}^3\mathbf{h}) \\ &= Ar_1{}^*r_1{}^n+Br_2{}^*r_2{}^n+Cr_3{}^*r_3{}^n. \end{aligned}$$

□

Theorem 10. *(Vajda-like Identity) Let m,n, and r be non-negative integers. Then we have*

$$\begin{aligned} &HGLe_{n+m}(x)HGLe_{n+r}(x)-HGLe_n(x)HGLe_{n+m+r}(x) \\ &\quad = AB(r_1r_2)^n(r_1{}^m-r_2{}^m)(r_1{}^*r_2{}^*r_2{}^r-r_2{}^*r_1{}^*r_1{}^r) \\ &\quad\quad +AC(r_1r_3)^n(r_1{}^m-r_3{}^m)(r_1{}^*r_3{}^*r_3{}^r-r_3{}^*r_1{}^*r_1{}^r) \\ &\quad\quad +BC(r_2r_3)^n(r_2{}^m-r_3{}^m)(r_2{}^*r_3{}^*r_3{}^r-r_3{}^*r_2{}^*r_2{}^r). \end{aligned} \quad (38)$$

Proof. By virtue of Equation (37), we have

$$\begin{aligned} &HGLe_{n+m}(x)HGLe_{n+r}(x)-HGLe_n(x)HGLe_{n+m+r}(x) \\ &= (Ar_1{}^*r_1{}^{n+m}+Br_2{}^*r_2{}^{n+m}+Cr_3{}^*r_3{}^{n+m})(Ar_1{}^*r_1{}^{n+r}+Br_2{}^*r_2{}^{n+r}+Cr_3{}^*r_3{}^{n+r}) \\ &\quad -(Ar_1{}^*r_1{}^n+Br_2{}^*r_2{}^n+Cr_3{}^*r_3{}^n)(Ar_1{}^*r_1{}^{n+m+r}+Br_2{}^*r_2{}^{n+m+r}+Cr_3{}^*r_3{}^{n+m+r}) \\ &= AB(r_1{}^*r_2{}^*r_1{}^{n+m}r_2{}^{n+r}+r_2{}^*r_1{}^*r_1{}^{n+r}r_2{}^{n+m}-r_1{}^*r_2{}^*r_1{}^nr_2{}^{n+m+r}-r_2{}^*r_1{}^*r_1{}^{n+m+r}r_2{}^n) \\ &\quad +AC(r_1{}^*r_3{}^*r_1{}^{n+m}r_3{}^{n+r}+r_3{}^*r_1{}^*r_1{}^{n+r}r_3{}^{n+m}-r_1{}^*r_3{}^*r_1{}^nr_3{}^{n+m+r}-r_3{}^*r_1{}^*r_1{}^{n+m+r}r_3{}^n) \\ &\quad +BC(r_2{}^*r_3{}^*r_2{}^{n+m}r_3{}^{n+r}+r_3{}^*r_2{}^*r_2{}^{n+r}r_3{}^{n+m}-r_2{}^*r_3{}^*r_2{}^nr_3{}^{n+m+r}-r_3{}^*r_2{}^*r_2{}^{n+m+r}r_3{}^n) \\ &= AB(r_1r_2)^n(r_1{}^m-r_2{}^m)(r_1{}^*r_2{}^*r_2{}^r-r_2{}^*r_1{}^*r_1{}^r) \\ &\quad +AC(r_1r_3)^n(r_1{}^m-r_3{}^m)(r_1{}^*r_3{}^*r_3{}^r-r_3{}^*r_1{}^*r_1{}^r) \\ &\quad +BC(r_2r_3)^n(r_2{}^m-r_3{}^m)(r_2{}^*r_3{}^*r_3{}^r-r_3{}^*r_2{}^*r_2{}^r). \end{aligned}$$

□

Corollary 4. *(Catalan-like Identity) If we put $m = -r$ in Equation (38), then we have*

$$\begin{aligned} HGLe_{n-r}(x)HGLe_{n+r}(x) - (HGLe_n(x))^2 &= AB(r_1r_2)^{n-r}({r_1}^r - {r_2}^r)({r_2}^*{r_1}^*{r_1}^r - {r_1}^*{r_2}^*{r_2}^r) \\ &+AC(r_1r_3)^{n-r}({r_1}^r - {r_3}^r)({r_3}^*{r_1}^*{r_1}^r - {r_1}^*{r_3}^*{r_3}^r) \\ &+BC(r_2r_3)^{n-r}({r_2}^r - {r_3}^r)({r_3}^*{r_2}^*{r_2}^r - {r_2}^*{r_3}^*{r_3}^r). \end{aligned}$$

Corollary 5. *(Cassini-like Identity) If we put $r = -m = 1$ in Equation (38), then we have*

$$\begin{aligned} HGLe_{n-1}(x)HGLe_{n+1}(x) - (HGLe_n(x))^2 &= AB(r_1r_2)^{n-1}(r_1 - r_2)({r_2}^*{r_1}^*r_1 - {r_1}^*{r_2}^*r_2) \\ &+AC(r_1r_3)^{n-1}(r_1 - r_3)({r_3}^*{r_1}^*r_1 - {r_1}^*{r_3}^*r_3) \\ &+BC(r_2r_3)^{n-1}(r_2 - r_3)({r_3}^*{r_2}^*r_2 - {r_2}^*{r_3}^*r_3). \end{aligned}$$

Corollary 6. *(d'Ocagne-like Identity) If we put $r = k - n$ and $m = 1$ in Equation (38), then we have*

$$\begin{aligned} &HGLe_{n+1}(x)HGLe_k(x) - HGLe_n(x)HGLe_{k+1}(x) \\ &= AB(r_1r_2)^n(r_1 - r_2)({r_1}^*{r_2}^*{r_2}^{k-n} - {r_2}^*{r_1}^*{r_1}^{k-n}) \\ &+AC(r_1r_3)^n(r_1 - r_3)({r_1}^*{r_3}^*{r_3}^{k-n} - {r_3}^*{r_1}^*{r_1}^{k-n}) \\ &+BC(r_2r_3)^n(r_2 - r_3)({r_2}^*{r_3}^*{r_3}^{k-n} - {r_3}^*{r_2}^*{r_2}^{k-n}). \end{aligned}$$

5. Conclusions

In this study, at first, some new identities involving the Gaussian Leonardo numbers and the Gaussian Leonardo hybrid numbers are given. Then a new polynomial sequence, called the Gaussian Leonardo polynomial sequence, is introduced and studied. Moreover, several properties, including the recurrence relation, Binet-like formula, generating function, and some identities, such as Vajda-like identity and Catalan-like identity for these polynomials, are derived. After that, a new hybrid sequence with Gaussian Leonardo polynomial coefficients, called the Gaussian Leonardo hybrid polynomial sequence, is studied. Furthermore, some properties of these hybrid polynomials are investigated.

The sequence of the Gaussian Leonardo polynomials is a generalization of the sequence of the Gaussian Leonardo numbers. Similarly, the sequence of the Gaussian Leonardo hybrid polynomials is a generalization of the Gaussian Leonardo hybrid numbers. Therefore, if we replace $x = 1$ in the nth Gaussian Leonardo polynomial $GLe_n(x)$, we obtain the nth Gaussian Leonardo number GLe_n in Refs. [33–35]. If we replace $x = 1$ in the nth Gaussian Leonardo hybrid polynomial $HGLe_n(x)$, we obtain the nth Gaussian Leonardo hybrid number $HGLe_n$ in [33].

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