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# Rigidity results for submanifolds in generalized Sasakian space forms

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**Abstract**

*In this article, generalized Saasakian space forms are discussed and invariant submanifolds of these space forms are examined. The curvature tensor chosen is of great importance when examining the characterization of a manifold. In this article, invariant submanifolds of generalized Sasakian space forms are characterized according to the $W_0^*-$curvature tensor and pseudoparallel submanifolds are investigated for these space forms.*

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## 1. Introduction

Let $\hat{N}$ be a $(2n+1)$-dimensional, differentiable manifold of class $C^{\infty}$. If the structural group of its tangent bundle reduces to $U(n)\times 1$, $\hat{N}$ is said to have an almost contact structure by J.W.Gray in [10]. Later, equivalently, S. Sasaki and S. Hatakeyama showed that an almost contact structure is given by a triple $(\phi, \xi, \eta)$ satisfying certain conditions in [15],[16]. Many different almost contact structures have been defined such as cosymplectic, Sasakian, almost cosymplectic, quasi Sasakian, normal, $\alpha-$Kenmotsu, $\alpha-$Sasakian, trans-Sasakian in [8] ,[9],[11],[14]. These types of structures bear sufficient resemblance to cosymplectic and Sasakian structures so that it is possible to generalize a portion of cosymplectie and Sasakian geometry to each type.

Let $\hat{N}(\phi, \xi, \eta, g)$ be the almost contact metric manifold. If there are functions $\Upsilon_1, \Upsilon_2, \Upsilon_3$ on $\hat{N}$ such that

$$\begin{aligned}
R(\beta_1,\beta_2)\beta_3 &= \Upsilon_1\left[g(\beta_2,\beta_3)\beta_1 - g(\beta_1,\beta_3)\beta_2\right] \\
&+\Upsilon_2\left[g(\beta_1,\phi\beta_3)\phi\beta_2 - g(\beta_2,\phi\beta_3)\phi\beta_1\right. \\
&\left.+2g(\beta_1,\phi\beta_2)\phi\beta_3\right] + \Upsilon_3\left[\eta(\beta_1)\eta(\beta_3)\beta_2\right. \\
&-\eta(\beta_2)\eta(\beta_3)\beta_1 + g(\beta_1,\beta_3)\eta(\beta_2)\xi \\
&\left.-g(\beta_2,\beta_3)\eta(\beta_1)\xi\right],
\end{aligned} \tag{1.1}$$

$\hat{N} = \hat{N}(\phi, \xi, \eta, g)$ is defined a generalized Sasakian space form and such a manifold is shown by $\hat{N}^{2n+1}(\Upsilon_1, \Upsilon_2, \Upsilon_3)$. For brevity, the generalized Sasakian space forms will now be represented as GS-space forms. Such manifolds were introduced by P. Alegre et al [1]. P. Alegre et al calculated the Riemann curvature tensor of a GS-space forms. In [17], GS-space forms are studied under some conditions related to projective curvature. In this work, U.C. De and A. Sarkar studied GS-space forms that provided $PS = 0$ and $PR = 0$. Again, in [3], the same authors studied quasi conformal flat, Ricci symmetric and Ricci semi-symmetric generalized Sasakian space forms. In [2], the curvatures of para-Sasakian manifolds are studied and in this study authors studied the curvatures of para-Sasakian manifolds. M. Atçeken studied and classified generalized Sasakian space forms for some curvature conditions related to concircular, Riemann, Ricci and projective curvature tensors in [7]. Again, many authors have worked on generalized Sasakian space forms ([4],[12],[13],[5],[6],[19]).

In this article, invariant submanifolds of GS-space forms are discussed. First, the parallel second fundamental form case of the GS-space form has examined. Then, pseudoparallel submanifolds of GS-space forms on the $W_0^*-$curvature tensor have investigated. GS-space forms have been characterized for $W_0^*-$ pseudoparallel, $W_0^*$ $2-$ pseudoparallel, $W_0^*-$Ricci generalized pseudoparallel and $W_0^*$ $2-$Ricci generalized pseudoparallel submanifolds.

## 2. Preliminary

Let's take an $(2n+1)-$dimensional differentiable $N$ manifold. If it admits a tensor field $\phi$ of type $(1,1)$, a vector field $\xi$ and a 1-form $\eta$ satisfying the following conditions;

$$\phi^2\beta_1 = -\beta_1 + \eta(\beta_1)\xi \text{ and } \eta(\xi) = 1,$$

then this $(\phi,\xi,\eta)$ is called an almost contact structure, and the $(N,\phi,\xi,\eta)$ is called an almost contact manifold. If there is a $g$ metric that satisfies the condition

$$g(\phi\beta_1,\phi\beta_2) = g(\beta_1,\beta_2) - \eta(\beta_1)\eta(\beta_2) \text{ and } g(\beta_1,\xi) = \eta(\beta_1),$$

for all $\beta_1,\beta_2 \in \chi(N)$ and $\xi \in \chi(N)$; $(\phi,\xi,\eta,g)$ is called almost contact metric structure and $(N,\phi,\xi,\eta,g)$ is called almost contact metric manifold. On the $(2n+1)$ dimensional $N$ manifold,

$$g(\phi\beta_1,\beta_2) = -g(\beta_1,\phi\beta_2)$$

for all $\beta_1,\beta_2 \in \chi(N)$, that is, $\phi$ is an anti-symmetric tensor field according to the $g$ metric. The $\Phi$ transformation defined as

$$\Phi(\beta_1,\beta_2) = g(\beta_1,\phi\beta_2)$$

for all $\beta_1,\beta_2 \in \chi(N)$, is called the fundamental 2-form of the $(\phi,\xi,\eta,g)$ almost contact metric structure, where

$$\eta \wedge \Phi^n \neq 0.$$

Sasakian space forms are very important for contact metric geometry. The curvature tensor for the Sasakian space form is defined as

$$R(\beta_1,\beta_2)\beta_3 = \left(\tfrac{c+3}{4}\right)[g(\beta_2,\beta_3)\beta_1 - g(\beta_1,\beta_3)\beta_2]$$

$$+\left(\tfrac{c-1}{4}\right)[g(\beta_1,\phi\beta_3)\phi\beta_2 - g(\beta_2,\phi\beta_3)\phi\beta_1$$

$$+2g(\beta_1,\phi\beta_2)\phi\beta_3 + \eta(\beta_1)\eta(\beta_3)\beta_2$$

$$-\eta(\beta_2)\eta(\beta_3)\beta_1 + g(\beta_1,\beta_3)\eta(\beta_2)\xi$$

$$-g(\beta_2,\beta_3)\eta(\beta_1)\xi],$$

If we choose $\Upsilon_1 = \frac{c+3}{4}, \Upsilon_2 = \Upsilon_3 = \frac{c-1}{4}$ in Sasakian space forms, then we get GS-space forms. Therefore such manifolds are a generalization of Saskian space forms.

For a $(2n+1)-$dimensional GS-space form $\hat{N}^{2n+1}(\Upsilon_1,\Upsilon_2,\Upsilon_3)$ the following equations are provided [17].

$$\tilde{\nabla}_{\beta_1}\xi = -(\Upsilon_1 - \Upsilon_3)\phi\beta_1, \tag{2.1}$$

$$\tilde{R}(\xi,\beta_2)\beta_3 = (\Upsilon_1 - \Upsilon_3)[g(\beta_2,\beta_3)\xi - \eta(\beta_3)\beta_2], \tag{2.2}$$

$$\tilde{R}(\beta_1,\beta_2)\xi = (\Upsilon_1 - \Upsilon_3)[\eta(\beta_2)\beta_1 - \eta(\beta_1)\beta_2], \tag{2.3}$$

$$S(\beta_1,\xi) = 2n(\Upsilon_1 - \Upsilon_3)\eta(\beta_1), \tag{2.4}$$

$$Q\xi = 2n(\Upsilon_1 - \Upsilon_3)\xi, \tag{2.5}$$

$$\left(\tilde{\nabla}_{\beta_1}\phi\right)\beta_2 = (\Upsilon_1 - \Upsilon_3)[g(\beta_1,\beta_2)\xi - \eta(\beta_2)\beta_1], \tag{2.6}$$

for all $\beta_1,\beta_2,\beta_3 \in \chi(N)$, where $Q$,$S$ are the Ricci operator, Ricci tensor of manifold $\hat{N}^{2n+1}(\Upsilon_1,\Upsilon_2,\Upsilon_3)$, respectively.

Let $N$ be the immersed submanifold of the $(2n+1)-$dimensional generalized Sasakian space form $\hat{N}^{2n+1}(\Upsilon_1,\Upsilon_2,\Upsilon_3)$. Let the tangent and normal subspaces of $N$ in $\hat{N}^{2n+1}(\Upsilon_1,\Upsilon_2,\Upsilon_3)$ be $\Gamma(TN)$ and $\Gamma\left(T^{\perp}N\right)$, respectively. Gauss and Weingarten formulas for $\Gamma(TN)$ and $\Gamma\left(T^{\perp}N\right)$ are

(2.7) $$\tilde{\nabla}_{\beta_1}\beta_2 = \nabla_{\beta_1}\beta_2 + \sigma(\beta_1, \beta_2),$$

(2.8) $$\tilde{\nabla}_{\beta_1}\beta_4 = -A_{\beta_4}\beta_1 + \nabla^{\perp}_{\beta_1}\beta_4,$$

respectively, for all $\beta_1, \beta_2 \in \Gamma(TN)$ and $\beta_4 \in \Gamma\left(T^{\perp}N\right)$, where $\nabla$ and $\nabla^{\perp}$ are the connections on $N$ and $\Gamma\left(T^{\perp}N\right)$, respectively, $\sigma$ and $A$ are the second fundamental form and the shape operator of $N$. There is a relation

$$g(A_{\beta_4}\beta_1, \beta_2) = g(\sigma(\beta_1, \beta_2), \beta_4)$$

between the second basic form and shape operator defined as above. The covariant derivative of the second fundamental form $\sigma$ is defined as

(2.9) $$\left(\tilde{\nabla}_{\beta_1}\sigma\right)(\beta_2, \beta_3) = \nabla^{\perp}_{\beta_1}\sigma(\beta_2, \beta_3) - \sigma(\nabla_{\beta_1}\beta_2, \beta_3) - \sigma(\beta_2, \nabla_{\beta_1}\beta_3).$$

Specifically, if $\tilde{\nabla}\sigma = 0$, $N$ is said to be is parallel second fundamental form.

Let $R$ be the Riemann curvature tensor of $N$. In this case, the Gauss equation can be expressed as

$$\tilde{R}(\beta_1, \beta_2)\beta_3 = R(\beta_1, \beta_2)\beta_3 + A_{\sigma(\beta_1,\beta_3)}\beta_2 - A_{\sigma(\beta_2,\beta_3)}\beta_1$$
$$+ \left(\tilde{\nabla}_{\beta_1}\sigma\right)(\beta_2, \beta_3) - \left(\tilde{\nabla}_{\beta_1}\sigma\right)(\beta_1, \beta_3).$$

Let $N$ be a Riemannian manifold, $T$ is $(0, k)-$type tensor field and $A$ is $(0, 2)-$type tensor field. In this case, Tachibana tensor field $Q(A, T)$ is defined as

(2.10) $$Q(A,T)(X_1, ..., X_{k-1}; \beta_1, \beta_2) = -T((\beta_1 \wedge_A \beta_2) X_1, ..., X_k)$$
$$-... - T(X_1, ..., X_{k-1}, (\beta_1 \wedge_A \beta_2) X_k),$$

where,

(2.11) $$(\beta_1 \wedge_A \beta_2)\beta_3 = A(\beta_2, \beta_3)\beta_1 - A(\beta_1, \beta_3)\beta_2,$$

$k \geq 1, X_1, X_2, ..., X_k, \beta_1, \beta_2 \in \Gamma(TN)$.

## 3. Invariant Pseudoparalel Submanifolds of Generalized Sasakian Space Forms

Let $N$ be the immersed submanifold of a $(2n+1)-$dimensional GS-space form $\hat{N}^{2n+1}(\Upsilon_1, \Upsilon_2, \Upsilon_3)$. If $\phi(T_{\beta_1}N) \subset T_{\beta_1}N$ in every $\beta_1$ point, the $N$ manifold is called invariant submanifold. From this section of the article, we will assume that the manifold $N$ is the invariant submanifold of the GS-space form $\hat{N}^{2n+1}(\Upsilon_1, \Upsilon_2, \Upsilon_3)$. So, it is clear that

$$\sigma(\phi\beta_1, \beta_2) = \sigma(\beta_1, \phi\beta_2) = \phi\sigma(\beta_1, \beta_2), \tag{3.1}$$

$$\sigma(\beta_1, \xi) = 0, \tag{3.2}$$

for all $\beta_1, \beta_2 \in \Gamma(TN)$.

**Lemma 1.** *Let $N$ be the invariant submanifold of the $(2n+1)-$dimensional GS-space form $\hat{N}^{2n+1}(\Upsilon_1, \Upsilon_2, \Upsilon_3)$. The second fundamental form $\sigma$ of $N$ is parallel if and only if $N$ is the total geodesic submanifold provided $\Upsilon_1 \neq \Upsilon_3$.*

**Proof.** Let's assume that $N$ is parallel second fundamental form. So, we can write

$$\left(\tilde{\nabla}_{\beta_1}\sigma\right)(\beta_2, \beta_3) = \nabla^{\perp}_{\beta_1}\sigma(\beta_2, \beta_3) - \sigma(\nabla_{\beta_1}\beta_2, \beta_3) - \sigma(\beta_2, \nabla_{\beta_1}\beta_3) = 0.$$

If we choose $\beta_3 = \xi$ in the last equation and use (2) and (4), we obtain

$$(\Upsilon_1 - \Upsilon_3)\phi\sigma(\beta_2, \beta_1) = 0.$$

From here it is clear that $N$ is the total geodesic provided $\Upsilon_1 \neq \Upsilon_3$. □

M. Tripathi and P. Gunam described a $T-$curvature tensor of the $(1,3)$ type in an $n$-dimensional $(N, g)$ semi-Riemann manifold [19]. This curvature tensor is defined as

$$\begin{aligned} T(\beta_1, \beta_2)\beta_3 &= a_0 R(\beta_1, \beta_2)\beta_3 + a_1 S(\beta_2, \beta_3)\beta_1 + a_2 S(\beta_1, \beta_3)\beta_2 \\ &+ a_3 S(\beta_1, \beta_2)\beta_3 + a_4 g(\beta_2, \beta_3) Q\beta_1 + a_5 g(\beta_1, \beta_3) Q\beta_2 \\ &+ a_6 g(\beta_1, \beta_2) Q\beta_3 + a_7 r\left[g(\beta_2, \beta_3)\beta_1 - g(\beta_1, \beta_3)\beta_2\right], \end{aligned} \tag{3.3}$$

where $R, S, Q$, and $r$ are Riemann curvature tensor, Ricci curvature tensor, Ricci operator, and scalar curvature of manifold $N$, respectively. According to the choosing of smooth functions $a_0, a_1, ..., a_7$ the curvature tensor $T$ is reduced to some special curvature tensors as follows.

**Definition 1.** *If* $a_0 = 1, a_1 = -a_5 = -\frac{1}{2n}, a_2 = a_3 = a_4 = a_6 = a_7 = 0$ *are chosen in* (15), *the* $W_0^*-$*curvature tensor is defined as*

$$W_0^*(\beta_1, \beta_2)\beta_3 = R(\beta_1, \beta_2)\beta_3 - \frac{1}{2n}\left[S(\beta_2, \beta_3)\beta_1 - g(\beta_1, \beta_3)Q\beta_2\right]. \tag{3.4}$$

For the $(2n+1)-$dimensional generalized Saskian space form, if we choose $\beta_1 = \xi, \beta_2 = \xi, \beta_3 = \xi$ respectively in $(16)$, then we get

$$W_0^*(\xi, \beta_2)\beta_3 = \frac{(1-2n)\Upsilon_3 - 3\Upsilon_2}{2n}\left[g(\beta_2, \beta_3)\xi - \eta(\beta_3)\beta_2\right], \tag{3.5}$$

$$W_0^*(\beta_1, \xi)\beta_3 = 0, \tag{3.6}$$

$$W_0^*(\beta_1, \beta_2)\xi = \frac{(2n-1)\Upsilon_3 + 3\Upsilon_2}{2n}\left[\eta(\beta_1)\beta_2 - \eta(\beta_1)\eta(\beta_2)\xi\right]. \tag{3.7}$$

Let us now examine the pseudoparallel submanifolds of $(2n+1)-$dimensional GS-space forms $\hat{N}^{2n+1}(\Upsilon_1, \Upsilon_2, \Upsilon_3)$ on the $W_0^*-$curvature tensor.

**Definition 2.** *Let* $N$ *be the invariant submanifold of the* $(2n+1)-$*dimensional GS-space form* $\hat{N}^{2n+1}(\Upsilon_1, \Upsilon_2, \Upsilon_3)$. *If* $W_0^* \cdot \sigma$ *and* $Q(g, \sigma)$ *are linearly dependent,* $N$ *is called* $W_0^*-$*pseudoparallel submanifold.*

Equivalent to this definition, it can be said that there is a function $k_1$ on the set $M_1 = \{x \in N | \sigma(x) \neq g(x)\}$ such that

$$W_0^* \cdot \sigma = k_1 Q(g, \sigma).$$

If $k_1 = 0$ specifically, $N$ is called a $W_0^*-$semiparallel submanifold.

Let $N$ be the invariant submanifold of the $(2n+1)-$dimensional GS-space form $\hat{N}^{2n+1}(\Upsilon_1, \Upsilon_2, \Upsilon_3)$. If $N$ is $W_0^*-$pseudoparallel submanifold, then $N$ is either a total geodesic or $k_1 = \frac{(2n-1)\Upsilon_3 + 3\Upsilon_2}{2n}$.

**Proof.** Let's assume that $N$ is a $W_0^*-$pseudoparallel submanifold. So, we can write

$$\left(W_0^*\left(\beta_1,\beta_2\right)\cdot\sigma\right)\left(\beta_5,\beta_4\right)=k_1Q\left(g,\sigma\right)\left(\beta_5,\beta_4;\beta_1,\beta_2\right),$$

for all $\beta_1,\beta_2,\beta_5,\beta_4\in\Gamma\left(TN\right)$. From, it is clear that

$$\begin{aligned}&R^{\perp}\left(\beta_1,\beta_2\right)\sigma\left(\beta_5,\beta_4\right)-\sigma\left(W_0^*\left(\beta_1,\beta_2\right)\beta_5,\beta_4\right)\\&-\sigma\left(\beta_5,W_0^*\left(\beta_1,\beta_2\right)\beta_4\right)=-k_1\left\{\sigma\left(\left(\beta_1\wedge_g\beta_2\right)\beta_5,\beta_4\right)\right.\\&\left.+\sigma\left(\beta_5,\left(\beta_1\wedge_g\beta_2\right)\beta_4\right)\right\}.\end{aligned}$$

If we use (12) in the last equation, we can write

$$\begin{aligned}&R^{\perp}\left(\beta_1,\beta_2\right)\sigma\left(\beta_5,\beta_4\right)-\sigma\left(W_0^*\left(\beta_1,\beta_2\right)\beta_5,\beta_4\right)\\&-\sigma\left(\beta_5,W_0^*\left(\beta_1,\beta_2\right)\beta_4\right)=-k_1\left\{g\left(\beta_2,\beta_5\right)\sigma\left(\beta_1,\beta_4\right)\right.\\&-g\left(\beta_1,\beta_5\right)\sigma\left(\beta_2,\beta_4\right)+g\left(\beta_2,\beta_4\right)\sigma\left(\beta_5,\beta_1\right)\\&\left.-g\left(\beta_1,\beta_4\right)\sigma\left(\beta_5,\beta_2\right)\right\}.\end{aligned}\tag{3.8}$$

If we choose $\beta_4=\xi$ in (20) and make use of $(14),(19)$, we get

$$\begin{aligned}&\tfrac{(2n-1)\Upsilon_3+3\Upsilon_2}{2n}\eta\left(\beta_1\right)\sigma\left(\beta_5,\beta_2\right)=\\&k_1\left\{\eta\left(\beta_2\right)\sigma\left(\beta_5,\beta_2\right)-\eta\left(\beta_1\right)\sigma\left(\beta_5,\beta_2\right)\right\}\end{aligned}\tag{3.9}$$

IIf we choose $\beta_1=\xi$ in (21) and make use of $(14)$, we obtain

$$\left[k_1-\frac{\left(2n-1\right)\Upsilon_3+3\Upsilon_2}{2n}\right]\sigma\left(\beta_5,\beta_2\right)=0.\tag{3.10}$$

It is clear from equation (22) that

$$\sigma\left(\beta_5,\beta_2\right)=0$$

or

$$k_1=\frac{\left(2n-1\right)\Upsilon_3+3\Upsilon_2}{2n}.$$

This completes the proof. □

**Corollary 1.** *Let $N$ be the invariant submanifold of the $(2n+1)-$dimensional GS-space form $\hat{N}^{2n+1}\left(\Upsilon_1,\Upsilon_2,\Upsilon_3\right)$. If $N$ is $W_0^*-$semiparallel submanifold, then $N$ is either a total geodesic or a real space form with constant section curvature $c=1$.*

**Proof.** Let's assume that $N$ is a $W_0^*-$semiparallel submanifold. In this case, there is a function $k_1 = 0$ on the set $M_1 = \{x \in N |\, \sigma(x) \neq g(x)\}$ such that

$$W_0^* \cdot \sigma = k_1 Q(g, \sigma).$$

Then, as it is easily seen from the proof of Theorem-1, we obtain

$$\frac{(2n-1)\Upsilon_3 + 3\Upsilon_2}{2n}\sigma(\beta_5, \beta_2) = 0.$$

It is clear from last equation that

$$\sigma(\beta_5, \beta_2) = 0$$

or

$$(2n-1)\Upsilon_3 + 3\Upsilon_2 = 0.$$

This completes the proof of the corollary. □

**Definition 3.** *Let $N$ be the invariant submanifold of the $(2n+1)-$dimensional GS-space form $\hat{N}^{2n+1}(\Upsilon_1, \Upsilon_2, \Upsilon_3)$. If $W_0^* \cdot \tilde{\nabla}\sigma$ and $Q\left(g, \tilde{\nabla}\sigma\right)$ are linearly dependent, then $N$ is called $W_0^*$ $2-$ pseudoparallel submanifold.*

In this case, it can be said that there is a function $k_2$ on the set $M_2 = \left\{x \in N |\, \tilde{\nabla}\sigma(x) \neq g(x)\right\}$ such that

$$W_0^* \cdot \tilde{\nabla}\sigma = k_2 Q\left(g, \tilde{\nabla}\sigma\right).$$

If $k_2 = 0$ specifically, $N$ is called a $W_0^*$ $2-$ semiparallel submanifold.

Let $N$ be the invariant submanifold of the $(2n+1)-$dimensional GS-space form $\hat{N}^{2n+1}(\Upsilon_1, \Upsilon_2, \Upsilon_3)$. If $N$ is $W_0^*$ $2-$ pseudoparallel submanifold, then $N$ is either a total geodesic or $k_2 = \dfrac{(1-2n)\Upsilon_3 - 3\Upsilon_2 + 2n}{4n}$.

**Proof.** Let's assume that $N$ is a $W_0^*$ $2-$ pseudoparallel submanifold. So, we can write

$$\left(W_0^*(\beta_1, \beta_2) \cdot \tilde{\nabla}\sigma\right)(\beta_5, \beta_4, \beta_3) = k_2 Q\left(g, \tilde{\nabla}\sigma\right)(\beta_5, \beta_4, \beta_3; \beta_1, \beta_2),$$

for all $\beta_1, \beta_2, \beta_5, \beta_4, \beta_3 \in \Gamma(TN)$. Easily from here, we can write

$$\begin{aligned} & R^{\perp}(\beta_1, \beta_2)\left(\tilde{\nabla}_{\beta_5}\sigma\right)(\beta_4, \beta_3) - \left(\tilde{\nabla}_{W_0^*(\beta_1,\beta_2)\beta_5}\sigma\right)(\beta_4, \beta_3) \\ & - \left(\tilde{\nabla}_{\beta_5}\sigma\right)(W_0^*(\beta_1, \beta_2)\beta_4, \beta_3) - \left(\tilde{\nabla}_{\beta_5}\sigma\right)(\beta_4, W_0^*(\beta_1, \beta_2)\beta_3) \\ & = -k_2\left\{\left(\tilde{\nabla}_{(\beta_1 \wedge_g \beta_2)\beta_5}\sigma\right)(\beta_4, \beta_3) + \left(\tilde{\nabla}_{\beta_5}\sigma\right)((\beta_1 \wedge_g \beta_2)\beta_4, \beta_3)\right. \\ & \left. + \left(\tilde{\nabla}_{\beta_5}\sigma\right)(\beta_4, (\beta_1 \wedge_g \beta_2)\beta_3)\right\}. \end{aligned} \tag{3.11}$$

If we choose $\beta_1 = \beta_3 = \xi$ in $(23)$, we can write

$$\begin{aligned} & R^{\perp}(\xi, \beta_2)\left(\tilde{\nabla}_{\beta_5}\sigma\right)(\beta_4, \xi) - \left(\tilde{\nabla}_{W_0^*(\xi,\beta_2)\beta_5}\sigma\right)(\beta_4, \xi) \\ & - \left(\tilde{\nabla}_{\beta_5}\sigma\right)(W_0^*(\xi, \beta_2)\beta_4, \xi) - \left(\tilde{\nabla}_{\beta_5}\sigma\right)(\beta_4, W_0^*(\xi, \beta_2)\xi) \\ & = -k_2\left\{\left(\tilde{\nabla}_{(\xi \wedge_g \beta_2)\beta_5}\sigma\right)(\beta_4, \xi) + \left(\tilde{\nabla}_{\beta_5}\sigma\right)((\xi \wedge_g \beta_2)\beta_4, \xi)\right. \\ & \left. + \left(\tilde{\nabla}_{\beta_5}\sigma\right)(\beta_4, (\xi \wedge_g \beta_2)\xi)\right\}. \end{aligned} \tag{3.12}$$

Let's calculate all the expressions in (24). So, we can write

$$\begin{aligned} & R^{\perp}(\xi, \beta_2)\left(\tilde{\nabla}_{\beta_5}\sigma\right)(\beta_4, \xi) = W_0^{*\perp}(\xi, \beta_2)\left\{\nabla^{\perp}_{\beta_5}\sigma(\beta_4, \xi)\right. \\ & \left. -\sigma(\nabla_{\beta_5}\beta_4, \xi) - \sigma(\beta_4, \nabla_{\beta_5}\xi)\right\} \\ & = (\Upsilon_1 - \Upsilon_3)R^{\perp}(\xi, \beta_2)\phi\sigma(\beta_4, \beta_5), \end{aligned} \tag{3.13}$$

$$\begin{aligned} & \left(\tilde{\nabla}_{W_0^*(\xi,\beta_2)\beta_5}\sigma\right)(\beta_4, \xi) = \nabla^{\perp}_{W_0^*(\xi,\beta_2)\beta_5}\sigma(\beta_4, \xi) \\ & -\sigma\left(\nabla_{W_0^*(\xi,\beta_2)\beta_5}\beta_4, \xi\right) - \sigma\left(\beta_4, \nabla_{W_0^*(\xi,\beta_2)\beta_5}\xi\right) \\ & = -\sigma(\beta_4, -(\Upsilon_1 - \Upsilon_3)^*_0(\xi, \beta_2)\beta_5) \\ & = -\tfrac{(1-2n)\Upsilon_3 - 3\Upsilon_2}{2n}(\Upsilon_1 - \Upsilon_3)\eta(\beta_5)\phi\sigma(\beta_4, \beta_2), \end{aligned} \tag{3.14}$$

$$\begin{aligned}
&\left(\tilde{\nabla}_{\beta_5}\sigma\right)\left(W_0^*\left(\xi,\beta_2\right)\beta_4,\xi\right)=\nabla^{\perp}_{\beta_5}\sigma\left(W_0^*\left(\xi,\beta_2\right)\beta_4,\xi\right)\\
&-\sigma\left(\nabla_{\beta_5}W_0^*\left(\xi,\beta_2\right)\beta_4,\xi\right)-\sigma\left(W_0^*\left(\xi,\beta_2\right)\beta_4,\nabla_{\beta_5}\xi\right)\\
&=-\sigma\left(\tfrac{(1-2n)\Upsilon_3-3\Upsilon_2}{2n}\left[g\left(\beta_2,\beta_4\right)\xi-\eta\left(\beta_4\right)\beta_2\right],-\left(\Upsilon_1-\Upsilon_3\right)\phi\beta_5\right)\\
&=-\tfrac{(1-2n)\Upsilon_3-3\Upsilon_2}{2n}\left(\Upsilon_1-\Upsilon_3\right)\eta\left(\beta_4\right)\phi\sigma\left(\beta_2,\beta_5\right),
\end{aligned} \tag{3.15}$$

$$\begin{aligned}
&\left(\tilde{\nabla}_{\beta_5}\sigma\right)\left(\beta_4,W_0^*\left(\xi,\beta_2\right)\xi\right)=\left(\tilde{\nabla}_{\beta_5}\sigma\right)\left(\beta_4,\eta\left(\beta_2\right)\xi-\beta_2\right)\\
&=\left(\tilde{\nabla}_{\beta_5}\sigma\right)\left(\beta_4,\eta\left(\beta_2\right)\xi\right)-\left(\tilde{\nabla}_{\beta_5}\sigma\right)\left(\beta_4,\beta_2\right)\\
&=-\sigma\left(\beta_4,\beta_5\eta\left(\beta_2\right)\xi+\eta\left(\beta_2\right)\nabla_{\beta_5}\xi\right)-\left(\tilde{\nabla}_{\beta_5}\sigma\right)\left(\beta_4,\beta_2\right)\\
&=\left(\Upsilon_1-\Upsilon_3\right)\eta\left(\beta_2\right)\phi\sigma\left(\beta_4,\beta_5\right)-\left(\tilde{\nabla}_{\beta_5}\sigma\right)\left(\beta_4,\beta_2\right),
\end{aligned} \tag{3.16}$$

$$\begin{aligned}
&\left(\tilde{\nabla}_{(\xi\wedge_g\beta_2)\beta_5}\sigma\right)\left(\beta_4,\xi\right)=\nabla^{\perp}_{(\xi\wedge_g\beta_2)\beta_5}\sigma\left(\beta_4,\xi\right)\\
&-\sigma\left(\nabla_{(\xi\wedge_g\beta_2)\beta_5}\beta_4,\xi\right)-\sigma\left(\beta_4,\nabla_{(\xi\wedge_g\beta_2)\beta_5}\xi\right)\\
&=-\sigma\left(\beta_4,\nabla_{g(\beta_2,\beta_5)\xi-g(\xi,\beta_5)\beta_2}\xi\right)\\
&=-\left(\Upsilon_1-\Upsilon_3\right)\eta\left(\beta_5\right)\phi\sigma\left(\beta_4,\beta_2\right),
\end{aligned} \tag{3.17}$$

$$\begin{aligned}
&\left(\tilde{\nabla}_{\beta_5}\sigma\right)\left(\left(\xi\wedge_g\beta_2\right)\beta_4,\xi\right)=\nabla^{\perp}_{\beta_5}\sigma\left(\left(\xi\wedge_g\beta_2\right)\beta_4,\xi\right)\\
&-\sigma\left(\nabla_{\beta_5}\left(\xi\wedge_g\beta_2\right)\beta_4,\xi\right)-\sigma\left(\left(\xi\wedge_g\beta_2\right)\beta_4,\nabla_{\beta_5}\xi\right)\\
&=-\sigma\left(g\left(\beta_2,\beta_4\right)\xi-g\left(\xi,\beta_4\right)\beta_2,-\left(\Upsilon_1-\Upsilon_3\right)\phi\beta_5\right)\\
&=-\left(\Upsilon_1-\Upsilon_3\right)\eta\left(\beta_4\right)\phi\sigma\left(\beta_2,\beta_5\right),
\end{aligned} \tag{3.18}$$

$$\begin{aligned}(3.19)\quad &\left(\tilde{\nabla}_{\beta_5}\sigma\right)(\beta_4,(\xi\wedge_g\beta_2)\xi)=\left(\tilde{\nabla}_{\beta_5}\sigma\right)(\beta_4,\eta(\beta_2)\xi-\beta_2)\\ &=\left(\tilde{\nabla}_{\beta_5}\sigma\right)(\beta_4,\eta(\beta_2)\xi)-\left(\tilde{\nabla}_{\beta_5}\sigma\right)(\beta_4,\beta_2)\\ &=-\sigma(\beta_4,\beta_5\eta(\beta_2)\xi+\eta(\beta_2)\nabla_{\beta_5}\xi)-\left(\tilde{\nabla}_{\beta_5}\sigma\right)(\beta_4,\beta_2)\\ &=(\Upsilon_1-\Upsilon_3)\eta(\beta_2)\phi\sigma(\beta_4,\beta_5)-\left(\tilde{\nabla}_{\beta_5}\sigma\right)(\beta_4,\beta_2).\end{aligned}$$

If we substitute (25) , (26) , (27) , (28) , (29) , (30) , (31) in (24) , we obtain

$$\begin{aligned}&R^{\perp}(\xi,\beta_2)(\Upsilon_1-\Upsilon_3)\phi\sigma(\beta_4,\beta_5)\\ &+\tfrac{(1-2n)\Upsilon_3-3\Upsilon_2}{2n}(\Upsilon_1-\Upsilon_3)\eta(\beta_5)\phi\sigma(\beta_4,\beta_2)\\ &+\tfrac{(1-2n)\Upsilon_3-3\Upsilon_2}{2n}(\Upsilon_1-\Upsilon_3)\eta(\beta_4)\phi\sigma(\beta_2,\beta_5)\\ (3.20)\quad &-(\Upsilon_1-\Upsilon_3)\eta(\beta_2)\phi\sigma(\beta_4,\beta_5)+\left(\tilde{\nabla}_{\beta_5}\sigma\right)(\beta_4,\beta_2)\\ &=-k_2\{-(\Upsilon_1-\Upsilon_3)\eta(\beta_5)\phi\sigma(\beta_4,\beta_2)\\ &-(\Upsilon_1-\Upsilon_3)\eta(\beta_4)\phi\sigma(\beta_2,\beta_5)+(\Upsilon_1-\Upsilon_3)\eta(\beta_2)\phi\sigma(\beta_4,\beta_5)\\ &-\left(\tilde{\nabla}_{\beta_5}\sigma\right)(\beta_4,\beta_2)\Big\}.\end{aligned}$$

If we choose $\beta_4=\xi$ in (32) , we get

$$\begin{aligned}(3.21)\quad &\tfrac{(1-2n)\Upsilon_3-3\Upsilon_2}{2n}(\Upsilon_1-\Upsilon_3)\phi\sigma(\beta_2,\beta_5)+\left(\tilde{\nabla}_{\beta_5}\sigma\right)(\xi,\beta_2)\\ &=-k_2\left\{-(\Upsilon_1-\Upsilon_3)\phi\sigma(\beta_2,\beta_5)-\left(\tilde{\nabla}_{\beta_5}\sigma\right)(\xi,\beta_2)\right\}.\end{aligned}$$

On the other hand, it is clear that

$$(3.22)\quad \left(\tilde{\nabla}_{\beta_5}\sigma\right)(\xi,\beta_2)=(\Upsilon_1-\Upsilon_3)\phi\sigma(\beta_2,\beta_5).$$

If (34) is written instead of (33), we obtain

$$\left\{\tfrac{(1-2n)\Upsilon_3-3\Upsilon_2}{2n}\left(\Upsilon_1-\Upsilon_3\right)+\left(\Upsilon_1-\Upsilon_3\right)\right. \\ \left.-2\left(\Upsilon_1-\Upsilon_3\right)k_2\right\}\phi\sigma\left(\beta_2,\beta_5\right)=0. \tag{3.23}$$

It is clear from equation (35) that

$$\sigma\left(\beta_2,\beta_5\right)=0,$$

or

$$k_2=\frac{(1-2n)\,\Upsilon_3-3\Upsilon_2+2n}{4n}.$$

This completes of the proof. □

**Corollary 2.** *Let $N$ be the invariant submanifold of the $(2n+1)-$dimensional GS-space form $\hat{N}^{2n+1}\left(\Upsilon_1,\Upsilon_2,\Upsilon_3\right)$. If $N$ is $W_0^*$ $2-$semiparallel submanifold, then $N$ is either a total geodesic or a real space form with constant section curvature $c=\dfrac{5n+1}{n+1}$.*

**Proof.** Let's assume that $N$ is a $W_0^*$ $2-$semiparallel submanifold. In this case, there is a function $k_2=0$ on the set $M_2=\left\{x\in N|\,\tilde{\bigtriangledown}\sigma\left(x\right)\neq g\left(x\right)\right\}$ such that

$$W_0^*\cdot\tilde{\bigtriangledown}\sigma=k_2Q\left(g,\tilde{\bigtriangledown}\sigma\right).$$

Then, as it is easily seen from the proof of Theorem-2, we obtain

$$\left[(1-2n)\,\Upsilon_3-3\Upsilon_2+2n\right]\left(\Upsilon_1-\Upsilon_3\right)\phi\sigma\left(\beta_5,\beta_2\right)=0.$$

It is clear from last equation that

$$\sigma\left(\beta_5,\beta_2\right)=0$$

or

$$(2n-1)\,\Upsilon_3+3\Upsilon_2-2n=0.$$

This completes the proof of the corollary. □

**Definition 4.** *Let $N$ be the invariant submanifold of the $(2n+1)-$dimensional GS-space form $\hat{N}^{2n+1}\left(\Upsilon_1,\Upsilon_2,\Upsilon_3\right)$. If $W_0^*\cdot\sigma$ and $Q\left(S,\sigma\right)$ are linearly dependent, $N$ is called $W_0^*-$Ricci generalized pseudoparallel submanifold.*

In this case, there is a function $k_3$ on the set $M_3 = \{x \in N|\, \sigma(x) \neq S(x)\}$ such that

$$W_0^* \cdot \sigma = k_3 Q(S, \sigma).$$

If $k_3 = 0$ specifically, $N$ is called a $W_0^*$ Ricci generalized semiparallel submanifold.

Let $N$ be the invariant submanifold of the $(2n+1)-$dimensional GS-space form $\hat{N}^{2n+1}(\Upsilon_1, \Upsilon_2, \Upsilon_3)$. If $N$ is $W_0^*-$Ricci generalized pseudoparallel submanifold, then $N$ is either a total geodesic or $k_3 = \dfrac{(1-2n)\Upsilon_3 - 3\Upsilon_2}{4n^2(\Upsilon_1 - \Upsilon_3)}$ provided $\Upsilon_1 \neq \Upsilon_3$.

**Proof.** Let's assume that $N$ is a $W_0^*-$Ricci generalized pseudoparallel submanifold. So, we can write

$$(W_0^*(\beta_1, \beta_2) \cdot \sigma)(\beta_5, \beta_4) = k_3 Q(S, \sigma)(\beta_5, \beta_4; \beta_1, \beta_2),$$

that is

$$\begin{aligned} & R^{\perp}(\beta_1, \beta_2)\sigma(\beta_5, \beta_4) - \sigma(W_0^*(\beta_1, \beta_2)\beta_5, \beta_4) \\ & -\sigma(\beta_5, W_0^*(\beta_1, \beta_2)\beta_4) = -k_3\{S(\beta_2, \beta_5)\sigma(\beta_1, \beta_4) \\ & -S(\beta_1, \beta_5)\sigma(\beta_2, \beta_4) + S(\beta_2, \beta_4)\sigma(\beta_5, \beta_1) \\ & -S(\beta_1, \beta_4)\sigma(\beta_5, \beta_2)\}. \end{aligned} \tag{3.24}$$

for all $\beta_1, \beta_2, \beta_5, \beta_4 \in \Gamma(TN)$. If we choose $\beta_1 = \beta_4 = \xi$ in (36), we get

$$-\sigma(\beta_5, W_0^*(\xi, \beta_2)\xi) = k_3 S(\xi, \xi)\sigma(\beta_5, \beta_2). \tag{3.25}$$

If we make use of (5) and (17) in (37), we obtain

$$\left[\frac{(1-2n)\Upsilon_3 - 3\Upsilon_2}{2n} - 2n(\Upsilon_1 - \Upsilon_3)k_3\right]\sigma(\beta_5, \beta_2) = 0 \tag{3.26}$$

It is clear from in (38) that either

$$\sigma(\beta_5, \beta_2) = 0,$$

or

$$k_3 = \frac{(1-2n)\Upsilon_3 - 3\Upsilon_2}{4n^2(\Upsilon_1 - \Upsilon_3)}.$$

This completes the proof. □

**Corollary 3.** *Let $N$ be the invariant submanifold of the $(2n+1)-$dimensional GS-space form $\hat{N}^{2n+1}(\Upsilon_1, \Upsilon_2, \Upsilon_3)$. If $N$ is $W_0^*$ Ricci generalized semiparallel submanifold, then $N$ is either a total geodesic or a real space form with constant section curvature $c = 1$.*

**Proof.** Let's assume that $N$ is a $W_0^*$ Ricci generalized semiparallel submanifold. In this case, there is a function $k_3 = 0$ on the set $M_3 = \{x \in N | \sigma(x) \neq S(x)\}$ such that

$$W_0^* \cdot \sigma = k_3 Q(S, \sigma).$$

Then, as it is easily seen from the proof of Theorem-3, we obtain

$$[(1-2n)\Upsilon_3 - 3\Upsilon_2]\sigma(\beta_5, \beta_2) = 0.$$

It is clear from last equation that

$$\sigma(\beta_5, \beta_2) = 0$$

or

$$(1-2n)\Upsilon_3 - 3\Upsilon_2 = 0.$$

This completes the proof of the corollary. □

**Definition 5.** *Let $N$ be the invariant submanifold of the $(2n+1)-$dimensional GS-space form $\hat{N}^{2n+1}(\Upsilon_1, \Upsilon_2, \Upsilon_3)$. If $W_0^* \cdot \tilde{\bigtriangledown}\sigma$ and $Q\left(S, \tilde{\bigtriangledown}\sigma\right)$ are linearly dependent, $N$ is called $W_0^*-2$ Ricci generalized pseudoparallel submanifold.*

Then, there is a function $k_4$ on the set $M_4 = \left\{x \in N | \tilde{\bigtriangledown}\sigma(x) \neq S(x)\right\}$ such that

$$W_0^* \cdot \tilde{\bigtriangledown}\sigma = k_4 Q\left(S, \tilde{\bigtriangledown}\sigma\right).$$

If $k_4 = 0$ specifically, $N$ is called a $W_0^*$ $2-$Ricci generalized semiparallel submanifold.

Let $N$ be the invariant submanifold of the $(2n+1)-$dimensional GS-space form $\hat{N}^{2n+1}(\Upsilon_1, \Upsilon_2, \Upsilon_3)$. If $N$ is $W_0^*$ $2-$Ricci generalized pseudoparallel submanifold, then $N$ is either a total geodesic or $k_4 = \dfrac{(1-2n)\Upsilon_3 - 3\Upsilon_2}{4n^2(\Upsilon_1 - \Upsilon_3)}$ provided $\Upsilon_1 \neq \Upsilon_3$.

**Proof.** Let's assume that $N$ is a $W_0^*$ 2−Ricci generalized pseudoparallel submanifold. So, we can write

$$\left(W_0^*\left(\beta_1,\beta_2\right)\cdot\tilde{\nabla}\sigma\right)\left(\beta_5,\beta_4,\beta_3\right)=k_4Q\left(S,\tilde{\nabla}\sigma\right)\left(\beta_5,\beta_4,\beta_3;\beta_1,\beta_2\right),$$

for all $\beta_1,\beta_2,\beta_5,\beta_4,\beta_3\in\Gamma\left(TN\right)$. Easily from here, we can write

$$\begin{aligned}&R^{\perp}\left(\beta_1,\beta_2\right)\left(\tilde{\nabla}_{\beta_5}\sigma\right)\left(\beta_4,\beta_3\right)-\left(\tilde{\nabla}_{W_0^*\left(\beta_1,\beta_2\right)\beta_5}\sigma\right)\left(\beta_4,\beta_3\right)\\&-\left(\tilde{\nabla}_{\beta_5}\sigma\right)\left(W_0^*\left(\beta_1,\beta_2\right)\beta_4,\beta_3\right)-\left(\tilde{\nabla}_{\beta_5}\sigma\right)\left(\beta_4,W_0^*\left(\beta_1,\beta_2\right)\beta_3\right)\\&=-k_4\left\{\left(\tilde{\nabla}_{\left(\beta_1\wedge_S\beta_2\right)\beta_5}\sigma\right)\left(\beta_4,\beta_3\right)+\left(\tilde{\nabla}_{\beta_5}\sigma\right)\left(\left(\beta_1\wedge_S\beta_2\right)\beta_4,\beta_3\right)\right.\\&\left.+\left(\tilde{\nabla}_{\beta_5}\sigma\right)\left(\beta_4,\left(\beta_1\wedge_S\beta_2\right)\beta_3\right)\right\}.\end{aligned}\tag{3.27}$$

If we choose $\beta_1=\beta_4=\xi$ in $(39)$, we can write

$$\begin{aligned}&R^{\perp}\left(\xi,\beta_2\right)\left(\tilde{\nabla}_{\beta_5}\sigma\right)\left(\xi,\beta_3\right)-\left(\tilde{\nabla}_{W_0^*\left(\xi,\beta_2\right)\beta_5}\sigma\right)\left(\xi,\beta_3\right)\\&-\left(\tilde{\nabla}_{\beta_5}\sigma\right)\left(W_0^*\left(\xi,\beta_2\right)\xi,\beta_3\right)-\left(\tilde{\nabla}_{\beta_5}\sigma\right)\left(\xi,W_0^*\left(\xi,\beta_2\right)\beta_3\right)\\&=-k_4\left\{\left(\tilde{\nabla}_{\left(\xi\wedge_S\beta_2\right)\beta_5}\sigma\right)\left(\xi,\beta_3\right)+\left(\tilde{\nabla}_{\beta_5}\sigma\right)\left(\left(\xi\wedge_S\beta_2\right)\xi,\beta_3\right)\right.\\&\left.+\left(\tilde{\nabla}_{\beta_5}\sigma\right)\left(\xi,\left(\xi\wedge_S\beta_2\right)\beta_3\right)\right\}.\end{aligned}\tag{3.28}$$

Let's calculate all the expressions in (40). Firstly, we can write

$$\begin{aligned}&R^{\perp}\left(\xi,\beta_2\right)\left(\tilde{\nabla}_{\beta_5}\sigma\right)\left(\xi,\beta_3\right)=W_0^{*\perp}\left(\xi,\beta_2\right)\left\{\nabla^{\perp}_{\beta_5}\sigma\left(\xi,\beta_3\right)\right.\\&\left.-\sigma\left(\nabla_{\beta_5}\beta_3,\xi\right)-\sigma\left(\beta_3,\nabla_{\beta_5}\xi\right)\right\}\\&=R^{\perp}\left(\xi,\beta_2\right)\left(\Upsilon_1-\Upsilon_3\right)\phi\sigma\left(\beta_3,\beta_5\right),\end{aligned}\tag{3.29}$$

$$
\begin{aligned}
&\left(\tilde{\nabla}_{W_0^*(\xi,\beta_2)\beta_5}\sigma\right)(\xi,\beta_3) = \nabla^{\perp}_{W_0^*(\xi,\beta_2)\beta_5}\sigma(\xi,\beta_3) \\
&-\sigma\left(\nabla_{W_0^*(\xi,\beta_2)\beta_5}\xi,\beta_3\right) - \sigma\left(\xi,\nabla_{W_0^*(\xi,\beta_2)\beta_5}\beta_3\right) \\
&= (\Upsilon_1-\Upsilon_3)\,\phi\sigma\left(W_0^*(\xi,\beta_2)\beta_5,\beta_3\right) \\
&= -\tfrac{(1-2n)\Upsilon_3-3\Upsilon_2}{2n}(\Upsilon_1-\Upsilon_3)\,\eta(\beta_5)\,\phi\sigma(\beta_2,\beta_3),
\end{aligned} \tag{3.30}
$$

$$
\begin{aligned}
&\left(\tilde{\nabla}_{\beta_5}\sigma\right)\left(W_0^*(\xi,\beta_2)\xi,\beta_3\right) = \left(\tilde{\nabla}_{\beta_5}\sigma\right)\left(\tfrac{(1-2n)\Upsilon_3-3\Upsilon_2}{2n}\left[\eta(\beta_2)\xi-\beta_2\right],\beta_3\right) \\
&= \tfrac{(1-2n)\Upsilon_3-3\Upsilon_2}{2n}\left\{\left(\tilde{\nabla}_{\beta_5}\sigma\right)(\eta(\beta_2)\xi,\beta_3) - \left(\tilde{\nabla}_{\beta_5}\sigma\right)(\beta_2,\beta_3)\right\} \\
&= \tfrac{(1-2n)\Upsilon_3-3\Upsilon_2}{2n}\left\{-\sigma\left(\beta_5\eta(\beta_2)\xi+\eta(\beta_2)\nabla_{\beta_5}\xi,\beta_3\right) - \left(\tilde{\nabla}_{\beta_5}\sigma\right)(\beta_2,\beta_3)\right\} \\
&= \tfrac{(1-2n)\Upsilon_3-3\Upsilon_2}{2n}\left\{(\Upsilon_1-\Upsilon_3)\,\eta(\beta_2)\,\phi\sigma(\beta_5,\beta_3) - \left(\tilde{\nabla}_{\beta_5}\sigma\right)(\beta_2,\beta_3)\right\},
\end{aligned} \tag{3.31}
$$

$$
\begin{aligned}
&\left(\tilde{\nabla}_{\beta_5}\sigma\right)\left(\xi,W_0^*(\xi,\beta_2)\beta_3\right) = \nabla^{\perp}_{\beta_5}\sigma\left(\xi,W_0^*(\xi,\beta_2)\beta_3\right) \\
&-\sigma\left(\nabla_{\beta_5}\xi,W_0^*(\xi,\beta_2)\beta_3\right) - \sigma\left(\xi,\nabla_{\beta_5}W_0^*(\xi,\beta_2)\beta_3\right) \\
&= -\sigma\left(-(\Upsilon_1-\Upsilon_3)\,\phi\beta_5,W_0^*(\xi,\beta_2)\beta_3\right) \\
&= -\tfrac{(1-2n)\Upsilon_3-3\Upsilon_2}{2n}(\Upsilon_1-\Upsilon_3)\,\eta(\beta_3)\,\phi\sigma(\beta_5,\beta_2)
\end{aligned} \tag{3.32}
$$

$$
\begin{aligned}
&\left(\tilde{\nabla}_{(\xi\wedge_S\beta_2)\beta_5}\sigma\right)(\xi,\beta_3) = \nabla^{\perp}_{(\xi\wedge_S\beta_2)\beta_5}\sigma(\xi,\beta_3) \\
&-\sigma\left(\nabla_{(\xi\wedge_S\beta_2)\beta_5}\xi,\beta_3\right) - \sigma\left(\xi,\nabla_{(\xi\wedge_S\beta_2)\beta_5}\beta_3\right) \\
&= -\sigma\left(S(\beta_2,\beta_5)\nabla_{\xi}\xi - S(\xi,\beta_5)\nabla_{\beta_2}\xi,\beta_3\right) \\
&= -2n(\Upsilon_1-\Upsilon_3)^2\,\eta(\beta_5)\,\phi\sigma(\beta_2,\beta_3),
\end{aligned} \tag{3.33}
$$

$$
\begin{aligned}
&\left(\tilde{\nabla}_{\beta_5}\sigma\right)\left(\left(\xi \wedge_S \beta_2\right)\xi, \beta_3\right) = \left(\tilde{\nabla}_{\beta_5}\sigma\right)\left(S\left(\beta_2,\xi\right)\xi - S\left(\xi,\xi\right)\beta_2, \beta_3\right) \\
&= \left(\tilde{\nabla}_{\beta_5}\sigma\right)\left(2n\left(\Upsilon_1-\Upsilon_3\right)\eta\left(\beta_2\right)\xi - 2n\left(\Upsilon_1-\Upsilon_3\right)\beta_2, \beta_3\right) \\
&= 2n\left(\Upsilon_1-\Upsilon_3\right)\Big\{\nabla^{\perp}_{\beta_5}\sigma\left(\eta\left(\beta_2\right)\xi,\beta_3\right) - \sigma\left(\nabla_{\beta_5}\eta\left(\beta_2\right)\xi,\beta_3\right) \\
&-\sigma\left(\eta\left(\beta_2\right)\xi, \nabla_{\beta_5}\beta_3\right) - \left(\tilde{\nabla}_{\beta_5}\sigma\right)\left(\beta_2,\beta_3\right)\Big\} \\
&= -2n\left(\Upsilon_1-\Upsilon_3\right)\left(\tilde{\nabla}_{\beta_5}\sigma\right)\left(\beta_2,\beta_3\right) \\
&+2n\left(\Upsilon_1-\Upsilon_3\right)^2\eta\left(\beta_2\right)\phi\sigma\left(\beta_5,\beta_3\right),
\end{aligned}
\tag{3.34}
$$

$$
\begin{aligned}
&\left(\tilde{\nabla}_{\beta_5}\sigma\right)\left(\xi, \left(\xi \wedge_S \beta_2\right)\beta_3\right) = \left(\tilde{\nabla}_{\beta_5}\sigma\right)\left(\xi, S\left(\beta_2,\beta_3\right)\xi - S\left(\xi,\beta_3\right)\beta_2\right) \\
&= \left(\tilde{\nabla}_{\beta_5}\sigma\right)\left(\xi, S\left(\beta_2,\beta_3\right)\xi\right) - 2n\left(\Upsilon_1-\Upsilon_3\right)\left(\tilde{\nabla}_{\beta_5}\sigma\right)\left(\xi, \eta\left(\beta_3\right)\beta_2\right) \\
&= -2n\left(\Upsilon_1-\Upsilon_3\right)^2\eta\left(\beta_3\right)\phi\sigma\left(\beta_5,\beta_2\right).
\end{aligned}
\tag{3.35}
$$

If we substitute $(41), (42), (43), (44), (45), (46), (47)$ in $(40)$, we obtain

$$
\begin{aligned}
& R^{\perp}(\xi,\beta_2)(\Upsilon_1-\Upsilon_3)\phi\sigma(\beta_3,\beta_5) \\
& +\tfrac{(1-2n)\Upsilon_3-3\Upsilon_2}{2n}(\Upsilon_1-\Upsilon_3)\eta(\beta_5)\phi\sigma(\beta_2,\beta_3) \\
& -\tfrac{(1-2n)\Upsilon_3-3\Upsilon_2}{2n}(\Upsilon_1-\Upsilon_3)\eta(\beta_2)\phi\sigma(\beta_5,\beta_3) \\
& +\tfrac{(1-2n)\Upsilon_3-3\Upsilon_2}{2n}(\Upsilon_1-\Upsilon_3)\eta(\beta_3)\phi\sigma(\beta_5,\beta_2) \\
& +\tfrac{(1-2n)\Upsilon_3-3\Upsilon_2}{2n}\left(\tilde{\nabla}_{\beta_5}\sigma\right)(\beta_2,\beta_3) \\
& =-k_4\left\{-2n(\Upsilon_1-\Upsilon_3)^2\eta(\beta_5)\phi\sigma(\beta_2,\beta_3)\right. \\
& +2n(\Upsilon_1-\Upsilon_3)^2\eta(\beta_2)\phi\sigma(\beta_5,\beta_3) \\
& -2n(\Upsilon_1-\Upsilon_3)^2\eta(\beta_3)\phi\sigma(\beta_5,\beta_2) \\
& \left.-2n(\Upsilon_1-\Upsilon_3)\left(\tilde{\nabla}_{\beta_5}\sigma\right)(\beta_2,\beta_3)\right\}.
\end{aligned}
\tag{3.36}
$$

If we choose $\beta_3=\xi$ in $(48)$, we get

$$
\begin{aligned}
& \tfrac{(1-2n)\Upsilon_3-3\Upsilon_2}{2n}\left(\tilde{\nabla}_{\beta_5}\sigma\right)(\beta_2,\xi)+\tfrac{(1-2n)\Upsilon_3-3\Upsilon_2}{2n}(\Upsilon_1-\Upsilon_3)\phi\sigma(\beta_5,\beta_2) \\
& =-k_4\left\{-2n(\Upsilon_1-\Upsilon_3)\left(\tilde{\nabla}_{\beta_5}\sigma\right)(\beta_2,\xi)\right.
\end{aligned}
\tag{3.37}
$$

On the other hand, it is clear that

$$\left(\tilde{\nabla}_{\beta_5}\sigma\right)(\xi,\beta_2)=(\Upsilon_1-\Upsilon_3)\phi\sigma(\beta_2,\beta_5).$$

If the last equation is written instead of $(49)$, we obtain

$$
\begin{aligned}
& \left[\tfrac{(1-2n)\Upsilon_3-3\Upsilon_2}{n}(\Upsilon_1-\Upsilon_3)\phi\right. \\
& \left.-4n(\Upsilon_1-\Upsilon_3)^2\phi k_4\right]\sigma(\beta_2,\beta_5)=0.
\end{aligned}
\tag{3.38}
$$

It is clear from the last equality

$$\sigma(\beta_2,\beta_5)=0 \text{ or } k_4=\frac{(1-2n)\Upsilon_3-3\Upsilon_2}{4n^2(\Upsilon_1-\Upsilon_3)}.$$

This completes the proof. □

**Corollary 4.** *Let $N$ be the invariant submanifold of the $(2n+1)-$dimensional GS-space form $\hat{N}^{2n+1}(\Upsilon_1, \Upsilon_2, \Upsilon_3)$. If $N$ is $W_0^*$ $2-$Ricci generalized semi-parallel submanifold, then $N$ is either a total geodesic or a real space form with constant section curvature $c = 1$.*

**Proof.** Let's assume that $N$ is a $W_0^*$ $2-$Ricci generalized semiparallel submanifold. In this case, there is a function $k_4 = 0$ on the set $M_4 = \left\{x \in N | \tilde{\bigtriangledown}\sigma(x) \neq S(x)\right\}$ such that

$$W_0^* \cdot \tilde{\bigtriangledown}\sigma = k_4 Q\left(S, \tilde{\bigtriangledown}\sigma\right).$$

Then, as it is easily seen from the proof of Theorem-4, we obtain

$$\frac{(1-2n)\Upsilon_3 - 3\Upsilon_2}{n}(\Upsilon_1 - \Upsilon_3)\phi\sigma(\beta_5, \beta_2) = 0.$$

It is clear from last equation that

$$\sigma(\beta_5, \beta_2) = 0$$

or

$$(1-2n)\Upsilon_3 - 3\Upsilon_2 = 0.$$

This completes the proof of the corollary. □

Let us now give an example that satisfies the theorems we have given above. In [18], A. Sarkar and N. Biswas took a $5-$dimensional GS-space forms and obtained its $3-$dimensional submanifold. Let us show that the same manifold satisfies the conditions of the theorems we proved above.

**Example 1.** *Let us consider the five-dimensional manifold $N^5 = \{(\beta_1, \beta_2, \beta_3, \beta_5, \beta_4) \in \mathbf{R}^5 | \beta_3 \neq 0\}$, where $(\beta_1, \beta_2, \beta_3, \beta_5, \beta_4)$ are the standard coordinates of $\mathbf{R}^5$. We choose the vector fields*

$$e_1 = \frac{\partial}{\partial\beta_1} + \beta_2\frac{\partial}{\partial\beta_3}, e_2 = \frac{\partial}{\partial\beta_2}, e_3 = 2\frac{\partial}{\partial\beta_3}, e_4 = \frac{\partial}{\partial\beta_5} + \beta_4\frac{\partial}{\partial\beta_3}, e_5 = \frac{\partial}{\partial\beta_4},$$

*which are linearly independent at each point of $N^5$. we define $g$ such that*

$$\begin{aligned} g(e_1, e_1) &= \tfrac{1}{4}, g(e_2, e_2) = \tfrac{1}{4}, \\ g(e_4, e_4) &= \tfrac{1}{4}, g(e_5, e_5) = \tfrac{1}{4}, \\ g(e_3, e_3) &= 1, g(e_i, e_j) = 0, \end{aligned}$$

*for the remaining* $i, j; i, j = 1, 2, 3, 4, 5$. *We consider a* $1-$*form* $\eta$ *defined by*

$$\eta(\beta_1) = g(\beta_1, e_3), \beta_1 \in \chi\left(N^5\right).$$

*So, we choose* $e_3 = \xi$. *We define the* $(1, 1)$ *tensor field* $\phi$ *by*

$$\phi(e_1) = -e_2, \phi(e_2) = e_1, \phi(e_3) = 0, \phi(e_4) = -e_5, \phi(e_5) = e_4.$$

*The linearity property of* $g$ *and* $\phi$ *shows that*

$$\eta(e_5) = 1, \phi^2(\beta_1) = -\beta_1 + \eta(\beta_1) e_5,$$

$$g(\phi\beta_1, \phi\beta_2) = g(\beta_1, \beta_2) - \eta(\beta_1)\eta(\beta_2),$$

*for any vector field* $\beta_1, \beta_2$. *So* $N^5(\phi, \xi, \eta, g)$ *defines am almost contact manifold with* $e_3 = \xi$. *Moreover, let* $\tilde{\nabla}$ *be the Levi-Civita connection with respect to metric* $g$. *Then we have*

$$[e_1, e_2] = [e_4, e_5] = -\frac{1}{2}e_3, [e_i, e_j] = 0, \text{ otherswise.}$$

*By Kozsul's formula, we obtain the following*

$$\begin{aligned}
&\tilde{\nabla}_{e_3} e_1 = e_2, && \tilde{\nabla}_{e_1} e_2 = -\tfrac{1}{4} e_3, && \tilde{\nabla}_{e_1} e_3 = e_2, \\
&\tilde{\nabla}_{e_2} e_1 = \tfrac{1}{4} e_3, && \tilde{\nabla}_{e_2} e_3 = -e_1, && \tilde{\nabla}_{e_3} e_2 = -e_1, \\
&\tilde{\nabla}_{e_3} e_4 = e_5, && \tilde{\nabla}_{e_4} e_5 = -\tfrac{1}{4} e_3, && \tilde{\nabla}_{e_4} e_3 = e_5, \\
&\tilde{\nabla}_{e_5} e_4 = \tfrac{1}{4} e_3, && \tilde{\nabla}_{e_5} e_3 = -e_4, && \tilde{\nabla}_{e_3} e_4 = e_5 \\
&\tilde{\nabla}_{e_3} e_5 = -e_4.
\end{aligned}$$

*For the remaining cases* $\tilde{\nabla}_{e_i} e_j = 0$.

*Now, from the definition of curvatue tensor, we obtain its non-vanishing components as follows;*

$$\begin{aligned}
&R(e_1, e_2) e_1 = \tfrac{3}{4} e_2, && R(e_1, e_3) e_1 = -\tfrac{1}{4}, && R(e_1, e_2) e_2 = -\tfrac{3}{4}, \\
&R(e_2, e_3) e_2 = -\tfrac{1}{4} e_3, && R(e_1, e_3) e_3 = e_1, && R(e_2, e_3) e_3 = e_2, \\
&R(e_4, e_5) e_4 = \tfrac{3}{4} e_5, && R(e_4, e_3) e_4 = -\tfrac{1}{4} e_3, && R(e_4, e_5) e_5 = -\tfrac{3}{4} e_1, \\
&R(e_5, e_3) e_5 = -\tfrac{1}{4} e_3, && R(e_4, e_3) e_3 = e_4, && R(e_5, e_3) e_3 = e_5.
\end{aligned}$$

*Thus $N^5$ is a GS-space form with $\Upsilon_1 = 0, \Upsilon_2 = -1, \Upsilon_3 = -1$.*

*Let $N$ be a subset of $N^5$ and consider the isometric immersion*

$$f : N \to N^5, f(\beta_1, \beta_2, \beta_3) = (\beta_1, \beta_2, \beta_3, 0, 0).$$

*It is easy to prove that $N = \{(\beta_1, \beta_2, \beta_3) \in \mathbf{R}^3 | \, \beta_3 \neq 0\}$ is a submanifold of $N^5$, where $(\beta_1, \beta_2, \beta_3)$ are the standart coordinates of $\mathbf{R}^3$.*

*We choose the vector fields*

$$e_1 = \frac{\partial}{\partial \beta_1} + \beta_2 \frac{\partial}{\partial \beta_3}, e_2 = \frac{\partial}{\partial \beta_2}, e_3 = 2\frac{\partial}{\partial \beta_3},$$

*which are linearly independent at each point of $N$. We use the restrictions of $\phi, \xi, \eta$ and $g$ to denote the structures on the submanifold. We take $g$ such that*

$$g(e_1, e_1) = \frac{1}{4}, g(e_2, e_2) = \frac{1}{4}, g(e_3, e_3) = 1,$$

*and $g(e_i, e_j) = 0$ for the remaining $i, j; i, j = 1, 2, 3$. We take the $1-$form $\eta$ defined by*

$$\eta(\beta_1) = g(\beta_1, e_3), \beta_1 \in \chi(N).$$

*So, we choose $e_3 = \xi$. We consider $\phi$ by using*

$$\phi(e_1) = -e_2, \phi(e_2) = e_1, \phi(e_3) = 0.$$

*The linearity property of $g$ and $\phi$ shows that*

$$\eta(e_5) = 1, \phi^2 \beta_1 = -\beta_1 + \eta(\beta_1) e_5,$$

$$g(\phi\beta_1, \phi\beta_2) = g(\beta_1, \beta_2) - \eta(\beta_1)\eta(\beta_2),$$

*for any vector field $\beta_1, \beta_2$ of $N$. So $N(\phi, \xi, \eta, g)$ defines an almost contact manifold with $e_3 = \xi$. It seen that $N$ is invariant. Moreover, let $\nabla$ be the Levi-Civita connection with respect to metric $g$ of the submanifold. Then we have*

$$[e_1, e_2] = -\frac{1}{2}e_3, [e_i, e_j] = 0, \text{ otherswise.}$$

*By using Kozsul's formula, we obtain*

$$\nabla_{e_1} e_2 = -\tfrac{1}{4}e_3, \quad \nabla_{e_1} e_3 = e_2, \quad \nabla_{e_2} e_1 = \tfrac{1}{4}e_3,$$

$$\nabla_{e_2} e_3 = -e_1, \quad \nabla_{e_3} e_1 = e_2, \quad \nabla_{e_3} e_2 = -e_1.$$

*Let any vector field* $\beta_5, \beta_4 \in \chi(N)$ *, then there exist some scalars* $\lambda_i$ *and* $\mu_i$ *such that*

$$\beta_5 = \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3$$

*and*

$$\beta_4 = \mu_1 e_1 + \mu_2 e_2 + \mu_3 e_3.$$

*Then, since* $\sigma$ *is linear, we get*

$$\sigma(\beta_5, \beta_4) = 0.$$

*This tell us that* $N$ *is a totally geodesic submanifold. Since it is a totally geodesic submanifold, the conditions given in the theorems and Lemma 1 satisfied. For Example, in Lemma 1, since* $\Upsilon_1 - \Upsilon_3 \neq 0$*, it implies that submanifold is totally geodesic. In same way, since the totally geodesic submanifolds are the simplest manifolds, all the conditions given in the theorems are satisfied.Thus the submanifold is totally geodesic. Also this tell us that* $N$ *is* $W_0^*-$*pseudoparallel,* $W_0^* - 2$ *pseudoparallel,* $W_0^*-$ *Ricci generalized pseudoparallel and* $W_0^* - 2$ *Ricci generalized pseudoparallel.*

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