Gulf Journal of Mathematics ISSN: 2309-4966
Vol 14, Issue 2 (2023) 101-114 https://doi.org/10.56947/gjom.v14i2.1179

SEMI-SYMMETRIC ALMOST $C(\alpha)$-MANIFOLD ON SOME CURVATURE TENSORS

TUĞBA MERT[1*] , MEHMET ATÇEKEN[2] AND PAKIZE UYGUN[3]

Abstract. In this article, semi-symmetry of almost $C(\alpha)$-manifold is investigated on some special curvature tensors. First, the behavior of the almost $C(\alpha)$-manifold is investigated when the special curvature tensors discussed are flat. Then, for these special curvature tensors, the behavior of the manifold in the semi-symmetric condition is observed and for some special curvature tensors, important properties such as the semi-symmetric almost $C(\alpha)$-manifold being Einstein and $\eta-$Einstein manifold are obtained.

1. Introduction

One of the most important concepts for Riemannian manifolds is the semi-symmetric metric connection. Yano described the semi-symmetric metric connection on a Riemann manifold in 1970 and showed that Riemannian curvature tensor with respect to a semi-symmetric metric connection vanishes if and only if the manifold is conformally flat [1]. Later, semi-symmetric metric connection on different manifolds was investigated and many geometrical properties were revealed by many geometers ([2]-[5]). It has been shown by Barua et al. that, under some special conditions, the conformal curvature tensor corresponding to the semi-symmetric metric connection on a Riemann manifold is invariant [6]. And again it was proved by U.C. De and J. Sengupta that the congruent and projective curvature tensors on an almost contact metric manifold are invariant with respect to the semi-symmetrical metric connection under certain constraints [7].

Semi-symmetric metric connection also has very important contributions to our daily life, physics and engineering. For example, if we are moving somewhere on the earth's surface, such as the North Pole, and we are always facing a certain point, this displacement is semi-symmetrical and metric [8].

Again, many authors have investigated curvatures on different manifolds such as para-contact metric manifolds, and many important properties such as semi-symmetry and pseudo-symmetry ([9]-[16]). In addition, many mathematicians in

Date: Received: May 1, 2022; Accepted: Sep 24, 2022.

* Corresponding author.

2010 *Mathematics Subject Classification.* 53C15; 53C44; 53D10.

Key words and phrases. $T-$Curvature Tensor, Semi-Symmetric Manifold, Almost $C(\alpha)$-manifold.

many different spaces have obtained important properties and characterizations of manifolds ([18]-[22]).

In this article, semi-symmetry of almost $C(\alpha)$-manifold is investigated on some special curvature tensors. First, the behavior of the almost $C(\alpha)$-manifold is investigated when the special curvature tensors discussed are flat. Then, for these special curvature tensors, the behavior of the manifold in the semi-symmetric condition is observed and for some special curvature tensors, important properties such as the semi-symmetric almost $C(\alpha)$-manifold being Einstein and $\eta-$Einstein manifold are obtained.

2. Preliminary

Let's take an $(2n+1)-$dimensional differentiable M manifold. If it admits a tensor field ϕ of type $(1,1)$, a vector field ξ and a 1-form η satisfying the following conditions;

$$\phi^2 V_1 = -V_1 + \eta(V_1)\xi \text{ and } \eta(\xi) = 1,$$

then this (ϕ, ξ, η) is called an almost contact structure, and the (M, ϕ, ξ, η) is called an almost contact manifold. If there is a g metric that satisfies the condition

$$g(\phi V_1, \phi V_2) = g(V_1, V_2) - \eta(V_1)\eta(V_2) \text{ and } g(V_1, \xi) = \eta(V_1),$$

for all $V_1, V_2 \in \chi(M)$ and $\xi \in \chi(M)$; (ϕ, ξ, η, g) is called almost contact metric structure and (M, ϕ, ξ, η, g) is called almost contact metric manifold. On the $(2n+1)$ dimensional M manifold,

$$g(\phi V_1, V_2) = -g(V_1, \phi V_2)$$

for all $V_1, V_2 \in \chi(M)$, that is, ϕ is an anti-symmetric tensor field according to the g metric. The Φ transformation defined as

$$\Phi(V_1, V_2) = g(V_1, \phi V_2)$$

for all $V_1, V_2 \in \chi(M)$, is called the fundamental 2-form of the (ϕ, ξ, η, g) almost contact metric structure, where

$$\eta \wedge \Phi^n \neq 0.$$

If the Riemann curvature tensor R of the M almost contact metric manifold satisfies the condition

$$\begin{aligned} R(V_1, V_2)V_3 &= \left(\frac{c+3\alpha}{4}\right)\{g(V_2, V_3)V_1 - g(V_1, V_3)V_2\} \\ &+ \left(\frac{c-\alpha}{4}\right)\{g(V_1, \phi V_3)\phi V_2 - g(V_2, \phi V_3)\phi V_1 \\ &+ 2g(V_1, \phi V_2)\phi V_3 + \eta(V_2)\eta(V_3)V_1 \\ &+ g(V_1, V_3)\eta(V_2)\xi - g(V_2, V_3)\eta(V_1)\xi\}. \end{aligned} \quad (1)$$

for all $V_1, V_2, V_3 \in \chi(M)$, at least one $\alpha \in \mathbb{R}$, then M is almost $C(\alpha)-$manifold with $c-$constant sectional curvature. If we choose $V_1 = \xi, V_2 = \xi, V_3 = \xi$ respectively in (1), then we get

$$R(\xi, V_2)V_3 = \alpha[g(V_2, V_3)\xi - \eta(V_3)V_2], \tag{2}$$

$$R(V_1, \xi)V_3 = \alpha[-g(V_1, V_3)\xi + \eta(V_3)V_1], \tag{3}$$

$$R(V_1, V_2)\xi = \alpha[\eta(V_2)V_1 - \eta(V_1)V_2]. \tag{4}$$

If we take the inner product of both sides of (1) by $\xi \in \chi(M)$, we have

$$\eta(R(V_1, V_2)V_3) = \alpha[g(V_2, V_3)\eta(V_1) - g(V_1, V_3)\eta(V_2)]. \tag{5}$$

Lemma 2.1. *For a $(2n+1)-$dimensional M almost $C(\alpha)-$manifold, the following equations are provided.*

$$S(V_1, V_2) = \left[\frac{\alpha(3n-1) + c(n+1)}{2}\right] g(V_1, V_2) + \frac{(\alpha - c)(n+1)}{2}\eta(V_1)\eta(V_2), \tag{6}$$

$$S(V_1, \xi) = 2n\alpha\eta(V_1), \tag{7}$$

$$QV_1 = \left[\frac{\alpha(3n-1) + c(n+1)}{2}\right] V_1 + \frac{(\alpha - c)(n+1)}{2}\eta(V_1)\xi, \tag{8}$$

$$Q\xi = 2n\alpha\xi. \tag{9}$$

for all $V_1, V_2 \in \chi(M)$, where Q and S are the Ricci operator and Ricci tensor of manifold M, respectively.

M. Tripathi and P. Gunam described a T curvature tensor of the $(1,3)$ type in an n-dimensional (M, g) semi-Riemann manifold [17]. This curvature tensor is defined as

$$\begin{aligned} T(V_1, V_2)V_3 &= a_0 R(V_1, V_2)V_3 + a_1 S(V_2, V_3)V_1 + a_2 S(V_1, V_3)V_2 \\ &+ a_3 S(V_1, V_2)V_3 + a_4 g(V_2, V_3)QV_1 + a_5 g(V_1, V_3)QV_2 \\ &+ a_6 g(V_1, V_2)QV_3 + a_7 r[g(V_2, V_3)V_1 - g(V_1, V_3)V_2], \end{aligned} \tag{10}$$

where R, S, Q, and r are Riemann curvature tensor, Ricci curvature tensor, Ricci operator, and scalar curvature of manifold M, respectively. According to the choosing of smooth functions $a_0, a_1, ..., a_7$ the curvature tensor T is reduced to some special curvature tensors as follows.

If $a_0 = 1, a_1 = -a_6 = -\frac{1}{2n}, a_2 = a_3 = a_4 = a_5 = a_7 = 0$ are chosen in (10), the T_1-curvature tensor is defined as

$$T_1(V_1, V_2)V_3 = R(V_1, V_2)V_3 - \frac{1}{2n}\left[S(V_2, V_3)V_1 - g(V_1, V_2)QV_3\right]. \tag{11}$$

If we choose $V_1 = \xi, V_2 = \xi, V_3 = \xi$ respectively in (11), then we get

$$\begin{aligned} T_1(\xi, V_2)V_3 &= \tfrac{(\alpha-c)(n+1)}{4n} g(V_2, V_3)\xi + \tfrac{\alpha(3n-1)+c(n+1)}{4n}\eta(V_2)V_3 \\ &-\alpha\eta(V_3)V_2, \end{aligned} \tag{12}$$

$$\begin{aligned} T_1(V_1, \xi)V_3 &= -\alpha g(V_1, V_3)\xi + \tfrac{\alpha(3n-1)+c(n+1)}{4n}\eta(V_1)V_3 \\ &+\tfrac{(\alpha-c)(n+1)}{4n}\eta(V_1)\eta(V_3)\xi, \end{aligned} \tag{13}$$

$$T_1(V_1, V_2)\xi = \alpha\left[g(V_1, V_2)\xi - \eta(V_1)V_2\right]. \tag{14}$$

If $a_0 = 1, a_1 = -a_4 = -\frac{1}{2n}, a_2 = a_3 = a_5 = a_6 = a_7 = 0$ are chosen in (10), the T_2-curvature tensor is defined as

$$T_2(V_1, V_2)V_3 = R(V_1, V_2)V_3 - \frac{1}{2n}\left[S(V_2, V_3)V_1 - g(V_2, V_3)QV_1\right]. \tag{15}$$

If we choose $V_1 = \xi, V_2 = \xi, V_3 = \xi$ respectively in (15), then we get

$$\begin{aligned} T_2(\xi, V_2)V_3 &= \tfrac{\alpha(5n+1)-c(n+1)}{4n} g(V_2, V_3)\xi - \alpha\eta(V_3)V_2 \\ &-\tfrac{(\alpha-c)(n+1)}{4n}\eta(V_2)\eta(V_3)\xi, \end{aligned} \tag{16}$$

$$\begin{aligned} T_2(V_1, \xi)V_3 &= -\alpha g(V_1, V_3)\xi + \tfrac{\alpha(3n-1)+c(n+1)}{4n}\eta(V_3)V_1 \\ &+\tfrac{(\alpha-c)(n+1)}{4n}\eta(V_1)\eta(V_3)\xi, \end{aligned} \tag{17}$$

$$\begin{aligned} T_2(V_1, V_2)\xi &= -\alpha\eta(V_1)V_2 + \tfrac{\alpha(3n-1)+c(n+1)}{4n}\eta(V_2)V_1 \\ &+\tfrac{(\alpha-c)(n+1)}{4n}\eta(V_1)\eta(V_2)\xi. \end{aligned} \tag{18}$$

If $a_0 = 1, a_1 = -a_3 = -\frac{1}{2n}, a_2 = a_4 = a_5 = a_6 = a_7 = 0$ are chosen in (10), the T_3-curvature tensor is defined as

$$T_3(V_1, V_2)V_3 = R(V_1, V_2)V_3 - \frac{1}{2n}\left[S(V_2, V_3)V_1 - S(V_1, V_2)V_3\right]. \tag{19}$$

If we choose $V_1 = \xi, V_2 = \xi, V_3 = \xi$ respectively in (19), then we get

$$\begin{aligned} T_3(\xi, V_2)V_3 &= \tfrac{(\alpha-c)(n+1)}{4n}\left[g(V_2, V_3)\xi - \eta(V_2)\eta(V_3)\xi\right] \\ &-\alpha\left[\eta(V_3)V_2 - \eta(V_2)V_3\right], \end{aligned} \tag{20}$$

$$T_3(V_1, \xi)V_3 = \alpha\left[-g(V_1, V_3)\xi + \eta(V_1)V_3\right], \tag{21}$$

$$\begin{aligned} T_3(V_1,V_2)\xi &= \tfrac{\alpha(3n-1)+c(n+1)}{4n} g(V_1,V_2)\xi - \alpha\eta(V_1)V_2 \\ &+ \tfrac{(\alpha-c)(n+1)}{4n}\eta(V_1)\eta(V_2)\xi. \end{aligned} \tag{22}$$

If $a_0 = 1, a_3 = -a_4 = \frac{1}{2n}, a_1 = a_2 = a_5 = a_6 = a_7 = 0$ are chosen in (10), the T_4-curvature tensor is defined as

$$T_4(V_1,V_2)V_3 = R(V_1,V_2)V_3 + \frac{1}{2n}\left[S(V_1,V_2)V_3 - g(V_2,V_3)QV_1\right]. \tag{23}$$

If we choose $V_1 = \xi, V_2 = \xi, V_3 = \xi$ respectively in (23), then we get

$$T_4(\xi,V_2)V_3 = \alpha\left[\eta(V_2)V_3 - \eta(V_3)V_2\right], \tag{24}$$

$$\begin{aligned} T_4(V_1,\xi)V_3 &= \alpha\left[-g(V_1,V_3)\xi + \eta(V_1)V_3\right] \\ &+ \tfrac{(\alpha-c)(n+1)}{4n}\left[\eta(V_3)V_1 - \eta(V_1)\eta(V_3)\xi\right], \end{aligned} \tag{25}$$

$$\begin{aligned} T_4(V_1,V_2)\xi &= \tfrac{\alpha(3n-1)+c(n+1)}{4n} g(V_1,V_2)\xi - \alpha\eta(V_1)V_2 \\ &+ \tfrac{(\alpha-c)(n+1)}{4n}\eta(V_2)V_1. \end{aligned} \tag{26}$$

3. Flatness of $T-$Curvature Tensors on Almost $C(\alpha)$-Manifolds

In this section, let's investigate the flatness of the $T-$curvature tensors defined as above on almost $C(\alpha)-$manifold.

Theorem 3.1. *Let M be a $(2n+1)-$dimensional almost $C(\alpha)-$manifold. If M is T_1-flat, then M is an $\eta-$Einstein manifold.*

Proof. Let's assume that M is T_1-flat. So, we can write

$$T_1(V_1,V_2)V_3 = 0$$

for every $V_1, V_2, V_3 \in \chi(M)$. That is

$$R(V_1,V_2)V_3 = \frac{1}{2n}S(V_2,V_3)V_1 - \frac{1}{2n}g(V_1,V_2)QV_3. \tag{27}$$

If we choose $V_1 = \xi$ in (27), we get

$$R(\xi,V_2)V_3 = \frac{1}{2n}S(V_2,V_3)\xi - \frac{1}{2n}g(\xi,V_2)QV_3.$$

If we use (2) and (8) in the last equation, we have

$$\begin{aligned} \alpha g(V_2,V_3)\xi - \alpha\eta(V_3)V_2 &= \tfrac{1}{2n}S(V_2,V_3)\xi \\ &- \tfrac{\alpha(3n-1)+c(n+1)}{4n}\eta(V_2)V_3 - \tfrac{(\alpha-c)(n+1)}{4n}\eta(V_2)\eta(V_3)\xi. \end{aligned} \tag{28}$$

If we take inner product both sides of (28) by $\xi \in \chi(M)$ and make the necessary adjustments, we obtain

$$S(V_2,V_3) = 2n\alpha g(V_2,V_3) - \frac{(\alpha-c)(n+1)}{2}\eta(V_2)\eta(V_3).$$

This completes the proof. □

Theorem 3.2. *Let M be a $(2n+1)-$dimensional almost $C(\alpha)-$manifold. If M is T_2-flat, then M is an $\eta-$Einstein manifold.*

Proof. Let's assume that M is T_2-flat. So, we can write

$$T_2(V_1,V_2)V_3=0$$

for every $V_1,V_2,V_3\in\chi(M)$. That is

$$R(V_1,V_2)V_3=\frac{1}{2n}S(V_2,V_3)V_1-\frac{1}{2n}g(V_2,V_3)QV_1. \tag{29}$$

If we choose $V_1=\xi$ in (29), we get

$$R(\xi,V_2)V_3=\frac{1}{2n}S(V_2,V_3)\xi-\frac{1}{2n}g(V_2,V_3)Q\xi.$$

If we use (2) and (9) in the last equation, we have

$$\alpha g(V_2,V_3)\xi-\alpha\eta(V_3)V_2=\frac{1}{2n}S(V_2,V_3)\xi-\alpha g(V_2,V_3)\xi. \tag{30}$$

If we take inner product both sides of (30) by $\xi\in\chi(M)$ and make the necessary adjustments, we obtain

$$S(V_2,V_3)=4n\alpha g(V_2,V_3)-2n\alpha\eta(V_2)\eta(V_3).$$

This completes the proof. □

Theorem 3.3. *Let M be a $(2n+1)-$dimensional almost $C(\alpha)-$manifold. If M is T_3-flat, then M is an Einstein manifold.*

Proof. Let's assume that M is T_3-flat. So, we can write

$$T_3(V_1,V_2)V_3=0$$

for every $V_1,V_2,V_3\in\chi(M)$. That is

$$R(V_1,V_2)V_3=\frac{1}{2n}S(V_2,V_3)V_1-\frac{1}{2n}S(V_1,V_2)V_3. \tag{31}$$

If we choose $V_1=\xi$ in (31), we get

$$R(\xi,V_2)V_3=\frac{1}{2n}S(V_2,V_3)\xi-\frac{1}{2n}S(\xi,V_2)V_3.$$

If we use (2) and (7) in the last equation, we have

$$\alpha g(V_2,V_3)\xi-\alpha\eta(V_3)V_2=\frac{1}{2n}S(V_2,V_3)\xi-\alpha\eta(V_2)V_3. \tag{32}$$

If we take inner product both sides of (32) by $\xi\in\chi(M)$ and make the necessary adjustments, we obtain

$$S(V_2,V_3)=2n\alpha g(V_2,V_3).$$

This completes the proof. □

Theorem 3.4. *Let M be a $(2n+1)-$dimensional almost $C(\alpha)-$manifold. If M is T_4-flat, then M is an Einstein manifold.*

Proof. Let's assume that M is T_4-flat. So, we can write

$$T_4(V_1, V_2)V_3 = 0$$

for every $V_1, V_2, V_3 \in \chi(M)$. That is

$$R(V_1, V_2)V_3 = \frac{1}{2n}g(V_2, V_3)QV_1 - \frac{1}{2n}S(V_1, V_2)V_3. \tag{33}$$

If we choose $V_3 = \xi$ in (33), we get

$$R(V_1, V_2)\xi = \frac{1}{2n}g(V_2, \xi)QV_1 - \frac{1}{2n}S(V_1, V_2)\xi.$$

If we use (4) in the last equation, we have

$$\alpha\eta(V_2)V_1 - \alpha\eta(V_1)V_2 = \frac{1}{2n}\eta(V_2)QV_1 - \frac{1}{2n}S(V_1, V_2)\xi. \tag{34}$$

If we choose $V_2 = \xi$ in equation (34) and we take inner product of both sides of the equation by $V_3 \in \chi(M)$, we obtain

$$S(V_1, V_3) = 2n\alpha g(V_1, V_3).$$

This completes the proof. □

4. Semi-Symmetric Almost $C(\alpha)-$Manifold

In this section, the semi-symmetry condition of almost $C(\alpha)-$manifold will be investigated for some special $T-$curvature tensors described above.

Theorem 4.1. *Let M be the $(2n+1)-$dimensional almost $C(\alpha)-$manifold. If M is T_1-semi-symmetric, then M is either a co-Kaehler manifold or an $\eta-$Einstein manifold.*

Proof. Let's assume that M is T_1-semi symmetric manifold. Then we can write

$$(R(V_1, V_2)T_1)(V_4, V_5, V_3) = 0,$$

for each $V_1, V_2, V_4, V_5, V_3 \in \chi(M)$. That is, we can write

$$\begin{aligned} &R(V_1, V_2)T_1(V_4, V_5)V_3 - T_1(R(V_1, V_2)V_4, V_5)V_3 \\ &-T_1(V_4, R(V_1, V_2)V_5)V_3 - T_1(V_4, V_5)R(V_1, V_2)V_3 = 0. \end{aligned} \tag{35}$$

If we choose $V_1 = \xi$ in (35) and use (2), we get

$$\begin{aligned} &\alpha\{g(V_2, T_1(V_4, V_5)V_3)\xi - \eta(T_1(V_4, V_5)V_3)V_2 \\ &-g(V_2, V_4)T_1(\xi, V_5)V_3 + \eta(V_4)T_1(V_2, V_5)V_3 \\ &-g(V_2, V_5)T_1(V_4, \xi)V_3 + \eta(V_5)T_1(V_4, V_2)V_3 \\ &-g(V_2, V_3)T_1(V_4, V_5)\xi + \eta(V_3)T_1(V_4, V_5)V_2\} = 0. \end{aligned} \tag{36}$$

If we use (12),(13),(14) in (36), we have

$$\begin{aligned}
&\alpha\{g(V_2,T_1(V_4,V_5)V_3)\xi-\eta(T_1(V_4,V_5)V_3)V_2\\
&-\tfrac{(\alpha-c)(n+1)}{4n}g(V_2,V_4)g(V_5,V_3)\xi-\tfrac{\alpha(3n-1)+c(n+1)}{4n}g(V_2,V_4)\eta(V_5)V_3\\
&+\alpha g(V_2,V_4)\eta(V_3)V_5+\eta(V_4)T_1(V_2,V_5)V_3\\
&+\alpha g(V_2,V_5)g(V_4,V_3)\xi-\tfrac{\alpha(3n-1)+c(n+1)}{4n}g(V_2,V_5)\eta(V_4)V_3\\
&-\tfrac{(\alpha-c)(n+1)}{4n}g(V_2,V_5)\eta(V_4)\eta(V_3)\xi+\eta(V_5)T_1(V_4,V_2)V_3\\
&-\alpha g(V_2,V_3)g(V_4,V_5)\xi+\alpha g(V_2,V_3)\eta(V_4)V_5\\
&+\eta(V_3)T_1(V_4,V_5)V_2\}=0.
\end{aligned}\tag{37}$$

If we choose $V_4=\xi$ in (37) and make use of (12), we obtain

$$\begin{aligned}
&\alpha\Big\{-\tfrac{(\alpha-c)(n+1)}{4n}g(V_5,V_3)V_2+T_1(V_2,V_5)V_3\\
&-\tfrac{\alpha(3n-1)+c(n+1)}{4n}g(V_2,V_5)V_3+\alpha g(V_2,V_3)V_5\Big\}=0.
\end{aligned}\tag{38}$$

Substituting (11) in (38), we have

$$\begin{aligned}
&\alpha\Big\{-\tfrac{(\alpha-c)(n+1)}{4n}g(V_5,V_3)V_2+R(V_2,V_5)V_3\\
&-\tfrac{1}{2n}S(V_5,V_3)V_2+\tfrac{1}{2n}g(V_2,V_5)QV_3\\
&-\tfrac{\alpha(3n-1)+c(n+1)}{4n}g(V_2,V_5)V_3+\alpha g(V_2,V_3)V_5\Big\}=0.
\end{aligned}\tag{39}$$

If we choose $V_5=\xi$ in (39), we get

$$\begin{aligned}
&\alpha\Big\{-\tfrac{(\alpha-c)(n+1)}{4n}\eta(V_3)V_2+\tfrac{1}{2n}\eta(V_2)QV_3\\
&-\tfrac{\alpha(3n-1)+c(n+1)}{4n}\eta(V_2)V_3\Big\}=0.
\end{aligned}$$

If we choose $V_2=\xi$ in the last equation and take inner product both sides of the last equation by $V_5\in\chi(M)$, we can write

$$\begin{aligned}
&\alpha\Big\{\tfrac{1}{2n}S(V_3,V_5)-\tfrac{\alpha(3n-1)+c(n+1)}{4n}g(V_3,V_5)\\
&-\tfrac{(\alpha-c)(n+1)}{4n}\eta(V_3)\eta(V_5)\Big\}=0.
\end{aligned}\tag{40}$$

From (40), we have

$$S(V_3,V_5)=\frac{\alpha(3n-1)+c(n+1)}{2}g(V_3,V_5)+\frac{(\alpha-c)(n+1)}{2}\eta(V_3)\eta(V_5),$$

or

$$\alpha = 0.$$

This completes the proof. □

Theorem 4.2. *Let M be the $(2n+1)-$dimensional almost $C(\alpha)-$manifold. If M is T_2-semi-symmetric, then M is either a co-Kaehler manifold or an $\eta-$Einstein manifold.*

Proof. Let's assume that M is T_2-semi symmetric manifold. Then we can write

$$(R(V_1, V_2) T_2)(V_4, V_5, V_3) = 0,$$

for each $V_1, V_2, V_4, V_5, V_3 \in \chi(M)$. That is, we can write

$$\begin{aligned} & R(V_1, V_2) T_2(V_4, V_5) V_3 - T_2(R(V_1, V_2) V_4, V_5) V_3 \\ & -T_2(V_4, R(V_1, V_2) V_5) V_3 - T_2(V_4, V_5) R(V_1, V_2) V_3 = 0. \end{aligned} \tag{41}$$

If we choose $V_1 = \xi$ in (41) and use (2), we get

$$\begin{aligned} & \alpha \{g(V_2, T_2(V_4, V_5) V_3) \xi - \eta(T_2(V_4, V_5) V_3) V_2 \\ & -g(V_2, V_4) T_2(\xi, V_5) V_3 + \eta(V_4) T_2(V_2, V_5) V_3 \\ & -g(V_2, V_5) T_2(V_4, \xi) V_3 + \eta(V_5) T_2(V_4, V_2) V_3 \\ & -g(V_2, V_3) T_2(V_4, V_5) \xi + \eta(V_3) T_2(V_4, V_5) V_2\} = 0. \end{aligned} \tag{42}$$

If we use (16),(17),(18) in (42), we have

$$\begin{aligned} & \alpha \{g(V_2, T_2(V_4, V_5) V_3) \xi - \eta(T_2(V_4, V_5) V_3) V_2 \\ & -\tfrac{\alpha(5n+1)+c(n+1)}{4n} g(V_2, V_4) g(V_5, V_3) \xi - \alpha g(V_2, V_4) \eta(V_3) V_5 \\ & +\tfrac{(\alpha-c)(n+1)}{4n} g(V_2, V_4) \eta(V_5) \eta(V_3) \xi + \eta(V_4) T_2(V_2, V_5) V_3 \\ & +\alpha g(V_2, V_5) g(V_4, V_3) \xi - \tfrac{\alpha(3n-1)+c(n+1)}{4n} g(V_2, V_5) \eta(V_3) V_4 \\ & -\tfrac{(\alpha-c)(n+1)}{4n} g(V_2, V_5) \eta(V_4) \eta(V_3) \xi + \eta(V_5) T_2(V_4, V_2) V_3 \\ & +\alpha g(V_2, V_3) \eta(V_4) V_5 - \tfrac{\alpha(3n-1)+c(n+1)}{4n} g(V_2, V_3) \eta(V_5) V_4 \\ & -\tfrac{(\alpha-c)(n+1)}{4n} g(V_2, V_3) \eta(V_4) \eta(V_5) \xi + \eta(V_3) T_2(V_4, V_5) V_2 \Big\} = 0 \end{aligned} \tag{43}$$

If we choose $V_4 = \xi$ in (43) and make use of (16), we obtain

$$\begin{aligned} &\alpha\Big\{-\tfrac{\alpha(5n+1)+c(n+1)}{4n}g(V_5,V_3)V_2+\tfrac{(\alpha-c)(n+1)}{4n}\eta(V_5)\eta(V_3)V_2 \\ &+\tfrac{(\alpha+c)(n+1)}{4n}g(V_2,V_5)\eta(V_3)\xi+\tfrac{(\alpha+c)(n+1)}{4n}g(V_2,V_3)\eta(V_5)\xi \\ &-\tfrac{(\alpha-c)(n+1)}{2n}\eta(V_3)\eta(V_5)\eta(V_2)\xi+T_2(V_2,V_5)V_3 \\ &+\alpha g(V_2,V_3)V_5\Big\}=0. \end{aligned} \tag{44}$$

Substituting (15) in (44), we have

$$\begin{aligned} &\alpha\Big\{-\tfrac{\alpha(5n+1)+c(n+1)}{4n}g(V_5,V_3)V_2+\tfrac{(\alpha-c)(n+1)}{4n}\eta(V_5)\eta(V_3)V_2 \\ &+R(V_2,V_5)V_3-\tfrac{1}{2n}S(V_5,V_3)V_2+\tfrac{1}{2n}g(V_5,V_3)QV_2 \\ &+\tfrac{(\alpha+c)(n+1)}{4n}g(V_2,V_5)\eta(V_3)\xi+\tfrac{(\alpha+c)(n+1)}{4n}g(V_2,V_3)\eta(V_5)\xi \\ &-\tfrac{(\alpha-c)(n+1)}{2n}\eta(V_3)\eta(V_5)\eta(V_2)\xi+\alpha g(V_2,V_3)V_5\Big\}=0. \end{aligned} \tag{45}$$

If we choose $V_3 = \xi$ in (45), we get

$$\begin{aligned} &\alpha\Big\{-\tfrac{n\alpha+c(n+1)}{2n}\eta(V_5)V_2+\tfrac{1}{2n}\eta(V_5)QV_2+\tfrac{(\alpha+c)(n+1)}{4n}g(V_2,V_5)\xi \\ &+\tfrac{(\alpha+c)(n+1)}{4n}\eta(V_2)\eta(V_5)\xi-\tfrac{(\alpha-c)(n+1)}{2n}\eta(V_5)\eta(V_2)\xi\Big\}=0. \end{aligned}$$

If we choose $V_5 = \xi$ in the last equation and take inner product both sides of the last equation by $V_3 \in \chi(M)$, we can write

$$\alpha\left\{\frac{1}{2n}S(V_2,V_3)-\frac{n\alpha+c(n+1)}{2n}g(V_2,V_3)+\frac{c(n+1)}{n}\eta(V_2)\eta(V_3)\right\}=0.$$

From the last equality, we have

$$S(V_2,V_3)=[n\alpha+c(n+1)]\,g(V_2,V_3)-2c(n+1)\,\eta(V_2)\eta(V_3),$$

or

$$\alpha = 0.$$

This completes the proof. □

Theorem 4.3. *Let M be the $(2n+1)-$dimensional almost $C(\alpha)-$manifold. If M is T_3-semi-symmetric, then M is either a co-Kaehler manifold or an $\eta-$Einstein manifold.*

Proof. Let's assume that M is T_3-semi symmetric manifold. Then we can write

$$(R(V_1,V_2)T_3)(V_4,V_5,V_3)=0,$$

for each $V_1, V_2, V_4, V_5, V_3 \in \chi(M)$. That is, we can write

$$\begin{aligned} &R(V_1, V_2) T_3(V_4, V_5) V_3 - T_3(R(V_1, V_2) V_4, V_5) V_3 \\ &- T_3(V_4, R(V_1, V_2) V_5) V_3 - T_3(V_4, V_5) R(V_1, V_2) V_3 = 0. \end{aligned} \tag{46}$$

If we choose $V_1 = \xi$ in (46) and use (2), we get

$$\begin{aligned} &\alpha \{ g(V_2, T_3(V_4, V_5) V_3) \xi - \eta(T_3(V_4, V_5) V_3) V_2 \\ &- g(V_2, V_4) T_3(\xi, V_5) V_3 + \eta(V_4) T_3(V_2, V_5) V_3 \\ &- g(V_2, V_5) T_3(V_4, \xi) V_3 + \eta(V_5) T_3(V_4, V_2) V_3 \\ &- g(V_2, V_3) T_3(V_4, V_5) \xi + \eta(V_3) T_3(V_4, V_5) V_2 \} = 0. \end{aligned} \tag{47}$$

If we use (20),(21),(22) in (47), we have

$$\begin{aligned} &\alpha \{ g(V_2, T_3(V_4, V_5) V_3) \xi - \eta(T_3(V_4, V_5) V_3) V_2 \\ &- \tfrac{(\alpha-c)(n+1)}{4n} g(V_2, V_4) g(V_5, V_3) \xi + \alpha g(V_2, V_4) \eta(V_3) V_5 \\ &+ \tfrac{(\alpha-c)(n+1)}{4n} g(V_2, V_4) \eta(V_5) \eta(V_3) \xi - \alpha g(V_2, V_4) \eta(V_5) V_3 \\ &+ \eta(V_4) T_3(V_2, V_5) V_3 + \alpha g(V_2, V_5) g(V_4, V_3) \xi \\ &- \alpha g(V_2, V_5) \eta(V_4) V_3 + \eta(V_5) T_3(V_4, V_2) V_3 \\ &- \tfrac{\alpha(3n-1)+c(n+1)}{4n} g(V_2, V_3) g(V_4, V_5) \xi + \alpha g(V_2, V_3) \eta(V_4) V_5 \\ &+ \eta(V_3) T_3(V_4, V_5) V_2 - \tfrac{(\alpha-c)(n+1)}{4n} g(V_2, V_3) \eta(V_4) \eta(V_5) \xi \Big\} = 0. \end{aligned} \tag{48}$$

If we choose $V_4 = \xi$ in (48) and make use of (20), we obtain

$$\begin{aligned} &\alpha \Big\{ \tfrac{(\alpha-c)(n+1)}{4n} g(V_2, V_3) \eta(V_5) \xi - \tfrac{(\alpha-c)(n+1)}{2n} \eta(V_3) \eta(V_5) \eta(V_2) \xi \\ &+ \alpha g(V_2, V_3) V_5 + \tfrac{(\alpha-c)(n+1)}{4n} g(V_5, V_2) \eta(V_3) \xi \\ &+ \tfrac{(\alpha-c)(n+1)}{4n} \eta(V_5) \eta(V_3) V_2 + R(V_2, V_5) V_3 \\ &- \tfrac{1}{2n} S(V_5, V_3) V_2 + \tfrac{1}{2n} S(V_2, V_5) V_3 \\ &- \tfrac{(\alpha-c)(n+1)}{4n} g(V_5, V_3) V_2 \Big\} = 0. \end{aligned} \tag{49}$$

If we choose $V_3 = \xi$ in(49), we get

$$\alpha \left\{ -\frac{(\alpha-c)(n+1)}{4n} \eta(V_2) \eta(V_5) \xi + \frac{(\alpha-c)(n+1)}{4n} g(V_5, V_2) \xi + \frac{1}{2n} S(V_2, V_5) \xi \right\} = 0.$$

If we take inner product both sides of the last equation by $\xi \in \chi(M)$ in the last equation, we can write

$$\alpha\left\{-\frac{(\alpha-c)(n+1)}{4n}\eta(V_2)\eta(V_5)+\frac{(\alpha-c)(n+1)}{4n}g(V_5,V_2)+\frac{1}{2n}S(V_2,V_5)\right\}=0.$$

This completes the proof. □

Theorem 4.4. *Let M be the $(2n+1)-$dimensional almost $C(\alpha)-$manifold. If M is T_4-semi-symmetric, then M is either a co-Kaehler manifold or an Einstein manifold.*

Proof. Let's assume that M is T_4-semi symmetric manifold. Then we can write

$$(R(V_1,V_2)T_4)(V_4,V_5,V_3)=0,$$

for each $V_1,V_2,V_4,V_5,V_3 \in \chi(M)$. That is, we can write

$$\begin{aligned}&R(V_1,V_2)T_4(V_4,V_5)V_3-T_4(R(V_1,V_2)V_4,V_5)V_3\\&-T_4(V_4,R(V_1,V_2)V_5)V_3-T_4(V_4,V_5)R(V_1,V_2)V_3=0.\end{aligned}\tag{50}$$

If we choose $V_1=\xi$ in (50) and use (2), we get

$$\begin{aligned}&\alpha\{g(V_2,T_4(V_4,V_5)V_3)\xi-\eta(T_4(V_4,V_5)V_3)V_2\\&-g(V_2,V_4)T_4(\xi,V_5)V_3+\eta(V_4)T_4(V_2,V_5)V_3\\&-g(V_2,V_5)T_4(V_4,\xi)V_3+\eta(V_5)T_4(V_4,V_2)V_3\\&-g(V_2,V_3)T_4(V_4,V_5)\xi+\eta(V_3)T_4(V_4,V_5)V_2\}=0.\end{aligned}\tag{51}$$

If we use (24),(25),(26) in (51), we have

$$\begin{aligned}&\alpha\{g(V_2,T_4(V_4,V_5)V_3)\xi-\eta(T_4(V_4,V_5)V_3)V_2\\&-\alpha g(V_2,V_4)\eta(V_5)V_3+\alpha g(V_2,V_4)\eta(V_3)V_5\\&+\eta(V_4)T_4(V_2,V_5)V_3+\alpha g(V_2,V_5)g(V_4,V_3)\xi\\&-\tfrac{(\alpha-c)(n+1)}{4n}g(V_2,V_5)\eta(V_3)V_4-\alpha g(V_2,V_5)\eta(V_4)V_3\\&+\tfrac{(\alpha-c)(n+1)}{4n}g(V_2,V_5)\eta(V_4)\eta(V_3)\xi+\eta(V_5)T_4(V_4,V_2)V_3\\&-\tfrac{\alpha(3n-1)+c(n+1)}{4n}g(V_2,V_3)g(V_4,V_5)\xi+\alpha g(V_2,V_3)\eta(V_4)V_5\\&-\tfrac{(\alpha-c)(n+1)}{4n}g(V_2,V_3)\eta(V_5)V_4+\eta(V_3)T_4(V_4,V_5)V_2\Big\}=0.\end{aligned}\tag{52}$$

If we choose $V_4=\xi$ in (52) and make use of (24), we obtain

$$\alpha\{T_4(V_2,V_5)V_3-\alpha g(V_2,V_5)V_3+\alpha g(V_2,V_3)V_5\}=0\tag{53}$$

If we choose $V_3 = \xi$ in (53) and we take inner product both sides of the last equation by $\xi \in \chi(M)$, we get

$$\alpha \left\{ \frac{1}{2n} S(V_2, V_5) - \alpha g(V_2, V_5) \right\} = 0.$$

This completes the proof. □

References

1. K. Yano, *On semi-symmetric metric connection*, Rev. Roumaine Math. Pures Appl. **15** (1970), 1579–1586.
2. S. K. Chaubey and S. K. Yadav, *Study of Kenmotsu manifolds with semi-symmetric metric connection*, Universal J. Math. Appl. **1(2)** (2018), 89–97. https://doi.org/10.32323/ujma.427238
3. D. H. Jin, *Half lightlike submanifolds of a semi-Riemannian space form with a semisymmetric non-metric connection*, J. Korean Soc. Math. Educ. Ser. B Pure Appl. Math. **21(1)** (2014), 39–50. http://dx.doi.org/10.7468/jksmeb.2014.21.1.39
4. D. H. Jin and J. W. Lee, *Einstein half lightlike submanifolds of a Lorentzian space form with a semi-symmetric metric connection*, Quaest. Math. **37(4)** (2014), 485–505. https://doi.org/10.2989/16073606.2014.981686
5. F. Zengin, S. A. Demirbağ, S. A. Uysal, and H. B. Yilmaz, *Some vector fields on a Riemannian manifold with semi-symmetric metric connection*, Bull. Iranian Math. Soc.**38(2)** (2012), 479–490.
6. B. Barua and A. Kr. Ray, *Some properties of semisymmetric metric connection in a Riemannian manifold*, Indian J. Pure Appl. Math. **16(7)** (1985), 736–740.
7. U. C. De and J. Sengupta, *On a type of semi-symmetric metric connection on an almost-contact metric manifold*, Facta Univ. Ser. Math. Inform. **16** (2001), 87–96.
8. J. A. Schouten, *An Introduction to Tensor Analysis and Geometrical Applications*, Ricci-Calculus, Springer-Verlag, Berlin-Gottingen-Heidelberg, 1954.
9. S.Kaneyuki and F.L. Williams, *Almost paracontact and parahodge structures on manifolds*, Nagoya Math. J., **99** (1985), 173-187. https://doi.org/10.1017/S0027763000021565
10. S. Zamkovoy, *Canonical connections on paracontact manifolds*, Ann. Global Anal. Geom., **36** (2009), 37-60. https://doi.org/10.1007/s10455-008-9147-3
11. M. Atçeken, P. Uygun, *Characterizations for totally geodesic submanifolds of (k, μ) −paracontact metric manifolds*, Korean J. Math., **28** (2020), 555-571. https://doi.org/10.11568/kjm.2020.28.3.555
12. M. Atçeken, S. Dirik, *On the geometry of pseudo-slant submanifolds of a Kenmotsu manifold*, Gulf Journal of Mathematics, **2** (2014), 51-66. https://doi.org/10.56947/gjom.v2i2.196
13. Ü. Yıldırım, M. Atçeken and S. Dirik, *Pseudo projective curvature tensor satisfying some properties on a normal paracontact metric manifold*, Commun. Fac. Sci. Univ. Ank. Ser. A1, Math. Stat. **68** (2019)997-1006. https://doi.org/10.31801/cfsuasmas.501436
14. M. Atçeken, *On generalized Sasakian space forms satisfying certain conditions on the concircular curvature tensor*, Bulletin of Mathematical Analysis and Applications, **6** (2014), 1-8.
15. T. Mert and M. Atçeken, *Almost $C(\alpha)$ −Manifold on W_0^*− Curvature Tensor*, Applied Mathematical Sciences. **15(15)** (2021), 693–703. https://doi.org/10.12988/ams.2021.916556
16. T. Mert, *Characterization of Some Special Curvature Tensor on Almost $C(\alpha)$ −Manifold*, Asian Journal of Mathematics and Computer Research. **29(1)** (2022), 27-41. https://doi.org/10.56557/ajomcor/2022/v29i17629
17. M. Tripathi and P. Gupta, *τ− Curvature Tensor on A Semi-Riemannian Manifold*, J. Adv. Math. Studies, **4** (2011), 117-129.

18. P. Uygun and M. Atçeken, *On A $(k,\mu)-$Paracontact Manifold Satisfying Some Conditions On The Projective Curvature Tensor*, Gulf Journal of Mathematics, **11(2)** (2021), 27-35. https://doi.org/10.56947/gjom.v11i2.498
19. A. Mondal, *Three-Dimensional Para-Kenmotsu Manifold Admitting $\eta-$Ricci Solutions*, Gulf Journal of Mathematics, **11(2)** (2021), 44-52. https://doi.org/10.56947/gjom.v11i2.584
20. M. Atçeken, S. Dirik and Ü. Yıldırım, *Anti-Invariant Submanifolds Of A Normal Paracontact Metric Manifold*, Gulf Journal of Mathematics, **10(2)** (2021), 38-49. https://doi.org/10.56947/gjom.v10i2.475
21. S. Pahan, *On Geometry Of Warped Product Pseudo-Slant Submanifolds On Generalized Sasakian Space Forms*, Gulf Journal of Mathematics, **9(1)** (2020), 42-61. https://doi.org/10.56947/gjom.v9i1.450
22. R. Prasad and S. Kumar, *Semi-Slant Riemannian Maps From Cosymplectic Manifolds into Riemannian Manifolds*, Gulf Journal of Mathematics, **9(1)** (2020), 62-80. https://doi.org/10.56947/gjom.v9i1.451

[1] Department of Mathematics, Faculty of Science, University of Sivas Cumhuriyet, Sivas, Turkey.

Email address: tmert@cumhuriyet.edu.tr

[2] Department of Mathematics, Faculty of Art and Science, University of Aksaray, Aksaray, Turkey

Email address: mehmet.atceken382@gmail.com

[3] Department of Mathematics, Faculty of Arts and Sciences, Tokat University, Tokat, Turkey

Email address: pakizeuygun@hotmail.com