Sharp Interface Limit of Stochastic Cahn-Hilliard Equation with Singular Noise

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Abstract

We study the sharp interface limit of the two dimensional stochastic Cahn-Hilliard equation driven by two types of singular noise: a space-time white noise and a space-time singular divergence-type noise. We show that with appropriate scaling of the noise the solutions of the stochastic problems converge to the solutions of the determinisitic Mullins-Sekerka/Hele-Shaw problem.

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1 Introduction

We consider the stochastic Cahn-Hilliard equation with additive noise

$$
\begin{aligned}
\mathrm{d}u^\varepsilon &= \Delta\Big(-\varepsilon\Delta u^\varepsilon + \frac{1}{\varepsilon}f(u^\varepsilon)\Big)\mathrm{d}t + \varepsilon^\sigma\,\mathrm{d}\mathcal{W} && \text{in } \mathcal{D}_T := (0,T)\times\mathcal{D}\,, \\
\partial_n u^\varepsilon &= \partial_n \Delta u^\varepsilon = 0 && \text{on } (0,T)\times\partial\mathcal{D}\,, \\
u^\varepsilon(0) &= u_0^\varepsilon,
\end{aligned}
\tag{1.1}
$$

where $\mathcal{D} = [0,1]^2$, n is the outward unit normal to $\partial\mathcal{D}$, $\sigma > 0$ is a constant, $\varepsilon > 0$ is a small parameter and $\mathcal{W}$ is a singular space-time noise which will be specified later on. The nonlinearity in Eq. 1.1 is taken as $f(u) = F'(u)$ where $F(u) = \frac{1}{4}(u^2 - 1)^2$ is the double-well potential.

By introducing an additional variable, a chemical potential w^ε, it is possible to reformulate (1.1) as

$$
\begin{aligned}
\mathrm{d}u^\varepsilon &= \Delta w^\varepsilon\mathrm{d}t + \varepsilon^\sigma\,\mathrm{d}\mathcal{W} && \text{in } \mathcal{D}_T\,, \\
w^\varepsilon &= -\varepsilon\Delta u^\varepsilon + \frac{1}{\varepsilon}f(u^\varepsilon) && \text{in } \mathcal{D}_T\,, \\
\partial_n u^\varepsilon &= \partial_n w^\varepsilon = 0 && \text{on } (0,T)\times\partial\mathcal{D}\,, \\
u^\varepsilon(0) &= u_0^\varepsilon && \text{in } \mathcal{D}.
\end{aligned}
\tag{1.2}
$$

The deterministic Cahn-Hilliard equation (i.e. Eq. 1.2 with $\mathcal{W} \equiv 0$) reads as

$$
\begin{aligned}
\partial_t u_{\mathrm{D}}^\varepsilon &= \Delta w_{\mathrm{D}}^\varepsilon && \text{in } \mathcal{D}_T, \\
w_{\mathrm{D}}^\varepsilon &= -\varepsilon\Delta u_{\mathrm{D}}^\varepsilon + \frac{1}{\varepsilon}f(u_{\mathrm{D}}^\varepsilon) && \text{in } \mathcal{D}_T.
\end{aligned}
\tag{1.3}
$$

The Cahn-Hilliard equation is a model for the non-equilibrium dynamics of metastable states in phase transitions [6, 15, 17]. The parameter ε in Eq. 1.2 represents an "interaction length", which is typically very small, and u^ε is an order parameter (scaled concentration) which assumes the values $u^\varepsilon = 1$ and $u^\varepsilon = -1$, respectively, in the regions occupied by the pure phases. The phase separation consists of two stages a so-called spinodal decomposition which is followed by a coarsening process. Starting from a fully mixed state, e.g., a random perturbation around the initial mass, the system undergoes a short phase, so-called spinodal decomposition, during which the initial phases are formed. The solution quickly approaches the respective values $1, -1$ in the regions occupied by the pure phases. The pure phases are separated by a thin region with a width proportional to ε, so-called diffuse interface. Once the diffuse interface is fully formed, the evolution enters a second stage, so-called coarsening phase, during which the originally fine-grained structure coarsens, the geometric structure of the phase regions gradually becomes simpler and eventually tends to regions of minimum surface area with preserved volume.

A rigorous sharp interface limit of the deterministic Cahn-Hilliard (1.3) has been obtained in [1] Under the assumption that the interfaces have been formed, i.e., that there exists a smooth closed curve $\Gamma_{00} \subset\subset \mathcal{D}$ such that $u^\varepsilon(0) \approx -1$ in $\mathcal{D}^-$, in the region enclosed by Γ_{00}, and $u^\varepsilon(0) \approx 1$ in $\mathcal{D}^+ := \mathcal{D} \setminus (\Gamma_{00} \cup \mathcal{D}^-)$, it is shown in [1] that $v := \lim_{\varepsilon\to 0} w_{\mathrm{D}}^\varepsilon$

along with $\Gamma_t := \lim_{\varepsilon \to 0} \Gamma_t^\varepsilon := \{x \in \mathcal{D} : u_D^\varepsilon(t, x) = 0\}$, $t \in (0, T)$ satisfy the deterministic Mullins-Sekerka/Hele-Shaw problem:

$$
\begin{cases}
\Delta v = 0 \text{ in } \mathcal{D} \setminus \Gamma_t, \quad t > 0, \\
\partial_n v = 0 \quad \text{on} \quad \partial \mathcal{D}, \\
\quad v = \lambda H \text{ on } \Gamma_t, \\
\mathcal{V} = \frac{1}{2}(\partial_{n_\Gamma} v^+ - \partial_{n_\Gamma} v^-) \quad \text{on} \quad \Gamma_t, \\
\Gamma_0 = \Gamma_{00},
\end{cases}
\tag{1.4}
$$

where $\lambda = \frac{1}{\sqrt{2}} \int_{-1}^{1} F(s)\mathrm{d}s$, H is the mean curvature of Γ_t with the sign convention that convex hypersurfaces have positive mean curvature, $\mathcal{V}$ is the normal velocity of the interface with the sign convention that the normal velocity of expanding hypersurfaces is positive, $n_\Gamma : [0, T] \to \mathbb{S}^1$ is the normal to Γ_t and v^+, v^- are respectively the restriction of v on $\mathcal{D}_t^\pm$ (the exterior/interior of Γ_t in $\mathcal{D}$).

The sharp interface limit of stochastic Cahn-Hilliard equation with trace-class noise has been studied in [2]. There, the authors show that for sufficiently large σ the sharp interface limit of Eq. 1.2 satisfies the deterministic Mullins-Sekerka/Hele-Shaw problem (1.4). In the recent paper [3] the authors show convergence of structure preserving numerical approximation of the stochastic Cahn-Hilliard to the deterministic Mullins-Sekerka/Hele-Shaw problem. In addition [3] obtains a uniform convergence result which implies convergence of the zero-level set of the numerical solution to the free boundary $\Gamma := \cup_{0 \le t \le T}(\Gamma_t \times \{t\})$ of Eq. 1.4. The case of $\sigma \le 1$ remains an open problem.

In this paper we study the sharp interface limit of stochastic Cahn-Hilliard equation driven by singular noise. We consider the Cahn-Hilliard-Cook model, proposed by Cook, cf. [6] and [15]), which incorporates thermal fluctuations in the form of an additive noise $\mathcal{W}$ in Eq. 1.2. We choose the noise as $\mathcal{W} = W_1$ or $\mathcal{W} = \nabla \cdot W_2$, where W_1 is mass-conserved $L^2(\mathcal{D}, \mathbb{R})$-cylindrical Wiener process and W_2 is an $L^2(\mathcal{D}, \mathbb{R}^2)$-cylindrical Wiener process. We note that in the latter case the equation is known as the time-dependent Ginzburg-Landau (TDGL) equation and is also related to the stochastic quantization for $(\phi)_2^4$-quantum field. For the existence and uniqueness results for these two kinds of equations we refer to [8, 22] and the reference therein.

To analyze the sharp interface limit of the solution $(u^\varepsilon, w^\varepsilon)$ to Eq. 1.2 for the case of the space-time white noise $\mathcal{W} = W_1$, we adapt the approach of [2]. We estimate the difference of $(u^\varepsilon, w^\varepsilon)$ to an approximate solution $(u_A^\varepsilon, w_A^\varepsilon)$ which is constructed by the matched asymptotic expansion method such that the interface Γ_t is the zero level set of $u_A^\varepsilon(t)$, cf. [1]. The approximation satisfies a perturbed equation

$$
\begin{cases}
\partial_t u_A^\varepsilon = \Delta w_A^\varepsilon & \text{in } \mathcal{D}_T, \\
w_A^\varepsilon = \frac{1}{\varepsilon} f(u_A^\varepsilon) - \varepsilon \Delta u_A^\varepsilon + r_A^\varepsilon & \text{in } \mathcal{D}_T,
\end{cases}
\tag{1.5}
$$

along with the boundary conditions

$$
\frac{\partial u_A^\varepsilon}{\partial n} = \frac{\partial \Delta u_A^\varepsilon}{\partial n} = 0 \text{ on } \partial \mathcal{D}.
$$

Since the r_A^ε goes to zero as $\varepsilon \to 0$, we can show that for $\varepsilon \to 0$ the differences $u^\varepsilon - u_A^\varepsilon$, $w^\varepsilon - w_A^\varepsilon$ converge to 0 for $\sigma > \frac{107}{12}$. As a consequence the sharp interface limit of the Eq. 1.2 satisfies the deterministic Hele-Shaw model (1.4) for $\sigma > \frac{107}{12}$. We note that the low regularity of the considered noise, prohibits the direct application of Itô's formula to

estimate $u^\varepsilon - u^\varepsilon_A$. Hence the arguments of [2] are not directly transferable to our case. Instead, we make use of the idea of Da Prato-Debussche [10]: after introducing a variable $Z^\varepsilon = \varepsilon^\sigma \int_0^t e^{-\varepsilon(t-s)\Delta^2} d\mathcal{W}_s$ we study the translated difference $Y^\varepsilon = u^\varepsilon - u^\varepsilon_A - Z^\varepsilon$ which enjoys better regularity properties. By combining the estimates for Y^ε and Z^ε we bound the error $u^\varepsilon - u^\varepsilon_A$ and obtain the sharp-interface limit.

For the case singular divergence-type noise $\mathcal{W} = \nabla \cdot W_2$ the Eq. 1.2 is ill-posed in the classical sense, since the solution is not a function but a distribution. Hence, it does not make sense to consider the sharp interface limit of Eq. 1.2 directly. Instead we follow the renormalization approach: we employ a suitable regularization $W^h := W_2 * \rho_h$ of the the space-time white noise W_2 and consider the regularized equation:

$$du^{\varepsilon,h} = \Delta \left(-\varepsilon \Delta u^{\varepsilon,h} + \frac{1}{\varepsilon} \left(f(u^{\varepsilon,h}) - 3c^\varepsilon_{h,t} u^{\varepsilon,h} \right) \right) dt + \varepsilon^\sigma \nabla \cdot dW^h_t, \qquad (1.6)$$

where $3c^\varepsilon_{h,t} u^{\varepsilon,h}$ is a renormalization term (see Eq. 5.7) which ensures that $u^{\varepsilon,h}$ converges to u^ε for $h \to 0$, where u^ε is the unique solution of the renormalized version of Eq. 1.2, see Eq. 5.11. The analysis in the case of the divergence-type noise is complicated by the fact that for fixed $\varepsilon > 0$ the renormalization constant in Eq. 1.6 diverges, i.e., that $c^\varepsilon_{h,t} \to \infty$ as $h \to 0$.

By choosing $\varepsilon \lesssim h^\iota$ for some $\iota > 0$ and h goes to 0 (see Theorem 5.6) the constant $c^\varepsilon_{h,t}$ becomes small as $h \to 0$ which enables us to control the term $c^\varepsilon_{h,t} u^{\varepsilon,h}$. The remaining steps in the analysis of Eq. 1.6 are analogical to the first case: we obtain that the sharp interface limit of Eq. 1.6 for $\sigma > \frac{26}{3}$ is the deterministic Mullins-Sekerka problem (1.4).

The paper is organized as follows. In Section 2 we give an overview of existing results on sharp interface limits for related problems. In Section 3, we introduce the notation and state preliminary results. The sharp interface limit for the space-time white noise is stated in Section 4 and we prove it in Section 4.1. In Section 5 we use a similar argument as we used in Section 4.1 to prove the results for divergence-type noise.

2 Overview of Existing Results

For stochastic Cahn-Hilliard eqation, the authors in [2] prove that for large σ the sharp interface limit of Eq. 1.2 also satifies the deterministic Hele-Shaw model if $\dot{\mathcal{W}}$ is a trace-class noise. For $\sigma = 1$, the sharp interface limit is also conjectured to satisfy the following stochastic Hele-Shaw model:

$$\begin{cases} \Delta v = 0 \text{ in } \mathcal{D} \setminus \Gamma_t, \ t > 0, \\ \partial_n v = 0 \text{ on } \partial \mathcal{D}, \\ \quad v = \lambda H + \mathcal{W} \text{ on } \Gamma_t, \\ \mathcal{V} = \frac{1}{2}(\partial_n v^+ - \partial_n v^-) \text{ on } \Gamma_t, \\ \Gamma_0 = \Gamma_{00}, \end{cases} \qquad (2.1)$$

In [4], the authors prove that the sharp interface limit of generalized Cahn-Hilliard equation: $\partial_t u = \Delta(-\varepsilon \Delta u + \frac{1}{\varepsilon} f(u) - G_2^\varepsilon) + G_1^\varepsilon$ satisfies the following Hele-Shaw model:

$$\begin{cases} \Delta v = -\lim_{\varepsilon \to 0} G_1^\varepsilon \text{ in } \mathcal{D} \setminus \Gamma_t, \ t > 0, \\ \partial_n v = 0 \text{ on } \partial \mathcal{D}, \\ \quad v = \lambda H - \lim_{\varepsilon \to 0} G_2^\varepsilon \text{ on } \Gamma_t, \\ \mathcal{V} = \frac{1}{2}(\partial_n v^+ - \partial_n v^-) \text{ on } \Gamma_t, \\ \Gamma_0 = \Gamma_{00}, \end{cases} \tag{2.2}$$

Since they require some regularity conditions for G_1^ε, G_2^ε w.r.t time, which are not satisfied by Bronwnian motions, it is not clear how to obtain the stochastic Hele-Shaw model rigorously. Until now, the rigorous complete description of the motion of interfaces in dimensions two and three in stochastic case stands for many years as a wide open problem.

Another simpler model is the following Allen-Cahn equation

$$\partial_t u = \Delta u - \frac{1}{\varepsilon^2} f(u). \tag{2.3}$$

It is well-known that the movement of interface is characterized by mean curvature flow (see e.g. [9, 11, 16]). Unlike the solution to the Allen-Cahn equation, the solution to the Cahn-Hilliard (1.3) does not approach ± 1 away from the interface exponentially fast. The direct application of the method of asymptotic matching in [9] does not lead to the desired approximation solutions. In stochastic case which is also called Model A of [15]), the authors in [12] and [26] consider the following stochastic Allen-Cahn equation

$$\partial_t u = \Delta u - \frac{1}{\varepsilon^2} f(u) + \frac{1}{\varepsilon} \Xi_t^\varepsilon. \tag{2.4}$$

The noise Ξ^ε is constant in space and smooth in time. For $\varepsilon \to 0$ the correlation length goes to zero at a precise rate and $\int_0^t \Xi_s^\varepsilon ds$ converges to a Brownian motion pathwisely. They prove that the dynamics of the phase-separating hyperplane Γ_t appearing in the limit is given by stochastic mean curvature flow (see also in [13, Chapter 4]). For space-time white noise, in [25] the authors prove the "exponential loss of memory property". But for sharp interface limit, there is still no result for space-time white noise.

3 Notations and Preliminaries

Throughout the paper, we use the notation $a \lesssim b$ if there exists a constant $c > 0$ which is independent to ε and time T such that $a \leq cb$. If c is depend on T, we use the notation $a \lesssim_T b$. We write $a \simeq b$ if $a \lesssim b$ and $b \lesssim a$.

Let $\mathcal{D} := (0, 1)^2$, $\mathcal{D}_T := (0, T) \times \mathcal{D}$. In this paper, we always use $\langle \cdot, \cdot \rangle$ to denote the $L^2(\mathcal{D})$-inner product. For any $E \subset \mathcal{D}$, we denote by χ_E the characteristic function of E, i.e.

$$\chi_E(x) = \begin{cases} 1 \text{ if } x \in E, \\ 0 \text{ if } x \notin E. \end{cases}$$

We consider the Neumann Laplacian operator Δ on $L^2(\mathcal{D})$ with domain

$$D(\Delta) = \{u \in H^2(\mathcal{D}) : \frac{\partial u}{\partial n} = 0 \text{ on } \partial \mathcal{D}\}.$$

The operator $-\Delta$ is self-adjoint positive and has compact resolvent. It possesses a basis of eigenvectors $\{e_k\}_{k\in\mathbb{Z}^2}$ which is orthonormal in $L^2(\mathcal{D})$. In fact for $k = (k_1, k_2) \in \mathbb{Z}^2$, $e_k(x)$ is given by

$$e_0(x) := 1, e_{(k_1,0)}(x) = \sqrt{2}\cos\pi k_1 x_1, e_{(0,k_2)}(x) = \sqrt{2}\cos\pi k_2 x_2, ,$$
$$e_k(x) := 2\cos\pi k_1 x_1 \cdot \cos\pi k_2 x_2, k_1 k_2 \neq 0. \tag{3.1}$$

It is associated with the eigenvalues $\{\lambda_k\}$, where $\lambda_k \simeq |k|^2$.

We also introduce a notation for the average of $g \in L^2(\mathcal{D})$:

$$m(g) := \langle g, e_0 \rangle.$$

For any $\alpha \in \mathbb{R}$, we define V^α as the closure of $C^\infty(\mathcal{D})$ under the norm

$$\|g\|_{V^\alpha}^2 := m(g)^2 + \sum_k \lambda_k^\alpha \langle g, e_k \rangle^2.$$

It is easy to see that $(V^\alpha, \|\cdot\|_{V^\alpha})$ is a Hilbert space and $V^\alpha \simeq H^\alpha$, where H^α is the classical Sobolev space on domain $\mathcal{D}$ which can be defined as the closure of $C^\infty(\mathcal{D})$ under the norm

$$\|g\|_{H^\alpha}^2 = \sum_{k\in\mathbb{Z}^2} (1 + \lambda_k)^\alpha \langle g, e_k \rangle^2.$$

In the rest of this paper, we use the notation H^α to represent V^α for simiplicity.

Moreover for any $s, \alpha \in \mathbb{R}$, we can define a bounded operator $(-\Delta)^s : H^\alpha \to H^{\alpha-2s}$ by:

$$(-\Delta)^s u = \sum_{k\in\mathbb{Z}^2\setminus\{(0,0)\}} \lambda_k^s u_k e_k,$$

where $u = \sum_k u_k e_k \in H^\alpha$.

We also set

$$H_0^\alpha := \{g \in H^\alpha : \langle g, e_0 \rangle_{H^\alpha} = 0\},$$

where $\langle \cdot, \cdot \rangle_{H^\alpha}$ denote the inner product in H^α. Moreover we denote $L_0^2 := H_0^0$.

The analysis of this paper relies heavily on the existence of smooth solution to Eq. 1.4 which is guaranteed by the next theorem.

Theorem 3.1 [5, Theorem 1.1] *For any $\Gamma_{00} \in \mathcal{C}^{3+\alpha}$ for some $\alpha \in (0, 1)$, there exists a $T > 0$, such that Eq. 1.4 has a unique local solution $\{(v, \Gamma)\}_{t\in[0,T]}$, where $\Gamma \in C^{\frac{3+\alpha}{3}}([0, T]; \mathcal{C}^{3+\alpha})$.*

Throughout the paper we assume that Γ_{00} and T satisfy the conditions of Theorem 3.1, i.e., that the Mullins-Sekerka problem (1.4) admits a unique classical solution on $[0, T]$. Consequently, it is possible to construct an approximate solution that satisfies (1.5). The properties of the solution of Eq. 1.5 which are summarized in the theorem below are the consequence of Theorems 2.1 and 4.12 of [1].

Theorem 3.2 *Let $\{(v, \Gamma_t)\}_{t\in[0,T]}$ be the classical smooth solution to Eq. 1.4. For any $K > 0$ there exists a pair $(u_A^\varepsilon, w_A^\varepsilon)$ of solutions to Eq. 1.5, such that for a small enough $\varepsilon > 0$, u_A^ε is uniformly bounded and*

$$\|r_A^\varepsilon\|_{C(\mathcal{D}_T)} \lesssim \varepsilon^{K-2},$$

Moreover

$$\|w_A^\varepsilon - v\|_{C(\mathcal{D}_T)} \lesssim \varepsilon,$$

where v is the solution to Eq. 1.4 below.

Finally for any $x \in \mathcal{D} \setminus \Gamma_t$ such that $d(x, \Gamma_t) > C\varepsilon^{\alpha}$, where $0 < \alpha < 1$ and $d(x, \Gamma_t)$ is the distance of x to Γ_t and C is a constant that is independent to ε, it holds respectively that $|u_A^{\varepsilon}(t, x) - 1| \lesssim \varepsilon$ on $\mathcal{D}_t^+$, $|u_A^{\varepsilon}(t, x) + 1| \lesssim \varepsilon$ on $\mathcal{D}_t^-$, where $\mathcal{D}_t^-$ and $\mathcal{D}_t^+$ are respectively the interior and exterior of Γ_t in $\mathcal{D}$. u_A^{ε} also satisfies the following thin interface conditions:

$$\mathcal{L}\left(\{(t, x) \in \mathcal{D}_T : |u_A^{\varepsilon}| < 1\}\right) \lesssim \varepsilon,$$

where $\mathcal{L}$ is the Lebesgue measure.

4 The Sharp Interface Limit for Space-Time White Noise

Let $\mathcal{W} = W$ be an $L_0^2(\mathcal{D})$-cylindrical Wiener process on a fixed stochastic basis $(\Omega, \mathcal{F}, \mathbb{P})$, i.e. $\frac{dW_t}{dt}$ is the space-time white noise.

Theorem 4.1 *([8, Theorem 2.1])For $\mathbb{P} - a.s.$ ω, there exists a unique solution u^{ε} to Eq. 1.2 in $C([0, T]; H^{-1})$.*

We rewrite the Eq. 1.2 as

$$\begin{cases} du^{\varepsilon} = \Delta w^{\varepsilon} dt + \varepsilon^{\sigma} dW \text{ in } \mathcal{D}_T, \\ w^{\varepsilon} = \dfrac{1}{\varepsilon} f(u^{\varepsilon}) - \varepsilon \Delta u^{\varepsilon} \text{ in } \mathcal{D}_T. \end{cases} \tag{4.1}$$

We assume that the interface has been formed initially. That is, there exists a smooth closed curve $\Gamma_{00} \subset\subset \mathcal{D}$ such that $u^{\varepsilon}(0) \approx -1$ in $\mathcal{D}^-$, the region enclosed by Γ_{00}, and $u^{\varepsilon}(0) \approx 1$ in $\mathcal{D}^+ := \mathcal{D} \setminus (\Gamma_{00} \cup \mathcal{D}^-)$.

Our main theorem will show that as $\varepsilon \to 0$, w^{ε} tends to v, which, together with a free boundary $\Gamma \equiv \cup_{0 \le t \le T}(\Gamma_t \times \{t\})$, satisfies the deterministic Hele-Shaw problem (1.4).

We present now the following spectral estimate which is useful in our proof.

Proposition 4.2 *([1, Proposition 3.1]) Let u_A^{ε} be the approximation given in Theorem 3.2. Then for all $w \in H^1$ satisfying Neumann boundary conditions such that $\int_{\mathcal{D}} w = 0$, the following estimate is valid*

$$\varepsilon \|w\|_{H^1}^2 + \frac{1}{\varepsilon} \int f'(u_A^{\varepsilon}) w^2 \ge -C_0 \|w\|_{H^{-1}}^2.$$

We consider the residual

$$R^{\varepsilon} := u^{\varepsilon} - u_A^{\varepsilon}, \tag{4.2}$$

where u^{ε} is the unique solution to Eq. 4.1. We show bounds for this error R^{ε} in our main theorem below.

Theorem 4.3 (**Main Theorem**) *Let u_A^{ε} be defined in Theorem 3.2 with large enough K and let u^{ε} be the unique solution to Eq. 1.2 with initial value $u^{\varepsilon}(0) = u_A^{\varepsilon}(0)$. For any $\sigma^* > \delta > 0$,*

$$\begin{cases} \gamma > 13, \\ \sigma^* > \dfrac{1}{3}\gamma + \dfrac{13}{3} + 2\delta, \end{cases}$$

where $\sigma^* = \sigma - \frac{1}{4}$ is introduced in Lemma 4.6, there exist a generic constant $C > 0$ and a constant $C_\delta > 0$ for all $\delta > 0$ such that the following estimates hold

$$\mathbb{P}\left[\|R^\varepsilon\|_{L^3(\mathcal{D}_T)} \le C\varepsilon^{\frac{\gamma}{3}}\right] \ge 1 - C_\delta\varepsilon^\delta,$$

$$\mathbb{P}\left[\|R^\varepsilon\|^2_{L^\infty(0,T;H^{-1})} \le C\left(\varepsilon^{\gamma-1} + \varepsilon^{\sigma^*-1-2\delta+\frac{\gamma}{3}}\right)\right] \ge 1 - C_\delta\varepsilon^\delta,$$

$$\mathbb{P}\left[\|w^\varepsilon - w^\varepsilon_A\|^2_{L^1(0,T;H^{-2})} \le C\varepsilon^{\frac{\gamma}{3}-1}\right] \ge 1 - C_\delta\varepsilon^\delta.$$

Remark 4.4 Since δ can be arbitrarily small, the best choice is $\sigma > \frac{107}{12}$.

Corollary 4.5 *There exists a subsequence $\{\varepsilon_k\}_{k=1}^\infty$ such that for $\mathbb{P} - a.s.\ \omega \in \Omega$*

$$\lim_{k\to\infty} u^{\varepsilon_k} = 1 - 2\chi_{\mathcal{D}_t^-} \ in\ L^3(\mathcal{D}_T),$$

where $\mathcal{D}_t^-$ is the interior of Γ_t in $\mathcal{D}$.

Proof We note that by Theorem 4.3 the problem (1.4) has a unique strong solution on $[0, T]$. Hence, by the construction of u^ε_A, see [1], it holds uniformly $t \in [0, T]$ that

$$\lim_{\varepsilon\to0} u^\varepsilon_A = 1 - 2\chi_{\mathcal{D}_t^-} \ \text{uniformly on compact subsets.}$$

For any $\eta > 0$, choosing ε small enough such that $C\varepsilon^{\frac{\gamma}{3}} < \eta$, then we have

$$\mathbb{P}\left[\|R^\varepsilon\|_{L^3(\mathcal{D}_T)} > \eta\right] \le \mathbb{P}\left[\|R^\varepsilon\|_{L^3(\mathcal{D}_T)} > C\varepsilon^{\frac{\gamma}{3}}\right] \le C_\delta\varepsilon^\delta,$$

which implies that $\|R^\varepsilon\|_{L^3}$ converge in probability to 0. Thus there exists a subsequence (still denoted as ε), such that

$$\lim_{\varepsilon\to0} \|R^\varepsilon\|_{L^3(\mathcal{D}_T)} = 0 \ \ \mathbb{P} - a.s..$$

Since $R^\varepsilon = u^\varepsilon - u^\varepsilon_A$, we obtain the assertion. $\qquad\square$

4.1 The proof of the Main Theorem

4.1.1 The Decomposition of the Equation for the Error

On Combining (4.1), (1.5) and noting (4.2) we obtain

$$\begin{cases} dR^\varepsilon = -\varepsilon\Delta^2 R^\varepsilon dt + \dfrac{1}{\varepsilon}\Delta\left(f(u^\varepsilon_A + R^\varepsilon) - f(u^\varepsilon_A)\right)dt + \Delta r^\varepsilon_A dt + \varepsilon^\sigma dW, \\ \dfrac{\partial R^\varepsilon}{\partial n} = \dfrac{\partial\Delta R^\varepsilon}{\partial n} = 0 \ \text{on}\ \partial\mathcal{D}. \end{cases} \quad (4.3)$$

Let $Z^\varepsilon_t := \varepsilon^\sigma \int_0^t e^{-(t-s)\varepsilon\Delta^2} dW_s$, which is the mild solution to the linear equation:

$$\begin{cases} dZ^\varepsilon = -\varepsilon\Delta^2 Z^\varepsilon dt + \varepsilon^\sigma dW, \\ \dfrac{\partial Z^\varepsilon}{\partial n} = \dfrac{\partial\Delta Z^\varepsilon}{\partial n} = 0 \ \text{on}\ \partial\mathcal{D}. \end{cases} \quad (4.4)$$

Then $Y^\varepsilon := R^\varepsilon - Z^\varepsilon$ satisfies:

$$
\begin{cases}
dY^\varepsilon = -\varepsilon\Delta^2 Y^\varepsilon dt + \dfrac{1}{\varepsilon}\Delta\left(f'(u_A^\varepsilon)(Y^\varepsilon + Z^\varepsilon) + \mathcal{N}(u_A^\varepsilon, Y^\varepsilon + Z^\varepsilon)\right) dt + \Delta r_A^\varepsilon dt, \\[2mm]
\dfrac{\partial Y^\varepsilon}{\partial n} = \dfrac{\partial \Delta Y^\varepsilon}{\partial n} = 0 \quad \text{on } \partial\mathcal{D}.
\end{cases}
\tag{4.5}
$$

where $\mathcal{N}(u, v) := f(u + v) - f(u) - f'(u)v$.

Moreover, we define a stopping time T_ε by:

$$
T_\varepsilon := T \wedge \inf\{t > 0 : \int_0^t \|Y_s^\varepsilon\|_{L^3}^3 ds > \varepsilon^\gamma\},
\tag{4.6}
$$

for some $\gamma > 1$.

4.1.2 Estimate for Z^ε

Lemma 4.6 *For any $\delta > 0$, there exists a constant $C_\delta > 0$, such that*

$$
\mathbb{P}\left[\Omega_{\delta,\varepsilon}\right] > 1 - C_\delta \varepsilon^\delta,
$$

where $C_1 > 0$ is a universal constant, $\Omega_{\delta,\varepsilon} := \{\|Z^\varepsilon\|_{C(\mathcal{D}_T)} \leq C_1 \varepsilon^{\sigma^* - 2\delta}\}$, and $\sigma^* := \sigma - \frac{1}{4}$.

Proof By the factorization method in [7] we have that for $\kappa \in (0, 1)$

$$
Z^\varepsilon(t) = \varepsilon^\sigma \frac{\sin(\pi\kappa)}{\pi} \int_0^t (t - s)^{\kappa-1}\langle M(\varepsilon(t - s), x, \cdot), U(s)\rangle ds,
$$

where $M(\varepsilon t, x, y)$ is the kernel of the semigroup $\{e^{-\varepsilon t\Delta^2}\}$ and

$$
U^\varepsilon(s, \cdot) = \int_0^s (s - r)^{-\kappa} e^{-\varepsilon(s-r)\Delta^2} dW_r.
$$

Similarly to the proof of Lemma 2.12 in [7], we have that

$$
\mathbb{E}\left[\|Z^\varepsilon(t)\|_{C(\mathcal{D}_T)}\right] \lesssim_T \varepsilon^{\sigma - \frac{\kappa}{2}} \mathbb{E}\left[\|U^\varepsilon\|_{L^{2p}(\mathcal{D}_T)}\right].
\tag{4.7}
$$

It suffices to estimate $\mathbb{E}\left[\|U^\varepsilon\|_{L^{2p}(\mathcal{D}_T)}\right]$ for $p > \frac{1}{2\kappa}$.

In fact, we have that

$$
\begin{aligned}
\mathbb{E}\left[\|U^\varepsilon(s)\|_{L^{2p}(\mathcal{D}_T)}^{2p}\right] &\lesssim \int_{\mathcal{D}_T} \mathbb{E}\left[\left|\int_0^s (s - r)^{-\kappa} e^{-\varepsilon(s-r)\Delta^2} dW_r\right|^{2p}\right] dsdx \\
&\lesssim \int_{\mathcal{D}_T}\left(\mathbb{E}\left[\left|\int_0^s (s - r)^{-\kappa} e^{-\varepsilon(s-r)\Delta^2} dW_r\right|^2\right]\right)^p dsdx.
\end{aligned}
\tag{4.8}
$$

Here we used that $U^\varepsilon(x)$ belongs to the first order Wiener-chaos and Gaussian hypercontractivity (cf. [21, Section 1.4.3] and [20]) in the second inequality. Moreover, we obtain that

$$
\mathbb{E}\left[\left|\int_0^s (s - r)^{-\kappa} e^{-\varepsilon(s-r)\Delta^2} dW_r\right|^2\right] \lesssim \int_0^s \int_{\mathcal{D}} (s - r)^{-2\kappa} M(\varepsilon(s - r), x, y)^2 dyds. \tag{4.9}
$$

Since $M(t, x, y)$ is the kernel of $e^{-t\Delta^2}$, we have that for any $g \in L^2$

$$
\int_{\mathcal{D}} M(t, x, y)g(y)dy = e^{-t\Delta^2}g(x) \simeq \sum_k \langle g, e^{-t|k|^4} e_k\rangle e_k(x).
$$

Hence

$$M(t, x, y) \simeq \sum_k e^{-t|k|^4} e_k(x) e_k(y). \tag{4.10}$$

where e_k is defined in Eq. 3.1. Note that $e_k(x) e_k(y) = \frac{1}{2}(e_k(x - y) + e_k(x + y))$. Thus we obtain

$$M(t, x, y) \simeq \sum_k e^{-t|k|^4} (e_k(x - y) + e_k(x + y)) := P(t, x - y) + P(t, x + y), \tag{4.11}$$

Then Eq. 4.9 becomes

$$\mathbb{E}\left[\left|\int_0^s (s - r)^\kappa e^{-\varepsilon(s-r)\Delta^2} dW_r\right|^2\right] \lesssim \int_0^s \int_{\mathcal{D}} (s-r)^{-2\kappa} \left(P(\varepsilon(s - r), x - y)^2 + P(\varepsilon(s - r), x + y)^2\right) dy ds, . \tag{4.12}$$

By [24, p282, (c)], we have that

$$|P(t, x)| \lesssim |x|^{-2} e^{-\frac{t}{|x|^4}} \lesssim t^{-\frac{\eta}{4}} |x|^{-2+\eta}, \ \forall \eta \in [0, 2]. \tag{4.13}$$

Then taking (4.12) into (4.13), we deduce that

$$\mathbb{E}\left[\left|\int_0^s (s - r)^\kappa e^{-\varepsilon(s-r)\Delta^2} dW_r\right|^2\right] \lesssim \varepsilon^{-\frac{\eta}{2}} \int_0^s \int_{\mathcal{D}} (s - r)^{-2\kappa-\frac{\eta}{2}} \left(|x + y|^{-4+2\eta} + |x - y|^{-4+2\eta}\right) dy ds$$

$$\lesssim \varepsilon^{-\frac{\eta}{2}} s^{1-2\kappa-\frac{\eta}{2}} |x|^{-2+2\eta}. \tag{4.14}$$

Here we require that

$$1 - 2\kappa - \frac{\eta}{2} > 0, \quad -2 + 2\eta > 0,$$

that is

$$1 < \eta < 2 - 4\kappa, \tag{4.15}$$

which can be obtained by choosing small enough $\kappa > 0$. Hence by Eqs. 4.7 and 4.8, we obtain that for any $p \geq 1$

$$\mathbb{E}\left[\|U^\varepsilon\|_{L^{2p}(\mathcal{D}_T)}\right] \lesssim \varepsilon^{-\frac{\eta}{4}},$$

This implies that for any $2 > \eta > 1$ and small enough $\kappa > 0$,

$$\mathbb{E}\left[\|Z^\varepsilon\|_{\mathcal{C}(\mathcal{D}_T)}\right] \lesssim \varepsilon^{\sigma-\frac{\eta}{4}-\frac{\kappa}{2}}. \tag{4.16}$$

Hence the statement follows by Cheybeshev's inequatliy. □

4.1.3 Local-in-time estimate for Y^ε up to T_ε on the set $\Omega_{\delta,\varepsilon}$

In the remainder of the proof we fix $\omega \in \Omega_{\delta,\varepsilon}$ where $\Omega_{\delta,\varepsilon}$ is defined in Lemma 4.6 and work pathwise. We note that by definition $\|Z^\varepsilon(\omega)\|_{C(\mathcal{D}_T)} \lesssim \varepsilon^{\sigma^*-2\delta}$ for $\omega \in \Omega_{\delta,\varepsilon}$.

By taking inner product with $(-\Delta)^{-1} Y^\varepsilon$ in both side of Eq. 4.5 we have that

$$\frac{1}{2} \frac{d\|Y^\varepsilon\|_{H^{-1}}^2}{dt} + \varepsilon \|Y_t^\varepsilon\|_{H^1}^2 = -\frac{1}{\varepsilon} \langle f'(u_A^\varepsilon)(Y^\varepsilon + Z^\varepsilon) + \mathcal{N}(u_A^\varepsilon, Y^\varepsilon + Z^\varepsilon), Y^\varepsilon \rangle - \langle r_A^\varepsilon, Y^\varepsilon \rangle. \tag{4.17}$$

We estimate each term in right hand side of Eq. 4.17 separately. Using Proposition 4.2 we have that

$$-\frac{1}{\varepsilon} \langle f'(u_A^\varepsilon) Y^\varepsilon, Y^\varepsilon \rangle \leq \varepsilon \|Y^\varepsilon\|_{H^1}^2 + C_0 \|Y^\varepsilon\|_{H^{-1}}^2 \tag{4.18}$$

For $-\frac{1}{\varepsilon}\langle f''(u_A^\varepsilon)(Y^\varepsilon, Z^\varepsilon\rangle$ by Theorem 3.2 we know that u_A^ε is uniformly bounded in $\mathcal{D}_T$. Thus we have that

$$\frac{1}{\varepsilon}|\langle f''(u_A^\varepsilon)Y^\varepsilon, Z^\varepsilon\rangle| \lesssim \frac{1}{\varepsilon}\|Y^\varepsilon\|_{L^3}\|Z^\varepsilon\|_{L^{\frac{3}{2}}} \lesssim \varepsilon^{\sigma^*-1-2\delta}\|Y^\varepsilon\|_{L^3}, \qquad (4.19)$$

where we used Hölder's inequality in the first inequality and Lemma 4.6 in the last inequality.

It has been proved in [1, Lemma 2.2] that $v\mathcal{N}(u, v) \geq -C|v|^3$ for u in a bounded set which is the case in the lines below. Then

$$-\frac{1}{\varepsilon}\langle\mathcal{N}(u_A^\varepsilon, Y^\varepsilon + Z^\varepsilon), Y^\varepsilon\rangle = -\frac{1}{\varepsilon}\langle\mathcal{N}(u_A^\varepsilon, Y^\varepsilon + Z^\varepsilon), Y^\varepsilon + Z^\varepsilon\rangle + \frac{1}{\varepsilon}\langle\mathcal{N}(u_A^\varepsilon, Y^\varepsilon + Z^\varepsilon), Z^\varepsilon\rangle$$

$$\lesssim \frac{1}{\varepsilon}\|Y^\varepsilon + Z^\varepsilon\|_{L^3}^3 + \frac{1}{\varepsilon}|\langle\mathcal{N}(u_A^\varepsilon, Y^\varepsilon + Z^\varepsilon), Z^\varepsilon\rangle|$$

$$\lesssim \frac{1}{\varepsilon}\|Y\|_{L^3}^3 + \varepsilon^{3(\sigma^*-2\delta)-1} + \frac{1}{\varepsilon}|\langle\mathcal{N}(u_A^\varepsilon, Y^\varepsilon + Z^\varepsilon), Z^\varepsilon\rangle|, \qquad (4.20)$$

where we used Lemma 4.6 in the last inequality.

For $|\langle\mathcal{N}(u_A^\varepsilon, Y^\varepsilon + Z^\varepsilon), Z^\varepsilon\rangle|$, by the Taylor expansion, $\mathcal{N}(u_A^\varepsilon, Y^\varepsilon + Z^\varepsilon) = f''(u_A^\varepsilon + \theta(Y^\varepsilon + Z^\varepsilon))(Y^\varepsilon + Z^\varepsilon)^2 = 6(u_A^\varepsilon + \theta(Y^\varepsilon + Z^\varepsilon))(Y^\varepsilon + Z^\varepsilon)^2$, where $\theta \in (0, 1)$. Then we have

$$|\langle\mathcal{N}(u_A^\varepsilon, Y^\varepsilon + Z^\varepsilon), Z^\varepsilon\rangle| \lesssim \varepsilon^{\sigma^*-2\delta}\|\mathcal{N}(u_A^\varepsilon, Y^\varepsilon + Z^\varepsilon)\|_{L^1}$$

$$\lesssim \varepsilon^{\sigma^*-2\delta}(\|Y^\varepsilon + Z^\varepsilon\|_{L^3}^3 + \|Y^\varepsilon + Z^\varepsilon\|_{L^2}^2)$$

$$\lesssim \varepsilon^{3(\sigma^*-2\delta)} + \varepsilon^{4(\sigma^*-2\delta)} + \varepsilon^{\sigma^*-2\delta}\|Y^\varepsilon\|_{L^3}^2 + \varepsilon^{\sigma^*-2\delta}\|Y^\varepsilon\|_{L^3}^3, \qquad (4.21)$$

where we used the uniform boundness of u_A^ε in the second inequality and Lemma 4.6 in the first and the last inequality.

For $|\langle r_A^\varepsilon, Y^\varepsilon\rangle|$, by Theorem 3.2 we have

$$|\langle r_A^\varepsilon, Y^\varepsilon\rangle| \lesssim \varepsilon^{K-2}\|Y^\varepsilon\|_{L^1} \lesssim \varepsilon^{K-2}\|Y^\varepsilon\|_{L^3}. \qquad (4.22)$$

Let $\sigma^* > \delta$, $\varepsilon < 1$, δ be small enough and K large enough. Collecting (4.17)-(4.22) together, by using Hölder's inequality we have

$$\frac{d\|Y^\varepsilon(t)\|_{H^{-1}}^2}{dt} \lesssim \|Y^\varepsilon\|_{H^{-1}}^2 + \frac{1}{\varepsilon}\|Y^\varepsilon\|_{L^3}^3 + \varepsilon^{\sigma^*-1-2\delta}(\|Y^\varepsilon\|_{L^3} + \|Y^\varepsilon\|_{L^3}^2 + \|Y^\varepsilon\|_{L^3}^3) + \varepsilon^{3(\sigma^*-2\delta)-1}.$$

Then for any $t \leq T_\varepsilon$ we have

$$\|Y^\varepsilon(t)\|_{H^{-1}}^2 \lesssim \int_0^t e^{t-s}\left(\frac{1}{\varepsilon}\|Y^\varepsilon\|_{L^3}^3 + \varepsilon^{\sigma^*-1-2\delta}\|Y^\varepsilon\|_{L^3} + \varepsilon^{3(\sigma^*-2\delta)-1}\right)ds$$

$$\lesssim_T \frac{1}{\varepsilon}\int_0^t \|Y^\varepsilon\|_{L^3}^3 d\tau + \varepsilon^{\sigma^*-1-2\delta}\left(\int_0^t \|Y^\varepsilon\|_{L^3}^3 d\tau\right)^{\frac{1}{3}} + \varepsilon^{3(\sigma^*-2\delta)-1} \qquad (4.23)$$

$$\lesssim \varepsilon^{\gamma-1} + \varepsilon^{\sigma^*-1-2\delta+\frac{\gamma}{3}} + \varepsilon^{3(\sigma^*-2\delta)-1}.$$

To estimate $L^2(0, T_\varepsilon; H^1)$ norm of Y^ε, we use the estimate presented in [1, p.171]

$$-\frac{1}{\varepsilon}\int_0^t \int_{\mathcal{D}} f''(u_A^\varepsilon)g^2 dxds \lesssim \varepsilon^{-\frac{2}{3}}(\int_0^t \|g\|_{L^3}^3 ds)^{\frac{2}{3}}, \quad \forall g \in L^3.$$

Then

$$-\frac{1}{\varepsilon}\int_0^t \langle f''(u_A^\varepsilon)Y^\varepsilon, Y^\varepsilon\rangle ds \le \varepsilon^{-\frac{2}{3}}\Big(\int_0^t \|Y^\varepsilon\|_{L^3}^3 ds\Big)^{\frac{2}{3}} \lesssim \varepsilon^{\frac{2}{3}(\gamma-1)}. \tag{4.24}$$

Combining (4.17), (4.19)-(4.22) and (4.24) we have for any $t \le T_\varepsilon$

$$\int_0^t \|Y^\varepsilon\|_{H^1}^2 ds \lesssim \varepsilon^{\frac{2}{3}\gamma-\frac{5}{3}} + \varepsilon^{\sigma^*-2-2\delta+\frac{\gamma}{3}} + \varepsilon^{3(\sigma^*-2\delta)-2} + \varepsilon^{\gamma-2}. \tag{4.25}$$

4.1.4 Final step: Globalization $T_\varepsilon \equiv T$

Let

$$\gamma_1 := (\gamma-1) \wedge (3(\sigma^*-2\delta)-1) \wedge (\sigma^*-1-2\delta+\frac{\gamma}{3}),$$

$$\gamma_2 := (\frac{2}{3}\gamma-\frac{5}{3}) \wedge (3(\sigma^*-2\delta)-2) \wedge (\sigma^*-2-2\delta+\frac{\gamma}{3}) \wedge (\gamma-2) = (\frac{2}{3}\gamma-\frac{5}{3}) \wedge (\gamma_1-1),$$

then we have for any $t \le T_\varepsilon$

$$\sup_{s\in[0,t]} \|Y^\varepsilon\|_{H^{-1}}^2 \lesssim \varepsilon^{\gamma_1}, \qquad \int_0^t \|Y^\varepsilon\|_{H^1}^2 ds \lesssim \varepsilon^{\gamma_2}. \tag{4.26}$$

We use the Sobolev's embedding of H^β into L^p with $\beta := 2(\frac{1}{2}-\frac{1}{p}) = \frac{p-2}{p}$. Then by the interpolation we have

$$\|Y^\varepsilon\|_{L^3} \lesssim \|Y^\varepsilon\|_{H^{\frac{1}{3}}} \lesssim \|Y^\varepsilon\|_{H^1}^{\frac{2}{3}} \|Y^\varepsilon\|_{H^{-1}}^{\frac{1}{3}}.$$

For any $t \le T_\varepsilon$ by Eq. 4.26 we have

$$\int_0^t \|Y^\varepsilon\|_{L^3}^3 ds \lesssim \sup_{s\in[0,t]} \|Y^\varepsilon\|_{H^{-1}} \int_0^t \|Y^\varepsilon\|_{H^1}^2 ds$$

$$\lesssim \varepsilon^{\frac{\gamma_1}{2}+\gamma_2}. \tag{4.27}$$

Then we have that for ε small enough, $T_\varepsilon = T$, if $\gamma < \frac{\gamma_1}{2} + \gamma_2$.

Let $\gamma_1 > \frac{2}{3}\gamma - \frac{2}{3}$ such that $\gamma_2 = \frac{2}{3}\gamma - \frac{5}{3}$, then we only need

$$\gamma_1 > \frac{2}{3}\gamma + \frac{10}{3}.$$

i.e.

$$\begin{cases} \gamma - 1 > \dfrac{2}{3}\gamma + \dfrac{10}{3} \\[2mm] 3(\sigma^*-2\delta) - 1 > \dfrac{2}{3}\gamma + \dfrac{10}{3} \\[2mm] \sigma^* - 1 - 2\delta + \dfrac{\gamma}{3} > \dfrac{2}{3}\gamma + \dfrac{10}{3}. \end{cases}$$

A direct calculation yields that

$$\begin{cases} \gamma > 13 \\[2mm] \sigma^* > \dfrac{1}{3}\gamma + \dfrac{13}{3} + 2\delta, \end{cases} \tag{4.28}$$

which also implies $\gamma_1 = (\gamma-1) \wedge (\sigma^*-1-2\delta+\frac{\gamma}{3})$.

Since $R^\varepsilon = Y^\varepsilon + Z^\varepsilon$, by Lemma 4.6 we have for any $\omega \in \Omega_{\delta,\varepsilon}$

$$\|R^\varepsilon(\omega)\|_{L^3(\mathcal{D}_T)} \lesssim \varepsilon^{\frac{\gamma}{3}} + \varepsilon^{\sigma^*-2\delta} \lesssim \varepsilon^{\frac{\gamma}{3}},$$

$$\|R^\varepsilon(\omega)\|_{L^\infty(0,T;H^{-1})}^2 \lesssim \varepsilon^{\gamma-1} + \varepsilon^{\sigma^*-1-2\delta+\frac{\gamma}{3}}. \tag{4.29}$$

Hence we note that

$$w^\varepsilon - w_A^\varepsilon = \varepsilon\Delta(Y^\varepsilon + Z^\varepsilon) - \frac{1}{\varepsilon}\left(f(u_\varepsilon) - f(u_A^\varepsilon)\right).$$

Using the embedding $C(\mathcal{D}) \subset L^2$ we get

$$\|\Delta(Y^\varepsilon + Z^\varepsilon)\|_{L^1(0,T;H^{-2})} \lesssim \|Y^\varepsilon\|_{L^2(0,T;L^2)} + \|Z^\varepsilon\|_{C(\mathcal{D}_T)} \lesssim \varepsilon^{\frac{\gamma}{3}} + \varepsilon^{\sigma^* - 2\delta}.$$

Similarly as above

$$\begin{aligned}
f(u^\varepsilon) - f(u_A^\varepsilon) &= f'(u_A^\varepsilon)R^\varepsilon + \mathcal{N}(u_A^\varepsilon, R^\varepsilon) \\
&= f'(u_A^\varepsilon)R^\varepsilon + f''(u_A^\varepsilon + \theta(R^\varepsilon))(R^\varepsilon)^2 \\
&= f'(u_A^\varepsilon)R^\varepsilon + 6(u_A^\varepsilon + \theta(R^\varepsilon))(R^\varepsilon)^2.
\end{aligned}$$

Since $\{u_A^\varepsilon\}$ are uniformly bounded in ε and $\theta \in [0, 1]$, we have that

$$\begin{aligned}
\|f(u^\varepsilon) - f(u_A^\varepsilon)\|_{L^1(0,T;H^{-2})} &\lesssim \|f(u^\varepsilon) - f(u_A^\varepsilon)\|_{L^1(\mathcal{D}_T)} \\
&\lesssim \|(R^\varepsilon)^3\|_{L^1(\mathcal{D}_T)} + \|(R^\varepsilon)^2\|_{L^1(\mathcal{D}_T)} + \|R^\varepsilon\|_{L^1(\mathcal{D}_T)} \\
&\lesssim \|R^\varepsilon\|_{L^3(\mathcal{D}_T)} + \|R^\varepsilon\|_{L^3(\mathcal{D}_T)}^2 + \|R^\varepsilon\|_{L^3(\mathcal{D}_T)}^3 \\
&\lesssim \varepsilon^{\frac{\gamma}{3}} + \varepsilon^{\sigma^* - 2\delta} \\
&\lesssim \varepsilon^{\frac{\gamma}{3}},
\end{aligned}$$

where we use the Soblev embedding $L^1 \subset H^{-2}$ in the first inequality.

Hence we deduce that

$$\begin{aligned}
\|w^\varepsilon - w_A^\varepsilon\|_{L^1(0,T;H^{-2})} &\lesssim \varepsilon^{\frac{\gamma}{3}+1} + \varepsilon^{\frac{\gamma}{3}-1} \\
&\lesssim \varepsilon^{\frac{\gamma}{3}-1}.
\end{aligned} \tag{4.30}$$

The statement of the Theorem 4.3 then follows on combining the above inequality with Eq. 4.28.

5 Sharp Interface Limit for the Divergence-Type Noise

Throughout this section we consider the singular divergence-type noise $\mathcal{W} = \nabla \cdot W$, where W is an $L_0^2(\mathcal{D}, \mathbb{R}^2)$-cylindrical Wiener process on stochastic basis $(\Omega, \mathcal{F}, \mathbb{P})$. For $g \in L_0^2(\mathcal{D}, \mathbb{R}^2)$, we denote its component functions by $g_1, g_2 \in L_0^2(\mathcal{D})$, i.e. $g(x) = (g_1(x), g_2(x)), \forall x \in \mathcal{D}$. There exist two independent $L_0^2(\mathcal{D})$-cylindrical Wiener processes W^1 and W^2 such that $W = (W^1, W^2)$. Similarly as in [22, 23], it follows that the solution to Eq. 1.2 with the divergence-type noise is distribution-valued. It does not appear to be possible to obtain the sharp-interface limit by directly considering (5.11). Thus we study the sharp interface limit of the regularized (1.6) instead.

5.1 Existence and Uniqueness of Solutions to Eq. 1.6

In order to consider the convolution of the noise with an approximate delta function (the standard mollifier). we need to extend the noise to the whole space $\mathbb{R}^2$. Considering the Neumann boundary condition, it is reasonable to extend it evenly to $[-1, 1]^2$ first, then do a periodical extension to the whole space. That is, for any function g on $\mathcal{D}$ which satisfies

the Neumann boundary condition, we view it as a function $\bar{g}$ on $\mathbb{R}^2$ by

$$\bar{g}(x) := g(|x_1+k_1|, |x_2+k_2|), \quad \forall x = (x_1, x_2) \in \mathbb{R}^2, \quad \forall k = (k_1, k_2) \in \mathbb{Z}^2 \text{ when } x+k \in [-1, 1]^2.$$

Moreover, for $x \in \mathbb{R}^2$ and $t > 0$, define

$$\bar{M}(t, x) = -\mathcal{F}^{-1}(e^{-\frac{t}{2}|\pi \cdot|^4})(x),$$

where $\mathcal{F}^{-1}$ is the inverse Fourier transformation on $\mathbb{R}^2$. By Poisson summation formula, for any $(x, y) \in \mathcal{D}^2$

$$M(t, x, y) := \sum_{k \in \mathbb{Z}^2} \left(\bar{M}(t, x + y + 2k) + \bar{M}(t, x - y + 2k) \right)$$

is the kernel of $e^{-t\Delta^2}$ on $\mathcal{D}$, where Δ is the Neumann Laplacian operator on $\mathcal{D}$. A direct calculation yields that for any $g \in L^2(\mathcal{D})$

$$\int_{\mathcal{D}} M(t, x, y)g(y)dy = \int_{\mathbb{R}^2} \bar{M}(t, x - y)\bar{g}(y)dy. \tag{5.1}$$

Define

$$K(t, x, y) := -\nabla_y M(t, x, y) = \sum_{k \in \mathbb{Z}^2} \left(\bar{K}(t, x + y + 2k) - \bar{K}(t, x - y + 2k) \right),$$

where $\bar{K}(t, x) = (\bar{K}^1(t, x), \bar{K}^2(t, x)) := -\nabla \bar{M}(t, x)$, thus for any $t > 0$, $\bar{K}^j(t, \cdot)$ is the inverse Fourier transformation of the function $\eta \to -\pi i \eta_j e^{-\frac{t}{2}|\pi \eta|^4}$, i.e.

$$\bar{K}^j(t, x) := -\mathcal{F}^{-1}(\pi i \eta_j e^{-\frac{t}{2}|\pi \eta|^4})(x).$$

We use $\mathcal{S}(\mathbb{R}^2)$ to denote the Schwartz funtion on $\mathbb{R}^2$, $\mathcal{S}'(\mathbb{R}^2)$ to denote the Schwartz distribution on $\mathbb{R}^2$ and $_{\mathcal{S}'(\mathbb{R}^2)}\langle \cdot, \cdot \rangle_{\mathcal{S}(\mathbb{R}^2)}$ to denote the dual between $\mathcal{S}(\mathbb{R}^2)$ and $\mathcal{S}'(\mathbb{R}^2)$. Then we know that $\bar{K}^j(t, \cdot) \in \mathcal{S}(\mathbb{R}^2)$ for any $t > 0$. Moreover we define Z^ε by

$$Z^\varepsilon(t, x) := \varepsilon^\sigma \int_0^t \langle K(\varepsilon(t - s), x, \cdot), dW_s \rangle_{L^2(\mathcal{D}, \mathbb{R}^2)} = \varepsilon^\sigma \sum_{j=1}^2 \int_0^t {}_{\mathcal{S}'(\mathbb{R}^2)}\langle \bar{K}^j(\varepsilon(t - s), x - \cdot), d\bar{W}_s^j \rangle_{\mathcal{S}(\mathbb{R}^2)}. \tag{5.2}$$

Here $\bar{W} = (\bar{W}^1, \bar{W}^2)$, $\bar{W}^j$, $j = 1, 2$ is two i.i.d Wiener processes defined by

$$_{\mathcal{S}'(\mathbb{R}^2)}\langle \bar{W}^j, g \rangle_{\mathcal{S}(\mathbb{R}^2)} = \langle W, \tilde{g} \rangle_{L^2(\mathcal{D})},$$

for any $g \in \mathcal{S}(\mathbb{R}^2)$ and $\tilde{g} \in L^2(\mathcal{D})$ is defined as

$$\tilde{g}(x) := \sum_{k \in \mathbb{Z}^2} (g(x + 2k) - g(-x + 2k)), \quad x \in \mathcal{D}.$$

For simplicity we write

$$Z^\varepsilon(t, x) = \varepsilon^\sigma \sum_{j=1}^2 \int_0^t {}_{\mathcal{S}'(\mathbb{R}^2)}\langle \bar{K}^j(\varepsilon(t-s), x-\cdot), d\bar{W}_s^j \rangle_{\mathcal{S}(\mathbb{R}^2)} := \varepsilon^\sigma \int_0^t {}_{\mathcal{S}'}\langle \bar{K}(\varepsilon(t-s), x-\cdot), d\bar{W}_s \rangle_{\mathcal{S}}.$$

We also denote

$$\bar{Z}^\varepsilon := Z^\varepsilon + e^{-\varepsilon t \Delta^2} m(z), \tag{5.3}$$

where $z \in H^{-1}$, $m(z)$ is defined in Section 3. Then $\bar{Z}^\varepsilon$ is the mild solution to the linear equation

$$\begin{cases} d\bar{Z}^\varepsilon = -\varepsilon \Delta^2 \bar{Z}^\varepsilon + \varepsilon^\sigma B dW, \\ \bar{Z}^\varepsilon(0) \equiv m(z) \in \mathbb{R}, \end{cases}$$

with Neumann boundary conditions,

$$\frac{\partial \bar{Z}^\varepsilon}{\partial n} = \frac{\partial \Delta \bar{Z}^\varepsilon}{\partial n} = 0 \text{ on } \partial \mathcal{D},$$

where

$$D(B) = H^1(\mathcal{D}, \mathbb{R}^2), \ B = \text{div}, \ D(B^*) = H^1(\mathcal{D}), \ B^* = -\nabla. \tag{5.4}$$

Let ρ_h be an approximate delta function on $\mathbb{R}^2$ given by

$$\rho_h(x) = h^{-2} \rho(\frac{x}{h}), \quad \int \rho = 1.$$

Define for any $(t, x) \in \mathcal{D}_T$

$$\begin{aligned} Z^{\varepsilon,h}(t, x) &:= \varepsilon^\sigma \int_0^t {}_{\mathcal{S}'} \langle \bar{K}(\varepsilon(t-r), x - \cdot), d\bar{W}_s^h \rangle_{\mathcal{S}} \\ &= \varepsilon^\sigma \int_0^t {}_{\mathcal{S}'} \langle \bar{K}_h(\varepsilon(t-r), x - \cdot), d\bar{W}_s \rangle_{\mathcal{S}}, \end{aligned} \tag{5.5}$$

where $\bar{W}^h = \bar{W} * \rho_h$, and $\bar{K}_h(t, x) = (\bar{K}_h^1(t, x), \bar{K}_h^2(t, x))$,

$$\bar{K}_h^j(t, x) = \int_{\mathbb{R}^2} \bar{K}^j(t, x - y)\rho_h(y)dy.$$

For fixed $\varepsilon, h > 0$, let $\varphi^{\varepsilon,h}$ be a solution to the following equation on $\mathcal{D}$

$$\begin{cases} \dfrac{d\varphi^{\varepsilon,h}}{dt} = \Delta(-\varepsilon \Delta \varphi^{\varepsilon,h} + \dfrac{1}{\varepsilon} : f(\varphi^{\varepsilon,h} + \bar{Z}^{\varepsilon,h}) :) \\ \varphi^{\varepsilon,h}(0) = (z - m(z)) * \rho_h, \end{cases} \tag{5.6}$$

with Δ the Neumann Laplacian operator on $\mathcal{D}$. Here $: f(\varphi^{\varepsilon,h} + \bar{Z}^{\varepsilon,h}) :$ is the Wick power defined by

$$: f(\varphi^{\varepsilon,h} + \bar{Z}^{\varepsilon,h}) := \sum_{k=0}^{3} \binom{3}{k} : \left(\bar{Z}^{\varepsilon,h}\right)^{3-k} : \left(\varphi^{\varepsilon,h}\right)^k \tag{5.7}$$

where for any $k = 0, 1, 2, 3$

$$: \left(\bar{Z}^{\varepsilon,h}\right)^k := \sum_{l=0}^{3} \binom{3}{k} : \left(Z^{\varepsilon,h}\right)^{k-l} : \left(e^{-\varepsilon t \Delta^2} m(z)\right)^k,$$

$$: \left(Z^{\varepsilon,h}\right)^0 : = 1, \ : \left(Z^{\varepsilon,h}\right) := \left(Z^{\varepsilon,h}\right), \ : \left(Z^{\varepsilon,h}\right)^2 := \left(Z^{\varepsilon,h}\right)^2 - c_{h,t}^\varepsilon(x),$$

$$: \left(Z^{\varepsilon,h}\right)^3 : \qquad := \left(Z^{\varepsilon,h}\right)^3 - 3c_{h,t}^\varepsilon(x) \left(Z^{\varepsilon,h}\right).$$

and

$$c_{h,t}^\varepsilon(x) = \mathbb{E}\left[Z^{\varepsilon,h}(t, x)^2\right]. \tag{5.8}$$

Lemma 5.1 ([18, Example 5.2.27]) *For any $\varepsilon, h > 0$, there exists a unique solution $\varphi^{\varepsilon,h} \in C([0, T]; L^2(\mathcal{D}))$ to Eq. 5.6.*

Since $m(z) \in \mathbb{R}$, similar as in the proof in [19, 22, 23], for any $k = 1, 2, 3$, as $h \to 0$, $: (\bar{Z}^{\varepsilon,h})^k :$ converges in $C([0, T], \mathcal{C}^\alpha)$ for any $\alpha < 0$ whose limit is denoted as $: (\bar{Z}^\varepsilon)^k :$. Here $\mathcal{C}^\alpha$ is defined as the Besov space $B^\alpha_{\infty,\infty}$, see [23] and the reference therein for details.

Then we denote

$$
\begin{cases}
\dfrac{d\varphi^\varepsilon}{dt} = \Delta(-\varepsilon\Delta\varphi^\varepsilon + \dfrac{1}{\varepsilon} : f(\varphi^\varepsilon + \bar{Z}^\varepsilon) :), \\
\varphi^\varepsilon(0) = z - m(z) \in H_0^{-1},
\end{cases}
\tag{5.9}
$$

where

$$
: f(\varphi^\varepsilon + \bar{Z}^\varepsilon) :\!:= \sum_{k=0}^{3} \binom{3}{k} : (\bar{Z}^\varepsilon)^{3-k} : (\varphi^\varepsilon)^k
\tag{5.10}
$$

Theorem 5.2 [22, Theorem 4.4] *For $\mathbb{P} - a.s.$ ω, there exists a unique solution φ^ε to Eq. 5.9 in $C([0, T]; H_0^{-1})$ for any fixed $\varepsilon > 0$.*

Remark 5.3 We note that in [22] the authors consider the periodical boundary condition, which is different from the Neumann boundary condition. But by our extension method as we explained before, a similar proof follows.

In fact, $\varphi^\varepsilon = \lim_{h\to 0} \varphi^{\varepsilon,h}$ in $C([0, T]; H_0^{-1})$. Let $u^{\varepsilon,h} := \varphi^{\varepsilon,h} + Z^{\varepsilon,h}$, $u^{\varepsilon,h}$ also converges to u^ε in $C([0, T]; H^{-1})$, which is the unique solution to

$$
\begin{cases}
du^\varepsilon = \Delta(-\varepsilon\Delta u^\varepsilon + \dfrac{1}{\varepsilon} : f(u^\varepsilon) :)dt + \varepsilon^\sigma BdW, \\
u^\varepsilon(0) = z \in H^{-1},
\end{cases}
\tag{5.11}
$$

with Neumann boundary conditions,

$$
\frac{\partial u^\varepsilon}{\partial n} = \frac{\partial \Delta u^\varepsilon}{\partial n} = 0 \text{ on } \partial\mathcal{D},
\tag{5.12}
$$

where $: f(u^\varepsilon) :=: f(\varphi^\varepsilon + \bar{Z}^\varepsilon) :$ is defined in Eq. 5.10.

5.2 The Sharp Interface Limit of Eq. 1.6

Similarly as in the proof of Theorem 4.3 we prove that for a suitable choice $h(\varepsilon)$, the solutions to Eq. 5.11 will converge to the solution to deterministic Hele-Shaw model (1.4).

The method is a modification of the one in Section 4.1. We consider the residual

$$
R^{\varepsilon,h} := u^{\varepsilon,h} - u_A^\varepsilon.
\tag{5.13}
$$

Let $Y^{\varepsilon,h} = R^{\varepsilon,h} - Z^{\varepsilon,h}$, which satisfies

$$
dY^{\varepsilon,h} = -\varepsilon\Delta^2 Y^{\varepsilon,h}dt + \frac{1}{\varepsilon}\Delta\left(f'(u_A^\varepsilon)(Y^{\varepsilon,h} + Z^{\varepsilon,h}) + \mathcal{N}(u_A^\varepsilon, Y^{\varepsilon,h} + Z^{\varepsilon,h})\right)dt
$$
$$
- \frac{c_{h,t}^\varepsilon}{\varepsilon}\Delta(u_A^\varepsilon + Z^{\varepsilon,h} + Y^{\varepsilon,h}) + \Delta r_A^\varepsilon dt,
\tag{5.14}
$$

where $c_{h,t}^{\varepsilon}$ is defined in Eq. 5.8. For $Y^{\varepsilon,h}$ we also have the energy estimate:

$$\frac{1}{2}\frac{d\|Y^{\varepsilon,h}\|_{H^{-1}}^2}{dt} + \varepsilon\|Y^{\varepsilon,h}(t)\|_{H^1}^2 = -\frac{1}{\varepsilon}\langle f'(u_A^{\varepsilon})(Y^{\varepsilon,h} + Z^{\varepsilon,h}) + \mathcal{N}(u_A^{\varepsilon}, Y^{\varepsilon,h} + Z^{\varepsilon,h}), Y^{\varepsilon,h}\rangle$$
$$- \langle r_A^{\varepsilon}, Y^{\varepsilon,h}\rangle + \frac{c_{h,t}^{\varepsilon}}{\varepsilon}\langle u_A^{\varepsilon} + Y^{\varepsilon,h} + Z^{\varepsilon,h}, Y^{\varepsilon,h}\rangle.$$
(5.15)

In order estimate $Y^{\varepsilon,h}$, we still need the estimation of $Z^{\varepsilon,h}$ and $c_{h,t}^{\varepsilon}$. Analogously to Lemma 4.6 we have

Lemma 5.4 *There exists a consant $C_2 > 0$ such that for any $0 < \beta \leq 1$,*

$$\mathbb{E}\left[\|Z^{\varepsilon,h}\|_{C(\mathcal{D}_T)}\right] \leq C_2\varepsilon^{\sigma_*}h^{-2},$$

where $\sigma_* = \sigma - \frac{\beta}{4}$. Then for any $\delta > 0$, there exists a constant $C_\delta > 0$, such that

$$\mathbb{P}\left[\Omega_{\delta,\varepsilon,h}'\right] > 1 - C_\delta\varepsilon^{\delta},$$

where $\Omega_{\delta,\varepsilon,h}' = \{\|Z^{\varepsilon,h}\|_{C(\mathcal{D}_T)} \leq C_2\varepsilon^{\sigma_*-2\delta}h^{-2}\}$.

Proof We follow a similar proof as in Lemma 4.6. A factorization formula implies that

$$Z^{\varepsilon,h}(t,x) = \varepsilon^{\sigma}\frac{\sin\pi\kappa}{\pi}\int_0^t (t-s)^{\kappa-1}\langle M(\varepsilon(t-s), x - \cdot), U^{\varepsilon,h}(s)\rangle ds,$$

where $M(t,x,y)$ is the kernel of $e^{-t\Delta^2}$ and

$$U^{\varepsilon,h}(s,x) = \int_0^t \langle (t-r)^{-\kappa}K_h(\varepsilon(t-r), x, \cdot), dW_s\rangle_{L^2(\mathcal{D},\mathbb{R}^2)},$$

where K_h is defined in Eq. 5.21. Combined with Eq. 5.22, we have that

$$|K_h(\varepsilon t, x, y)| \lesssim (\varepsilon t)^{-\frac{\beta}{4}}h^{-\eta}\left(|x-y|^{-\zeta} + |x+y|^{-\zeta}\right),$$

where $\beta, \zeta, \eta \geq 0$ and $\beta + \zeta + \eta = 3$. Similarly to Eqs. 4.9-4.14 we have that

$$\mathbb{E}\left[\left|U^{\varepsilon,h}(s,x)\right|^2\right] \lesssim \varepsilon^{-\frac{\beta}{2}}h^{-2\eta}\int_0^s\int_{\mathcal{D}}(s-r)^{-2\kappa-\frac{\beta}{2}}\left(|x+y|^{-2\zeta} + |x-y|^{-2\zeta}\right)dyds$$
$$\lesssim \varepsilon^{-\frac{\beta}{2}}h^{-2\eta}s^{1-2\kappa-\frac{\beta}{2}}|x|^{2-2\zeta},$$
(5.16)

where we require that

$$1 - 2\kappa - \frac{\beta}{2} > 0, \quad \zeta < 1$$

Similarly to Eq. 4.8, we have that

$$\mathbb{E}\left[\|U^{\varepsilon,h}\|_{L^{2p}(\mathcal{D}_T)}\right] \lesssim \varepsilon^{-\frac{\beta}{4}}h^{-\eta}.$$

Let $\eta = 2$ and $\kappa > 0$ be small enough such that $\beta < 1 - \zeta < 2 - 4\kappa$, $\zeta < 1$. Similarly as in the proof of Lemma 2.7 in [7], we have that

$$\mathbb{E}\left[\|Z^{\varepsilon,h}(t)\|_{C(\mathcal{D}_T)}\right] \lesssim_T \varepsilon^{\sigma}\mathbb{E}\left[\|U^{\varepsilon,h}\|_{L^{2p}(\mathcal{D}_T)}\right]$$
$$\lesssim \varepsilon^{\sigma-\frac{\beta}{4}-\frac{\kappa}{2}}h^{-2}.$$
(5.17)

The statement then follows by Chebyshev's inequality. $\qquad\square$

For $c_{h,t}^\varepsilon$, we have the following estimate:

Lemma 5.5 *There exists a constant $C > 0$ such that for any $(t, x) \in \mathcal{D}_T$ and any $\varepsilon, h \in (0, 1)$,*

$$|c_{h,t}^\varepsilon(x)| \le -C\varepsilon^{2\sigma-1} \log h$$

Proof Following a similar argument as in Eqs. 4.10, 4.11 and 4.13, we obtain that for all $g \in (g_1, g_2) \in L^2(\mathcal{D}, \mathbb{R}^2)$

$$\int_\mathcal{D} K(t, x, y)g(y)dy = \int_\mathcal{D} K^1(t, x, y)g_1(y)dy + \int_\mathcal{D} K^2(t, x, y)g_2(y)dy$$
$$\simeq \sum_k \left(\langle g_1, |k_1|e_k \rangle + \langle g_2, |k_2|e_k \rangle \right) e^{-t|k|^4} e_k(x).$$

Hence

$$K(t, x, y) \simeq \sum_k |k|e^{-t|k|^4} e_k(x)e_k(y). \tag{5.18}$$

where e_k is defined in Eq. 3.1. Note that $e_k(x)e_k(y) = \frac{1}{2}\left(e_k(x - y) + e_k(x + y) \right)$. Thus we obtain

$$K(t, x, y) \simeq \sum_k |k|e^{-t|k|^4}(e_k(x - y) + e_k(x + y)) := P_2(t, x - y) + P_2(t, x + y). \tag{5.19}$$

By [24, p282, (c)], we have that for any $(t, x) \in \mathcal{D}_T$,

$$|P_2(t, x)| \lesssim |x|^{-3} e^{-\frac{t}{|x|^4}} \lesssim \left(t^{\frac{1}{3}} + |x| \right)^{-3}. \tag{5.20}$$

Thus we obtain for any $t \in [0, T]$, $x, y \in \mathcal{D}$,

$$|K(\varepsilon t, x, y)| \lesssim \left((\varepsilon t)^{\frac{1}{4}} + |x - y| \right)^{-3} + \left((\varepsilon t)^{\frac{1}{4}} + |x + y| \right)^{-3}.$$

We can extend the definition of $K(t, x, y)$ for $x, y \in \mathbb{R}^2$ with the same form as in Eq. 5.18, and denote

$$K_h(t, x, y) := \int_{\mathbb{R}^2} \rho_h(z) K(t, x, y - z)dz. \tag{5.21}$$

Therefore (5.5) becomes

$$Z^{\varepsilon,h}(t, x) = \varepsilon^\sigma \int_0^t \langle K_h(\varepsilon(t - r), x - \cdot), dW_s \rangle_{L^2(\mathcal{D},\mathbb{R}^2)}$$

Then by [14, Lemma 10.17] we have that

$$|K_h(\varepsilon t, x, y)| \lesssim \left((\varepsilon t)^{\frac{1}{4}} + |x - y| + h \right)^{-3} + \left((\varepsilon t)^{\frac{1}{4}} + |x + y| + h \right)^{-3}. \tag{5.22}$$

Then we have that for any $(t, x) \in \mathcal{D}_T$.

$$|c_{h,t}^\varepsilon(x)| \le \varepsilon^{2\sigma} \int_0^t \int_\mathcal{D} |K_h^\varepsilon(t - r, x, y)|^2 dr dy$$
$$\lesssim \varepsilon^{2\sigma-1} \int_0^{t\varepsilon} \int_\mathcal{D} \left(r^{\frac{1}{4}} + |x - y| + h \right)^{-6} dr dy + \varepsilon^{2\sigma-1} \int_0^{t\varepsilon} \int_\mathcal{D} \left(r^{\frac{1}{4}} + |x + y| + h \right)^{-6} dr dy$$
$$\lesssim -\varepsilon^{2\sigma-1} \log h.$$

$$\tag{5.23}$$

$\square$

The next theorem is the main result of this section.

Theorem 5.6 *Let $u^{\varepsilon,h}$ be the unique solution to Eq. 5.11 and u_A^ε be defined in Theorem 3.2 with large enough $K > 0$. For some $\theta > 0$ such that $\varepsilon^\theta \lesssim h^2$, we assume that*

$$\begin{cases} \gamma > 13, \\ \sigma > \dfrac{1}{3}\gamma + \dfrac{13}{3} + \theta. \end{cases} \tag{5.24}$$

Then there exist a generic constant $C > 0$ and a constant $C_\delta > 0$ for all $0 < \delta < \frac{\sigma}{2} - \frac{1}{6}\gamma - \frac{13}{6} - \frac{\theta}{2}$ such that the following estimates hold

$$\mathbb{P}\left[\|R^{\varepsilon,h}\|_{L^3(\mathcal{D}_T)} \le C\varepsilon^{\frac{\gamma}{3}}\right] \ge 1 - C_\delta \varepsilon^\delta,$$

$$\mathbb{P}\left[\|R^{\varepsilon,h}\|_{L^\infty(0,T;H^{-1})}^2 \le C\left(\varepsilon^{\gamma-1} + \varepsilon^{\sigma_* - 1 - 2\delta - \theta}\right)\right] \ge 1 - C_\delta \varepsilon^\delta, \tag{5.25}$$

$$\mathbb{P}\left[\|v^{\varepsilon,h} - w_A^\varepsilon\|_{L^1(0,T;H^{-2})}^2 \le C\varepsilon^{\frac{\gamma}{3}-1}\right] \ge 1 - C_\delta \varepsilon^\delta,$$

where

$$v^{\varepsilon,h} := -\varepsilon\Delta u^{\varepsilon,h} + \frac{1}{\varepsilon}\left(f(u^{\varepsilon,h}) - 3c_{h,t}^\varepsilon u^{\varepsilon,h}\right).$$

Proof We proceed similarly as in Section 4.1. We define a stopping time

$$T^{\varepsilon,h} := T \wedge \inf\{t > 0 : \int_0^t \|Y^{\varepsilon,h}(\tau)\|_{L^3}^3 d\tau > \varepsilon^\gamma\}. \tag{5.26}$$

Then let $t < T^{\varepsilon,h}$ and fix an $\omega \in \Omega'_{\delta,\varepsilon,h}$. Since

$$h^{-2} \lesssim \varepsilon^{-\theta} \tag{5.27}$$

for some $\theta > 0$. We have that

$$-\log h \lesssim -\frac{\theta}{2}\log\varepsilon \lesssim \varepsilon^{-\delta}, \quad |c_{h,t}^\varepsilon| \lesssim \varepsilon^{2\sigma-1-\delta}. \tag{5.28}$$

For $\frac{c_{h,t}^\varepsilon}{\varepsilon}\langle u_A^\varepsilon + Y^{\varepsilon,h} + Z^{\varepsilon,h}, Y^{\varepsilon,h}\rangle$ we have that for small enough ε

$$\int_0^t \frac{c_{h,t}^\varepsilon}{\varepsilon}|\langle u_A^\varepsilon + Y^{\varepsilon,h} + Z^{\varepsilon,h}, Y^{\varepsilon,h}\rangle|d\tau \lesssim \varepsilon^{\frac{\gamma}{3}-1}c_{h,t}^\varepsilon \lesssim \varepsilon^{2\sigma+\frac{\gamma}{3}-2-\delta}.$$

For the rest of the terms on the right hand side of Eq. 5.15, we follow the proof in Section 4.1 by repalcing the estimate for Z^ε with the estimate of $Z^{\varepsilon,h}$ in Lemma 5.4. Thus we have that for small enough ε and $t \le T^{\varepsilon,h}$

$$\sup_{\tau\in[0,t]} \|Y^{\varepsilon,h}(\tau)\|_{H^{-1}}^2 d\tau \lesssim \varepsilon^{2\sigma+\frac{\gamma}{3}-2-\delta} + \varepsilon^{\gamma-1} + \varepsilon^{\sigma_*-1-2\delta+\frac{\gamma}{3}-\theta} + \varepsilon^{3(\sigma_*-2\delta-\theta)-1}.$$

Also,

$$\int_0^t \|Y^{\varepsilon,h}(\tau)\|_{H^1}^2 d\tau \lesssim \varepsilon^{2\sigma+\frac{\gamma}{3}-3-\delta} + \varepsilon^{\frac{2}{3}(\gamma-1)-1} + \varepsilon^{\sigma_*-2-2\delta+\frac{\gamma}{3}-\theta} + \varepsilon^{3(\sigma_*-2\delta-\theta)-2} + \varepsilon^{\gamma-2}.$$

Hence we have

$$\sup_{\tau\in[0,t]} \|Y^{\varepsilon,h}(\tau)\|_{H^{-1}}^2 d\tau \lesssim \varepsilon^{\gamma_1}, \quad \int_0^t \|Y^{\varepsilon,h}(\tau)\|_{H^1}^2 d\tau \lesssim \varepsilon^{\gamma_2}, \tag{5.29}$$

where

$$\gamma_1 := (2\sigma + \frac{\gamma}{3} - 2 - \delta) \wedge (\sigma_* - 1 - 2\delta - \theta + \frac{\gamma}{3}) \wedge (3(\sigma_* - 2\delta - \theta) - 1) \wedge (\gamma - 1),$$

$$\gamma_2 := (\gamma_1 - 1) \wedge (\frac{2}{3}\gamma - \frac{5}{3}).$$

Similarly to Eq. 4.27, we have

$$\int_0^t \|Y^{\varepsilon,h}\|_{L^3}^3 d\tau \lesssim \varepsilon^{\frac{\gamma_1}{2} + \gamma_2}.$$

In order to prove $T^{\varepsilon,N} = T$ for small enough ε, we need to prove $\gamma < \frac{1}{2}\gamma_1 + \gamma_2$. First we assume that $\gamma_2 = \frac{2}{3}\gamma - \frac{5}{3}$, i.e.

$$\gamma_1 > \frac{2}{3}\gamma - \frac{2}{3}.$$

Then $\frac{1}{2}\gamma_1 + \gamma_2 > \gamma$ yields

$$\gamma_1 > \frac{2}{3}\gamma + \frac{10}{3}.$$

A direct calculation yields that

$$\begin{cases} \gamma > 13, \\ \sigma > \frac{1}{3}\gamma + \frac{13}{3} + 2\delta + \theta + \frac{\beta}{4}, \end{cases}$$

which implies that

$$\gamma_1 = (\sigma_* - 1 - 2\delta - \theta + \frac{\gamma}{3}) \wedge (\gamma - 1).$$

Since $\delta, \beta > 0$ can be arbitrarily small, we can only assume that Eq. 5.24 hold and let $0 < 2\delta < \sigma - \frac{1}{3}\gamma - \frac{13}{3} - \theta$.

Since $R^{\varepsilon,h} = Y^{\varepsilon,h} + Z^{\varepsilon,h}$, and $H^1 \subset L^3$, we can obtain the estimate of $R^{\varepsilon,h}$ which is similar to Eq. 4.29. Moreover since

$$v^{\varepsilon,h} = -\varepsilon \Delta u^{\varepsilon,h} + \frac{1}{\varepsilon} \left(f(u^{\varepsilon,h}) - 3c_{h,t}^\varepsilon u^{\varepsilon,h} \right),$$

similarly to Eq. 4.30, we obtain that

$$\|v^{\varepsilon,h} - w_A^\varepsilon\|_{L^1(0,T;H^{-2})}^2 \lesssim \varepsilon \|R^{\varepsilon,h}\|_{L^2(\mathcal{D}_T)} + \frac{1}{\varepsilon}\|R^{\varepsilon,h}\|_{L^1(\mathcal{D}_T)} + \frac{1}{\varepsilon}\|c_{h,t}^\varepsilon u^{\varepsilon,h}\|_{L^1(0,T;H^{-2})}^2$$

$$\lesssim \varepsilon \|R^{\varepsilon,h}\|_{L^3(\mathcal{D}_T)} + \frac{1}{\varepsilon}\|R^{\varepsilon,h}\|_{L^3(\mathcal{D}_T)} + \frac{1}{\varepsilon}\|c_{h,t}^\varepsilon\|_{L^\infty}\|R^{\varepsilon,h} + u_A^\varepsilon\|_{L^3(\mathcal{D}_T)}$$

$$\lesssim \varepsilon^{\frac{\gamma}{3}-1} - \varepsilon^{2\sigma-2}\log h \lesssim \varepsilon^{\frac{\gamma}{3}-1}.$$

$$\square$$

Remark 5.7 It is easy to see that Eq. 5.24 implies the condition $\sigma > \frac{26}{3} + \theta$. This implies that the slower h converges to 0 than ε, the smaller σ could be. Since θ can be arbitrarily small, the lower bound for σ is $\frac{26}{3}$.

The next corollary is a simple consequence of Theorem 5.6 and can be shown as Corollary 4.5.

Corollary 5.8 *There exist a subsequence $\{\varepsilon_k\}_{k=1}^{\infty}$ and $\{h_k\}_{k=1}^{\infty}$ with $\varepsilon_k^{\theta} \lesssim h_k^2$ such that for*
$\mathbb{P} - a.s.\ \omega \in \Omega$

$$\lim_{k \to \infty} u^{\varepsilon_k, h_k} = 1 - 2\chi_{\mathcal{D}_t^-} \quad in \quad L^3(\mathcal{D}_T),$$

where $\mathcal{D}_t^-$ is the interior of Γ_t in $\mathcal{D}$.

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