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Space-time PU-DWR error control and adaptivity for the heat equation

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In this work, space-time goal-oriented a posteriori error estimation using a partition-of-unity localization is applied to the linear heat equation. The algorithmic developments are substantiated with a numerical example.

1 Introduction

The dual weighted residual (DWR) method [2] for deriving goal-oriented error estimates using an adjoint equation has been used sucessfully for both stationary and nonstationary methods. In order to employ the approach for mesh adaptivity, error indicators must be localized. One of the first approaches for obtaining (cell-)local estimators is by applying integration-by-parts to the global estimator. This results in the strong form of the estimator, which holds point-wise, allowing for the calculation of cell-wise estimators and the use of a marking strategy. In [5] a new approach was proposed for stationary problems, using a DoF-wise partition of unity (PU). In this work, we apply this approach to the nonstationary linear heat equation and compare the results to the classical approach using integration by parts. Our implementation is based on the DTM package dwr-diffusion [4] and the finite element library deal.II [1].

2 Discretization and Goal Oriented Estimator

We work with a space-time formulation, and refer the reader to [6] for a detailed description of the procedure. In the following, we will use $(\cdot,\cdot)_H$ for the spatial L^2 scalar product $(\cdot,\cdot)_{L^2(\Omega)}$. The discretization is split into the temporal and spatial part to allow for different finite element spaces. The time interval $I = (0, T)$ is split into M subintervals and the spatial domain Ω is discretized on each subinterval using quadrilaterals yielding M triangulations $\mathcal{T}_h^m$. The temporal function spaces are obtained by taking polynomials of degree r on each subinterval I_m. Using quadrilateral finite elements of degree s and allowing for different spatial triangulations on each temporal subinterval we obtain two fully discrete function spaces $X_{k,h}^{r,s}$ and $\tilde{X}_{k,h}^{r,s}$. The latter function space allows for discontinuities on the discrete time points. Using discontinuous elements of degree 0 in time and linear elements in space we obtain the weak formulation of the heat equation. Find $u_{kh} \in \tilde{X}_{k,h}^{0,1}$, such that $A(u_{kh}, \varphi_{kh}) = F(\varphi)$ for all $\varphi_{kh} \in \tilde{X}_{k,h}^{0,1}$ with

$$A(u_{kh}, \varphi_{kh}) := \sum_{m=1}^{M} \int_{I_m} (\partial_t u_{kh}, \varphi_{kh})_H + (\nabla u_{kh}, \nabla \varphi_{kh})_H \, dt + \sum_{m=0}^{M} ([u_{kh}]_m, \varphi_{kh,m}^{+})_H + (u_{kh,0}, \varphi_{kh,0}^{-})_H$$

$$F(\varphi) := \sum_{m=1}^{M} \int_{I_m} (f, \varphi_{kh})_H \, dt + (u^0, \varphi_{kh,0}^{-})_H.$$

For a given global goal functional $J(u) = \int_0^T \tilde{J}(u(t)) \, dt$ we can now apply the Lagrange formalism to obtain an auxiliary dual problem which has to be solved backward in time.
Find $z_{kh} \in X_{k,h}^{1,2}$, such that

$$\sum_{m=1}^{M} \int_{I_m} (\psi_{kh}, -\partial_t u_{kh})_H + (\nabla \psi_{kh}, \nabla u_{kh})_H \, dt = J_u'(u_{kh})(\psi) \forall \psi \in \tilde{X}_{k,h}^{0,1} \tag{1}$$

Since the heat equation is linear we obtain the following (see [2])

Proposition 2.1 *It holds the error identity*

$$J(u) - J(u_{kh}) = F(z - z_{kh}) - A(u_{kh}, z - z_{kh})$$

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In practice z is usually unknown and is replaced by a higher order solution, which is then interpolated to obtain the approximate solution z_{kh}. In our case this leads to $J(u) - J(u_{kh}) \approx \eta_{kh} := F(z_{kh} - i_{kh} z_{kh}) - A(u_{kh}, z_{kh} - i_{kh} z_{kh})$ with the interpolation operator $i_{kh} : X^{1,2}_{k,h} \to \tilde{X}^{0,1}_{k,h}$.

3 Error Localization with the PU Approach

The fundamental idea is the multiplication of the interpolation difference with a smart choice of 1 in all terms of the estimator. For many finite element families the sum over all test functions forms a partition of unity (PU), such that most, if not all, common finite element libraries bring all the ingredients for this approach with them.
The simplest choice in a space-time setting is $\chi_j \in \tilde{X}^{0,1}_{kh}$, which results in one spatial PU $(\chi_{i,m})_{i=1}^{\#DoFs(\mathcal{T}_h^m)}$ in each time interval I_m. Insertion into the error estimator yields

$$\eta_{kh} = \sum_{m=1}^{M} \sum_{i=1}^{\#DoFs(\mathcal{T}_h^m)} \eta_i^m, \text{ with } \eta_i^m := F([z_{kh} - i_{kh} z_{kh}]\chi_{i,m}) - A(u_{kh}, [z_{kh} - i_{kh} z_{kh}]\chi_{i,m}). \tag{2}$$

4 Numerical Example

This test case, introduced in [3], is a moving hill circling around the center of the domain $\Omega = (0,1)^2$ with homogeneous Dirichlet boundary conditions on all four sides. Insertion of the manufactured solution

$$u(x,y,t) = \frac{1}{(1 + (x - 0.5 - 0.25\cos(2\pi t))^2 + (y - 0.5 - 0.25\sin(2\pi t))^2}$$

into the heat equation yields the right hand side and the initial condition.
The functional of interest is the global L^2 error over the space-time domain $Q = (0,T) \times \Omega$.

Table 1: Error, indicators and effectivity indices.

loop	$N_{\max}$	$K_{\max}$	$\|e\|_{L^2(Q)}$	η_{kh}	I_{eff}
1	10	4	1.78372e-01	1.93504e-01	1.085
2	15	7	1.19508e-01	1.45724e-01	1.219
⋮					
8	163	133	1.03063e-02	7.44191e-03	0.722
9	244	166	7.69145e-03	6.26862e-03	0.815
10	366	205	5.81066e-03	5.51932e-03	0.950
11	549	256	4.88177e-03	4.64812e-03	0.952
12	823	322	3.81006e-03	4.06347e-03	1.067
13	1234	415	3.14812e-03	3.09928e-03	0.984
14	1851	532	2.64354e-03	2.58486e-03	0.978

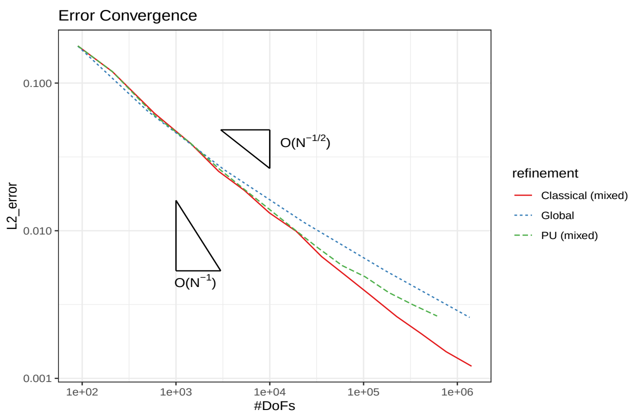

Fig. 1: L^2 error convergence for refinement strategies

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