Received: 24 June 2021 | Accepted: 20 September 2021

DOI: 10.1002/pamm.202100175

A residual-based error estimator and mesh adaptivity for the time harmonic Maxwell equations applied to a Y-beam splitter

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In this work, local mesh adaptivity for the time harmonic Maxwell equations is studied. The main purpose is to apply a known a posteriori residual-based error estimator from the literature and to investigate its performance for a Y-beam splitter setting. This configuration is an important prototype for the design of optical systems within the excellence cluster PhoenixD. Specifically, the branching region is of interest and requires a high accuracy of the numerical simulation. One numerical example shows the performance of our approach.

1 Introduction

The Maxwell equations are well known for describing electromagnetic phenomena [5, 6]. One field of current interest are optical technologies and the design of novel optical systems as for instance Y-splitters [8]. However, the branching region of the Y-splitter requires in numerical approximations a high accuracy of the solution. For cost-complexity reasons, error-controlled local mesh adaptivity is a tool at hand. In this work, we concentrate on a re-implementation of a residual-based error estimator presented in [4]; for early rigorous work we refer also to [9]. To this end, for the discretization Nédélec finite elements are employed, e.g., [7], and the numerical solution strategy is based on a domain decomposition method [3] with our own results provided in [2]. The implementation is done in the open-source finite element software deal.II [1].

2 Time harmonic Maxwell equations

Let $\Omega \subset \mathbb{R}^d,\ d \in \{2,3\}$ be a bounded domain with the sufficiently smooth boundary $\Gamma = \Gamma^\infty \cup \Gamma^{\text{inc}}$. Find $\mathbf{E} \in \mathbf{H}\,(\text{curl}, \Omega) := \{v \in \mathcal{L}^2(\Omega),\ \text{curl}\,(v) \in \mathcal{L}^2(\Omega)\}$ such that

$$\left\{\begin{array}{lll} \text{curl}\left(\mu^{-1}\text{curl}\,(\mathbf{E})\right) - \omega^2\mathbf{E} & = \mathbf{0} & \text{in } \Omega \\ \mu^{-1}\gamma^t\,(\text{curl}\,(\mathbf{E})) - i\omega\gamma^T\,(\mathbf{E}) & = \mathbf{0} & \text{on } \Gamma^\infty \\ \gamma^T\,(\mathbf{E}) & = \gamma^T\left(\mathbf{E}^{\text{inc}}\right) & \text{on } \Gamma^{\text{inc}} \end{array}\right. , \tag{1}$$

where $\mathbf{E}^{\text{inc}} : \mathbb{R}^d \to \mathbb{C}^d,\ d \in \{2,3\}$ is the incident field. Furthermore we define the traces $\gamma^t\,(\mathbf{v}) = \mathbf{n} \times \mathbf{v}$ and $\gamma^T\,(\mathbf{v}) = \mathbf{n} \times (\mathbf{v} \times \mathbf{n})$ where $\mathbf{n}$ denotes the normal to Ω. We define $\omega = \epsilon\frac{2\pi}{\lambda}$, where $\lambda > 0$ is the wave length, ϵ is the relative permittivity and μ is the relative permeability. For a more detailed description see [2, 7].

3 Discretization and adaptive mesh refinement

Problem (1) is discretized by means of the h-version of the finite element method with Nedelec elements of a fixed polynomial degree $p = 2$ on adaptively refined quadrilateral/hexehedral meshes with hanging nodes. To compute the error between the continuous solution $\mathbf{E}$ and the finite element approximation $\mathbf{E}_h$, we concentrate on a residual-based strategy, estimating a global-norm error of the form $\|\mathbf{E} - \mathbf{E}_h\|$. The resulting residual-based a posteriori error estimator is denoted by $\eta(\mathbf{E}_h)$ and which can be localized to single mesh elements K of the governing triangulation. Following [4], we specific form of localized indicators reads

$$\eta_K(\mathbf{E}_h)^2 = \eta_{R,K}(\mathbf{E}_h)^2 + \eta_{J,K}(\mathbf{E}_h)^2. \tag{2}$$

As usual, these indicators consists of two parts: an element-based residual term and face terms, respectively:

$$\eta_{R,K}(\mathbf{E}_h)^2 := \frac{h_K^2}{p^2}\Big(\left\|\text{curl}\left(\mu^{-1}\text{curl}\,(\mathbf{E}_h)\right) - \omega^2\mathbf{E}_h - \mathbf{s}\right\|^2_{L^2(K)} + \left\|\text{div}\left(\omega^2\mathbf{E}_h\right)\right\|^2_{L^2(K)} \Big),$$

$$\eta_{J,K}(\mathbf{E}_h)^2 := \frac{1}{2}\sum_{f\,\in\mathcal{F}} \frac{h_f}{p}\Big(\left\|\left[\gamma^T\left(\mu^{-1}\text{curl}\,(\mathbf{E}_h)\right)\right]\right\|^2_{L^2(f)} + \left\|\left[n_f \cdot \left(\omega^2\mathbf{E}_h + \mathbf{s}\right)\right]\right\|^2_{L^2(f)} \Big),$$

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where K denotes the current element with diameter h_K and f denotes element faces with diameter h_f. Moreover, n_f denotes the normal vector of f and $\mathbf{s}$ denotes the right hand side of the weak form.

4 Numerical results

In this section, we realize the previously presented a posteriori error estimator using a Y-beam splitter configuration. Therefore, we consider the Y-beam splitter shown in figure 1, which is made out of a material with the refrective index $n = 1.4906$ and is surrounded by air $n_{\text{air}} = 1.0$ and the wavelength $\lambda = 660$ nm was used.

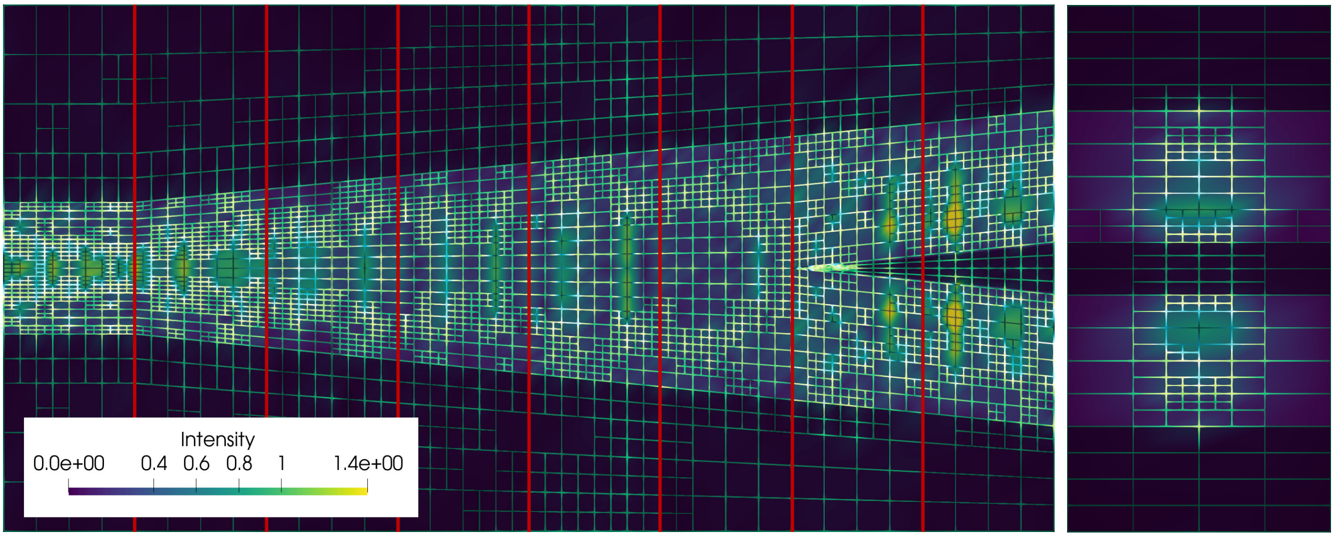

Fig. 1: On the left side, the intensity plot of the $x-y$ plane of the Y-beam splitter is shown. There, the red lines mark the interfaces between the different domains. On the right side, the intensity at the output is visualized. The surrounding air is displayed in dark color.

One major benefit of the combination of domain decomposition and adaptive grid refinement is that the first iteration steps of the domain decomposition method can be done on a coarse grid and only in the last steps, the mesh will be refinened adaptively, therefore speeding up the computations. As hypothesized, specifically the splitting region is adaptivitely refined, which is however driven automatically by the residual-based error estimator.

Acknowledgements This work is funded by the Deutsche Forschungsgemeinschaft (DFG) under Germany's Excellence Strategy within the Cluster of Excellence PhoenixD (EXC 2122, Project ID 390833453).
Open access funding enabled and organized by Projekt DEAL.

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