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Families of curves with Higgs field of arbitrarily large kernel

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Abstract

In this article, we consider the flat bundle $\mathcal{U}$ and the kernel $\mathcal{K}$ of the Higgs field naturally associated to any (polarized) variation of Hodge structures of weight 1. We study how strict the inclusion $\mathcal{U} \subseteq \mathcal{K}$ can be in the geometric case. More precisely, for any smooth projective curve C of genus $g \geqslant 2$ and any $r = 0, \ldots, g-1$, we construct non-isotrivial deformations of C over a quasi-projective base such that $\operatorname{rk}\mathcal{K} = r$ and $\operatorname{rk}\mathcal{U} \leqslant \frac{g+1}{2}$.

1. *Introduction and notations*

The Hodge bundle $\mathcal{H}^{1,0} = f_*\omega_f$ of a one-parametric semistable family $f\colon S \to B$ of complex projective curves of genus g (or more generally, of a polarized variation of Hodge structures of weight one) carries two natural vector subbundles: the flat unitary summand $\mathcal{U}$ of the second Fujita decomposition [**4, 5, 12**] and the kernel $\mathcal{K}$ of the associated Higgs field (see Section 3 for more details). By definition, there is an inclusion $\mathcal{U} \subseteq \mathcal{K}$, which must be an equality if $\mathcal{K} = \mathcal{H}^{1,0}$. Besides this trivial case, it is not difficult to explicitly construct (non-geometric) variations of Hodge structures over a disk where both $\operatorname{rk}\mathcal{U}$ and $\operatorname{rk}\mathcal{K}$ can be chosen arbitrarily (satisfying $\operatorname{rk}\mathcal{U} \leqslant \operatorname{rk}\mathcal{K} < g$). However, it is not clear whether this construction can provide geometric variations of Hodge structures, that is, arising from a semistable family of curves, or on the contrary such geometric variations have some restrictions on the ranks of $\mathcal{U}$ and $\mathcal{K}$. In particular, it is not clear when the equality $\mathcal{U} = \mathcal{K}$ holds in the geometric case.

We shortly highlight the importance of both bundles in the literature. The flat unitary summand $\mathcal{U}$ is the obstruction to the ampleness of the Hodge bundle, which has very strong consequences for the theory of moduli of algebraic varieties. In a recent paper [**2**], Catanese and Detweiler showed that $\mathcal{U}$ might be not even semiample, answering in the negative a question posed by Fujita. As for the kernel bundle $\mathcal{K}$, it is very closely related to the study of (strictly) maximal Higgs fields addressed, for example, by Viehweg and Zuo [**22–24**], for which some Arakelov type inequality is actually an equality.

The main result of this note is that $\mathcal{K}$ can have any rank (between 0 and $g-1$) also in geometric cases, with families containing an arbitrarily chosen curve, and even over (quasi-)projective base. If moreover the chosen curve has simple Jacobian variety, the family can be chosen with $\mathcal{U} = 0$. More precisely, we prove:

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Theorem 1.1. *Let C be any smooth projective curve of genus $g \geqslant 2$. Then for any $0 \leqslant r < g$ there is $f: \mathcal{C} \to B$, a non-isotrivial semistable one-dimensional family of deformations of C over a projective base B, such that $\operatorname{rk}\mathcal{K} = r$ and $\operatorname{rk}\mathcal{U} \leqslant \frac{g+1}{2}$.*

Corollary 1.2. *If C is a smooth projective curve of genus $g \geqslant 2$ with simple jacobian variety, then for $0 < r < g$ there is a deformation as in Theorem 1.1 with $\mathcal{U} = 0$, hence $\mathcal{U} \subsetneq \mathcal{K}$.*

Our motivation to study this question stems from the classification of fibered (irregular) surfaces. Indeed, in the recent work [**15**] an upper bound for the rank of $\mathcal{U}$ is obtained, depending on geometric invariants of the fibers like their genus and the general Clifford index, generalizing a previous result of [**1**] on the relative irregularity. A closer look at the proof of that result shows that in some cases the inequality $\operatorname{rk}\mathcal{U} \leqslant \frac{g+1}{2}$ can be proved using Massey products of sections of $\mathcal{U}$ [**14, 21**] combined with Castelnuovo–de Franchis fibration type theorems. Similar constructions are used in [**18**] to study hyperelliptic fibrations. In the remaining cases, the upper bound on $\operatorname{rk}\mathcal{U}$ is actually a bound for the rank of $\mathcal{K}$. Therefore a better understanding of the inclusion $\mathcal{U} \subseteq \mathcal{K}$ could lead to improvements of the main result in [**15**].

A second possible application is the so-called Coleman–Oort conjecture: roughly speaking, for high enough genus, the Torelli locus in $\mathcal{A}_g$ should not contain positive-dimensional Shimura subvarieties. A curve $X \subseteq \mathcal{A}_g$ carries a natural variation of Hodge structures with a flat unitary subbundle $\mathcal{U}$. In a first study [**19**], Lu and Zuo excluded the existence of certain types of Shimura curves in the Torelli locus using properties of $\mathcal{U}$. In the subsequent works [**6, 7**], Chen, Lu and Zuo proved that if $\operatorname{rk}\mathcal{U} \geqslant \frac{4g+2}{5}$ or $\operatorname{rk}\mathcal{U} \leqslant \frac{2g-22}{7}$, then X is not generically contained in the Torelli locus (that is, X intersects the Torelli locus at most in isolated points). Therefore, Shimura curves in the Torelli locus cannot have $\mathcal{U}$ of too big or too small rank. Since both bundles $\mathcal{U}$ and $\mathcal{K}$ for a curve X in $\mathcal{A}_g$ reflect the local structure of $X \subset \mathcal{A}_g$, there could be a similar statement with $\operatorname{rk}\mathcal{K}$ instead of $\operatorname{rk}\mathcal{U}$. The relation between $\mathcal{U}$ and $\mathcal{K}$ with Massey products has also recently been used by Ghigi, Pirola and the second author in [**13**] to prove that any Shimura subvariety generically contained in the Torelli locus can have dimension at most $\frac{7g-2}{3}$. Altogether this supports the idea that a better understanding of the inclusion $\mathcal{U} \subseteq \mathcal{K}$ might lead to new insights for the Coleman–Oort conjecture.

Let us devote a few words to our techniques. Our main tool to estimate the ranks of $\mathcal{U}$ and $\mathcal{K}$ is Lemma 3.3, which leads us to focus on families that are supported on relatively rigid divisors (see Definition 2.1). Roughly speaking, on a general fiber the first-order infinitesimal deformation is described by a rigid divisor of the fiber, and these divisors glue along the family. However, supporting divisors are not canonically definable, not even the minimal ones. Indeed, any divisor of degree greater than $2g-2$ supports every deformation (for example, $D = (2g-1)p$ for any point p), and thus any deformation has a minimal supporting divisor concentrated at any given point (with multiplicity). Nonetheless, for families obtained by deforming a branched finite covering, the theory developed by Horikawa in [**16**] allows to construct some natural minimal supporting divisors (see Lemma 2.3).

At first sight, one might expect that $\mathcal{U}$ and $\mathcal{K}$ coincide locally, and that a strict inequality $\mathcal{U} \subsetneq \mathcal{K}$ would be caused by monodromy if the base is not simply connected. But this is false, as the local nature of Lemma 3.3 shows. This fact is strongly highlighted in Theorem 3.5 where some *ad hoc* local examples have been constructed. We note that the set of rigid divisors of a given degree of a curve is open and Zariski-dense in the Picard variety of the fixed degree, hence many families can be constructed in this way.

The proof of Theorem 1.1 follows this line. We take a smooth projective curve C of genus g and for any $0 \leqslant r < g$ we construct a covering $C \to \mathbb{P}^1$ suitably ramified on a chosen rigid divisor $D \subset C$ of degree $g-r$. Then we consider a family of coverings obtained by moving D, which can be extended to a quasi-projective base. At this point the proof concludes as a straightforward application of Lemmas 2.3 and 2.2.

The proof of Corollary 1.2 follows immediately by Theorem 1.1 because the monodromy of the flat bundle $\mathcal{U}$ of those families is finite by a result of [**21**]. Thus a non-vanishing $\mathcal{U}$ would define a subvariety of the Jacobian of a general fiber, contradicting its simplicity.

Although the constructions as given in the proof are already very explicit, in Section 4 we study in more detail some deformations of cyclic coverings inspired by the study on $\mathcal{U}$ done in [**3, 17**]. Our interest on these examples is motivated by the fact that the corresponding $\mathcal{U}$ has infinite monodromy, rank bigger than $(g+1)/2$ and moreover $\mathcal{K}=\mathcal{U}$, hence they look very different from our case where $\mathcal{U}$ is smaller than $\mathcal{K}$, has rank less than $(g+1)/2$ and finite monodromy. This kind of examples has been intensively studied with different approaches and objectives (see [**8, 20**]). We note that they are also interesting in our study since they admit a non-vanishing flat bundle, which does not occur for a very general curve (see [**11**, Theorem 3.13]) and therefore we spend a few lines rephrasing some of their results using our tools.

The paper is organized as follows. In Section 2, we relate the theories of supporting divisors and deformations of maps and prove Lemma 2.3, which constructs a natural minimal supporting divisor by means of Horikawa's theory. In Section 3, we analyze the case of rigid supporting divisors (Lemma 2.2) and construct local families with any $\operatorname{rk}\mathcal{K}$ (Theorem 3.5). In Section 4, we consider in more detail deformations of cyclic coverings and compare them to those of [**3, 17**]. Finally in Section 5 we prove Theorem 1.1 and Corollary 1.2.

2. *Horikawa's deformation theory and supporting divisors*

In this section, we relate the theories of supporting divisors (see [**1**]) and of deformation of maps (see [**16**]) to produce a somehow canonical supporting divisor for families of morphisms, which we use to estimate the ranks of $\mathcal{U}$ and $\mathcal{K}$. Let $f:\mathcal{C}\to B$ be a smooth family of projective curves of genus $g\geqslant 2$ over a disk B.

Definition 2.1 (Supporting divisors). Let C be a smooth projective curve and $\xi\in H^1(C,T_C)$ a first-order infinitesimal deformation of C. An effective divisor D in C is a *supporting divisor* of ξ if

$$\xi\in\ker\left(H^1(C,T_C)\longrightarrow H^1(C,T_C(D))\right)=\operatorname{im}\left(H^0\left(D,T_C(D)_{|D}\right)\longrightarrow H^1(C,T_C)\right). \tag{2.1}$$

A *minimal* supporting divisor is a supporting divisor D with the extra property that any effective strict subdivisor $D'<D$ does not support ξ.

A (*minimal*) *supporting divisor* of a smooth family of curves $f:\mathcal{C}\to B$ is an effective divisor $\mathcal{D}\subset\mathcal{C}$ such that on a general fiber $C_b=f^{-1}(b)$ the restriction $D_b=\mathcal{D}_{|C_b}$ is a (minimal) supporting divisor of the infinitesimal deformation ξ_b of C_b induced by f.

For instance, when ξ is supported on a single point $D=P$, ξ is a Schiffer variation.

In the case of a family, up to shrinking B, we can always assume that a supporting divisor consists of sections of f (possibly with coefficients).

For any divisor D on a curve C we denote by $r(D)=\dim|D|=h^0(C,\mathcal{O}_C(D))-1$ the dimension of its complete linear series, and by $\operatorname{Cliff}(D)=\deg D-2r(D)$ its Clifford index.

The following result is our basic tool to estimate the ranks of $\mathcal{U}$ and $\mathcal{K}$ in terms of invariants of a supporting divisor.

Lemma 2.2 ([**1**, Lemma 2.3 and Theorem 2.4] or [**15**, Theorem 2.9]). *Let C be a projective curve of genus g, $\xi\in H^1(C,T_C)$ a first-order infinitesimal deformation and $\cup\xi:H^0(C,\omega_C)\to H^1(C,\mathcal{O}_C)$ the map induced by cup-product.*

(1) *If D is a divisor (in C) supporting ξ, then $H^0(C,\omega_C(-D))\subseteq\ker(\cup\xi)$ and hence*

$$\dim\ker(\cup\xi)\geqslant g-(\deg D-r(D)).$$

(2) *If further D supports ξ* minimally, *then*

$$\dim \ker(\cup\xi) \leqslant g - (\deg D - 2r(D)) = g - \mathrm{Cliff}(D).$$

We note that in particular, when a minimal supporting divisor D is rigid (that is, $h^0(C, \mathcal{O}_C(D)) = 1$), the estimates in Lemma 2.2 lead to the equality

$$\dim \ker(\cup\xi) = g - \deg D. \tag{2.2}$$

In order to apply Lemma 2.2, one has to construct a divisor minimally supporting f, but unfortunately such divisors are not unique and in general there is no canonical choice. In the case of families of curves f arising as deformations of morphisms onto a fixed curve, Horikawa's theory as developed in [**16**] gives a natural way to construct a supporting divisor using the so-called Horikawa characteristic class.

We shortly recall the construction of the characteristic map and the relation to the Kodaira–Spencer class. Let C' be a smooth projective curve. A family of morphisms of curves onto C' is a morphism $(f, \Phi) : \mathcal{C} \to B \times C'$ such that $f : \mathcal{C} \to B$ is a family of curves, and for any $b \in B$ the restriction $\pi_b = \Phi \circ i_b : C_b \to C'$ given by the inclusion $i_b : C_b = f^{-1}(b) \hookrightarrow \mathcal{C}$ is a non-constant morphism of curves. For any fixed $b \in B$, the morphism $\pi = \pi_b : C = C_b \to C'$ defines a short exact sequence

$$0 \longrightarrow T_C \xrightarrow{d\pi} \pi^* T_{C'} \xrightarrow{p_\pi} \mathcal{T}_\pi \longrightarrow 0. \tag{2.3}$$

We can fix local coordinate systems $(\mathcal{U}_i, (z_i, t))$ of $\mathcal{C}$ and (V_i, w_i) of C' by choosing Stein open sets such that $\Phi(\mathcal{U}_i) \subset V_i$ and where t is the pull-back of a local coordinate of B around b. We denote by Φ_i the local expression of Φ with respect to these coordinate systems, that is, $w_i = \Phi_i(z_i, t)$, and define a 0-cochain of $\pi^* T_{C'}$ by setting

$$s_i = \left(\frac{\partial \Phi_i}{\partial t}\right)_{|t=b} \frac{\partial}{\partial w_i}$$

on $U_i = \mathcal{U}_i \cap C$. By applying p_π we obtain a 0-cochain of $\mathcal{T}_\pi$ given by $\tau_i = p_\pi(s_i)$ on U_i. These sections turn out to agree in the intersections $U_i \cap U_j$, giving rise to a section $\tau \in H^0(C, \mathcal{T}_\pi)$ that is called *characteristic class* of π. The characteristic map

$$\tau : T_b B \to H^0(C, \mathcal{T}_\pi)$$

is the map that sends the generator $\frac{\partial}{\partial t} \in T_b B$ to the characteristic class $\tau \in H^0(C, \mathcal{T}_\pi)$ defined as above. By [**16**, Proposition 1.4], the Kodaira–Spencer map $KS : T_b B \to H^1(C, T_C)$ factors through the characteristic map as in

$$\begin{tikzcd} T_b B \arrow[r, "\tau"] \arrow[dr, "KS"'] & H^0(C, \mathcal{T}_\pi) \arrow[d, "\delta"] \\ & H^1(C, T_C) \end{tikzcd} \tag{2.4}$$

where $\delta : H^0(C, \mathcal{T}_\pi) \to H^1(C, T_C)$ is the connecting homomorphism associated to (2.3). By construction this gives a one-to-one correspondence between the vector space $H^0(C, \mathcal{T}_\pi)$ and the set of equivalence classes of first-order deformations of the morphism $\pi : C \to C'$ (leaving C' fixed).

The sheaf $\mathcal{T}_\pi$ can be more explicitly described through the ramification divisor R of π. Indeed, by definition of the ramification divisor there is an isomorphism $\pi^* T_{C'} \cong T_C(R)$ identifying $d\pi$ with the natural inclusion $T_C \hookrightarrow T_C(R)$. This in turn induces an isomorphism $\mathcal{T}_\pi \cong T_C(R)_{|R}$, which we use to construct a divisor minimally supporting f in some cases.

In the previous setting, we say that the family f is obtained from some $\pi = \pi_o : C \to C'$ by moving some (distinct) branch points $q_1, \ldots, q_k \in C'$ (while keeping the remaining branch points $q_{k+1},\dots,q_n$ fixed) if there are some maps $\widetilde{q_1},\dots,\widetilde{q_k}:B\to C'$ injective around $b=o$ and such that each $\pi_b:C_b\to C'$ is ramified over $\widetilde{q_1}(b),\dots,\widetilde{q_k}(b),q_{k+1},\dots,q_n$ with the same ramification type as for $b=o$.

LEMMA 2.3. *Keeping the above notations, suppose furthermore that for $i=1,\dots,k$ there is only one ramification point p_i over q_i, let r_i+1 be its ramification index and set $D=\sum_{i=1}^k r_ip_i$. If $H^0(C,T_C(D))=0$, then any deformation of π obtained moving $q_1,\dots,q_k$ is minimally supported in D.*

Proof. We consider the extension class $\xi\in H^1(C,T_C)$ induced by f on $C=f^{-1}(o)$ and we prove that this is minimally supported over $D=\sum_{i=1}^k r_ip_i$. To do so, we compute ξ by using the Horikawa characteristic map. Fix first a local coordinate t of B centered in $o\in B$ and for each $i=1,\dots,k$ choose local coordinates z_i, respectively, w_i, centered on $p_i\in C$, respectively, $q_i=\pi(p_i)\in C'$, such that $w_i=f(z_i,t)=z_i^{r_i+1}+t$. Then ξ is given as

$$\xi=KS\left(\frac{\partial}{\partial t}\right)=\delta\left(\sum_{i=1}^k\frac{\partial}{\partial w_i}\right)=\delta\left(\sum_{i=1}^k\frac{1}{(r_i+1)z_i^{r_i}}\frac{\partial}{\partial z_i}\right).$$

Since $\sum_{i=1}^k\frac{1}{(r_i+1)z_i^{r_i}}\frac{\partial}{\partial z_i}$ is an element in $H^0(D,T_C(D)_{|D})\subset H^0(R,T_C(R)_{|R})$, this proves that

$$\xi\in\mathrm{Im}(H^0(C,T_C(D)_{|D})\longrightarrow H^1(C,T_C))$$

and so that D supports ξ. We now prove that D is minimally supporting ξ, that is, that any effective subdivisor $D'<D$ does not support it. To do this, it is enough to consider a subdivisor $D'=D-p_i$ of D obtained by removing a point p_i and then check this is not supporting f. We consider the short exact sequences $0\to T_C\to T_C(D')\to T_C(D')_{|D'}\to 0$ and $0\to T_C\to T_C(D)\to T_C(D)_{|D}\to 0$ induced by D and D' and we compare them through the inclusion $D'<D$. Since we have assumed $H^0(C,T_C(D))=0$, the map $H^0(C,T_C(D)_{|D})\longrightarrow H^1(C,T_C)$ is injective, hence it is enough to check that $\frac{1}{z_i^{r_i}}\frac{\partial}{\partial z_i}$ does not lie in $H^0(D',T_C(D')_{|D'})\subseteq H^0(D,T_C(D)_{|D})$. Indeed with the induced trivializations, the map $H^0(D',T_C(D')_{|D'})\to H^0(D,T_C(D)_{|D})$ is given by multiplication with z_i, and sends the subset $<1,z_i,\dots,z_i^{r_i-2}>\otimes\{\frac{1}{z_i^{r_i-1}}\frac{\partial}{\partial z_i}\}$ of $H^0(D',T_C(D')_{|D'})$ to the subset $<z_i,z_i^2,\dots,z_i^{r_i-1}>\otimes\{\frac{1}{z_i^{r_i}}\frac{\partial}{\partial z_i}\}\subset H^0(D,T_C(D)_{|D})$, which obviously does not contain $\frac{1}{z_i^{r_i}}\frac{\partial}{\partial z_i}$. □

REMARK 2.4. Note that in the above setting, if $k\geqslant 1$, there is a non-zero minimal supporting divisor. This implies that the family is not isotrivial, since the infinitesimal deformation is not zero.

3. *The case of rigid supporting divisors*

In this section, we study the ranks of the unitary flat and kernel bundles for families supported on (relatively) rigid divisors and we also analyze the monodromy of the unitary flat bundle. In particular, we construct families of curves with $\mathcal{K}$ of any given rank between 0 and $g-1$. On the other hand, we show that $\mathrm{rk}\,\mathcal{U}\leqslant\frac{g+1}{2}$, hence in particular we can construct (local) families with $\mathcal{U}\subsetneq\mathcal{K}$. Note that $\mathrm{rk}\,\mathcal{K}=g$ happens if and only if the family is isotrivial, and hence also $\mathcal{U}=\mathcal{K}$.

We start recalling the basic definitions around these bundles. Let B be a complex curve and $f:\mathcal{C}\to B$ a non-isotrivial semistable family of projective curves of genus $g\geqslant 2$. Consider the Hodge bundle $f_*\omega_f$, where $\omega_f=\omega_{\mathcal{C}}\otimes f^*\omega_B^\vee$. The Fujita decomposition [12] factors it as a direct sum $f_*\omega_f=\mathcal{U}\oplus\mathcal{A}$, with $\mathcal{U}$ unitary flat and $\mathcal{A}$ ample. If $\Gamma\subset B$ denotes the set of critical values (corresponding to singular fibers) and $\Upsilon = f^*\Gamma$, we can also consider the short exact sequence

$$0 \to f^*\omega_B(\log\Gamma) \to \Omega^1_{\mathcal{C}}(\log\Upsilon) \to \Omega^1_{\mathcal{C}/B}(\log\Upsilon) \to 0.$$

Pushing it forward and using the canonical isomorphism $f_*\omega_f \simeq f_*\Omega^1_{\mathcal{C}/B}(\log\Upsilon)$ we obtain a long exact sequence with connecting homomorphism

$$\theta : f_*\omega_f \simeq f_*\Omega^1_{\mathcal{C}/B}(\log\Upsilon) \longrightarrow R^1 f_* f^*\omega_B(\log\Gamma) \simeq R^1 f_*\mathcal{O}_{\mathcal{C}} \otimes \omega_B(\log\Gamma). \tag{3.1}$$

It is a morphism of vector bundles whose kernel $\mathcal{K} = \ker\theta$ is a vector subbundle of $f_*\omega_f$. Indeed, by definition of $\mathcal{K}$ there is an inclusion of sheaves $f_*\omega_f/\mathcal{K} \hookrightarrow R^1 f_*\mathcal{O}_{\mathcal{C}} \otimes \omega_B(\log\Gamma)$. Since $R^1 f_*\mathcal{O}_{\mathcal{C}}$ is a locally free (hence torsion-free) sheaf on B, the quotient $f_*\omega_f/\mathcal{K}$ is also torsion-free, and thus locally free because B is a curve. This shows that $\mathcal{K} \subseteq f_*\omega_f$ is actually a vector subbundle.

Definition 3.1. We call the bundles $\mathcal{U}$ and $\mathcal{K}$ as defined above the *unitary flat bundle* and *kernel bundle of f*, respectively.

By construction there are inclusions $\mathcal{U} \subseteq \mathcal{K} \subseteq f_*\omega_f$, which combined with the splitting $f_*\omega_f \cong \mathcal{U} \oplus \mathcal{A}$ give an exact sequence

$$0 \to \mathcal{K}/\mathcal{U} \to \mathcal{A} \to f_*\omega_f/\mathcal{K} \to 0,$$

exhibiting $\mathcal{K}/\mathcal{U}$ as a vector subbundle of $\mathcal{A}$. If $\mathcal{U} \neq \mathcal{K}$, $\mathcal{K}/\mathcal{U}$ has negative curvature by [**25**], hence $f_*\omega_f/\mathcal{K}$ has bigger degree than $\mathcal{A}$.

If f is not semistable, the Fujita decomposition still exists and thus $\mathcal{U}$ can be defined, but we lack (in general) of a global definition of θ as (3.1), hence $\mathcal{K}$ might not be defined. Nonetheless, we can always perform a semistable reduction and obtain a new fibration where both $\mathcal{U}$ and $\mathcal{K}$ are defined, although the new $\mathcal{U}$ might have bigger rank than the original one because of the monodromy around the non-semistable families (see, for example, [**15**]). Moreover, as we will explain next, these $\mathcal{U}$ and $\mathcal{K}$ will be determined by the polarized variation of the Hodge structures on the smooth fibers of the new fibration, hence also by the smooth fibers of the original, possibly non-semistable one.

With a little abuse of notation, suppose thus for this paragraph that f is smooth. In this case, the Hodge bundle is

$$\mathcal{H}^{1,0} = f_*\omega_f \subset \mathcal{H}^1 = R^1 f_*\mathbb{C} \otimes_{\mathbb{C}} \mathcal{O}_B,$$

where $\omega_f = \omega_{\mathcal{C}} \otimes f^*\omega_B^\vee \cong \Omega^1_{\mathcal{C}/B}$ because f is smooth. The Gauß–Manin connection restricts to $\nabla_{\mathcal{H}^{1,0}} : \mathcal{H}^{1,0} \to \mathcal{H}^1 \otimes \omega_B$, the *unitary flat local system* over $\mathcal{H}^{1,0}$ is $\mathbb{U} = \ker\nabla_{\mathcal{H}^{1,0}}$ and the *unitary flat subbundle* is $\mathcal{U} = \mathbb{U} \otimes_{\mathbb{C}} \mathcal{O}_B$. The Higgs field

$$\theta = p \circ \nabla_{\mathcal{H}^{1,0}} : \mathcal{H}^{1,0} \to \mathcal{H}^1 \otimes \omega_B \to \left(\mathcal{H}^1/\mathcal{H}^{1,0}\right) \otimes \omega_B \cong R^1 f_*\mathcal{O}_{\mathcal{C}} \otimes \omega_B$$

coincides with the connecting homomorphism

$$\theta : \mathcal{H}^{1,0} = f_*\Omega^1_{\mathcal{C}/B} \to R^1 f_*(f^*\omega_B) = R^1 f_*\mathcal{O}_{\mathcal{C}} \otimes \omega_B$$

arising by pushing forward the exact sequence $0 \to f^*\omega_B \to \Omega^1_{\mathcal{C}} \to \Omega^1_{\mathcal{C}/B} \to 0$, and the *kernel bundle* is $\mathcal{K} = \ker\theta$.

The link of this construction with the cup-products with Kodaira–Spencer classes discussed in the previous section is that, for any $b \in B$ with smooth fiber C_b and Kodaira–Spencer class $\xi_b \in H^1(C_b, T_{C_b})$, the Higgs field

$$\theta(b) : f_*\Omega^1_{\mathcal{C}/B} \otimes \mathbb{C}(b) = H^0(C_b, \omega_{C_b}) \longrightarrow R^1 f_*\mathcal{O}_{\mathcal{C}} \otimes \omega_B \otimes \mathbb{C}(b) = H^1(C_b, \mathcal{O}_{C_b}) \otimes T^\vee_{B,b}$$

coincides with $\cup\xi_b$ (up to non-zero scalar, depending on the choice of an isomorphism $T^\vee_{B,b} \cong \mathbb{C}$).

REMARK 3.2. The equality $\mathcal{K} = \ker\theta$ gives inclusions $\mathcal{K}\otimes\mathbb{C}(b) \subseteq \ker(\theta(b))$ for every b. These inclusions are equalities for general $b \in B$, but might be strict even for some b corresponding to smooth fibers. Indeed, denoting $\mathcal{G} = \operatorname{coker}(\theta)$, there is a short exact sequence

$$0 \longrightarrow \mathcal{K}\otimes\mathbb{C}(b) \longrightarrow \ker(\theta(b)) \longrightarrow \operatorname{Tor}_1^{\mathcal{O}_B}(\mathcal{G}, \mathbb{C}(b)) \longrightarrow 0.$$

Thus $\mathcal{K}\otimes\mathbb{C}(b) \subsetneq \ker(\theta(b))$ precisely at the points where $\mathcal{G}$ is not locally free (that is, the points where θ drops rank).

Our main tool in order to understand how $\mathcal{K}$ can be larger than $\mathcal{U}$ is given by the following

LEMMA 3.3. *Let f be minimally supported on a divisor $\mathcal{D}$ with $\mathcal{D}\cdot C_b = d$ and $h^0(C_b, \mathcal{O}_{C_b}(\mathcal{D}_{|C_b})) = 1$ for general $b \in B$ (that is, $\mathcal{D}$ is relatively rigid). Then* (1) $\operatorname{rk}\mathcal{K} = g - d$ *and* (2) $\operatorname{rk}\mathcal{U} \leqslant \frac{g+1}{2}$.

Proof. (1) It follows from Lemma 2.2. For a general $b \in B$, we indeed have

$$\mathcal{K}\otimes\mathbb{C}(b) = \ker\left(\cup\xi_b : H^0(C_b, \omega_{C_b}) \longrightarrow H^1(C_b, \mathcal{O}_{C_b})\right).$$

Since $h^0(C_b, \mathcal{O}_{C_b}(\mathcal{D}_{|C_b})) = 1$, then

$$\operatorname{rk}\mathcal{K} = \dim\mathcal{K}\otimes\mathbb{C}(b) = \dim\ker(\cup\xi_b) = g - \deg(\mathcal{D}_{|C_b}) = g - d.$$

(2) The argument follows the line of [**15**, section 3.1, case 1] (see also [**21**, Lemma 3.2]). Assume that $\operatorname{rank}\mathcal{U} \geqslant 2$, otherwise there is nothing to prove. Since $\operatorname{rk}\mathcal{U}$ is determined by any open subset of smooth fibers, we can also assume that B is a disk and $f\colon S \to B$ is smooth. By [**15**, Lemma 2.12], there is a factorization $\omega_f(-\mathcal{D}) \hookrightarrow \Omega^1_S \to \omega_f$, and moreover $\mathcal{K} = f_*\omega_f(-\mathcal{D})$ ([**15**, pp. 8674]). Then, we can lift a basis $\eta_1, \ldots, \eta_{u_f}$ of flat sections of $\mathcal{U} \subseteq \mathcal{K}$, to a set $\omega_1, \ldots, \omega_{u_f} \in H^0(\mathcal{C}, \Omega^1_{\mathcal{C}})$ of linearly independent *closed* 1-forms which are sections of the line bundle $\omega_f(-\mathcal{D}) \subseteq \Omega^1_S$, hence any two of these forms wedge to zero. Applying the 'Tubular Castelnuovo–de Franchis' (see [**15**, Theorem 1.4]), we get a family $\varphi_b : C_b \to C$ of morphisms from the general fiber C_b of f to a fixed curve C of genus $g(C) \geqslant u_f = \operatorname{rk}\mathcal{U}$. By the Riemann–Hurwitz formula,

$$2g - 2 = \deg\varphi_b(2g(C) - 2) + \deg R_b \geqslant 2\deg\varphi_b(u_f - 1),$$

where R_b is the ramification divisor of φ_b. In particular,

$$g - 1 \geqslant \deg\varphi_b(u_f - 1),$$

and so for $u_f > \frac{g+1}{2}$ and $g \geqslant 2$, one has $\deg\varphi_b = 1$ and hence a isotrivial family. □

LEMMA 3.4. *Let f be minimally supported on a relatively rigid divisor. Then $\mathcal{U}$ has finite monodromy.*

Proof. We can assume $\operatorname{rank}\mathcal{U} \geqslant 2$ (in the case of rank 1, the monodromy is finite since the line bundle must be torsion, proven, for example, in [**9**, Corollary 4.2.8]). Repeating the argument given in (2) of the proof of Lemma 3.3, we have that our bundle satisfies the assumptions of [**21**, Theorem 0.2] and thus has finite monodromy. □

We end this section by providing a way to construct non-isotrivial local families of curves with $\mathcal{K}$ of any given rank between 1 and $g - 1$.

THEOREM 3.5. *Let C be any curve of genus $g \geqslant 3$. Then for any $0 \leqslant r \leqslant g - 1$ there are one-dimensional deformations of C with $\operatorname{rk}\mathcal{K} = r$.*

Proof. Let us first consider a more geometric interpretation of supporting divisors. Let C be a curve of genus g and $\phi : C \to \mathbb{P}(H^0(C, \omega_C^{\otimes 2})^\vee) \cong \mathbb{P}^{3g-4}$ its bicanonical embedding. Given an effective divisor D in C, we define its *span* as

$$\langle D \rangle := \cap_{D \leqslant \phi^* H} H = \mathbb{P}\big(H^0(C, \omega_C^{\otimes 2}(-D))^\perp\big),$$

that is, the intersection of all hyperplanes cutting out a divisor on C that contains D, which coincides with the projectivization of the annihilator of $H^0(C, \omega_C^{\otimes 2}(-D))$. In particular, if $\deg D < \deg \omega_C = 2g-2$, then Riemann–Roch gives $\dim \langle D \rangle = \deg D - 1$.

Let now $\xi \in H^1(C, T_C)$ be a non-zero first-order infinitesimal deformation, which defines a point $[\xi] \in \mathbb{P}(H^1(C, T_C)) \cong \mathbb{P}(H^0(C, \omega_C^{\otimes 2})^\vee)$. It is just a reformulation of the definitions that a divisor D supports ξ if and only if $[\xi] \in \langle D \rangle$. Thus first-order deformations supported on a divisor D correspond to points in $\langle D \rangle$. Furthermore, $\langle D' \rangle \subsetneq \langle D \rangle$ for any $0 \leqslant D' < D$ if and only if $\omega_C^{\otimes 2}(-D)$ has no base points, for example, if $\deg D \leqslant 2g-4$. In this case, the first-order deformations *minimally* supported in D form a non-empty Zariski-open subset $\langle D \rangle^\circ$ of $\langle D \rangle$, namely the complement of the spans of the finitely many strict subdivisors of D.

We want to focus on the deformations supported on rigid divisors of a given degree $d \in \{1, \ldots, g\}$. For any such d, the map $C^{(d)} = Div^d(C) \to Pic^d(C)$ is generically one-to-one, thus the rigid divisors form a non-empty Zariski-open set $V_d \subseteq C^{(d)}$. Let

$$X_d = \{(D, [\xi]) \mid \deg D = d, \, [\xi] \in \langle D \rangle\} = \bigcup_{D \in C^{(d)}} \{D\} \times \langle D \rangle \subset C^{(d)} \times \mathbb{P}(H^0(C, \omega_C^{\otimes 2})^\vee)$$

be the obvious incidence variety, which is irreducible of dimension $2d-1$ because $\dim\langle D \rangle = d-1$ for $d < 2g-2$. The subset

$$X_d^\circ = \bigcup_{D \in V_d} \{D\} \times \langle D \rangle^\circ \subset X_d$$

is a dense open subset. Indeed, its complement $X_d \setminus X_d^\circ$ is contained in the union of:

(1) the Zariski-closed strict subset $(V_d \setminus V_{d-1}) \times \mathbb{P}(H^0(C, \omega_C^{\otimes 2})^\vee)$; and
(2) the image of $X_{d-1} \times C \to X_d$, defined by $(D', [\xi], p) \mapsto (D' + p, [\xi])$, which has dimension at most

$$\dim X_{d-1} + \dim C = 2(d-1) - 1 + 1 = 2d - 2 < \dim X_d.$$

Set also $Y_d = p_2(X_d) \subset \mathbb{P}(H^0(C, \omega_C^{\otimes 2})^\vee)$, which by the above discussion corresponds to the (closed) set of infinitesimal deformations supported on some divisor of degree d. Of course, Y_d coincides with the dth secant variety of $C \subset \mathbb{P}(H^0(C, \omega_C^{\otimes 2})^\vee)$. Define also the dense subset

$$Y_d^\circ = p_2(X_d^\circ) \subset Y_d,$$

which corresponds to the first-order deformations *minimally* supported on some divisor of degree d. Thus, for any $[\xi] \in Y_d^\circ$, there is some minimal supporting divisor D of degree d and $r(D) = 0$, and hence by Lemma 2.2

$$\dim \ker(\cup\xi : H^0(C, \omega_C) \to H^1(C, \mathcal{O}_C)) = g - d.$$

Let now $\pi : \mathcal{C} \to \Delta$ be a semiuniversal deformation of C over some $(3g-3)$-dimensional polydisk Δ, $\mathbb{P} = \mathbb{P}_\Delta(\pi_* \omega_\pi^{\otimes 2}) \to \Delta$ and $\phi : \mathcal{C} \to \mathbb{P}$ the relative bicanonical map. We can mimic the above construction on every fiber of π and obtain a non-empty locally closed subset $\mathcal{Y}_d^\circ \subset \mathbb{P}$ that surjects onto Δ. Indeed, if $\mathcal{C}_\Delta^{(d)} = Div^d(\mathcal{C}/\Delta)$ denotes the relative symmetric dth product of $\mathcal{C}$, we can consider the Zariski-open subset $\mathcal{V}_d \subseteq \mathcal{C}_\Delta^{(d)}$ corresponding to rigid divisors and the incidence subvariety

$$\mathcal{X}_d = \big\{(D, \xi, t) \mid t \in \Delta, D \in Div^d(C_t), \xi \in H^1(C_t, T_{C_t}), [\xi] \in \langle D \rangle\big\} \subseteq \mathcal{C}_\Delta^{(d)} \times_\Delta \mathbb{P}.$$

The announced $\mathcal{Y}_d^\circ$ is then the image by the projection to $\mathbb{P}$ of the (non-empty) open subset

$$\mathcal{X}_d^o = (\mathcal{X}_d \setminus (\mathcal{X}_{d-1} \times_\Delta \mathcal{C})) \cap (\mathcal{V}_d \times_\Delta \mathbb{P}).$$

Up to shrinking Δ, we can find a section $\sigma : \Delta \to \mathcal{Y}_d^\circ$, which thus at every point $b \in B$ defines (up to scalar) a first-order deformation ξ_b minimally supported on a divisor of degree d.

Since $T_b\Delta \cong H^1(C_b, T_{C_b})$ for any $b \in \Delta$, the relative bicanonical space $\mathbb{P}$ can be identified with the projectivization of the tangent bundle of Δ. In this way, any section $\sigma : \Delta \to \mathcal{Y}_d^\circ$ can be thought of as a rank-one (hence automatically integrable) distribution on Δ. If $B \subset \Delta$ is any integral curve of a given σ, the restriction $f = \pi_B : \pi^{-1}(B) \to B$ gives the desired family. □

These families are constructed over a disk. One could thus wonder, if such examples can exist over a quasi-projective base B. The answer is yes, as our main results and also some more explicit examples constructed in Section 4 show.

4. *Semistable families of cyclic coverings of $\mathbb{P}^1$ with $\mathcal{K}$ larger than $\mathcal{U}$*

In this section we construct semistable families of curves over a projective base with $\mathcal{U} \subsetneq \mathcal{K}$ by moving few branch points of a low degree covering. The largest range for $\operatorname{rk}\mathcal{K}$ is achieved by families of hyperelliptic curves. Our main tool is the following

Proposition 4.1. *Let $\pi : C \to \mathbb{P}^1$ be a simple cyclic covering of degree n with reduced branch divisor $B = q_1 + \cdots + q_m$ $(n \mid m)$ and suppose $g(C) = g \geqslant 2$. Let $f : \mathcal{C} \to \Delta$ be a deformation of C obtained by moving the branch points $q_1, \ldots, q_k$. If $k < \frac{m}{n}$, then*

$$\operatorname{rk}\mathcal{K} = g - (n-1)k = \frac{(n-1)(m-2-2k)}{2} \quad \text{and} \quad \operatorname{rk}\mathcal{U} \leqslant \frac{g+1}{2}.$$

In particular, if $k < \frac{g-1}{2(n-1)} = \frac{mn-2n-m}{4(n-1)}$, then $\mathcal{U} \subsetneq \mathcal{K}$.

Proof. For each $i = 1, \ldots, k$, let $p_i = \pi^{-1}(q_i)$ be the ramification point above q_i, and set $D = (n-1)(p_1 + \cdots + p_k)$, the *variable ramification divisor*. We will show that D is a rigid divisor that supports f minimally, and Lemma 3.3 gives the final assertions.

In order to show that D is a rigid divisor, let us consider first the divisor $D' = \frac{n}{n-1}D = n(p_1 + \cdots + p_k) = \pi^*(q_1 + \cdots + q_k)$. It holds then

$$H^0(C, \mathcal{O}_C(D')) = H^0(C, \pi^*\mathcal{O}_{\mathbb{P}^1}(q_1 + \cdots + q_k)) \overset{\pi^*}{\cong} H^0\big(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(q_1 + \cdots + q_k) \otimes (\pi_*\mathcal{O}_C)\big)$$

$$\cong \bigoplus_{i=0}^{n-1} H^0\Big(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(q_1 + \cdots + q_k) \otimes \mathcal{O}_{\mathbb{P}^1}\Big(-i\frac{m}{n}\Big)\Big) = H^0\big(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(q_1 + \cdots + q_k)\big),$$

where the last equality follows from the hypothesis $k < \frac{m}{n}$, hence

$$\deg\Big(\mathcal{O}_{\mathbb{P}^1}(q_1 + \cdots + q_k) \otimes \mathcal{O}_{\mathbb{P}^1}\Big(-i\frac{m}{n}\Big)\Big) < 0$$

for any $i > 0$.

This shows that any meromorphic function in $H^0(C, \mathcal{O}_C(D'))$ is the pull-back of a meromorphic function in $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(q_1 + \cdots + q_k))$. In particular, any non-constant function in $H^0(C, \mathcal{O}_C(D'))$ has poles of order exactly n at some p_i, and hence $H^0(C, \mathcal{O}_C(D)) \subseteq H^0(C, \mathcal{O}_C(D'))$ consists only of the constant functions, that is, D is rigid.

It remains to show that D is a minimal supporting divisor of f. The genus of C is $g = \frac{(m-2)(n-1)}{2}$, and thus $\deg(T_C(D)) \leqslant 0$ with equality if and only if $n = k = g = 2$ (hence $m = 6$).

In this last case, an argument along the lines above shows that $H^0(C, T_C(D)) = 0$. Lemma 2.3 can thus be applied in any case, giving that D is a minimal rigid supporting divisor. □

THEOREM 4.2. *Let n, m and k be positive integers such that $n \mid m$ and $k < \frac{m}{n}$. Then there is a semistable fibration $f : Z \to B$ over a projective curve B whose general fiber has genus $g = \frac{(n-1)(m-2)}{2}$, $\mathcal{U}$ has finite monodromy group, $\operatorname{rk}\mathcal{U} \leqslant \frac{g+1}{2}$, and moreover*

$$\operatorname{rk}\mathcal{K} = g - (n-1)k = \frac{(n-1)(m-2-2k)}{2}.$$

Proof. We construct the family as in Proposition 4.1, deforming a simple cyclic covering $\pi : C \to \mathbb{P}^1$ of degree n with reduced branch divisor $B = q_1 + \cdots + q_m$. In order to obtain a projective base, fix two points $0, \infty \in \mathbb{P}^1$ and for each $i = 1, \ldots, k$ let $L_i \subset \mathbb{P}^1 \times \mathbb{P}^1$ be a curve of bidegree (1,1) with $L_i \cap \{0\} \times \mathbb{P}^1 = \{(0, q_i)\}$, or equivalently, the graph of an automorphism $\phi_i : \mathbb{P}^1 \to \mathbb{P}^1$ with $\phi_i(0) = q_i$. For $i = k+1, \ldots, m$, let $L_i = \mathbb{P}^1 \times \{q_i\} \subset \mathbb{P}^1 \times \mathbb{P}^1$.

For a general choice of the lines L_i we can assume that all of them intersect transversely in $k(m-k) + 2\binom{k}{2} = k(m-1)$ different points $t_1, \ldots, t_{k(m-1)}$, none of them lying on $M = \{\infty\} \times \mathbb{P}^1$. In this case, the divisor $M + \sum_{i=1}^m L_i$ has simple normal crossings.

If $r \in \mathbb{Z}_{\geqslant 0}$ is such that $k + r$ is a multiple of n, then $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(\sum_{i=1}^m L_i + rM) = \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(k + r, m)$ is divisible by n. We can thus consider the degree n cyclic covering of $\mathbb{P}^1 \times \mathbb{P}^1$ branched along $\sum_{i=1}^m L_i + rM$, so that the family defined by the projection onto the first $\mathbb{P}^1$ looks like the deformations in Proposition 4.1 around the smooth fibers. The claimed fibration can be constructed as the semistable reduction of the minimal desingularization of such cyclic covering. The claims on the ranks of $\mathcal{U}$ and $\mathcal{K}$ follow at once from Proposition 4.1.

The assertion on the finite monodromy of $\mathcal{U}$ follows from Lemma 3.4, since the supporting divisor of the family is relatively rigid (proof of Proposition 4.1). □

This construction is inspired by a series of examples studied by Catanese and Dettweiler in [**3**], where also degree n cyclic coverings are considered, but ramified only over four points (with multiplicities). Although they do a more general analysis, we will here focus on what they call 'standard case', which has $\gcd(n, 6) = 1$, three of the branch points have multiplicity one and the fourth one has multiplicity $n - 3$. Moving one of the ramification points defines a family over $\mathbb{P}^1$, which becomes semistable after a degree n covering of the base (and a desingularization). Let $f : S \to B$ be the resulting fibration. The genus of the smooth fibers is $g = n - 1$ and the singular fibers consist of two curves of genus $\frac{n-1}{2}$ meeting transversely in one point. It holds $q(S) = g(B) = \frac{n-1}{2}$, hence f is the Albanese map of S. More details can be found in [**3**, Section 4]. These families provide examples where $\mathcal{U}$ has infinite monodromy group, so they behave very differently from ours, where we have seen that the monodromy is finite.

The rank of their flat unitary summand $\mathcal{U}$ has been studied by Lu in [**17**], where arbitrary $n \geqslant 4$ is also considered, proving the lower bounds

$$\operatorname{rk}\mathcal{U} \geqslant \begin{cases} \left\lfloor \frac{2g+1}{3} \right\rfloor & \text{if } n \not\equiv 2 \mod 3 \\ \left\lfloor \frac{2g-2}{3} \right\rfloor & \text{if } n \equiv 2 \mod 3. \end{cases} \tag{4.1}$$

A straightforward application of Riemann–Hurwitz gives the relation

$$g = \begin{cases} n-2 & \text{if } n \equiv 0 \mod 3 \\ n-1 & \text{if } n \not\equiv 0 \mod 3. \end{cases} \tag{4.2}$$

With our techniques we are able to prove furthermore the following.

PROPOSITION 4.3. *Let $f : S \to B$ be as in the 'standard cases' of* [3]. *Then $\mathcal{U} = \mathcal{K}$ and equality holds in* (4.1).

Proof. Since $\mathcal{U} \subseteq \mathcal{K}$, we only have to prove $\operatorname{rk}\mathcal{K} \leqslant \lfloor \frac{2g+1}{3} \rfloor$ if $n \equiv 1 \mod 3$, and $\operatorname{rk}\mathcal{K} \leqslant \lfloor \frac{2g-2}{3} \rfloor$ otherwise.

By construction, f is a family obtained by moving one branch point of a morphism from a general fiber to $\mathbb{P}^1$. So we can apply Lemma 2.3 to obtain a minimally supporting divisor $\mathcal{D} = (n-1)\mathcal{P} \subseteq S$, where $\mathcal{P}$ is the section defined by moving the branch point. By Lemma 2.2, we have that

$$\operatorname{rk}\mathcal{K} \leqslant g - \deg D + 2r(D),$$

where $D = \mathcal{D}_{|C} = (n-1)P$ is the restriction to a general fiber C of f. In order to compute $r(D) = r((n-1)P)$, note first that $r((n-1)P) = r(nP) - 1$ because P is not a base point of $|nP|$. Indeed, since $nP = \pi^*Q$ for some $Q \in \mathbb{P}^1$, the pull-back of any other $Q' \in \mathbb{P}^1$ is a divisor linearly equivalent to nP not containing P. Second, since $C \to \mathbb{P}^1$ is a morphism of degree n ramified over a divisor of the form $R = (n-3)0 + 1 + Q + \infty$, that is, also of degree n, we can apply [10, Corollary 3.11] with $\mathcal{L} = \mathcal{O}_{\mathbb{P}^1}(1)$ and obtain

$$\pi_*\mathcal{O}_C = \bigoplus_{i=0}^{n-1} \mathcal{O}_{\mathbb{P}^1}\left(-i + \left\lfloor \frac{i}{n} R \right\rfloor\right) = \bigoplus_{i=0}^{n-1} \mathcal{O}_{\mathbb{P}^1}\left(-i + \left\lfloor \frac{i(n-3)}{n} \right\rfloor\right) = \bigoplus_{i=0}^{n-1} \mathcal{O}_{\mathbb{P}^1}\left(\left\lfloor \frac{-3i}{n} \right\rfloor\right).$$

This implies

$$\begin{aligned} h^0(\mathcal{O}_C(nP)) &= h^0(\pi^*\mathcal{O}_{\mathbb{P}^1}(1)) = h^0(\pi_*\pi^*\mathcal{O}_{\mathbb{P}^1}(1)) = h^0(\mathcal{O}_{\mathbb{P}^1}(1) \otimes \pi_*\mathcal{O}_C) \\ &= \sum_{i=0}^{n-1} h^0\left(\mathcal{O}_{\mathbb{P}^1}\left(1 + \left\lfloor \frac{-3i}{n} \right\rfloor\right)\right) = \left\lfloor \frac{n}{3} \right\rfloor + 2, \end{aligned}$$

and thus $r(D) = r(nP) - 1 = \lfloor \frac{n}{3} \rfloor$. Explicitly writing $\lfloor \frac{n}{3} \rfloor$ in terms of g according to (4.2) leads to the desired upper bound for $\operatorname{rk}\mathcal{K}$. □

5. *Proof of the main theorems*

In this section, we give the proof of Theorem 1.1 and Corollary 1.2.

Proof of Theorem 1.1. We say that a ramified covering $\pi\colon C \to \mathbb{P}^1$ is *simply ramified* at $p \in C$ if p is the only ramification point on its fiber and its ramification index is 2.

We first show that for any subset $\{p_1, \ldots, p_g\}$ of g distinct points of C there is a covering $\pi\colon C \to \mathbb{P}^1$ simply ramified at each $p_1, \ldots, p_g$. To this aim, we fix an embedding $C \hookrightarrow \mathbb{P}^n = \mathbb{P}(H^0(C, L))$ given by a complete linear system $|L|$ of degree $d \geqslant 5g + 3$ and consider morphisms $\pi_H\colon C \to \mathbb{P}^1$ given by projection from a linear subspace $H \subset \mathbb{P}^n$ of codimension 2 and disjoint from C.

The condition on the degree assures that $h^0(C, L(-D)) = h^0(C, L) - \deg D$ for any effective divisor D with $\deg D \leqslant 3g + 2$. In particular, for any $p \in C$ the tangent line $L_p = T_pC$ and the osculating plane Π_p are given by

$$L_p = \mathbb{P}\Big(H^0(C, L(-2p))^\perp\Big) \cong \mathbb{P}^1 \qquad \text{and} \qquad \Pi_p = \mathbb{P}\Big(H^0(C, L(-3p))^\perp\Big) \cong \mathbb{P}^2,$$

where $\perp$ denotes the annihilator inside $H^0(C, L)^\vee$. Moreover, for any distinct $p_1, \ldots, p_g, p \in C$ the osculating planes $\Pi_{p_1}, \ldots, \Pi_{p_g}$ and the tangent line L_p are independent, in the sense that the linear span

$$\langle \Pi_{p_1}, \ldots, \Pi_{p_g}, L_p \rangle \subset \mathbb{P}^n$$

has dimension $3g + 2$, the maximal possible.

Consider now a linear subspace $H \subset \mathbb{P}^n$ of codimension 2 and disjoint from C. Then π_H is ramified at p if and only if $L_p \subseteq \langle p, H\rangle$, (that is, if $L_p \cap H \neq \emptyset$) and the ramification index is exactly 2 if and only if $\Pi_p \not\subset \langle p, H\rangle$ (that is, if $\Pi_p \cap H = L_p \cap H$). On the other hand, it holds $\pi_H(p) = \pi_H(q)$ for $p \neq q$ if and only if $\langle p, H\rangle = \langle q, H\rangle$, or equivalently H intersects the line $\overline{pq}$.

Let now $p_1, \dots, p_g \in C$ be arbitrary distinct points and for each $i = 1, \dots, g$ pick $q_i \in L_{p_i}, q_i \neq p_i$. It is now easy to show that the set of codimension-2 linear subspaces H containing $q_1, \dots, q_g$ and such that π_H is ramified at each p_i with index 2 and $\pi_H(p_i) \neq \pi_H(p_j)$ for $i \neq j$ form a Zariski-open subset of the Grassmannian $\mathbb{G}$ of codimension-2 subspaces containing the $q_1, \dots, q_g$. It remains to achieve the simple ramification at each $p_1, \dots, p_g$, that is, no other ramification point has the same image as any p_i. By the above discussion, the covering π_H is ramified at another given point $p \in C$ if and only $L_p \cap H \neq \emptyset$, which is a codimension-2 condition on $\mathbb{G}$ (because of the condition $\deg L \geqslant 5g + 3$). By moving $p \in C$, we see that the set of 'bad' subspaces H (such that π_H is not simply ramified at $p_1, \dots, p_g$) has codimension at least 1 in $\mathbb{G}$, hence a general $H \in \mathbb{G}$ defines a covering simply ramified at $p_1, \dots, p_g$, as wanted.

Suppose now in addition that the points $p_1, \dots, p_g$ form a rigid divisor on C (which happens for a set of g points in general position on C) and pick a covering $\pi\colon C \to \mathbb{P}^1$ as above, simply ramified at $p_1, \dots, p_g$. Denote by $b_1 = \pi(p_1), \dots, b_g = \pi(p_g), b_{g+1}, \dots, b_k$ the branch points of π. To finish the proof, we construct a one-dimensional family $f\colon \mathcal{C} \to B$ of deformations of C over a quasi-projective base B, moving r of the branch points $b_1, \dots, b_g$ as follows.

For $i = 1, \dots, r$, let $\Delta_i \subset \mathbb{P}^1$ be a disk centered in b_i, small enough so that $\Delta_i \cap \Delta_j = \emptyset$ for $i \neq j$, and also $b_j \notin \Delta_i$ for $i = 1, \dots, r$ and $j = r+1, \dots, k$. By the Riemann-existence theorem, for any $t = (t_1, \dots, t_r) \in \prod_{i=1}^r \Delta_i = \Delta$ there is a covering $\pi_t : C_t \to \mathbb{P}^1$ branched on $\{t_1, t_2, \dots, t_r, q_{r+1}, \dots, q_k\}$ with the same ramification data as π. These coverings vary holomorphically over the polydisk Δ, and thus define an r-dimensional family $f : \mathcal{C} \to \Delta$ with $f^{-1}(t) = C_t$. Because of monodromy reasons, this family might not extend automatically over the quasi-projective variety $X = (\mathbb{P}^1)^r \setminus Z$, where

$$Z = \{(t_1, \dots, t_r) \in (\mathbb{P}^1)^r \mid t_i = t_j \text{ or } t_i = b_j \text{ for some } i \neq j\}$$

is the set where two branch points collide. We can anyway extend it over any simply connected open set containing Δ and so in particular over the universal covering $\psi : \tilde{X} \to X$, which is, however, not quasi-projective.

Nevertheless, for given $t \in X$ there are only finitely many coverings (up to isomorphism) branched over $\{t_1, t_2, \dots, t_r, b_{r+1}, \dots, b_k\}$, and the fundamental group $\pi_1(X)$ acts naturally on this finite set. The kernel G of the induced group homomorphism $\rho : \pi_1(X) \to \Sigma_N$ into the symmetric group Σ_N (for some appropriate N) has therefore finite index in $\pi_1(X)$ and is independent of $t \in X$ general. The family over $\tilde{X}$ induces thus a family $f\colon \mathcal{C} \to Y$ over $Y = \tilde{X}/G$, which is a finite covering of the quasi-projective variety X, hence quasi-projective itself.

To finish the proof, we consider a quasi-projective curve $B \subset Y$ through a point t_0 of Y above $(b_1, \dots, b_r) \in X$ corresponding to π, and transverse to the 'coordinate hypersurfaces' $\{t_i = b_i\}$. Possibly after a finite base change this family can be extended to a semistable one over a projective base. The fact that $\operatorname{rk} \mathcal{K} = g - r$ follows directly from Lemmas 2.3 and 2.2. Indeed, Lemma 2.3 shows that for $t_0 \in B$ the infinitesimal deformation is minimally supported on $D = p_1 + \cdots + p_r$, hence in particular is not isotrivial (see Remark 2.4). By construction of π, the divisor D is rigid and Lemma 3.3 gives both $\operatorname{rk} \mathcal{K} = g - r$ and $\operatorname{rk} \mathcal{U} \leqslant \frac{g+1}{2}$. □

Proof of Corollary 1.2. The proof is a straightforward application of Theorem 1.1 together with the following argument about the monodromy of the unitary flat summand. Since C is a smooth curve with simple Jacobian variety $J(C)$, the unitary flat bundle $\mathcal{U}$ of any one-dimensional family $f : \mathcal{C} \to B$ through C must be either zero or have infinite monodromy. Otherwise, $\mathcal{U}$ would become trivial after a finite étale base change, defining an abelian subvariety of $J(C)$ and contradicting its simplicity. However, the family $f : \mathcal{C} \to B$ as constructed in the proof of Theorem 1.1 is minimally supported on a relatively rigid divisor. Lemma 3.4 implies that $\mathcal{U}$ has finite monodromy, hence it must be zero by the above discussion. □

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