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CERTAIN SUBCLASSES OF ANALYTIC FUNCTION BY SĂLĂGEAN q-DIFFERENTIAL OPERATOR

DILEEP L.[1*], DIVYA RASHMI S. V.[1], §

Abstract. The theory of q-analysis has many applications in various sub-fields of mathematics and quantum physics. In the present article, we define the class $\mathcal{T}_n(\alpha, \lambda; q)$ using the Sălăgean q-differential operator. For functions belonging to this class we obtain coefficient estimates, extreme points and integral preserving properties .

Keywords: Univalent functions, Sălăgean, q-derivative, coefficient estimate.

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1. Introduction

The class of all analytic univalent functions denoted by $\mathcal{A}$ is of the form

$$f(z) = z + \sum_{m=2}^{\infty} a_m z^m \tag{1}$$

defined in the unit disc $\mathbb{U} = \{z : |z| < 1\}$.
Let $\mathcal{T}$ denote the subclass of $\mathcal{A}$ in $\mathbb{U}$, consisting of analytic functions whose non-zero coefficients from the second onwards are negative. That is, an analytic function $f \in \mathcal{T}$ if it has a Taylor expansion of the form

$$f(z) = z - \sum_{m=2}^{\infty} a_m z^m \quad (a_m \geq 0) \tag{2}$$

which are univalent in the open disc $\mathbb{U}$.
For functions $f \in \mathcal{A}$ of the form (1), Govindaraj and S Sivasubramanian [2] introduced the following operator $\mathcal{S}_q^n$ which is called as Sălăgean q-differential operator.

$$\mathcal{S}_q^0 f(z) = f(z), \qquad \mathcal{S}_q^1 f(z) = z\partial_q f(z), \quad \cdots, \qquad \mathcal{S}_q^n f(z) = z\partial_q(\mathcal{S}_q^{n-1} f(z)).$$

A simple calculation implies

$$\mathcal{S}_q^n f(z) = f(z) * G_{q,n}(z),$$

[1] Department of Mathematics, Vidyavardhaka College of Engineering, Mysuru, India.
e-mail: Dileepl84@gmail.com; ORCID: https://orcid.org/0000-0002-1059-0118.
[*] Corresponding author.
e-mail: rashmi.divya@gmail.com; ORCID: https://orcid.org/0000-0002-2712-9168.
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where

$$G_{q,n}(z) = z + \sum_{m=2}^{\infty} [m]_q^n z^m, \quad (n \in \mathbb{N}),$$

where $[m]_q = \dfrac{1-q^m}{1-q}$.
The power series of $\mathcal{S}_q^n f(z)$ for functions $f \in \mathcal{A}$ of the form (1), is given by

$$\mathcal{S}_q^n f(z) = z + \sum_{m=2}^{\infty} [m]_q^n a_m z^m. \tag{3}$$

Note that

$$\lim_{q \to 1-} G_{q,n}(z) = z + \sum_{m=2}^{\infty} m^n z^m$$

and

$$\lim_{q \to 1-} \mathcal{S}_q^n f(z) = f(z) * \left(z + \sum_{m=2}^{\infty} m^n z^m \right)$$

which is the familiar Sălăgean derivative [5].

Now we define the following subclass of $\mathcal{T}$ by using Sălăgean q-differential operator.

Let $\mathcal{T}_n(\alpha, \lambda; q)$ be the subclass of $\mathcal{T}$ consisting of functions which satisfy the conditions

$$\Re \left\{ \frac{z(\mathcal{S}_q^n f)'}{\lambda z(\mathcal{S}_q^n f)' + (1-\lambda)\mathcal{S}_q^n f} \right\} > \alpha, \tag{4}$$

for some $\alpha,\ \lambda \quad (0 \le \alpha, \lambda < 1)$ and $n \in \mathbf{N}_0$.
If $q \to 1$, we get the class studied by Dileep L and Latha S [1]. For different parametric values of n and $q \to 1$ we get the classes studied by Mostafa [3].

2. Main Results

Theorem 2.1. *A function f defined by (1.2) is in the class $\mathcal{T}_n(\alpha, \lambda; q)$ if and only if*

$$\sum_{m=2}^{\infty} [m]_q^n a_m [m - \alpha + \alpha\lambda - \alpha\lambda m] < 1 - \alpha, \tag{5}$$

where $\alpha,\ \lambda \quad (0 \le \alpha, \lambda < 1)$ and $n \in \mathbf{N}_0$.

Proof. Suppose $f \in \mathcal{T}_n(\alpha, \lambda; q)$. Then

$$\Re \left\{ \frac{z(\mathcal{S}_q^n f)'}{\lambda z(\mathcal{S}_q^n f)' + (1-\lambda)\mathcal{S}_q^n f} \right\} > \alpha,$$

$$\Re \left\{ \frac{z - \sum_{m=2}^{\infty} m[m]_q^n a_m z^m}{\lambda \left[z - \sum_{m=2}^{\infty} [m]_q^n m a_m z^m \right] + (1-\lambda) \left[z - \sum_{m=2}^{\infty} [m]_q^n a_m z^m \right]} \right\} > \alpha.$$

$$\Re \left\{ \frac{z - \sum_{m=2}^{\infty} m[m]_q^n a_m z^m}{z - \sum_{m=2}^{\infty} [m]_q^n a_m z^m \left[\lambda(m-1) + 1 \right]} \right\} > \alpha.$$

Letting $z \to 1$, then we get,

$$1-\sum_{m=2}^{\infty}[m]_q^n a_m m > \alpha\left\{1-\sum_{m=2}^{\infty}[m]_q^n a_m\left[\lambda(m-1)+1\right]\right\}$$

$$\sum_{m=2}^{\infty}[m]_q^n a_m m - \alpha\sum_{m=2}^{\infty}[m]_q^n a_m\left[\lambda(m-1)+1\right] < (1-\alpha)$$

$$\sum_{m=2}^{\infty}[m]_q^n a_m\left[m-\alpha\lambda m+\alpha\lambda-\alpha\right] < (1-\alpha).$$

Conversely, assume that (5) be true. We have to show that (4) is satisfied or equivalently

$$\left|\left\{\frac{z(\mathcal{S}_q^n f)'}{\lambda z(\mathcal{S}_q^n f)'+(1-\lambda)\mathcal{S}_q^n f}\right\}-1\right| < 1-\alpha.$$

But

$$\left|\left\{\frac{z-\sum_{m=2}^{\infty}m[m]_q^n a_m z^m}{z-\sum_{m=2}^{\infty}[m]_q^n a_m z^m\left[\lambda(m-1)+1\right]}\right\}-1\right| =$$

$$\left|\frac{\sum_{m=2}^{\infty}[m]_q^n a_m(m-1)(\lambda-1)z^m}{z-\sum_{m=2}^{\infty}[m]_q^n a_m[\lambda(m-1)+1]z^m}\right|$$

$$\le \frac{\sum_{m=2}^{\infty}[m]_q^n a_m(m-1)(\lambda-1)|z^m|}{|z|-\sum_{m=2}^{\infty}[m]_q^n a_m[\lambda(m-1)+1]|z^m|}$$

$$\le \frac{\sum_{m=2}^{\infty}[m]_q^n a_m(m-1)(\lambda-1)}{1-\sum_{m=2}^{\infty}[m]_q^n a_m[\lambda(m-1)+1]}.$$

The last expression is bounded above by $1-\alpha$ if

$$\sum_{m=2}^{\infty}[m]_q^n a_m(m-1)(\lambda-1) \le (1-\alpha)\left(1-\sum_{m=2}^{\infty}[m]_q^n a_m[\lambda(m-1)+1]\right),$$

or

$$\sum_{m=2}^{\infty}[m]_q^n a_m[m-\alpha+\alpha\,\lambda-\alpha\,\lambda m] < 1-\alpha,$$

which is true by hypothesis. This completes the assertion of Theorem 2.1.

□

Corollary 2.1. *If* $f \in \mathcal{T}_n(\alpha,\lambda;q)$ *then*

$$|a_m| \le \frac{1-\alpha}{[m]_q^n\left[m-\alpha\lambda m+\alpha\lambda-\alpha\right]}.$$

Theorem 2.2. *Let* $0 \leq \alpha < 1, \quad 0 \leq \lambda_1 \leq \lambda_2 < 1, \quad n \in \mathbb{N}_0,$ *then* $\mathcal{T}_n(\alpha, \lambda_2; q) \subset \mathcal{T}_n(\alpha, \lambda_1; q)$.

Proof. From Theorem 2.1,

$$\sum_{m=2}^{\infty} [m]_q^n \left[m - \alpha\lambda_2 m + \alpha\lambda_2 - \alpha\right] a_m$$

$$\leq \sum_{m=2}^{\infty} [m]_q^n \left[m - \alpha\lambda_1 m + \alpha\lambda_1 - \alpha\right] a_m$$

$$\leq (1-\alpha).$$

For $f(z) \in \mathcal{T}_n(\alpha, \lambda_2; q)$. Hence $f(z) \in \mathcal{T}_n(\alpha, \lambda_1; q)$. □

Theorem 2.3. *Let* $f(z) \in \mathcal{T}_n(\alpha, \lambda; q)$. *Define* $f_1(z) = z$ *and*

$$f_m(z) = z + \frac{1-\alpha}{[m]_q^n \left[m - \alpha\lambda m + \alpha\lambda - \alpha\right]} z^m, \quad m = 2, 3, \cdots,$$

for some α, $\lambda\,(0 \leq \lambda < 1), n \in \mathbb{N}_0$ *and* $z \in \mathbb{U}$. *Then* $f \in \mathcal{T}_n(\alpha, \lambda; q)$ *if and only if* f *can be expressed as* $f(z) = \sum_{m=1}^{\infty} \mu_m f_m(z)$ *where* $\mu_m \geq 0$ *and* $\sum_{m=1}^{\infty} \mu_m = 1$.

Proof. If $f(z) = \sum_{m=1}^{\infty} \mu_m f_m(z)$ with $\sum_{m=1}^{\infty} \mu_m = 1, \quad \mu_m \geq 0$, then

$$\sum_{m=2}^{\infty} \frac{[m]_q^n \left[m - \alpha\lambda m + \alpha\lambda - \alpha\right] \mu_m}{[m]_q^n \left[m - \alpha\lambda m + \alpha\lambda - \alpha\right]} (1-\alpha)$$

$$= \sum_{m=2}^{\infty} \mu_m (1-\alpha) = (1-\mu_1)(1-\alpha)$$

$$\leq (1-\alpha).$$

Hence $f \in \mathcal{T}_n(\alpha, \lambda; q)$.

Conversely, let $f(z) = z - \sum_{m=2}^{\infty} a_m z^m \in \mathcal{T}_n(\alpha, \lambda; q)$, define

$$\mu_m = \frac{[m]_q^n \left[m - \alpha\lambda m + \alpha\lambda - \alpha\right] |a_m|}{(1-\alpha)}, \quad m = 2, 3, \cdots,$$

and define $\mu_1 = 1 - \sum_{m=2}^{\infty} \mu_m$. From Theorem 2.1, $\sum_{m=2}^{\infty} \mu_m \leq 1$ and so $\mu_1 \geq 0$.
Since $\mu_m f_m(z) = \mu_m f + a_m z^m$,

$$\sum_{m=1}^{\infty} \mu_m f_m(z) = z - \sum_{m=2}^{\infty} a_m z^m = f(z).$$ □

Theorem 2.4. *The class* $\mathcal{T}_n(\alpha, \lambda; q)$ *is closed under convex linear combination.*

Proof. Let f, $g \in \mathcal{T}_n(\alpha, \lambda; q)$ and let

$$f(z) = z - \sum_{m=2}^{\infty} a_m z^m, \quad g(z) = z - \sum_{m=2}^{\infty} b_m z^m.$$

For η such that $0 \leq \eta \leq 1$, it suffices to show that the function defined by $h(z) = (1-\eta)f(z) + \eta g(z), \quad z \in \mathbb{U}$ belongs to $\mathcal{T}_n(\alpha, \lambda; q)$. Now

$$h(z) = z - \sum_{m=2}^{\infty} [(1-\eta)a_m + \eta b_m] z^m.$$

Applying Theorem 2.1, to $f, \ g \in \mathcal{T}_n(\alpha, \lambda; q)$, we have

$$\begin{aligned}
&\sum_{m=2}^{\infty} [m]_q^n \left[m - \alpha\lambda m + \alpha\lambda - \alpha\right] [(1-\eta)a_m + \eta b_m] \\
&\quad = (1-\eta) \sum_{m=2}^{\infty} [m]_q^n \left[m - \alpha\lambda m + \alpha\lambda - \alpha\right] a_m + \eta \sum_{m=2}^{\infty} [m]_q^n \left[m - \alpha\lambda m + \alpha\lambda - \alpha\right] b_m \\
&\quad \leq (1-\eta)(1-\alpha) + \eta(1-\alpha) = (1-\alpha).
\end{aligned}$$

This implies that $h \in \mathcal{T}_n(\alpha, \lambda)$. □

Corollary 2.2. *If $f_1(z), \ f_2(z)$ are in $\mathcal{T}_n(\alpha, \lambda; q)$ then the function defined by $g(z) = \frac{1}{2}[f_1(z) + f_2(z)]$ is also in $\mathcal{T}_n(\alpha, \lambda; q)$.*

Theorem 2.5. *Let for $j = 1, 2, \cdots, m, \quad f_j(z) = z - \sum_{m=2}^{\infty} a_{m,j} z^m \in \mathcal{T}_n(\alpha, \lambda; q)$ and $0 < \lambda_j < 1$ such that $\sum_{j=1}^{m} \lambda_j = 1$, then the function $F(z)$ defined by $F(z) = \sum_{j=1}^{m} \lambda_j f_j(z)$ is also in $\mathcal{T}_n(\alpha, \lambda; q)$.*

Proof. For each $j \in \{1, 2, 3, \cdots, m\}$ we obtain

$$\sum_{m=2}^{\infty} [m]_q^n \left[m - \alpha\lambda m + \alpha\lambda - \alpha\right] |a_m| < (1-\alpha).$$

Since
$$\begin{aligned}
F(z) &= \sum_{j=1}^{m} \lambda_j (z - \sum_{m=2}^{\infty} a_{m,j} z^m) \\
&= z - \sum_{m=2}^{\infty} (\sum_{j=1}^{m} \lambda_j a_{m,j}) z^m.
\end{aligned}$$

$$\begin{aligned}
&\sum_{m=2}^{\infty} [m]_q^n \left[m - \alpha\lambda m + \alpha\lambda - \alpha\right] \left[\sum_{j=1}^{m} \lambda_j a_{m,j}\right] \\
&\quad = \sum_{j=1}^{m} \lambda_j \left[\sum_{m=2}^{\infty} [m]_q^n \left[m - \alpha\lambda m + \alpha\lambda - \alpha\right]\right] \\
&\quad < \sum_{j=1}^{m} \lambda_j (1-\alpha) < (1-\alpha).
\end{aligned}$$

Therefore $F(z) \in \mathcal{T}_n(\alpha, \lambda; q)$. □

Theorem 2.6. *Let $f(z) \in \mathcal{T}_n(\alpha, \lambda; q)$. Komato operator of f is defined by*

$$k(z) = \int_0^1 \frac{(c+1)^\gamma}{\Gamma(\gamma)} t^c \left(\log \frac{1}{t}\right)^{\gamma-1} \frac{f(tz)}{t} dt,$$

$c > -1, \quad \gamma \geq 0$ *then* $k(z) \in \mathcal{T}_n(\alpha, \lambda; q)$.

Proof. We have

$$\int_0^1 t^c \left(\log \frac{1}{t}\right)^{\gamma-1} dt = \frac{\Gamma(\gamma)}{(c+1)^\gamma}$$
$$\int_0^1 t^{m+c-1} \left(\log \frac{1}{t}\right)^{\gamma-1} dt = \frac{\Gamma(\gamma)}{(c+1)^\gamma}, \quad m = 2, 3, \cdots,$$

$$k(z) = \frac{(c+1)^\gamma}{\Gamma(\gamma)} \left[\int_0^1 t^c \left(\log \frac{1}{t}\right)^{\gamma-1} z dt - \sum_{m=2}^{\infty} z^m \int_0^1 a_m t^{m+c-1} \left(\log \frac{1}{t}\right)^{\gamma-1} dt\right]$$
$$= z - \sum_{m=2}^{\infty} \left(\frac{c+1}{c+m}\right)^\gamma a_m z^m.$$

Since $f \in \mathcal{T}_n(\alpha, \lambda; q)$ and since $\left(\frac{c+1}{c+m}\right)^\gamma < 1$, we have

$$\sum_{m=2}^{\infty} [m]_q^n \left[m - \alpha\lambda m + \alpha\lambda - \alpha\right] \left(\frac{c+1}{c+m}\right)^\gamma a_m < (1-\alpha).$$

□

Theorem 2.7. *Let* $f \in \mathcal{T}_n(\alpha, \lambda; q)$, *then for every* $0 \leq \delta < 1$ *the function*

$$\mathcal{H}_\delta(z) = (1-\delta) f(z) + \delta \int_0^z \frac{f(t)}{t} dt.$$

Proof. We have $\mathcal{H}_\delta(z) = z - \sum_{m=2}^{\infty} \left(1 + \frac{\delta}{m} - \delta\right) a_m z^m$.

Since $\left(1 + \frac{\delta}{m} - \delta\right) < 1, \quad m \geq 2$, so by Theorem 2.1,

$$\sum_{m=2}^{\infty} \left(1 + \frac{\delta}{m} - \delta\right) [m]_q^n \left[m - \alpha\lambda m + \alpha\lambda - \alpha\right] a_m$$
$$< \sum_{m=2}^{\infty} [m]_q^n \left[m - \alpha\lambda m + \alpha\lambda - \alpha\right] a_m$$
$$< (1-\alpha).$$

Therefore $\mathcal{H}_\delta(z) \in \mathcal{T}_n(\alpha, \lambda; q)$. □

3. Conclusions

Here, in our present investigation, we have successfully introduced a new subclass of analytic functions $\mathcal{T}_n(\alpha, \lambda; q)$ using the Sălăgean q- differential operator. Many properties and characteristics of this newly-defined function class such as coefficient estimates, extreme points, integral theorem have been studied.

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Dr. Dileep L. completed his MSc in 2007 and his PhD. in 2015 at the University of Mysore, Mysuru. Currently he is working as an Associate Professor in the Department of Mathematics, Vidyavardhaka College of Engineering, Mysuru. His areas of interests include Complex analysis, Graph Theory and Number Theory.

Dr. Divya Rashmi S. V. completed her MSc. in 2005 at Kuvempu University, Shimoga, M. Phil in 2010 at Venkateshwara university, Thirupathi, Andhra Pradesh and her PhD. in 2017 at Kalasalingam University, Krishnankoil, TN. Currently, she is working as an Assistant Professor in the Department of Mathematics, Vidyavardhaka College of Engineering, Mysuru. Her areas of interests include Complex analysis, Graph Theory and Number Theory.