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ON A REGULARIZED SOLUTION OF THE CAUCHY PROBLEM FOR MATRIX FACTORIZATIONS OF THE HELMHOLTZ EQUATION

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Abstract. In this paper, we consider the problem of recovering solutions for matrix factorizations of the Helmholtz equation in a multidimensional bounded domain from their values on a part of the boundary of this domain, i.e., the Cauchy problem. An approximate solution to this problem is constructed based on the Carleman matrix method.

Keywords: Ill-Posed Cauchy Problems, regularized solution, approximate solution, matrix factorization, elliptical system.

AMS Subject Classification: 35J46, 35J56

1. Introduction

Many scientific and applied problems, studied at the world level, in many cases are reduced to the study of ill-posed boundary value problems for partial differential equations. Applied research on conditional correctness and construction of an approximate solution for given values on a part of the boundary of the region, for equations of elliptical type, are especially important in hydrodynamics, geophysics and electrodynamics. The study of a family of regularizing solutions to ill-posed problems served as an impetus for the beginning of studies of the well-posedness class when narrowed to a compact set. Therefore, the study of ill-posed problems for linear elliptic systems of the first order is one of the topical problems in the theory of partial differential equations. At present, in the world, in the study of ill-posed boundary value problems for linear elliptic systems of the first order,

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the construction of a regularized solution plays a special role. The Cauchy problem for elliptic equations is ill-posed (example Hadamard, see for instance [18], p. 39).

At present, special attention is paid to topical aspects of differential equations and mathematical physics, which have scientific and practical applications in the fundamental sciences. In particular, special attention is paid to the study of various ill-posed boundary value problems for partial differential equations of elliptic type, which have practical application in applied sciences. As a result, significant results were obtained in studies of ill-posed boundary value problems for partial differential equations, that is, approximate solutions were constructed using Carleman matrices in explicit form from approximate data in special domains, estimates of conditional stability and solvability criteria were established. The first results, from the point of view of practical importance, for ill-posed problems and for reducing the class of possible solutions to a compact set and reducing problems to stable ones were obtained in the works of A.N. Tikhonov (see [2]). In the works of M.M. Lavrent'ev, estimates were obtained that characterize the stability of the spatial problem in the class of bounded solutions of the Cauchy problem for the Laplace equation and some other ill-posed problems of mathematical physics in a straight cylinder, as well as for an arbitrary spatial domain with a sufficiently smooth boundary (see, for instance [20]-[21]).

In this work, based on the results of works [20]-[21], [27]-[30], based on the Cauchy problem for the Laplace and Helmholtz equations, an explicit Carleman matrix was constructed and, on its basis, a regularized solution of the Cauchy problem for the matrix factorization of the Helmholtz equation. In work [32], the calculation of double integrals with the help of some connection between wave equation and ODE system was considered.

The problem of reconstructing the solution for matrix factorization of the Helmholtz equation (see, for instance [3], [4], [5], [6], [7], [8], [9], [10], [11], [12] and [13]), is one of the topical problems in the theory of differential equations.

At present, there is still interest in classical ill-posed problems of mathematical physics. This direction in the study of the properties of solutions of Cauchy problem for Laplace equation was started in [31], [20]-[21], [1], [27]-[30] and subsequently developed in [14]-[15], [19], [23]-[24], [17], [3]-[13].

2. Basic information and statement of the Cauchy problem

Let $\mathbb{R}^m$, $(m = 2k,\ k \geq 1)$ be a $m-$dimensional real Euclidean space,

$$x = (x_1, ..., x_m) \in \mathbb{R}^m,\ y = (y_1, ..., y_m) \in \mathbb{R}^m,$$

$$x' = (x_1, ..., x_{m-1}) \in \mathbb{R}^{m-1},\ y' = (y_1, ..., y_{m-1}) \in \mathbb{R}^{m-1}.$$

We introduce the following notation:

$$r = |y - x|\,, \alpha = \left|y' - x'\right|, w = i\tau\sqrt{u^2 + \alpha^2} + \beta,\ w_0 = i\tau\alpha + \beta,$$

$$\beta = \tau y_m,\ \tau = tg\frac{\pi}{2\rho},\ \rho > 1,\ u \geq 0,\ s = \alpha^2,$$

$$G_\rho = \left\{y:\ \left|y'\right| < \tau y_m,\, y_m > 0\right\},\ \partial G_\rho = \left\{y:\ \left|y'\right| = \tau y_m,\, y_m > 0\right\},$$

$$\frac{\partial}{\partial x} = \left(\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_m}\right)^T,\ \frac{\partial}{\partial x} = \xi^T,\ \xi^T = \begin{pmatrix} \xi_1 \\ ... \\ \xi_m \end{pmatrix} \text{- transposed vector } \xi,$$

$$U(x) = (U_1(x), ..., U_n(x))^T,\ u^0 = (1, ..., 1) \in \mathbb{R}^n,\ n = 2^m,\ m \geq 2,$$

$$E(z) = \left\| \begin{matrix} z_1 \dots 0 \\ \dots\dots\dots \\ 0 \dots z_n \end{matrix} \right\| \text{- diagonal matrix, } z = (z_1, \dots, z_n) \in \mathbb{R}^n.$$

$G_\rho \subset \mathbb{R}^m$, $(m = 2k,\ k \geq 1)$ be a bounded simply-connected domain, the boundary of which consists of the surface of the cone ∂G_ρ, and a smooth piece of the surface S, lying in the cone G_ρ, i.e., $\partial G_\rho = S \bigcup T,\ T = \partial G_\rho \backslash S$. Let $(0, 0, \dots, x_m) \in G_\rho,\ x_m > 0$.

Let $D(\xi^T)$ be a $(n \times n)-$ dimensional matrix with elements consisting of a set of linear functions with constant coefficients of the complex plane for which the following condition is satisfied:

$$D^*(\xi^T)D(\xi^T) = E((|\xi|^2 + \lambda^2)u^0),$$

where $D^*(\xi^T)$ is the Hermitian conjugate matrix $D(\xi^T)$, $\lambda-$ is a real number.

We consider a system of differential equations in the region G

$$D\left(\frac{\partial}{\partial x}\right)U(x) = 0, \tag{1}$$

where $D\left(\frac{\partial}{\partial x}\right)$ is the matrix of first-order differential operators.

We denote by $A(G_\rho)$ the class of vector functions in the domain G_ρ continuous on $\overline{G}_\rho = G_\rho \bigcup \partial G_\rho$ and satisfying system (1).

3. Construction of the Carleman matrix and the Cauchy problem

Formulation of the problem. Suppose $U(y) \in A(G_\rho)$ and

$$U(y)|_S = f(y),\ y \in S. \tag{2}$$

Here, $f(y)$ a given continuous vector-function on S. It is required to restore the vector function $U(y)$ in the domain G_ρ, based on it's values $f(y)$ on S.

If $U(y) \in A(G_\rho)$, then the following integral formula of Cauchy type is valid

$$U(x) = \int\limits_{\partial G_\rho} N(y, x; \lambda)U(y)ds_y,\ \ x \in G, \tag{3}$$

where

$$N(y, x; \lambda) = \left(E\left(\varphi_m(\lambda r)u^0\right)D^*\left(\frac{\partial}{\partial x}\right)\right)D(t^T).$$

Here $t = (t_1, \dots, t_m)-$is the unit exterior normal, drawn at a point y, the surface ∂G_ρ, $\varphi_m(\lambda r)-$ is the fundamental solution of the Helmholtz equation in $\mathbb{R}^m$, $(m = 2k,\ k \geq 1)$, where $\varphi_m(\lambda r)$ defined by the following formula:

$$\begin{gathered} \varphi_m(\lambda r) = P_m \lambda^{(m-2)/2} \frac{H^{(1)}_{(m-2)/2}(\lambda r)}{r^{(m-2)/2}}, \\ P_m = \frac{1}{2i(2\pi)^{(m-2)/2}},\ m = 2k,\ k \geq 1. \end{gathered} \tag{4}$$

Here $H^{(1)}_{(m-2)/2}(\lambda r)-$ is the Hankel function of the first kind of $(m-2)/2-$ th order (see, for instance [25]).

We denote by $K(w)$ is an entire function taking real values for real w, $(w = u + iv,\ u, v-$real numbers) and satisfying the following conditions:

$$K(u) \neq 0,\ \sup_{v\geq 1} \left| v^p K^{(p)}(w) \right| = B(u,p) < \infty, \qquad (5)$$

$$-\infty < u < \infty,\ p = 0, 1, ..., m.$$

We define the function $\Phi(y, x; \lambda)$ at $y \neq x$ by the following equality

$$\Phi(y, x; \lambda) = \frac{1}{c_m K(x_m)} \frac{\partial^{k-1}}{\partial s^{k-1}} \int_0^\infty \operatorname{Im} \left[\frac{K(w)}{w - x_m} \right] \frac{u I_0(\lambda u)}{\sqrt{u^2 + \alpha^2}} du, \qquad (6)$$

$$m = 2k,\ k \geq 1,$$

where $c_2 = -2\pi$, $c_m = (-1)^{k-1}(m-2)(k-1)!\omega_m$; $I_0(\lambda u) = J_0(i\lambda u)-$is the Bessel function of the first kind of zero order (see, [1]), ω_m- area of a unit sphere in space $\mathbb{R}^m$.

In the formula (6), choosing

$$K(w) = E_\rho(\sigma^{1/\rho} w),\ K(x_m) = E_\rho(\sigma^{1/\rho}\gamma),\ \gamma = \tau x_m,\ \sigma > 0, \qquad (7)$$

we get

$$\Phi_\sigma(y, x; \lambda) = \frac{E_\rho(\sigma^{1/\rho}\gamma)}{c_m} \frac{\partial^{k-1}}{\partial s^{k-1}} \int_0^\infty \operatorname{Im} \left[\frac{E_\rho(\sigma^{1/\rho} w)}{w - x_m} \right] \frac{u I_0(\lambda u)}{\sqrt{u^2 + \alpha^2}} du. \qquad (8)$$

Here $E_\rho(\sigma^{1/\rho} w)-$ is the entire Mittag-Leffler function (see, [22]). In [26], using the S-generalized beta function, a new generalization of the Mittag-Leffler function and its properties is presented.

The formula (3) is true if instead $\varphi_m(\lambda r)$ of substituting the function

$$\Phi_\sigma(y, x; \lambda) = \varphi_m(\lambda r) + g_\sigma(y, x; \lambda), \qquad (9)$$

where $g_\sigma(y, x)-$ is the regular solution of the Helmholtz equation with respect to the variable y, including the point $y = x$.

Then the integral formula has the form:

$$U(x) = \int_{\partial G_\rho} N_\sigma(y, x; \lambda) U(y) ds_y,\ \ x \in G, \qquad (10)$$

where

$$N_\sigma(y, x; \lambda) = \left(E\left(\Phi_\sigma(y, x; \lambda) u^0\right) D^*\left(\frac{\partial}{\partial x}\right)\right) D(t^T).$$

Recall the basic properties of the Mittag-Leffler function. The entire function of Mittag-Leffler is defined by a series.

$$\sum_{n=1}^{\infty} \frac{w^n}{\Gamma(1 + \rho^{-1} n)} = E_\rho(w),\ w = u + iv,$$

where $\Gamma(s)-$ is the Euler gamma function.

We denote by $\gamma_\varepsilon(\beta_0)(\varepsilon > 0,\ 0 < \beta_0 < \pi)$ the contour in the complex plane ζ, run in the direction of non-decreasing $\arg\zeta$ and consisting of the following parts:

1. The beam $\arg\zeta = -\beta_0,\ |\zeta| \geq \varepsilon$;
2. The arc $-\beta_0 < \arg\zeta < \beta_0$ of circle $|\zeta| = \varepsilon$;
3. The beam $\arg\zeta = \beta_0,\ |\zeta| \geq \varepsilon$.

The contour $\gamma_\varepsilon(\beta_0)$ divides the plane ζ into two unbounded simply connected domains G_ρ^- and G_ρ^+ lying to the left and to the right of $\gamma_\varepsilon(\beta_0)$, respectively.

Let $\rho > 1,\ \dfrac{\pi}{2\rho} < \beta_0 < \dfrac{\pi}{\rho}$.

Denote

$$\psi_\rho(w) = \frac{1}{2\pi i} \int\limits_{\gamma_\varepsilon(\beta_0)} \frac{\exp(\zeta^\rho)}{\zeta - w} d\zeta, \tag{11}$$

Then the following integral representations are valid:

$$E_\rho(w) = \psi_\rho(w), \quad z \in G_\rho^-, \tag{12}$$

$$E_\rho(w) = \rho \exp(w^\rho) + \psi_\rho(w), \quad z \in G_\rho^+, \tag{13}$$

From these formulas we find

$$\left.\begin{array}{l} |E_\rho(w)| \le \rho \exp(\operatorname{Re} w^\rho) + |\psi_\rho(w)|\,, \ \ |\arg w| \le \dfrac{\pi}{2\rho} + \eta_0, \\ |E_\rho(w)| \le |\psi_\rho(w)|\,, \ \ \dfrac{\pi}{2\rho} + \eta_0 \le |\arg w| \le \pi, \ \ \eta_0 > 0 \end{array}\right\} \tag{14}$$

$$|\psi_\rho(w)| \le \frac{M}{1 + |w|},\ M = const \tag{15}$$

$$E_\rho(w) \approx \rho \exp(w^\rho), \ \ w > 0, \ \ w \to \infty, \tag{16}$$

Further, since $E_\rho(w)$ is real with real w, then

$$\operatorname{Re} \psi_\rho(w) = \frac{\rho}{2\pi i} \int\limits_{\gamma_\varepsilon(\beta_0)} \frac{2\zeta - \operatorname{Re} w}{(\zeta - w)\zeta - \overline{w})} \exp(\zeta^\rho) d\zeta,$$

$$\operatorname{Im} \psi_\rho(w) = \frac{\rho \operatorname{Im}(w)}{2\pi i} \int\limits_{\gamma_\varepsilon(\beta_0)} \frac{\exp(\zeta^\rho)}{(\zeta - w)\zeta - \overline{w})} d\zeta,$$

The information given here concerning the function $E_\rho(w)$ is taken from (see, for instance [5] and [9]).

In what follows, to prove the main theorems, we need the following estimates for the function $\Phi_\sigma(y, x; \lambda$.

Lemma 3.1. *Let $x = (x_1, ..., x_m) \in G_\rho,\ y \ne x,\ \sigma \ge \lambda + \sigma_0,\ \sigma_0 > 0$, then*

1) at $\beta \le \alpha$ inequalities are satisfied

$$|\Phi_\sigma(y, x; \lambda)| \le C(\rho, \lambda) \frac{\sigma^{m-3}}{r^{m-2}} \exp(-\sigma\gamma^\rho)\,,\ \sigma > 1,\ x \in G_\rho, \tag{17}$$

$$\left|\frac{\partial \Phi_\sigma(y, x; \lambda)}{\partial y_j}\right| \le C(\rho, \lambda) \frac{\sigma^m}{r^{m-1}} \exp(-\sigma\gamma^\rho)\,,\ \sigma > 1,\ x \in G_\rho,\ j = 1, ..., m. \tag{18}$$

$$\left|\frac{\partial \Phi_\sigma(y, x; \lambda)}{\partial x_j}\right| \le C(\rho, \lambda) \frac{\sigma^m}{r^{m-1}} \exp(-\sigma\gamma^\rho)\,,\ \sigma > 1,\ x \in G_\rho,\ j = 1, ..., m. \tag{19}$$

2) at $\beta > \alpha$ inequalities are satisfied

$$|\Phi_\sigma(y, x; \lambda)| \le C(\rho, \lambda) \frac{\sigma^{m-3}}{r^{m-2}} \exp(-\sigma\gamma^\rho + \sigma \operatorname{Re} w_0^\rho)\,,\ \sigma > 1,\ x \in G_\rho, \tag{20}$$

$$\left|\frac{\partial \Phi_\sigma(y, x; \lambda)}{\partial y_j}\right| \le C(\rho, \lambda) \frac{\sigma^m}{r^{m-1}} \exp(-\sigma\gamma^\rho + \sigma \operatorname{Re} w_0^\rho)\,,\ \sigma > 1,\ x \in G_\rho,\ j = 1, ..., m. \tag{21}$$

$$\left|\frac{\partial \Phi_\sigma(y, x; \lambda)}{\partial x_j}\right| \le C(\rho, \lambda) \frac{\sigma^m}{r^{m-1}} \exp(-\sigma\gamma^\rho + \sigma \operatorname{Re} w_0^\rho)\,,\ \sigma > 1,\ x \in G_\rho,\ j = 1, ..., m. \tag{22}$$

Here $C(\rho, \lambda)$ is the function depending on ρ and λ.

For a fixed $x \in G_\rho$ we denote by S^* the part of S on which $\beta \geq \alpha$. If $x \in G_\rho$, then $S = S^*$ (in this case, $\beta = \tau y_m$ and the inequality $\beta \geq \alpha$ means that y lies inside or on the surface cone).

4. The continuation formula and regularization according to M.M. Lavrent'ev's

Theorem 4.1. *Let $U(y) \in A(G_\rho)$ it satisfy the inequality*

$$|U(y)| \leq M, \ y \in T = \partial G_\rho \backslash S^*. \tag{23}$$

If

$$U_\sigma(x) = \int\limits_{S^*} N_\sigma(y, x; \lambda) U(y) ds_y, \ \ x \in G_\rho, \tag{24}$$

then the following estimates are true

$$|U(x) - U_\sigma(x)| \leq M C_\rho(\lambda, x) \sigma^k \exp(-\sigma \gamma^\rho), \ \ \sigma > 1, \ x \in G_\rho. \tag{25}$$

$$\left| \frac{\partial U(x)}{\partial x_j} - \frac{\partial U_\sigma(x)}{\partial x_j} \right| \leq M C_\rho(\lambda, x) \sigma^k \exp(-\sigma \gamma^\rho), \ \ \sigma > 1, \ x \in G_\rho, \ j = 1, ..., m. \tag{26}$$

Here and below functions bounded on compact subsets of the domain G_ρ, we denote by $C_\rho(\lambda, x)$.

Proof. Let us first estimate inequality (25). Using the integral formula (10) and the equality (24), we obtain

$$U(x) = \int\limits_{S^*} N_\sigma(y, x; \lambda) U(y) ds_y + \int\limits_{\partial G_\rho \backslash S^*} N_\sigma(y, x; \lambda) U(y) ds_y =$$

$$= U_\sigma(x) + \int\limits_{\partial G_\rho \backslash S^*} N_\sigma(y, x; \lambda) U(y) ds_y, \ \ x \in G_\rho.$$

Taking into account the inequality (23), we estimate the following

$$\begin{aligned} |U(x) - U_\sigma(x)| &\leq \left| \int\limits_{\partial G_\rho \backslash S^*} N_\sigma(y, x; \lambda) U(y) ds_y \right| \leq \\ &\leq \int\limits_{\partial G_\rho \backslash S^*} |N_\sigma(y, x; \lambda)| \, |U(y)| \, ds_y \leq M \int\limits_{\partial G_\rho \backslash S^*} |N_\sigma(y, x; \lambda)| \, ds_y, \ \ x \in G_\rho. \end{aligned} \tag{27}$$

To do this, we estimate the integrals $\int\limits_{\partial G_\rho \backslash S^*} |\Phi_\sigma(y, x; \lambda)| \, ds_y$, $\int\limits_{\partial G_\rho \backslash S^*} \left| \frac{\partial \Phi_\sigma(y, x; \lambda)}{\partial y_j} \right| ds_y$ and $\int\limits_{\partial G_\rho \backslash S^*} \left| \frac{\partial \Phi_\sigma(y, x; \lambda)}{\partial y_m} \right| ds_y$ on the part $\partial G_\rho \backslash S^*$ of the plane $y_m = 0$ $(j = 1, 2, ..., m-1)$.

Separating the imaginary part of (8), we obtain

$$\Phi_\sigma(y,x;\lambda)=\frac{E_\rho(\sigma^{1/\rho}\gamma)}{c_m}\left[\frac{\partial^{k-1}}{\partial s^{k-1}}\int\limits_0^\infty\frac{(y_m-x_m)\mathrm{Im}E_\rho(\sigma^{1/\rho}w)}{u^2+r^2}\,\frac{uI_0(\lambda u)}{\sqrt{u^2+\alpha^2}}du-\right.$$
$$\left.-\frac{\partial^{k-1}}{\partial s^{k-1}}\int\limits_0^\infty\frac{u\mathrm{Re}\,E_\rho(\sigma^{1/\rho}w)}{u^2+r^2}I_0(\lambda u)\,du\right],\ y\neq x,\ x_m>0. \tag{28}$$

Given (28) and the inequality

$$I_0(\lambda u)\le\sqrt{\frac{2}{\lambda\pi u}}, \tag{29}$$

we have

$$\int\limits_{\partial G_\rho\backslash S^*}|\Phi_\sigma(y,x;\lambda)|\,ds_y\le C_\rho(\lambda,x)\sigma^k\exp(-\sigma\gamma^\rho),\ \ \sigma>1,\ x\in G_\rho, \tag{30}$$

To estimate the second integral, we use the equality

$$\frac{\partial\Phi_\sigma(y,x;\lambda)}{\partial y_j}=\frac{\partial\Phi_\sigma(y,x;\lambda)}{\partial s}\frac{\partial s}{\partial y_j}=2(y_j-x_j)\frac{\partial\Phi_\sigma(y,x;\lambda)}{\partial s},$$
$$s=\alpha^2,\ j=1,2,...,m-1. \tag{31}$$

Given equality (28), inequality (29) and equality (31), we obtain

$$\int\limits_{\partial G_\rho\backslash S^*}\left|\frac{\partial\Phi_\sigma(y,x;\lambda)}{\partial y_j}\right|ds_y\le C_\rho(\lambda,x)\sigma^k\exp(-\sigma\gamma^\rho),\ \ \sigma>1,\ x\in G_\rho,$$
$$j=1,2,...,m-1. \tag{32}$$

Now, we estimate the integral $\int\limits_{\partial G_\rho\backslash S^*}\left|\frac{\partial\Phi_\sigma(y,x;\lambda)}{\partial y_m}\right|ds_y.$

Taking into account equality (28) and inequality (29), we obtain

$$\int\limits_{\partial G_\rho\backslash S^*}\left|\frac{\partial\Phi_\sigma(y,x;\lambda)}{\partial y_m}\right|ds_y\le C_\rho(\lambda,x)\sigma^k\exp(-\sigma\gamma^\rho),\ \ \sigma>1,\ x\in G_\rho, \tag{33}$$

From inequalities (30), (32), (33) and (27), we obtain an estimate (25).

Now let us prove inequality (26). To do this, we take the derivatives from equalities (10) and (24) with respect to $x_j,\ j=1,...,m$, then we obtain the following:

$$\frac{\partial U(x)}{\partial x_j}=\int\limits_{S^*}\frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j}U(y)ds_y+\int\limits_{\partial G_\rho\backslash S^*}\frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j}U(y)ds_y,$$
$$\frac{\partial U_\sigma(x)}{\partial x_j}=\int\limits_{S^*}\frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j}U(y)ds_y,\ \ x\in G_\rho,\ j=1,...,m. \tag{34}$$

Taking into account the (34) and inequality (23), we estimate the following

$$\left|\frac{\partial U(x)}{\partial x_j}-\frac{\partial_\sigma U(x)}{\partial x_j}\right|\leq\left|\int\limits_{\partial G_\rho\backslash S^*}\frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j}U(y)ds_y\right|\leq$$

$$\leq\int\limits_{\partial G_\rho\backslash S^*}\left|\frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j}\right||U(y)|\,ds_y\leq M\int\limits_{\partial G_\rho\backslash S^*}\left|\frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j}\right|ds_y, \tag{35}$$

$$x\in G_\rho,\ j=1,...,m.$$

To do this, we estimate the integrals $\int\limits_{\partial G_\rho\backslash S^*}\left|\frac{\partial \Phi_\sigma(y,x;\lambda)}{\partial x_j}\right|ds_y$, $(j=1,2,...,m-1)$ and $\int\limits_{\partial G_\rho\backslash S^*}\left|\frac{\partial \Phi_\sigma(y,x;\lambda)}{\partial x_m}\right|ds_y$ on the part $\partial G_\rho\backslash S^*$ of the plane $y_m=0$.

To estimate the first integrals, we use the equality

$$\frac{\partial \Phi_\sigma(y,x;\lambda)}{\partial x_j}=\frac{\partial \Phi_\sigma(y,x;\lambda)}{\partial s}\frac{\partial s}{\partial x_j}=-2(y_j-x_j)\frac{\partial \Phi_\sigma(y,x;\lambda)}{\partial s},$$
$$s=\alpha^2,\ j=1,2,...,m-1. \tag{36}$$

Given equality (28), inequality (29) and equality (36), we obtain

$$\int\limits_{\partial G_\rho\backslash S^*}\left|\frac{\partial \Phi_\sigma(y,x;\lambda)}{\partial x_j}\right|ds_y\leq C_\rho(\lambda,x)\sigma^k\exp(-\sigma\gamma^\rho),\ \sigma>1,\ x\in G_\rho,$$
$$j=1,2,...,m-1. \tag{37}$$

Now, we estimate the integral $\int\limits_{\partial G_\rho\backslash S^*}\left|\frac{\partial \Phi_\sigma(y,x;\lambda)}{\partial x_m}\right|ds_y$.

Taking into account equality (28) and inequality (29), we obtain

$$\int\limits_{\partial G_\rho\backslash S^*}\left|\frac{\partial \Phi_\sigma(y,x;\lambda)}{\partial x_m}\right|ds_y\leq C_\rho(\lambda,x)\sigma^k\exp(-\sigma\gamma^\rho),\ \sigma>1,\ x\in G_\rho, \tag{38}$$

From inequalities (35), (37) and (38), we obtain an estimate (26).
Theorem 4.1 is proved. □

Corollary 4.1. *For each $x\in G_\rho$, the equalities are true*

$$\lim_{\sigma\to\infty}U_\sigma(x)=U(x),\ \lim_{\sigma\to\infty}\frac{\partial U_\sigma(x)}{\partial x_j}=\frac{\partial U(x)}{\partial x_j},\ j=1,...,m.$$

We denote by $\overline{G}_\varepsilon$ the set

$$\overline{G}_\varepsilon=\left\{(x_1,...,x_m)\in G_\rho,\ a>x_m\geq\varepsilon,\ a=\max_T\psi(x'),\ 0<\varepsilon<a\right\}.$$

Here, at $m=2$, $\psi(x_1)$ - is a curve, and at $m=2k,\ k\geq1$, $\psi(x')$ - is a surface. It is easy to see that the set $\overline{G}_\varepsilon\subset G_\rho$ is compact.

Corollary 4.2. *If $x \in \overline{G}_\varepsilon$, then the families of functions $\{U_\sigma(x)\}$ and $\left\{\dfrac{\partial U_\sigma(x)}{\partial x_j}\right\}$ converge uniformly for $\sigma \to \infty$, i.e.:*

$$U_\sigma(x) \rightrightarrows U(x),\ \frac{\partial U_\sigma(x)}{\partial x_j} \rightrightarrows \frac{\partial U(x)}{\partial x_j},\ j = 1, ..., m.$$

It should be noted that the set $E_\varepsilon = G_\rho \backslash \overline{G}_\varepsilon$ serves as a boundary layer for this problem, as in the theory of singular perturbations, where there is no uniform convergence.

5. Estimation of the stability of the solution to the Cauchy problem

Suppose that the surface S (or the curve at $m = 2$) is given by the equation

$$y_m = \psi(y'),\ y' \in \mathbb{R}^{m-1},$$

where $\psi(y')$ is a single-valued function satisfying the Lyapunov conditions.

We put

$$a = \max_T \psi(y'),\ b = \max_T \sqrt{1 + \psi'^2(y')}.$$

Theorem 5.1. *Let $U(y) \in A(G_\rho)$ satisfy condition (23), and on a smooth surface S the inequality*

$$|U(y)| \leq \delta,\ 0 < \delta < 1. \tag{39}$$

Then the following estimates are true

$$|U(x)| \leq C_\rho(\lambda, x)\sigma^k M^{1-\left(\frac{\gamma}{a}\right)^\rho} \delta^{\left(\frac{\gamma}{a}\right)^\rho},\ \sigma > 1,\ x \in G_\rho. \tag{40}$$

$$\left|\frac{\partial U(x)}{\partial x_j}\right| \leq C_\rho(\lambda, x)\sigma^k M^{1-\left(\frac{\gamma}{a}\right)^\rho} \delta^{\left(\frac{\gamma}{a}\right)^\rho},\ \sigma > 1,\ x \in G_\rho,\ j = 1, .., m. \tag{41}$$

Here is $a^\rho = \max_{y \in S} \operatorname{Re} w_0^\rho$.

Proof. Let us first estimate inequality (40). Using the integral formula (10), we have

$$U(x) = \int_{S^*} N_\sigma(y, x; \lambda) U(y) ds_y + \int_{\partial G_\rho \backslash S^*} N_\sigma(y, x; \lambda)) U(y) ds_y,\ x \in G_\rho. \tag{42}$$

We estimate the following

$$|U(x)| \leq \left|\int_{S^*} N_\sigma(y, x; \lambda) U(y) ds_y\right| + \left|\int_{\partial G_\rho \backslash S^*} N_\sigma(y, x; \lambda) U(y) ds_y\right|,\ x \in G_\rho. \tag{43}$$

Given inequality (39), we estimate the first integral of inequality (43).

$$\begin{gathered}\left|\int_{S^*} N_\sigma(y, x; \lambda) U(y) ds_y\right| \leq \int_{S^*} |N_\sigma(y, x; \lambda)| \, |U(y)| \, ds_y \leq \\ \leq \delta \int_{S^*} |N_\sigma(y, x; \lambda)| \, ds_y,\ x \in G_\rho.\end{gathered} \tag{44}$$

To do this, we estimate the integrals $\int\limits_{S^*} |\Phi_\sigma(y,x;\lambda)|\, ds_y$, $\int\limits_{S^*} \left|\frac{\partial\Phi_\sigma(y,x;\lambda)}{\partial y_j}\right| ds_y$, $(j = 1, 2, ..., m-1)$ and $\int\limits_{S^*} \left|\frac{\partial\Phi_\sigma(y,x;\lambda)}{\partial y_m}\right| ds_y$ on a smooth surface S.

Given equality (28) and the inequality (29), we have

$$\int\limits_{S^*} |\Phi_\sigma(y,x;\lambda)|\, ds_y \leq C_\rho(\lambda,x)\sigma^k \exp\sigma(\tau^\rho a^\rho - \gamma^\rho),\ \sigma > 1,\ x \in G_\rho. \tag{45}$$

To estimate the second integral, using equalities (28) and (31) as well as inequality (29), we obtain

$$\int\limits_{S^*} \left|\frac{\partial\Phi_\sigma(y,x;\lambda)}{\partial y_j}\right| ds_y \leq C_\rho(\lambda,x)\sigma^k \exp\sigma(\tau^\rho a^\rho - \gamma^\rho),\ \sigma > 1,\ x \in G_\rho,$$
$$j = 1, ..., m-1. \tag{46}$$

To estimate the integral $\int\limits_{S^*} \left|\frac{\partial\Phi_\sigma(y,x;\lambda)}{\partial y_m}\right| ds_y$, using equality (28) and inequality (29), we obtain

$$\int\limits_{S^*} \left|\frac{\partial\Phi_\sigma(y,x;\lambda)}{\partial y_m}\right| ds_y \leq C_\rho(\lambda,x)\sigma^k \exp\sigma(\tau^\rho a^\rho - \gamma^\rho),\ \sigma > 1,\ x \in G_\rho. \tag{47}$$

From (45) - (47), we obtain

$$\left|\int\limits_{S^*} N_\sigma(y,x;\lambda)U(y)ds_y\right| \leq C_\rho(\lambda,x)\sigma^k\delta\ \exp\sigma(\tau^\rho a^\rho - \gamma^\rho),\ \sigma > 1,\ x \in G_\rho. \tag{48}$$

The following is known

$$\left|\int\limits_{\partial G_\rho\backslash S^*} N_\sigma(y,x;\lambda)U(y)ds_y\right| \leq C_\rho(\lambda,x)\sigma^k M \exp(-\sigma\gamma^\rho),\ \sigma > 1,\ x \in G_\rho. \tag{49}$$

Now taking into account (48) - (49), we have

$$|U(x)| \leq \frac{C_\rho(\lambda,x)\sigma^k}{2}(\delta\exp(\sigma\tau^\rho a^\rho) + M)\exp(-\sigma\gamma^\rho),\ \sigma > 1,\ x \in G_\rho. \tag{50}$$

Choosing σ from the equality

$$\sigma = \frac{1}{a^\rho}\ln\frac{M}{\delta}, \tag{51}$$

we obtain an estimate (40).

Now let us prove inequality (41). To do this, we find the partial derivative from the integral formula (10) with respect to the variable $x_j,\ j = 1, ..., m-1$:

$$\frac{\partial U(x)}{\partial x_j} = \int\limits_{S^*} \frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j}U(y)ds_y + \int\limits_{\partial G_\rho\backslash S^*} \frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j}U(y)ds_y +$$
$$+\frac{\partial U_\sigma(x)}{\partial x_j} + \int\limits_{\partial G_\rho\backslash S^*} \frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j}U(y)ds_y,\ x \in G_\rho,\ j = 1, ..., m. \tag{52}$$

Here

$$\frac{\partial U_\sigma(x)}{\partial x_j} = \int\limits_{S^*} \frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j} U(y) ds_y. \tag{53}$$

We estimate the following

$$\begin{gathered}\left|\frac{\partial U(x)}{\partial x_j}\right| \le \left|\int\limits_{S^*} \frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j} U(y) ds_y\right| + \left|\int\limits_{\partial G_\rho \backslash S^*} \frac{\partial N_\sigma(y,x;\lambda))}{\partial x_j} U(y) ds_y\right| \le \\ \le \left|\frac{\partial U_\sigma(x)}{\partial x_j}\right| + \left|\int\limits_{\partial G_\rho \backslash S^*} \frac{\partial N_\sigma(y,x;\lambda))}{\partial x_j} U(y) ds_y\right|,\ x \in G_\rho,\ j=1,...,m.\end{gathered} \tag{54}$$

Given inequality (39), we estimate the first integral of inequality (54).

$$\begin{gathered}\left|\int\limits_{S^*} \frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j} U(y) ds_y\right| \le \int\limits_{S^*} \left|\frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j}\right| |U(y)|\, ds_y \le \\ \le \delta \int\limits_{S^*} \left|\frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j}\right| ds_y,\ x \in G_\rho,\ j=1,...,m.\end{gathered} \tag{55}$$

To do this, we estimate the integrals $\int\limits_{S^*} \left|\frac{\partial \Phi_\sigma(y,x;\lambda)}{\partial x_j}\right| ds_y$, $(j = 1,2,...,m-1)$ and $\int\limits_{S^*} \left|\frac{\partial \Phi_\sigma(y,x;\lambda)}{\partial x_m}\right| ds_y$ on a smooth surface S.

Given equality (28), inequality (29) and equality (36), we obtain

$$\begin{gathered}\int\limits_{S^*} \left|\frac{\partial \Phi_\sigma(y,x;\lambda)}{\partial x_j}\right| ds_y \le C_\rho(\lambda,x)\sigma^k \exp\sigma(\tau^\rho a^\rho - \gamma^\rho),\ \sigma > 1,\ x \in G_\rho, \\ j=1,2,...,m-1.\end{gathered} \tag{56}$$

Now, we estimate the integral $\int\limits_{S^*} \left|\frac{\partial \Phi_\sigma(y,x;\lambda)}{\partial x_m}\right| ds_y$.

Taking into account equality (28) and inequality (29), we obtain

$$\int\limits_{S^*} \left|\frac{\partial \Phi_\sigma(y,x;\lambda)}{\partial x_m}\right| ds_y \le C_\rho(\lambda,x)\sigma^k \delta \exp\sigma(\tau^\rho a^\rho - \gamma^\rho),\ \sigma > 1,\ x \in G_\rho, \tag{57}$$

From (56) - (57), we obtain

$$\begin{gathered}\left|\int\limits_{S^*} \frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j} U(y)\right| \le C_\rho(\lambda,x)\sigma^k \delta\, \exp\sigma(\tau^\rho a^\rho - \gamma^\rho),\ \sigma > 1,\ x \in G_\rho, \\ j=1,...,m.\end{gathered} \tag{58}$$

The following is known

$$\left|\int\limits_{\partial G_\rho \backslash S^*} \frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j} U(y) ds_y\right| \leq C_\rho(\lambda,x)\sigma^k M \exp(-\sigma\gamma^\rho),\ \sigma > 1,\ x \in G_\rho, \quad j = 1,...,m. \tag{59}$$

Now taking into account (58) - (59), we have

$$\left|\frac{\partial U(x)}{\partial x_j}\right| \leq \frac{C_\rho(\lambda,x)\sigma^k}{2}(\delta \exp(\sigma\tau^\rho a^\rho) + M)\exp(-\sigma\gamma^\rho),\ \sigma > 1,\ x \in G_\rho, \quad j = 1,...,m. \tag{60}$$

Choosing σ from the equality (51) we obtain an estimate (41).
Theorem 5.1 is proved. □

Let $U(y) \in A(G_\rho)$ and instead $U(y)$ on S with its approximation $f_\delta(y)$, respectively, with an error $0 < \delta < 1$,

$$\max_S |U(y) - f_\delta(y)| \leq \delta. \tag{61}$$

We put

$$U_{\sigma(\delta)}(x) = \int\limits_{S^*} N_\sigma(y,x;\lambda) f_\delta(y) ds_y,\ x \in G_\rho. \tag{62}$$

Theorem 5.2. *Let $U(y) \in A(G_\rho)$ on the part of the plane $y_m = 0$ satisfy condition (23). Then the following estimates is true*

$$\left|U(x) - U_{\sigma(\delta)}(x)\right| \leq C_\rho(\lambda,x)\sigma^k M^{1-\left(\frac{\gamma}{a}\right)^\rho} \delta^{\left(\frac{\gamma}{a}\right)^\rho},\ \sigma > 1,\ x \in G_\rho. \tag{63}$$

$$\left|\frac{\partial U(x)}{\partial x_j} - \frac{\partial U_{\sigma(\delta)}(x)}{\partial x_j}\right| \leq C_\rho(\lambda,x)\sigma^k M^{1-\left(\frac{\gamma}{a}\right)^\rho} \delta^{\left(\frac{\gamma}{a}\right)^\rho},\ \sigma > 1,\ x \in G_\rho, \quad j = 1,...,m. \tag{64}$$

Proof. From the integral formulas (10) and (62), we have

$$U(x) - U_{\sigma(\delta)}(x) = \int\limits_{\partial G_\rho} N_\sigma(y,x;\lambda) U(y) ds_y - \int\limits_{S^*} N_\sigma(y,x;\lambda) f_\delta(y) ds_y =$$

$$= \int\limits_{S^*} N_\sigma(y,x;\lambda) U(y) ds_y + \int\limits_{\partial G_\rho \backslash S^*} N_\sigma(y,x;\lambda) U(y) ds_y - \int\limits_{S^*} N_\sigma(y,x;\lambda) f_\delta(y) ds_y =$$

$$= \int\limits_{S^*} N_\sigma(y,x;\lambda) \left\{U(y) - f_\delta(y)\right\} ds_y + \int\limits_{\partial G_\rho \backslash S^*} N_\sigma(y,x;\lambda) U(y) ds_y.$$

and

$$\frac{\partial U(x)}{\partial x_j} - \frac{\partial U_{\sigma(\delta)}(x)}{\partial x_j} = \int\limits_{\partial G_\rho} \frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j} U(y) ds_y - \int\limits_{S^*} \frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j} f_\delta(y) ds_y =$$

$$= \int\limits_{S^*} \frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j} U(y) ds_y + \int\limits_{\partial G_\rho \backslash S^*} \frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j} U(y) ds_y - \int\limits_{S^*} \frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j} f_\delta(y) ds_y =$$

$$= \int\limits_{S^*} \frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j} \left\{U(y) - f_\delta(y)\right\} ds_y + \int\limits_{\partial G_\rho \backslash S^*} \frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j} U(y) ds_y, \; j = 1, ..., m.$$

Using conditions (23) and (61), we estimate the following:

$$\left|U(x) - U_{\sigma(\delta)}(x)\right| = \left| \int\limits_{S^*} N_\sigma(y,x;\lambda) \left\{U(y) - f_\delta(y)\right\} ds_y \right| +$$

$$+ \left| \int\limits_{\partial G_\rho \backslash S^*} N_\sigma(y,x;\lambda) U(y) ds_y \right| \le \int\limits_{S^*} \left|N_\sigma(y,x;\lambda)\right| \left|\left\{U(y) - f_\delta(y)\right\}\right| ds_y +$$

$$+ \int\limits_{\partial G_\rho \backslash S^*} \left|N_\sigma(y,x;\lambda)\right| \left|U(y)\right| ds_y \le \delta \int\limits_{S^*} \left|N_\sigma(y,x;\lambda)\right| ds_y + M \int\limits_{\partial G_\rho \backslash S^*} \left|N_\sigma(y,x;\lambda)\right| ds_y.$$

and

$$\left|\frac{\partial U(x)}{\partial x_j} - \frac{\partial U_{\sigma(\delta)}(x)}{\partial x_j}\right| = \left| \int\limits_{S^*} \frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j} \left\{U(y) - f_\delta(y)\right\} ds_y \right| +$$

$$+ \left| \int\limits_{\partial G_\rho \backslash S^*} \frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j} U(y) ds_y \right| \le \int\limits_{S^*} \left| \frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j} \right| \left|\left\{U(y) - f_\delta(y)\right\}\right| ds_y +$$

$$+ \int\limits_{\partial G_\rho \backslash S^*} \left| \frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j} \right| \left|U(y)\right| ds_y \le \delta \int\limits_{S^*} \left| \frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j} \right| ds_y +$$

$$+ M \int\limits_{\partial G_\rho \backslash S^*} \left| \frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j} \right| ds_y, \; j = 1, ..., m.$$

Now, repeating the proof of Theorems 4.1 and 5.1, we obtain

$$\left|U(x) - U_{\sigma(\delta)}(x)\right| \le \frac{C_\rho(\lambda,x)\sigma^k}{2} (\delta \exp(\sigma \tau^\rho a^\rho) + M) \exp(-\sigma \gamma^\rho),$$

$$\left|\frac{\partial U(x)}{\partial x_j} - \frac{U_{\sigma(\delta)}(x)}{\partial x_j}\right| \le \frac{C_\rho(\lambda,x)\sigma^k}{2} (\delta \exp(\sigma \tau^\rho a^\rho) + M) \exp(-\sigma \gamma^\rho), \; j = 1, ..., m.$$

From here, choosing σ from equality (51), we obtain an estimates (63) and (64).

Theorem 5.2 is proved. □

Corollary 5.1. *For each $x \in G_\rho$, the equalities are true*

$$\lim_{\delta\to 0} U_{\sigma(\delta)}(x) = U(x),\ \lim_{\delta\to 0} \frac{\partial U_{\sigma(\delta)}(x)}{\partial x_j} = \frac{\partial U(x)}{\partial x_j},\ j = 1, ..., m.$$

Corollary 5.2. *If $x \in \overline{G}_\varepsilon$, then the families of functions $\left\{U_{\sigma(\delta)}(x)\right\}$ and $\left\{\frac{\partial U_{\sigma(\delta)}(x)}{\partial x_j}\right\}$ converge uniformly for $\delta \to 0$, i.e.:*

$$U_{\sigma(\delta)}(x) \rightrightarrows U(x),\ \frac{\partial U_{\sigma(\delta)}(x)}{\partial x_j} \rightrightarrows \frac{\partial U(x)}{\partial x_j},\ j = 1, ..., m.$$

6. Conclusion

This article obtained the following results:

Using the Carleman function, a formula is obtained for the continuation of the solution of linear elliptic systems of the first order with constant coefficients in a spatial bounded domain $\mathbb{R}^m$, $(m = 2k,\ k \geq 1)$. The resulting formula is an analogue of the classical formula of B. Riemann, W. Voltaire and J. Hadamard, which they constructed to solve the Cauchy problem in the theory of hyperbolic equations. An estimate of the stability of the solution of the Cauchy problem in the classical sense for matrix factorizations of the Helmholtz equation is given. The problem is considered in which instead of the exact data of the Cauchy problem; their approximations with a given deviation in the uniform metric are given and under the assumption that the solution of the Cauchy problem is bounded on part T of the boundary of the domain G_ρ; an explicit regularization formula is obtained.

We note that when solving applied problems, one should find the approximate values of $U(x)$ and $\frac{\partial U(x)}{\partial x_j}$, $x \in G_\rho$, $j = 1, ..., m$. In this paper, we construct a family of vector-functions $U(x, f_\delta) = U_{\sigma(\delta)}(x)$ and $\frac{\partial U(x, f_\delta)}{\partial x_j} = \frac{\partial U_{\sigma(\delta)}(x)}{\partial x_j}$, $j = 1, ..., m$ depending on a parameter σ, and prove that under certain conditions and a special choice of the parameter $\sigma = \sigma(\delta)$, at $\delta \to 0$, the family $U_{\sigma(\delta)}(x)$ and $\frac{\partial U_{\sigma(\delta)}(x)}{\partial x_j}$ converges in the usual sense to a solution $U(x)$ and its derivative $\frac{\partial U(x)}{\partial x_j}$ at a point $x \in G_\rho$.

Following A.N. Tikhonov (see [2]), a family of vector-valued functions $U_{\sigma(\delta)}(x)$ and $\frac{\partial U_{\sigma(\delta)}(x)}{\partial x_j}$ is called a regularized solution of the problem. A regularized solution determines a stable method of approximate solution of the problem.

Thus, functionals $U_{\sigma(\delta)}(x)$ and $\frac{\partial U_{\sigma(\delta)}(x)}{\partial x_j}$ determines the regularization of the solution of problem (1)–(2).

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