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A SUBCLASS OF BI-UNIVALENT FUNCTIONS RELATED TO SHELL-LIKE CURVES CONNECTED WITH FIBONACCI NUMBERS ASSOCIATED WITH (p,q)-DERIVATIVE

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Abstract. In this paper, we define a new subclass of bi-univalent functions related to shell-like curves connected with Fibonacci numbers by using (p,q)-derivative and the coefficient estimates, Fekete-Szego inequalities are discussed for the functions belonging to this class.

Keywords: Bi-univalent functions, Fekete-Szego inequality, Fibonacci numbers, Shell-like curves and (p,q)-derivative.

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1. Introduction

Let $\mathcal{A}$ be the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the open unit disc $\mathcal{D} = \{z \in \mathbb{C}; |z| < 1\}$ with normalization $f(0) = f'(0) - 1 = 0$. By $\mathcal{S}$ we mean the class of all functions $\mathcal{A}$ which are univalent in $\mathcal{D}$. Also let $\mathcal{P}$ be the class of Carathéodory functions $p : \mathcal{D} \to \mathbb{C}$ of the form $p(z) = 1 + c_1 z + c_2 z^2 + ...$, $z \in \mathcal{D}$ such that $\Re\{p(z)\} > 0$. We say that f is subordinate to g in $\mathcal{D}$, written as $f \prec g$ provided there is an analytic function w in $\mathcal{D}$ with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$. It follows from Schwarz Lemma that

$$f(z) \prec g(z) \iff f(0) = g(0) \quad and \quad f(\mathcal{D}) \subset g(\mathcal{D}), \quad z \in \mathcal{D}.$$

For $0 < q < p \leq 1$, the (p,q)-analogue of Jackson derivative [3] is given by

$$D_{p,q} f(z) = \frac{f(pz) - f(qz)}{(p-q)z}, \quad z \neq 0.$$

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Therefore for f as in (1), we have

$$D_{p,q}f(z) = 1 + \sum_{n=2}^{\infty}[n]_{p,q}a_n z^{n-1},$$

where $[n]_{p,q} = \frac{p^n - q^n}{p-q}$, $(0 < q < p \leq 1)$.

By the Kӧebe's one quarter theorem [2], we know that the image of $\mathcal{D}$ under every univalent function $f \in \mathcal{A}$ contains a disk of radius $1/4$. Therefore, every univalent function f has an inverse f^{-1} satisfying:
$f^{-1}(f(z)) = z, (z \in \mathcal{D})$ and $f(f^{-1}(w)) = w, (|w| < r_0(f), r_0(f) \geq \frac{1}{4})$.
It is easy to see that the inverse function has the form

$$g(w) = f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + ... \tag{2}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathcal{D}$ if both f and its inverse map $g = f^{-1}$ are univalent in $\mathcal{D}$. Let Σ denote the class of bi-univalent functions in $\mathcal{D}$ given by the Taylor's-Macluarin series expansion (1).

For $f \in \mathcal{A}$ the class $\mathcal{SL}$ of shell-like functions which is the subclass of the class $\mathcal{S}^*$ of starlike functions was first introduced by Sokol [11], in 1999 as below

Definition 1.1. *The function $f \in \mathcal{A}$ belongs to the class $\mathcal{SL}$ of starlike shell-like functions if it satisfies the condition that*

$$\frac{zf'(z)}{f(z)} \prec \tilde{p}(z) = \frac{1+\tau^2z^2}{1-\tau z-\tau^2z^2},$$

where $\tau = (1-\sqrt{5})/2 \approx -0.618$.

In the year 2011, Dziok et al. [4], introduced the class $\mathcal{KSL}$ of convex functions related to a shell-like curves as follows:

Definition 1.2. *The function $f \in \mathcal{A}$ belongs to the class $\mathcal{KSL}$ of convex shell-like functions if it satisfies the condition that*

$$1 + \frac{zf''(z)}{f'(z)} \prec \tilde{p}(z) = \frac{1+\tau^2z^2}{1-\tau z-\tau^2z^2},$$

where $\tau = (1-\sqrt{5})/2 \approx -0.618$.

Again Dziok et al. [5] in the year 2011, defined the following class $\mathcal{SLM}_\alpha$ of α-convex shell-like functions.

Definition 1.3. *The function $f \in \mathcal{A}$ belongs to the class $\mathcal{SLM}_\alpha$, $(0 \leq \alpha \leq 1)$ if it satisfies the condition that*

$$\alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) + (1-\alpha)\frac{zf'(z)}{f(z)} \prec \tilde{p}(z) = \frac{1+\tau^2z^2}{1-\tau z-\tau^2z^2},$$

where $\tau = (1-\sqrt{5})/2 \approx -0.618$.

We note that $\mathcal{SLM}_0 \equiv \mathcal{SL}$, $\mathcal{SLM}_1 \equiv \mathcal{KSL}$ and $\mathcal{SLM}_\alpha \neq \mathcal{KSL}$ for $\alpha \neq 1$.

The function $\tilde{p}$ is not univalent in $\mathcal{D}$, but it is univalent in the disc $|z| < (3-\sqrt{5})/2 \approx 0.38$. For example, $\tilde{p}(0) = \tilde{p}\left(\frac{-1}{2\tau}\right) = 1$ and $\tilde{p}\left(e^{\mp iarcos(1/4)}\right) = \frac{\sqrt{5}}{5}$, and it may also be noticed that

$$\frac{1}{|\tau|} = \frac{|\tau|}{1-|\tau|},$$

which shows that the number $|\tau|$ divides $[0,1]$ such that it fulfils the golden section. The image of the unit circle $|z|=1$ under $\tilde{p}$ is a curve described by the equation given by

$$(10x-\sqrt{5})y^2=(\sqrt{5}-2x)(\sqrt{5}x-1)^2,$$

which is translated and revolved trisectrix of Maclaurin. The curve $\tilde{p}(re^{it})$ is a closed curve without any loops for $0<r\leq r_0=(3-\sqrt{5})/2\approx 0.38$. For $r_0<r<1$, it has a loop and for $r=1$, it has a vertical asymptote. Since τ satisfies the equation $\tau^2=1+\tau$, this expression can be used to obtain higher powers τ^n as a linear function of lower powers, which in turn can be decomposed all the way down to a linear combination of τ and 1. The resulting recurrence relationships yield Fibonacci numbers u_n:

$$\tau^n=u_n\tau+u_{n-1}.$$

In [8], taking $\tau z=t$, Raina and Sokol showed that

$$\tilde{p}(z)=\frac{1+\tau^2z^2}{1-\tau z-\tau^2z^2}=1+\sum_{n=1}^{\infty}(u_{n-1}+u_{n+1})\tau^n z^n,$$

where $u_n=\dfrac{(1-\tau)^n-\tau^n}{\sqrt{5}}, \quad \tau=\dfrac{1-\sqrt{5}}{2}, \quad (n=1,2,...).$

This shows that the relevant connection of $\tilde{p}$ with the sequence of Fibonacci numbers u_n, such that $u_0=0, \quad u_1=1, \quad u_{n+2}=u_n+u_{n+1} \quad for \quad n=0,1,2,...$.

Hence

$$\tilde{p}(z)=1+\tau z+3\tau^2z^2+4\tau^3z^3+...$$

Motivated by these works we define a new subclass of bi-univalent functions related to shell-like curves connected to Fibonacci number using (p,q)- derivative.

Definition 1.4. *For* $0<q<p\leq 1$ *and* $0\leq\alpha\leq 1$, *a function* $f\in\Sigma$ *given by (1) is said to be in the class* $\mathcal{SLM}_{\alpha,\Sigma}(p,q,\tilde{p}(z))$ *if it satisfies the following conditions:*

$$\frac{(1-\alpha)zD_{p,q}f(z)+\alpha zD_{p,q}(zD_{p,q}f(z))}{(1-\alpha)f(z)+\alpha zD_{p,q}f(z)}\prec\tilde{p}(z)=\frac{1+\tau^2z^2}{1-\tau z-\tau^2z^2} \tag{3}$$

and

$$\frac{(1-\alpha)wD_{p,q}g(w)+\alpha wD_{p,q}(wD_{p,q}g(w))}{(1-\alpha)g(w)+\alpha wD_{p,q}g(w)}\prec\tilde{p}(w)=\frac{1+\tau^2w^2}{1-\tau w-\tau^2w^2}, \tag{4}$$

where where $\tau=(1-\sqrt{5})/2\approx-0.618$, $g=f^{-1}$ *given by (2) and* $z,w\in\mathcal{D}$.

Specializing the parameter $\alpha=0$ and $\alpha=1$ we have the following respectively:

Definition 1.5. *A function* $f\in\Sigma$ *of the form(1) is said to be in the class* $\mathcal{SL}_{\Sigma}(p,q,\tilde{p}(z))$ *if it satisfies the following conditions:*

$$\frac{zD_{p,q}f(z)}{f(z)}\prec\tilde{p}(z)=\frac{1+\tau^2z^2}{1-\tau z-\tau^2z^2}$$

and

$$\frac{wD_{p,q}g(w)}{g(w)}\prec\tilde{p}(w)=\frac{1+\tau^2w^2}{1-\tau w-\tau^2w^2},$$

where $\tau=(1-\sqrt{5})/2\approx-0.618$, $g=f^{-1}$ *given by (2) and* $z,w\in\mathcal{D}$.

Definition 1.6. *A function* $f\in\Sigma$ *of the form (1) is said to be in the class* $\mathcal{KSL}_{\Sigma}(p,q,\tilde{p}(z))$ *if it satisfies the following conditions:*

$$\frac{D_{p,q}(zD_{p,q}f(z))}{D_{p,q}f(z)}\prec\tilde{p}(z)=\frac{1+\tau^2z^2}{1-\tau z-\tau^2z^2}$$

and

$$\frac{D_{p,q}(wD_{p,q}g(w))}{D_{p,q}g(w)} \prec \tilde{p}(w) = \frac{1+\tau^2w^2}{1-\tau w-\tau^2w^2},$$

where $\tau = {(1-\sqrt{5})}/{2} \approx -0.618$, $g = f^{-1}$ *given by (2) and* $z, w \in \mathcal{D}$.

Remarks 1.1.

(i) $\mathcal{SLM}_{0,\Sigma}(1,q,\tilde{p}(z)) = q-\mathcal{SL}_\Sigma$ *and* $\mathcal{SLM}_{1,\Sigma}(1,q,\tilde{p}(z)) = q-\mathcal{KSL}_\Sigma$, *the classes of q-bi-univalent functions established by Ahuja* [1].

(ii) $\mathcal{SLM}_{\alpha,\Sigma,}(1,1,\tilde{p}(z)) = \mathcal{SLM}^{(\alpha,\Sigma)}(\tilde{p}(z))$, *the class of bi-univalent functions defined by Gurmeet Singh* [9].

(iii) $\mathcal{SLM}_{0,\Sigma,}(1,1,\tilde{p}(z)) = \mathcal{SL}_\Sigma(\tilde{p}(z))$ *and* $\mathcal{SLM}_{1,\Sigma,}(1,1,\tilde{p}(z)) = \mathcal{KSL}_\Sigma(\tilde{p}(z))$ *the classes of bi-univalent functions studied by Guney* [6].

In order to prove our results we need the following lemma.

Lemma 1.1. [7] *If* $p \in \mathcal{P}$ *with* $p(z) = 1 + c_1z + c_2z^2 + ...$, *then*

$$|c_n| \le 2, \quad n \ge 1.$$

In the next section we obtain the initial Taylor coefficients $|a_2|$ and $|a_3|$ for the function class $\mathcal{SLM}_{\alpha,\Sigma}(p,q,\tilde{p}(z))$. Later we will reduce these bounds to other classes for special cases.

2. Coefficient estimates

Theorem 2.1. *For* $0 < q < p \le 1$, $0 \le \alpha \le 1$, *let* $f \in \mathcal{SLM}_{\alpha,\Sigma}(p,q,\tilde{p}(z))$. *Then*

$$|a_2| \le \frac{|\tau|}{\sqrt{|(\eta-\psi)\tau + (1-3\tau)\zeta|}} \tag{5}$$

and

$$|a_3| \le \frac{|\tau|\left\{|(\eta-\psi)\tau + (1-3\tau)\zeta| + \eta|\tau|\right\}}{\eta|(\eta-\psi)\tau + (1-3\tau)\zeta|}, \tag{6}$$

where

$$\eta = ([3]_{p,q}-1)\left[1+\alpha([3]_{p,q}-1)\right], \tag{7}$$

$$\psi = ([2]_{p,q}-1)\left[1+\alpha([2]_{p,q}-1)\right]^2, \tag{8}$$

$$\zeta = ([2]_{p,q}-1)^2\left[1+\alpha([2]_{p,q}-1)\right]^2. \tag{9}$$

Proof. Let f be given by (1). As $f \in \mathcal{SLM}_{\alpha,\Sigma}(p,q,\widetilde{p(z)})$, so by definition 1.4 and using the concept of subordination, there exists Schwarz functions $u, v : \mathcal{D} \to \mathcal{D}$ with $u(0) = 0 = v(0)$, such that

$$\frac{(1-\alpha)zD_{p,q}f(z) + \alpha zD_{p,q}(zD_{p,q}f(z))}{(1-\alpha)f(z) + \alpha zD_{p,q}f(z)} = \tilde{p}(u(z)) \tag{10}$$

and

$$\frac{(1-\alpha)wD_{p,q}g(w) + \alpha wD_{p,q}(wD_{p,q}g(w))}{(1-\alpha)g(w) + \alpha wD_{p,q}g(w)} = \tilde{p}(v(w)). \tag{11}$$

Now define the function,

$$h(z) = \frac{1+u(z)}{1-u(z)} = 1 + c_1z + c_2z^2 + c_3z^3 + ...$$

Then

$$\tilde{p}(u(z)) = 1 + \frac{c_1}{2}\tau z + \frac{1}{2}\left(c_2 - \frac{c_1^2}{2} + \frac{3c_1^2}{2}\tau\right)\tau z^2 + ... \tag{12}$$

Similarly we define the function,

$$k(w) = \frac{1+v(w)}{1-v(w)} = 1 + d_1 w + d_2 w^2 + d_3 w^3 + ...$$

Then

$$\tilde{p}(v(w)) = 1 + \frac{d_1}{2}\tau w + \frac{1}{2}\left(d_2 - \frac{d_1^2}{2} + \frac{3d_1^2}{2}\tau\right)\tau w^2 + ... \tag{13}$$

and by considering the LHS of (10) and (11), we have

$$\frac{(1-\alpha)zD_{p,q}f(z) + \alpha z D_{p,q}(zD_{p,q}f(z))}{(1-\alpha)f(z) + \alpha z D_{p,q}f(z)}$$
$$=1 + ([2]_{p,q}-1)(1+\alpha([2]_{p,q}-1))a_2 z +$$
$$\left\{([3]_{p,q}-1)(1+\alpha([3]_{p,q}-1))a_3 - ([2]_{p,q}-1)(1+\alpha([2]_{p,q}-1))^2 a_2^2\right\} z^2 + ...$$

and

$$\frac{(1-\alpha)wD_{p,q}g(w) + \alpha w D_{p,q}(wD_{p,q}g(w))}{(1-\alpha)g(w) + \alpha w D_{p,q}g(w)}$$
$$=1 - ([2]_{p,q}-1)(1+\alpha([2]_{p,q}-1))a_2 w +$$
$$\left\{2([3]_{p,q}-1)\left(1+\alpha([3]_{p,q}-1) - ([2]_{p,q}-1)(1+\alpha([2]_{p,q}-1))^2\right) a_2^2\right.$$
$$\left. -([3]_{p,q}-1)(1+\alpha([3]_{p,q}-1))a_3\right\} w^2 + . \quad . \quad .$$

Using (12),(13) and the above two equations in (10) and (11) and equating the coefficients of z, z^2, w and w^2 we get

$$([2]_{p,q}-1)(1+\alpha([2]_{p,q}-1))a_2 = \frac{c_1}{2}\tau, \tag{14}$$

$$([3]_{p,q}-1)(1+\alpha([3]_{p,q}-1))a_3 - \left\{([2]_{p,q}-1)(1+\alpha([2]_{p,q}-1))^2\right\} a_2^2$$
$$= \frac{1}{2}\left(c_2 - \frac{c_1^2}{2}\right)\tau + \frac{3c_1^2}{4}\tau^2, \tag{15}$$

$$-([2]_{p,q-1})(1+\alpha([2]_{p,q}-1))a_2 = \frac{d_1}{2}\tau, \tag{16}$$

and

$$\left\{2([3]_{p,q}-1)\left(1+\alpha([3]_{p,q}-1) - ([2]_{p,q}-1)(1+\alpha([2]_{p,q}-1))^2\right)\right\} a_2^2$$
$$- \left\{([3]_{p,q}-1)(1+\alpha([3]_{p,q}-1))\right\} a_3 = \frac{1}{2}\left(d_2 - \frac{d_1^2}{2}\right)\tau + \frac{3d_1^2}{4}\tau^2. \tag{17}$$

From (14) and (16), we have

$$c_1 = -d_1 \tag{18}$$

and also

$$2([2]_{p,q}-1)^2(1+\alpha([2]_{p,q}-1))^2 a_2^2 = \frac{(c_1^2+d_1^2)\tau^2}{4} \tag{19}$$

$$a_2^2 = \frac{(c_1^2+d_1^2)\tau^2}{8([2]_{p,q}-1)^2(1+\alpha([2]_{p,q}-1))^2}. \tag{20}$$

Adding (15) and (17), we get

$$2\left\{([3]_{p,q}-1)(1+\alpha([3]_{p,q}-1)) - ([2]_{p,q}-1)(1+\alpha([2]_{p,q}-1))^2\right\} a_2^2$$
$$= \frac{1}{2}(c_2+d_2)\tau - \frac{1}{4}(c_1^2+d_1^2)\tau + \frac{3}{4}(c_1^2+d_1^2)\tau^2. \tag{21}$$

Using (20) in the above equation, we get

$$4a_2^2 = \frac{(c_2+d_2)\tau^2}{[(\eta-\psi)\tau+(1-3\tau)\zeta]}, \tag{22}$$

where η, ψ and ζ are given by (7), (8) and (9) respectively. Using Lemma 1.1, we obtain the required inequality for $|a_2|$.
To find $|a_3|$ first we subtract (17) from (15) and then by using (18), we get

$$2([3]_{p,q}-1)\left[1+\alpha([3]_{p,q}-1)\right](a_3-a_2^2) = \frac{1}{2}(c_2-d_2)\tau$$

$$a_3 = \frac{(c_2-d_2)\tau}{4([3]_{p,q}-1)\left[1+\alpha([3]_{p,q}-1)\right]} + a_2^2. \tag{23}$$

Now by using (22) in (23) and Lemma 1.1, we get the coefficient bound for $|a_3|$.

□

If we can take the parameter $\alpha = 0$ and $\alpha = 1$ in the above theorem, we have the following the initial Taylor coefficients $|a_2|$ and $|a_3|$ for the function classes $\mathcal{SL}_\Sigma(p,q,\tilde{p}(z))$ and $\mathcal{KSL}_\Sigma(p,q,\tilde{p}(z))$, respectively.

Corollary 2.1. *For $0<q<p\leq 1$, let $f\in \mathcal{SL}_\Sigma(p,q,\tilde{p}(z))$. Then*

$$|a_2| \leq \frac{|\tau|}{\sqrt{|([3]_{p,q}-[2]_{p,q})\tau+(1-3\tau)([2]_{p,q}-1)^2|}}$$

and

$$|a_3| \leq \frac{|\tau|\left\{|([3]_{p,q}-[2]_{p,q})\tau+(1-3\tau)([2]_{p,q}-1)^2|+([3]_{p,q}-1)|\tau|\right\}}{([3]_{p,q}-1)|([3]_{p,q}-[2]_{p,q})\tau+(1-3\tau)([2]_{p,q}-1)^2|}.$$

Corollary 2.2. *For $0<q<p\leq 1$, let $f\in \mathcal{KSL}_\Sigma(p,q,\tilde{p}(z))$. Then*

$$|a_2| \leq \frac{|\tau|}{\sqrt{|([3]_{p,q}([3]_{p,q}-1)-[2]_{p,q}^2([2]_{p,q}-1))\tau+(1-3\tau)[2]_{p,q}^2([2]_{p,q}-1)^2|}}$$

and

$$|a_3| \leq \frac{|\tau|\left\{|([3]_{p,q}([3]_{p,q}-1)-[2]_{p,q}^2([2]_{p,q}-1))\tau+(1-3\tau)[2]_{p,q}^2([2]_{p,q}-1)^2|+|[3]_{p,q}([3]_{p,q}-1)\tau|\right\}}{[3]_{p,q}([3]_{p,q}-1)|([3]_{p,q}([3]_{p,q}-1)-[2]_{p,q}^2([2]_{p,q}-1))\tau+(1-3\tau)[2]_{p,q}^2([2]_{p,q}-1)^2|}.$$

Remark 2.1. *For $p=1, \alpha=0$ and $p=1, \alpha=1$, Theorem 2.1 gives the initial Taylor coefficients $|a_2|$ and $|a_3|$ for the function classes $q-\mathcal{SL}_\Sigma$ and $q-\mathcal{KSL}_\Sigma$, respectively defined by Ahuja* [1].

Letting $p=1$ and $q\to 1$ in Theorem 2.1 we obtain the following result.

Corollary 2.3. *If $f\in \mathcal{SLM}^{(\alpha,\Sigma)}(\tilde{p}(z))$, then*

$$|a_2| \leq \frac{|\tau|}{\sqrt{(1+\alpha)^2-(2+4\alpha+4\alpha^2)\tau}}$$

and

$$|a_3| \leq \frac{|\tau|\left[(1+\alpha)^2-(4+8\alpha+4\alpha^2)\tau\right]}{2(1+2\alpha)\left[(1+\alpha)^2-(2+4\alpha+4\alpha^2)\tau\right]}.$$

Remark 2.2. *For $p=1, q\to 1, \alpha=0$ and $p=1, q\to 1, \alpha=1$, Theorem 2.1 gives the initial coefficients $|a_2|$ and $|a_3|$ for the function classes $\mathcal{SL}_\Sigma(\tilde{p}(z))$ and $\mathcal{KSL}_\Sigma(\tilde{p}(z))$, respectively defined by Guney* [6].

In the next section we obtain the Fekete-Szego inequalities for the function class $\mathcal{SLM}_{\alpha,\Sigma}(p,q,\tilde{p}(z))$.

3. Fekete-Szego inequality

Theorem 3.1. *Let f given by (1) be in the class $\mathcal{SLM}_{\alpha,\Sigma}(p,q,\tilde{p}(z))$ and $\mu \in \mathcal{R}$. Then*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{|\tau|}{\eta}, & |\mu - 1| \leq \dfrac{|\tau(\eta-\psi)+(1-3\tau)\zeta|}{|\tau|\eta} \\ \dfrac{|\mu-1||\tau|^2}{|\tau(\eta-\psi)+(1-3\tau)\zeta|}, & |\mu-1| \geq \dfrac{|\tau(\eta-\psi)+(1-3\tau)\zeta|}{|\tau|\eta}, \end{cases}$$

where η, ψ and ζ are given by (7),(8) and (9) respectively.

Proof. From (22) and (23), we obtain

$$\begin{aligned} a_3 - \mu a_2^2 = &\left(h(\mu) + \frac{\tau}{4\left(([3]_{p,q}-1)(1+\alpha([3]_{p,q}-1))\right)}\right)c_2 + \\ &\left(h(\mu) - \frac{\tau}{4\left(([3]_{p,q}-1)(1+\alpha([3]_{p,q}-1))\right)}\right)d_2 \\ = &\left(h(\mu)+\frac{\tau}{4\eta}\right)c_2 + \left(h(\mu)-\frac{\tau}{4\eta}\right)d_2 \end{aligned} \tag{24}$$

where

$$h(\mu) = \frac{(1-\mu)\tau^2}{4\left((\eta-\psi)\tau + (1-3\tau)\zeta\right)}.$$

By taking modulus of (24) and using Lemma 1.1, we get

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{|\tau|}{\eta}, & |h(\mu)| \leq \frac{|\tau|}{4\eta} \\ 4|h(\mu)|, & |h(\mu)| \geq \frac{|\tau|}{4\eta}. \end{cases}.$$

This gives the desired result. □

Taking $\mu = 1$, we have the following result.

Corollary 3.1. *If $f \in \mathcal{SLM}_{\alpha,\Sigma}(p,q,\tilde{p}(z))$, then*

$$|a_3 - a_2^2| \leq \frac{|\tau|}{\eta}.$$

If we can take the parameter $\alpha = 0$ and $\alpha = 1$ in the above theorem,we have the following Fekete-Szego inequality for the function classes $\mathcal{SL}_\Sigma(p,q,\tilde{p}(z))$ and $\mathcal{KSL}_\Sigma(p,q,\tilde{p}(z))$, respectively.

Corollary 3.2. *Let f given by (1) be in the class $\mathcal{SL}_\Sigma(p,q,\tilde{p}(z))$ and $\mu \in \mathcal{R}$. Then*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{|\tau|}{([3]_{p,q}-1)}, & |\mu-1| \leq \dfrac{|\tau([3]_{p,q}-[2]_{p,q})+(1-3\tau)([2]_{p,q}-1)^2|}{|\tau|([3]_{p,q}-1)} \\ \dfrac{|\mu-1||\tau|^2}{|\tau([3]_{p,q}-[2]_{p,q})+(1-3\tau)([2]_{p,q}-1)^2|}, & |\mu-1| \geq \dfrac{|\tau([3]_{p,q}-[2]_{p,q})+(1-3\tau)([2]_{p,q}-1)^2|}{|\tau|([3]_{p,q}-1)}. \end{cases}$$

Corollary 3.3. *Let f given by (1) be in the class $\mathcal{KSL}_\Sigma(p,q,\tilde{p}(z))$ and $\mu \in \mathcal{R}$. Then*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{|\tau|}{A}, & |\mu-1| \leq \dfrac{|B|}{|\tau|A} \\ \frac{|\mu-1||\tau|^2}{|B|}, & |\mu-1| \geq \dfrac{|B|}{|\tau|A}, \end{cases}$$

where $A = [3]_{p,q}([3]_{p,q}-1)$ and
$B = \tau\left([3]_{p,q}([3]_{p,q}-1) - [2]_{p,q}^2([2]_{p,q}-1)\right) + (1-3\tau)[2]_{p,q}^2([2]_{p,q}-1)^2.$

For $p = 1$ and $q \to 1$ Theorem 3.1 agrees with the following result proved by Gurmeet Singh [9], (see Theorem 3):

Corollary 3.4. *If $f \in \mathcal{SLM}^{(\alpha,\Sigma)}(\tilde{p}(z))$ and $\mu \in \mathcal{R}$. Then*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \dfrac{|\tau|}{2(1+2\alpha)}, & |\mu - 1| \leq \dfrac{\left[(1+\alpha)^2 - (2+4\alpha+4\alpha^2)\tau\right]}{2(1+2\alpha)|\tau|} \\ \dfrac{|(1-\mu)|\tau^2}{\left[(1+\alpha)^2 - (2+4\alpha+4\alpha^2)\tau\right]}, & |\mu - 1| \geq \dfrac{\left[(1+\alpha)^2 - (2+4\alpha+4\alpha^2)\tau\right]}{2(1+2\alpha)|\tau|}. \end{cases}$$

Remark 3.1. *For $p = 1, q \to 1, \alpha = 0$ and $p = 1, q \to 1, \alpha = 1$, Theorem 3.1 gives the Fekete-Szego inequality for the function classes $\mathcal{SL}_\Sigma(\tilde{p}(z))$ and $\mathcal{KSL}_\Sigma(\tilde{p}(z))$ respectively, defined by Guney* [6].

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