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Refining the Lorenz-ranking of rules for claims problems on restricted domains

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Abstract

The comparison of the central rules for claims problems, according to the Lorenz order, has been studied not only on the entire set of problems but also on some restricted domains. We provide new characterizations of the adjusted proportional rule as being Lorenz-maximal or Lorenz-minimal within a class of rules on the half-domains. Using this result, we rank the adjusted proportional, the minimal overlap, and the average-of-awards rules by analyzing whether or not these rules satisfy progressivity and regressivity on the half-domains. We also find that the adjusted proportional rule violates two well-known claim monotonicity properties.

KEYWORDS

adjusted proportional rule, average-of-awards rule, claims problems, Lorenz ranking, minimal overlap rule

JEL CLASSIFICATION

C71, D63, G33

1 | INTRODUCTION

A claims problem arises when an amount has to be divided among a set of agents with claims that, in aggregate, exceed what is available. A rule is a way of selecting a division among the claimants. The definition of rules and the study of different approaches to evaluate and compare them started with O'Neill (1982) and Aumann and Maschler (1985), and has produced ever since a vast literature. The model has many applications that include bankruptcy problems, taxation systems, rationing problems, or the distribution of the carbon budget. For a thorough survey on this subject refer to Thomson (2019).

The best-known rule is the proportional rule that simply shares the scarce resource proportional to claims. Primarily, this paper focuses on the adjusted proportional, the minimal overlap, and the average-of-awards rules. The adjusted proportional rule was defined and studied by Curiel et al. (1987). This rule first allocates to each claimant his minimal right, the part of the amount that is left after each other individual is fully compensated. Each agent's claim is revised down to the minimum of the remainder and the difference between his initial claim and his minimal right. Finally, the resulting problem is solved using the proportional rule.

In the 12th century, the talmudic scholar Ibn Ezra described a problem consisting in dividing an estate among four sons. The recommendation that he presented was a particular case of a method proposed by Rabad, also in the 12th century, defined for problems such that no claim exceeds the estate. This incompletely specified rule was extended for an arbitrary claims problem by O'Neill (1982), and named the minimal overlap rule by Thomson (2003). Imagine that the amount available consists of distinct parts, and that each agent, instead of expressing his claim in some abstract way, claims specific parts of the total amount equal to his claim. The minimal overlap rule chooses awards vectors that minimize "extent of conflict" over each unit available. Alcalde et al. (2005, 2008), Chun and Thomson (2005), and Hendrickx et al. (2007) have given implicit formulae and new representations and interpretations of the minimal overlap rule.

A division rule must satisfy three natural requirements: no claimant is asked to pay; no claimant receives more than his claim; and the entire endowment is allocated. The set of all the allocations that meet these basic properties is the set of awards vectors for the claims problem. The average-of-awards rule, introduced by Mirás Calvo et al. (2022b), selects for each claims problem the expected value of the (continuous) uniform distribution over its set of awards vectors. O'Neill (1982) associates to each claims problem a coalitional game whose core is the set of awards vectors of the problem. González-Díaz and Sánchez-Rodríguez (2007) introduce, for the class of coalitional games with a non-empty core, the core-center solution: the centroid of the core. Therefore, the average-of-awards rule corresponds to the core-center solution for the associated coalitional game.

But the inventory of rules is rich. In this paper, we also considered: the constrained equal awards, the constrained equal losses, Piniles', the Talmud, the constrained egalitarian, and the random arrival rules. A rule might be selected by the appeal of its own definition and by the properties that it satisfies or violates. In fact, a rule might be characterized as the only one that satisfies certain properties, or axioms. When a rule violates a property, it is relevant to know if the rule satisfies it when restricted to a subdomain of problems. Some meaningful subdomains have already received attention. Aumann and Maschler (1985) argue that the half-sum of the claims is an important point (a watershed). In fact, the definition of the Talmud rule, for example, depends on whether or not the endowment is lower or bigger than the half-sum of the claims. These sets of problems are called the lower-half and higher-half domains respectively. Their intersection, the midpoint domain, is the set of problems for which the endowment is equal to the half-sum of the claims. Thomson (2019) discusses the domain of simple claims problems, those for which each claim is at most as large as the endowment.

Rules can be compared and ranked. The Lorenz criterion is widely used for this purpose. In order to compare a pair of awards vectors, rearrange the coordinates of each vector in a non-decreasing order. One vector Lorenz-dominates the other if the first coordinate and all the cumulative sums of the rearranged coordinates are greater with the former than with the latter. Many authors have contributed to the ranking of rules. Hougaard and Thorlund-Petersen (2001) and Moreno Ternero and Villar (2006) are the first papers exploring that line. A convenient approach is to characterize a rule as being maximal or minimal with respect to the Lorenz relation within a class of rules. Bosmans and Lauwers (2011), improving upon previous results, present Lorenz-based characterization of the constrained egalitarian, the constrained equal awards, Piniles', the minimal overlap, and the Talmud rules. One important aspect of their analysis is that, since some rules are not Lorenz-comparable, they restrict the comparison to the lower-half and higher-half domains. Naturally, on the restricted domains the ranking of rules is richer than on the full domain. Thomson (2012) develops three general methods to perform Lorenz comparisons of rules: giving conditions such that two members of a certain family of rules can be compared; providing criteria to deduce Lorenz-domination for arbitrarily many claimants from Lorenz-domination in the two-claimant case; and analyzing conditions under which operators preserve or reverse the Lorenz order. As a corollary of all the different approaches, Bosmans and Lauwers (2011) and Thomson (2019) provide a diagram reflecting the ranking of nine of the central rules discussed in the literature on the full domain. Mirás Calvo et al. (2022a) incorporate the average-of-awards rule to this ranking. Bosmans and Lauwers (2011) also present a diagram showing the ranking of the nine central rules on the restricted domains. Our goal in this paper is to justify the refined ranking summarized in Figure 4.

First, we show that the Talmud, the random arrival, the adjusted proportional, and the average-of-awards rules, that are extensions of the concede-and-divide rule, recommend the same division for any problem that belongs to a particular subclass that we called the middle domain. That is a consequence of the fact that, for this type of problems, the corresponding set of awards vectors presents a very simple structure similar to the configuration in the two-claimant case.

Suppose that an agent claim increases. Claim monotonicity implies that his award should not decrease. Other-regarding claim monotonicity requires each of the other claimants to receive at most as much as initially. If there are at least three claimants, order preservation under claims variations says that given any two claimants whose claim remains the same, the change in the award to the smaller one should be at most as large as the change in the award to the larger one. We prove that, contrary to the other central rules, the adjusted proportional rule violates both other-regarding claim monotonicity and order preservation under claims variations. That prevents us from using the existing Lorenz-based characterization of the minimal overlap rule to compare these two rules.

We provide new Lorenz-based characterizations of the adjusted proportional rule on the half domains. Progressivity and regressivity on these subdomains are the key requirements. These properties are very natural in taxation problems and were studied in Ju and Moreno Ternero (2008, 2011). Progressivity requires that a taxpayer with a higher income should pay at least as much rate of tax as a taxpayer with a lower income. When the problem is to share a scarce resource progressivity means that the claimants with big claims should receive more per unit of claim. Regressivity is the opposite requirement. Naturally, in order to apply the Lorenz-based characterizations of the adjusted proportional rule, we must establish whether or not the minimal overlap and the average-of-awards rules are progressive or regressive on the restricted domains. We prove that the minimal overlap rule is progressive on the higher-half domain and that the average-of-awards rule is regressive on the lower-half domain and progressive on the higher-half domain. The first important implication is that we are able to show that the adjusted proportional rule Lorenz-dominates the minimal overlap rule on the entire domain.

A second consequence is that we can derive a complete ranking of the average-of-awards rule on the restricted domains. If only as a "central" point of reference inside the set of awards vectors, it is worthy comparing the average-of-awards rule with the other rules. Basically, the geometric center of the set of awards vectors ranks between the Talmud rule and the adjusted proportional rule in both half domains, but, of course in different directions. In fact, the average-of-awards rule Lorenz-dominates the adjusted proportional rule and is Lorenz-dominated by the Talmud rule on the lower-half domain. Only the random arrival rule is not Lorenz-comparable to the average-of-awards rule on the restricted domains.

Figure 4 is not only an update of the ranking of the ten central rules on the restricted domains. It also provides a dynamic view of how the ranking, for a fixed vector of claims, changes as the endowment increases from 0 to the sum of the claims. The ranking for three-claimant problems, illustrated in Figure 6, presents some particularities.

In Section 2 we introduce notations, properties of rules, and the relevant domains. Equalities of rules on the midpoint and middle domains are studied in Section 3. We devote Section 4 to show whether or not the adjusted proportional and the minimal overlap rules satisfy some additional properties that include others-regarding claim monotonicity, order preservation under claims variations, progressivity, and regressivity. The Lorenz-based characterizations of the adjusted proportional rule are given in Section 5. Finally, the ranking of the ten rules is updated in Section 6. We include an Appendix with the results that are just technical in nature. The computations and figures in the examples were carried out using the *ClaimsProblems* R package (Núñez Lugilde et al., 2022).

2 | PRELIMINARIES

Let $\mathcal{N}$ be the set of all finite subsets of the natural numbers $\mathbb{N}$. Given $N \in \mathcal{N}, x \in \mathbb{R}^N$, and $S \in 2^N$ let $|N| = n$ be the number of elements of N and $x(S) = \sum_{i \in S} x_i$. If $N' \subset N \in \mathcal{N}$ and $x \in \mathbb{R}^N$, let $x_{N'} = (x_i)_{i \in N'} \in \mathbb{R}^{N'}$ be the projection of x onto $\mathbb{R}^{N'}$. In particular denote $x_{-i} = x_{N\backslash\{i\}} \in \mathbb{R}^{N\backslash\{i\}}$ the vector obtained by neglecting the ith-coordinate of x, that is, $x_{-i} = (x_1, ..., x_{i-1}, x_{i+1}, ..., x_n)$. For simplicity, we will write $x = (x_{-i}, x_i)$.

A claims problem with set of claimants $N \in \mathcal{N}$ is a pair (E, d) where $E \geq 0$ is the endowment to be divided and $d \in \mathbb{R}^N$ is the vector of claims satisfying $d_i \geq 0$ for all $i \in N$ and $d(N) \geq E$. We denote the class of claims problems with set of players N by C^N.

For each $(E, d) \in C^N$ and each $i \in N$ let $D_{-i} = d(N) - d_i = d(N\backslash\{i\})$. The minimal right and truncated claim of claimant $i \in N$ in $(E, d) \in C^N$ are the quantities $m_i(E, d) = \max\{0, E - D_{-i}\}$ and $t_i(E, d) = \min\{E, d_i\}$, respectively. Let $m(E, d) = (m_i(E, d))_{i \in N}$ and $t(E, d) = (t_i(E, d))_{i \in N}$. Let us write $t = t(E, d)$ and $m = m(E, d)$ if no confusion is possible.

Let $\mathbb{R}^n_{\leq}$ be the set of nonnegative n-dimensional vectors $x = (x_1, ..., x_n)$ with coordinates ordered from small to large, that is, $0 \leq x_1 \leq \cdots \leq x_n$. For simplicity, given $(E, d) \in C^N$ with $|N| = n$, we will assume throughout the paper that $N = \{1, ..., n\}$ and that $d \in \mathbb{R}^n_{\leq}$. As a consequence of such an arrangement of the claims we have that $d_i \leq D_{-i}, D_{-i} \geq D_{-(i+1)}$ and $m_i \leq m_{i+1}$ for all $i \in N\backslash\{n\}$. Nevertheless, as it is illustrated in Figure 1, we can either have $d_n \leq D_{-n}$ or $D_{-n} \leq d_n$. In any case, $\frac{1}{2}d(N)$ is the middle point of the line segment with endpoints d_n and D_{-n}. In fact, $\frac{1}{2}d(N)$ is also the middle point of the intervals $[d_i, D_{-i}]$ for all $i \in N\backslash\{n\}$.

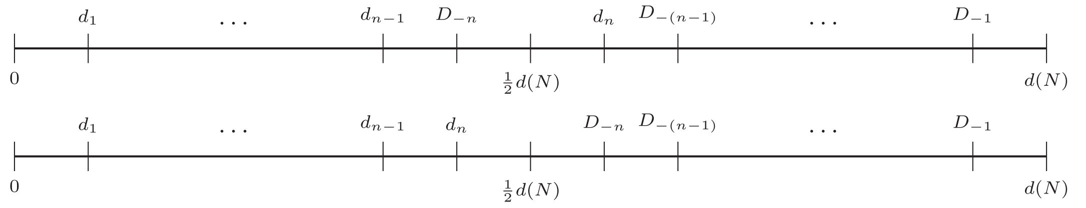

FIGURE 1 Claims arranged in ascending order on the interval $[0, d(N)]$

The claims problems $(E, d) \in C^N$ and $(d(N) - E, d) \in C^N$ are dual claims problems. Given a domain of claims problems $\Omega \subset C^N$ the domain of dual claims problems $\Omega^* = \{(d(N) - E, d) \in C^N : (E, d) \in \Omega\}$ is the dual domain of Ω. The lower-half domain $C_L^N = \{(E, d) \in C^N : E \leq \frac{1}{2}d(N)\}$ is the subdomain of claims problems for which the endowment is lower than the half-sum of claims. The subdomain of claims problems for which the endowment is bigger than the half-sum of claims, $C_H^N = \{(E, d) \in C^N : E \geq \frac{1}{2}d(N)\}$, is called the higher-half domain. Naturally, the lower-half domain and the higher-half domain are dual. Let us called the intersection $C_L^N \cap C_H^N = \{(E, d) \in C^N : E = \frac{1}{2}d(N)\}$ the midpoint domain: the class of claims problems for which the amount to divide is exactly the half-sum of the claims.

We say that a claims problem $(E, d) \in C^N$ belongs to the middle domain C_M^N if one of the following conditions holds: (1) $D_{-n} \leq E \leq d_n$ or (2) $d_n \leq D_{-n}$ and $E = \frac{1}{2}d(N)$. Therefore,

$$C_M^N = \left\{(E, d) \in C^N : \min\left\{D_{-n}, \frac{1}{2}d(N)\right\} \leq E \leq \max\left\{d_n, \frac{1}{2}d(N)\right\}\right\}.$$

The middle domain includes the claims problems for which either it is feasible to satisfy all the agents' claims except for the one with the highest claim or, otherwise, the endowment coincides with the half-sum of claims. The intersections of the middle domain with the lower-half and higher-half domains are denoted:

$$C_{ML}^N = C_M^N \cap C_L^N = \left\{(E, d) \in C^N : \min\left\{D_{-n}, \tfrac{1}{2}d(N)\right\} \leq E \leq \tfrac{1}{2}d(N)\right\}$$
$$C_{MH}^N = C_M^N \cap C_H^N = \left\{(E, d) \in C^N : \tfrac{1}{2}d(N) \leq E \leq \max\left\{d_n, \tfrac{1}{2}d(N)\right\}\right\}.$$

Clearly, C_{ML}^N and C_{MH}^N are dual domains.

Thomson (2019) discusses another domain: claims problems such that no claim exceeds the endowment. A claims problem $(E, d) \in C^N$ is a simple claims problem if $E \geq d_i$ for all $i \in N$. Let us denote the domain of simple claims problems by $C_S^N = \{(E, d) \in C^N : E \geq d_i \text{ for all } i \in N\}$.

Given a vector of claims $d \in \mathbb{R}^n_{\leq}$, Figure 2 shows schematically the intervals in which the endowment E has to be so that the claims problem (E, d) belongs to each of the subdomains that we have defined.

A vector $x \in \mathbb{R}^N$ is an awards vector for $(E, d) \in C^N$ if $0 \leq x_i \leq d_i$ for all $i \in N$ and $x(N) = E$. Let $X(E, d)$ be the set of awards vectors for $(E, d) \in C^N$. A rule is a function $\mathcal{R} : C^N \to \mathbb{R}^N$ assigning to each claims problem $(E, d) \in C^N$ an awards vector

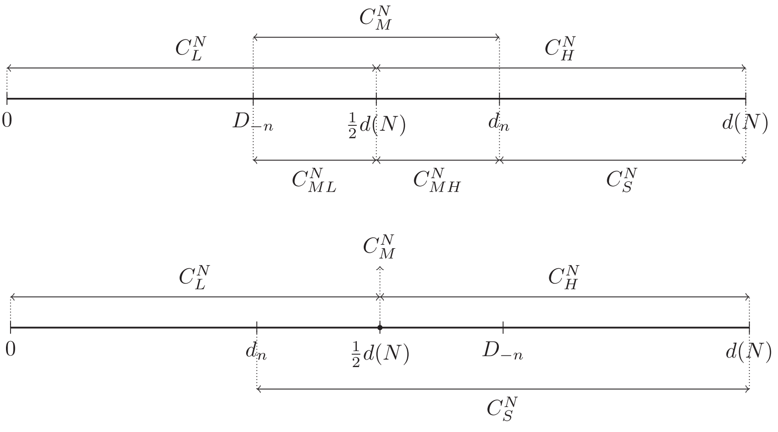

FIGURE 2 The subdomains of claims problems relevant to our study

$\mathcal{R}(E,d) \in X(E,d)$. That is, a rule is a way of associating with each claims problem a division among the claimants of the amount available satisfying three natural requirements: non-negativity (no claimant is asked to pay); claims boundedness (no claimant receives more than his claim); and the balance requirement (the entire endowment is allocated). It turns out, see Thomson (2019) for example, that the set of awards vectors for a claims problem $(E,d) \in C^N$ is the set of allocations satisfying the balance requirement that are bounded from below by the minimal rights vector and bounded from above by the truncated claims vector:[1]

$$\begin{aligned} X(E,d) &= \{x \in \mathbb{R}^N : 0 \le x_i \le d_i \text{ for all } i \in N, x(N) = E\} \\ &= \{x \in \mathbb{R}^N : m_i(E,d) \le x_i \le t_i(E,d) \text{ for all } i \in N, x(N) = E\}. \end{aligned}$$

We present a basic list of properties of rules. We say that a rule $\mathcal{R}$ satisfies:

- *anonymity* if for each $(E,d) \in C^N$, each $\pi \in \Pi^N$, and each $i \in N$, we have $\mathcal{R}_{\pi(i)}(E,(d_{\pi(i)})) = \mathcal{R}_i(E,d)$, where Π^N is the class of bijections from N into itself.
- *the midpoint property* if $\mathcal{R}(\frac{1}{2}d(N),d) = \frac{d}{2}$.
- *self-duality* if for each $(E,d) \in C^N$ we have $\mathcal{R}(E,d) = d - \mathcal{R}(d(N)-E,d)$.
- *minimal rights first* if for each $(E,d) \in C^N$ we have $\mathcal{R}(E,d) = m + \mathcal{R}\Big(E - \sum_{i \in N} m_i, d-m\Big)$.
- *claims truncation invariance* if for each $(E,d) \in C^N$ we have $\mathcal{R}(E,d) = \mathcal{R}(E,t)$.
- *order preservation in awards* if for each $(E,d) \in C^N$ we have $\mathcal{R}_i(E,d) \le \mathcal{R}_{i+1}(E,d)$ for all $i \in N \backslash \{n\}$.
- *order preservation in losses* if for each $(E,d) \in C^N$ we have $d_i - \mathcal{R}_i(E,d) \le d_{i+1} - \mathcal{R}_{i+1}(E,d)$ for all $i \in N \backslash \{n\}$.
- $\frac{1}{|N|}$*-truncated-claims lower bounds on awards*, if for each $(E,d) \in C^N$ then $\mathcal{R}_i(E,d) \ge \frac{1}{|N|} t_i(E,d)$ for all $i \in N$.

[1] For each $(E,d) \in C^N$ consider the coalitional game with set of players N and characteristic function $v(S) = \max\{0, E - d(N \backslash S)\}, S \in 2^N$. Then $X(E,d)$ coincides with the core of the coalitional game v.

- *endowment monotonicity* if for each $(E, d) \in C^N$ and each $E' \geq E$, if $d(N) \geq E' \geq E$ we have $\mathcal{R}_i(E', d) \geq \mathcal{R}_i(E, d)$ for all $i \in N$.
- *claim monotonicity* if for each $(E, d) \in C^N$, each $i \in N$, and each $d_i' > d_i$, then $\mathcal{R}_i(E, (d_{-i}, d_i')) \geq \mathcal{R}_i(E, d)$.

Order preservation is the property obtained as the combination of order preservation in awards and order preservation in losses. With each rule $\mathcal{R}$ we can associate a unique dual rule $\mathcal{R}^*$, defined by $\mathcal{R}^*(E, d) = d - \mathcal{R}(d(N) - E, d)$. Two properties are dual if, whenever a rule satisfies one of them, its dual satisfies the other. A property is self-dual if it coincides with its dual. Clearly, if a rule $\mathcal{R}$ satisfies a property when restricted to a domain $\Omega \subset C^N$ then its dual rule $\mathcal{R}^*$ satisfies the dual property on the dual domain Ω^*. Throughout this paper we consider ten rules: the proportional rule (PRO), the constrained equal awards rule (CEA), the constrained equal losses rule (CEL), the constrained egalitarian rule (CE), Piniles' rule (PIN), the Talmud rule (T), the random arrival rule (RA), the average-of-awards rule (AA), the adjusted proportional rule (APRO), and the minimal overlap rule (MO). The formal definitions are given in Appendix A. Table 1, adapted from Thomson (2019) and Mirás Calvo et al. (2022b), summarizes which of the above properties are satisfied by these rules. A check mark, ✓, in a cell means that the property in the row is satisfied by the rule indexing the column. A minus sing, –, means the opposite.

We have assumed that given a claims problem $(E, d) \in C^N$ the coordinates of the vector of claims are ordered from small to large, that is, $d \in \mathbb{R}^n_{\leq}$. Since the ten rules that we discuss in this paper are anonymous this assumption is fully justified. Moreover, if $\mathcal{R}$ is a rule that satisfies order preservation in awards (the ten rules satisfy this property) then $\mathcal{R}(E, d) \in \mathbb{R}^n_{\leq}$.

3 | RULES THAT COINCIDE ON THE MIDDLE DOMAIN

First of all, note that, by definition, the CE, T, and PIN rules make the same recommendation when restricted to claims problems that belong to the lower-half domain, that is, $\text{CE}(E, d) = T(E, d) = \text{PIN}(E, d)$ for all $(E, d) \in C_L^N$. Obviously, the rules satisfying the midpoint property coincide on the midpoint domain $C_L^N \cap C_H^N$. Therefore,

TABLE 1 Main properties satisfied by the 10 rules

	PRO	APRO	MO	CEA	CEL	CE	PIN	T	RA	AA
Anonymity	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
Midpoint	✓	✓	–	–	–	✓	✓	✓	✓	✓
Self-duality	✓	✓	–	–	–	–	–	✓	✓	✓
Minimal rights first	–	✓	✓	–	✓	–	–	✓	✓	✓
Claims truncation invariance	–	✓	✓	✓	–	–	–	✓	✓	✓
Order preservation	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
$\frac{1}{\mid N \mid}$-truncated-claims lower bounds	–	✓	✓	✓	–	✓	✓	✓	✓	✓
Endowment monotonicity	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
Claim monotonicity	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓

$\mathrm{CE}(E,d)=\mathrm{PIN}(E,d)=T(E,d)=\mathrm{RA}(E,d)=\mathrm{AA}(E,d)=\mathrm{APRO}(E,d)=\mathrm{PRO}(E,d)$ whenever $(E,d)\in C_L^N\cap C_H^N$. We devote this section to identify other domains where some of the central rules coincide.

Given a two-claimant problem $(E,d)\in C^N$ with $N=\{1,2\}, d=(d_1,d_2)\in\mathbb{R}^2_{\leq}$, that is, $0\leq D_{-2}=d_1\leq d_2$, the set of awards vectors $X(E,d)$ is the line segment with endpoints $(m_1,E-m_1)$ and $(E-m_2,m_2)$, where $m_1=\max\{0,E-d_2\}$ and $m_2=\max\{0,E-d_1\}$. Therefore, for two-claimant problems the middle-domain is $C_M^N=\{(E,(d_1,d_2))\in C^N: d_1\leq E\leq d_2\}$ which strictly contains the midpoint domain unless $d_1=d_2$. The concede-and-divide rule (CD) is the two-claimant rule that first assigns to each claimant the difference between the endowment and the other agent's claim (or 0 if this difference is negative), and divides the remainder equally:

$$\mathrm{CD}(E,(d_1,d_2))=\begin{cases}\left(\frac{E}{2},\frac{E}{2}\right) & \text{if } 0\leq E\leq d_1\\ \left(\frac{d_1}{2},E-\frac{d_1}{2}\right) & \text{if } d_1\leq E\leq d_2\\ \left(\frac{E+d_1-d_2}{2},\frac{E-d_1+d_2}{2}\right) & \text{if } d_2\leq E\leq d_1+d_2\end{cases}.$$

Geometrically, the CD rule selects the middle point of the line segment $X(E,d)$ so it coincides with the AA rule. Since, for $N=\{1,2\}$, the CD rule is the unique two-claimant rule satisfying the midpoint property, minimal rights first, and claims truncation invariance, then $\mathrm{CD}(E,d)=\mathrm{AA}(E,d)=\mathrm{APRO}(E,d)=T(E,d)=\mathrm{RA}(E,d)=\mathrm{MO}(E,d)$ for all $(E,d)\in C^N$. Moreover, for each claims problem in the middle domain $(E,d)\in C_M^N=\{(E,(d_1,d_2))\in C^N: d_1\leq E\leq d_2\}$ we have that $m_1=0$ and $m_2=E-d_1$ so $X(E,d)=\{(x_1,x_2)\in\mathbb{R}^2: 0\leq x_1\leq d_1, x_2=E-x_1\}$. Therefore, for all claims problem in the middle domain $(E,d)\in C_M^N$ the CD rule recommends the constant amount $\frac{d_1}{2}$ to the first claimant and the remainder $E-\frac{d_1}{2}$ to the other agent.

Let us see how, for an arbitrary population, the two-claimant structure of the set of awards vectors for problems on the middle domain is replicated. Let N be an arbitrary set of claimants and $(E,d)\in C_M^N$ such that $D_{-n}\leq E\leq d_n$. Then the set of awards vectors $X(E,d)$ has a very simple structure:

$$X(E,d)=\left\{x\in\mathbb{R}^n: x_{-n}\in\prod_{i=1}^{n-1}[0,d_i], x_n=E-x(N\backslash\{n\})\right\}.$$

So, as in the two-claimant case, a natural recommendation for a claims problem that belongs to the middle domain $(E,d)\in C_M^N$ is to assign the division that gives to all the claimants, except the last, the geometric center of the $(n-1)$-rectangle $\prod_{i=1}^{n-1}[0,d_i]$, and to the last claimant what is left. Naturally, this is the division selected by the AA rule:

$$\mathrm{AA}(E,d)=\left(\frac{d_1}{2},\ldots,\frac{d_{n-1}}{2},E-\frac{D_{-n}}{2}\right)\in X(E,d).$$

The rules that satisfy the midpoint property coincide on the midpoint domain and, with the exception of the MO rule, the rules that extend the CD rule coincide on the middle domain.

Proposition 3.1. *Let $(E, d) \in C^N$ and $\mathcal{R}$ be a rule that satisfies the midpoint property. We have that:*

1. *$\mathcal{R} = \mathrm{AA}$ on C^N_{ML} if either $\mathcal{R}$ satisfies claims truncation invariance and endowment monotonicity on C^N_{ML} or if $\mathcal{R}$ satisfies claims truncation invariance and minimal rights first on C^N_{ML}.*
2. *$\mathcal{R} = \mathrm{AA}$ on C^N_{MH} if either $\mathcal{R}$ satisfies minimal rights first and endowment monotonicity on C^N_{MH} or if $\mathcal{R}$ satisfies claims truncation invariance and minimal rights first on C^N_{MH}.*
3. *If $(E, d) \in C^N_M$ then $T(E, d) = \mathrm{RA}(E, d) = \mathrm{APRO}(E, d) = \mathrm{AA}(E, d)$.*
4. *If $(E, d) \in C^N_{ML}$ then $\mathrm{CE}(E, d) = \mathrm{PIN}(E, d) = T(E, d) = \mathrm{RA}(E, d) = \mathrm{APRO}(E, d) = \mathrm{AA}(E, d)$.*

Proof. Let $\mathcal{R}$ be a rule that satisfies claims truncation invariance and endowment monotonicity on C^N_{ML}, and also the midpoint property. Take $(E, d) \in C^N_{ML}$. If $\min\{D_{-n}, \frac{1}{2}d(N)\} = \frac{1}{2}d(N)$ then $E = \frac{1}{2}d(N)$, and, by the midpoint property, $\mathcal{R}(\frac{1}{2}d(N), d) = \mathrm{AA}(\frac{1}{2}d(N), d) = \frac{d}{2}$. On the other hand, if $\min\{D_{-n}, \frac{1}{2}d(N)\} = D_{-n}$ then $E \in [D_{-n}, \frac{1}{2}d(N)]$ and $E \le d_n$. By claims truncation invariance and the midpoint property,

$$\mathcal{R}(D_{-n}, d) = \mathcal{R}(D_{-n}, (d_{-n}, D_{-n})) = \left(\frac{d_1}{2}, \ldots, \frac{d_{n-1}}{2}, \frac{D_{-n}}{2}\right),$$
$$\mathcal{R}\left(\frac{1}{2}d(N), d\right) = \left(\frac{d_1}{2}, \ldots, \frac{d_{n-1}}{2}, \frac{d_n}{2}\right).$$

But, since $\mathcal{R}$ satisfies endowment monotonicity on C^N_{ML}, for each $E \in (D_{-n}, \frac{1}{2}d(N))$ and each $j \in N \backslash \{n\}$, we have $\mathcal{R}_j(D_{-n}, d) \le \mathcal{R}_j(E, d) \le \mathcal{R}_j(\frac{1}{2}d(N), d)$, so, necessarily, $\mathcal{R}_j(E, d) = \frac{d_j}{2}$. Therefore, $\mathcal{R}(E, d) = \mathrm{AA}(E, d)$.

Assume now that $\mathcal{R}$ is a rule that satisfies claims truncation invariance and minimal rights first on C^N_{ML}, and the midpoint property. Let $(E, d) \in C^N_{ML}$. When $\min\{D_{-n}, \frac{1}{2}d(N)\} = \frac{1}{2}d(N)$, the result is obvious. If $\min\{D_{-n}, \frac{1}{2}d(N)\} = D_{-n}$ then $E \in [D_{-n}, \frac{1}{2}d(N)]$ and $m(E, d) = (0, \ldots, 0, E - D_{-n})$. By claims truncation invariance, minimal rights first, and the midpoint property,

$$\mathcal{R}(E, d) = \mathcal{R}(E, t) = m + \mathcal{R}(D_{-n}, (d_{-n}, D_{-n})) = (0, \ldots, 0, E - D_{-n}) + \left(\frac{d_1}{2}, \ldots, \frac{d_{n-1}}{2}, \frac{D_{-n}}{2}\right) = \mathrm{AA}(E, d).$$

Therefore, the first statement holds. Finally, since claims truncation invariance and minimal rights first are dual properties and C^N_{ML} and C^N_{MH} are dual domains, the second statement follows at once.

The T, RA, and APRO rules satisfy the midpoint property, claims truncation invariance, minimal rights first, and endowment monotonicity. Therefore, $T(E, d) = \mathrm{RA}(E, d) = \mathrm{APRO}(E, d) = \mathrm{AA}(E, d)$ for all $(E, d) \in C^N_M$. Moreover, by definition, $\mathrm{CE}(E, d) = \mathrm{PIN}(E, d) = T(E, d)$ if $(E, d) \in C^N_{ML}$. □

4 | PROPERTIES OF THE APRO AND MO RULES ON SUBDOMAINS

Let us state some extra properties, all of them well-known in the literature, that have being used to characterize some of our basic rules. They will be relevant, in what follows, to compare the awards vectors recommended by the APRO, MO, and AA rules. In the remainder of the section, we analyze whether or not the APRO and MO rules satisfy these additional properties on the half domains.

We know that our 10 rules satisfy claim monotonicity, that is, if agent i's claim increases then his award should not decrease. Let us state two related properties. Other-regarding claim monotonicity requires that if agent i's claim increases then each of the other claimants should receive at most as much as initially. If there are at least three claimants, order preservation under claims variations says that given any two claimants whose claim remains the same, the change in the award to the smaller one should be at most as large as the change in the award to the larger one. A rule $\mathcal{R}$ satisfies:

- *other-regarding claim monotonicity* if for each $(E, d) \in C^N$, each $i \in N$, and each $d_i' > d_i$, then $\mathcal{R}_j(E, (d_{-i}, d_i')) \leq \mathcal{R}_j(E, d)$ for all $j \in N\backslash\{i\}$.
- *order preservation under claims variations* if for each $(E, d) \in C^N$ with $|N| \geq 3$, each $i \in N$, each $d_i' > d_i$, and each pair $\{j, k\} \subset N\backslash\{i\}$ such that $d_j \leq d_k$, then $\mathcal{R}_j(E, d) - \mathcal{R}_j(E, (d_{-i}, d_i')) \leq \mathcal{R}_k(E, d) - \mathcal{R}_k(E, (d_{-i}, d_i'))$.

Consider now situations in which the population of claimants involved may vary. In this case, a claims problem is defined by first specifying $N \in \mathcal{N}$, then a pair $(E, d) \in C^N$. So, a rule is a function defined on $\bigcup_{N\in\mathcal{N}} C^N$ that associates with each $N \in \mathcal{N}$ and each $(E, d) \in C^N$ an awards vector for (E, d). We say that a rule $\mathcal{R}$ satisfies:

- *null claims consistency*, if for each $N \in \mathcal{N}$, each $(E, d) \in C^N$ and each $N' \subset N$, if $d_i = 0$ for all $i \in N\backslash N'$, then $\mathcal{R}_{N'}(E, d) = \mathcal{R}(E, d_{N'})$.
- *population monotonicity*, if for each $N \in \mathcal{N}$, each $(E, d) \in C^N$, and each $N' \subset N$, if $d(N') \geq E$ then $\mathcal{R}_j(E, d) \leq \mathcal{R}_j(E, d_{N'})$ for all $j \in N'$.
- *order preservation under population variations*, if for each $(E, d) \in C^N$, each $i \in N$ with $E < d(N\backslash\{i\})$ and each pair $\{j, k\} \subseteq N\backslash\{i\}$, if $d_j \leq d_k$, then $\mathcal{R}_j(E, d_{-i}) - \mathcal{R}_j(E, d) \leq \mathcal{R}_k(E, d_{-i}) - \mathcal{R}_k(E, d)$.

It is well known that our ten rules satisfy null claims consistency, since the departure of agents whose claims are 0 has no effect on what the other agents are awarded. Population monotonicity implies that if the population of claimants enlarges but the endowment stays the same, then each of the claimants initially present should receive at most as much as initially. Order preservation under population variations says that when population decreases, given two remaining claimants the difference between the smaller claimant new and initial awards should be at least as large as the corresponding difference for the largest claimant.

Finally we recall a pair of dual properties. Progressivity requires that if the claim of agent i is at most as large as the claim of agent j, agent i should receive proportionally at most as much as agent j. The dual requirement is regressivity. A rule $\mathcal{R}$ satisfies:

- *progressivity* if for each $(E, d) \in C^N$ and each pair $\{i, j\} \subset N$, if $0 < d_i \leq d_j$ then $\frac{\mathcal{R}_i(E, d)}{d_i} \leq \frac{\mathcal{R}_j(E, d)}{d_j}$.

- *regressivity* if for each $(E, d) \in C^N$ and each pair $\{i, j\} \subset N$, if $0 < d_i \leq d_j$ then $\frac{\mathcal{R}_i(E,d)}{d_i} \geq \frac{\mathcal{R}_j(E,d)}{d_j}$.

Clearly, the PRO rule is the only rule to be both regressive and progressive. The CEA rule is regressive and the CEL rule is progressive. Of course, there are rules that violate both properties even when restricted to the lower-half or higher-half domains.[2] The combination of regressivity on the lower half-domain and progressivity on the higher half-domain has a clear economic interpretation. If a rule satisfies regressivity on the lower-half domain, for each problem with an endowment smaller than the half-sum of claims, given any two claimants, the one with the smaller claim receives at least the same amount (per unit of claim) than the one with the higher claim. Basically, agents with lower claims benefit when the endowment is small. But, when the endowment is higher than the half-sum of claims, a progressive rule on the higher-half domain is biased toward the agents with larger claims.

Table 2 summarizes the behavior of the APRO and the MO rules with respect to the additional properties on both half-domains. Again, a check mark, ✓, in a cell means that the property is satisfied by the rule and a minus sing, −, means the opposite. Moreover, some cells include the reference to the result where the corresponding mark is established. In any case, all signs are discussed below.

4.1 | APRO rule

A simple expression for the allocation selected by the APRO rule restricted to claims problems in C_L^N is given in Lemma B.1. Based on this formula, the positive results in Table 2 for the APRO rule are established in Appendix B.[3]

Now, let us justify the negative marks. Since progresivity and regressivity are dual properties, the APRO rule is self-dual, and the PRO rule is the only rule to be both progressive and regressive on C^N, then the APRO rule fails progressivity on the lower-half domain and regressivity on the higher-half domain.

Grahn and Voorneveld (2002) present a four-claimant example where the APRO rule violates population monotonicity.[4] Our next example illustrates that the APRO rule violates not only population monotonicity but also order preservation under population variation.

[2]For instance, the RA rule. Indeed, let $N = \{1, 2, 3\}$ and $d = (1, 5, 5)$. Now, $(2, d) \in C_L^N$ and $(4, d) \in C_L^N$. But, $\frac{\text{RA}_1(2,d)}{d_1} = \frac{1}{3} > \frac{1}{6} = \frac{\text{RA}_2(2,d)}{d_2}$ and $\frac{\text{RA}_1(4,d)}{d_1} = \frac{1}{3} < \frac{11}{30} = \frac{\text{RA}_2(4,d)}{d_2}$. As a consequence, the RA rule is neither regressive nor progressive on C_L^N. Since the RA rule is self-dual we conclude that it is neither regressive nor progressive on C_H^N.

[3]Linked endowment-population monotonicity is the dual property of population monotonicity. Since C_L^N and C_H^N are dual domains, and the APRO rule is self-dual, then the APRO rule satisfies linked endowment-population monotonicity on C_H^N. A similar argument shows that the APRO rule satisfies the dual properties of other-regarding claim monotonicity, order preservation under claims variations, and order preservation under population variations on C_H^N. These dual properties are described in Thomson (2019) but they are not given a specific name.

[4]In fact, these authors show that the APRO rule does not satisfy linked-endowment population monotonicity, that they called the thieve property. Let $N = \{1, 2, 3, 4\}$, $E = 12$, and $d = (1, 2, 9, 10)$. Then $\text{APRO}(E, d) = \left(\frac{6}{11}, \frac{12}{11}, \frac{54}{11}, \frac{60}{11}\right)$ and $\text{APRO}(E, d_{-2}) = \left(\frac{9}{17}, \frac{89}{17}, \frac{106}{17}\right)$. So, when claimant 2 leaves, claimant 1 receives less than initially, and hence the APRO rule is not population monotonic.

TABLE 2 Properties satisfied by the APRO and MO rules on the half domains

	APRO		MO	
	C_L^N	C_H^N	C_L^N	C_H^N
Null claims consistency	✓	✓	✓	✓
Other-regarding claim monotonicity	✓ (B.3)	– (4.2)	✓	✓
Order preservation under claims variations	✓ (B.3)	– (4.2)	✓	✓
Population monotonicity	✓ (B.4)	– (4.1)	✓	✓
Order preservation under population variation	✓ (B.5)	– (4.1)	✓	✓
Progressivity	–	✓ (B.2)	– (4.4)	✓ (C.3)
Regressivity	✓ (B.2)	–	– (4.4)	–

Example 4.1. Let $N = \{1, 2, 3, 4, 5\}, E = 17,$ and $d = (1, 2, 3, 8, 10)$. Consider the problems $(E, d) \in C_H^N$ and $(E, d_{-3}) \in C_H^{N\backslash\{3\}}$. Then

$$\text{APRO}(E, d) = \left(\frac{13}{20}, \frac{26}{20}, \frac{39}{20}, \frac{111}{20}, \frac{151}{20}\right), \text{APRO}(E, d_{-3}) = \left(\frac{7}{11}, \frac{14}{11}, \frac{72}{11}, \frac{94}{11}\right).$$

So, when claimant 3 leaves, claimants 1 and 2 receive less than initially, and hence the APRO rule is not population monotonic. Moreover, $\text{APRO}_1(E, d_{-3}) - \text{APRO}_1(E, d) = -\frac{3}{220} > -\frac{3}{110} = \text{APRO}_2(E, d_{-3}) - \text{APRO}_2(E, d)$, so the APRO rule does not satisfy order preservation under population variation.

Most of the central rules satisfy other-regarding claim monotonicity and order preservation under claims variations but not the APRO rule.

Proposition 4.2. *The adjusted proportional rule satisfies neither other-regarding claim monotonicity nor order preservation under claims variations.*

Proof. Thomson (2019) shows that if a rule $\mathcal{R}$ satisfies null claims consistency and other-regarding claim monotonicity then $\mathcal{R}$ satisfies population monotonicity. Similarly, if a rule $\mathcal{R}$ satisfies null claims consistency and order preservation under claims variation then $\mathcal{R}$ satisfies order preservation under population variation. Indeed, let $N \in \mathcal{N}, (E, d) \in C^N, i \in N$ with $E < d(N\backslash\{i\})$, and $\{j, k\} \subseteq N\backslash\{i\}$ such that $d_j \leq d_k$. Then,

$$\begin{aligned}\mathcal{R}_k(E, d) - \mathcal{R}_k(E, d_{-i}) &= \mathcal{R}_k(E, d) - \mathcal{R}_k(E, (d_{-i}, 0)) \\ &\leq \mathcal{R}_j(E, d) - \mathcal{R}_j(E, (d_{-i}, 0)) = \mathcal{R}_j(E, d) - \mathcal{R}_j(E, d_{-i}),\end{aligned}$$

where the inequality holds by order preservation under claims variations, and the equalities by null claims consistency. Now, the APRO rule satisfies null claims consistency but, as shown in Example 4.1, violates both population monotonicity and order preservation under population variation. Therefore, the APRO rule fails other-regarding claim monotonicity and order preservation under claims variations. □

The statement of Proposition 4.2 is somehow surprising.[5] If agent i's claim increases, his minimal right doesn't change, but the minimal rights of the other claimants decrease in an order preserving manner. That is, if $(E, d) \in C^N$, $i \in N$, $d_i < d_i'$, and $d' = (d_{-i}, d_i')$ then $m(E, d) \geq m(E, d')$ and $m_j(E, d) - m_j(E,d') \leq m_k(E, d) - m_k(E,d')$ for all $j, k \in N \backslash \{i\}$ such that $j \leq k$. Therefore, $M = \sum_{j \in N} m_j(E, d) \geq M' = \sum_{j \in N} m_j(E,d')$. Therefore $M = \sum_{j \in N} m_j(E, d) \geq M' = \sum_{j \in N} m_j(E, d')$. Now, $\text{APRO}(E, d) = m(E, d) + \text{PRO}(E - M, t(E - M, d - m(E, d)))$ and $\text{APRO}(E, d') = m(E, d') + \text{PRO}(E - M', t(E - M', d' - m(E, d')))$. The problems $(E - M, t(E - M, d - m(E, d)))$ and $(E - M', t(E - M', d' - m(E, d')))$ differ not only on agent i's claim but also on the endowment and on the claims of some other agents. So even though the PRO rule satisfies other-regarding claim monotonicity and order preservation under claims variations, we can not conclude that, when applied to the revised problems, we obtain the wanted inequalities for the APRO rule. It is easy to check that the APRO rule satisfies other-regarding claim monotonicity for three-claimant problems and order preservation under claims variations for four-claimant problems.[6] The following example not only presents a problem that illustrates the conclusions of Proposition 4.2, but it also shows that other-regarding claim monotonicity and order preservation under claims variations are not preserved under the attribution of minimal rights operator.

Example 4.3. Let $N = \{1, 2, 3, 4, 5\}$, $E = 17$, $d = (1, 2, 3, 8, 10)$, and $d' = (1, 2, 4, 8, 10)$. Clearly, $(E, d) \in C_H^N$, $(E, d') \in C_H^N$, $m(E, d) = (0, 0, 0, 1, 3)$, and $m(E, d') = (0, 0, 0, 0, 2)$, so

$$\text{APRO}(E, d) = (0, 0, 0, 1, 3) + \text{PRO}(13, (1, 2, 3, 7, 7)) = \left(\frac{13}{20}, \frac{26}{20}, \frac{39}{20}, \frac{111}{20}, \frac{151}{20}\right)$$
$$\text{APRO}(E, d') = (0, 0, 0, 0, 2) + \text{PRO}(15, (1, 2, 4, 8, 8)) = \left(\frac{15}{23}, \frac{30}{23}, \frac{60}{23}, \frac{120}{23}, \frac{166}{23}\right).$$

Since $\text{APRO}_1(E, d') > \text{APRO}_1(E, d)$, the APRO rule violates other-regarding claim monotonicity. Moreover,

$$\text{APRO}_1(E, d) - \text{APRO}_1(E, d') = -\frac{1}{460} > -\frac{1}{230} = \text{APRO}_2(E, d) - \text{APRO}_2(E, d').$$

Therefore, order preservation under claims variations is also violated by the APRO rule.

For each $x \in [3, 5]$ let $d_x = (1, 2, x, 8, 10)$ and consider, for each $i \in \{1, 2\}$, the increments $\Delta_i(x) = \text{APRO}_i(E, d) - \text{APRO}_i(E, d_x)$. As Figure 3 illustrates, $\Delta_2(x) < \Delta_1(x) < 0$ for all $x \in \left(3, \frac{25}{6}\right)$. Then, in fact, the APRO rule violates both properties for the problems (E, d) and (E, d_x) with x in the range $\left(3, \frac{25}{6}\right)$.

[5]For instance, Bosmans and Lauwers (2011) state that the APRO rule satisfies order preservation under claims variations.

[6]Here is an example of a four-claimant problem for which the APRO rule fails other-regarding claim monotonicity. Let $N = \{1, 2, 3, 4\}$, $E = 12$, $d = (0.5, 1, 9, 10)$, and $d' = (1, 1, 9, 10)$. Then, $\text{APRO}_2(E, d') = \frac{11}{20} > \frac{20}{37} = \text{APRO}_2(E, d)$.

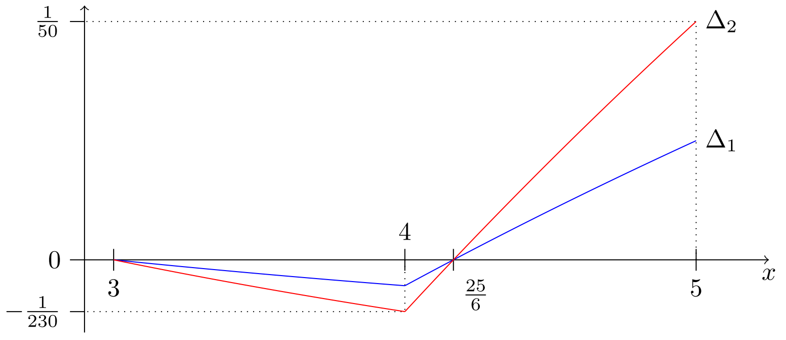

FIGURE 3 The increments $\Delta_1(x)$ and $\Delta_2(x)$ for $x \in [3, 5]$ [Color figure can be viewed at wileyonlinelibrary.com]

4.2 | MO rule

As we already point out in the proof of Proposition 4.2, null claims consistency and other-regarding claim monotonicity (or order preservation under claims variation) together imply population monotonicity (respectively, order preservation under population variation). It is known, see Bosmans and Lauwers (2011) and Thomson (2019), that the MO rule satisfies both other-regarding claim monotonicity and order preservation under claims variations. Therefore, the MO rule also satisfies population monotonicity and order preservation under population variation.

Next, we show that the MO rule is neither regressive nor progressive on the lower-half domain.

Example 4.4. Let $N = \{1, 2, 3\}$, $E = 5.3$, and $d = (1, 4, 7)$. Clearly, $(E, d) \in C_L^N$. Since, $d_2 < E < d_3$, we have that $\text{MO}(E, d) = \left(\frac{d_1}{3}, \frac{d_1}{3} + \frac{d_2 - d_1}{2}, \frac{d_1}{3} + \frac{d_2 - d_1}{2} + E - d_2\right) = \left(\frac{1}{3}, \frac{11}{6}, \frac{47}{15}\right)$. Since $\frac{\text{MO}_1(E,d)}{d_1} = \frac{1}{3} < \frac{11}{24} = \frac{\text{MO}_2(E,d)}{d_2}$ the MO rule is not regressive. But, $\frac{\text{MO}_2(E,d)}{d_2} = \frac{11}{24} > \frac{47}{105} = \frac{\text{MO}_3(E,d)}{d_3}$ so the MO rule fails progressivity.

Recall that the MO rule is not self-dual. Therefore, Example 4.4 provides no information about whether or not the MO rule is progressive or regressive on the higher-half domain. But, since the MO and APRO rules coincide with the CD rule for two-claimant problems and, as we have seen, the APRO rule is not regressive on the higher-half domain we conclude that the MO rule fails regressivity on C_H^N. The domains C_S^N and C_H^N have non-empty intersection but they are not comparable by inclusion. In any case, we prove in Proposition C.2 that the MO rule satisfies progressivity on the domain C_S^N of simple claims problems. We also show, see Proposition C.3, that the MO rule satisfies progressivity on the higher-half domain.

Remark 4.5. Let $N = \{1, 2, 3\}$, $E = 5.3$, and $d = (1, 4, 7)$ as in Example 4.4. Since $(E, d) \in C_{ML}^N$ and $\frac{\text{MO}_2(E,d)}{d_2} = \frac{11}{24} > \frac{47}{105} = \frac{\text{MO}_3(E,d)}{d_3}$, we have an example that shows that progressivity of the MO rule can not be extended beyond the domains C_S^N and C_H^N.

As Thomson (2019) notes, progressivity and regressivity are not preserved under claims truncation. Nevertheless, given $(E, d) \in C^N$ then the truncated problem belongs to the domain of simple claims problems, that is, $(E, t(E, d)) \in C_S^N$. Since the MO rule is progressive on C_S^N, the MO rule satisfies progressivity on the truncated problem. Now, the MO rule is claims truncation invariant. Therefore, for each $(E, d) \in C^N$ and each pair

$\{i, j\} \subset N$, if $0 < d_i \leq d_j$ then $\frac{\mathrm{MO}_i(E,d)}{t_i} \leq \frac{\mathrm{MO}_j(E,d)}{t_j}$. That is, if the claim of agent i is at most as large as the claim of agent j, agent i should receive per unit of truncated claim at most as much as agent j. In fact, in our running example, the truncated claims vector is $t(E, d) = (1, 4, 5.3)$ and $\frac{\mathrm{MO}_1(E,d)}{t_1} = \frac{1}{3} < \frac{\mathrm{MO}_2(E,d)}{t_2} = \frac{11}{24} < \frac{\mathrm{MO}_3(E,d)}{t_3} = \frac{94}{159}$.

5 | LORENZ-BASED CHARACTERIZATIONS

The CE, CEA, PIN, MO, and T rules have being characterized as being maximal or minimal with respect to the Lorenz relation within a class of rules. We provide in this section a new characterization of the APRO rule as being Lorenz-maximal or Lorenz-minimal within a class of rules on the half-domains. Using this result, we are able to rank the APRO, MO, and AA rules.

Given a claims problem $(E, d) \in C^N$, an awards vector $x \in X(E, d)$ Lorenz-dominates an awards vector $y \in X(E, d)$ if all the cumulative sums of the rearranged coordinates are greater with x than with y.

Definition 5.1. Let $x, y \in \mathbb{R}^n_{\leq}$. We say that x Lorenz-dominates y, and write $x \succcurlyeq y$, if for each $k = 1, ..., n - 1$,

$$\sum_{j=1}^{k} x_j \geq \sum_{j=1}^{k} y_j \text{ and } \sum_{j=1}^{n} x_j = \sum_{j=1}^{n} y_j.$$

The Lorenz order is a partial order in $\mathbb{R}^n_{\leq}$, so it is a binary relation that is reflexive, antisymmetric, and transitive. If x Lorenz-dominates y and $x \neq y$, then at least one of the $n - 1$ inequalities is strict.

Recall that, if $(E, d) \in C^N$ and $\mathcal{R}$ is any of the ten rules, since $d \in \mathbb{R}^n_{\leq}$ we have that $\mathcal{R}(E, d) \in \mathbb{R}^n_{\leq}$. Therefore, we can use the Lorenz order to check whether a rule is more favorable to smaller claimants relative to larger claimants than other.

Definition 5.2. Let $\mathcal{R}$ and $\mathcal{R}'$ be two rules that satisfy order preservation in awards. We say that $\mathcal{R}$ Lorenz-dominates $\mathcal{R}'$ on the domain $\Omega \subset C^N$, and we write $\mathcal{R} \succcurlyeq \mathcal{R}'$, if $\mathcal{R}(E, d) \succcurlyeq \mathcal{R}'(E, d)$ for all $(E, d) \in \Omega$.

It is well known that the duality operator reverses the Lorenz order, so $\mathcal{R} \succcurlyeq \mathcal{R}'$ on Ω if and only if $(\mathcal{R}')^* \succcurlyeq \mathcal{R}^*$ on Ω^*. The CEA rule is Lorenz-maximal in the set of rules that preserve the order of awards. Then, its dual, the CEL rule, is Lorenz-minimal in the set of rules that satisfy order preservation. If two rules $\mathcal{R}$ and $\mathcal{R}'$ satisfy self-duality, then $\mathcal{R}$ and $\mathcal{R}'$ cannot be compared on the entire domain C^N.

Let us show that given a subdomain $\Omega \subset C^N$, the PRO rule Lorenz-dominates on Ω any rule that is progressive on Ω. Naturally, the PRO rule is Lorenz-dominated on Ω by any rule that is regressive on Ω.

Proposition 5.3. *Let $\Omega \subset C^N$ and let $\mathcal{R}$ be a rule that satisfies order preservation in awards.*

1. *If $\mathcal{R}$ is progressive on $\Omega \subset C^N$ then the proportional rule Lorenz-dominates $\mathcal{R}$ on Ω.*
2. *If $\mathcal{R}$ is regressive on $\Omega \subset C^N$ then the proportional rule is Lorenz-dominated by $\mathcal{R}$ on Ω.*

Proof. Let $\mathcal{R}$ be a rule that satisfies order preservation in awards and that is progressive on $\Omega \subset C^N$. Given $(E, d) \in \Omega$ and $k \in N \backslash \{n\}$ we have to prove that $\sum_{j=1}^{k} \mathcal{R}_j(E, d) \leq \sum_{j=1}^{k} \text{PRO}_j(E, d) = \frac{E}{d(N)} \sum_{j=1}^{k} d_j$ or, equivalently, $d(N) \sum_{j=1}^{k} \mathcal{R}_j(E, d) \leq E \sum_{j=1}^{k} d_j$. Since $E = \sum_{j=1}^{n} \mathcal{R}_j(E, d)$, the last inequality can be written as

$$\left(\sum_{j=1}^{k} d_j + \sum_{i=k+1}^{n} d_i\right) \sum_{j=1}^{k} \mathcal{R}_j(E, d) \leq \left(\sum_{j=1}^{k} \mathcal{R}_j(E, d) + \sum_{i=k+1}^{n} \mathcal{R}_i(E, d)\right) \sum_{j=1}^{k} d_j.$$

Therefore, we have to show that $\sum_{j=1}^{k} \mathcal{R}_j(E, d) \sum_{i=k+1}^{n} d_i \leq \sum_{i=k+1}^{n} \mathcal{R}_i(E, d) \sum_{j=1}^{k} d_j$. If $\mathcal{R}$ is progressive on $\Omega \subset C^N$ then $d_i \mathcal{R}_j(E, d) \leq \mathcal{R}_i(E, d) d_j$ for all $j \in \{1, \ldots k\}$ and all $i \in \{k+1, \ldots n\}$. Therefore, $\mathcal{R}_j(E, t) \sum_{i=k+1}^{n} d_i \leq d_j \sum_{i=k+1}^{n} \mathcal{R}_i(E, d)$ for all $j \in \{1, \ldots, k\}$. But then $\sum_{j=1}^{k} \mathcal{R}_j(E, d) \sum_{i=k+1}^{n} d_i \leq \sum_{i=k+1}^{n} \mathcal{R}_i(E, d) \sum_{j=1}^{k} d_j$ and, indeed, $\text{PRO}(E, d) \succcurlyeq \mathcal{R}(E, d)$. On the other hand, if $\mathcal{R}$ is regressive on $\Omega \subset C^N$, a similar argument shows that $\mathcal{R}(E, d) \succcurlyeq \text{PRO}(E, d)$, which proves the second statement. □

Chun et al. (2001) characterize the CE rule as being Lorenz-maximal within the subclass of rules that satisfy order preservation in awards, endowment monotonicity, and the midpoint property. Since the T rule is self-dual, the CE and the T rules are equal on C_L^N, and the domains C_L^N and C_H^N are dual, it is easy to conclude that, when restricted to C_H^N, the T rule is Lorenz-minimal within the subclass of rules that satisfy order preservation in losses, endowment monotonicity, and the midpoint property. Let us state these two results and also the Lorenz-based characterization of the MO rule proved by Bosmans and Lauwers (2011).

1. Let $\mathcal{S}$ be the set of rules that satisfy order preservation in awards, endowment monotonicity, and the midpoint property. The CE rule is the only rule in $\mathcal{S}$ that Lorenz-dominates each rule in $\mathcal{S}$.
2. Let $\mathcal{S}$ be the set of rules that satisfy order preservation in awards, order preservation in losses, endowment monotonicity, and the midpoint property. The T rule is the only rule in $\mathcal{S}$ that is Lorenz-dominated by each rule in $\mathcal{S}$ on the subdomain C_H^N.
3. Let $\mathcal{S}$ be the set of rules that satisfy order preservation in awards, order preservation in losses, order preservation under claims variations, null claims consistency, and $\frac{1}{|N|}$-truncated-claims lower bounds on awards. The MO rule is the only rule in $\mathcal{S}$ that is Lorenz-dominated by each rule in $\mathcal{S}$.

We know from Proposition 4.2 that the APRO rule does not satisfy order preservation under claims variations, so we can not apply the characterization of the MO rule to compare it with the APRO rule. In order to rank these rules we provide a Lorenz-based characterization of the APRO rule. According to Table 2, the APRO rule is regressive on the lower-half domain and progressive on the higher half-domain.

Theorem 5.4. *Let $\mathcal{S}_1$ be the set of rules satisfying the midpoint property, minimal rights first, claims truncation invariance, order preservation in awards, and regressivity on C_L^N. Let $\mathcal{S}_2$ be the set of rules satisfying the midpoint property, minimal rights first, claims truncation invariance, order preservation in losses, and progressivity on C_H^N.*

1. *The* APRO *rule is the only rule in* $\mathcal{S}_1$ *that is Lorenz-dominated by each rule in* $\mathcal{S}_1$ *on* C_L^N.
2. *The* APRO *rule is the only rule in* $\mathcal{S}_2$ *that Lorenz-dominates each rule in* $\mathcal{S}_2$ *on* C_H^N.

Proof. Let $\mathcal{S}_1$ be the set of rules satisfying the midpoint property, minimal rights first, claims truncation invariance, order preservation in awards, and regressivity on C_L^N. First, let us show that the APRO rule is the only rule in $\mathcal{S}_1$ that is Lorenz-dominated by each rule in $\mathcal{S}_1$ on C_L^N. Clearly, APRO$\in \mathcal{S}_1$. Let $(E, d) \in C_L^N$ and $\mathcal{R} \in \mathcal{S}_1$. By claims truncation invariance, $\mathcal{R}(E, d) = \mathcal{R}(E, t)$. If $E \leq D_{-n}$ then $\text{APRO}(E, d) = \text{PRO}(E, t)$. But, $\mathcal{R}_1(E, t) \geq \text{PRO}_1(E, t)$ if and only if $\mathcal{R}_1(E, t)\sum_{j=2}^n t_j \geq t_1\sum_{j=2}^n \mathcal{R}_j(E, t)$. This inequality holds because $\mathcal{R}$ is regressive. Let $\mathcal{T} = \sum_{j\in N} t_j$. Now, for each $k \in N\setminus\{n\}$, $\sum_{j=1}^k \mathcal{R}_j(E, t) \geq \sum_{j=1}^k \text{PRO}_j(E, t) = \frac{E}{\mathcal{T}}\sum_{j=1}^k t_j$ if and only if $\sum_{j=1}^k \mathcal{R}_j(E, t)\sum_{j=k+1}^n t_j \geq \sum_{j=k+1}^n \mathcal{R}_j(E, t)\sum_{j=1}^k t_j$. Again, by regressivity, $\mathcal{R}_j(E, t)\sum_{j=k+1}^n t_j \geq t_j\sum_{j=k+1}^n \mathcal{R}_j(E, t)$ for $j \in \{1, ..., k\}$. If $E \in [D_{-n}, \frac{1}{2}d(N)]$ then $m = (0, ..., 0, E - D_{-n})$. By the midpoint property, $\mathcal{R}(E, t) = m + \mathcal{R}(D_{-n}, (d_{-n}, D_{-n})) = m + \text{APRO}(D_{-n}, (d_{-n}, D_{-n})) = \text{APRO}(E, t)$.

Let $\mathcal{S}_2$ be the set of rules satisfying the midpoint property, minimal rights first, claims truncation invariance, order preservation in losses, and progressivity on C_H^N. We claim that the APRO rule is the only rule in $\mathcal{S}_2$ that Lorenz-dominates each rule in $\mathcal{S}_2$ on C_H^N. Indeed, APRO$\in \mathcal{S}_2$. But, order preservation in awards and order preservation in losses are dual properties, and the same happens with claims truncation invariance and minimal rights first, and with regressivity and progressivity. Besides, C_L^N and C_H^N are dual domains. Therefore, the characterization on C_H^N is the dual result of the one just proven above. □

According to Theorem 5.4, among the rules that satisfy the midpoint property, minimal rights first, claims truncation invariance, and order preservation on the entire domain, those that in addition are progressive on the lower-half domain Lorenz-dominate the APRO rule restricted to that domain, while those that in turn are regressive on the higher-half domain are Lorenz-dominated by the APRO rule on that domain.

6 | RANKING OF RULES

Our aim in this Section is to establish the ranking of the ten rules summarized in Figure 4. An arrow (or a sequence of arrows) from a rule $\mathcal{R}$ to a rule $\mathcal{R}'$ specifies that $\mathcal{R}$ Lorenz-dominates $\mathcal{R}'$, and the absence of an arrow (or of a sequence of arrows) specifies that there is no relationship. The restricted domain where each ranking holds is schematically indicated by the vertical lines at both sides of the diagram. As in Figure 2, they show the intervals where the endowment E has to be so that the claims problem (E, d) belongs to each particular subdomain. The vertical line at the left reflects the case where $D_{-n} \leq d_n$, the line to the right the case when $d_n \leq D_{-n}$. Therefore, the diagram provides a dynamic view of how the ranking, for a fixed vector of claims, changes as the endowment increases from 0 to the sum of the claims $d(N)$.

Most of the arrows (and absence of arrows) have already been studied by several authors. They were compiled, with the corresponding references, by Bosmans and Lauwers (2011) and Thomson (2019). But Figure 4 is a refinement of the diagrams work out by these authors because it presents several new features: it includes the middle domain, it incorporates the AA rule, and it completes the ranking of the APRO and the MO rules on the restricted domains. In

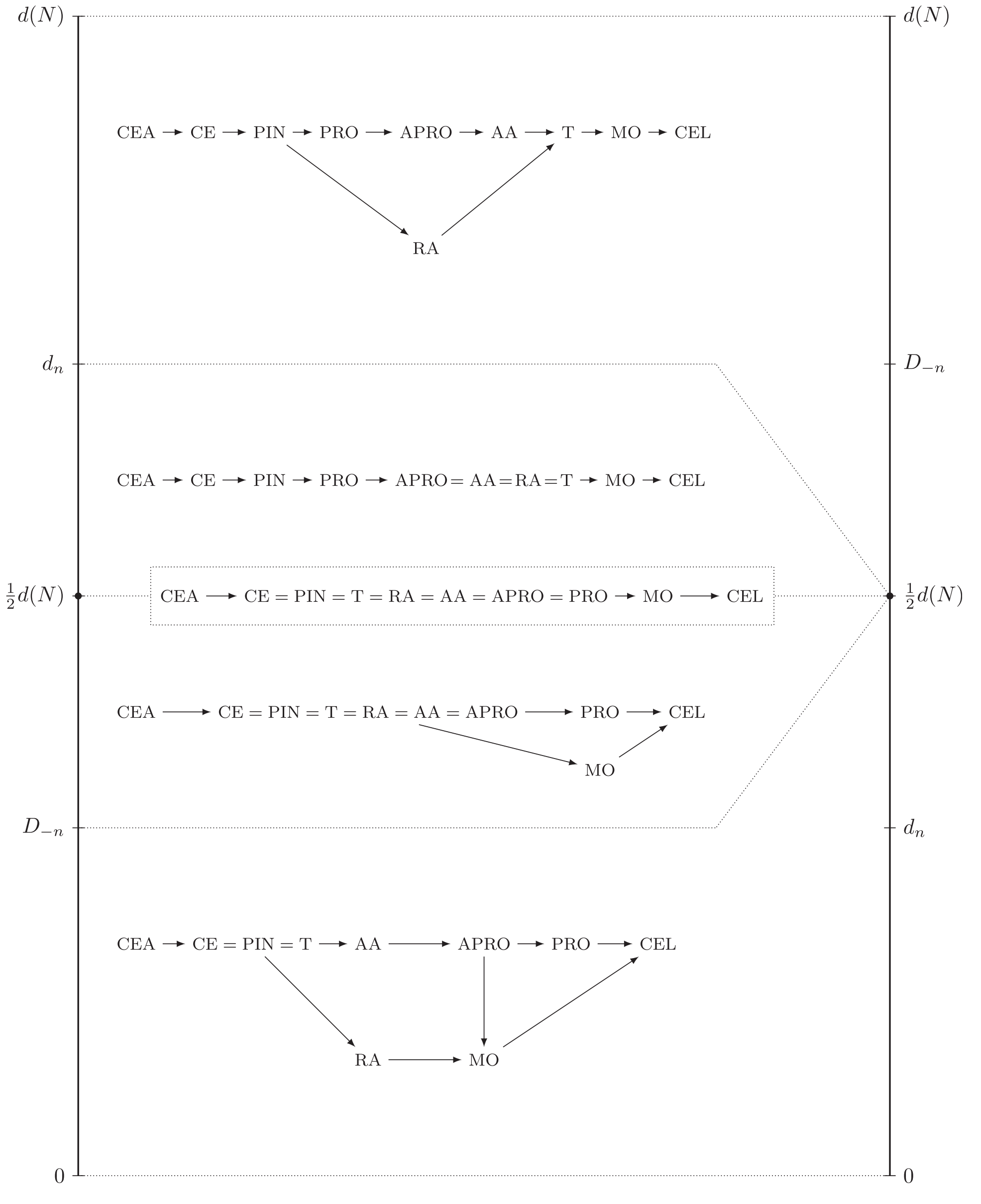

FIGURE 4 Ranking of rules on the lower-half, middle, midpoint, and higher-half domains

addition, we particularize the ranking of the ten rules for two-claimant and three-claimant populations (Figures 5 and 6, respectively).

6.1 | Ranking of the adjusted proportional and the minimal overlap rules

The Lorenz relationship between the APRO and the MO rule has already been stated. As far as we know, the comparison has been established applying the Lorenz-based characterization of

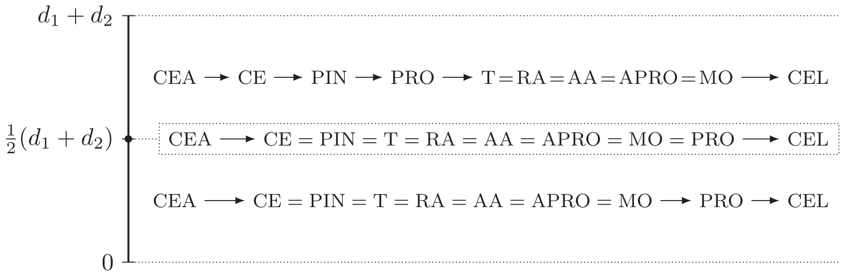

FIGURE 5 Ranking of rules when $|N| = 2$

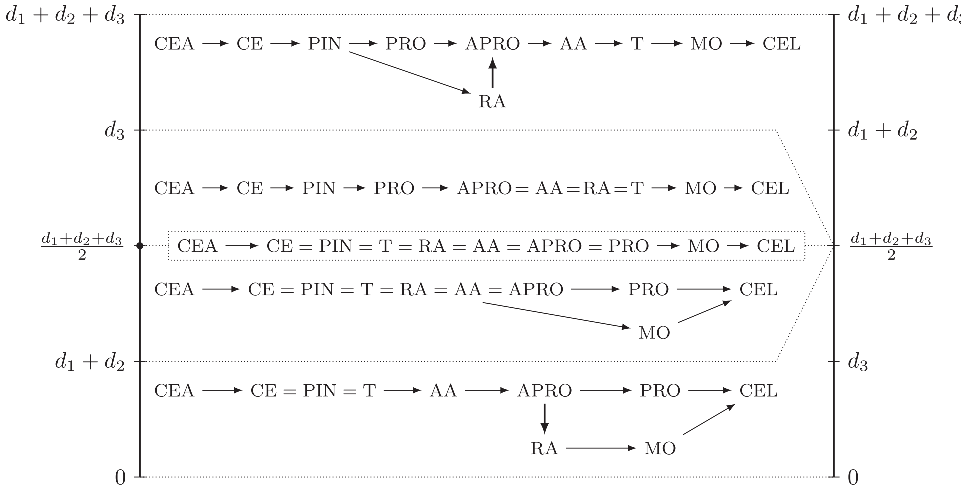

FIGURE 6 Ranking of rules when $|N| = 3$

the MO rule, in spite of the fact that the APRO rule does not satisfy order preservation under claims variations. In any case, we show next that the APRO rule Lorenz-dominates the MO rule on the entire domain of claims problems.

Theorem 6.1. *The adjusted proportional rule Lorenz-dominates the minimal overlap rule.*

Proof. We are going to establish the relationship separately on both half-domains. First, we restrict to the lower-half domain. Both the APRO and MO rules satisfy claims truncation invariance, therefore if $(E, d) \in C_L^N$ then $\text{APRO}(E, d) = \text{APRO}(E, t)$ and $\text{MO}(E, d) = \text{MO}(E, t)$. Note that $(E, t) \in C_S^N$ is a simple claims problem because $E \geq t_n(E, d) = \min\{E, d_n\}$. Therefore, since the MO rule is progressive on C_S^N (Proposition C.2), by Proposition 5.3 we have that $\text{PRO}(E, t) \succcurlyeq \text{MO}(E, t)$. We can have two cases. If $E \leq \min\{D_{-n}, \frac{1}{2}d(N)\}$ then, according to Lemma B.1, $\text{APRO}(E, d) = \text{PRO}(E, t)$. Hence, $\text{APRO}(E, d) = \text{PRO}(E, t) \succcurlyeq \text{MO}(E, t) = \text{MO}(E, d)$. On the other hand, if $E \in [D_{-n}, \frac{1}{2}d(N)]$ then (E, d) belongs to the middle domain so

$\mathrm{APRO}(E, d) = T(E, d)$. But, the T rule Lorenz-dominates the MO rule,[7] so $\mathrm{APRO}(E, d) = T(E, d) \succcurlyeq \mathrm{MO}(E, d)$. In any case, we conclude that $\mathrm{APRO} \succcurlyeq \mathrm{MO}$ on C_L^N.

We turn to the higher-half domain. We know, see Table 1, that the T rule satisfies the midpoint property, minimal rights first, claims truncation invariance, and order preservation. Moreover, we know from Proposition C.3 that the MO rule is progressive on C_H^N. Applying Theorem 5.4, we have that $\mathrm{APRO} \succcurlyeq T$ on C_H^N. Since $T \succcurlyeq \mathrm{MO}$, the transitive property leads to $\mathrm{APRO} \succcurlyeq \mathrm{MO}$ on C_H^N. Then, indeed, the APRO rule Lorenz-dominates the MO rule, that is, $\mathrm{APRO} \succcurlyeq \mathrm{MO}$ on C^N. □

6.2 | Ranking of the average-of-awards rule

We turn our attention to the ranking of the AA rule on the half-domains. We show that the AA rule ranks between the T and the APRO rules in both half domains, but, of course in different directions.[8]

Theorem 6.2. *The average-of-awards rule Lorenz-dominates the adjusted proportional rule and is Lorenz-dominated by the Talmud rule on C_L^N. Reciprocally, the average-of-awards rule Lorenz-dominates the Talmud rule and is Lorenz-dominated by the adjusted proportional rule on C_H^N.*

Proof. According to Table 1, the AA rule satisfies order preservation in awards, order preservation in losses, endowment monotonicity, and the midpoint property. Therefore, by the characterizations of the CE and the T rules, we conclude that $\mathrm{CE} \succcurlyeq \mathrm{AA}$ on C^N and $\mathrm{AA} \succcurlyeq T$ on C_H^N. As a corollary, restricted to the lower-half domain, the AA rule is Lorenz-dominated by the T rule, that is, $T = \mathrm{CE} \succcurlyeq \mathrm{AA}$ on C_L^N.

The AA rule, as it is proven in Appendix C, satisfies regressivity when restricted to the lower-half domain and, by self-duality, progressivity when restricted to the higher-half domain. Since it also satisfies the midpoint property, minimal rights first, claims truncation invariance, and order preservation, we deduce, from the characterizations of the APRO rule, that $\mathrm{AA} \succcurlyeq \mathrm{APRO}$ on C_L^N and that $\mathrm{APRO} \succcurlyeq \mathrm{AA}$ on C_H^N. □

Next, we present an example to illustrate that the RA and the AA rules are not Lorenz-comparable neither on the lower-half domain nor on the higher-half domain.

Example 6.3. Let $N = \{1, 2, 3, 4\}, E = 16$, and $d = (3, 10, 12, 13) \in \mathbb{R}^4_{\leq}$. Since $d(N) = 38$ then $(E, d) \in C_L^N$. We have that $\mathrm{RA}(E, d) = \left(\frac{3}{2}, 4, 5, \frac{11}{2}\right)$ and $\mathrm{AA}(E, d) = \left(\frac{29}{20}, \frac{43}{10}, 5, \frac{21}{4}\right)$. Therefore, $\mathrm{AA}_1(E, d) < \mathrm{RA}_1(E, d)$ but

$$\mathrm{AA}_1(E, d) + \mathrm{AA}_2(E, d) = \frac{23}{4} > \frac{11}{2} = \mathrm{RA}_1(E, d) + \mathrm{RA}_2(E, d).$$

[7]For instance, Thomson (2019) gives a direct proof of this fact.

[8]In contrast, Mirás Calvo et al. (2016) show that for the airport problem the core-center solution Lorenz-dominates the Shapley value and is Lorenz-dominated by the nucleolus.

Hence, $\mathrm{AA}(E,d)$ and $\mathrm{RA}(E,d)$ are not comparable. Naturally, the dual problem $(d(N)-E,d)\in C_H^N$ and, since the average of awards and the random arrival rules are both self-dual, $\mathrm{AA}(d(N)-E,d)$ and $\mathrm{RA}(d(N)-E,d)$ are not comparable.

In summary, $\mathrm{CE}=\mathrm{PIN}=T\succcurlyeq\mathrm{AA}\succcurlyeq\mathrm{APRO}$ on C_L^N and $\mathrm{APRO}\succcurlyeq\mathrm{AA}\succcurlyeq T$ on C_H^N, so the AA rule is Lorenz-comparable on the half domains to all the rules except the RA rule. To end, we conclude, directly from Figure 4, that $\mathrm{PIN}\succcurlyeq\mathrm{AA}\succcurlyeq\mathrm{MO}$ on the entire domain of claims problems. This result was already established by Mirás Calvo et al. (2022a) using a different approach.

6.3 | Ranking on the middle domain

So far, in Section 3, we have analyzed the rules that coincide when restricted to the middle domain. According to Table 2, the APRO rule is progressive on C_{MH}^N and regressive on C_{ML}^N, so by Proposition 5.3, $\mathrm{PRO}\to\mathrm{APRO}$ on C_{MH}^N and $\mathrm{APRO}\to\mathrm{PRO}$ on C_{ML}^N (a fact already established by Bosmans and Lauwers [2011]). Now, we show that the PRO and the MO rules are not comparable on C_{ML}^N.

Example 6.4. Let $N=\{1,2,3\}$, $E=5.1$, and $d=(1,4,7)\in\mathbb{R}^3_{\leq}$. Since $d(N)=12$ then $(E,d)\in C_{ML}^N$. We have that $\mathrm{PRO}(E,d)=\left(\frac{17}{40},\frac{17}{10},\frac{119}{40}\right)$ and $\mathrm{MO}(E,d)=\left(\frac{1}{3},\frac{11}{6},\frac{44}{15}\right)$. Then $\mathrm{PRO}_1(E,d)>\mathrm{MO}_1(E,d)$ and

$$\mathrm{PRO}_1(E,d)+\mathrm{PRO}_2(E,d)=\frac{17}{8}<\frac{13}{6}=\mathrm{MO}_1(E,d)+\mathrm{MO}_2(E,d).$$

Therefore, $\mathrm{PRO}(E,d)$ and $\mathrm{MO}(E,d)$ are not comparable.

6.4 | Ranking for two-claimant populations

It is well known that, when $|N|=2$, the AA, T, APRO, MO, and RA rules coincide with the CD rule. In addition, $D_{-2}=d_1\leq d_2$ so $(E,d)\in C_M^N$ if and only if $d_1\leq E\leq d_2$. Figure 5 summarizes the ranking of the ten rules for two-claimant problems.

6.5 | Ranking for three-claimant populations

Let us focus next on claims problems with three claimants. Take $N=\{1,2,3\}$ and $d=(d_1,d_2,d_3)\in\mathbb{R}^3_{\leq}$. First, observe that given two rules $\mathcal{R}$ and $\mathcal{R}'$ satisfying order preservation in awards and a problem $(E,d)\in C^N$ then $\mathcal{R}(E,d)$ Lorenz-dominates $\mathcal{R}'(E,d)$ if and only if $\mathcal{R}_1(E,d)\geq\mathcal{R}_1'(E,d)$ and $\mathcal{R}_3(E,d)\leq\mathcal{R}_3'(E,d)$. We present three-claimant examples to show that the RA and the PRO rules are incomparable on both half domains and that the MO and the PRO rules are not Lorenz-comparable on the lower-half domain.

Example 6.5. Let $N=\{1,2,3\}$, $E=4$, and $d=(1,4,5)\in\mathbb{R}^3_{\leq}$. Then $(E,d)\in C_L^N$, PRO $(E,d)=\left(\frac{2}{5},\frac{8}{5},2\right)$, and $\mathrm{RA}(E,d)=\mathrm{MO}(E,d)=\left(\frac{1}{3},\frac{11}{6},\frac{11}{6}\right)$.

Clearly, $\mathrm{RA}_1(E,d)<\mathrm{PRO}_1(E,d)$ and $\mathrm{RA}_3(E,d)<\mathrm{PRO}_3(E,d)$ so $\mathrm{RA}(E,d)=\mathrm{MO}(E,d)$ and $\mathrm{PRO}(E,d)$ are not Lorenz-comparable. Therefore, the RA and the MO rules are not comparable with the PRO rule on C_L^N. The corresponding dual problem $(d(N)-E,d)=(6,(1,4,5))\in C_H^N$, $PRO(d(N)-E,d)=\left(\frac{3}{5},\frac{12}{5},3\right)$ and $RA(d(N)-E,d)=\left(\frac{2}{3},\frac{13}{6},\frac{19}{6}\right)$.

Since $\mathrm{RA}_1(d(N)-E,d)>\mathrm{PRO}_1(d(N)-E,d)$ and $\mathrm{RA}_3(d(N)-E,d)>\mathrm{PRO}_3(d(N)-E,d)$, the RA and the PRO rules are incomparable on C_H^N.

Bosmans and Lauwers (2011) also give examples showing that the RA and the APRO rule are not comparable neither on C_L^N nor on C_H^N, but the examples involved problems with four claimants. In fact, we prove in Appendix D that these two rules are comparable on the half domains when there are just three claimants. Indeed, if $|N|=3$ then the APRO rule Lorenz-dominates the RA rule on the lower-half domain and the APRO rule is Lorenz-dominated by the RA rule on the higher-half domain. The ranking of the ten rules for problems with just three claimants on the half-domains is summarized in Figure 6. Observe that the RA rule can be compared with all the rules except the PRO rule.

6.6 | Final comments on the refined ranking

First, let us give an example of a three-claimant problem to illustrate the results that we have established on the lower-half, higher-half, and middle domains.

Example 6.6. Let $N=\{1,2,3\}$ and $d=(1,4,7)\in\mathbb{R}^3_{\leq}$. Then $d(N)=12$, $\frac{1}{2}d(N)=6$, and $D_{-3}=d_1+d_2=5$, so $D_{-3}<\frac{1}{2}d(N)<d_3$. If $E\in[5,7]$ then $(E,d)\in C_M^N$ so $T(E,d)=\mathrm{APRO}(E,d)=\mathrm{AA}(E,d)=(\frac{1}{2},2,E-\frac{5}{2})$. Now, let $\mathcal{R}$ be any of these five rules: PRO, APRO, T, AA, or MO. Consider the function $\mathcal{R}_1(\cdot,d):[0,12]\to\mathbb{R}$ that assigns to each $E\in[0,12]$ the value $\mathcal{R}_1(E,d)$, the award given to the first claimant by $\mathcal{R}$ in the claims problem (E,d). The plot of the function $\mathcal{R}_1(\cdot,d)$ is called the schedule of awards of the rule for the claims vector d corresponding to the first claimant. Figure 7 shows the schedules of awards of the five rules.

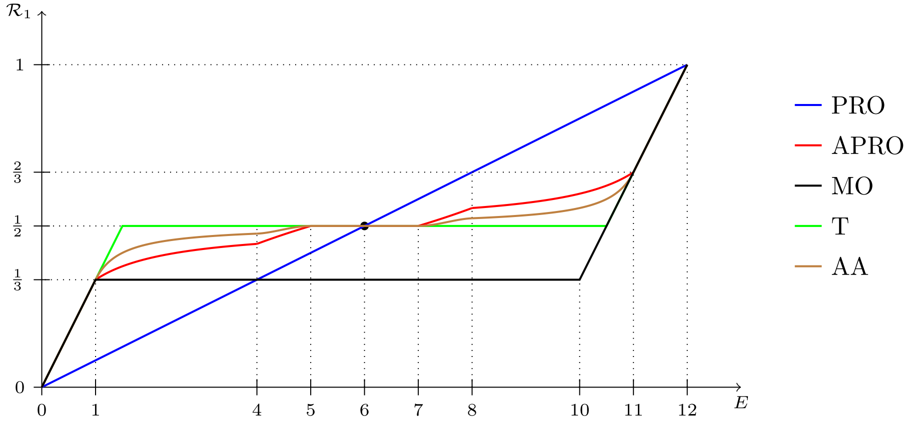

FIGURE 7 The schedules of awards $\mathcal{R}_1(\cdot,d)$ of several rules for $d=(1,4,7)$ [Color figure can be viewed at wileyonlinelibrary.com]

Since the APRO, T, and AA rules Lorenz-dominate the MO rule on the entire domain, we see that the schedules of awards of APRO, T, and AA lie above the schedule of awards of the MO rule. Nevertheless, the paths of $PRO_1(\cdot, d)$ and $MO_1(\cdot, d)$ cross each other on the interval $[0, 6]$ indicating that these rules are not Lorenz-comparable on the lower-half domain. Now, we have that $T \succcurlyeq AA \succcurlyeq APRO \succcurlyeq PRO$ on C_L^N but $PRO \succcurlyeq APRO \succcurlyeq AA \succcurlyeq T$ on C_H^N which implies that $T_1(E, d) \geq AA_1(E, d) \geq APRO_1(E, d) \geq PRO_1(E, d)$ for all $E \in [0, 6]$ and $PRO_1(E, d) \geq APRO_1(E, d) \geq AA_1(E, d) \geq T_1(E, d)$ if $E \in [6, 12]$.

The Lorenz order is a partial order, so it is noteworthy to obtain results regarding the Lorenz ranking of rules. The ranking illustrated in Figure 6 can be helpful, for instance, to discard three-claimant problems as a source of counterexamples for some axioms and rules. As for the big picture, summarized in Figure 4, the analysis outlined in Example 6.6 can be extended to larger populations and can provide helpful information in situations when the initial resource varies with time. One of the many applications of claims problems is the allocation of CO_2 emissions. In Giménez-Gómez et al. (2016) model, the endowment, E_0, is the available carbon budget and the claimants are the countries/regions that claim their current quota of CO_2 emissions. Now, in order to comply with the international agreements on climate change, the allowed global CO_2 emissions in the next decade must drop 7.6 per cent per year before crossing a dangerous threshold. Mirás Calvo et al. (2020) present a dynamic model by selecting a set of 20 emitters, $N = \{1, \ldots, 20\}$, with fixed claims $d \in \mathbb{R}^N$ (their current emissions), and considering a sequence of claims problems $(E_t, d) \in C^N$, with $t \in \{1, \ldots, 10\}$, where $E_t = (1 - 0.076)^t E_0$. They analyze several rules that provide different distribution patterns of the emissions reduction among the polluters. If two rules are Lorenz-comparable then the one that Lorenz-dominates demands a lesser effort to the top polluters than the one that is Lorenz-dominated. But, since the endowment is reduced by 7.6 percent in each period, the claims problems at the beginning of the decade belong to the higher-half domain while toward the end of the period they belong to the lower-half domain. Therefore, according to the refined ranking, some rules that demand a lesser effort to the top polluters in the first years will ask for a bigger effort to those countries at the end of the period, and vice versa.[9] These qualitative differences are of great interest when negotiating the policy to be implemented.

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CONFLICT OF INTEREST

The authors declare no conflicts of interest.

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[9]In a hypothetical case with a "super-polluter" (one country whose emissions are bigger that the emissions of the others combined), at some point in time we may have problems that belong to the middle domain.

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APPENDIX A: DEFINITION OF RULES

- Proportional rule (PRO): For each $(E, d) \in C^N$ and each $i \in N$, $\text{PRO}_i(E, d) = \frac{d_i}{d(N)}E$ if $d(N) \neq 0$ and $\text{PRO}_i(E, 0) = 0$, otherwise.

- Adjusted proportional rule (APRO): For each $(E, d) \in C^N$ and each $i \in N$,

$$\text{APRO}_i(E, d) = m_i + \text{PRO}_i(E - M, (\min\{d_j - m_j, E - M\})_{j \in N}),$$

  where $M = \sum_{j \in N} m_j$.

- Minimal overlap rule (MO): Let $d_0 = 0$. For each $(E, d) \in C^N$ and each $i \in N$,
  i) If $E \leq d_n$ then $\text{MO}_i(E, d) = \frac{t_1}{n} + \frac{t_2 - t_1}{n-1} + \cdots + \frac{t_i - t_{i-1}}{n-i+1}$.

  ii) If $E > d_n$, let $s^* \in (d_{k^*}, d_{k^*+1}]$, with $k^* \in \{0, 1, \ldots, n-2\}$, be the unique solution to the equation $\sum_{j \in N} \max\{d_j - s, 0\} = E - s$. Then,

$$\text{MO}_i(E, d) = \begin{cases} \frac{d_1}{n} + \frac{d_2 - d_1}{n-1} + \cdots + \frac{d_i - d_{i-1}}{n-i+1} & \text{if } i \in \{1, \ldots, k^*\} \\ \text{MO}_i(s^*, d) + d_i - s^* & \text{if } i \in \{k^* + 1, \ldots, n\} \end{cases}.$$

- Constrained equal awards rule (CEA): For each $(E, d) \in C^N$ and each $i \in N$, $\text{CEA}_i(E, d) = \min\{\alpha, d_i\}$, where $\alpha \geq 0$ is chosen such that $E = \sum_{i \in N} \text{CEA}_i(E, d)$.
- Constrained equal losses rule (CEL): For each $(E, d) \in C^N$ and each $i \in N$, $\text{CEL}_i(E, d) = \max\{0, d_i - \beta\}$, where $\beta \geq 0$ is chosen such that $E = \sum_{i \in N} \text{CEL}_i(E, d)$.
- Talmud rule (T): For each $(E, d) \in C^N$ and each $i \in N$,

$$T_i(E, d) = \begin{cases} \text{CEA}_i\left(E, \frac{d}{2}\right) & \text{if } E \leq \frac{1}{2}d(N) \\ d_i - \text{CEA}_i\left(d(N) - E, \frac{d}{2}\right) & \text{if } E \geq \frac{1}{2}d(N) \end{cases}.$$

- Piniles' rule (PIN): For each $(E, d) \in C^N$ and each $i \in N$,

$$\mathrm{PIN}_i(E,d)=\begin{cases}\mathrm{CEA}_i\left(E,\frac{d}{2}\right) & \text{if } E\le \frac{1}{2}d(N)\\ \frac{d_i}{2}+\mathrm{CEA}_i\left(E-\frac{1}{2}d(N),\frac{d}{2}\right) & \text{if } E\ge \frac{1}{2}d(N)\end{cases}$$

- Constrained egalitarian rule (CE): For each $(E,d)\in C^N$ and each $i\in N$,

$$\mathrm{CE}_i(E,d)=\begin{cases}\mathrm{CEA}_i\left(E,\frac{d}{2}\right) & \text{if } E\le \frac{1}{2}d(N)\\ \max\left\{\frac{d_i}{2},\min\{d_i,\lambda\}\right\} & \text{if } E\ge \frac{1}{2}d(N)\end{cases}$$

  where $\lambda\ge 0$ is chosen such that $\sum_{i\in N}\max\left\{\frac{d_i}{2},\min\{d_i,\lambda\}\right\}=E$.
- Random arrival rule (RA): For each $(E,d)\in C^N$ and each $i\in N$,

$$\mathrm{RA}_i(E,d)=\frac{1}{|N|!}\sum_{\pi\in\Pi^N}\min\{d_i,\max\{0,E-d(P_\pi(i))\}\},$$

  where Π^N is the set of strict orders on N and $P_\pi(i)=\{j\in N:\pi(j)<\pi(i)\}$ for $\pi\in\Pi^N$.
- Average-of-awards rule (AA): For each $(E,d)\in C^N$ the average-of-awards rule, $\mathrm{AA}(E,d)$, selects the centroid of the set of awards vectors $X(E,d)$. Let μ be the $(n-1)$-dimensional Lebesgue measure and denote $V(E,d)=\mu(X(E,d))$ the volume (measure) of the set of awards vectors. If $V(E,d)>0$ then for each $i\in N$,

$$\mathrm{AA}_i(E,d)=\frac{1}{V(E,d)}\int_{X(E,d)}x_i\,d\mu.$$

APPENDIX B: PROPERTIES OF THE APRO RULE

Lemma B.1. *Let $(E,d)\in C_L^N$ and $d_0=0$. We have that*

1. *If* $E\le\min\{D_{-n},\frac{1}{2}d(N)\}$, *let* $k_0=|\{k\in N:d_k\le E\}|$ *and* $\mathcal{T}=\sum_{s=0}^{k_0}d_s+(n-k_0)E$. *Then*

$$\mathrm{APRO}_j(E,d)=\mathrm{PRO}_j(E,t)=\frac{t_jE}{\mathcal{T}}=\begin{cases}\frac{d_jE}{\mathcal{T}} & \text{if } j\le k_0\\ \frac{E^2}{\mathcal{T}} & \text{if } j>k_0\end{cases}.$$

2. *If* $E\in[D_{-n},\frac{1}{2}d(N)]$ *then* $\mathrm{APRO}(E,d)=\left(\frac{d_1}{2},\ldots,\frac{d_{n-1}}{2},E-\frac{D_{-n}}{2}\right)$.

Proof. Let $(E,d)\in C_L^N$. If $(E,d)\in C_{ML}^N$ then $\mathrm{APRO}(E,d)=\mathrm{AA}(E,d)$. Now, if $E\le\min\{D_{-n},\frac{1}{2}d(N)\}$ then $m_j=0$ for all $j\in N$ and $\mathrm{APRO}(E,d)=\mathrm{PRO}(E,t)$. Observe that, if $k_0=|\{k\in N:d_k\le E\}|$ then $\mathcal{T}=\sum_{i\in N}t_i(E,d)=\sum_{s=0}^{k_0}d_s+(n-k_0)E$. The result in this case is straightforward. □

Proposition B.2. *The* APRO *rule satisfies regressivity on* C_L^N *and progressivity on* C_H^N.

Proof. Let $(E, d) \in C_L^N$ and denote $\mathcal{T} = \sum_{i \in N} t_i(E, d)$. Then

$$\frac{\text{APRO}_i(E, d)}{d_i} = \begin{cases} \frac{t_i E}{d_i \mathcal{T}} & \text{if } E \le \min\left\{D_{-n}, \frac{1}{2}d(N)\right\} \\ \frac{1}{2} & \text{if } D_{-n} \le E \le \frac{1}{2}d(N) \text{ and } i \in N\backslash\{n\}. \\ \frac{2E - D_{-n}}{2d_n} & \text{if } D_{-n} \le E \le \frac{1}{2}d(N) \text{ and } i = n \end{cases}$$

It follows at once that the APRO rule is regressive on C_L^N and, by self-duality, progressive on C_H^N. □

Proposition B.3. *The* APRO *rule satisfies both other-regarding claim monotonicity and order preservation under claims variations on* C_L^N.

Proof. Let $(E, d) \in C_L^N, i \in N, d_i' > d_i, d' = (d_{-i}, d_i')$, and denote $\Delta_j = \text{APRO}_j(E, d) - \text{APRO}_j(E, d')$ for $j \in N\backslash\{i\}$.We have to prove that $0 \le \Delta_j \le \Delta_k$ for all $\{j, k\} \subset N\backslash\{i\}$ with $d_j \le d_k$. Since the APRO rule satisfies anonymity, it is sufficient to prove the result when $d_i < d_i' \le d_{i+1}$. Now, since $(E, d) \in C_L^N$ then $(E, d') \in C_L^N$ and $t_j = t_j'$ for $j \in N\backslash\{i\}$. Let $\mathcal{T} = \sum_{k \in N} t_k$ and $\mathcal{T}' = \sum_{k \in N} t_k'$. If $E \le \min\{D_{-n}, \frac{1}{2}d(N)\}$ then $E \le \min\left\{D_{-n}', \frac{1}{2}d'(N)\right\}$, so $\Delta_j = \left(\frac{1}{\mathcal{T}} - \frac{1}{\mathcal{T}'}\right)t_j E$ for each $j \ne i$, and the result follows immediately. Suppose $E \in [D_{-n}, \frac{1}{2}d(N)]$. If $i = n$ the result also holds because $D_{-n}' = D_{-n}, E \in \left[D_{-n}', \frac{1}{2}d'(N)\right]$, and $\Delta_j = 0$ for $j < n$. Suppose $i < n$:

Case 1: If $E \le D_{-n}'$ then

$$\Delta_j = \begin{cases} \left(\frac{1}{2} - \frac{E}{\mathcal{T}'}\right)d_j & \text{if } j \in N\backslash\{i, n\} \\ E - \frac{D_{-n}}{2} - \frac{E}{\mathcal{T}'}E & \text{if } j = n \end{cases}.$$

Take $j, k \in N\backslash\{i, n\}, j \ne k$. Now $\mathcal{T}' = D_{-n}' + E$ so $E < \frac{1}{2}\mathcal{T}'$ and $0 \le \Delta_j \le \Delta_k$. Moreover, $\Delta_n = \left(1 - \frac{E}{\mathcal{T}'}\right)E - \frac{1}{2}D_{-n} \ge 0$ because $E \ge D_{-n}$ and $\Delta_j \le \Delta_n$ since $\mathcal{T}' \ge D_{-n}$.

Case 2: If $E \ge D_{-n}'$ then

$$\Delta_j = \begin{cases} 0 & \text{if } j \in N\backslash\{i, n\} \\ \frac{1}{2}\left(D_{-n}' - D_{-n}\right) & \text{if } j = n \end{cases}.$$

The result holds since $D_{-n}' \ge D_{-n}$. □

Proposition B.4. *The* APRO *rule satisfies population monotonicity on* C_L^N.

Proof. Let $N \in \mathcal{N}, (E, d) \in C_L^N$, and $N' \subset N$ such that $d(N') \geq E$. If $(E, d_{N'}) \in C_L^{N'}$, then by Proposition B.3, $\text{APRO}_{N'}(E, d) \leq \text{APRO}_{N'}(E, (d_{N'}, 0_{N \setminus N'}))$. But, the APRO rule satisfies null claims consistency, so $\text{APRO}_{N'}(E, (d_{N'}, 0_{N \setminus N'})) = \text{APRO}_{N'}(E, d_{N'})$. Therefore, $\text{APRO}_{N'}(E, d) \leq \text{APRO}_{N'}(E, d_{N'})$. On the contrary, if $(E, d_{N'}) \in C_H^{N'}$ then, for each $j \in N'$,

$$\text{APRO}_j(E, d) \leq \text{APRO}_j\left(\tfrac{1}{2}d(N), d\right) = \tfrac{d_j}{2} = \text{APRO}_j\left(\tfrac{1}{2}d(N'), d_{N'}\right) \leq \text{APRO}_j(E, d_{N'})$$

where, we have applied that $E \leq \frac{1}{2}d(N)$ and that the adjusted proportional rule satisfies endowment monotonicity and the midpoint property. Hence, we conclude that the adjusted proportional rule satisfies population monotonicity on C_L^N. □

Proposition B.5. *The* APRO *rule satisfies order preservation under population variation on* C_L^N.

Proof. Let $(E, d) \in C_L^N$, $i \in N$ with $E < d(N \setminus \{i\})$ and a pair $\{j, k\} \subseteq N \setminus \{i\}$ where $d_j \leq d_k$, we have to prove that $\text{APRO}_k(E, d) - \text{APRO}_j(E, d) \leq \text{APRO}_k(E, d_{-i}) - \text{APRO}_j(E, d_{-i})$. When $E \leq \frac{1}{2}D_{-i}$, the property holds directly by Proposition B.3 since the APRO rule satisfies null claims consistency. Now, if $E \geq \frac{1}{2}D_{-i}$, we have that

$$\begin{aligned}
\text{APRO}_k(E, d_{-i}) - \text{APRO}_j(E, d_{-i}) &\geq \text{APRO}_j(E, d_{-i})\tfrac{d_k}{d_j} - \text{APRO}_j(E, d_{-i}) \\
&= \left(\frac{d_k - d_j}{d_j}\right)\text{APRO}_j(E, d_{-i}) \\
&\geq \left(\frac{d_k - d_j}{d_j}\right)\text{APRO}_j(E, d) \\
&= \text{APRO}_j(E, d)\tfrac{d_k}{d_j} - \text{APRO}_j(E, d) \\
&\geq \text{APRO}_k(E, d) - \text{APRO}_j(E, d).
\end{aligned}$$

The first inequality holds because, by Proposition B.2, the APRO rule is progressive on $C_H^{N \setminus \{i\}}$. The second inequality is an application of Proposition B.4, and the last one follows from Proposition B.2. □

APPENDIX C: PROGRESSIVITY AND REGRESSIVITY OF THE MO AND AA RULES

Lemma C.1. *Let* $(E, d) \in C^N$. *If* $\text{MO}_{i+1}(E, d) = \text{MO}_i(E, d) + \frac{t_{i+1} - t_i}{n - i}$ *for some* $i \in N \setminus \{n\}$ *then* $\frac{\text{MO}_i(E, d)}{t_i} \leq \frac{\text{MO}_{i+1}(E, d)}{t_{i+1}}$.

Proof. Let $i \in N \backslash \{n\}$ such that $\text{MO}_{i+1}(E,d) = \text{MO}_i(E,d) + \frac{t_{i+1}-t_i}{n-i}$. Simple calculations show that $\frac{\text{MO}_i(E,d)}{t_i} \le \frac{\text{MO}_{i+1}(E,d)}{t_{i+1}}$ if and only if $(t_{i+1}-t_i)\left(\frac{t_i}{n-i} - \text{MO}_i(E,d)\right) \ge 0$. But, $\left(\frac{t_i}{n-i} - \text{MO}_i(E,d)\right) \ge 0$, because:

$$(n-i)\text{MO}_i(E,d) = \frac{n-i}{n}t_1 + \cdots + \frac{n-i}{n-i+1}(t_i - t_{i-1})$$
$$\le t_1 + (t_2 - t_1) + \cdots + (t_i - t_{i-1}) = t_i.$$

Since $(t_{i+1}-t_i) \ge 0$, we have that $(t_{i+1}-t_i)\left(\frac{t_i}{n-i} - \text{MO}_i(E,d)\right) \ge 0$. □

Proposition C.2. *The* MO *rule satisfies progressivity on* C_S^N.

Proof. When $|N| = 2$ the result is clear. So, let $|N| \ge 3$ and $(E,d) \in C_S^N$. It suffices to prove that $\frac{\text{MO}_i(E,d)}{d_i} \le \frac{\text{MO}_{i+1}(E,d)}{d_{i+1}}$ for all $i \in N\backslash\{n\}$. Since $(E,d) \in C_S^N$ then $E \ge d_n$ and $t(E,d) = d$. If $E = d_n$, the result follows directly from Lemma C.1. If $E > d_n$, let $s^* \in (d_{k*}, d_{k^*+1}]$, with $k^* \in \{0,1,\ldots,n-2\}$, be the solution to the equation $\sum_{i\in N} \max\{d_i - s, 0\} = E - s$. By Lemma C.1, we have that $\frac{\text{MO}_i(E,d)}{d_i} \le \frac{\text{MO}_{i+1}(E,d)}{d_{i+1}}$ for all $i \in \{1,\ldots,k^*-1\}$. Now, since $s^* \ge d_{k^*}$ we have that $\text{MO}_{k^*}(E,d) = \text{MO}_{k^*}(s^*,d)$. In addition,

$$\begin{aligned}\text{MO}_{k^*+1}(E,d) &= \text{MO}_{k^*+1}(s^*,d) + (d_{k^*+1} - s^*) = \text{MO}_{k^*}(s^*,d)\\ &+ \frac{s^* - d_k^*}{n-k^*} + (d_{k^*+1} - s^*).\end{aligned} \tag{C1}$$

But, again by Lemma C.1,

$$\frac{\text{MO}_{k^*}(E,d)}{d_{k^*}} = \frac{\text{MO}_{k^*}(s^*,d)}{d_{k^*}} \le \frac{\text{MO}_{k^*+1}(s^*,d)}{d_{k^*+1}} = \frac{\text{MO}_{k^*}(E,d) + \frac{s^*-d_{k^*}}{n-k*}}{d_{k^*+1}}.$$

Then, since $\frac{d_{k^*+1}-s^*}{d_{k^*+1}} \ge 0$, we have that

$$\frac{\text{MO}_{k^*}(E,d)}{d_{k^*}} \le \frac{\text{MO}_{k^*}(E,d) + \frac{s^*-d_{k^*}}{n-k^*}}{d_{k^*+1}} + \frac{d_{k^*+1} - s^*}{d_{k^*+1}} = \frac{\text{MO}_{k^*+1}(E,d)}{d_{k^*+1}},$$

where the last equality follows from (C1). Finally, if $i \ge k^* + 1$ then $\frac{\text{MO}_i(E,d)}{d_i} \le \frac{\text{MO}_{i+1}(E,d)}{d_{i+1}}$ if and only if

$$\left(\text{MO}_{k^*}(E,d) + \frac{s^*-d_{k^*}}{n-k^*} + d_i - s^*\right)d_{i+1} \le \left(\text{MO}_{k^*}(E,d) + \frac{s^*-d_{k^*}}{n-k^*} + d_{i+1} - s^*\right)d_i,$$

or, equivalently,

$$(d_{i+1} - d_i)\left(\text{MO}_{k^*}(E, d) + \frac{s^* - d_{k^*}}{n - k^*} - s^*\right) \leq 0.$$

Since $d_{i+1} - d_i \geq 0$, we have to prove that

$$\text{MO}_{k^*}(E, d) + \frac{s^* - d_{k^*}}{n - k^*} - s^* = \frac{(n - k^*)\text{MO}_{k^*}(E, d) - d_{k^*} - (n - k^* - 1)s^*}{n - k^*} \leq 0.$$

Indeed, the inequality holds since $n - k^* - 1 \geq 0$, because $k^* \leq n - 1$, and $(n - k^*)\text{MO}_{k^*}(E, d) \leq d_{k^*}$, because $(n - k^*)\text{MO}_{k^*}(E, d) = \frac{n-k^*}{n}d_1 + \dots + \frac{n-k^*}{n-k^*+1}\left(d_{k^*} - d_{k^*-1}\right) \leq d_1 + d_2 - d_1 + \dots + d_{k^*} - d_{k^*-1} = d_{k^*}$. □

Proposition C.3. *The* MO *rule satisfies progressivity on* C_H^N.

Proof. When $|N| = 2$ the result is clear. So, let $|N| \geq 3$ and $(E, d) \in C_H^N$. It suffices to prove that $\frac{\text{MO}_i(E,d)}{d_i} \leq \frac{\text{MO}_{i+1}(E,d)}{d_{i+1}}$ for all $i \in N\backslash\{n\}$. If $E \geq d_n$, the result follows directly from Proposition C.2. As a consequence, we just have to prove the result if $\frac{1}{2}d(N) \leq E < d_n$. But then $d_i \leq D_{-n} \leq E$ for all $i \in N\backslash\{n\}$, so $t(E, d) = (d_1, \dots, d_{n-1}, E)$ and $\text{MO}_{i+1}(E, d) = \text{MO}_i(E, d) + \frac{t_{i+1} - t_i}{n-i}$. By Lemma C.1 we have that $\frac{\text{MO}_i(E,d)}{d_i} \leq \frac{\text{MO}_{i+1}(E,d)}{d_{i+1}}$ for each $i \in N\backslash\{n, n-1\}$. It remains to be proved that $\text{MO}_n(E, d)d_{n-1} \geq \text{MO}_{n-1}(E, d)d_n$. Since $\text{MO}_n(E, d) = \text{MO}_{n-1}(E, d) + E - d_{n-1}$, we have to prove that $(d_n - d_{n-1})\text{MO}_{n-1}(E, d) \leq (E - d_{n-1})d_{n-1}$. But,

$$\begin{aligned}\text{MO}_{n-1}(E, d) = \frac{d_1}{n} + \frac{d_2 - d_1}{n-1} + \dots + \frac{d_{n-1} - d_{n-2}}{2} \leq \frac{d_1}{2} + \frac{d_2 - d_1}{2} \\ + \dots + \frac{d_{n-1} - d_{n-2}}{2} = \frac{d_{n-1}}{2}.\end{aligned}$$

Moreover, it is easy to check that $\frac{d_{n-1}}{2}(d_n - d_{n-1}) \leq d_{n-1}\left(\frac{1}{2}d(N) - d_{n-1}\right)$. Then, $(d_n - d_{n-1})\text{MO}_{n-1}(E, d) \leq \frac{d_{n-1}}{2}(d_n - d_{n-1}) \leq d_{n-1}\left(\frac{1}{2}d(N) - d_{n-1}\right) \leq (E - d_{n-1})d_{n-1}$. □

Lemma C.4. *The* CD *rule satisfies regressivity on* C_L^N *and progressivity on* C_H^N.

Proof. Let $N = \{1, 2\}$. Since progressivity and regressivity are dual properties and the concede-and-divide rule is self-dual, it suffices to prove that the concede-and-divide rule is regressive on C_L^N. But, if $0 \leq E \leq d_1$, then $\frac{\text{CD}_1(E,d)}{d_1} \geq \frac{\text{CD}_2(E,d)}{d_2}$ if and only if $\frac{E}{2d_1} \geq \frac{E}{2d_2}$, which holds because $d_2 \geq d_1$. On the other hand, if $d_1 \leq E \leq \frac{d_1 + d_2}{2}$ then $\frac{\text{CD}_1(E,d)}{d_1} \geq \frac{\text{CD}_2(E,d)}{d_2}$ if and only if $\frac{1}{2} \geq \frac{2E - d_1}{2d_2}$ which obviously holds since $\frac{d_1 + d_2}{2} \geq E$. □

Proposition C.5. *The* AA *rule is regressive on* C_L^N *and progressive on* C_H^N.

Proof. Progressivity and regressivity are dual properties and the average-of-awards rule is self-dual, so it suffices to show that it is regressive on C_L^N. We proceed by induction on the number of claimants $n = |N|$. First, the average-of-awards rule coincides with the concede-and-divide rule when $|N| = 2$, and, by Lemma C.4, the concede-and-divide rule satisfies regressivity on C_L^N.

Now, let $|N| = n \geq 3, (E, d) \in C_L^N$, and $i, j \in N$ such that $d_i \leq d_j$. By the induction hypothesis, assume that the average-of-awards rule is regressive on the lower-half domain for all problems with $n-1$ claimants. We have to prove that $\frac{\mathrm{AA}_j(E,d)}{d_j} \leq \frac{\mathrm{AA}_i(E,d)}{d_i}$. Clearly, $D_{-n} \leq \cdots \leq D_{-1}$ and if $k \in N\backslash\{n\}$ then $d_k \leq D_{-k}$. As for the relative position of d_n and D_{-n}, both situations, $d_n \leq D_{-n}$ and $D_{-n} \leq d_n$, are possible (see Figure 1). In any case, $\frac{1}{2}d(N)$ is the middle point of the interval with extreme points d_k and D_{-k} for all $k \in N$. We distinguish three cases:

Case 1: Let $E \in [0, d_1]$. For each $k \in N$ denote $a^k \in \mathbb{R}^N$ the vector with E in the kth-coordinate and 0's elsewhere. Then $X(E, d)$ is the regular simplex spanned by the points $a^k, k \in N$, so the centroid of $X(E, d)$ is the arithmetic mean of its extreme points: $\mathrm{AA}_k(E, d) = \frac{E}{n}$ for each $k \in N$. Therefore, $\frac{\mathrm{AA}_j(E,d)}{d_j} - \frac{\mathrm{AA}_i(E,d)}{d_i} = \frac{E}{nd_j} - \frac{E}{nd_i} \leq 0$.

Case 2: If $E \in [D_{-n}, \frac{1}{2}d(N)]$, then $(E, d) \in C_M^N$ so $\mathrm{AA}_j(E, d) = \frac{d_j}{2}$ for all $j \in N\backslash\{n\}$ and $\mathrm{AA}_n(E, d) = E - \frac{D_{-n}}{2}$. Hence, $\frac{\mathrm{AA}_i(E,d)}{d_i} - \frac{\mathrm{AA}_j(E,d)}{d_j} = 0$ whenever $j < n$. But, $\frac{\mathrm{AA}_n(E,d)}{d_n} - \frac{\mathrm{AA}_i(E,d)}{d_i} = \frac{2E - D_{-n}}{2d_n} - \frac{1}{2} = \frac{2E - d(N)}{2d_n} \leq 0$ because $2E \leq d(N)$.

Case 3: Let $E \in [d_1, \min\{D_{-n}, \frac{1}{2}d(N)\}]$. Take $k \in N\backslash\{i, j\}$. Let $g_k : (0, d(N)) \times [0, D_{-k}] \longrightarrow \mathbb{R}$ be the function defined by $g_k(E, u) = \frac{\sqrt{n}}{\sqrt{n-1}} \frac{V(u, d_{-k})}{V(E, d)}, (E, u) \in (0, d(N)) \times [0, D_{-k}]$. Mirás Calvo et al. (2022b) show that, for all $\ell \in N\backslash\{k\}$,

$$\mathrm{AA}_\ell(E, d) = \int_{r_k(E,d)}^{R_k(E,d)} \mathrm{AA}_\ell(u, d_{-k}) g_k(E, u) du,$$

where $r_k(E, d) = \max\{0, E - d_k\}$ and $R_k(E, d) = \min\{E, D_{-k}\}$. But, $E \leq D_{-k}$ and $R_k(E, d) = E$. Then, since $i, j \in N\backslash\{k\}$, we can write:

$$\frac{\mathrm{AA}_j(E, d)}{d_j} - \frac{\mathrm{AA}_i(E, d)}{d_i} = \int_{r_k(E,d)}^{E} \left(\frac{\mathrm{AA}_j(u, d_{-k})}{d_j} - \frac{\mathrm{AA}_i(u, d_{-k})}{d_i}\right) g_k(E, u) du. \tag{C2}$$

Now, if $E \in [d_1, \frac{1}{2}D_{-k}]$ then $E \leq \frac{1}{2}D_{-k} \leq D_{-k}$. Therefore $(u, d_{-k}) \in C_L^{N\backslash\{k\}}$ for all $u \in [r_k(E, d), E]$, so, by the induction hypothesis, $\frac{\mathrm{AA}_j(u,d_{-k})}{d_j} - \frac{\mathrm{AA}_i(u,d_{-k})}{d_i} \leq 0$. Since $g_k(E, u) \geq 0$ we have $\frac{\mathrm{AA}_j(E,d)}{d_j} - \frac{\mathrm{AA}_i(E,d)}{d_i} \leq 0$.

On the other hand, if $E \in [\frac{1}{2}D_{-k}, \frac{1}{2}d(N)]$ then equality (C2) can be express as

$$\frac{AA_j(E,d)}{d_j} - \frac{AA_i(E,d)}{d_i} = \int_{r_k(E,d)}^{\frac{1}{2}D_{-k}} \left(\frac{AA_j(u,d_{-k})}{d_j} - \frac{AA_i(u,d_{-k})}{d_i}\right) g_k(E,u)du + \int_{\frac{1}{2}D_{-k}}^{E} \left(\frac{AA_j(u,d_{-k})}{d_j} - \frac{AA_i(u,d_{-k})}{d_i}\right) g_k(E,u)du. \tag{C3}$$

By self-duality,

$$\frac{AA_j(u,d_{-k})}{d_j} - \frac{AA_i(u,d_{-k})}{d_i} = \frac{d_j - AA_j(D_{-k}-u,d_{-k})}{d_j} - \frac{d_i - AA_i(D_{-k}-u,d_{-k})}{d_i} = \frac{AA_i(D_{-k}-u,d_{-k})}{d_i} - \frac{AA_j(D_{-k}-u,d_{-k})}{d_j}.$$

It is easy to see that $X(E,d_{-k}) = D_{-k} - X(D_{-k}-E,d_{-k})$. Then $V(E,d_{-k}) = V(D_{-k}-E,d_{-k})$ and applying the change of variable $s = D_{-k} - u$,

$$\int_{\frac{1}{2}D_{-k}}^{E} \left(\frac{AA_j(u,d_{-k})}{d_j} - \frac{AA_i(u,d_{-k})}{d_i}\right) g_k(E,u)du = \int_{D_{-k}-E}^{\frac{1}{2}D_{-k}} \left(\frac{AA_i(s,d_{-k})}{d_i} - \frac{AA_j(s,d_{-k})}{d_j}\right) g_k(E,s)ds. \tag{C4}$$

Since $\frac{1}{2}D_{-k} \le E \le \frac{1}{2}d(N) \le D_{-k}$, we have that $r_k(E,d) \le D_{-k} - E \le \frac{1}{2}D_{-k}$. Then,

$$\int_{r_k(E,d)}^{\frac{1}{2}D_{-k}} \left(\frac{AA_j(u,d_{-k})}{d_j} - \frac{AA_i(u,d_{-k})}{d_i}\right) g_k(E,u)du = \int_{r_k(E,d)}^{D_{-k}-E} \left(\frac{AA_j(u,d_{-k})}{d_j} - \frac{AA_i(u,d_{-k})}{d_i}\right) g_k(E,u)du + \int_{D_{-k}-E}^{\frac{1}{2}D_{-k}} \left(\frac{AA_j(u,d_{-k})}{d_j} - \frac{AA_i(u,d_{-k})}{d_i}\right) g_k(E,u)du. \tag{C5}$$

Now combining (C3) with the equalities (C4) and (C5),

$$\frac{AA_j(E,d)}{d_j} - \frac{AA_i(E,d)}{d_i} = \int_{r_k(E,d)}^{D_{-k}-E} \left(\frac{AA_j(u,d_{-k})}{d_j} - \frac{AA_i(u,d_{-k})}{d_i}\right) g_k(E,u)du.$$

Finally, $(u,d_{-k}) \in C_L^{N\setminus\{k\}}$ for all $u \in [r_k(E,d), D_{-k}-E]$, so, by the induction hypothesis, $\frac{AA_j(u,d_{-k})}{d_j} - \frac{AA_i(u,d_{-k})}{d_i} \le 0$. Therefore, $\frac{AA_j(E,d)}{d_j} - \frac{AA_i(E,d)}{d_i} \le 0$. □

APPENDIX D: RANKING THE RA AND APRO RULES FOR THREE-CLAIMANT PROBLEMS

Let $N = \{1, 2, 3\}$. We want to prove that APRO$\succcurlyeq$RA on C_L^N and that RA$\succcurlyeq$APRO on C_H^N. Let $(E, d) \in C^N$ with $d = (d_1, d_2, d_3) \in \mathbb{R}^3_{\leq}$. Since both the APRO rule and the RA rule are self-dual we just have to show that $\text{APRO}(E, d) \succcurlyeq \text{RA}(E, d)$ whenever $(E, d) \in C_L^N$. Then, assume that $E \leq \frac{1}{2}(d_1 + d_2 + d_3)$ and let us prove that $\text{APRO}_1(E, d) \geq \text{RA}_1(E, d)$ and $\text{APRO}_3(E, d) \leq \text{RA}_3(E, d)$. We distinguish five cases.

Case 1: If $E \leq d_1$, then $\text{APRO}(E, d) = \text{RA}(E, d)$ and the result is trivial.

Case 2: If $d_1 < E \leq d_2$, then $\text{APRO}(E, d) = \left(\frac{d_1 E}{d_1 + 2E}, \frac{E^2}{d_1 + 2E}, \frac{E^2}{d_1 + 2E}\right)$ and $\text{RA}(E, d) = \left(\frac{d_1}{3}, \frac{3E - d_1}{6}, \frac{3E - d_1}{6}\right)$. Easy computations show that $\text{APRO}_1(E, d) \geq \text{RA}_1(E, d)$ and $\text{APRO}_3(E, d) \leq \text{RA}_3(E, d)$.

Case 3: If $d_2 < E \leq d_3 \leq \frac{1}{2}d(N)$ or $d_2 < E \leq d_1 + d_2 \leq \frac{1}{2}d(N)$ then:

$$\text{APRO}(E, d) = \left(\frac{d_1 E}{d_1 + d_2 + E}, \frac{d_2 E}{d_1 + d_2 + E}, \frac{E^2}{d_1 + d_2 + E}\right)$$
$$\text{RA}(E, d) = \left(\frac{E + 2d_1 - d_2}{6}, \frac{E - d_1 + 2d_2}{6}, \frac{4E - d_1 - d_2}{6}\right).$$

Clearly, $\text{APRO}_1(E, d) \geq \text{RA}_1(E, d)$ if and only if $E^2 - AE + B \leq 0$ where $A = 3d_1$ and $B = 2d_1^2 + d_1 d_2 - d_2^2$. But $E^2 - AE + B = (E - (2d_1 - d_2))(E - (d_1 + d_2)) \leq 0$ since $E \leq d_1 + d_2$ and $E \geq d_2 \geq 2d_1 - d_2$. Analogously, $\text{APRO}_3(E, d) \leq \text{RA}_3(E, d)$ if and only if $2E^2 - 3CE + C^2 \leq 0$ where $C = d_1 + d_2$. But $2E^2 - 3CE + C^2 = (E - C)(2E - C) \leq 0$ because $E \leq C$ and $\frac{C}{2} \leq d_2 \leq E$.

Case 4: If $d_1 + d_2 \leq E \leq \frac{1}{2}d(N)$ then $\text{RA}(E, d) = \text{APRO}(E, d) = \left(\frac{d_1}{2}, \frac{d_2}{2}, E - \frac{d_1 + d_2}{2}\right)$.

Case 5: If $d_3 \leq E \leq \frac{1}{2}d(N)$ then:

$$\text{APRO}(E, d) = \left(\frac{d_1 E}{d_1 + d_2 + d_3}, \frac{d_2 E}{d_1 + d_2 + d_3}, \frac{d_3 E}{d_1 + d_2 + d_3}\right)$$
$$\text{RA}(E, d) = \left(\frac{2E + 2d_1 - d_2 - d_3}{6}, \frac{2E - d_1 + 2d_2 - d_3}{6}, \frac{2E - d_1 - d_2 + 2d_3}{6}\right).$$

Basic computations show that $\text{APRO}_1(E, d) \geq \text{RA}_1(E, d)$ if and only if $M = 6d_1 E - (d_1 + d_2 + d_3)(2E + 2d_1 - d_2 - d_3) \geq 0$. But $M = (2E - (d_1 + d_2 + d_3))(2d_1 - d_2 - d_3) \geq 0$ since $E \leq \frac{1}{2}d(N)$ and $2d_1 \leq d_2 + d_3$. Now, $\text{APRO}_3(E, d) \leq \text{RA}_3(E, d)$ if and only if $Q = 6d_3 E - (d_1 + d_2 + d_3)(2E - d_1 - d_2 + 2d_3) \leq 0$. But $Q = (2E - (d_1 + d_2 + d_3))(2d_3 - d_1 - d_2) \leq 0$ since $E \leq \frac{1}{2}d(N)$ and $2d_3 \geq d_1 + d_2$.