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APPLICATION OF MITTAG–LEFFLER FUNCTION ON CERTAIN SUBCLASSES OF ANALYTIC FUNCTIONS

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Abstract. The purpose of the present paper is to find the sufficient conditions for some subclasses of analytic functions associated with Mittag-Leffler function to be in subclasses of $k-\mathbf{ST}[A,B]$ and $k-\mathbf{UCV}[A,B]$ involving Jack's Lemma. Further, we discuss certain inclusion results and characteristic properties of an integral operator related to Mittag-Leffler function.

Keywords: Analytic functions, univalent function, starlike function, Hadamard product, Mittag–Leffler-function.

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1. Introduction and definitions

Let $\mathbf{A}$ denote the class of analytic functions in the open unit disk

$$\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\},$$

of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

and are normalized by the conditions $f(0) = 0 = f'(0)-1$. Further, let $\mathbf{T}$ [30] be a subclass of $\mathbf{A}$ consisting of functions of the form,

$$f(z) = z - \sum_{n=2}^{\infty} |a_n|\, z^n, \qquad z \in \mathbb{U}. \tag{2}$$

For given $g(z) = z + \sum\limits_{n=2}^{\infty} b_n z^n \in \mathbf{A}$, the Hadamard product of f and g is defined by

$$(f*g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n = (g*f)(z). \tag{3}$$

We note that $f*g \in \mathbf{A}$ is analytic and univalent in the open disc $\mathbb{U}$.

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A function $f \in \mathbf{S}$ is said to be *bounded turning* in $\mathbb{U}$ if and only if

$$\Re\left(f'(z)\right) > 0, \quad (z \in \mathbb{U})$$

denoted by $\mathbf{R}$.
A function $f \in \mathbf{S}$ is said to be *starlike* in $\mathbb{U}$ if and only if

$$\Re\Big(\frac{zf'(z)}{f(z)}\Big) > 0, \quad (z \in \mathbb{U})$$

and on the other hand , a function $f \in \mathbf{S}$ is said to be *convex* in Δ if and only if

$$\Re\Big(1 + \frac{zf''(z)}{f'(z)}\Big) > 0, \quad (z \in \mathbb{U})$$

denoted by $\mathbf{S}^*$ and $\mathbf{C}$ respectively.

A function $f(z)$ is of the form given in (1) is said to be in the class $\mathbf{SP}_P(\rho)$, if it satisfies the analytic characterization

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \left|\frac{zf'(z)}{f(z)} - 1\right| + \rho, \ (0 \leq \rho < 1)$$

and is said to be in the class $\mathbf{UCV}_P(\rho)$, if it satisfies the inequality

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \left|\frac{zf''(z)}{f'(z)}\right| + \rho, \ (0 \leq \rho < 1)$$

and $f \in \mathbf{UCV}_P(\rho)$ if and only if $zf' \in \mathbf{SP}_P(\rho)$.

Due to Noor and Malik [23], we let k-$\mathbf{ST}[A, B]$ and k-$\mathbf{UCV}[A, B]$ be subclasses of $\mathbf{A}$ as given below:
A function $f \in \mathbf{A}$ is said to be in the class k-$\mathbf{ST}[A, B]$, $k \geq 0$, $-1 \leq B < A \leq 1$, if and only if,

$$\Re\left(\frac{(B-1)\frac{zf'(z)}{f(z)} - (A-1)}{(B+1)\frac{zf'(z)}{f(z)} - (A+1)}\right) > k\left|\frac{(B-1)\frac{zf'(z)}{f(z)} - (A-1)}{(B+1)\frac{zf'(z)}{f(z)} - (A+1)} - 1\right|.$$

A function $f \in \mathbf{A}$ is said to be in the class k-$\mathbf{UCV}[A, B]$, $k \geq 0$, $-1 \leq B < A \leq 1$, if and only if,

$$\Re\left(\frac{(B-1)\frac{(zf'(z))'}{f'(z)} - (A-1)}{(B+1)\frac{(zf'(z))'}{f'(z)} - (A+1)}\right) > k\left|\frac{(B-1)\frac{(zf'(z))'}{f'(z)} - (A-1)}{(B+1)\frac{(zf'(z))'}{f'(z)} - (A+1)} - 1\right|.$$

It can be easily seen that

$$f(z) \in k\text{-}\mathbf{UCV}[A, B] \Leftrightarrow zf'(z) \in k\text{-}\mathbf{ST}[A, B].$$

Suitably specializing the parameters, we note that

(1) k-$\mathbf{ST}[1, -1] = k$-$\mathbf{ST}$ and k-$\mathbf{UCV}[1, -1] = k$-$\mathbf{UCV}$, the well-known classes of k-starlike and k-uniformly convex functions respectively, introduced by Kanas and Wisniowska [15, 16] (see also,[26]).
(2) k-$\mathbf{ST}[1 - 2\rho, -1] = k$-$\mathbf{SD}[k, \rho]$ and k-$\mathbf{UCV}[1 - 2\rho, -1] = k$-$\mathbf{KD}[k, \rho]$, the classes introduced by Shams et al. in [29].
(3) By fixing $k = 0$, 0-$\mathbf{ST}[A, B] = \mathbf{S}^*[A, B]$ and 0-$\mathbf{UCV}[A, B] = \mathbf{C}[A, B]$, the well-known classes of Janowski starlike and Janowski convex functions respectively, introduced by Janowski [14].
(4) By fixing $k = 0$, 0-$\mathbf{ST}[1 - 2\rho, -1] = \mathbf{S}^*(\rho)$ and 0-$\mathbf{UCV}[1 - 2\rho, -1] = \mathbf{C}(\rho)$, the well-known classes of starlike functions of order $\rho(0 \leq \rho < 1)$ and convex functions of order $\rho(0 \leq \rho < 1)$ respectively, (see [30]).

Geometrically, if a function $f(z) \in k\text{-}\mathbf{ST}[A,B]$ then

$$w = \frac{(B-1)\frac{zf'(z)}{f(z)} - (A-1)}{(B+1)\frac{zf'(z)}{f(z)} - (A+1)}$$

takes all values from the domain Ω_k, $k \geq 0$ as

$$\begin{aligned} \Omega_k &= \{w : \Re\,(w) > k\,|w-1|\} \\ &= \left\{u + iv : u > k\sqrt{(u-1)^2 + v^2}\right\}. \end{aligned}$$

The domain Ω_k represents the right half plane for $k = 0$; a hyperbola for $0 < k < 1$; a parabola for $k = 1$ and an ellipse for $k > 1$, (see [23]). Further the class of such functions which map the open unit disk onto this modified conic domain is discussed. Also the classes of $k-$uniformly Janowski convex and $k-$Janowski starlike functions are defined and their coefficient inequalities are obtained as in the following results:

Lemma 1.1. [23]*A function f of the form (1) is in the class k-$\mathbf{ST}[A,B]$, if it satisfies the condition*

$$\sum_{n=2}^{\infty}[2(k+1)(n-1) + |n(B+1) - (A+1)|]\,|a_n| \leq |B-A| \tag{4}$$

where $-1 \leq B < A \leq 1$ and $k \geq 0$.

Lemma 1.2. [23]*A function f of the form (1) is in the class k-$\mathbf{UCV}[A,B]$, if it satisfies the condition*

$$\sum_{n=2}^{\infty} n[2(k+1)(n-1) + |n(B+1) - (A+1)|]\,|a_n| \leq |B-A| \tag{5}$$

where $-1 \leq B < A \leq 1$ and $k \geq 0$.

1.1. **Mittag-Leffler type function (MLF).** Recently, there has been awesome attention in the study of Mittag-Leffler functions. The Mittag-Leffler type functions have extensive applications in physics, biology, chemistry, engineering, and some other applied sciences and other aspects of the applications of these functions can be seen in fractional differ en-tail equations, stochastic systems, dynamical systems, statistical distributions, and chaotic systems.(for more applications see [27] pp. 269-296) . The Mittag-Leffler function arises naturally in the solution of fractional order differential and integral equations, and especially in the investigations of fractional generalization of kinetic equation, random walks, Lévy flights, super-diffusive transport and in the study of complex systems. The geometric properties like starlikeness, convexity, and close-to-convexity of these functions have been examined on a large scale by a number of researchers. Several properties of Mittag-Leffler function(MFLs) and generalized Mittag-Leffler function(MFLs) can be found e.g. in [2, 5, 8, 11, 12, 17]. We can easily see the direct usage of these functions to many techniques of fractional calculus. Much later,theoretical applications at the study of integral equations and more practical applications to the modeling of"non-standard"processes have been found for the Mittag-Lefer function. Now we recall the well Mittag-Leffler function $\mathbb{E}_\alpha(z)$ studied by Mittag-Leffler [21] and given by

$$\mathbb{E}_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \quad (z \in \mathbb{C}, \Re(\alpha) > 0\ )$$

converges in the whole complex plane for all $\Re(\alpha) > 0$, where as it diverges every where on $\mathbb{C} \setminus \{0\}$ for all $\Re(\alpha) < 0$. We also note that for $\Re(\alpha) = 0$, its radius of convergence is $R = e^{\pi/2|Im(z)|}$.

A more general function $\mathbb{E}_{\alpha,\beta}$ generalizing $E_\alpha(z)$ was introduced by Wiman [35, 36] and defined by

$$\mathbb{E}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \quad (z, \alpha, \beta \in \mathbb{C},\ \Re(\beta) > 0,\ \Re(\alpha) > 0). \tag{6}$$

For different values of parameters,the conditions of convergence vary. When α and β are positive and real,the above given series converges in the entire complex plane. Another important and interesting fact about Mittag-Leffler function is that the one and two-parametric Mittag-Leffler function are fractional extensions of the basic functions. That is the function $\mathbb{E}_{\alpha,\beta}$ contains many well-known functions as its special case, for example, $\mathbb{E}_{2,1}(z) = z\cosh\sqrt{z}$, $\mathbb{E}_{2,2}(z) = \sqrt{z}\sinh\sqrt{z}$, $\mathbb{E}_{2,3}(z) = 2[\cosh\sqrt{z} - 1]$ and $\mathbb{E}_{2,4}(z) = 6[\sinh\sqrt{z} - \sqrt{z}]/\sqrt{z}$.

The early and lengthy work focused on the essential properties of the Mittag-Leffler function as entire functions and remained on the theoretical level of pure mathematics. Three decades later,the Mittag-Leffler function application the solution for a fractional differential equation period has finally arrived lately many applications and special cases of these functions lead many applications in solving differential equations (see[3, 6, 19, 22, 25, 33]and references cited therein).The(classical)Mittag-Lefer function has been introduced to give an answer to a classical question of complex analysis,namely,to describe the procedure of analytic continuation of power series outside the disc of their convergence. Further it has been observed that Mittag-Leffler function $E_{\alpha,\beta}$ does not belong to the family $\mathbf{A}$. Therefore, we consider the following normalization of the Mittag-Leffler function (see,[5, 24])

$$\begin{aligned} E_{\alpha,\beta}(z) &= \Gamma(\beta) z \mathbb{E}_{\alpha,\beta}(z) \\ &= z + \sum_{n=2}^{\infty} \frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)} z^n, \quad z, \alpha, \beta \in \mathbb{C},\ \Re(\beta) > 0,\ \Re(\alpha) > 0. \end{aligned} \tag{7}$$

Whilst formula (7) holds for complex-valued α, β and $z \in \mathbb{C}$, however in this paper, we shall restrict our attention to the case of real-valued α, β and $z \in \mathbb{U}$.

Now we define the following new linear operator based on convolution (or Hadamard) product.

For positive real parameters α, β and $E_{\alpha,\beta}$ be given by (7), we define the linear operator $\Lambda_{\alpha,\beta} : \mathbf{A} \to \mathbf{A}$ with the aid of the convolution product

$$\begin{aligned} \Lambda_{\alpha,\beta} f(z) : &= f(z) * E_{\alpha,\beta}(z) \\ &= z + \sum_{n=2}^{\infty} \frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)} a_n z^n,\ z \in \mathbb{U}, \beta > 0, \alpha > 0 \end{aligned} \tag{8}$$

where $*$ denote the convolution or Hadamard product given by (3). Lately ,the mathematicians paid more attention to the behavior of the Mittag-Leffler function and extended their results to the complex domain. Until now, many properties of the highlighted special function have been derived in the literature,[3, 6, 19, 22, 25, 33] for some other perspective in this paper inspired by several previous results on connections between several subclasses of analytic and univalent functions, using hypergeometric functions (see for example, [1, 10, 20, 31]), generalized Bessel functions (see for example, [4, 9]), Struve functions (see for example, [7, 32]) we regulate new outcomes of analytic functions linked with operator $\Lambda_{\alpha,\beta}(z)$, defined by Mittag–Leffler function such as sufficient conditions for $\Lambda_{\alpha,\beta}f(z)$ to be in the classes k-$\mathbf{ST}[A,B]$ and $k-\mathbf{UCV}[A,B]$. Furthermore, we estimate certain sufficient conditions and inclusion relations among the classes $\mathbf{R}^{\tau}(C,D)$ and $k-\mathbf{UCV}[A,B]$. Finally, we found conditions for an integral operator $\mathbf{G}_{\alpha,\beta}(z)=\int_0^z \frac{\Lambda_{\alpha,\beta}f(t)}{t}dt$ to be in the class $k-\mathbf{UCV}[A,B]$.

2. Sufficient conditions for MLFs involving Jack's Lemma

In the present section, we determine certain sufficient conditions involving Jack's Lemma for $\Lambda_{\alpha,\beta}(z)$, $z(\Lambda_{\alpha,\beta}(z))'$ to be in the class of Janowski starlike and Janowski convex functions. To prove the main theorems we need the following:

Lemma 2.1. [13] *Let $\omega(z)$ be regular in the unit disk $\mathbb{U}$ with $\omega(0)=0$. If $|\omega(z)|$ attains a maximum value on the circle $|z|=r\,(0\leq r<1)$ at a point z, then $z_1\omega'(z_1)=m\omega(z_1)$ where m is real and $m\geq 1$.*

Lemma 2.2. [28] *Let a function f of the form (1) satisfy*

$$\left|\frac{zf'(z)}{f(z)}-1\right|^{1-\vartheta}\left|\frac{zf''(z)}{f'(z)}\right|^{\vartheta}<\frac{(A-B)(2+A+A^2)^{\alpha}}{(1+|B|)(1+A)^{2\vartheta}} \tag{9}$$

for fixed constants A, B and α such that $-1\leq B<A\leq 1$, $\vartheta\geq 0$ and $\forall z\in\mathbb{U}$. Then $f\in\mathbf{S}^*[A,B]$.

Theorem 2.1. *Let $f\in\mathbf{A}$. If*

$$\left|(\Lambda_{\alpha,\beta}f(z))'-1\right|^{1-\vartheta}\left|\frac{z(\Lambda_{\alpha,\beta}f(z))''}{(\Lambda_{\alpha,\beta}f(z))'}\right|^{\vartheta}<\frac{1}{2^{\vartheta}}\quad(\vartheta\geq 0). \tag{10}$$

Then $\Lambda_{\alpha,\beta}f$ is univalent in $\mathbb{U}$.

Proof. We know that

$$\Lambda_{\alpha,\beta}f(z)=z+\sum_{n=2}^{\infty}\frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)}a_nz^n,\ z\in\mathbb{U},$$

in $\mathbf{A}$. Define ω by $\omega(z)=(\Lambda_{\alpha,\beta}f(z))'-1$ for $z\in\mathbb{U}$. Then it follows that ω is analytic in $\mathbb{U}$ with $\omega(0)=0$. In view of (10)

$$|\omega(z)|^{1-\vartheta}\left|\frac{z\omega'(z)}{1+\omega(z)}\right|^{\vartheta}=|\omega(z)|\left|\frac{z\omega'(z)}{\omega(z)}\frac{1}{1+\omega(z)}\right|^{\vartheta}<\frac{1}{2^{\vartheta}}. \tag{11}$$

Suppose that there exists a point $z_1\in\mathbb{U}$ such that $\max_{|z|\leq|z_1|}|\omega(z)|=|\omega(z_1)|=1$. Then, by Lemma 2.1, we can put $\frac{z_1\omega'(z_1)}{\omega(z_1)}=m\geq 1$. Therefore, we obtain

$$|\omega(z_1)|\left|\frac{z_1\omega'(z_1)}{\omega(z_1)}\frac{1}{1+\omega(z)}\right|^{\vartheta}\geq\left(\frac{m}{2}\right)^{\vartheta}\geq\frac{1}{2^{\vartheta}}$$

which contradicts the condition (11). This shows that $|\omega(z)|=|(\Lambda_{\alpha,\beta}f(z))'-1|<1$ which implies that $\Re(\Lambda_{\alpha,\beta}f(z))'>0$ for $z\in\mathbb{U}$. Therefore, by the Noshiro-warschawski theorem [?], $\Lambda_{\alpha,\beta}f(z)$ is univalent in $\mathbb{U}$. □

Theorem 2.2. *Let $f\in\mathbf{A}$, and let $\Lambda_{\alpha,\beta}(z)$ defined by*

$$\Lambda_{\alpha,\beta}(z):=z+\sum_{n=2}^{\infty}\frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)}z^n,\ z\in\mathbb{D}, \tag{12}$$

satisfy the inequality

$$\left|\frac{z\Lambda'_{\alpha,\beta}(z)}{\Lambda_{\alpha,\beta}(z)}\right| < \frac{A-B}{1+|B|} \tag{13}$$

where $-1 \leq B < A < 1$ *and* $-1 \leq B \leq 0, z \in \mathbb{U}$. *Then* $z\,\Lambda_{\alpha,\beta}(z) \in \mathbf{S}^*[A,B]$.

Proof. Let us define a function $P(z)$ by

$$P(z) = z\Lambda_{\alpha,\beta}(z),\ (z \in \mathbb{U}).$$

In view of (13), we have

$$\left|\frac{zP'(z)}{P(z)} - 1\right| < \frac{A-B}{1+|B|}. \tag{14}$$

An application Lemma 2.2 with $\vartheta = 0$ proves Theorem 2.2. □

Theorem 2.3. *Let* $f \in \mathbf{A}$, *and* $\Lambda_{\alpha,\beta}(z)$ *given by* (12) *satisfy the inequality*

$$\left|\frac{z\Lambda''_{\alpha,\beta}(z)}{\Lambda'_{\alpha,\beta}(z)}\right| < \frac{(A-B)(2+A+A^2)}{(1+|B|)(1+A)^2} \tag{15}$$

where $-1 \leq B < A < 1$ *and* $-1 \leq B \leq 0$. *Then* $\Lambda_{\alpha,\beta}(z)$ *is starlike of order* $\frac{A+B}{2B}$ *and type* $|B|$ *with respect to* 1.

Proof. Let $H : \mathbb{U} \to \mathbb{C}$ be defined by $H(z) = \frac{\Lambda_{\alpha,\beta}(z) - b_o}{b_1}$. Then $H \in \mathbf{A}$ and $H(z)$ satisfies

$$\begin{aligned}\left|\frac{zH''(z)}{H'(z)}\right| &= \left|\frac{z\Lambda''_{\alpha,\beta}(z)}{\Lambda'_{\alpha,\beta}(z)}\right| \\ &< \frac{(A-B)(2+A+A^2)}{(1+|B|)(1+A)^2}.\end{aligned}$$

An application of Lemma 2.2 with $\vartheta = 1$ implies that $\Lambda_{\alpha,\beta}(z)$ is starlike of order $\frac{A+B}{2B}$ and type $|B|$ with respect to 0 as the value of $b_0 = 1$. □

3. Sufficient conditions for MLF to be in k-$\mathbf{ST}[A,B]$, and k-$\mathbf{ST}[A,B]$

Unless otherwise mentioned, we shall assume in this paper that

$$\alpha, \beta > 0, \quad k \geq 0 \quad and \quad -1 \leq B < A \leq 1$$

and we will use the notation (7), therefore

$$E_{\alpha,\beta}(1) - 1 = \sum_{n=2}^{\infty} \frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)}, \tag{16}$$

$$E'_{\alpha,\beta}(1) - 1 = \sum_{n=2}^{\infty} \frac{n\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)}, \tag{17}$$

$$E''_{\alpha,\beta}(1) = \sum_{n=2}^{\infty} \frac{n(n-1)\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)} \tag{18}$$

$$\int_0^1 \frac{E_{\alpha,\beta}(t)}{t}dt = \sum_{n=2}^{\infty} \frac{\Gamma(\beta)}{n\Gamma(\alpha(n-1)+\beta)} \tag{19}$$

Firstly, we obtain the necessary and sufficient conditions for $E_{\alpha,\beta}$ to be in the class k-$\mathbf{ST}[A,B]$.

Theorem 3.1. *If*

$$\begin{aligned} & (2k+B+3)E'_{\alpha,\beta}(1)+(A-2k-1)E_{\alpha,\beta}(1)-(B+A+2) \\ \leq \; & |B-A| \end{aligned} \tag{20}$$

then $E_{\alpha,\beta} \in k$-$\mathbf{ST}[A,B]$.

Proof. In view of Lemma 1.1 and (4) it suffices to show that

$$Q_1 := \sum_{n=2}^{\infty}[2(k+1)(n-1)+|n(B+1)-(A+1)|]\frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)} \leq |B-A|\,.$$

Now

$$\begin{aligned} Q_1 \;&\leq\; \sum_{n=2}^{\infty}[2(k+1)(n-1)+n(B+1)+(A+1)]\frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)} \\ &=\; \sum_{n=2}^{\infty}\left[(2k+B+3)n+(A-2k-1)\right]\frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)} \\ &=\; \sum_{n=2}^{\infty}(2k+B+3)\frac{n\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)}+\sum_{n=2}^{\infty}(A-2k-1)\frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)} \\ &=\; (2k+B+3)(E'_{\alpha,\beta}(1)-1)+(A-2k-1)(E_{\alpha,\beta}(1)-1) \\ &=\; (2k+B+3)E'_{\alpha,\beta}(1)+(A-2k-1)E_{\alpha,\beta}(1)-(B+A+2) \\ &\leq\; |B-A|\,, \end{aligned}$$

by the given hypothesis (20). This complete the proof of Theorem 3.1. □

Theorem 3.2. *If*

$$\begin{aligned} & (2k+B+3)E''_{\alpha,\beta}(1)+(B+A+2)E'_{\alpha,\beta}(1)-(B+A+2) \\ \leq \; & |B-A| \end{aligned} \tag{21}$$

then $E_{\alpha,\beta} \in k$-$\mathbf{UCV}[A,B]$.

Proof. In view of Lemma 1.2 and (5) it suffices to show that

$$Q_2 := \sum_{n=2}^{\infty}n[2(k+1)(n-1)+|n(B+1)-(A+1)|]\frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)} \leq |B-A|\,.$$

Now

$$\begin{aligned} Q_2 \;&\leq\; \sum_{n=2}^{\infty}n\left[(2k+B+3)n+(A-2k-1)\right]\frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)} \\ &=\; \sum_{n=2}^{\infty}[n^2(2k+B+3)+n(A-2k-1)]\frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)} \\ &=\; \sum_{n=2}^{\infty}\left[(2k+B+3)n(n-1)+n(B+A+2)\right]\frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)} \\ &=\; (2k+B+3)\sum_{n=2}^{\infty}\frac{n(n-1)\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)}+(B+A+2)\sum_{n=2}^{\infty}\frac{n\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)} \\ &=\; (2k+B+3)E''_{\alpha,\beta}(1)+(B+A+2)(E'_{\alpha,\beta}(1)-1) \\ &\leq\; |B-A|\,, \end{aligned}$$

by the given hypothesis (21). This complete the proof of Theorem 3.2. □

4. Inclusion Properties

The next class $\mathbf{R}^{\tau}(\rho,\delta)$ was introduced earlier by Swaminathan [34], and for special cases see the references cited there in:

Definition 4.1. *A function $f \in \mathbf{A}$ is said to be in the class $\mathbf{R}^{\tau}(\rho,\delta)$, where $\tau \in \mathbb{C}\setminus\{0\}$, $0<\rho\leq 1$, and $\delta<1$, if it satisfies the inequality*

$$\left|\frac{(1-\rho)\frac{f(z)}{z}+\rho f'(z)-1}{2\tau(1-\delta)+(1-\rho)\frac{f(z)}{z}+\rho f'(z)-1}\right|<1,\ z\in\mathbb{D}.$$

Lemma 4.1. [34] *If $f\in\mathbf{R}^{\tau}(\rho,\delta)$ is of the form (1), then*

$$|a_n| \leq \frac{2\,|\tau|\,(1-\delta)}{1+\rho(n-1)},\ n\in\mathbb{N}\setminus\{1\}. \tag{22}$$

The bounds given in (22) is sharp for

$$f(z)=\frac{1}{\rho z^{1-\frac{1}{\rho}}}\int_0^z t^{1-\frac{1}{\rho}}\left[1+\frac{2(1-\delta)\tau\ t^{n-1}}{1-2^{n-1}}\right]dt.$$

Making use of Lemma 4.1 we will study the action of the Mittag–Leffler-functions on the class k-$\mathbf{UCV}[A,B]$.

Theorem 4.1. *If*

$$\begin{aligned}\frac{2\,|\tau|\,(1-\delta)}{\rho}&\left[(2k+B+3)(E_{\alpha,\beta}(1)-1)+(A-2k-1)\int_0^1\frac{E_{\alpha,\beta}(t)}{t}dt\right]\\ \leq\ & |B-A|\end{aligned} \tag{23}$$

then $\Lambda_{\alpha,\beta}(\mathbf{R}^{\tau}(\rho,\delta))\subset k-\mathbf{ST}[A,B]$.

Proof. In view of Lemma 1.1 and (4) it suffices to show that

$$\sum_{n=2}^{\infty}[2(k+1)(n-1)+|n(B+1)-(A+1)|]\frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)}\,|a_n|\leq|B-A|\,.$$

$$\begin{aligned}Q_3 &\leq \sum_{n=2}^{\infty}[2(k+1)(n-1)+n(B+1)+(A+1)]\frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)}|a_n|\\ &= \sum_{n=2}^{\infty}[(2k+B+3)n+(A-2k-1)]\,\frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)}|a_n|\\ &= \sum_{n=2}^{\infty}(2k+B+3)\frac{n\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)}|a_n|+\sum_{n=2}^{\infty}(A-2k-1)\frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)}|a_n|.\end{aligned}$$

Since, $f\in\mathbf{R}^{\tau}(\rho,\delta)$, in virtue of Lemma 4.1,

$$|a_n|\leq\frac{2\,|\tau|\,(1-\delta)}{1+\rho(n-1)},\ n\in\mathbb{N}\setminus\{1\},\quad\text{and}\quad 1+\rho(n-1)\geq\rho n.$$

Therefore, it is enough to show that

$$\begin{aligned}
& \frac{2\left|\tau\right|(1-\delta)}{\rho}\left[\sum_{n=2}^{\infty}(2k+B+3)\frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)}+\sum_{n=2}^{\infty}\frac{(A-2k-1)}{n}\frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)}\right] \\
= \quad & \frac{2\left|\tau\right|(1-\delta)}{\rho}\left[(2k+B+3)(E_{\alpha,\beta}(1)-1)+(A-2k-1)\int_0^1\frac{E_{\alpha,\beta}(t)}{t}dt\right] \\
\leq \quad & \left|B-A\right|.
\end{aligned}$$

□

Theorem 4.2. *If*

$$\begin{aligned}
& \frac{2\left|\tau\right|(1-\delta)}{\rho}\left[(2k+B+3)E'_{\alpha,\beta}(1)+(A-2k-1)E_{\alpha,\beta}(1)-(B+A+2)\right] \\
\leq \quad & \left|B-A\right| \qquad (24)
\end{aligned}$$

then $\Lambda_{\alpha,\beta}(\mathbf{R}^{\tau}(\rho,\delta))\subset k-\mathbf{UCV}[A,B]$.

Proof. In view of Lemma 1.2 and (5) it suffices to show that

$$\sum_{n=2}^{\infty}n[2(k+1)(n-1)+|n(B+1)-(A+1)|]\frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)}\left|a_n\right|\leq\left|B-A\right|.$$

Since, $f\in\mathbf{R}^{\tau}(\rho,\delta)$, in virtue of Lemma 4.1,

$$\left|a_n\right|\leq\frac{2\left|\tau\right|(1-\delta)}{1+\rho(n-1)},\ n\in\mathbb{N}\setminus\{1\},\quad\text{and}\quad 1+\rho(n-1)\geq\rho n,$$

therefore, it is enough to show that

$$\begin{aligned}
& \frac{2\left|\tau\right|(1-\delta)}{\rho}\left[\sum_{n=2}^{\infty}[2(k+1)(n-1)+|n(B+1)-(A+1)|]\frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)}\right] \\
\leq \quad & \left|B-A\right|.
\end{aligned}$$

By a similar proof like those of Theorem 3.1 we get that $\Lambda_{\alpha,\beta}f\in k-\mathbf{UCV}[A,B]$ if (24) holds. □

5. An integral operator

Advances in the subject have led to consideration of more difficult transformations preserving the class **S**. The first form of this integral transform was introduced and studied by Alexander[18].

Theorem 5.1. *If the integral operator*

$$\mathbf{G}_{\alpha,\beta}(z):=\int_0^z\frac{\Lambda_{\alpha,\beta}(t)}{t}dt,\ z\in\mathbb{U}, \qquad (25)$$

is in k-$\mathbf{ST}[A,B]$ *if the inequality*

$$\begin{aligned}
& (2k+B+3)(E_{\alpha,\beta}(1)-1)+(A-2k-1)\int_0^1\frac{E_{\alpha,\beta}(t)}{t}dt \\
\leq \quad & \left|B-A\right|. \qquad (26)
\end{aligned}$$

is satisfied.

Proof. According to (8) it follows that

$$\mathbf{G}_{\alpha,\beta}(z) = z - \sum_{n=2}^{\infty} \frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)} \frac{z^n}{n}.$$

Using Lemma 1.1 and (4), the integral operator $\mathbf{G}_{\alpha,\beta}(z)$ belongs to $k-\mathbf{ST}[A,B]$ if

$$\sum_{n=2}^{\infty}[2(k+1)(n-1)+|n(B+1)-(A+1)|]\frac{\Gamma(\beta)}{n\Gamma(\alpha(n-1)+\beta)} \leq |B-A|.$$

Let $Q_4 = \sum_{n=2}^{\infty}[2(k+1)(n-1)+|n(B+1)-(A+1)|]\frac{\Gamma(\beta)}{n\Gamma(\alpha(n-1)+\beta)}$,then

$$\begin{aligned} Q_4 &\leq \sum_{n=2}^{\infty}[2(k+1)(n-1)+n(B+1)+(A+1)]\frac{\Gamma(\beta)}{n\Gamma(\alpha(n-1)+\beta)} \\ &= \sum_{n=2}^{\infty}[(2k+B+3)n+(A-2k-1)]\frac{\Gamma(\beta)}{n\Gamma(\alpha(n-1)+\beta)}. \end{aligned}$$

$$\begin{aligned} &\sum_{n=2}^{\infty}(2k+B+3)\frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)} + \sum_{n=2}^{\infty}\frac{(A-2k-1)}{n}\frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)} \\ = &\ (2k+B+3)(E_{\alpha,\beta}(1)-1)+(A-2k-1)\int_0^1 \frac{E_{\alpha,\beta}(t)}{t}dt \end{aligned}$$

and the above equation is bounded by $|B-A|$, if (26) holds. This completes the proof of Theorem 5.1. □

Theorem 5.2. *If the integral operator*

$$\mathbf{G}_{\alpha,\beta}(z) := \int_0^z \frac{\Lambda_{\alpha,\beta}(t)}{t}dt, \ z \in \mathbb{U},$$

is in $k-\mathbf{UCV}[A,B]$ *if the inequality* (20) *is satisfied.*

Proof. According to (8) it follows that

$$\mathbf{G}_{\alpha,\beta}(z) = z - \sum_{n=2}^{\infty} \frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)} \frac{z^n}{n}.$$

Using Lemma 1.2 and (5), the integral operator $\mathbf{G}_{\alpha,\beta}(z)$ belongs to $k-\mathbf{UCV}[A,B]$ if

$$\sum_{n=2}^{\infty}[2(k+1)(n-1)+|n(B+1)-(A+1)|]\frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)} \leq |B-A|.$$

By a similar proof like those of Theorem 4.2 we get that $\mathbf{G}_{\alpha,\beta} \in k-\mathbf{UCV}[A,B]$ if (20) holds. □

6. Conclusions

By specializing the parameters, we can easily state the (sufficient conditions and inclusion properties) results analogues to Theorems 3.1 to 5.2 proved in this paper associated with Mittag-Leffler function .By fixing $A = 1-2\rho$, and $B = -1$ with $0 \leq \rho < 1$, one can state the results for the subclasses k-$\mathbf{SD}[k,\rho]$ and k-$\mathbf{KD}[k,\rho]$ studied by Shams et al.,[29] and by fixing $A = 1$, and $B = -1$ one can deduce the results for the well-known classes of k-starlike and k-uniformly convex functions respectively, introduced by Kanas and Wisniowska [15, 16] (see also,[26]). Further by taking $k = 0$ we state the results for the function classes $\mathbf{S}^*[A, B]$ and $\mathbf{C}[A, B]$, introduced by Janowski [14] and also for the function classes $\mathbf{S}^*(\rho), \mathbf{C}(\rho)$, studied in [30]. Further,by fixing $\alpha = 1/2$ and $\beta = 1$ we get

$$\mathbf{E}_{\frac{1}{2},1}(z) = e^{z^2} \cdot \operatorname{erfc}(-z) = e^{z^2}\left(1 + \frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(2n+1)}z^{2n+1}\right)$$

which yields the new inclusion results as in Theorems 3.1 to 5.2 associated with error functions, for the function classes k-$\mathbf{ST}[A, B]$ and $k - \mathbf{UCV}[A, B]$ which has not been studied so far,we left this as an exercise to interested readers.

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G. Murugusundaramoorthy for the photography and short autobiography, see TWMS J. App. and Eng. Math. V.11, N.2.
