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Uniqueness for a high order ill posed problem

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In this work, we study a high order derivative in time problem. First, we show that there exists a sequence of elements of the spectrum which tends to infinity and therefore, it is ill posed. Then, we prove the uniqueness of solutions for this problem by adapting the logarithmic arguments to this situation. Finally, the results are applied to the backward in time problem for the generalized linear Burgers' fluid, a couple of heat conduction problems and a viscoelastic model.

Keywords: High order PDE; uniqueness; ill-posedness; logarithmic convexity; Burgers' fluid; dual-phase lag; viscoelasticity

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1. Introduction

It is easy to find equations and systems which lead to ill-posed problems (in the sense of Hadamard) when different models arisen in the applied mathematics are studied. Maybe the simplest way to find this problem is when we consider the backward in time heat equation, or the system of elasto-dynamics when the elasticity tensor is not positive definite. The spectra of these problems contain sequences whose elements have a real part which tends to infinity. Therefore, we cannot obtain the continuous dependence with respect to the initial data and, moreover, a certain norm of some solutions can blow up at finite time. So, it is important to know if, at least, we can ensure the uniqueness of solutions.

There are several techniques to study this kind of problems. We can cite the books of Ames and Straughan [1] or Flavin and Rionero [5], where the authors recall different techniques to analyse them. They also study a series of problems arisen in applied mathematics. We can find in these books a huge quantity of references where qualitative properties of ill-posed problems are studied. One of the most used arguments in these works is the so-called logarithmic convexity.

We can also recall the works of Knops and Payne [7], where they obtained results of uniqueness and instability for elasto-dynamic problems (see also [6]), and which were extended by other authors to consider different thermoelastic theories [15, 18]. In the book of Ames and Straughan [1] we can find a full description of this method for the backward in time heat equation.

In the last years, a huge interest has been developed in the study of equations involving high order derivative in time. These appear in a natural way when we want to study different problems in the applied mathematics. Usually, parabolic and hyperbolic equations have been considered, leading to well-posed problems, and some results have been obtained [2, 11, 13]. However, few attention has been devoted to ill-posed problems associated to high order equations [4]. In this work, we aim to study one problem of this type. That is, we consider a kind of problem which is ill-posed and associated to high order equations. We think that this is the first contribution in this line.

Here, we consider an ill-posed problem (see equation (2.1) below) and we want to obtain a result concerning the uniqueness of solutions. Our argument is based on the method of logarithmic convexity. We have found two main difficulties to prove the main result. The first one is that it was not clear the function which we had to use. The second difficulty was that we had to bound different terms which must be controlled. In this work, we have overcome these difficulties with the help of a combination of integrals with respect to the time.

The plan for this paper is the following: in the next section we propose the problem to be studied and we state some few properties to be used later. In section three we prove the uniqueness result and, in section four, we recall three different situations where the above result can be applied. In the last section we propose further comments which allow to extend the results of the third section.

2. Preliminaries

Let us assume that B represents a bounded domain in $\mathbb{R}^d$, for $d = 1, 2, 3$.

In this work, we are going to study uniqueness issues for the problem determined by the equation

$$a_1\dot{u} + a_2\ddot{u} + \ldots + a_n u^{(n)} + u^{(n+1)} = -k(b_1\Delta u + \ldots + b_n\Delta u^{(n-1)} + \Delta u^{(n)}), \quad (2.1)$$

where a_i and b_i are real numbers, n is a natural number greater than zero[1] and $k > 0$, with the boundary condition

$$u = 0 \quad \text{on} \quad \partial B, \quad (2.2)$$

and the initial conditions, for a.e. $\boldsymbol{x} \in B$,

$$u(\boldsymbol{x}, 0) = u^0(\boldsymbol{x}), \quad \dot{u}(\boldsymbol{x}, 0) = u^1(\boldsymbol{x}), \quad \ldots \quad , u^{(n)}(\boldsymbol{x}, 0) = u^n(\boldsymbol{x}). \quad (2.3)$$

[1]The case $n = 0$ corresponds to the backward in time version of the usual heat equation based on the Fourier law. It is well known that it corresponds to an ill posed problem in the sense of Hadamard. The uniqueness of solution in this case is well known (see[1]).

First, we must observe that this problem is not well posed in the sense of Hadamard. In particular, we will see that there exists a sequence of real numbers ξ_n which tend to $+\infty$ and belong to the point spectrum of this problem.

In fact, if we consider solutions of the form

$$u(\boldsymbol{x}, t) = e^{\omega t}\Phi_n(\boldsymbol{x}) \quad \text{for a.e. } \boldsymbol{x} \in B, \tag{2.4}$$

where the function $\Phi_n(\boldsymbol{x})$ is the solution to the problem

$$\Delta\Phi_n + \lambda_n\Phi_n = 0 \quad \text{in } B, \quad \Phi_n = 0 \quad \text{on} \quad \partial B, \tag{2.5}$$

then we find that the following relation

$$a_1\omega + a_2\omega^2 + \ldots + a_n\omega^n + \omega^{n+1} = k(b_1\lambda_n + b_2\omega\lambda_n + \ldots + b_n\omega^{n-1}\lambda_n + \omega^n\lambda_n)$$

holds.

That is, ω satisfies the equation:

$$x^{n+1} + (a_n - k\lambda_k)x^n + (a_{n-1} - kb_n\lambda_k)x^{n-1} + \ldots + (a_1 - kb_2\lambda_k) - kb_1\lambda_k = 0.$$

Our aim is to see that, when λ_n tends to infinity, there exists a sequence of real numbers which are solutions to this equation ξ_n satisfying the condition $\lambda_n^{1/2} \leqslant \xi_n < \infty$ such that $\xi_n \to \infty$ when $\lambda_n \to \infty$.

First, we fix the value of λ_n. Clearly, the function

$$P_k(x) = x^{n+1} + (a_n - k\lambda_k)x^n + \ldots + (a_1 - kb_2\lambda_k) - kb_1\lambda_k$$

tends to $+\infty$ when $x \to \infty$.

On the other hand, we can take the value of $P_k(x)$ for $x = \lambda_k^{1/2}$ and so, we have

$$P_k(\lambda_k^{1/2}) = \lambda_k^{\frac{n+1}{2}} + (a_n - k\lambda_k)\lambda_k^{n/2} + \ldots + (a_1 - kb_2\lambda_k)\lambda_k^{1/2} - kb_1\lambda_k.$$

It is obvious that the term of highest order of the above polynomial is $-k\lambda_k^{\frac{n+2}{2}}$.

Therefore, if λ_k is large enough, we find that $P_k(\lambda_k^{1/2}) < 0$ and so, every $P_r(\lambda_r^{1/2})$, for $r \geqslant k$, are all negative. Now, applying mean value theorem we conclude that there exists ξ_k, $\lambda_k^{1/2} \leqslant \xi_k < \infty$, such that $P_k(\xi_k) = 0$.

Clearly, for each large value of λ_k we can choose ξ_k and, since $\lambda_k^{1/2}$ tends to infinity, we may conclude that ξ_k also tends to infinity. Therefore, we have proved that problem (2.1)–(2.3) is ill-posed in the sense of Hadamard.

Remark 1. This analysis also applies if we consider the equation

$$a_1\dot{u} + a_2\ddot{u} + \ldots + a_n u^{(n)} + u^{(n+1)} = k(b_1 Au + \ldots + b_n Au^{(n-1)} + Au^{(n)}),$$

in a Hilbert space $\mathcal{H}$, where A is a symmetric and positive definite operator such that there exists an infinite sequence of eigenvalues $\lambda_n \to \infty$.

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Now, since we have shown that the problem is not well-posed, we can ask ourselves about the uniqueness of solutions. Hence, it will be enough to prove that the unique solution when we consider the initial conditions

$$u(\boldsymbol{x},0)=0, \quad \dot{u}(\boldsymbol{x},0)=0, \quad \ldots \quad , u^{(n)}(\boldsymbol{x},0)=0 \quad \text{for a.e. } \boldsymbol{x}\in B \tag{2.6}$$

is the null solution.

Thus, our aim in the next section will be to prove that, under some assumptions, the unique solution to problems (2.1), (2.2) and (2.6) is the null solution. Therefore, it will be useful to recall some properties.

First, we recall the Poincaré-like inequality which states that the following estimate

$$\int_0^t u^2(\xi)\, d\xi \leqslant \frac{4t^2}{\pi^2}\int_0^t |\dot{u}(\xi)|^2\, d\xi \tag{2.7}$$

holds, whenever $u(0)=0$.

It will be also convenient to remark that

$$\begin{aligned} u^{(n+1)}u^{(n-k)} &= \frac{\mathrm{d}}{\mathrm{d}t}\left[u^{(n)}u^{(n-k)}\right] - u^{(n)}u^{(n-k+1)} \\ &= \frac{\mathrm{d}^2}{\mathrm{d}t^2}\left[u^{(n-1)}u^{(n-k)}\right] - u^{(n-1)}u^{(n-k+1)} - u^{(n)}u^{(n-k+1)} \\ &= \ldots \\ &= \frac{1}{2}\frac{\mathrm{d}^{k+1}}{\mathrm{d}t^{k+1}}\left[|u^{(n-k)}|^2\right] - W_{n-k}(u^{(n)},u^{(n-1)},\ldots,u^{(n-k+1)}), \end{aligned} \tag{2.8}$$

for $1\leqslant k<n$, where W_{n-k} is a quadratic function in its arguments.

The relations (2.7) and (2.8) will be a key point in our study. From them, by using Hölder inequality we can conclude that

$$\left|\int_0^t \nabla u^{(i)}\nabla u^{(j)}\,\mathrm{d}s\right| \leqslant C^* t\int_0^t |\nabla u^{(n)}|^2\,\mathrm{d}s, \tag{2.9}$$

whenever $0\leqslant i,\, j\leqslant n,\; i+j<2n$ and (2.6) holds, where C^* is a computable constant.

From (2.7) and (2.9) we note that

$$\begin{aligned} &\int_0^t\int_0^{s_n}\cdots\int_0^{s_1}\int_B\left(\nabla G_1\nabla G_2 - |\nabla u^{(n)}|^2\right)\mathrm{d}v\mathrm{d}\tau\mathrm{d}s_1\ldots\mathrm{d}s_n \\ &\quad\leqslant C_1 t\int_0^t\int_0^{s_n}\cdots\int_0^{s_1}\int_B |\nabla u^{(n)}|^2\,\mathrm{d}v\mathrm{d}\tau\mathrm{d}s_1\ldots\mathrm{d}s_n, \end{aligned} \tag{2.10}$$

where

$$G_1 = a_1 u + a_2\dot{u} + \ldots + a_n u^{(n-1)} + u^{(n)},$$
$$G_2 = b_1 u + b_2\dot{u} + \ldots + b_n u^{(n-1)} + u^{(n)},$$

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C_1 is a positive calculable constant, and we have made a systematic use of the inequality (2.9), and for every constants a_i and b_i.

Finally, it is worth noting that if $u(0) = 0$ then

$$u^2(t) = 2\int_0^t u\dot{u}\,\mathrm{d}s \leqslant 2\left(\int_0^t u^2\,\mathrm{d}s\right)^{1/2}\left(\int_0^t |\dot{u}|^2\,\mathrm{d}s\right)^{1/2} \leqslant \frac{4t}{\pi}\int_0^t |\dot{u}|^2\,\mathrm{d}s.$$

In general, if we assume that conditions (2.6) are fulfilled, we can see that

$$\begin{aligned}\int_0^t\int_0^{s_n}\ldots\int_0^{s_{j+3}} &|u^{(n-j)}|^2\,\mathrm{d}s_{j+2}\ldots\mathrm{d}s_n\\ &\leqslant \frac{4t}{\pi}\int_0^t\int_0^{s_n}\ldots\int_0^{s_{j+2}}|u^{(n-j+1)}|^2\,\mathrm{d}s_{j+1}\ldots\mathrm{d}s_n\end{aligned} \tag{2.11}$$

for $2 \leqslant j \leqslant n$.

3. Uniqueness of solutions

The objective of this section is to obtain an uniqueness result to problem (2.1)–(2.3).

In order to simplify the notation, we can rewrite equation (2.1) in the form:

$$\dot{\tilde{u}} = -k\Delta\hat{u},$$

where $\tilde{u} = a_1u + a_2\dot{u} + \ldots + a_nu^{(n-1)} + u^{(n)}$ and $\hat{u} = b_1u + b_2\dot{u} + \ldots + b_nu^{(n-1)} + u^{(n)}$.

The main idea to prove the result will be to use the function

$$\begin{aligned}F(t) = \frac{1}{2}&\int_0^t\int_0^{s_n}\int_0^{s_{n-1}}\ldots\int_0^{s_1}\int_B|\tilde{u}|^2\,\mathrm{d}v\mathrm{d}\tau\mathrm{d}s_1\mathrm{d}s_2\ldots\mathrm{d}s_n\\ &+\frac{\omega}{2}\int_0^t\int_0^{s_n}\int_0^{s_{n-1}}\ldots\int_0^{s_1}\int_B|\nabla u^{(n-1)}|^2\,\mathrm{d}v\mathrm{d}\tau\mathrm{d}s_1\mathrm{d}s_2\ldots\mathrm{d}s_n,\end{aligned} \tag{3.1}$$

where ω is a positive constant which will be chosen later.

We have

$$\begin{aligned}\dot{F}(t) = &\int_0^t\int_0^{s_n}\ldots\int_0^{s_1}\int_B\tilde{u}\dot{\tilde{u}}\,\mathrm{d}v\mathrm{d}\tau\mathrm{d}s_1\mathrm{d}s_2\ldots\mathrm{d}s_n\\ &\quad+\frac{\omega}{2}\int_0^t\int_0^{s_n}\ldots\int_0^{s_2}\int_B|\nabla u^{(n-1)}|^2\,\mathrm{d}v\mathrm{d}\tau\mathrm{d}s_2\ldots\mathrm{d}s_n\\ &=\int_0^t\int_0^{s_n}\int_0^{s_{n-1}}\ldots\int_0^{s_1}\int_B\tilde{u}\dot{\tilde{u}}\,\mathrm{d}v\mathrm{d}\tau\mathrm{d}s_1\ldots\mathrm{d}s_n\\ &\quad+\omega\int_0^t\int_0^{s_n}\ldots\int_0^{s_1}\int_B\nabla u^{(n)}\nabla u^{(n-1)}\,\mathrm{d}v\mathrm{d}\tau\mathrm{d}s_1\ldots\mathrm{d}s_n\end{aligned}$$

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$$
\begin{aligned}
&= -\int_0^t \int_0^{s_n} \int_0^{s_{n-1}} \dots \int_0^{s_1} \int_B k\tilde{u}\Delta\hat{u}\, \mathrm{d}v\mathrm{d}\tau\mathrm{d}s_1 \dots \mathrm{d}s_n \\
&\quad + \omega \int_0^t \int_0^{s_n} \dots \int_0^{s_1} \int_B \nabla u^{(n)} \nabla u^{(n-1)}\, \mathrm{d}v\mathrm{d}\tau\mathrm{d}s_1 \dots \mathrm{d}s_n \\
&= k \int_0^t \int_0^{s_n} \int_0^{s_{n-1}} \dots \int_0^{s_1} \int_B \nabla\tilde{u}\nabla\hat{u}\, \mathrm{d}v\mathrm{d}\tau\mathrm{d}s_1 \dots \mathrm{d}s_n \\
&\quad + \omega \int_0^t \int_0^{s_n} \dots \int_0^{s_1} \int_B \nabla u^{(n)} \nabla u^{(n-1)}\, \mathrm{d}v\mathrm{d}\tau\mathrm{d}s_1 \dots \mathrm{d}s_n \\
&\geqslant k(1 - C_1^* t) \int_0^t \int_0^{s_n} \int_0^{s_{n-1}} \dots \int_0^{s_1} \int_B |\nabla u^{(n)}|^2\, \mathrm{d}v\mathrm{d}\tau\mathrm{d}s_1 \dots \mathrm{d}s_n,
\end{aligned}
$$

where we recall that C_1^* is a positive calculable constant, and we have made a systematic use of the inequality (2.9).

We note that we can choose T small enough to guarantee that

$$
\dot{F}(t) \geqslant \frac{k}{2} \int_0^t \int_0^{s_n} \int_0^{s_{n-1}} \dots \int_0^{s_1} \int_B |\nabla u^{(n)}|^2\, \mathrm{d}v\mathrm{d}\tau\mathrm{d}s_1 \dots \mathrm{d}s_n, \tag{3.2}
$$

for every $t \leqslant T$.

Now, we analyse the second derivative of the function F. It follows that[2]

$$
\begin{aligned}
\ddot{F}(t) &= k \int_0^t \int_0^{s_n} \dots \int_0^{s_1} \int_B \left(\nabla\dot{\tilde{u}}\nabla\hat{u} + \nabla\tilde{u}\nabla\dot{\hat{u}} \right) \mathrm{d}v\mathrm{d}\tau\mathrm{d}s_1 \dots \mathrm{d}s_n \\
&\quad + \frac{\omega}{2} \int_0^t \int_0^{s_n} \dots \int_0^{s_3} \int_B |\nabla u^{(n-1)}|^2\, \mathrm{d}v\mathrm{d}\tau\mathrm{d}s_3 \dots \mathrm{d}s_n.
\end{aligned}
$$

We can easily find that

$$
\begin{aligned}
&k \int_0^t \int_0^{s_n} \dots \int_0^{s_1} \int_B \nabla\dot{\tilde{u}}\nabla\hat{u}\, \mathrm{d}v\mathrm{d}\tau\mathrm{d}s_1 \dots \mathrm{d}s_n \\
&\qquad = -k \int_0^t \int_0^{s_n} \dots \int_0^{s_1} \int_B \dot{\tilde{u}}\Delta\hat{u}\, \mathrm{d}v\mathrm{d}\tau\mathrm{d}s_1 \dots \mathrm{d}s_n \\
&\qquad = \int_0^t \int_0^{s_n} \dots \int_0^{s_1} \int_B |\dot{\tilde{u}}|^2\, \mathrm{d}v\mathrm{d}\tau\mathrm{d}s_1 \dots \mathrm{d}s_n.
\end{aligned}
$$

However, the second summand of the first integral in $\ddot{F}$ is more difficult to handle. We obtain that

$$
\begin{aligned}
&k \int_0^t \int_0^{s_n} \dots \int_0^{s_1} \int_B \nabla\tilde{u}\nabla\dot{\hat{u}}\, \mathrm{d}v\mathrm{d}\tau\mathrm{d}s_1 \dots \mathrm{d}s_n \\
&\quad = k \int_0^t \int_0^{s_n} \dots \int_0^{s_1} \int_B \nabla(\tilde{u} - \hat{u} + \hat{u})\nabla(\dot{\hat{u}} - \dot{\tilde{u}} + \dot{\tilde{u}})\, \mathrm{d}v\mathrm{d}\tau\mathrm{d}s_1 \dots \mathrm{d}s_n \\
&\quad = I_1 + I_2 + I_3,
\end{aligned} \tag{3.3}
$$

[2]When $n = 1$ the second integral on the right-hand side is $\int_B |\nabla u|^2\, dv$.

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where

$$I_1 = k\int_0^t\int_0^{s_n}\ldots\int_0^{s_1}\int_B \nabla\hat{u}\nabla\dot{\tilde{u}}\,\mathrm{d}v\mathrm{d}\tau\mathrm{d}s_1\ldots\mathrm{d}s_n,$$
$$I_2 = k\int_0^t\int_0^{s_n}\ldots\int_0^{s_1}\int_B \nabla(\tilde{u}-\hat{u})\nabla\dot{\tilde{u}}\,\mathrm{d}v\mathrm{d}\tau\mathrm{d}s_1\ldots\mathrm{d}s_n, \tag{3.4}$$
$$I_3 = k\int_0^t\int_0^{s_n}\ldots\int_0^{s_1}\int_B \nabla\tilde{u}\nabla(\dot{\hat{u}}-\dot{\tilde{u}})\,\mathrm{d}v\mathrm{d}\tau\mathrm{d}s_1\ldots\mathrm{d}s_n.$$

We find that

$$I_1 = \int_0^t\int_0^{s_n}\ldots\int_0^{s_1}\int_B |\dot{\tilde{u}}|^2\,\mathrm{d}v\mathrm{d}\tau\mathrm{d}s_1\ldots\mathrm{d}s_n \tag{3.5}$$

as we have seen previously.

On the other hand, we also have

$$\begin{aligned}I_2 &= k\int_0^t\int_0^{s_n}\ldots\int_0^{s_1}\int_B \nabla(\tilde{u}-\hat{u})\nabla\dot{\tilde{u}}\,\mathrm{d}v\mathrm{d}\tau\mathrm{d}s_1\ldots\mathrm{d}s_n\\ &= k\int_0^t\int_0^{s_n}\ldots\int_0^{s_1}\int_B \nabla F_1\nabla F_2\,\mathrm{d}v\mathrm{d}\tau\mathrm{d}s_1\ldots\mathrm{d}s_n,\end{aligned}$$

where

$$F_1 = (a_1-b_1)u + (a_2-b_2)\dot{u} + \ldots + (a_n-b_n)u^{(n-1)},$$
$$F_2 = a_1\dot{u} + a_2\ddot{u} + \ldots + a_n u^{(n)} + u^{(n+1)}.$$

We can bound the integrals of the form

$$\int_0^t\int_0^{s_n}\ldots\int_0^{s_1}\int_B (a_i-b_i)\nabla u^{(i-1)}a_j\nabla u^{(j)}\,\mathrm{d}v\mathrm{d}\tau\mathrm{d}s_1\ldots\mathrm{d}s_n,\quad i,j=1,\ldots,n,$$

by the integral

$$Kt\int_0^t\int_0^{s_n}\ldots\int_0^{s_1}\int_B |\nabla u^{(n)}|^2\,\mathrm{d}v\mathrm{d}\tau\mathrm{d}s_1\ldots\mathrm{d}s_n,$$

where K is a computable constant, after a repetitive use of the inequality (2.9).

The terms more difficult to bound are those of the form

$$k\int_0^t\int_0^{s_n}\ldots\int_0^{s_1}\int_B (a_i-b_i)\nabla u^{(i-1)}\nabla u^{(n+1)}\,\mathrm{d}v\mathrm{d}\tau\mathrm{d}s_1\ldots\mathrm{d}s_n\quad \text{for } i=1,\ldots,n.$$

If we take into account that, for $j=1,\ \ldots,\ n-1$,

$$\nabla u^{(j)}\nabla u^{(n+1)} = \frac{1}{2}\frac{\mathrm{d}^{(n-j+1)}}{\mathrm{d}t^{n-j+1}}\left[|\nabla u^{(j)}|^2\right] - W_j(\nabla u^{(n)},\ldots,\nabla u^{(j+1)}),$$

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then we obtain

$$
\begin{aligned}
& k\int_0^t\int_0^{s_n}\ldots\int_0^{s_1}\int_B \nabla F_1\nabla u^{(n+1)}\,\mathrm{d}v\mathrm{d}\tau\mathrm{d}s_1\ldots\mathrm{d}s_n \\
&\quad + \frac{\omega}{2}\int_0^t\int_0^{s_n}\ldots\int_0^{s_3}\int_B |\nabla u^{(n-1)}|^2\mathrm{d}v\mathrm{d}\tau\mathrm{d}s_3\ldots\mathrm{d}s_n \\
&= \frac{k}{2}\int_0^t\int_0^{s_n}\ldots\int_0^{s_1}\int_B G(\tau)\,\mathrm{d}v\mathrm{d}\tau\mathrm{d}s_1\ldots\mathrm{d}s_n \\
&\quad + \frac{\omega}{2}\int_0^t\int_0^{s_n}\ldots\int_0^{s_3}\int_B |\nabla u^{(n-1)}|^2\mathrm{d}v\mathrm{d}\tau\mathrm{d}s_3\ldots\mathrm{d}s_n \\
&\quad - k\sum_{i=0}^{n-1}\int_0^t\int_0^{s_n}\ldots\int_0^{s_1}\int_B (a_{i+1}-b_{i+1}) \\
&\quad \times W_i(\nabla u^{(n)},\ldots,\nabla u^{(i+1)})\,\mathrm{d}v\mathrm{d}\tau\mathrm{d}s_1\ldots\mathrm{d}s_n, \qquad (3.6)
\end{aligned}
$$

where

$$G(t) = \sum_{i=0}^{n-1}(a_{i+1}-b_{i+1})\frac{\mathrm{d}^{n-i+1}}{\mathrm{d}\tau^{n-i+1}}|\nabla u^{(i)}|^2$$

and W_i is a quadratic expression of its arguments.

In view of the estimate (2.11) we see that the addition of the first and second terms on the right-hand side of (3.6) is positive whenever $t \leqslant T$ and T sufficiently small, and ω large enough. Moreover, the third term on the right-hand side of (3.6) will be greater or equal to

$$-(C_2 + C_2^* t)\int_0^t\int_0^{s_n}\ldots\int_0^{s_1}\int_B |\nabla u^{(n)}|^2\,\mathrm{d}v\mathrm{d}\tau\mathrm{d}s_1\ldots\mathrm{d}s_n,$$

where constants C_2 and C_2^* can be easily calculated. Therefore, we find that

$$
\begin{aligned}
& I_2 + \frac{\omega}{2}\int_0^t\int_0^{s_n}\ldots\int_0^{s_3}|\nabla u^{(n-1)}|^2\,\mathrm{d}v\mathrm{d}\tau\mathrm{d}s_3\ldots\mathrm{d}s_n \\
&\quad \geqslant -C_3\int_0^t\int_0^{s_n}\ldots\int_0^{s_1}\int_B |\nabla u^{(n)}|^2\,\mathrm{d}v\mathrm{d}\tau\mathrm{d}s_1\ldots\mathrm{d}s_n,
\end{aligned}
$$

where C_3 is a computable constant for every $t \leqslant T$ and T is small enough.

Thus, we have proved that

$$I_2 + \frac{\omega}{2}\int_0^t\int_0^{s_n}\ldots\int_0^{s_3}|\nabla u^{(n-1)}|^2\,\mathrm{d}v\mathrm{d}\tau\mathrm{d}s_3\ldots\mathrm{d}s_n \geqslant -\frac{2C_3}{k}\dot{F}(t).$$

We bound now the integral I_3. Proceeding in an analogous way, we also have

$$I_3 \geqslant -C_4\int_0^t\int_0^{s_n}\ldots\int_0^{s_1}\int_B |\nabla u^{(n)}|^2\,\mathrm{d}v\mathrm{d}\tau\mathrm{d}s_1\ldots\mathrm{d}s_n \geqslant \frac{-2C_4}{k}\dot{F}(t), \qquad (3.7)$$

where C_4 is again a computable constant for every $t \leqslant T$, where T is sufficiently small.

Combining all these estimates, we conclude that, for every $t \leqslant T$,

$$\begin{aligned}\ddot{F}(t) \geqslant\ & 2\int_0^t\int_0^{s_n}\ldots\int_0^{s_1}\int_B |\dot{\tilde{u}}|^2\,\mathrm{d}v\mathrm{d}\tau\mathrm{d}s_1\ldots\mathrm{d}s_n\\ & - C_5\int_0^t\int_0^{s_n}\ldots\int_0^{s_1}\int_B |\nabla u^{(n)}|^2\,\mathrm{d}v\mathrm{d}\tau\mathrm{d}s_1\ldots\mathrm{d}s_n\\ \geqslant\ & 2\int_0^t\int_0^{s_n}\ldots\int_0^{s_1}\int_B |\dot{\tilde{u}}|^2\,\mathrm{d}v\mathrm{d}\tau\mathrm{d}s_1\ldots\mathrm{d}s_n - \frac{2C_5}{k}\dot{F}(t).\end{aligned}$$

Therefore, if we assume that $t \leqslant T$ is small enough then we find that

$$\begin{aligned}F\ddot{F}-(\dot{F})^2 \geqslant & \left[\frac{1}{2}\int_0^t\int_0^{s_n}\ldots\int_0^{s_1}\int_B (|\tilde{u}|^2+\omega|\nabla u^{(n-1)}|^2)\,\mathrm{d}v\mathrm{d}\tau\mathrm{d}s_1\ldots\mathrm{d}s_n\right]\\ & \left[2\int_0^t\int_0^{s_n}\ldots\int_0^{s_1}\int_B |\dot{\tilde{u}}|^2\,\mathrm{d}v\mathrm{d}\tau\mathrm{d}s_1\ldots\mathrm{d}s_n\right.\\ & \left. - l_1\int_0^t\int_0^{s_n}\ldots\int_0^{s_1}\int_B |\nabla u^{(n)}|^2\,\mathrm{d}v\mathrm{d}\tau\mathrm{d}s_1\ldots\mathrm{d}s_n\right]\\ & - \left(\int_0^t\int_0^{s_n}\ldots\int_0^{s_1}\int_B (\tilde{u}\dot{\tilde{u}}+\omega\nabla u^{(n)}\nabla u^{(n-1)})\,\mathrm{d}v\mathrm{d}\tau\mathrm{d}s_1\ldots\mathrm{d}s_n\right)^2\\ = & \left[\frac{1}{2}\int_0^t\int_0^{s_n}\ldots\int_0^{s_1}\int_B (|\tilde{u}|^2+\omega|\nabla u^{(n-1)}|^2)\,\mathrm{d}v\mathrm{d}\tau\mathrm{d}s_1\ldots\mathrm{d}s_n\right]\\ \times & \left[2\int_0^t\int_0^{s_n}\ldots\int_0^{s_1}\int_B (|\dot{\tilde{u}}|^2+\omega|\nabla u^{(n)}|^2)\,\mathrm{d}v\mathrm{d}\tau\mathrm{d}s_1\ldots\mathrm{d}s_n\right.\\ & \left. - l^*\int_0^t\int_0^{s_n}\ldots\int_0^{s_1}\int_B |\nabla u^{(n)}|^2\,\mathrm{d}v\mathrm{d}\tau\mathrm{d}s_1\ldots\mathrm{d}s_n\right]\\ & - \left(\int_0^t\int_0^{s_n}\ldots\int_0^{s_1}\int_B (\tilde{u}\dot{\tilde{u}}+\omega\nabla u^{(n)}\nabla u^{(n-1)})\,\mathrm{d}v\mathrm{d}\tau\mathrm{d}s_1\ldots\mathrm{d}s_n\right)^2\\ \geqslant & -lF\dot{F},\end{aligned} \tag{3.8}$$

where l_1, l and l^* are computable constants.

The inequality (3.8) is well-known in the study of the qualitative properties of the solutions to ill-posed problems. We find that (see [**1**, **5**])

$$F(t) \leqslant F(0)^{\frac{\sigma-\sigma_2}{1-\sigma_2}}F(T)^{\frac{1-\sigma}{1-\sigma_2}} \quad \text{for a.e. } t\in[0,T],$$

where $\sigma = e^{-lt}$ and $\sigma_2 = e^{-lT}$. If we assume that the initial conditions are zero, then we obtain that $F(0) = 0$ and so, $F(t) = 0$ for a.e. $t \in [0, T]$. This implies that $u(t) = 0$ for a.e. $t \in [0, T]$.

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This process can be repeated successively in the interval $[T, 2T]$ and so on. Therefore, we obtain that problem (2.1)–(2.3) has a unique solution.

Remark 2. Our analysis could be adapted to the problem proposed in remark 1. An elemental example could be

$$a_1\dot{u} + a_2\ddot{u} + \ldots + a_n u^{(n)} + u^{(n+1)} = k\Delta^2(b_1 u + \ldots + b_n u^{(n-1)} + u^{(n)}),$$

with the boundary conditions:

$$u = \Delta u = 0 \quad \text{on} \quad \partial B.$$

Another interesting example could be again the problem proposed in remark 1 when $Au = (b_{ij}(\boldsymbol{x})u_{,i})_{,j}$, where $b_{ij}(\boldsymbol{x})$ is a symmetric positive definite tensor.

4. Applications to some special problems

In this section, we focus on the application of the result given in the previous section, related to the uniqueness of solution, to the generalized Burgers' fluid, a couple of heat conduction problems and a viscoelastic problem. In particular, we aim to prove that, under certain conditions, it is not possible that the solutions to these three problems are localized in time. It means that, if the solution vanishes after a finite time $t_0 \geqslant 0$, then this solution is null. It is convenient to note that this property is equivalent to show that the backward in time problem has a unique solution.

4.1. Generalized Burgers' fluid

In the paper [**14**] the authors proposed in a natural form the system which determines the evolution of the linearized form for the generalized Burgers' fluid. We recall that it is written as follows,

$$\begin{aligned} &\rho(\dot{\boldsymbol{v}} + \lambda_1\ddot{\boldsymbol{v}} + \lambda_2\dddot{\boldsymbol{v}}) = -\nabla q + \eta_1\Delta\boldsymbol{v} + \eta_2\Delta\dot{\boldsymbol{v}} + \eta_3\Delta\ddot{\boldsymbol{v}},\\ &\text{div } \boldsymbol{v} = 0. \end{aligned}$$

In the previous system, $\boldsymbol{v}$ is the velocity and $\lambda_1, \lambda_2, \eta_1, \eta_2, \eta_3$ are positive constants. It is easy to rewrite this system as

$$\frac{1}{\lambda_2}\dot{\boldsymbol{v}} + \frac{\lambda_1}{\lambda_2}\ddot{\boldsymbol{v}} + \dddot{\boldsymbol{v}} = \frac{\eta_3}{\lambda_2\rho}\left(\frac{\eta_1}{\eta_3}\Delta\boldsymbol{v} + \frac{\eta_2}{\eta_3}\Delta\dot{\boldsymbol{v}} + \Delta\ddot{\boldsymbol{v}}\right) - \nabla q^*.$$

The backward in time problem is written in the following form:

$$-\lambda_2^{-1}\dot{\boldsymbol{v}} + \lambda_1\lambda_2^{-1}\ddot{\boldsymbol{v}} - \dddot{\boldsymbol{v}} = \frac{\eta_3}{\lambda_2\rho}\left(\eta_1\eta_3^{-1}\Delta\boldsymbol{v} - \eta_2\eta_3^{-1}\Delta\dot{\boldsymbol{v}} + \Delta\ddot{\boldsymbol{v}}\right) - \nabla q^*.$$

Therefore, it leads to the following system:

$$\lambda_2^{-1}\dot{\boldsymbol{v}} - \lambda_1\lambda_2^{-1}\ddot{\boldsymbol{v}} + \dddot{\boldsymbol{v}} = -\frac{\eta_3}{\lambda_2\rho}\left(\eta_1\eta_3^{-1}\Delta\boldsymbol{v} - \eta_2\eta_3^{-1}\Delta\dot{\boldsymbol{v}} + \Delta\ddot{\boldsymbol{v}}\right) + \nabla q^*.$$

This system is of the form:

$$\dot{\hat{\boldsymbol{v}}} = -k\Delta\hat{\boldsymbol{v}} + \nabla q^*, \tag{4.1}$$

where $k = \frac{\eta_3}{\lambda_2\rho}$, $a_1 = \lambda_2^{-1}$, $a_2 = -\lambda_1\lambda_2^{-1}$, $b_1 = \eta_1\eta_3^{-1}$ and $b_2 = -\eta_2\eta_3^{-1}$, and we adjoin null boundary conditions. Since we assume that ρ, λ_2 and η_3 are positive, then problem (4.1) is ill posed in the sense of Hadamard.

Thus, the arguments proposed previously can be adapted easily to this situation and we can conclude the uniqueness of solutions to the backward in time problem.

4.2. Dual-phase-lag and three-phase-lag heat conduction

One of the theories proposed by Tzou [**17**] considers the heat conduction equation:

$$\frac{\tau_q^2}{2}\dddot{T} + \tau_q\ddot{T} + \dot{T} = k\left(\frac{\tau_T^2}{2}\Delta\ddot{T} + \tau_T\Delta\dot{T} + \Delta T\right),$$

where $k > 0$ and τ_q, τ_T are two positive relaxation parameters. The backward in time version of this equation is

$$-\frac{\tau_q^2}{2}\dddot{T} + \tau_q\ddot{T} - \dot{T} = k\left(\frac{\tau_T^2}{2}\Delta\ddot{T} - \tau_T\Delta\dot{T} + \Delta T\right).$$

We can write

$$\dddot{T} - \frac{2}{\tau_q}\ddot{T} + \frac{2}{\tau_q^2}\dot{T} = -k\left(\frac{\tau_T}{\tau_q}\right)\left(\Delta\ddot{T} - \frac{2}{\tau_T}\Delta\dot{T} + \frac{2}{\tau_T^2}\Delta T\right).$$

In view of the arguments of the previous section we can guarantee the uniqueness of solutions to this last equation with homogeneous Dirichlet boundary conditions.

In 2007, Roy Chouduri [**3**] proposed the heat equation:

$$\tau_q\dddot{T} + \ddot{T} = k^*\Delta T + \tau_\nu^*\Delta\dot{T} + k\tau_T\Delta\ddot{T},$$

where k and k^* are two positive parameters, $\tau_\nu^* = k^*\tau_\nu + k$ and τ_q, τ_ν and τ_T are three relaxation parameters. The backward in time version of this equation becomes

$$\dddot{T} - \tau_q^{-1}\ddot{T} = -\frac{k\tau_T}{\tau_q}\left(\Delta\ddot{T} - \frac{\tau_\nu^*}{k\tau_T}\Delta\dot{T} + \frac{k^*}{k\tau_T}T\right).$$

Therefore, we can guarantee the uniqueness of solutions to the problem determined by this equation with homogeneous Dirichlet boundary conditions.

4.3. Viscoelasticity

Lebedev and Gladwell [**10**] proposed a constitutive equation of the form:

$$C(\partial/\partial t)\sigma_{ij} = A(\partial/\partial t)\varepsilon_{kk}\delta_{ij} + 2B(\partial/\partial t)\varepsilon_{ij},$$

where A, B and C are polynomials, σ_{ij} is the stress tensor, ε_{ij} is the strain tensor and δ_{ij} is the Kronecker symbol.

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In the case that we consider anti-plane shear deformations and we assume that $degree(C) = degree(B) - 1$, we can obtain an equation of the form:

$$u^{(n+1)} + a_n u^{(n)} + \ldots + a_2 u^{(2)} = \mu(b_0 \Delta u + \ldots + b_n \Delta u^{(n-1)} + \Delta u^{(n)}), \tag{4.2}$$

where $\mu > 0$. We can apply the results of the previous section to the backward in time version of this equation to conclude the impossibility of localization for the solutions to the equation (4.2). In fact, we could use the results for a more general version of viscoelasticity.

Viscoelastic fluids have deserved much attention in the last years. An interesting class of them can be found in [**12**]. Linear and nonlinear versions have been studied [**8**, **9**, **16**]. For some of them, the study we have developed can be used. For instance, the ones called of Oldroyd can be written as

$$\ddot{v}_i + \gamma \dot{v}_i + p_{,i} = \mu \Delta \dot{v}_i + (\beta + \mu\gamma)\Delta v_i, \quad v_{i,i} = 0,$$

where the parameters μ, γ and β are positive. The backward in time version of this equation can be written as

$$\ddot{v}_i - \gamma \dot{v}_i + p_{,i} = -\mu(\Delta \dot{v}_i - \frac{\beta + \mu\gamma}{\mu} \Delta v_i), \quad v_{i,i} = 0.$$

Therefore, our results apply to this case. However, it is suitable to recognize that our arguments cannot be used to study others viscoelastic fluids as the ones known as Kelvin–Voigt type.

5. Further comments

In the previous sections, we have assumed that a_i and b_j are constants. The reason was that we needed it to prove that problem (2.1)–(2.3) is ill posed in the sense of Hadamard. However, in order to prove the uniqueness of solutions this assumption is not required as we will see below.

In what follows, we point out the changes in the proof that we need in the case that a_i and b_j can depend on the point; however, we still assume that k is a constant. Anyway, we impose that a_i and b_j are C^1 functions with respect to variable $\boldsymbol{x}$.

It will be relevant to take into account an easy extension of the equality (2.8). We also have

$$\begin{aligned} f^{(n+1)} g^{(n-k)} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(f^{(n)} g^{(n-k)}\right) - f^{(n)} g^{(n-k+1)} \\ &= \frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(f^{(n-1)} g^{(n-k)}\right) - f^{(n-1)} g^{(n-k+1)} - f^{(n)} g^{(n-k+1)} \\ &= \ldots \\ &= \frac{\mathrm{d}^{k+1}}{\mathrm{d}t^{k+1}}\left(f^{(n-k)} g^{(n-k)}\right) - \left[f^{(n-k)} + f^{(n-k+1)} + \ldots + f^{(n)}\right] g^{(n-k+1)}. \end{aligned} \tag{5.1}$$

Now, we develop the analysis. First, we note that the inequality (2.10) also holds in our case. We can define the function $F(t)$ as we have done in (3.1). Therefore, estimate (3.2) also holds. The unique difference is that we need to apply the Poincaré inequality at several points. We also have equality (3.3) where I_i are given in (3.4). Moreover, the equality (3.5) is satisfied but, again, the most difficult point is to estimate I_2. We have:

$$\nabla F_1 \nabla F_2 = \sum_{i=1}^{6} M_i,$$

where

$$M_1 = \sum_{i,j=1}^{n} \nabla(a_i - b_i) u^{(i-1)} a_j \nabla u^{(j)}, \quad M_2 = \sum_{i,j=1}^{n} (a_i - b_i) \nabla u^{(i-1)} \nabla a_j u^{(j)},$$

$$M_3 = \sum_{i,j=1}^{n} \nabla(a_i - b_i) \nabla a_j u^{(i-1)} u^{(j)}, \quad M_4 = \sum_{i,j=1}^{n} (a_i - b_i) a_j \nabla u^{(i-1)} \nabla u^{(j)},$$

$$M_5 = \sum_{i=1}^{n} \nabla(a_i - b_i) u^{(i-1)} \nabla u^{(n+1)}, \quad M_6 = \sum_{i=1}^{n} (a_i - b_i) \nabla u^{(i-1)} \nabla u^{(n+1)}.$$

Again, integrals M_i, $i = 1, \ldots, 4$, can be controlled by expressions of the form:

$$Kt \int_0^t \int_0^{s_n} \ldots \int_0^{s_1} \int_B |\nabla u^{(n)}|^2 \, \mathrm{d}v \mathrm{d}\tau \mathrm{d}s_1 \ldots \mathrm{d}s_n.$$

To estimate integral M_6 we can follow the same argument that in estimate (3.6) to obtain

$$\begin{aligned} M_6 &+ \frac{\omega_1}{2} \int_0^t \int_0^{s_n} \ldots \int_0^{s_1} \int_B |\nabla u^{(n-1)}|^2 \, \mathrm{d}v \mathrm{d}\tau \mathrm{d}s_1 \ldots \mathrm{d}s_n \\ &\geqslant -(K_1 + K_1^* t) \int_0^t \int_0^{s_n} \ldots \int_0^{s_1} \int_B |\nabla u^{(n)}|^2 \, \mathrm{d}v \mathrm{d}\tau \mathrm{d}s_1 \ldots \mathrm{d}s_n, \end{aligned}$$

where ω_1 is large enough and K_1 and K_1^* are constants whenever $t \leqslant T$ and T is sufficiently small.

Using the equality (5.1) and the Poincaré inequality we can find that

$$\begin{aligned} M_5 &+ \frac{\omega_2}{2} \int_0^t \int_0^{s_n} \ldots \int_0^{s_1} \int_B |\nabla u^{(n-1)}|^2 \, \mathrm{d}v \mathrm{d}\tau \mathrm{d}s_1 \ldots \mathrm{d}s_n \\ &\geqslant -(K_2 + K_2^* t) \int_0^t \int_0^{s_n} \ldots \int_0^{s_1} \int_B |\nabla u^{(n)}|^2 \, \mathrm{d}v \mathrm{d}\tau \mathrm{d}s_1 \ldots \mathrm{d}s_n, \end{aligned}$$

where, again, ω_2 is large enough and K_2 and K_2^* are constants when $t \leqslant T$ for T sufficiently small.

Moreover, we can estimate I_3 as in (3.7) after the use of the Poincaré inequality.

Therefore, we can obtain once again an inequality of the type of (3.8) and so, we can conclude the uniqueness result proceeding as in § 3.

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