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Research paper

Asymptotic behavior and numerical approximation of a double-suspended bridge system

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ABSTRACT

This paper is devoted to introduce and analyze a new non linear problem describing the vibrations of a double suspended bridge system. The road bed is modeled as a double beam of Woinowsky-Krieger type and the two cables, each connected to a single beam by a distributed system of elastic springs, are modeled as one-sided elastic strings. We achieve the existence and uniqueness of solutions by using the semigroup theory and the exponential decay property is also proved. Then, the model is numerically analyzed, through a variational formulation, by using the finite element method and a first-order time integration scheme. A priori error estimates are obtained and the linear convergence is derived under some suitable additional regularity conditions. Finally, some numerical experiments are performed to verify the behavior of the numerical method.

1. Introduction

In this paper we aim to study, from an analytic and numerical point of view, the damped transverse vibrations of a dynamical system which models a *double suspended bridge system* under compressive axial loads. The interest in the dynamic of suspension bridges relies on the fact that the analysis of their nonlinear vibrations is crucial for understanding the stabilization of the oscillations, especially when the coupling between the road bed and the suspension main cable is taken into account. In the literature, starting from the pioneering papers [1–3], the dynamic response of the bridge was mainly investigated with models where the road bed has been simpler considered as a vibrating one-dimensional beam (see for instance [4–17]). The main cable holding the cable stays, instead, is treated as a vibrating string and it is coupled with the road bed by different types of springs. For such a models, doubly nonlinear elastic and viscoelastic coupled systems were analyzed and the longtime behavior of solutions deeply investigated in [18–22].

Here, we consider a more realistic system, since the road bed is modeled as a sandwich structure composed of two lateral beams connected by an elastic rug [23,24]. The dynamics of each beam is coupled with a single cable, also considered as a vibrating string, and the coupling is carried out by a distributed system of vertical one-sided elastic springs (see Fig. 1).

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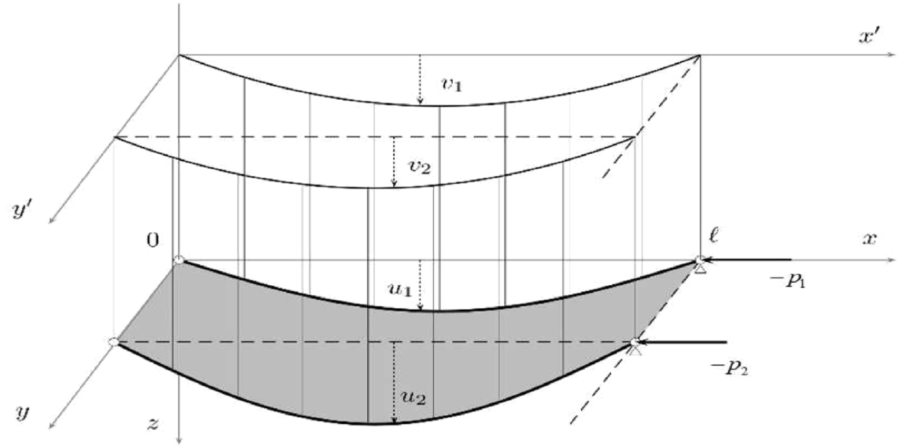

Fig. 1. The double suspended bridge model: two lateral beams connected by an elastic rug model the road bed and the coupling of each beam, with a single cable, is obtained through one-sided elastic springs.

In order to exhibit the mathematical model, we suppose that the two beams describing the mechanical structure of the road bed are equal and complied with the nonlinear model of Woinowsky-Krieger [25], so that large deformations are allowed. The beams are supposed to have the same natural length $\ell > 0$, constant mass density, and sectional dimensions which are negligible in comparison with their length (for the sake of convenience, in the following we will assume that $\ell = 1$). At their ends, they are simply supported and subjected to evenly distributed axial loads. A system of linear springs models the elastic filler connecting the beams: when the system lies in its natural configuration, the beams are straight and parallel. The distance between the beams is equal to the free lengths of the springs. Each suspended beam acts on its cable just through the suspenders (a distributed system of non-linear springs), so yielding a transversal distributed load on it. This nonlinear part connecting each beam to its string is able to pull the cable down and to hold the road bed up, but not the reverse. Hence, collecting all the previous observations, we propose the following dimensionless, doubly-nonlinear, coupled system

$$\left.\begin{aligned}
&\partial_{tt}u_1(t)+\partial_{xxxx}u_1(t)+\partial_t u_1+\big(p_1-\|\partial_x u_1\|^2_{L^2(0,1)}\big)\partial_{xx}u_1(t)\\
&\qquad+\kappa[u_1(t)-v_1(t)]^+ +\kappa^*[u_1(t)-u_2(t)]=f_1,\\
&\partial_{tt}v_1(t)-\partial_{xx}v_1(t)+\partial_t v_1-\kappa[u_1(t)-v_1(t)]^+=g_1,\\
&\partial_{tt}u_2(t)+\partial_{xxxx}u_2(t)+\partial_t u_2+\big(p_2-\|\partial_x u_2\|^2_{L^2(0,1)}\big)\partial_{xx}u_2(t)\\
&\qquad+\kappa[u_2(t)-v_2(t)]^+ -\kappa^*[u_1(t)-u_2(t)]=f_2,\\
&\partial_{tt}v_2(t)-\partial_{xx}v_2(t)+\partial_t v_2-\kappa[u_2(t)-v_2(t)]^+=g_2,
\end{aligned}\right\} \tag{1}$$

where the unknown variables $u_i : [0, 1] \times \mathbb{R}^+ \to \mathbb{R}$ $(i = 1, 2)$ represent the downward deflection (in the vertical plane) of each beam midline, $v_i : [0, 1] \times \mathbb{R}^+ \to \mathbb{R}$ $(i = 1, 2)$ the vertical displacement of each string with respect to its reference configuration (of unitary length) at rest. The sources, f_i and g_i, are the (given) vertical load distributions. In addition, the beams are connected by linear springs with common stiffness $\kappa^* > 0$, and the suspender cables (ties) are assumed to be one-sided elastic springs with common stiffness $\kappa > 0$, where w^+ stands for the positive part of w. Finally, every parameter p_i summarizes the effect of an axial force acting at one end of each beam: it is negative when both beams are stretched, positive when compressed. Both beams are hinged at their ends and both main cables have fixed ends, namely

$$\left.\begin{aligned}
&u_i(0,t)=u_i(1,t)=\partial_{xx}u_i(0,t)=\partial_{xx}u_i(1,t)=0,\quad t\in[0,\infty),\ i=1,2,\\
&v_i(0,t)=v_i(1,t)=0,\quad t\in[0,\infty),\ i=1,2.
\end{aligned}\right\} \tag{2}$$

In addition, the unknown fields u_i and v_i, $i = 1, 2$, are required to satisfy the following initial conditions:

$$\left.\begin{aligned}
&u_i(x,0)=\bar{u}_i(x),\quad \partial_t u_i(x,0)=\tilde{u}_i(x),\quad x\in[0,1],\ i=1,2,\\
&v_i(x,0)=\bar{v}_i(x),\quad \partial_t v_i(x,0)=\tilde{v}_i(x),\quad x\in[0,1],\ i=1,2,
\end{aligned}\right\} \tag{3}$$

where $\bar{u}_i$, $\bar{v}_i$, $\tilde{u}_i$ and $\tilde{v}_i$, $i = 1, 2$, are given functions which fulfill conditions (2).

In this framework, we present original results concerning the existence and uniqueness to problem (1)–(3) and its longtime behavior. To this aim, we write the original problem as an ordinary differential one through a suitable operator and we apply the theory of strongly continuous semigroups. Regarding the time asymptotic behavior, it is proved that the solution decays exponentially to zero when some restrictions on each p_i are imposed. A numerical analysis is also performed by using the finite element method to approximate the spatial variable and the implicit Euler scheme to discretize the time derivatives. An a priori error estimates result and numerical simulations are shown.

Finally, we remark that, even if the introduced mathematical model is characterized by some approximations and, hence, it does not take into account all the features of the complex behavior of an actual bridge, it exhibits some important dynamical properties and it gives the advantage of rigorous results within a precise mathematical setting. This justify the interest in the proposed mechanical problem. In addition, since a wide literature can be found for double beam systems (see, for instance, [26–32] and references therein), especially for their recent applications in nanostructures [33,34], to our knowledge this seems to be the first paper where the proposed kind of coupling is considered and mathematically analyzed.

The outline of the paper is the following. In Section 2 the abstract formulation of the problem is presented. The well-posedness result and the exponential decay are analyzed in Section 3. A numerical algorithm for the introduced mechanical problem is described and studied in Section 4, providing a main a priori error estimates result. Finally, in Section 5 some numerical results are presented to verify the behavior of the described numerical method.

2. Abstract space setting and a priori estimations

In order to apply the powerful tools of the theory of strongly continuous semigroup, it is convenient to recast the original system (1) into a space setting. To this aim, let us introduce a suitable functional framework and formulate the abstract problem. So, following the classical notation, let $(H, \langle\cdot,\cdot\rangle, \|\cdot\|)$ be a real Hilbert space, and let

$$A : \mathfrak{D}(A) \Subset H \to H$$

be a strictly positive self-adjoint operator. For $\tau \in \mathbb{R}$, we consider the family of Hilbert spaces with inner products and norms given by

$$H_\tau = \mathfrak{D}(A^{\tau/4}), \qquad \langle u, v\rangle_\tau = \langle A^{\tau/4}u, A^{\tau/4}v\rangle, \qquad \|u\|_\tau = \|A^{\tau/4}u\|.$$

The index τ will be always omitted whenever zero. For $\tau > 0$, $H_{-\tau}$ denotes the dual space of H_τ and the symbol $\langle\cdot,\cdot\rangle$ will also be used to represent the duality pairing between H_τ and its dual space $H_{-\tau}$. In addition, we have the compact embeddings $H_{\tau+1} \Subset H_\tau$, along with the generalized Poincaré inequalities (see, for instance, [35])

$$\lambda_1\|w\|_\tau^4 \le \|w\|_{\tau+1}^4, \qquad \forall w \in H_{\tau+1}, \tag{4}$$

where $\lambda_1 > 0$ is the first eigenvalue of A. Then, letting

$$\lambda = \min\{\lambda_1, \sqrt{\lambda_1}\},$$

from (4) it follows that

$$\lambda\left(\|u\|_\tau^2 + \|v\|_{\tau/2}^2\right) \le \|u\|_{\tau+2}^2 + \|v\|_{(\tau+2)/2}^2, \quad \forall u \in H_{\tau+2}, v \in H_{(\tau+2)/2}. \tag{5}$$

With this notation, we define the family of product Hilbert spaces

$$\mathcal{H} = H_{\tau+2} \times H_\tau \times H_{\tau+2} \times H_\tau \times H_{(\tau+2)/2} \times H_{\tau/2} \times H_{(\tau+2)/2} \times H_{\tau/2}, \qquad \tau \in [0, 2].$$

We are now ready to introduce the abstract Cauchy problem related to problem (1)–(3). Precisely, on $\mathcal{H}$, and for all $p_i \in \mathbb{R}$, $i = 1, 2$, we can write

$$\left.\begin{aligned}
&\partial_{tt}u_1 + Au_1 + \partial_t u_1 - (p_1 - \|u_1\|_1^2)A^{1/2}u_1 + \kappa(u_1 - v_1)^+\\
&\qquad +\kappa^*(u_1 - u_2) = f_1,\\
&\partial_{tt}v_1 + A^{1/2}v_1 + \partial_t v_1 - \kappa(u_1 - v_1)^+ = g_1,\\
&\partial_{tt}u_2 + Au_2 + \partial_t u_2 - (p_2 - \|u_2\|_1^2)A^{1/2}u_2 + \kappa(u_2 - v_2)^+\\
&\qquad -\kappa^*(u_1 - u_2) = f_2,\\
&\partial_{tt}v_2 + A^{1/2}v_2 + \partial_t v_2 - \kappa(u_2 - v_2)^+ = g_2,
\end{aligned}\right\} \tag{6}$$

with the following initial conditions

$$\begin{aligned}
&(u_1(0), \partial_t u_1(0), u_2(0), \partial_t u_2(0), v_1(0), \partial_t v_1(0), v_2(0), \partial_t v_2(0))\\
&\qquad = (\bar{u}_1, \tilde{u}_1, \bar{u}_2, \tilde{u}_2, \bar{v}_1, \tilde{v}_1, \bar{v}_2, \tilde{v}_2) \in \mathcal{H}.
\end{aligned}$$

Henceforth, a solution to the above problem will be denoted by $\sigma : \mathbb{R}^+ \to \mathcal{H}$, where

$$\sigma(t) = (u_1(t), \partial_t u_1(t), u_2(t), \partial_t u_2(t), v_1(t), \partial_t v_1(t), v_2(t), \partial_t v_2(t)),$$

and $z = (\bar{u}_1, \tilde{u}_1, \bar{u}_2, \tilde{u}_2, \bar{v}_1, \tilde{v}_1, \bar{v}_2, \tilde{v}_2) \in \mathcal{H}$ represents its initial data. Unless otherwise indicated, initial data of the problem are assumed to belong to a ball of radius R in $\mathcal{H}$, namely $\|z\|_{\mathcal{H}} \le R$.

We can also define the *total energy* of σ as

$$E(\sigma(t)) = \mathcal{E}(\sigma(t)) + \frac{1}{2}\sum_{i=1}^{2}\left(\|u_i(t)\|_1^2 - p_i\right)^2 + \kappa\sum_{i=1}^{2}\|(u_i(t) - v_i(t))^+\|^2 + \kappa^*\|u_1(t) - u_2(t)\|^2,$$

where $\mathcal{E}$ represents the *energy norm* of σ in $\mathcal{H}$, namely

$$\mathcal{E}(\sigma(t)) = \|\sigma(t)\|^2_{\mathcal{H}} = \sum_{i=1}^{2}\big(\|u_i(t)\|_2^2 + \|\partial_t u_i(t)\|^2 + \|v_i(t)\|_1^2 + \|\partial_t v_i(t)\|^2\big).$$

The total energy verifies the following relation, generally called *energy identity*:

$$\begin{aligned}&\frac{d}{dt}E(\sigma(t)) + 2\|\partial_t u_1(t)\|^2 + 2\|\partial_t u_2(t)\|^2 + 2\|\partial_t v_1(t)\|^2 + 2\|\partial_t v_2(t)\|^2\\ &\quad = 2\langle \partial_t u_1(t), f_1\rangle + 2\langle \partial_t u_2(t), f_2\rangle + 2\langle \partial_t v_1(t), g_1\rangle + 2\langle \partial_t v_2(t), g_2\rangle.\end{aligned} \tag{7}$$

It is obtained by multiplying (6)$_1$ by $\partial_t u_1$, (6)$_2$ by $\partial_t v_1$, (6)$_3$ by $\partial_t u_2$, (6)$_4$ by $\partial_t v_2$ in H, and taking into account the (formal) relation:

$$\langle (u_i(t) - v_i(t))^+, \partial_t(u_i(t) - v_i(t))\rangle = \frac{1}{2}\frac{d}{dt}\left(\|(u_i(t) - v_i(t))^+\|^2\right), \quad i = 1, 2.$$

The usefulness of formulation (6) is based on the fact that, for the particular choice $H = L^2(0, 1)$ and $A = \partial_{xxxx}$ with proper boundary conditions, the original problem (1)–(2) can be viewed as a special case of (6), where a single operator at different power is present. This justifies our choice of boundary conditions: the coupled system can be described by means of a single operator A only if the deck is assumed to be hinged at its ends. In fact, the abstract formulation (6) cannot be applied either when different boundary conditions are given (clamped–clamped and hinged–clamped ends) or if a two-dimensional model is considered. In addition, the operator $A = \partial_{xxxx}$ is strictly positive, self-adjoint, with compact inverse. Its domain is

$$\mathcal{D}(A) = \{w \in H^4(0, 1) \quad ; \quad w(0) = w(1) = \partial_{xx}w(0) = \partial_{xx}w(1) = 0\},$$

and its discrete spectrum is given by $\lambda_n = n^4\pi^4$, $n \in \mathbb{N}$.

We note that the boundedness of the energy norm can be obtained, which is summarized in the following.

Lemma 1. *Let $f_i \in H_{-2}$, $g_i \in H_{-1}$ and $p_i \in \mathbb{R}$, $i = 1, 2$. For all $t > 0$ and initial data $z \in \mathcal{H}$ with $\|z\|_{\mathcal{H}} \le R$, we have*

$$\mathcal{E}(\sigma(t)) \le Q(R),$$

where $Q : \mathbb{R}_0^+ \to \mathbb{R}^+$ denotes a generic increasing monotone function of radius R.

Proof. We introduce the functional $Ł : \mathcal{H} \to \mathbb{R}$ given by

$$Ł(\sigma) = E(\sigma) - 2\langle u_1, f_1\rangle - 2\langle v_1, g_1\rangle - 2\langle u_2, f_2\rangle - 2\langle v_2, g_2\rangle . \tag{8}$$

Along with any solution $\sigma(t)$ to (6), the time function $Ł(\sigma(t))$ is non increasing. Actually, from the energy identity (7) it follows that

$$\frac{d}{dt}Ł(\sigma(t)) = -2\|\partial_t u_1(t)\|^2 - 2\|\partial_t u_2(t)\|^2 - 2\|\partial_t v_1(t)\|^2 - 2\|\partial_t v_2(t)\|^2 \le 0, \tag{9}$$

which ensures that

$$Ł(\sigma(t)) \le Ł(z) \le Q(R),$$

for all $z \in \mathcal{H}$ with $\|z\|_{\mathcal{H}} \le R$. Note that the function $Q(R)$ explicitly depends only on R, but implicitly it also depends on the structural constants of the problem, and its expression may change even within the same line of a given equation.

Since $E \ge \mathcal{E}$, from (8) we obtain the estimate

$$\begin{aligned}Ł &\ge \mathcal{E} - 2\langle u_1, f_1\rangle - 2\langle u_2, f_2\rangle - 2\langle v_1, g_1\rangle - 2\langle v_2, g_2\rangle\\ &\ge \frac{1}{2}\mathcal{E} - 2(\|f_1\|_{-2}^2 + \|f_2\|_{-2}^2) - 2(\|g_1\|_{-1}^2 + \|g_2\|_{-1}^2),\end{aligned}$$

which finally gets

$$\mathcal{E}(\sigma(t)) \le 2Ł(\sigma(t)) + 4(\|f_1\|_{-2}^2 + \|f_2\|_{-2}^2 + \|g_1\|_{-1}^2 + \|g_2\|_{-1}^2) \le Q(R).$$

We conclude the section by reminding, for further convenience, the following result regarding the linear part of the differential operator acting on u_i. This result is a slight generalization of Lemma 4.5 in [36].

Lemma 2. *Let $p_i < \sqrt{\lambda_1}$ and $\mathcal{F}_{p_i}(u_i) = Au_i - p_iA^{1/2}u_i$, $i = 1, 2$. Then, we find that*

$$\left\langle \mathcal{F}_{p_i}(u_i)\,,\ u_i\right\rangle \ge C(p_i)\,\|u_i\|_2^2\,, \tag{10}$$

where $i = 1, 2$ and

$$C(p_i) = \begin{cases} 1 & \text{if } \ p_i \le 0\,, \\ \left(1 - \dfrac{p_i}{\sqrt{\lambda_1}}\right) & \text{if } \ 0 < p_i < \sqrt{\lambda_1}. \end{cases} \tag{11}$$

3. Well posedness and exponential stability

This section provides the main analytical results of the paper. They are related to the well posedness of the system and to its exponential stability provided that the parameters p_i are smaller than a critical value. Precisely, the proof of exponential stability is firstly based on the general result of Lemma 2 that justifies the restrictive assumptions on $p_i, \quad i = 1, 2$.

Proposition 1. *Let $f_i \in H_{-2}$, $g_i \in H_{-1}$ and $p_i \in \mathbb{R}$, $i = 1, 2$. For all initial data $z \in \mathcal{H}$, the abstract Cauchy problem (6) admits a unique solution*

$$\sigma = (u_1, \partial_t u_1, u_2, \partial_t u_2, v_1, \partial_t v_1, v_2, \partial_t v_2) \in \mathcal{C}([0, T]; \mathcal{H}),$$

which depends continuously on the initial data.

Proof. We omit the proof of the existence result, which is standard. In particular, one can apply a standard Faedo–Galerkin approximation procedure (see, for instance, [13,15,37,38]), together with a slight generalization of the usual Gronwall lemma. Indeed, the uniform-in-time estimates needed to obtain the global existence are exactly the same as in the previous Lemma 1. On the contrary, the uniqueness part deserves a detailed discussion.

Let us assume that there exist two weak solutions:

$$\sigma^1 = (u_1^1, \partial_t u_1^1, u_2^1, \partial_t u_2^1, v_1^1, \partial_t v_1^1, v_2^1, \partial_t v_2^1),$$
$$\sigma^2 = (u_1^2, \partial_t u_1^2, u_2^2, \partial_t u_2^2, v_1^2, \partial_t v_1^2, v_2^2, \partial_t v_2^2),$$

which both solve abstract problem (6) on the time interval $(0, T)$ with the same initial data z. This in turn implies that each solution fulfills Lemma 1, namely

$$\mathcal{E}(\sigma^i(t)) \le Q(R), \qquad \|z\|_{\mathcal{H}} \le R, \ \ i = 1, 2.$$

Let ω denote their difference

$$\omega = \sigma^1 - \sigma^2 = (\theta_1, \partial_t\theta_1, \theta_2, \partial_t\theta_2, \xi_1, \partial_t\xi_1, \xi_2, \partial_t\xi_2)$$

with

$$\theta_1 = u_1^1 - u_1^2, \qquad \theta_2 = u_2^1 - u_2^2, \qquad \xi_1 = v_1^1 - v_1^2, \qquad \xi_2 = v_2^1 - v_2^2.$$

Hence, ω takes a vanishing initial condition and solves the following homogeneous abstract problem:

$$\begin{aligned}
&\partial_{tt}\theta_1 + A\theta_1 + \partial_t\theta_1 - p_1 A^{1/2}\theta_1 + \kappa^*(\theta_1 - \theta_2) + N(u_1^1) - N(u_1^2)\\
&\qquad + F(u_1^1 - v_1^1) - F(u_1^2 - v_1^2) = 0,\\
&\partial_{tt}\xi_1 + A^{1/2}\xi_1 + \partial_t\xi_1 - F(u_1^1 - v_1^1) + F(u_1^2 - v_1^2) = 0,\\
&\partial_{tt}\theta_2 + A\theta_2 + \partial_t\theta_2 - p_2 A^{1/2}\theta_2 - \kappa^*(\theta_1 - \theta_2) + N(u_2^1) - N(u_2^2)\\
&\qquad + F(u_2^1 - v_2^1) - F(u_2^2 - v_2^2) = 0,\\
&\partial_{tt}\xi_2 + A^{1/2}\xi_2 + \partial_t\xi_2 - F(u_2^1 - v_2^1) + F(u_2^2 - v_2^2) = 0,
\end{aligned} \tag{12}$$

where $N(u) = \|u\|_1^2 A^{1/2}u$ and $F(w) = \kappa w^+$.

Proceeding as in [20] and taking advantage of Lemma 1, we obtain the following estimates:

$$\begin{aligned}
&\left|\left\langle N(u_1^1) - N(u_1^2),\ \partial_t\theta_1\right\rangle\right| \le Q(R)\,\mathcal{E}(\omega),\\
&\left|\left\langle N(u_2^1) - N(u_2^2),\ \partial_t\theta_2\right\rangle\right| \le Q(R)\,\mathcal{E}(\omega),\\
&\left|\left\langle F(u_1^1 - v_1^1) - F(u_1^2 - v_1^2), \partial_t\theta_1 - \partial_t\xi_1\right\rangle\right|\\
&\qquad \le c\,(\|\theta_1\| + \|\xi_1\|)\,(\|\partial_t\theta_1\| + \|\partial_t\xi_1\|) \le c\,\mathcal{E}(\omega),\\
&\left|\left\langle F(u_2^1 - v_2^1) - F(u_2^2 - v_2^2), \partial_t\theta_2 - \partial_t\xi_2\right\rangle\right|\\
&\qquad \le c\,(\|\theta_2\| + \|\xi_2\|)\,(\|\partial_t\theta_2\| + \|\partial_t\xi_2\|) \le c\,\mathcal{E}(\omega),\\
&|\langle \theta_1 - \theta_2, \partial_t\theta_1 - \partial_t\theta_2\rangle| \le c\,\mathcal{E}(\omega),
\end{aligned}$$

where c denotes a *generic* positive constant, which possibly (but implicitly) depends on the structural constants of the problem, and whose value may change even within the same line of a given equation.

After multiplying $(12)_1$ by $\partial_t\theta_1$, $(12)_2$ by ξ_1, $(12)_3$ by $\partial_t\theta_2$ and $(12)_4$ by ξ_2 in H, then adding the resulting equations and taking into account the previous estimates, we obtain

$$\frac{d}{dt}\Big[\mathcal{E}(\omega(t)) - \sum_{i=1}^{2} p_i\,\|\theta_i(t)\|_1^2\Big] \le Q(R)\mathcal{E}(\omega(t)). \tag{13}$$

Then, adding and subtracting the terms $\alpha_1 \langle\theta_1, \partial_t\theta_1\rangle$ and $\alpha_2 \langle\theta_2, \partial_t\theta_2\rangle$, with $\alpha_1, \alpha_2 > 0$, we have

$$\frac{d}{dt}\mathcal{G}(\omega(t)) \leq Q(R)\mathcal{E}(\omega(t)), \tag{14}$$

where

$$\mathcal{G}(\omega) = \mathcal{E}(\omega) - \sum_{i=1}^{2}\big(p_i \|\theta_i\|_1^2 - \alpha_i \|\theta_i\|^2\big).$$

By virtue of the interpolation inequality, we have $\|\theta_i\|_1^2 \leq \|\theta_i\|\|\theta_i\|_2$ for $i = 1, 2$, and we can infer that

$$\frac{1}{2}\|\theta_i\|_2^2 \leq \|\theta_i\|_2^2 - p_i\|\theta_i\|_1^2 + \alpha_i\|\theta_i\|^2 \leq b_i\|\theta_i\|_2^2, \quad i = 1, 2,$$

holds for any $p_i \in \mathbb{R}$ provided that α_i is large enough and $b_i = b_i(p_i, \alpha_i) > 1$. Accordingly, we find that

$$\frac{1}{2}\,\mathcal{E}(\omega) \leq \mathcal{G}(\omega) \leq b\,\mathcal{E}(\omega), \tag{15}$$

where $b = \max\{b_1, b_2\}$. Then, from (14) it follows that

$$\frac{d}{dt}\mathcal{G}(\omega(t)) \leq Q(R)\mathcal{G}(\omega(t)).$$

Now, since $\omega(0) = (0, 0, 0, 0, 0, 0, 0, 0)$ and $\mathcal{G}(\omega(0)) = 0$, an application of the Gronwall lemma leads to $\mathcal{G}(\omega(t)) = 0$ for all $t > 0$. By virtue of inequality (15), $\mathcal{E}(\omega(t)) = \|\omega(t)\|_{\mathcal{H}}^2 = 0$, so that $\sigma^1(t) = \sigma^2(t)$ and the uniqueness follows.

The same strategy leads to the continuous dependence of the solution with respect to the initial data in $\mathcal{H}$. Indeed, if σ^1 and σ^2 are two solutions corresponding to initial data z_1 and z_2, respectively, then estimate (13) holds with $\omega(0) = z_1 - z_2$ and

$$\left\|\sigma^1(t) - \sigma^2(t)\right\|_{\mathcal{H}}^2 \leq e^{Q(R)T}\ \|z_1 - z_2\|_{\mathcal{H}}^2, \qquad \forall t \in (0, T),$$

which implies the continuous dependence on the initial data.

According to Proposition 1, abstract system (6) generates a strongly continuous semigroup (or *dynamical system*) $S(t)$ on $\mathcal{H}$. That is to say, for a given initial data $z \in \mathcal{H}$, $\sigma(t) = S(t)z$ and $\mathcal{E}(t) = \|S(t)z\|_{\mathcal{H}}^2$ are the unique weak solution to (6) and its related energy norm, respectively.

Now, we consider the abstract Cauchy problem (6) under the restrictive assumptions $f_i = g_i = 0$ $(i = 1, 2)$, and let $S_0(t)$ be the semigroup generated under vanishing external forces. In the following, we will prove that $S_0(t)$ decays exponentially.

Theorem 3. *Let $z \in \mathcal{H}$ such that $\|z\|_{\mathcal{H}} \leq R$. Provided that $p_i < \sqrt{\lambda_1}$, $i = 1, 2$, all solutions $S_0(t)z$ decay exponentially, i.e.*

$$\mathcal{E}(t) \ \leq\ Q(R)\,e^{-\varepsilon_0 t},$$

where ε_0 is a suitable positive constant.

Proof. Let us introduce the functional

$$\Phi = E + \varepsilon\Big[\sum_{i=1}^{2}\big(\langle u_i, \partial_t u_i\rangle + \langle v_i, \partial_t v_i\rangle\big)\Big] - \frac{1}{2}\sum_{i=1}^{2}(p_i)^2.$$

Taking into account (11), the constant

$$\varepsilon = \min\{\lambda C(p_1), \lambda C(p_2), \frac{2\lambda}{1+\lambda}, 2\lambda, 1\}, \tag{16}$$

is positive provided that $p_i < \sqrt{\lambda_1}$, $i = 1, 2$.

After recalling the definition of E, we remark that

$$\begin{aligned}\Phi =\ & \sum_{i=1}^{2}\langle\mathcal{F}_{p_i}(u_i), u_i\rangle + \sum_{i=1}^{2}\big(\|\partial_t u_i\|^2 + \|\partial_t v_i\|^2\big) + \frac{1}{2}\sum_{i=1}^{2}\|u_i\|_1^4 + \sum_{i=1}^{2}\|v_i\|_1^2 \\ & + \kappa\sum_{i=1}^{2}\left\|(u_i - v_i)^+\right\|^2 + \varepsilon\Big[\sum_{i=1}^{2}\big(\langle u_i, \partial_t u_i\rangle + \langle v_i, \partial_t v_i\rangle\big)\Big].\end{aligned}$$

Now, by virtue of (5), (10) and (16), a lower bound is obtained as follows:

$$\begin{aligned}\Phi &\geq \left(1-\frac{\varepsilon}{2}\right)\sum_{i=1}^{2}\left(\|\partial_t u_i\|^2+\|\partial_t v_i\|^2\right)+\sum_{i=1}^{2}\left(C(p_i)-\frac{\varepsilon}{2\lambda}\right)\|u_i\|_2^2\\ &\quad+\left(1-\frac{\varepsilon}{2\lambda}\right)\sum_{i=1}^{2}\|v_i\|_1^2\geq\frac{\varepsilon}{2\lambda}\,\mathcal{E}.\end{aligned}\tag{17}$$

On the other hand, by applying Young inequality, using (5) and applying Lemma 1 restricted to $S_0(t)$, we can write the upper bound of Φ as

$$\Phi\leq\left[1+\frac{(\kappa+\kappa^*+|p|)}{\lambda}+\frac{1}{2\lambda}+\frac{\varepsilon^2}{2}+\frac{Q(R)}{2\lambda^2}\right]\mathcal{E}=Q(R)\,\mathcal{E},\tag{18}$$

where $|p|=\max\{|p_1|,|p_2|\}$.

Estimates (17)–(18) prove the equivalence between $\mathcal{E}$ and Φ, that is

$$\frac{\varepsilon}{2\lambda}\,\mathcal{E}\leq\Phi\leq Q(R)\,\mathcal{E}.\tag{19}$$

The last step is to show the exponential decay of Φ. To this aim, we obtain the identity:

$$\begin{aligned}&\frac{d}{dt}\,\Phi+\varepsilon\Phi+2\,(1-\varepsilon)\sum_{i=1}^{2}\left(\|\partial_t u_i\|^2+\|\partial_t v_i\|^2\right)+\frac{\varepsilon}{2}\sum_{i=1}^{2}\|u_i\|_1^4\\ &\qquad+\varepsilon\,(1-\varepsilon)\left[\sum_{i=1}^{2}\left(\langle\partial_t u_i\,,\,u_i\rangle+\langle\partial_t v_i\,,\,v_i\rangle\right)\right]=0,\end{aligned}$$

where ε is given by (16). Exploiting the Young inequality, (4) and (19), we have

$$\begin{aligned}\frac{d}{dt}\,\Phi+\varepsilon\Phi+(1-\varepsilon)\sum_{i=1}^{2}\left(\|\partial_t u_i\|^2+\|\partial_t v_i\|^2\right)&\leq\frac{\varepsilon^2\,(1-\varepsilon)}{4\lambda}\sum_{i=1}^{2}\left(\|u_i\|_2^2+\|v_i\|_1^2\right)\\ &\leq\frac{\varepsilon\,(1-\varepsilon)}{2}\,\Phi,\end{aligned}$$

from which it follows that

$$\frac{d}{dt}\Phi+\frac{\varepsilon(1+\varepsilon)}{2}\Phi\leq 0.$$

Letting $\varepsilon_0=\varepsilon\,(1+\varepsilon)/2$, we find that

$$\frac{\varepsilon}{2\,\lambda}\,\mathcal{E}(t)\leq\Phi\,(t)\leq\Phi\,(0)\;e^{-\varepsilon_0\,t}\leq Q(R)\,e^{-\varepsilon_0\,t},$$

which concludes the proof.

4. Fully discrete approximation and an a priori error analysis

In this section, we will study the numerical approximation of system (1) with boundary conditions (2) and initial conditions (3). However, in order to obtain the variational formulation of this problem, we must replace boundary conditions (2) by a convenient modification as follows:

$$\begin{aligned}&u_i(0,t)=u_i(1,t)=\partial_x u_i(0,t)=\partial_x u_i(1,t)=0,\quad t\in[0,T],\ i=1,2,\\ &v_i(0,t)=v_i(1,t)=0,\quad t\in[0,T],\ i=1,2,\end{aligned}\tag{20}$$

where, from now on, we denote by $[0,T]$, $T>0$, the time interval of interest.

Hence, we can now provide a weak formulation and so, we define the variational spaces $Y=L^2(0,1)$, $E=H_0^1(0,1)$ and $V=H_0^2(0,1)$.

Multiplying the equations of system (1) by adequate test functions, using the new boundary conditions (20) and applying integration by parts, we obtain the following variational formulation of our problem in terms of the downward velocities $e_i=\partial_t u_i$ and the vertical velocities $c_i=\partial_t v_i$ for $i=1,2$.

Find the downward velocities $e_1:[0,T]\to V$ and $e_2:[0,T]\to V$, and the vertical velocities $c_1:[0,T]\to E$ and $c_2:[0,T]\to E$ such that $e_1(0)=\tilde{u}_1$, $e_2(0)=\tilde{u}_2$, $c_1(0)=\tilde{v}_1$ and $c_2(0)=\tilde{v}_2$, and for a.e. $t\in[0,T]$ and $w,s\in V$, $r,z\in E$,

$$\begin{aligned}&(\partial_t e_1(t),w)+(\partial_{xx}u_1(t),\partial_{xx}w)+(e_1(t),w)+p_1(\partial_{xx}u_1(t),w)\\ &\quad+\|\partial_x u_1(t)\|^2(\partial_x u_1(t),\partial_x w)+\kappa((u_1(t)-u_2(t))^+,w)\\ &\quad+\kappa^*(u_1(t)-u_2(t),w)=(f_1(t),w),\end{aligned}\tag{21}$$

$$(\partial_t c_1(t), r) + (\partial_x v_1(t), \partial_x r) + (c_1(t), r) - \kappa((u_1(t) - v_1(t))^+, r) = (g_1(t), r), \tag{22}$$

$$\begin{aligned}&(\partial_t e_2(t), s) + (\partial_{xx} u_2(t), \partial_{xx} s) + (e_2(t), s) + p_2(\partial_{xx} u_2(t), s)\\&\quad + \|\partial_x u_2(t)\|^2 (\partial_x u_2(t), \partial_x s) + \kappa((u_2(t) - v_2(t))^+, s)\\&\quad + \kappa^*(u_2(t) - u_1(t), s) = (f_2(t), s),\end{aligned} \tag{23}$$

$$(\partial_t c_2(t), z) + (\partial_x v_2(t), \partial_x z) + (c_2(t), z) + \kappa((v_2(t) - u_2(t))^+, z) = (g_2(t), z), \tag{24}$$

where the downward deflections and the vertical displacements are obtained from the following relations:

$$u_1(t) = \int_0^t e_1(s)\, ds + \bar{u}_1, \quad u_2(t) = \int_0^t e_2(s)\, ds + \bar{u}_2, \tag{25}$$

$$v_1(t) = \int_0^t c_1(s)\, ds + \bar{v}_1, \quad v_2(t) = \int_0^t c_2(s)\, ds + \bar{v}_2. \tag{26}$$

Now, we will consider the numerical approximation of the above weak problem. As usual, we will do it in two steps. First, we define a spatial approximation and so, let us construct the finite dimensional spaces $E^h \subset E$ and $V^h \subset V$ as follows:

$$E^h = \{e^h \in \mathcal{C}([0,\ell]) \cap E \ ;\ e^h_{|[a_i,a_{i+1}]} \in P_1([a_i, a_{i+1}]) \quad \text{for } i = 0, \dots, N-1\}, \tag{27}$$

$$V^h = \{v^h \in \mathcal{C}^1([0,\ell]) \cap V \ ;\ v^h_{|[a_i,a_{i+1}]} \in P_3([a_i, a_{i+1}]) \quad \text{for } i = 0, \dots, N-1\}, \tag{28}$$

where we have used the uniform partition of the interval $[0, 1]$ divided into M subintervals denoted by $a_0 = 0 < a_1 < \cdots < a_M = 1$ with a uniform length $h = a_{i+1} - a_i = 1/M$. Here, $P_r([a_i, a_{i+1}])$ $(r = 1, 3)$ is the space of polynomials of degree less or equal to r for each subinterval $[a_i, a_{i+1}]$; that is, the finite element space E^h is composed of continuous and piecewise affine functions and the finite element space V^h is made of C^1 and piecewise cubic functions. Moreover, as usual, $h > 0$ denotes the spatial discretization parameter. Then, by using the finite element projection operators over E^h and V^h (see, for instance, the work of Clément [39]), we can define an approximation of the initial conditions given as

$$\begin{aligned}&u_1^{0h} = \mathcal{P}_1^h \bar{u}_1, \quad u_2^{0h} = \mathcal{P}_1^h \bar{u}_2, \quad e_1^{0h} = \mathcal{P}_1^h \tilde{u}_1, \quad e_2^{0h} = \mathcal{P}_1^h \tilde{u}_2,\\&v_1^{0h} = \mathcal{P}_2^h \bar{v}_1, \quad v_2^{0h} = \mathcal{P}_1^h \bar{v}_2, \quad c_1^{0h} = \mathcal{P}_2^h \tilde{v}_1, \quad c_2^{0h} = \mathcal{P}_2^h \tilde{v}_2,\end{aligned} \tag{29}$$

where $\mathcal{P}_1^h$ and $\mathcal{P}_2^h$ denote the projection operator over the finite element spaces V^h and E^h, respectively.

Secondly, we obtain the discretization of the time derivatives and so, we use a uniform partition of the time interval $[0, T]$, denoted by $0 = t_0 < t_1 < \cdots < t_N = T$, with a time step size $k = T/N$ and nodes $t_n = n\,k$ for $n = 0, 1, \dots, N$. For a continuous function $f(t)$ let $f^n = f(t_n)$ and, for a sequence $\{w^n\}_{n=0}^N$, let us denote $\delta w^n = (w^n - w^{n-1})/k$ its divided differences.

Therefore, by using the well-known implicit Euler scheme we derive the following fully discrete approximation of problem (21)–(26).

Find the discrete downward velocities $\{e_1^{hk,n}\}_{n=0}^N \subset V^h$ and $\{e_2^{hk,n}\}_{n=0}^N \subset V^h$, and the discrete vertical velocities $\{c_1^{hk,n}\}_{n=0}^N \subset E^h$ and $\{c_1^{hk,n}\}_{n=0}^N \subset E^h$ such that $e_1^{hk,0} = e_1^{0h}$, $e_2^{hk,0} = e_2^{0h}$, $c_1^{hk,0} = c_1^{0h}$ and $c_2^{hk,0} = c_2^{0h}$, and for all $n = 1, \dots, N$ and $w^h, s^h \in V^h$, $r^h, z^h \in E^h$,

$$\begin{aligned}&(\delta e_1^{hk,n}, w^h) + (\partial_{xx} u_1^{hk,n}, \partial_{xx} w^h) + (e_1^{hk,n}, w^h) + p_1(\partial_{xx} u_1^{hk,n}, w^h)\\&\quad + \|\partial_x u_1^{hk,n}\|^2 (\partial_x u_1^{hk,n}, \partial_x w^h) + \kappa((u_1^{hk,n} - u_2^{hk,n})^+, w^h)\\&\quad + \kappa^*(u_1^{hk,n} - u_2^{hk,n}, w^h) = (f_1^n, w^h),\end{aligned} \tag{30}$$

$$\begin{aligned}&(\delta c_1^{hk,n}, r^h) + (\partial_x v_1^{hk,n}, \partial_x r^h) + (c_1^{hk,n}, r^h) - \kappa((u_1^{hk,n} - v_1^{hk,n})^+, r^h)\\&\quad = (g_1^n, r^h),\end{aligned} \tag{31}$$

$$\begin{aligned}&(\delta e_2^{hk,n}, s^h) + (\partial_{xx} u_2^{hk,n}, \partial_{xx} s^h) + (e_2^{hk,n}, s^h) + p_2(\partial_{xx} u_2^{hk,n}, s^h)\\&\quad + \|\partial_x u_2^{hk,n}\|^2 (\partial_x u_2^{hk,n}, \partial_x s^h) + \kappa((u_2^{hk,n} - v_2^{hk,n})^+, s^h)\\&\quad + \kappa^*(u_2^{hk,n} - u_1^{hk,n}, s^h) = (f_2^n, s^h),\end{aligned} \tag{32}$$

$$\begin{aligned}&(\delta c_2^{hk,n}, z^h) + (\partial_x v_2^{hk,n}, \partial_x z^h) + (c_2^{hk,n}, z^h) + \kappa((v_2^{hk,n} - u_2^{hk,n})^+, z^h)\\&\quad = (g_2^n, z^h),\end{aligned} \tag{33}$$

where the discrete downward deflections and the discrete vertical displacements are obtained from the following relations:

$$u_1^{hk,n} = k \sum_{j=1}^n e_1^{hk,j} + u_1^{0h}, \quad u_2^{hk,n} = k \sum_{j=1}^n e_2^{hk,j} + u_2^{0h}, \tag{34}$$

$$v_1^{hk,n} = k \sum_{j=1}^n c_1^{hk,j} + v_1^{0h}, \quad v_2^{hk,n} = k \sum_{j=1}^n c_2^{hk,j} + v_2^{0h}. \tag{35}$$

From the assumptions required on the constitutive coefficients, proceeding as in the proof of the existence and uniqueness result we can obtain that this fully discrete problem has a unique solution.

In the rest of this section, we will derive the numerical analysis of this problem.

We recall an inequality which will be very useful to prove the results provided in this section (see [40] for details regarding its proof).

Lemma 4. *For each pair of functions $r, s \in H^1(0,1)$ we find that*

$$\|r_x\|^2(r_x, s_x - r_x) \leq -\frac{1}{4}\|r_x\|^4 + \frac{1}{4}\|s_x\|^4. \tag{36}$$

As a first result, we will prove a discrete stability property.

Lemma 5. *Under the conditions required in Section 1, we obtain that the sequences $\{u_1^{hk,n}, u_2^{hk,n}, v_1^{hk,n}, v_2^{hk,n}, e_1^{hk,n}, e_2^{hk,n}, c_1^{hk,n}, c_2^{hk,n}\}_{n=0}^N$, generated by discrete problem VP^{hk}, satisfy the stability estimate:*

$$\begin{aligned}&\|e_1^{hk,n}\|^2 + \|\partial_{xx}u_1^{hk,n}\|^2 + \|u_1^{hk,n}\|^2 + \|u_2^{hk,n}\|^2 + \|e_2^{hk,n}\|^2 + \|\partial_{xx}u_2^{hk,n}\|^2 + \|c_1^{hk,n}\|^2\\ &\quad + \|\partial_x u_1^{hk,n}\|^4 + \|\partial_x u_2^{hk,n}\|^4 + \|\partial_x v_1^{hk,n}\|^4 + \|c_2^{hk,n}\|^2 + \|\partial_x v_2^{hk,n}\|^4 \leq C,\end{aligned}$$

where C is a positive constant which is independent of the discretization parameters h and k.

Proof. Taking $w^h = e_1^{hk,n}$ as a test function in (30) we have

$$\begin{aligned}&(\delta e_1^{hk,n}, e_1^{hk,n}) + (\partial_{xx}u_1^{hk,n}, \partial_{xx}e_1^{hk,n}) + p_1(\partial_{xx}u_1^{hk,n}, e_1^{hk,n}) + (e_1^{hk,n}, e_1^{hk,n})\\ &\quad + \|\partial_x u_1^{hk,n}\|^2(\partial_x u_1^{hk,n}, \partial_x e_1^{hk,n}) + \kappa((u_1^{hk,n} - u_2^{hk,n})^+, e_1^{hk,n})\\ &\quad + \kappa^*(u_1^{hk,n} - u_2^{hk,n}, e_1^{hk,n}) = (f_1^n, e_1^{hk,n}).\end{aligned}$$

Now, keeping in mind that

$$\begin{aligned}&(\delta e_1^{hk,n}, e_1^{hk,n}) \geq \frac{1}{2k}\left\{\|e_1^{hk,n}\|^2 - \|e_1^{hk,n-1}\|^2\right\},\\ &(\partial_{xx}u_1^{hk,n}, \partial_{xx}e_1^{hk,n}) \geq \frac{1}{2k}\left\{\|\partial_{xx}u_1^{hk,n}\|^2 - \|\partial_{xx}u_1^{hk,n-1}\|^2\right\},\\ &(u_1^{hk,n}, e_1^{hk,n}) \geq \frac{1}{2k}\left\{\|u_1^{hk,n}\|^2 - \|u_1^{hk,n-1}\|^2\right\},\\ &\|\partial_x u_1^{hk,n}\|^2(\partial_x u_1^{hk,n}, \partial_x e_1^{hk,n}) = \frac{1}{k}\|\partial_x u_1^{hk,n}\|^2(\partial_x u_1^{hk,n}, \partial_x u_1^{hk,n} - \partial_x u_1^{hk,n-1})\\ &\qquad \geq \frac{1}{4k}\left\{\|\partial_x u_1^{hk,n}\|^4 - \|\partial_x u_1^{hk,n-1}\|^4\right\},\end{aligned}$$

where we have used inequality (36), applying the Cauchy–Schwarz and Cauchy's inequalities we find that

$$\begin{aligned}&\frac{1}{2k}\left\{\|e_1^{hk,n}\|^2 - \|e_1^{hk,n-1}\|^2\right\} + \frac{1}{2k}\left\{\|\partial_{xx}u_1^{hk,n}\|^2 - \|\partial_{xx}u_1^{hk,n-1}\|^2\right\}\\ &\quad + \frac{1}{2k}\left\{\|u_1^{hk,n}\|^2 - \|u_1^{hk,n-1}\|^2\right\} + \frac{1}{4k}\left\{\|\partial_x u_1^{hk,n}\|^4 - \|\partial_x u_1^{hk,n-1}\|^4\right\}\\ &\quad \leq C\left(1 + \|e_1^{hk,n}\|^2 + \|\partial_{xx}u_1^{hk,n}\|^2 + \|u_1^{hk,n}\|^2 + \|u_2^{hk,n}\|^2\right).\end{aligned}$$

Proceeding in a similar form, we obtain the estimates for the second downward velocity:

$$\begin{aligned}&\frac{1}{2k}\left\{\|e_2^{hk,n}\|^2 - \|e_2^{hk,n-1}\|^2\right\} + \frac{1}{2k}\left\{\|\partial_{xx}u_2^{hk,n}\|^2 - \|\partial_{xx}u_2^{hk,n-1}\|^2\right\}\\ &\quad + \frac{1}{2k}\left\{\|u_2^{hk,n}\|^2 - \|u_2^{hk,n-1}\|^2\right\} + \frac{1}{4k}\left\{\|\partial_x u_2^{hk,n}\|^4 - \|\partial_x u_2^{hk,n-1}\|^4\right\}\\ &\quad \leq C\left(1 + \|e_2^{hk,n}\|^2 + \|\partial_{xx}u_2^{hk,n}\|^2 + \|u_1^{hk,n}\|^2 + \|u_2^{hk,n}\|^2\right).\end{aligned}$$

Now, we derive the estimates for the first vertical velocity. Taking $r^h = c_1^{hk,n}$ as a test function in (31) it follows that

$$(\delta c_1^{hk,n}, c_1^{hk,n}) + (\partial_x v_1^{hk,n}, \partial_x c_1^{hk,n}) + (c_1^{hk,n}, c_1^{hk,n}) - \kappa((u_1^{hk,n} - v_1^{hk,n})^+, c_1^{hk,n}) = (g_{1n}, c_1^{hk,n}),$$

and so, keeping in mind that

$$\begin{aligned}&(\delta c_1^{hk,n}, c_1^{hk,n}) \geq \frac{1}{2k}\left\{\|v_1^{hk,n}\|^2 - \|v_1^{hk,n-1}\|^2\right\},\\ &(\partial_x v_1^{hk,n}, \partial_x c_1^{hk,n}) \geq \frac{1}{2k}\left\{\|\partial_x v_1^{hk,n}\|^2 - \|\partial_x v_1^{hk,n-1}\|^2\right\},\end{aligned}$$

we have

$$\frac{1}{2k}\left\{\|c_1^{hk,n}\|^2-\|c_1^{hk,n-1}\|^2\right\}+\frac{1}{2k}\left\{\|\partial_x v_1^{hk,n}\|^2-\|\partial_x v_1^{hk,n-1}\|^2\right\}$$
$$\leq C\left(1+\|c_1^{hk,n}\|^2+\|u_1^{hk,n}\|^2+\|v_1^{hk,n}\|^2\right).$$

Proceeding in an analogous way, it leads to the estimates for the second vertical velocity:

$$\frac{1}{2k}\left\{\|c_2^{hk,n}\|^2-\|c_2^{hk,n-1}\|^2\right\}+\frac{1}{2k}\left\{\|\partial_x v_2^{hk,n}\|^2-\|\partial_x v_2^{hk,n-1}\|^2\right\}$$
$$\leq C\left(1+\|c_2^{hk,n}\|^2+\|u_2^{hk,n}\|^2+\|v_2^{hk,n}\|^2\right).$$

Combining all these estimates, we find that

$$\begin{aligned}
&\frac{1}{2k}\left\{\|e_1^{hk,n}\|^2-\|e_1^{hk,n-1}\|^2\right\}+\frac{1}{2k}\left\{\|\partial_{xx}u_1^{hk,n}\|^2-\|\partial_{xx}u_1^{hk,n-1}\|^2\right\}\\
&\quad+\frac{1}{2k}\left\{\|u_1^{hk,n}\|^2-\|u_1^{hk,n-1}\|^2\right\}+\frac{1}{2k}\left\{\|u_2^{hk,n}\|^2-\|u_2^{hk,n-1}\|^2\right\}\\
&\quad+\frac{1}{2k}\left\{\|e_2^{hk,n}\|^2-\|e_2^{hk,n-1}\|^2\right\}+\frac{1}{2k}\left\{\|\partial_{xx}u_2^{hk,n}\|^2-\|\partial_{xx}u_2^{hk,n-1}\|^2\right\}\\
&\quad+\frac{1}{2k}\left\{\|c_1^{hk,n}\|^2-\|c_1^{hk,n-1}\|^2\right\}+\frac{1}{2k}\left\{\|\partial_x v_1^{hk,n}\|^2-\|\partial_x v_1^{hk,n-1}\|^2\right\}\\
&\quad+\frac{1}{2k}\left\{\|c_2^{hk,n}\|^2-\|c_2^{hk,n-1}\|^2\right\}+\frac{1}{2k}\left\{\|\partial_x v_2^{hk,n}\|^2-\|\partial_x v_2^{hk,n-1}\|^2\right\}\\
&\quad+\frac{1}{4k}\left\{\|\partial_x u_1^{hk,n}\|^4-\|\partial_x u_1^{hk,n-1}\|^4\right\}+\frac{1}{4k}\left\{\|\partial_x u_2^{hk,n}\|^4-\|\partial_x u_2^{hk,n-1}\|^4\right\}\\
&\leq C\left(1+\|e_1^{hk,n}\|^2+\|\partial_{xx}u_1^{hk,n}\|^2+\|u_1^{hk,n}\|^2+\|u_2^{hk,n}\|^2+\|e_2^{hk,n}\|^2+\|\partial_{xx}u_2^{hk,n}\|^2\right.\\
&\quad\left.+\|c_1^{hk,n}\|^2+\|v_1^{hk,n}\|^2+\|c_2^{hk,n}\|^2+\|v_2^{hk,n}\|^2\right).
\end{aligned}$$

Multiplying the above estimates by k and summing up to n, we obtain

$$\begin{aligned}
&\|e_1^{hk,n}\|^2+\|\partial_{xx}u_1^{hk,n}\|^2+\|u_1^{hk,n}\|^2+\|u_2^{hk,n}\|^2+\|e_2^{hk,n}\|^2+\|\partial_{xx}u_2^{hk,n}\|^2+\|c_1^{hk,n}\|^2\\
&\quad+\|\partial_x v_1^{hk,n}\|^2+\|c_2^{hk,n}\|^2+\|\partial_x v_2^{hk,n}\|^2+\|\partial_x u_1^{hk,n}\|^4+\|\partial_x u_2^{hk,n}\|^4\\
&\leq Ck\sum_{j=1}^{n}\left(1+\|e_1^{hk,j}\|^2+\|\partial_{xx}u_1^{hk,j}\|^2+\|u_1^{hk,j}\|^2+\|u_2^{hk,j}\|^2+\|e_2^{hk,j}\|^2+\|\partial_{xx}u_2^{hk,j}\|^2\right.\\
&\quad\left.+\|c_1^{hk,j}\|^2+\|v_1^{hk,j}\|^2+\|c_2^{hk,j}\|^2+\|v_2^{hk,j}\|^2\right)+C\left(\|e_1^{0h}\|^2+\|e_2^{0h}\|^2+\|u_1^{0h}\|^2_{H^2(0,1)}\right.\\
&\quad\left.+\|u_2^{0h}\|^2_{H^2(0,1)}+\|c_1^{0h}\|^2+\|c_2^{0h}\|^2+\|v_1^{0h}\|^2_{H^1(0,1)}+\|v_2^{0h}\|^2_{H^1(0,1)}\right).
\end{aligned}$$

Finally, using a discrete version of Gronwall's inequality (see, for instance, [41,42]), we conclude the desired stability property.

Now, we will prove some a priori error estimates on the numerical errors $e_1^n-e_1^{hk,n}$, $e_2^n-e_2^{hk,n}$, $c_1^n-c_1^{hk,n}$, and $c_2^n-c_2^{hk,n}$. First, subtracting variational Eq. (21) at time $t=t_n$ for a test function $w=w^h$ and discrete variational Eq. (30) we have:

$$\begin{aligned}
&(\partial_t e_1^n-\delta e_1^{hk,n},w^h)+(\partial_{xx}(u_1^n-u_1^{hk,n}),\partial_{xx}w^h)+(e_1^n-e_1^{hk,n},w^h)+p_1(\partial_{xx}(u_1^n-u_1^{hk,n}),w^h)\\
&\quad+(\|\partial_x u_1^n\|^2\partial_x u_1^n-\|\partial_x u_1^{hk,n}\|^2\partial_x u_1^{hk,n},\partial_x w^h)\\
&\quad+\kappa((u_1^n-u_2^n)^+-(u_1^{hk,n}-u_2^{hk,n})^+,w^h)+\kappa^*(u_1^n-u_1^{hk,n}-(u_2^n-u_2^{hk,n}),w^h)=0,
\end{aligned}$$

and so, we find that, for all $w^h\in V^h$,

$$\begin{aligned}
&(\partial_t e_1^n-\delta e_1^{hk,n},e_1^n-e_1^{hk,n})+(\partial_{xx}(u_1^n-u_1^{hk,n}),\partial_{xx}(e_1^n-e_1^{hk,n}))+(e_1^n-e_1^{hk,n},e_1^n-e_1^{hk,n})\\
&\quad+p_1(\partial_{xx}(u_1^n-u_1^{hk,n}),e_1^n-e_1^{hk,n})+(\|\partial_x u_1^n\|^2\partial_x u_1^n-\|\partial_x u_1^{hk,n}\|^2\partial_x u_1^{hk,n},\partial_x(e_1^n-e_1^{hk,n}))\\
&\quad+\kappa((u_1^n-u_2^n)^+-(u_1^{hk,n}-u_2^{hk,n})^+,e_1^n-e_1^{hk,n})+\kappa^*(u_1^n-u_1^{hk,n}-(u_2^n-u_2^{hk,n}),e_1^n-e_1^{hk,n})\\
&=(\partial_t e_1^n-\delta e_1^{hk,n},e_1^n-w^h)+(\partial_{xx}(u_1^n-u_1^{hk,n}),\partial_{xx}(e_1^n-w^h))+(e_1^n-e_1^{hk,n},e_1^n-w^h)\\
&\quad+p_1(\partial_{xx}(u_1^n-u_1^{hk,n}),e_1^n-w^h)+(\|\partial_x u_1^n\|^2\partial_x u_1^n-\|\partial_x u_1^{hk,n}\|^2\partial_x u_1^{hk,n},\partial_x(e_1^n-w^h))\\
&\quad+\kappa((u_1^n-u_2^n)^+-(u_1^{hk,n}-u_2^{hk,n})^+,e_1^n-w^h)+\kappa^*(u_1^n-u_1^{hk,n}-(u_2^n-u_2^{hk,n}),e_1^n-w^h).
\end{aligned}$$

Now, taking into account that

$$
\begin{aligned}
&(\partial_t e_1^n - \delta e_1^{hk,n}, e_1^n - e_1^{hk,n}) \geq (\partial_t e_1^n - \delta e_1^n, e_1^n - e_1^{hk,n}) \\
&\qquad + \frac{1}{2k}\left\{\|e_1^n - e_1^{hk,n}\|^2 - \|e_1^{n-1} - e_1^{hk,n-1}\|^2\right\}, \\
&(\partial_{xx}(u_1^n - u_1^{hk,n}), \partial_{xx}(e_1^n - e_1^{hk,n})) \geq (\partial_{xx}(u_1^n - u_1^{hk,n}), \partial_{xx}(\partial_t u_1^n - \delta u_1^n)) \\
&\qquad + \frac{1}{2k}\left\{\|\partial_{xx}(u_1^n - u_1^{hk,n})\|^2 - \|\partial_{xx}(u_1^{n-1} - u_1^{hk,n-1})\|^2\right\}, \\
&(u_1^n - u_1^{hk,n}, e_1^n - e_1^{hk,n}) \geq (u_1^n - u_1^{hk,n}, \partial_t u_1^n - \delta u_1^n) \\
&\qquad + \frac{1}{2k}\left\{\|u_1^n - u_1^{hk,n}\|^2 - \|u_1^{n-1} - u_1^{hk,n-1}\|^2\right\}, \\
&(\|\partial_x u_1^n\|^2 \partial_x u_1^n - \|\partial_x u_1^{hk,n}\|^2 \partial_x u_1^{hk,n}, \partial_x w) = -(\|\partial_x u_1^n\|^2 \partial_{xx} u_1^n - \|\partial_x u_1^{hk,n}\|^2 \partial_{xx} u_1^{hk,n}, w), \\
&(\|\partial_x u_1^n\|^2 \partial_{xx} u_1^n - \|\partial_x u_1^{hk,n}\|^2 \partial_{xx} u_1^{hk,n}, w) = ((\|\partial_x u_1^n\|^2 - \|\partial_x u_1^{hk,n}\|^2)\partial_{xx} u_1^n, w) \\
&\qquad + (\|\partial_x u_1^{hk,n}\|^2 \partial_{xx}(u_1^n - u_1^{hk,n}), w), \\
&(\|\partial_x u_1^{hk,n}\|^2 \partial_{xx}(u_1^n - u_1^{hk,n}), w) \leq C(\|\partial_{xx}(u_1^n - u_1^{hk,n})\|^2 + \|w\|^2), \\
&((\|\partial_x u_1^n\|^2 - \|\partial_x u_1^{hk,n}\|^2)\partial_{xx} u_1^n, w) \leq C\|w\|^2 + C(\|\partial_x u_1^n\|^2 - \|\partial_x u_1^{hk,n}\|^2)^2 \\
&\qquad \leq C(\|w\|^2 + \|\partial_x(u_1^n - u_1^{hk,n})\|^2),
\end{aligned}
$$

it follows that, for all $w^h \in V^h$,

$$
\begin{aligned}
&\frac{1}{2k}\left\{\|e_1^n - e_1^{hk,n}\|^2 - \|e_1^{n-1} - e_1^{hk,n-1}\|^2\right\} + \frac{1}{2k}\left\{\|u_1^n - u_1^{hk,n}\|^2 - \|u_1^{n-1} - u_1^{hk,n-1}\|^2\right\} \\
&\quad + \frac{1}{2k}\left\{\|\partial_{xx}(u_1^n - u_1^{hk,n})\|^2 - \|\partial_{xx}(u_1^{n-1} - u_1^{hk,n-1})\|^2\right\} \\
&\quad \leq C\Big(\|\partial_t e_1^n - \delta e_1^n\|^2 + \|\partial_t u_1^n - \delta u_1^n\|_V^2 + \|e_1^n - w^h\|_V^2 + \|e_1^n - e_1^{hk,n}\|^2 \\
&\quad + \|\partial_{xx}(u_1^n - u_1^{hk,n})\|^2 + \|\partial_x(u_1^n - u_1^{hk,n})\|^2 + \|u_1^n - u_1^{hk,n}\|^2 + \|u_2^n - u_2^{hk,n}\|^2 \\
&\quad + (\delta e_1^n - \delta e_1^{hk,n}, e_1^n - w^h) \Big),
\end{aligned}
$$

where, from now on, for a Hilbert space X let $\|\cdot\|_X$ be the corresponding usual norm.

Proceeding in a similar form for the second downward velocity we have, for all $s^h \in V^h$,

$$
\begin{aligned}
&\frac{1}{2k}\left\{\|e_{2n} - e_2^{hk,n}\|^2 - \|e_2^{n-1} - e_2^{hk,n-1}\|^2\right\} + \frac{1}{2k}\left\{\|u_2^n - u_2^{hk,n}\|^2 - \|u_2^{n-1} - u_2^{hk,n-1}\|^2\right\} \\
&\quad + \frac{1}{2k}\left\{\|\partial_{xx}(u_2^n - u_2^{hk,n})\|^2 - \|\partial_{xx}(u_2^{n-1} - u_2^{hk,n-1})\|^2\right\} \\
&\quad \leq C\Big(\|\partial_t e_2^n - \delta e_2^n\|^2 + \|\partial_t u_2^n - \delta u_2^n\|_V^2 + \|e_2^n - s^h\|_V^2 + \|e_2^n - e_2^{hk,n}\|^2 \\
&\quad + \|\partial_{xx}(u_2^n - u_2^{hk,n})\|^2 + \|\partial_x(u_2^n - u_2^{hk,n})\|^2 + \|u_1^n - u_1^{hk,n}\|^2 + \|u_2^n - u_2^{hk,n}\|^2 \\
&\quad + (\delta e_2^n - \delta e_2^{hk,n}, e_2^n - s^h) \Big).
\end{aligned}
$$

Now, we obtain the estimates for the first vertical velocity. Subtracting variational Eq. (22) for a test function $r^h \in E^h$, at time $t = t_n$, and discrete variational Eq. (31) we find that

$$
\begin{aligned}
&(\partial_t c_1^n - \delta c_1^{hk,n}, r^h) + (\partial_x(v_1^n - v_1^{hk,n}), \partial_x r^h) + (c_1^n - c_1^{hk,n}, r^h) \\
&\qquad - \kappa((u_1^n - v_1^n)^+ - (u_1^{hk,n} - v_1^{hk,n})^+, r^h) = 0,
\end{aligned}
$$

and so, we have, for all $r^h \in E^h$,

$$
\begin{aligned}
&(\partial_t c_1^n - \delta c_1^{hk,n}, c_1^n - c_1^{hk,n}) + (\partial_x(v_1^n - v_1^{hk,n}), \partial_x(c_1^n - c_1^{hk,n})) + (c_1^n - c_1^{hk,n}, c_1^n - c_1^{hk,n}) \\
&\qquad - \kappa((u_1^n - v_1^n)^+ - (u_1^{hk,n} - v_1^{hk,n})^+, c_1^n - c_1^{hk,n}) \\
&\quad = (\partial_t c_1^n - \delta c_1^{hk,n}, c_1^n - r^h) + (\partial_x(v_1^n - v_1^{hk,n}), \partial_x(c_1^n - r^h)) + (c_1^n - c_1^{hk,n}, c_1^n - r^h) \\
&\qquad - \kappa((u_1^n - v_1^n)^+ - (u_1^{hk,n} - v_1^{hk,n})^+, c_1^n - r^h).
\end{aligned}
$$

Keeping in mind that

$$
\begin{aligned}
&(\partial_t c_1^n - \delta c_1^{hk,n}, c_1^n - c_1^{hk,n}) \geq (\partial_t c_1^n - \delta c_1^n, c_1^n - c_1^{hk,n}) \\
&\qquad + \frac{1}{2k}\left\{\|c_1^n - c_1^{hk,n}\|^2 - \|c_1^{n-1} - c_1^{hk,n-1}\|^2\right\}, \\
&(\partial_x(v_1^n - v_1^{hk,n}), \partial_x(c_1^n - c_1^{hk,n})) \geq (\partial_x(v_1^n - v_1^{hk,n}), \partial_x(\partial_t v_1^n - \delta v_1^n)) \\
&\qquad + \frac{1}{2k}\left\{\|\partial_x(v_1^n - v_1^{hk,n})\|^2 - \|\partial_{xx}(v_1^{n-1} - v_1^{hk,n-1})\|^2\right\},
\end{aligned}
$$

it follows that, for all $r^h \in E^h$,

$$\frac{1}{2k}\left\{\|c_1^n - c_1^{hk,n}\|^2 - \|c_1^{n-1} - c_1^{hk,n-1}\|^2\right\} + \frac{1}{2k}\left\{\|\partial_x(v_1^n - v_1^{hk,n})\|^2 - \|\partial_x(v_1^{n-1} - v_1^{hk,n-1})\|^2\right\}$$
$$\leq C\left(\|\partial_t c_1^n - \delta c_1^n\|^2 + \|\partial_t v_1^n - \delta v_1^n\|_E^2 + \|c_1^n - r^h\|_E^2 + \|c_1^n - c_1^{hk,n}\|^2 + \|\partial_x(v_1^n - v_1^{hk,n})\|^2 + \|u_1^n - u_1^{hk,n}\|^2 + \|v_1 - v_1^{hk,n}\|^2 + (\delta c_1^n - \delta c_1^{hk,n}, c_1^n - r^h)\right).$$

Proceeding analogously, we obtain the following estimates for the second vertical velocity, for all $z^h \in E^h$,

$$\frac{1}{2k}\left\{\|c_2^n - c_2^{hk,n}\|^2 - \|c_2^{n-1} - c_2^{hk,n-1}\|^2\right\} + \frac{1}{2k}\left\{\|\partial_x(v_2^n - v_2^{hk,n})\|^2 - \|\partial_x(v_2^{n-1} - v_2^{hk,n-1})\|^2\right\}$$
$$\leq C\left(\|\partial_t c_2^n - \delta c_2^n\|^2 + \|\partial_t v_2^n - \delta v_2^n\|_E^2 + \|c_2^n - z^h\|_E^2 + \|c_2^n - c_2^{hk,n}\|^2 + \|\partial_x(v_2^n - v_2^{hk,n})\|^2 + \|u_2^n - u_2^{hk,n}\|^2 + \|v_2 - v_2^{hk,n}\|^2 + (\delta c_2^n - \delta c_2^{hk,n}, c_2^n - z^h)\right).$$

Combining the above estimates, we have, for all $w^h, s^h \in V^h$ and $r^h, z^h \in E^h$,

$$\begin{aligned}
&\frac{1}{2k}\left\{\|e_1^n - e_1^{hk,n}\|^2 - \|e_1^{n-1} - e_1^{hk,n-1}\|^2\right\} + \frac{1}{2k}\left\{\|u_1^n - u_1^{hk,n}\|^2 - \|u_1^{n-1} - u_1^{hk,n-1}\|^2\right\}\\
&\quad + \frac{1}{2k}\left\{\|\partial_{xx}(u_1^n - u_1^{hk,n})\|^2 - \|\partial_{xx}(u_1^{n-1} - u_1^{hk,n-1})\|^2\right\}\\
&\quad + \frac{1}{2k}\left\{\|e_{2n} - e_2^{hk,n}\|^2 - \|e_2^{n-1} - e_2^{hk,n-1}\|^2\right\} + \frac{1}{2k}\left\{\|u_2^n - u_2^{hk,n}\|^2 - \|u_2^{n-1} - u_2^{hk,n-1}\|^2\right\}\\
&\quad + \frac{1}{2k}\left\{\|\partial_{xx}(u_2^n - u_2^{hk,n})\|^2 - \|\partial_{xx}(u_2^{n-1} - u_2^{hk,n-1})\|^2\right\}\\
&\quad + \frac{1}{2k}\left\{\|c_1^n - c_1^{hk,n}\|^2 - \|c_1^{n-1} - c_1^{hk,n-1}\|^2\right\} + \frac{1}{2k}\left\{\|\partial_x(v_1^n - v_1^{hk,n})\|^2 - \|\partial_x(v_1^{n-1} - v_1^{hk,n-1})\|^2\right\}\\
&\quad + \frac{1}{2k}\left\{\|c_2^n - c_2^{hk,n}\|^2 - \|c_2^{n-1} - c_2^{hk,n-1}\|^2\right\} + \frac{1}{2k}\left\{\|\partial_x(v_2^n - v_2^{hk,n})\|^2 - \|\partial_x(v_2^{n-1} - v_2^{hk,n-1})\|^2\right\}\\
&\leq C\Big(\|\partial_t e_1^n - \delta e_1^n\|^2 + \|\partial_t u_1^n - \delta u_1^n\|_V^2 + \|e_1^n - w^h\|_V^2 + \|e_1^n - e_1^{hk,n}\|^2\\
&\quad + \|\partial_{xx}(u_1^n - u_1^{hk,n})\|^2 + \|\partial_x(u_1^n - u_1^{hk,n})\|^2 + \|u_1^n - u_1^{hk,n}\|^2 + \|u_2^n - u_2^{hk,n}\|^2\\
&\quad + (\delta e_1^n - \delta e_1^{hk,n}, e_1^n - w^h) + \|\partial_t e_2^n - \delta e_2^n\|^2 + \|\partial_t u_2^n - \delta u_2^n\|_V^2 + \|e_2^n - s^h\|_V^2 + \|e_2^n - e_2^{hk,n}\|^2\\
&\quad + \|\partial_{xx}(u_2^n - u_2^{hk,n})\|^2 + \|\partial_x(u_2^n - u_2^{hk,n})\|^2 + (\delta e_2^n - \delta e_2^{hk,n}, e_2^n - s^h) + \|\partial_t c_1^n - \delta c_1^n\|^2\\
&\quad + \|\partial_t v_1^n - \delta v_1^n\|_E^2 + \|c_1^n - r^h\|_E^2 + \|c_1^n - c_1^{hk,n}\|^2 + \|\partial_x(v_1^n - v_1^{hk,n})\|^2\\
&\quad + \|v_1 - v_1^{hk,n}\|^2 + (\delta c_1^n - \delta c_1^{hk,n}, c_1^n - r^h) + \|\partial_t c_2^n - \delta c_2^n\|^2 + \|\partial_t v_2^n - \delta v_2^n\|_E^2\\
&\quad + \|c_2^n - z^h\|_E^2 + \|c_2^n - c_2^{hk,n}\|^2 + \|\partial_x(v_2^n - v_2^{hk,n})\|^2 + \|v_2 - v_2^{hk,n}\|^2 + (\delta c_2^n - \delta c_2^{hk,n}, c_2^n - z^h)\Big).
\end{aligned}$$

Multiplying the above estimates by k and summing up to n, we have, for all $\{w^{h,j}, s^{h,j}\}_{j=1}^n \in V^h$ and $\{r^{h,j}, z^{h,j}\}_{j=1}^n \in E^h$,

$$\begin{aligned}
&\|e_1^n - e_1^{hk,n}\|^2 + \|u_1^n - u_1^{hk,n}\|^2 + \|\partial_{xx}(u_1^n - u_1^{hk,n})\|^2 + \|e_{2n} - e_2^{hk,n}\|^2\\
&\quad + \|u_2^n - u_2^{hk,n}\|^2 + \|\partial_{xx}(u_2^n - u_2^{hk,n})\|^2 + \|c_1^n - c_1^{hk,n}\|^2 + \|\partial_x(v_1^n - v_1^{hk,n})\|^2\\
&\quad + \|c_2^n - c_2^{hk,n}\|^2 + \|\partial_x(v_2^n - v_2^{hk,n})\|^2\\
&\leq Ck\sum_{j=1}^n \Big(\|\partial_t e_1^j - \delta e_1^j\|^2 + \|\partial_t u_1^j - \delta u_1^j\|_V^2 + \|e_1^n - w^{h,j}\|_V^2 + \|e_1^j - e_1^{hk,j}\|^2\\
&\quad + \|\partial_{xx}(u_1^j - u_1^{hk,j})\|^2 + \|\partial_x(u_1^j - u_1^{hk,j})\|^2 + \|u_1^j - u_1^{hk,j}\|^2 + \|u_2^j - u_2^{hk,j}\|^2\\
&\quad + (\delta e_1^j - \delta e_1^{hk,j}, e_1^j - w^{h,j}) + \|\partial_t e_2^j - \delta e_2^j\|^2 + \|\partial_t u_2^j - \delta u_2^j\|_V^2 + \|e_2^j - s^{h,j}\|_V^2 + \|e_2^j - e_2^{hk,j}\|^2\\
&\quad + \|\partial_{xx}(u_2^j - u_2^{hk,j})\|^2 + \|\partial_x(u_2^j - u_2^{hk,j})\|^2 + (\delta e_2^j - \delta e_2^{hk,j}, e_2^j - s^{h,j}) + \|\partial_t c_1^j - \delta c_1^j\|^2\\
&\quad + \|\partial_t v_1^j - \delta v_1^j\|_E^2 + \|c_1^n - r^{h,j}\|_E^2 + \|c_1^j - c_1^{hk,j}\|^2 + \|\partial_x(v_1^j - v_1^{hk,j})\|^2\\
&\quad + \|v_1^j - v_1^{hk,j}\|^2 + (\delta c_1^j - \delta c_1^{hk,j}, c_1^j - r^{h,j}) + \|\partial_t c_2^j - \delta c_2^j\|^2 + \|\partial_t v_2^j - \delta v_2^j\|_E^2\\
&\quad + \|c_2^j - z^{h,j}\|_E^2 + \|c_2^j - c_2^{hk,j}\|^2 + \|\partial_x(v_2^j - v_2^{hk,j})\|^2 + \|v_2^j - v_2^{hk,j}\|^2\\
&\quad + (\delta c_2^j - \delta c_2^{hk,j}, c_2^j - z^{h,j})\Big) + C\Big(\|\tilde{u}_1 - e_1^{0h}\|^2 + \|\bar{u}_1 - u_1^{0h}\|_V^2 + \|\tilde{u}_2 - e_2^{0h}\|^2\\
&\quad + \|\bar{u}_2 - u_2^{0h}\|_V^2 + \|\tilde{v}_1 - c_1^{0h}\|^2 + \|\bar{v}_1 - v_1^{0h}\|_E^2 + \|\tilde{v}_2 - c_2^{0h}\|^2 + \|\bar{v}_2 - v_2^{0h}\|_E^2\Big).
\end{aligned}$$

Finally, keeping in mind that

$$\begin{aligned}
k\sum_{j=1}^{n}(\delta e_1^j-\delta e_1^{hk,j},e_1^j-w^{h,j})&=(e_1^n-e_1^{hk,n},e_1^n-w^{h,n})+(e_1^{0h}-\tilde{u}_1,e_1^1-w^{h,1})\\
&\quad+\sum_{j=1}^{n-1}(e_1^j-e_1^{hk,j},e_1^j-w^{h,j}-(e_1^{j+1}-w^{h,j+1})),\\
k\sum_{j=1}^{n}(\delta e_2^j-\delta e_2^{hk,j},e_2^j-s^{h,j})&=(e_2^n-e_2^{hk,n},e_2^n-s^{h,n})+(e_2^{0h}-\tilde{u}_2,e_2^1-s^{h,1})\\
&\quad+\sum_{j=1}^{n-1}(e_2^j-e_2^{hk,j},e_2^j-s^{h,j}-(e_2^{j+1}-s^{h,j+1})),\\
k\sum_{j=1}^{n}(\delta c_1^j-\delta c_1^{hk,j},c_1^j-r^{h,j})&=(c_1^n-c_1^{hk,n},c_1^n-r^{h,n})+(c_1^{0h}-\tilde{v}_1,c_1^1-r^{h,1})\\
&\quad+\sum_{j=1}^{n-1}(c_1^j-c_1^{hk,j},c_1^j-r^{h,j}-(c_1^{j+1}-r^{h,j+1})),\\
k\sum_{j=1}^{n}(\delta c_2^j-\delta c_2^{hk,j},c_2^j-z^{h,j})&=(c_2^n-c_2^{hk,n},c_2^n-z^{h,n})+(c_2^{0h}-\tilde{v}_2,c_2^1-z^{h,1})\\
&\quad+\sum_{j=1}^{n-1}(c_2^j-c_2^{hk,j},c_2^j-z^{h,j}-(c_2^{j+1}-z^{h,j+1})),
\end{aligned}$$

using again a discrete version of Gronwall's inequality (see [41,42]) we conclude the following a priori error estimates result.

Theorem 6. *Let the assumptions of Lemma 5 still hold. If we denote by (e_1,e_2,c_1,c_2) the solution to Problem VP and by $\{e_1^{hk,n},e_2^{hk,n},c_1^{hk,n},c_2^{hk,n}\}_{n=0}^N$ the solution to Problem VP^{hk}, then we have the following a priori error estimates, for all $\{w^{h,j}\}_{j=0}^N$, $\{s^{h,j}\}_{j=0}^N\subset V^h$ and $\{r^{h,j}\}_{j=0}^N$, $\{z^{h,j}\}_{j=0}^N\subset E^h$,*

$$\begin{aligned}
&\max_{0\le n\le N}\Big\{\|e_1^n-e_1^{hk,n}\|^2+\|u_1^n-u_1^{hk,n}\|_V^2+\|e_{2n}-e_2^{hk,n}\|^2+\|u_2^n-u_2^{hk,n}\|_V^2\\
&\qquad+\|c_1^n-c_1^{hk,n}\|^2+\|v_1^n-v_1^{hk,n}\|_E^2+\|c_2^n-c_2^{hk,n}\|^2+\|v_2^n-v_2^{hk,n}\|_E^2\Big\}\\
&\le Ck\sum_{j=1}^{N}\Big(\|\partial_t e_1^j-\delta e_1^j\|^2+\|\partial_t u_1^j-\delta u_1^j\|_V^2+\|e_1^n-w^{h,j}\|_V^2+\|\partial_t e_2^j-\delta e_2^j\|^2\\
&\qquad+\|\partial_t u_2^j-\delta u_2^j\|_V^2+\|e_2^j-s^{h,j}\|_V^2+\|\partial_t c_1^j-\delta c_1^j\|^2+\|\partial_t v_1^j-\delta v_1^j\|_E^2\\
&\qquad+\|c_1^n-r^{h,j}\|_E^2+\|\partial_t c_2^j-\delta c_2^j\|^2+\|\partial_t v_2^j-\delta v_2^j\|_E^2+\|c_2^j-z^{h,j}\|_E^2\Big)\\
&\qquad+C\max_{0\le n\le N}\Big\{\|e_1^n-w^{h,n}\|^2+\|e_2^n-s^{h,n}\|^2+\|c_1^n-r^{h,n}\|^2+\|c_2^n-z^{h,n}\|^2\Big\}\\
&\qquad+\frac{C}{k}\sum_{j=1}^{N-1}\Big(\|e_1^j-w^{h,j}-(e_1^{j+1}-w^{h,j+1})\|^2+\|e_2^j-s^{h,j}-(e_2^{j+1}-s^{h,j+1})\|^2\\
&\qquad+\|c_1^j-r^{h,j}-(c_1^{j+1}-r^{h,j+1})\|^2+\|c_2^j-z^{h,j}-(c_2^{j+1}-z^{h,j+1})\|^2\Big)\\
&\qquad+C\Big(\|\tilde{u}_1-e_1^{0h}\|^2+\|\bar{u}_1-u_1^{0h}\|_V^2+\|\tilde{u}_2-e_2^{0h}\|^2+\|\bar{u}_2-u_2^{0h}\|_V^2\\
&\qquad+\|\tilde{v}_1-c_1^{0h}\|^2+\|\bar{v}_1-v_1^{0h}\|_E^2+\|\tilde{v}_2-c_2^{0h}\|^2+\|\bar{v}_2-v_2^{0h}\|_E^2\Big),
\end{aligned}$$

where C is again a positive constant which does not depend on parameters h and k.

From the previous error estimates, we can analyze the convergence order under suitable additional regularity conditions. For instance, if we assume that the solution to problem *VP* has the additional regularity:

$$\begin{aligned}
&e_1,e_2\in C^1([0,T];H^3(0,\ell))\cap H^2(0,T;V)\cap H^3(0,T;Y),\\
&c_1,c_2\in C^1([0,T];H^2(0,\ell))\cap H^2(0,T;E)\cap H^3(0,T;Y),
\end{aligned}$$

then there exists a positive constant C, assumed to be again independent of the discretization parameters h and k, such that

$$\begin{aligned}
&\max_{0\le n\le N}\Big\{\|e_1^n-e_1^{hk,n}\|+\|u_1^n-u_1^{hk,n}\|_V+\|e_{2n}-e_2^{hk,n}\|+\|u_2^n-u_2^{hk,n}\|_V\\
&\qquad+\|c_1^n-c_1^{hk,n}\|+\|v_1^n-v_1^{hk,n}\|_E+\|c_2^n-c_2^{hk,n}\|+\|v_2^n-v_2^{hk,n}\|_E\Big\}\le C(h+k),
\end{aligned}$$

and so, we can conclude the linear convergence of the approximations.

5. Numerical results

In this final section, we describe the numerical scheme implemented in MATLAB for solving problem (30)–(35), and we show some numerical examples to demonstrate the accuracy of the approximations and the behavior of the solution with respect to a coupling parameter.

5.1. Numerical scheme

As a first step, given the solution $e_1^{hk,n-1}$, $c_1^{hk,n-1}$, $e_2^{hk,n-1}$ and $c_2^{hk,n-1}$, at time t_{n-1}, variables $e_1^{hk,n}$, $c_1^{hk,n}$, $e_2^{hk,n}$ and $c_2^{hk,n}$ are obtained by solving the discrete linear system, for all w^h, $s^h \in V^h$, and r^h, $z^h \in E^h$.

$$
\begin{aligned}
(\frac{1}{k}e_1^{hk,n}, w^h) \quad & +(\partial_{xx}e_1^{hk,n}, \partial_{xx}w^h) + (e_1^{hk,n}, w^h) + p_1 k(\partial_{xx}e_1^{hk,n}, w^h) + \|\partial_x u_1^{hk,n}\|^2_{L^2(0,1)} k(\partial_x e_1^{hk,n}, \partial_x w^h) + \kappa^* k(e_1^{hk,n}, w^h) \\
& = (\frac{1}{k}e_1^{hk,n-1}, w^h) - (\partial_{xx}u_1^{hk,n-1}, \partial_{xx}w^h) - p_1(\partial_{xx}u_1^{hk,n-1}, w^h) - \|\partial_x u_1^{hk,n}\|^2_{L^2(0,1)}(\partial_x u_1^{hk,n-1}, \partial_x w^h) \\
& -\kappa([u_1^{hk,n} - v_1^{hk,n}]^+, w^h) - \kappa^*(u_1^{hk,n-1} - v_1^{hk,n}, w^h) + (f_1^n, w^h), \\
(\frac{1}{k}c_1^{hk,n}, r^h) \quad & +k(\partial_x c_1^{hk,n}, \partial_x r^h) + (c_1^{hk,n}, r^h) = (\frac{1}{k}c_1^{hk,n-1}, r^h) - (\partial_x v_1^{hk,n-1}, \partial_x r^h) + \kappa([u_1^{hk,n} - v_1^{hk,n}]^+, r^h) + (g_1^n, r^h), \\
(\frac{1}{k}e_2^{hk,n}, s^h) \quad & +(\partial_{xx}e_2^{hk,n}, \partial_{xx}s^h) + (e_2^{hk,n}, s^h) + p_2 k(\partial_{xx}e_2^{hk,n}, s^h) + \|\partial_x u_2^{hk,n}\|^2_{L^2(0,1)} k(\partial_x e_2^{hk,n}, \partial_x s^h) + \kappa^* k(e_2^{hk,n}, s^h) \\
& = (\frac{1}{k}e_2^{hk,n-1}, s^h) - (\partial_{xx}u_2^{hk,n-1}, \partial_{xx}s^h) - p_2(\partial_{xx}u_2^{hk,n-1}, s^h) - \|\partial_x u_2^{hk,n}\|^2_{L^2(0,1)}(\partial_x u_2^{hk,n-1}, \partial_x s^h) \\
& -\kappa([u_2^{hk,n} - v_2^{hk,n}]^+, s^h) - \kappa^*(u_2^{hk,n-1} - v_2^{hk,n}, s^h) + (f_2^n, s^h), \\
(\frac{1}{k}c_2^{hk,n}, z^h) \quad & +k(\partial_x c_2^{hk,n}, \partial_x z^h) + (c_2^{hk,n}, z^h) = (\frac{1}{k}c_2^{hk,n-1}, z^h) - (\partial_x v_2^{hk,n-1}, \partial_x z^h) + \kappa([u_2^{hk,n} - v_2^{hk,n}]^+, z^h) + (g_2^n, z^h).
\end{aligned}
$$

This numerical scheme was implemented on a 3.2 GHz PC using MATLAB by using a Newton iterative scheme, and we note that a typical run (using the discretization parameters $h = k = 0.001$) took about $1.35\,s$ of CPU time.

5.2. Numerical convergence and discrete energy decay

In order to show the accuracy of the approximations, we solve problem (1), (2) and (3) with the following data:

$$p_1 = 1, \quad p_2 = 1, \quad \kappa = 1, \quad \kappa^* = 1,$$

the initial conditions, for all $x \in (0, 1)$,

$$
\begin{aligned}
&\bar{u}_i(x) = \tilde{u}_i(x) = x^3(x-1)^3 \quad \text{for } i = 1, 2, \\
&\bar{v}_i(x) = \tilde{v}_i(x) = x(x-1) \quad \text{for } i = 1, 2,
\end{aligned}
$$

and assuming homogeneous Dirichlet boundary conditions. Moreover, we also suppose that the supply terms vanish, i.e. $f_1,\ g_1,\ f_2,\ g_2 = 0$.

Since the problem is nonlinear, we cannot calculate the exact solution and so, we take instead it the one obtained with the discretization parameters $h = 1/2^{14}$ and $k = 10^{-6}$. Thus, the approximation errors estimated by

$$
\begin{aligned}
\max_{0\le n\le N} \Big\{ & \|e_1^n - e_1^{n,hk}\| + \|u_1^n - u_1^{n,hk}\|_V + \|e_2^n - e_2^{n,hk}\| + \|u_2^n - u_2^{n,hk}\|_V + \|c_1^n - c_1^{n,hk}\| + \|v_1^n - v_1^{n,hk}\|_E \\
& + \|c_2^n - c_2^{n,hk}\| + \|v_2 n - v_2^{n,hk}\|_E \Big\}
\end{aligned}
$$

are presented in Table 1 (multiplicated by 10^2) for several values of the discretization parameters h and k. Moreover, the evolution of the error depending on the parameter $h + k$ is plotted in Fig. 2. We notice that the convergence of the algorithm is observed although we note that, for a fixed mesh size, the numerical error increases considerably when the time step decreases. Moreover, the linear convergence, stated in the previous section, is achieved.

If we use the final time $T = 10$, the same data than in the previous example and the initial conditions:

$$
\begin{aligned}
&\bar{u}_i(x) = \tilde{u}_i(x) = x^3(x-1)^3 \quad \text{for } i = 1, 2, \\
&\bar{v}_i(x) = \tilde{v}_i(x) = 0 \quad \text{for } i = 1, 2,
\end{aligned}
$$

taking the discretization parameters $h = k = 0.001$, the evolution in time of the discrete energy is plotted in Fig. 3 (in both natural and semi-log scales). As can be seen, it converges to zero and an exponential decay seems to be achieved.

5.3. Dependence of the solution with respect to the coupling parameter κ

In this last example, we will investigate the dependence on the coupling parameter κ for the solution to problem (1), (2) and (3).

Table 1
Example 1: Numerical errors ($\times 10^{-2}$) for some values of h and k.

$h \downarrow k \rightarrow$	0.01	0.005	0.002	0.001	0.0005	0.0002	0.0001
$1/2^4$	0.219722	0.733403	1.504026	1.914019	2.160085	2.327872	2.399697
$1/2^5$	0.111566	0.368263	0.753451	0.958395	1.081410	1.165508	1.201979
$1/2^6$	0.057543	0.185743	0.378205	0.480617	0.542101	0.584184	0.602455
$1/2^7$	0.030568	0.094515	0.190611	0.241745	0.272449	0.293509	0.302662
$1/2^8$	0.017144	0.048925	0.096824	0.122306	0.137607	0.148144	0.152735
$1/2^9$	0.010552	0.026169	0.049926	0.062557	0.070138	0.075400	0.077711
$1/2^{10}$	0.007462	0.014868	0.026465	0.032613	0.036294	0.038899	0.040075
$1/2^{11}$	0.006177	0.009358	0.014708	0.017503	0.019155	0.020391	0.021015

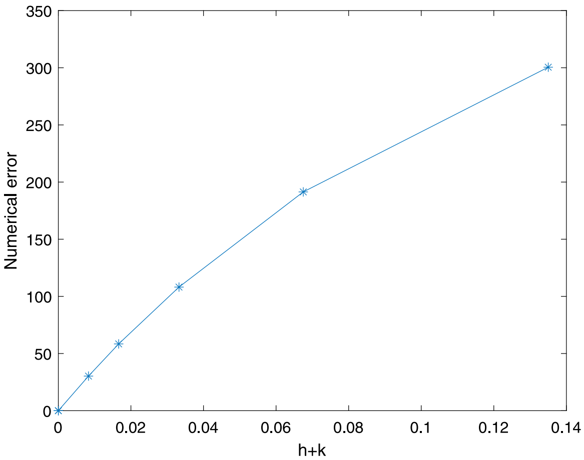

Fig. 2. Example 1: Asymptotic constant error.

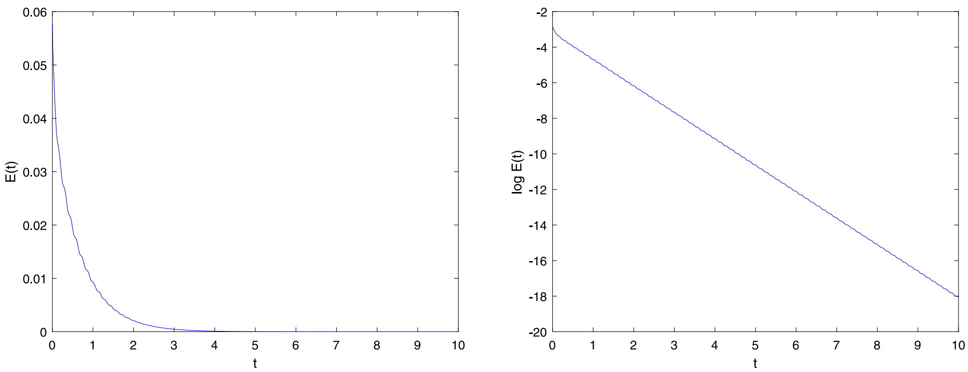

Fig. 3. Example 1: Evolution in time of the discrete energy (natural and semi-log scales).

Now, we assume that there are not supply terms, and we use the final time $T = 1$, the data

$$p_1 = 1, \quad p_2 = 1, \quad \kappa^* = 1,$$

and the initial conditions, for all $x \in (0, 1)$,

$$\bar{u}_i(x) = \tilde{u}_i(x) = x^3(x-1)^3 \quad \text{for } i = 1, 2,$$
$$\bar{v}_i(x) = \tilde{v}_i(x) = 0 \quad \text{for } i = 1, 2.$$

Taking the discretization parameters $h = k = 0.001$, the downward deflection u_1 and the downward velocity e_1 of the first component are plotted at final time in Fig. 4 for some values of the coupling parameter κ. We can see that the solution is rather different for each variable, being the differences greater for the downward velocity. As expected, when κ tends to zero the solution also converges to zero.

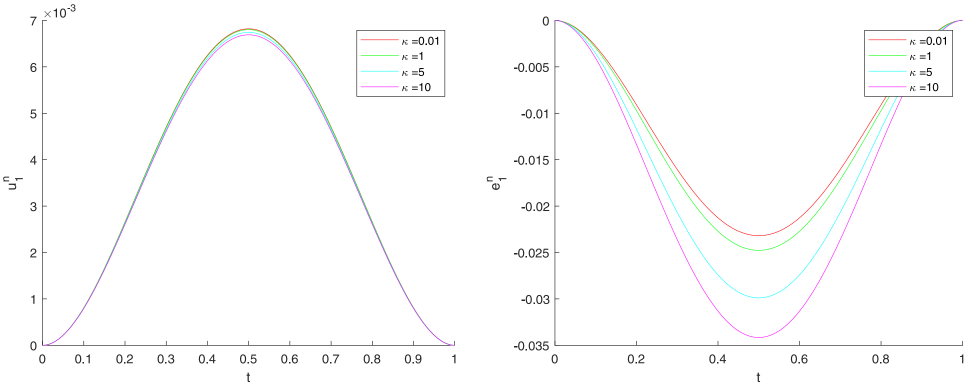

Fig. 4. Example 2: Downward deflection and downward velocity of the first component for different values of κ.

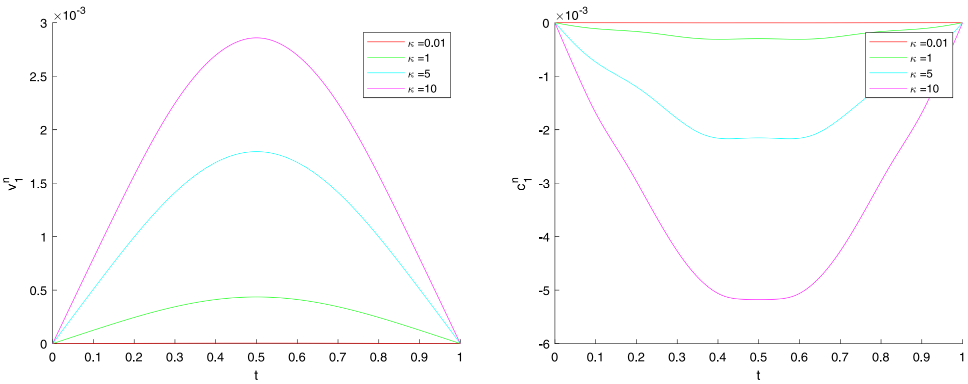

Fig. 5. Example 2: Vertical displacement and vertical velocity of the first component for different values of κ.

Moreover, in Fig. 5 the vertical displacement v_1 and the vertical velocity c_1 of the first component are shown, at final time, for different values of the coupling coefficient κ. Now, we can see again that both solutions are rather different and the differences are really important. Moreover, for small values of the parameter the solution almost vanishes.

CRediT authorship contribution statement

N. Bazarra: Software, Methodology, Writing – original draft. **I. Bochicchio:** Visualization, Investigation, Writing – review & editing, Formal analysis. **J.R. Fernández:** Writing – original draft, Investigation, Supervision. **E. Vuk:** Writing – original draft, Investigation, Formal analysis, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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