Annals of Operations Research
https://doi.org/10.1007/s10479-023-05214-8

ORIGINAL RESEARCH

An algorithm to compute the average-of-awards rule for claims problems with an application to the allocation of CO_2 emissions

Miguel Ángel Mirás Calvo[1] · **Iago Núñez Lugilde**[2] · **Carmen Quinteiro Sandomingo**[3] · **Estela Sánchez-Rodríguez**[2]

Accepted: 23 January 2023

Abstract
The set of awards vectors for a claims problem coincides with the core of the associated coalitional game. We analyze the structure of this set by defining for each group of claimants a, so called, utopia game, whose core comprises the most advantageous imputations available for the group. We show that, given a claims problem, the imputation set of the associated coalitional game can be partitioned by the cores of the utopia games. A rule selects for each claims problem a unique allocation from the set of awards vectors. The average-of-awards rule associates to each claims problem the geometric center of the corresponding set of awards vectors. Based on the decomposition of the imputation set, we obtain an interpretation of the average-of-awards rule as a point of fairness between stable and utopia imputations and provide a backward recurrence algorithm to compute it. To illustrate our analysis, we present an application to the distribution of CO_2 emissions.

Keywords Claims problems · Average-of-awards rule · Utopia games · Global carbon budget

1 Introduction

A claims problem (Aumann & Maschler, 1985; O'Neill, 1982) arises when a finite group of agents claim a scarce resource, the endowment, which is insufficient to honor the aggregate claim. A rule proposes a way to divide the endowment among the claimants. For each claims

This paper was presented in the European Meeting on Game Theory-SING16 (Granada, 2021), under the title *On properties of the average-of-awards rule with an application to the allocation of CO_2 emissions*.

✉ Estela Sánchez-Rodríguez
esanchez@uvigo.es

[1] RGEAF, Departamento de Matemáticas, Universidade de Vigo, 36310 Vigo, Spain

[2] SIDOR, Departamento de Estatística e Investigación Operativa, CINBIO, Universidade de Vigo, 36310 Vigo, Spain

[3] Departamento de Matemáticas, Universidade de Vigo, 36310 Vigo, Spain

Published online: 10 February 2023

problem, the allocation selected by a rule must belong to the set of awards vectors: it must be non-negative, bounded from above by the claims, and such that the sum of the awards is equal to the endowment. Division rules are classified and compared according to different principles that one may want them to satisfy. Thomson (2019) reviews the vast literature on claims problems, including the most important rules and axioms.

Following O'Neill (1982), to each claims problem one can associate a coalitional game, with the creditors as players, and such that the characteristic function assigns to each coalition the difference between the endowment and the sum of the claims of the complementary coalition whenever this difference is positive, and zero otherwise. Curiel et al. (1987) note that the coalitional game associated with a claims problem is a convex game, so its core is a non-empty convex set. In fact, it can be shown (see for instance Thomson (2019)) that the core of the coalitional game associated with a claims problem coincides with the corresponding set of awards vectors.

An intuitive and simple way of selecting an allocation from the set of awards vectors for a claims problem is to assume that all the awards vectors are equally likely and therefore choosing their "average". Mirás Calvo et al. (2022b) called this selection the average-of-awards rule (AA). Obviously, for each claims problem the AA rule selects the center of gravity (centroid) of the core of the associated game, so the AA rule corresponds to the core-center solution defined by González-Díaz & Sánchez-Rodríguez (2007) for balanced games. Mirás Calvo et al. (2022b) show that the AA rule satisfies a good number of properties so as to be included in the inventory of rules. Mirás Calvo et al. (2022a) compare the AA rule with the most important rules using the Lorenz order. In this paper we address the issue of computing the recommendation made by the AA rule for a problem.

Archimedes of Syracuse introduced the concept of center of gravity in his work *On the Equilibrium of Planes*. Archimedes derived the law of the lever and calculated the center of gravity of parallelograms, triangles, and trapeziums. Implicit in the work is the idea that the center of gravity is a point of equilibrium: a point from which a freely hanging body is stable. Today a more general concept, the centroid, is used in many different fields: physics, engineering, etc. The core-center is just another application of this concept, in this case to cooperative game theory.

It is easy to calculate the centroid for a symmetric shape, but for an irregular object it could be a challenging task. Again Archimedes paved the way. One of the techniques used in obtaining the centroid of a compound shape is the method of geometric decomposition. The method works by dividing the shape into a number of parts, that share no common volume, and then finding the overall centroid as the average of the centroid of each part weighted by its relative measure.

We rely on the method of geometric decomposition to obtain an algorithm to compute the AA rule for claims problems. Let us explain the intuitions behind our analysis with the help of a simple diagram. Consider a claims problem with just three claimants. Figure 1 shows a sketch of the imputation set of the associated coalitional game and a core with the maximum number of extreme points. If all of the claims are bigger than the endowment, the core coincides with the imputation set. In this situation, most of the rules recommend the egalitarian division among the claimants: the barycenter of the imputation triangle.

But, if at least one claim is less than the endowment then there are imputations that are not stable. In particular, when the three claims do not exceed the endowment, these imputations belong to the three interior equilateral triangles in the picture. The imputations in the triangle at bottom left, assign to claimant 1 at least its claim, so the allocations in this region are clearly the most favorable to this agent. This triangle happens to be the core of the coalitional game associated with a particular claims problem, that will be called the utopia game for claimant

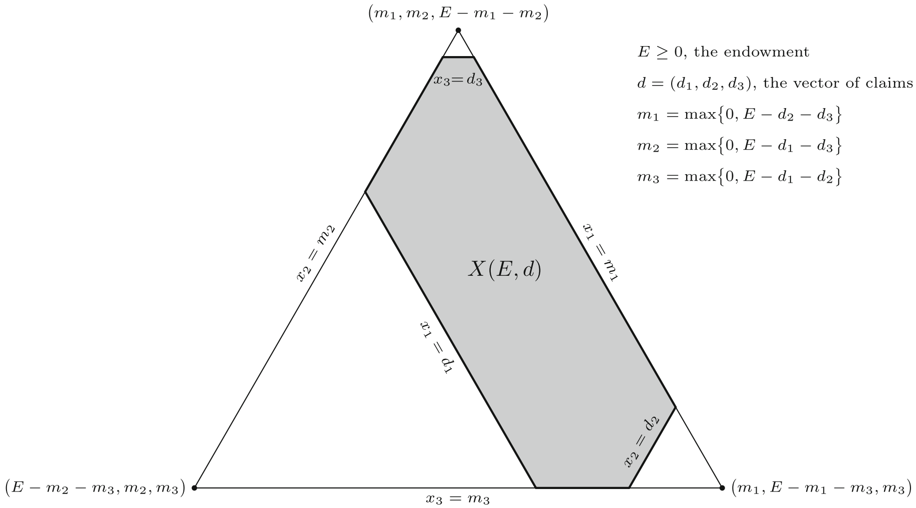


Fig. 1 A generic three-claimant set of awards vectors $X(E, d)$

1. Analogously, we can define utopia games for agents 2 and 3, whose respective cores are the other two interior triangles. Therefore, a reasonable selection in each of these equilateral triangles will be, again, the egalitarian division. Now, each utopia region represents a given percentage of the total imputation set. Since the big triangle can be decomposed as the union of the smaller triangles (the utopia regions for the claimants) and the core, the initial egalitarian division is the weighted average of the utopia egalitarian selections and the centroid of the core (the AA rule). So the AA rule can be computed using this decomposition: a partition of the imputations set by cores of coalitional games associated with claims problems.

Our aim is to show that these ideas can be extended to claims problems with an arbitrary number of claimants. Given a group of claimants, we want to identify the allocations that assign to the group at least the joint claim, provided that all the minimal rights have already been allocated. It turns out that these allocations are the set of awards vectors for a claims problem that we name the utopia problem for that group of claimants. We refer to the associated coalitional game as the utopia game for that coalition.

The contribution of this paper is threefold. First, we show that given a claims problem, an allocation that belongs to the imputation set of the associated coalitional game is either a stable allocation for that game or it is a stable allocation for the utopia game of some coalition. Secondly, we provide a backward recurrence algorithm to compute the allocation selected by the AA rule. Finally, we present an application to a real-world problem involving 20 claimants.

Our main result establishes that the imputation set of the game associated with a claims problem is the union of the cores of the utopia games for all the proper coalitions, and that any two pieces of this decomposition have negligible intersection. Therefore, applying the method of geometric decomposition, we develop an algorithm that computes the allocation selected by the AA rule. The algorithm can be greatly simplified by taking into consideration some of the properties satisfied by the rule. In doing so, we come out with an explicit formula for the AA rule. Based on the algorithm and the formula we provide interpretations of the allocation selected by the AA rule.

Certainly, our algorithm is specific for claims problems and therefore can not be applied to compute the core-center solution for arbitrary balanced games. Nevertheless, in addition to the class of bankruptcy games, it can be easily adapted to give the core-center solution for the class of two-bound core games (Gong et al., 2022), that includes, 1-convex games (Driessen, 1986), big boss games (Muto et al., 1988), and clan games (Potters et al., 1989). Moreover, our analysis extends to the computation of the centroid of any core-like polyhedron that is the intersection of an efficiency-type hyperplane with a rectangle, for instance, the core cover set (Tijs & Lipperts, 1982).

One of the many applications of claims problems is the allocation of CO_2 emissions, see Giménez-Gómez et al. (2016), Duro Moreno et al. (2020), Heo and Lee (2022), and the references therein. The endowment is the available carbon budget (the allowed global CO_2 emissions before crossing a dangerous threshold), and the claimants are the countries, or groups of countries, that are typically going to claim a larger quota of CO_2 emissions. How can the global carbon budget be distributed among the emitters? According to the United Nations Environment Program (2019) in order to get in line with the Paris Agreement, emissions must drop 7.6 per cent per year from 2020 to 2030. Based on this conclusion, we present a dynamic model that analyzes the year-by-year reductions proposed by the proportional, the Talmud, the random arrival, and the average-of-awards rules for the 2020–2030 period and the top 20 world emitters. This example illustrates the behavior of the AA rule and highlights some similarities and discrepancies with the other rules.

In Sect. 2 we introduce the basic definitions and notations. The utopia games are defined and analyzed in Sect. 3. In Sect. 4 we derive our decomposition result: the union of the cores of the utopia games comprises the imputation set. In Sect. 5, we develop a backward recurrence algorithm to compute the AA rule. Section 6 is devoted to the CO_2 emissions example. In Sect. 7 we briefly point out how the method can be applied to the computation of the core-center solution for some particular classes of games. We leave to the Appendix the proofs of the results. The computations in all the examples and applications throughout the paper were carried out with the *ClaimsProblems* R package Núñez Lugilde et al. (2023), Mirás Calvo et al. (2023).

2 Preliminaries

Let N be a finite subset of natural numbers. Given $z \in \mathbb{R}^N$ and $S \in 2^N$, let $|S|$ be the number of elements of S and $z(S) = \sum_{i \in S} z_i$. Given $N' \subset N$, let $z_{N'} = (z_i)_{i \in N'} \in \mathbb{R}^{N'}$ be the projection of z onto $\mathbb{R}^{N'}$. A claims problem with set of claimants N is a pair (E, d) where $E \geq 0$ is the endowment to be divided and $d \in \mathbb{R}^N$ is the vector of claims satisfying $d_i \geq 0$ for all $i \in N$ and $d(N) \geq E$. We denote the class of claims problems with set of claimants N by C^N. The minimal right of claimant $i \in N$ in $(E, d) \in C^N$ is the quantity $m_i(E, d) = \max\{0, E - d(N \backslash \{i\})\}$. The truncated claim of claimant $i \in N$ in $(E, d) \in C^N$ is $t_i(E, d) = \min\{E, d_i\}$. Let $m(E, d) = (m_i(E, d))_{i \in N}$ and $t(E, d) = (t_i(E, d))_{i \in N}$. Sometimes we write $t = t(E, d)$ and $m = m(E, d)$ if no confusion is possible.

A vector $x \in \mathbb{R}^N$ is an awards vector for $(E, d) \in C^N$ if $0 \leq x_i \leq d_i$ for all $i \in N$ and $x(N) = E$. Let $X(E, d)$ be the set of awards vectors for $(E, d) \in C^N$, that is, $X(E, d) = \{x \in \mathbb{R}^N : 0 \leq x_i \leq d_i \text{ for all } i \in N,\ x(N) = E\}$. Therefore, $X(E, d)$ is the intersection of the n-rectangle $\prod_{i \in N} [0, d_i]$ with the hyperplane $H(E, d) = \{x \in \mathbb{R}^N : x(N) = E\}$, so $X(E, d)$ is a nonempty compact convex polytope that has, at most, dimension $n - 1$. Let

$I(E, d) = \{x \in H(E, d) \colon x_i \geq m_i(E, d)\}$ be the set of allocations that share the endowment honoring the minimal rights.

A rule is a function $\mathcal{R} \colon C^N \to \mathbb{R}^N$ assigning to each claims problem $(E, d) \in C^N$ an awards vector $\mathcal{R}(E, d) \in X(E, d)$, that is, a way of associating with each claims problem a division among the claimants of the amount available.

A coalitional game with set of players N is a function $v \colon 2^N \to \mathbb{R}$ such that $v(\emptyset) = 0$. Let G^N be the set of all coalitional games with player set N. For simplicity, we will write $v(i)$ instead of $v(\{i\})$ for $i \in N$. An allocation $x \in \mathbb{R}^N$ is said to be efficient, or a preimputation, for a game $v \in G^N$ if $x(N) = v(N)$. The set of all efficient allocations for game v is the hyperplane $H(v) = \{x \in \mathbb{R}^N : x(N) = v(N)\}$. Given some class of games $\mathcal{G} \subset G^N$, a solution on $\mathcal{G}$ is a mapping $\varphi \colon \mathcal{G} \to \mathbb{R}^N$ that associates with each game $v \in \mathcal{G}$ a preimputation $\varphi(v) \in H(v)$.

Given a convex polytope $K \subset H(v)$ denote by $\mathrm{Vol}_{n-1}(K)$, or simply $\mathrm{Vol}(K)$ if no confusion is possible, its $(n-1)$-dimensional Lebesgue measure and by $\mu(K)$ its centroid. Convexity of K ensures that $\mu(K) \in K$. Also $\mu(a + K) = a + \mu(K)$ for all $a \in \mathbb{R}^N$. By the method of geometric decomposition, if $K = K_1 \cup K_2$, $\mathrm{Vol}\big(K_1 \cap K_2\big) = 0$, and $\rho = \frac{\mathrm{Vol}(K_1)}{\mathrm{Vol}(K)}$, then $\mu(K) = \rho\mu(K_1) + (1 - \rho)\mu(K_2)$.

The set of imputations of a game $v \in G^N$ is defined as $I(v) = \big\{x \in H(v) : x_i \geq v(i) \text{ for all } i \in N\big\}$. Clearly, $I(v)$ is nonempty if and only if $\Delta = v(N) - \sum_{k \in N} v(k) \geq 0$. In that case, $I(v) \subset H(v)$ is the regular simplex spanned by the points $a^i = (a^i_1, \ldots, a^i_n) \in \mathbb{R}^N$, $i \in N$, where $a^i_j = \begin{cases} v(j) & \text{if } j \in N \backslash \{i\} \\ v(N) - \displaystyle\sum_{k \neq i} v(k) & \text{if } j = i \end{cases}$. When $v(N) = \sum_{k \in N} v(k)$, the imputation set is a singleton, $I(v) = \{(v(1), \ldots, v(n))\}$. Otherwise, $\sqrt{2}\Delta$ is the common edge length and the $(n-1)$-volume of $I(v)$ is $\mathrm{Vol}(I(v)) = \frac{\sqrt{n}}{(n-1)!}\Delta^{n-1}$. The center of gravity of $I(v)$ is the arithmetic mean of its extreme points, $\mu(I(v)) = \sum_{i=1}^{n} \frac{a^i}{n}$, so $\mu_i(I(v)) = v(i) + \frac{\Delta}{n}$ for all $i \in N$.

The core of a game $v \in G^N$ is the set $C(v) = \big\{x \in I(v) : x(S) \geq v(S) \text{ for all } S \in 2^N\big\}$. A game $v \in G^N$ is called balanced if its core is non-empty, i.e., $C(v) \neq \emptyset$. The allocations that belong to the core are called stable allocations. A game $v \in G^N$ is additive if $v(S) = \sum_{i \in S} v(i)$ for all $S \in 2^N$, in which case $C(v) = I(v) = \{\big(v(i)\big)_{i \in N}\}$. Thus, an additive game $v \in G^N$ is characterized by the vector $a = \big(v(i)\big)_{i \in N} \in \mathbb{R}^N$. To simplify the notation, we identify by the same letter both the vector and the additive game. Two games $v_1, v_2 \in G^N$ are said to be strategically equivalent if there exists $k > 0$ and an additive game $a \in G^N$ such that $v_1 = a + kv_2$. Observe that if the scale factor is $k = 1$, then v_1 is the translate of game v_2 by the vector a. A game $v \in G^N$ is zero-normalized if $v(i) = 0$ for each player $i \in N$. Given a game $v \in G^N$ the zero-normalization of v is the game $v_0 \in G^N$ defined by $v_0(S) = v(S) - \sum_{i \in S} v(i)$, $S \in 2^N$. Clearly, a game $v \in G^N$ and its zero-normalization $v_0 \in G^N$ are strategically equivalent, in fact, $v = a + v_0$ where $a = \big(v(i)\big)_{i \in N}$. A game $v \in G^N$ is convex if $v(S \cup T) + v(S \cap T) \geq v(S) + v(T)$ for all $S, T \in 2^N$. It is known that convex games are balanced, so the core $C(v)$ of a convex game $v \in G^N$ is a non-empty convex polytope.

The core-center (González-Díaz & Sánchez-Rodríguez, 2007) is the solution that associates to each balanced game $v \in G^N$ the stable allocation $\mu(v) = \mu(C(v))$, the centroid of the core. Clearly, the core-center of a balanced game $v \in G^N$ is the expectation of the uniform distribution over the core of the game: the average stable payoff.

O'Neill (1982) associates with each claims problem $(E, d) \in C^N$ a coalitional game $v \in G^N$ defined as $v(S) = \max\{0,\ E - d(N\backslash S)\}$ for all $S \in 2^N$. Note that, for each $i \in N$, $m_i(E, d) = v(i)$ and $t_i(E, d) = v(N) - v(N\backslash\{i\})$. Naturally, $H(v) = H(E, d)$ and $I(v) = I(E, d)$. Thomson (2019) shows that for each claims problem $(E, d) \in C^N$, the core of the associated coalitional game is its set of awards vectors, that is, $C(v) = X(E, d)$. Curiel et al. (1987) note that the coalitional game associated with a claims problem is convex. If $a = \big(v(i)\big)_{i \in N}$ then the coalitional game $v_0 \in G^N$ associated with the claims problem $\big(E - a(N), d - a\big) \in C^N$ is the zero-normalization of v.

3 The utopia games

Let $(E, d) \in C^N$ be a claims problem and $v \in G^N$ the associated coalitional game. Curiel et al. (1987) show that the core $C(v)$ consists of all efficient allocations which are bounded from below by the minimal rights and bounded from above by the truncated claims:

$$C(v) = X(E, d) = \big\{x \in H(E, d) : m_i(E, d) \leq x_i \leq t_i(E, d) \text{ for all } i \in N\big\}.$$

For instance, for a claims problem $(E, d) \in C^N$ with two claimants, $N = \{1, 2\}$, and $d = (d_1, d_2) \in \mathbb{R}^N$ such that $0 \leq d_1 \leq d_2$, it is clear that $X(E, d) = I(E, d)$ is the line segment with endpoints $\big(m_1(E, d), E - m_1(E, d)\big)$ and $\big(E - m_2(E, d), m_2(E, d)\big)$. Figure 1 shows a set of awards vectors for a three-claimant problem with the maximum number of extreme points. The next result, whose proof follows at once from the representation of $X(E, d)$ given above, identifies the claims problems, with at least three claimants, for which the set of awards vectors is either a regular simplex or it is not full dimensional.

Proposition 3.1 *Let $(E, d) \in C^N$ be a claims problem with $|N| \geq 3$. Then, $X(E, d) = I(E, d)$ if and only if $E \leq d_i$ for all $i \in N$ or $E = d(N)$. Moreover,* $\mathrm{Vol}(X(E, d)) = 0$ *if and only if $E = 0$ or $E = d(N)$ or $d_i = 0$ for some $i \in N$.*

Let $C_+^N = \{(E, d) \in C^N : 0 < E < d(N), d_i > 0 \text{ for all } i \in N\}$ be the class of claims problems with set of claimants N such that $\mathrm{Vol}(X(E, d)) > 0$. Our analysis is mainly concerned with claims problems that belong to C_+^N, that is, problems for which the set of awards vectors is full dimensional. For such problems either $X(E, d) = I(E, d)$ or, as we argued in the simple example presented in the Introduction, there are vectors belonging to a region with positive measure that award the minimal rights to each claimant but that clearly benefit a certain creditor or group of creditors.

Denote $\mathcal{P} = \{S \in 2^N : S \neq N\}$ the collection of all the proper coalitions of N. Given a claims problem $(E, d) \in C^N$ and a group of claimants $T \in \mathcal{P}$ we want to identify all the allocations that guarantee to the claimants in T at least $d(T)$ provided that the minimal rights of all the claimants are honored, that is, allocations $x \in I(E, d)$ such that $x(T) \geq d(T)$. Certainly, these are ideal, or utopian, payoffs for the members of T as a group, because they receive an award larger than their initial claim. We achieve this goal by associating to each proper coalition T a so called T-utopia claims problem. If the sum of the claims of the creditors in T is bigger or equal than the endowment, $d(T) \geq E$, then the T-utopia claims problem is just the problem where the claimants in $N\backslash T$ withdraw their claims.[1] But, if the aggregate claim of the agents in T is less than the endowment, $d(T) < E$, then first we give to the members of T their initial claims and to the others their minimal rights. So, the new

[1] In this case, the associated coalitional game corresponds to the T-face game defined by Mirás Calvo et al. (2020).

endowment is the remainder, each agent in T claims that endowment while the others claim what was not already satisfied.

Definition 3.2 Let $(E, d) \in C^N$ be a claims problem and $T \in \mathcal{P}$. The T-utopia claims problem $(\tilde{E}_T, \tilde{d}_T) \in C^N$ and the T-utopia additive game $a_T \in G^N$ are given by:

	$d(T) \geq E$	$d(T) < E$
$\tilde{E}_T$	E	$E - d(T) - \sum_{\ell \in N \backslash T} m_\ell(E, d)$
$\tilde{d}_T$	$(\tilde{d}_T)_i = \begin{cases} d_i & \text{if } i \in T \\ 0 & \text{if } i \notin T \end{cases}$	$(\tilde{d}_T)_i = \begin{cases} \tilde{E}_T & \text{if } i \in T \\ d_i - m_i(E, d) & \text{if } i \notin T \end{cases}$
a_T	$a_T(i) = 0, \ i \in N$	$a_T(i) = \begin{cases} d_i & \text{if } i \in T \\ m_i(E, d) & \text{if } i \notin T \end{cases}$

Let $\tilde{v}_T \in G^N$ be the coalitional game associated with the T-utopia claims problem $(\tilde{E}_T, \tilde{d}_T) \in C^N$. The T-utopia game is defined as $v_T = a_T + \tilde{v}_T \in G^N$.

Let us see that the T-utopia problem given in Definition 3.2 is in fact a claims problem. Clearly $(\tilde{E}_T, \tilde{d}_T) \in C^N$ when $d(T) \geq E$. Now, suppose that $d(T) < E$ and let $v \in G^N$ be the coalitional game associated with $(E, d) \in C^N$. Then $\tilde{E}_T \geq 0$, because, by convexity of game v, $E - d(T) = v(N \backslash T) \geq \sum_{\ell \in N \backslash T} v(\ell) = \sum_{\ell \in N \backslash T} m_l(E, d)$. In addition, $(\tilde{d}_T)_i \geq 0$ for all $i \in N$ and $\tilde{d}_T(N) = |T|\tilde{E}_T + \tilde{d}_T(N \backslash T) \geq \tilde{E}_T$.

In Appendix A we analyze several properties of the T-utopia game and its core. We prove that v_T is a convex game, $v_T(N) = E$, and $v_T(T) = E - v(N \backslash T)$. Certainly, when $d(T) \geq E$ the core of the T-utopia game has at most dimension $n - 2$, so it is not full dimensional. If $T = \emptyset$ then $v_T = v$ and $\tilde{v}_T$ is the zero-normalization of v. The T-utopia game v_T for any coalition T with $n - 1$ claimants and such that $d(T) < E$ is additive, so $v_T = a_T$ and its core is a singleton, $C(v_T) = \{a_T\}$. In summary, for each coalition $T \in \mathcal{P}$ such that either $d(T) \geq E$ or $|T| = n - 1$ the core of the T-utopia game is not full dimensional. Given a claims problem $(E, d) \in C_+^N$ consider the family of coalitions with at most $n - 2$ claimants for which the aggregate sum of claims does not exceed the endowment,

$$\mathcal{F} = \left\{T \in \mathcal{P} \colon |T| \leq n - 2, d(T) < E\right\}.$$

Note that the empty set belongs to $\mathcal{F}$. Proposition A.2 shows that the core of the T-utopia game v_T is full dimensional if and only if coalition T belongs to family $\mathcal{F}$.

4 A decomposition of the imputation set

Our main result states that the imputation set of a coalitional game associated with a claims problem that has full dimensional core is the union of the full dimensional cores of the T-utopia games, and that any two pieces of this decomposition have negligible intersection. The proof is given in Appendix B.[2]

[2] Obviously, it is also true that the imputation set is the union of the cores of all the T-utopia games, $I(E, d) = \bigcup_{T \in \mathcal{P}} C(v_T)$, because if $T \notin \mathcal{F}$ then $C(v_T)$ is not full dimensional. Therefore, these pieces of the decomposition are, to our purposes, redundant.

Theorem 4.1 *Let* $(E, d) \in C_+^N$ *and for each* $T \in \mathcal{F}$ *let* $v_T \in G^N$ *be the* T*-utopia game. Then*

$$I(E, d) = \bigcup_{T \in \mathcal{F}} C(v_T).$$

Moreover, if $T, R \in \mathcal{F}$, $T \neq R$, *then* $\text{Vol}_{n-1}\big(C(v_T) \cap C(v_R)\big) = 0$.

Given a claims problem, any allocation that shares the entire endowment and assigns to each claimant at least the minimal rights must either belong to the set of awards vectors for the claims problem or being a stable allocation for the T-utopia game of a particular group of claimants T. Moreover, the intersection of the cores of two utopia games corresponding to two different coalitions in $\mathcal{F}$ has null volume. We also show in Appendix B that the core of the T-utopia game of an inclusion-wise maximal set T of $\mathcal{F}$ coincides with the imputation set of that game, and therefore it is a regular simplex, that is, if T is a maximal element of $\mathcal{F}$ then $C(v_T) = I(v_T)$.

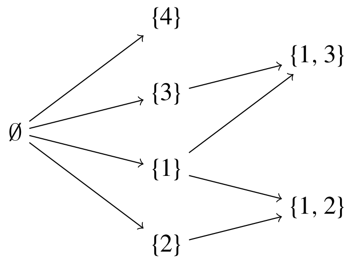
For each coalition $T \in \mathcal{F}$, the T-utopia game v_T is strategically equivalent to the coalitional game $\tilde{v}_T \in G^N$ associated with the T-utopia claims problem $(\tilde{E}_T, \tilde{d}_T) \in C_+^N$. Therefore, we can apply Theorem 4.1 to the T-utopia claims problem to obtain a decomposition of the imputation set of $\tilde{v}_T$ by the cores of the corresponding utopia games.

Let $T \in \mathcal{F}$ and denote $\mathcal{F}_T = \big\{S \in \mathcal{F} \colon S \supset T\big\}$. Observe that T is a maximal coalition of $\mathcal{F}$ if and only if $\mathcal{F}_T = \{T\}$. In that case $I(v_T) = C(v_T)$. In general, the imputation set of the T-utopia game v_T is the union of the cores of the utopia games corresponding to those coalitions that belong to $\mathcal{F}_T$. The proof is left to Appendix B.

Theorem 4.2 *Let* $(E, d) \in C_+^N$. *If* $T \in \mathcal{F}$ *then* $I(v_T) = \bigcup_{S \in \mathcal{F}_T} C(v_S)$.

Therefore, the tree structure of the inclusion relation on $\mathcal{F}$ determines the structure of the decomposition of the imputation set $I(v)$ through the cores of the T-utopia games. The inclusion-wise maximal sets of $\mathcal{F}$ contribute to the partition with pieces that are regular simplices. These pieces in turn are part of the decompositions of the utopia games corresponding to the subsets of the maximal elements. Repeating the process, going down the tree branches, we finally reach the empty set. Since the core of its utopia game is the set of awards vectors for the claims problem, $X(E, d) = C(v_\emptyset)$, the decomposition is then completed. We illustrate the process in the following example.

Example 4.3 Let $N = \{1, 2, 3, 4\}$ and consider the claims problem $(E, d) \in C_+^N$ with $E = 10$ and $d = (2, 4, 7, 9)$. The associated coalitional game $v \in G^N$ is given by: $v(i) = 0$ for all $i \in N$; $v(\{1, 2\}) = v(\{1, 3\}) = v(\{1, 4\}) = v(\{2, 3\}) = 0$; $v(\{2, 4\}) = 1$; $v(\{3, 4\}) = 4$; $v(\{1, 2, 3\}) = 1$; $v(\{1, 2, 4\}) = 3$; $v(\{1, 3, 4\}) = 6$; $v(\{2, 3, 4\}) = 8$; and $v(N) = 10$. Clearly, $\mathcal{F} = \big\{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{1, 3\}\big\}$. The following diagram represents the tree structure of family $\mathcal{F}$ ordered by inclusion.



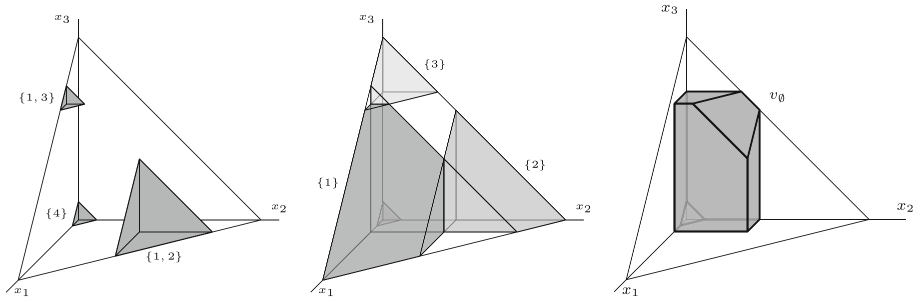


Fig. 2 The projection onto $\mathbb{R}^3$ of the T-utopia cores for $T \in \mathcal{F}$

The maximal elements of $\mathcal{F}$ are the coalitions {4}, {1, 2}, and {1, 3}. Therefore, $C(v_{\{4\}}) = I(v_{\{4\}})$, $C(v_{\{1,2\}}) = I(v_{\{1,2\}})$, and $C(v_{\{1,3\}}) = I(v_{\{1,3\}})$, whose projections onto $\mathbb{R}^3$ are the tetrahedrons shown in Fig. 2 (left). For any coalition $T \in \mathcal{F}$, the T-utopia additive game $a_T \in G^N$ and the T-utopia problem $(\tilde{E}_T, \tilde{d}_T) \in C_+^N$ are given in the next table:

	{1}	{2}	{3}	{4}	{1, 2}	{1, 3}
$\tilde{E}_T$	8	6	3	1	4	1
$\tilde{d}_T$	(8, 4, 7, 9)	(2, 6, 7, 9)	(2, 4, 3, 9)	(2, 4, 7, 1)	(4, 4, 7, 9)	(1, 4, 1, 9)
a_T	(2, 0, 0, 0)	(0, 4, 0, 0)	(0, 0, 7, 0)	(0, 0, 0, 9)	(2, 4, 0, 0)	(2, 0, 7, 0)

We have that $I(v_{\{1\}}) = C(v_{\{1\}}) \cup C(v_{\{1,2\}}) \cup C(v_{\{1,3\}})$, $I(v_{\{2\}}) = C(v_{\{2\}}) \cup C(v_{\{1,2\}})$, and $I(v_{\{3\}}) = C(v_{\{3\}}) \cup C(v_{\{1,3\}})$. The projections of the cores $C(v_{\{1\}})$, $C(v_{\{2\}})$, and $C(v_{\{3\}})$ are depicted in Fig. 2 (middle). Finally, since v is a zero-normalized game we have that $v_\emptyset = v$ and $I(v) = C(v) \cup C(v_{\{1\}}) \cup C(v_{\{2\}}) \cup C(v_{\{3\}}) \cup C(v_{\{4\}}) \cup C(v_{\{1,2\}}) \cup C(v_{\{1,3\}})$. The projected core of game v is shown in Fig. 2 (right).

5 The average-of-awards rule algorithm

One way of defining meaningful rules for claims problems is by applying game theoretical solutions to the associated coalitional game. The average-of-awards rule is the rule $\mathrm{AA}\colon C^N \to \mathbb{R}^N$ that assigns to each claims problem $(E, d) \in C^N$ the core-center of its associated coalitional game $v \in G^N$, that is, $\mathrm{AA}(E, d) = \mu(v) = \mu(C(v)) \in X(E, d)$. Since the core of the associated coalitional game coincides with the set of awards vectors for a claims problem, the average of awards rule is the expected value of the (continuous) uniform distribution over the set of awards vectors. This is an intuitive and simple way of selecting an allocation from this set: assume that all the awards vectors are equally likely and choose their expected value. Mirás Calvo et al. (2022b) analyze the AA rule in detail showing that it satisfies a good number of properties. Let us recall here the ones that are relevant for our purposes that, by the way, are straightforward. The AA rule satisfies:

- *Anonimity*: if for each $(E, d) \in C^N$, each bijection f from N into itself, and each $i \in N$, $\mathrm{AA}_i(E, d) = \mathrm{AA}_{f(i)}\big(E, (d_{f(i)})_{i \in N}\big)$.
- *Equal treatment of equals*[3]: if for each $(E, d) \in C^N$ such that $d_i = d_j$ we have $\mathrm{AA}_i(E, d) = \mathrm{AA}_j(E, d)$.

[3] Anonymity implies equal treatment of equals.

- *Null claims consistency*: if for each $(E, d) \in C^N$, and each $N' \subset N$, if $d(N \backslash N') = 0$ we have $\mathrm{AA}_{N'}(E, d) = \mathrm{AA}(E, d_{N'})$.
- *Self-duality*: if for each $(E, d) \in C^N$ we have $\mathrm{AA}(E, d) = d - \mathrm{AA}\big(d(N) - E, d\big)$.

Obviously, $\mathrm{AA}(0, d) = 0$, $\mathrm{AA}(d(N), d) = d$, and $\mathrm{AA}_i(E, d) = 0$ for each $i \in N$ such that $d_i = 0$. Moreover, by null claims consistency, in order to compute the AA rule, the agents whose claims are 0 can be removed, so we can restrict our analysis to the class C_+^N. But, if $(E, d) \in C_+^N$ then Theorems 4.1 and 4.2 allow us to apply the method of geometric decomposition and derive a backward recurrence algorithm to compute $\mathrm{AA}(E, d)$.

Let $(E, d) \in C_+^N$, $T \in \mathcal{F}$, $v_T \in G^N$ the T-utopia game, $a_T \in G^N$ the T-utopia additive game, and $\tilde{v}_T \in G^N$ the coalitional game associated with the T-utopia claims problem $(\tilde{E}_T, \tilde{d}_T) \in C_+^N$. The volumes of the imputation set and the core of game v_T, scaled by the factor $\alpha = \frac{\sqrt{n}}{(n-1)!}$, will be denoted $p_T^I = \frac{1}{\alpha}\operatorname{Vol}(I(v_T))$ and $p_T = \frac{1}{\alpha}\operatorname{Vol}(C(v_T))$.

First, we focus on the imputation set of the T-utopia game, $I(v_T)$. Consider the claims problem with initial endowment $\tilde{E}_T$ and n equal claims $(\tilde{E}_T, \ldots, \tilde{E}_T)$ whose associated coalitional game is given by $\tilde{v}_T^I(N) = \tilde{E}_T$ and $\tilde{v}_T^I(S) = 0$ if $S \neq N$. Then $v_T^I = a_T + \tilde{v}_T^I$; $I(v_T) = I(v_T^I) = C(v_T^I) = a_T + I(\tilde{v}_T^I)$; $p_T^I = (\tilde{E}_T)^{n-1}$; and $\mu_i(I(v_T)) = \mu_i(v_T^I) = a_T(i) + \frac{\tilde{E}_T}{n}$, for all $i \in N$.

Next, we turn our attention to the core of the T-utopia game, $C(v_T)$. According to Theorem 4.1 and Theorem 4.2, we can apply the method of geometric decomposition to obtain:

$$p_T^I = p_T + \sum_{\substack{S \in \mathcal{F}_T \\ S \neq T}} p_S \quad \text{and} \quad p_T^I \mu(v_T^I) = p_T \mu(v_T) + \sum_{\substack{S \in \mathcal{F}_T \\ S \neq T}} p_S \mu(v_S). \tag{1}$$

Denote $c = \max\{|T| \colon T \in \mathcal{F}\}$ the biggest cardinality of the coalitions in the family $\mathcal{F}$. If $c = 0$ then $\mathcal{F} = \{\emptyset\}$, $C(v) = I(v)$, and $\mathrm{AA}_i(E, d) = \frac{E}{n}$ for all $i \in N$. If $c > 1$, then for each $T \in \mathcal{F}$ with $|T| = c$, T is maximal so $C(v_T) = I(v_T)$, $p_T^I = p_T = (\tilde{E}_T)^{n-1}$, and $\mu_i(v_T) = a_T(i) + \frac{\tilde{E}_T}{n}$, for all $i \in N$. Then, we proceed backwards on the cardinality of the coalitions in $\mathcal{F}$. For each $T \in \mathcal{F}$ such that $|T| = t < c$, either T is maximal and so $C(v_T) = I(v_T)$ as before, or we can apply the equalities in (1) to compute p_T and $\mu(v_T)$, because $|S| > t$ whenever $S \in \mathcal{F}_T$, $S \neq T$. The procedure ends when $t = 0$, that is $T = \emptyset$, because $\mathrm{AA}(E, d) = \mu(C(v)) = \mu(C(v_\emptyset))$. The procedure is described in pseudocode form in Algorithm 1.

Example 5.1 Let $N = \{1, 2, 3\}$ and consider the claims problem $(E, d) \in C_+^N$ with $E = 4$ and $d = (3, 5, 5)$. Then $m(E, d) = (0, 0, 0)$ and $\mathcal{F} = \big\{\emptyset, \{1\}\big\}$, so $\{1\}$ is a maximal coalition of $\mathcal{F}$.

Therefore:

- $\tilde{E}_{\{1\}} = E - d_1 = 1$ and $a_{\{1\}} = (d_1, 0, 0) = (3, 0, 0)$.
- $p_{\{1\}}^I = (\tilde{E}_{\{1\}})^2 = 1$ and $\mu(v_{\{1\}}) = a_{\{1\}} + \big(\frac{\tilde{E}_{\{1\}}}{3}, \frac{\tilde{E}_{\{1\}}}{3}, \frac{\tilde{E}_{\{1\}}}{3}\big) = \big(\frac{10}{3}, \frac{1}{3}, \frac{1}{3}\big)$.
- $\tilde{E}_\emptyset = E = 4$, $a_\emptyset = (0, 0, 0)$, $p_\emptyset^I = (\tilde{E}_\emptyset)^2 = 16$ and $\mu(v_\emptyset^I) = \big(\frac{4}{3}, \frac{4}{3}, \frac{4}{3}\big)$.
- Since $c = \max\{|T| \colon T \in \mathcal{F}\} = 1$ then $p_{\{1\}} = p_{\{1\}}^I = 1$ and $p_\emptyset = p_\emptyset^I - p_{\{1\}} = 16 - 1 = 15$.

Finally,

$$AA(E, d) = \frac{1}{p_\emptyset}\big(p_\emptyset^I \mu(v_\emptyset^I) - p_{\{1\}}\mu(v_{\{1\}})\big) = \big(\tfrac{6}{5}, \tfrac{7}{5}, \tfrac{7}{5}\big).$$

The set of awards vectors $X(E, d)$ and the allocation $\mathrm{AA}(E, d)$ are shown in Fig. 3.

Algorithm 1 AA rule algorithm

procedure AA(E, d) ▷ The AA rule for $(E, d) \in C_+^N$
  $n \leftarrow |d|$ ▷ The number of claims
  $F \leftarrow T : d(T) < E, |T| \leq n-2$ ▷ The family $\mathcal{F}$
  $c \leftarrow \max\{|T| : T \in F\}$ ▷ The biggest cardinal
  for all $T \in F$ **do**
    $U(T) \leftarrow \left(\tilde{E}_T, a_T\right)$ ▷ The T-utopia game
    $p(T, I) \leftarrow \left(\tilde{E}_T\right)^{n-1}$ ▷ The imputation set volume: p_T^I
    $g(T, I) \leftarrow a_T + \frac{1}{n}\tilde{E}_T$ ▷ The imputation set center: μ_T^I
  end for
  for $t = c, c-1, \ldots, 0$ **do** ▷ Backward recurrence
    for all $T \in F : |T| = t$ **do**
      $p(T) \leftarrow p(T, I) - \sum_{\substack{S \in F \\ S \supset T}} p(S)$ ▷ The volume p_T
      $g(T) \leftarrow \frac{1}{p(T)}\left(p(T, I)\mu(T, I) - \sum_{\substack{S \in F \\ S \supset T}} p(S)g(S)\right)$ ▷ The center of gravity μ_T
    end for
  end for
  return $g(\emptyset)$ ▷ AA(E, d) is $\mu(\emptyset)$
end procedure

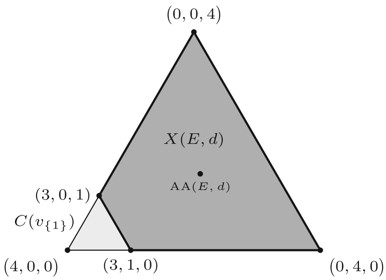


Fig. 3 The set of awards vectors and the AA allocation for the problem $\left(4, (3, 5, 5)\right)$

The AA rule is the unique rule for which the egalitarian division of the initial endowment E is the weighted sum of the allocations that the rule assigns to the T-utopia games with full dimensional core. The weights are the ratios of T-utopia allocations to stable allocations.

Working out the computations we obtain alternative expressions for all of the elements of the algorithm. In Appendix C we show that the AA rule is the core-center of a game, $v_\emptyset^* \in G^N$, that is strategically equivalent to the coalitional game $\tilde{v}_\emptyset^* \in G^N$ associated with a claims problem, $(\tilde{E}_\emptyset^*, \tilde{d}_\emptyset^*) \in C^N$ for which all of the claims are equal to the endowment. Therefore, since $\mu(\tilde{v}_\emptyset^*)$ coincides with the equalitarian division, initially, the AA rule divides the amount $\tilde{E}_\emptyset^*$ equally among the claimants. Then, each claimant's award is readjusted by adding the vector $a_\emptyset^*$.

Certainly, the algorithm can be improved upon by making use of some properties satisfied by the AA rule. For instance, by anonymity, we can assume that given a claims problem $(E, d) \in C^N$, the vector of claims $d = (d_1, \ldots, d_n) \in \mathbb{R}^N$ is sorted in ascending order, i.e., $d_1 \leq \cdots \leq d_n$. The AA rule is self-dual, so if the endowment is greater than the half-sum of claims, $E \geq \frac{1}{2}d(N)$, then $\text{AA}(E, d) = d - \text{AA}(d(N) - E, d)$ and $d(N) - E \leq \frac{1}{2}d(N)$. Therefore, we can restrict the algorithm to the claims problems for which $E \leq \frac{1}{2}d(N)$. But Mirás Calvo et al. (2022b) show that if $d(N \backslash \{n\}) \leq E \leq d_n$ then $\text{AA}_j(E, d) = \frac{d_j}{2}$

for all $j \in N\backslash\{n\}$ and $\text{AA}_n(E, d) = E - \frac{1}{2}d(N\backslash\{n\})$. As a consequence, we can always reduce the computation of the AA rule of a claims problem $(E, d) \in C_+^N$ to the case where $0 < E \leq \min\{\frac{1}{2}d(N), d(N\backslash\{n\})\}$. Let $\mathcal{I} = \{i \in N : \{i\} \in \mathcal{F}\}$ and $\chi(i) = 1$ if $\{i\} \in \mathcal{F}$ and $\chi(i) = 0$ otherwise. We further simplify the notation by writing p_i and p instead of $p_{\{i\}}$ and $p_\emptyset$, respectively.

Theorem 5.2 *Let $(E, d) \in C_+^N$ such that $E \leq \min\{\frac{1}{2}d(N), d(N\backslash\{n\})\}$. Then, for all $i \in N$, we have that*

$$\text{AA}_i(E, d) = \tfrac{1}{n}\Big(E + \sum_{j\in\mathcal{I}} \tfrac{p_j}{p} d_j\Big) - \chi(i)\tfrac{p_i}{p} d_i.$$

According to Theorem 5.2, in order to obtain $\text{AA}(E, d)$ we have to compute the ratios $\frac{p_j}{p}$ for the claimants such that $E > d_j$. Therefore, $\frac{p_j}{p}$ represents the ratio of utopia allocations for claimant j to awards vectors for the problem (E, d). We call this proportion the utopia ratio for claimant j. Now, let us increase the initial endowment by adding, for each claimant whose claim does not exceed the endowment, an amount equal to the utopia ratio times the claim, $E + \sum_{j\in\mathcal{I}} \frac{p_j}{p} d_j$. Then, the AA rule first acts as an egalitarian rule, dividing the endowment enlarged by the utopia ratios equally among the agents. Then, it takes away from each claimant the extra amount initially granted.

Example 5.3 Let $N = \{1, 2, 3, 4\}$ and consider the claims problem of Example 4.3, that is, $(E, d) \in C^N$ with $E = 10$ and $d = (2, 4, 7, 9)$. Recall that $\mathcal{F} = \big\{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{1, 3\}\big\}$ so $\mathcal{I} = \{\{1\}, \{2\}, \{3\}, \{4\}\}$. First, we compute p_T for the maximal elements of $\mathcal{F}$ that, as shown in Fig. 2, are tetrahedrons:

- $C(v_{\{4\}}) = I(v_{\{4\}})$ so $p_{\{4\}} = (E - d_4)^3 = (10 - 9)^3 = 1$.
- $C(v_{\{1,2\}}) = I(v_{\{1,2\}})$, so $p_{\{1,2\}} = (E - d_1 - d_2)^3 = (10 - 2 - 4)^3 = 64$.
- $C(v_{\{1,3\}}) = I(v_{\{1,3\}})$, so $p_{\{1,3\}} = (E - d_1 - d_3)^3 = (10 - 2 - 7)^3 = 1$.

Secondly, from the decompositions obtained in Example 4.3 we have:

- $I(v_{\{1\}}) = C(v_{\{1\}}) \cup C(v_{\{1,2\}}) \cup C(v_{\{1,3\}})$, so $(E - d_1)^3 = p_{\{1\}} + p_{\{1,2\}} + p_{\{1,3\}}$. Then $p_{\{1\}} = (10 - 2)^3 - 64 - 1 = 447$.
- $I(v_{\{2\}}) = C(v_{\{2\}}) \cup C(v_{\{1,2\}})$, so $(E - d_2)^3 = p_{\{2\}} + p_{\{1,2\}}$. Then $p_{\{2\}} = (10 - 4)^3 - 64 = 152$.
- $I(v_{\{3\}}) = C(v_{\{3\}}) \cup C(v_{\{1,3\}})$, so $(E - d_3)^3 = p_{\{3\}} + p_{\{1,3\}}$. Then $p_{\{3\}} = (10 - 7)^3 - 1 = 26$.

Finally, $I(v) = C(v) \cup C(v_{\{1\}}) \cup C(v_{\{2\}}) \cup C(v_{\{3\}}) \cup C(v_{\{4\}}) \cup C(v_{\{1,2\}}) \cup C(v_{\{1,3\}})$ and $p = p_\emptyset = E^3 - \sum_{T\in\mathcal{F}} p_T = 10^3 - 691 = 309$. Then, applying Theorem 5.2,

$$\begin{aligned}\text{AA}_1(E, d) =& \tfrac{1}{n}\Big(E + \sum_{j\in\mathcal{I}} \tfrac{p_j}{p} d_j\Big) - \chi(1)\tfrac{p_1}{p} d_1 \\ =& \frac{1}{4}\left(10 + \frac{447 \times 2 + 152 \times 4 + 26 \times 7 + 1 \times 9}{309}\right) - \frac{447 \times 2}{309} = \tfrac{1207}{1236}.\end{aligned}$$

Similar computations show that,

$$\text{AA}(E, d) = \left(\tfrac{1207}{1236}, \tfrac{2351}{1236}, \tfrac{1414}{431}, \tfrac{1494}{389}\right) = (0.9765, 1.9021, 3.2807, 3.8406).$$

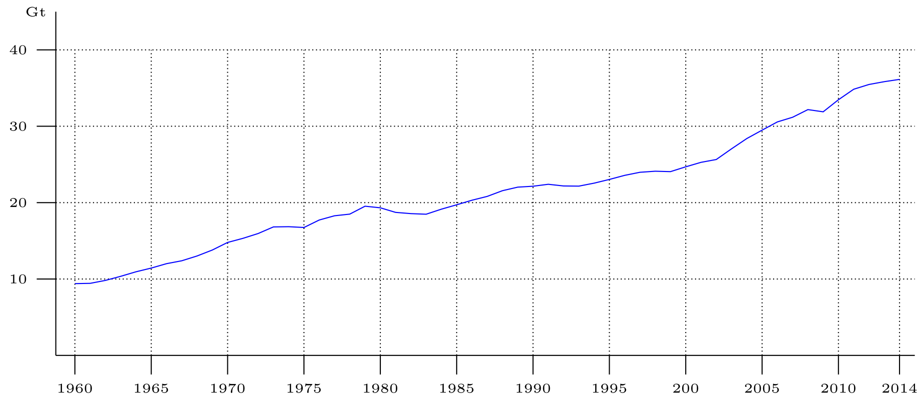


Fig. 4 World CO_2 emissions from 1960 to 2014

6 An application: CO_2 emissions

The United Nations Framework Convention on Climate Change (UNFCCC), created in 1992, is an international treaty that basically seeks to combat climate change by limiting average global temperature increases. Initially, 154 nations signed the UNFCCC (there are now 197 parties) that have met annually since the first Conference of the Parties (COP) took place on April 1995 in Berlin. At the 21st Conference of the Parties (COP 21) held in Paris on December 2015, parties to the UNFCCC reached a landmark agreement to combat climate change. The Paris Agreement central aim is: "to strengthen the global response to the threat of climate change, in the context of sustainable development and efforts to eradicate poverty, by holding the increase in the global average temperature to well below 2 degrees Celsius above pre-industrial levels and to pursuing efforts to limit the temperature increase to 1.5 degrees Celsius above pre-industrial levels, recognizing that this would significantly reduce the risk and impacts of climate change". The Paris Agreement entered into force on 4 November 2016. Figure 4 shows the World emissions of CO_2, measure in gigatonnes (Gt),[4] from 1960 to 2014 (source: Climate Change Data, World Bank Group).

In November 2019, the United Nations Environment Programme issued the tenth annual Emissions Gap Report: "It provides the latest assessment of scientific studies on current and estimated future greenhouse gas emissions and compares these with the emission levels permissible for the world to progress on a least-cost pathway to achieve the goals of the Paris Agreement. This difference between 'where we are likely to be and where we need to be' has become known as the emissions gap". The report tells us that total emissions reach a record high of 55.3 Gt of CO_2 equivalent in 2018. To get in line with the Paris Agreement, emissions must drop 7.6 per cent per year from 2020 to 2030 for the 1.5 °C goal and 2.7% per year for the 2 °C goal.

The world's countries emit vastly different amounts of greenhouse gases into the atmosphere. Even if all of them were fully committed to achieve the 1.5 °C goal, how to find an equitable share of the 7.6% per year drop on emissions? Giménez-Gómez et al. (2016) argue that: "framing climate negotiations as a classical conflicting claims problem may provide for an effective climate policy". In their analysis, the endowment is the available carbon budget and the claimants are the emitting countries. Duro Moreno et al. (2020) also use a

[4] $1Gt = 10^6 kt = 10^9 t = 10^{12} kg$.

Table 1 Selected countries/regions CO_2 emissions (kt) in 2014

1-China	*2-USA*	*3-India*	*4-Rest EU*	*5-Rest Asia*
10 291 926.878	5 225 412.661	2 232 729.957	2 095 334.801	1 848 538.367
6-Russia	*7-Western Asia*	*8-Japan*	*9-Rest Europe*	*10-Rest America*
1 736 984.560	1 256 361.871	1 206 674.021	1 131 240.164	919 404.908
11-Rest Africa	*12-Germany*	*13-Iran*	*14-Saudi Arabia*	*15-Republic of Korea*
897 886.952	720 363.815	652 392.303	601 046.969	587 156.373
16-Canada	*17-Brazil*	*18-South Africa*	*19-Mexico*	*20-Oceania*
540 614.809	533 530.165	484 495.041	481 499.102	413 861.287

claims approach to analyze some theoretical solutions through the establishment of equity and stability criteria. Heo and Lee (2022) present a dynamic claims problem and analyze CO_2 allocations over time. Following these models, we consider the 13 countries that emitted the most carbon dioxide in 2014: China, USA, India, Russia, Japan, Germany, Iran, Saudi Arabia, Republic of Korea, Canada, Brazil, South Africa, and Mexico. The year 2014 is the last for which there are data of CO_2 emissions available for all the world's countries from the Climate Change Data, World Bank Group. The remaining countries are grouped in 7 geographical regions: Rest of the European Union, Rest of Europe, Western Asia, Rest of Asia, Rest of America, Rest of Africa, and Oceania. Table 1 shows the estimated carbon dioxide emissions, in kilotons (kt), by the selected 20 claimants. We assume that, at least, each country commits to maintain its annual CO_2 emissions below the 2014 amount. Therefore, each emitter's claim d_j, $j \in N = \{1, \ldots, 20\}$, corresponds to its estimated emission in 2014. Naturally, we denote $d = (d_1, \ldots, d_{20})$.

According to the data presented in Table 1, the sum of CO_2 emissions in 2014 is $E_0 =$ 33 857 455.004 kt. We take the 2014 emissions as the ones valid for the year 2020 (whenever updated data become available, the analysis can be carried over with the new information). Now, the Emissions Gap Report points out that each year from 2021 to 2030, the total emissions must drop 7.6%. Therefore, for each $i \in \{1, \ldots, 10\}$, we consider the claims problem $(E_i, d) \in C^N$, where $E_i = (1 - 0.076)^i E_0$, that is:

E_1	E_2	E_3	E_4	E_5
31 284 288.424	28 906 682.503	26 709 774.633	24 679 831.761	22 804 164.547

E_6	E_7	E_8	E_9	E_{10}
21 071 048.042	19 469 648.391	17 989 955.113	16 622 718.524	15 359 391.916

Now, for each problem $(E_i, d) \in C^N$, we compute the recommendations made by the proportional rule (PRO), the Talmud rule (T), the random arrival rule (RA), and the average-of-awards rule (AA).[5]

The four rules provide different ways to share the emissions reduction among the polluters. We chose three countries (China, USA, and Saudi Arabia) and three regions (the rest of the European Union, the rest of Europe, and Oceania) to illustrate the results. Figure 5 shows the evolution of the CO_2 emissions reduction, from 2021 to 2030, recommended by the four rules for the aforementioned CO_2 emitters. We observe some clear patterns. The PRO and RA rules demand big reductions in the first years to the top polluters, China and the USA for example, while the T and AA rules dictate an initial lesser effort from these countries. Naturally, the

[5] The definitions of the PRO, T, and RA rules can be found in Appendix D.

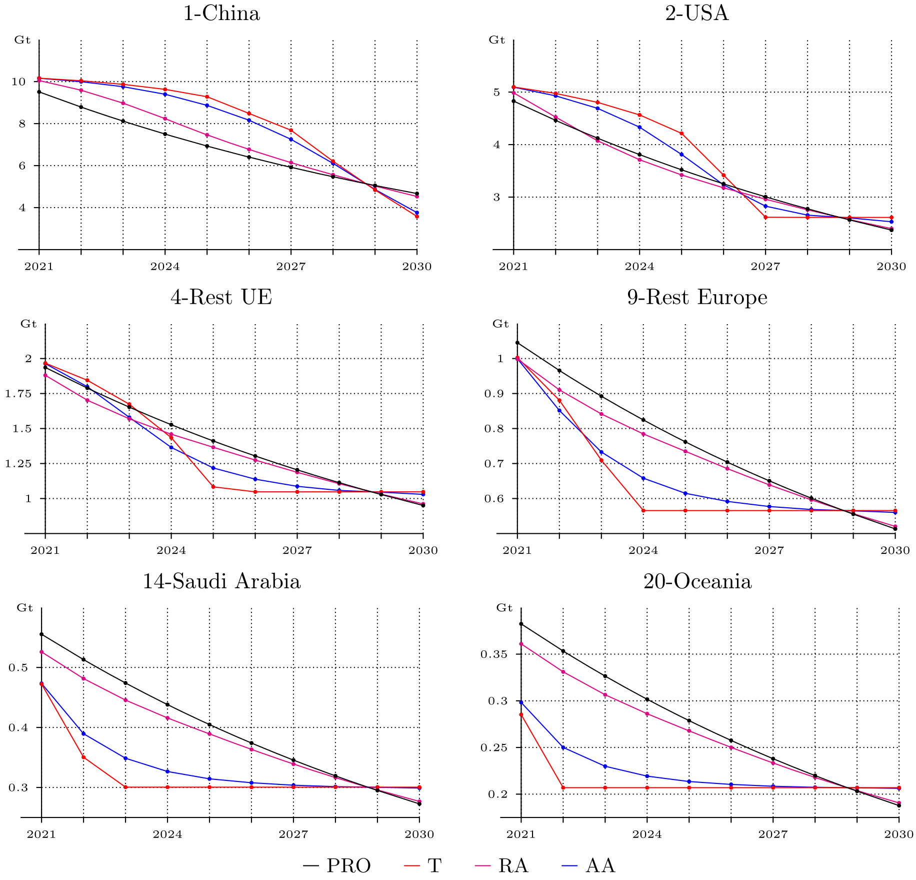


Fig. 5 Different emissions reduction patterns given by the four rules

situation is reversed for the countries/regions with the lowest emissions claims: the reductions in the first years are very severe with the T and AA rules but steadily decreasing with the other rules. As we see in the graphic corresponding to the rest of the European Union (the fourth polluter in the ranking) the behavior of the four rules is somehow similar. But, as we consider regions with lesser emissions claims, the rest of Europe for instance, the pattern of stricter reductions implied by the T and AA rules stars to emerge. Note that for all $i \in \{1, \dots, 8\}$, the endowment E_i is bigger than the half-sum of the claims, while $E_{10} < E_9 < \frac{1}{2}d(N)$. That explains why the paths depicted in each picture of Fig. 5 crossed right before the year 2029.

One can think of a continuous time version of our model by just considering for each $t \in [0, 10]$ the claims problem $(E_t, d) \in C^N$, where $E_t = E_0 e^{-0.076t}$. The path followed by the awards vector chosen by any of the four rules, say $\mathcal{R}$, as the time increases from 0 to 10, that is, the function $\mathcal{R}(t) = \mathcal{R}(E_t, d)$ is a dynamic strategy of emissions reduction. Now, a rule $\mathcal{R}$ satisfies endowment differentiability if $\mathcal{R}(\cdot, d)$ is a differentiable function of the endowment for all claims vector. Obviously, the PRO rule satisfies this property. When there are more than two claimants, Mirás Calvo et al. (2022b) show that the AA rule is endowment differentiable. Nevertheless, both the T and RA rules violate it. As a consequence, the path of emissions reduction corresponding to the PRO and AA rules do not present brisk changes in the rate of reduction. The graphs shown in Fig. 5 correspond to the discrete time approach but one can easily observe that the T rule does not vary smoothly.

Even though the stakes are very high, most of the countries do not comply with the commitments made. In particular, the top emitters tend to be more hesitant. We think that, qualitative, rules that demand a lesser effort to the top polluters in the first years and a bigger effort at the end of the period, such as the T and AA rules, are more realistic.

7 Concluding remarks

Theorem 5.2 provides an expression to compute the allocation recommended by the AA rule in terms of the claims problem initial data (the endowment E and the vector of claims d) by calculating the utopia ratios $\frac{p_i}{p}$. The algorithm that we present in this paper is a mechanism to obtain these ratios. But, one can rely on any other alternative method to compute the volumes p_i and p, for instance applying a general algorithm to compute the volume of a convex polyhedron such as Lasserre (1983).

Our algorithm can be easily adapted to compute the core-center solution for the class of two-bound core games (Gong et al., 2022). A balanced game $v \in G^N$ is a two-bound core game if there exist $l, u \in \mathbb{R}^N$ such that $C(v) = \{x \in \mathbb{R}^N : l_i \leq x_i \leq u_i \text{ for all } i \in N,\ x(N) = v(N)\big\}$, that is, if $C(v)$ is the intersection of the n-rectangle $\prod_{i \in N}[l_i, u_i]$ with the hyperplane $H(v)$. Given a two-bound core game $v \in G^N$ and a player $i \in N$, let $l_i^*(v) = \min_{x \in C(v)} x_i$ and $u_i^*(v) = \max_{x \in C(v)} x_i$. The vectors $l^*(v) = \big(l_i^*(v)\big)_{i \in N}$ and $u^*(v) = \big(u_i^*(v)\big)_{i \in N}$ are called the lower exact core bound and the upper exact core bound, respectively. Two-bound core games are closely related to claims problems. In fact, Gong et al. (2022) prove that if $v \in G^N$ is a two-bound core game then $C(v) = l^*(v) + X(E, d)$, where the claims problem $(E, d) \in C^N$ is given by $E = v(N) - \sum_{i \in N} l_i^*(v)$ and $d = u^*(v) - l^*(v)$. Consequently, the core-center of a two-bound core game $v \in G^N$ can be computed as:

$$\mu(v) = l^*(v) + \text{AA}\Big(v(N) - \sum_{i \in N} l_i^*(v), u^*(v) - l^*(v)\Big).$$

The class of two-bound core games includes, among others, 1-convex games (Driessen, 1986), big boss games (Muto et al., 1988), clan games (Potters et al., 1989), compromise stable games (Quant et al., 2005), and reasonable stable games (Dietzenbacher, 2018).

The analysis presented in this paper also applies to the computation of the centroid of any convex polytope that is the intersection of a rectangle with an efficiency-type hyperplane. That is the case, in the theory of coalitional games, of the core cover set (Tijs & Lipperts, 1982) and the reasonable set (Gerard-Varet & Zamir, 1987). The core cover of a game $v \in G^N$ is the set $\mathcal{CC}(v) = \big(\prod_{i \in N} [m_i(v), M_i(v)]\big) \cap H(v)$, where, for each $i \in N$, $M_i(v) = v(N) - v(N \backslash \{i\})$ and $m_i(v) = \max_{S \subset N:\, i \in S} \big\{v(S) - \sum_{j \in S \backslash \{i\}} M_j(v)\big\}$. A game $v \in G^N$ is compromise admissible if the core cover set is nonempty.[6] Estévez-Fernández et al. (2012) prove that if $v \in G^N$ is compromise admissible then $\mathcal{CC}(v) = m(v) + X\big(v(N) - \sum_{j \in N} m_j(v), M(v) - m(v)\big)$, where $m(v) = \big(m_i(v)\big)_{i \in N}$ and $M(v) = \big(M_i(v)\big)_{i \in N}$. Therefore, the centroid of the core cover set is $\mu(\mathcal{CC}(v)) = m(v) + \text{AA}\big(v(N) - \sum_{j \in N} m_j(v), M(v) - m(v)\big)$.

[6] A balanced game $v \in G^N$ is compromise stable if $C(v) = \mathcal{CC}(v)$.

A permutation on N is a bijection $\pi : \{1, \ldots, n\} \to N$, where $\pi(k)$ denotes the player at position k. The set of all permutations of N is denoted by Π^N. Let $v \in G^N$. For a given permutation $\pi \in \Pi^N$ of the player set N the marginal worth vector $m^\pi(v)$ is defined, for each $k \in N$, by $m_k^\pi(v) = v(S_k^\pi) - v(S_{k-1}^\pi)$, where $S_0^\pi = \emptyset$ and $S_k^\pi = \{\pi(j) : j \le k\}$. The reasonable set of $v \in G^N$ is given by $R(v) = \left\{x \in \mathbb{R}^N : \min_{\pi \in \Pi^N} m_i^\pi(v) \le x_i \le \max_{\pi \in \Pi^N} m_i^\pi(v)\right\} \cap H(v)$. Again, the centroid of the reasonable set can be computed by means of our algorithm.[7]

The algorithm developed in this paper works for the classes of games just mentioned above. González-Díaz et al. (2016) describe another particular procedure, based on a quite different approach, to calculate the core-center solution for airport games. But, as far as we know, no specific algorithm to compute the core-center solution for arbitrary balanced games is available, so one has to rely on general methods for the computation of the centroid of convex polyhedrons. It is an open question if the technique presented in this paper can be extended to other subclasses of balanced games, in particular, to convex games. Certainly, the core of a convex game has a more complex structure than the core of a compromise admissible game but it still has good properties (González-Díaz & Sánchez-Rodríguez, 2008). Recall that there are two key aspects that allow us to develop the algorithm to compute the centroid of the set of awards vectors for a claims problem, relying on the method of geometric decomposition. First, the imputation set of the associated game can be partitioned by the cores of the utopia games corresponding to the relevant coalitions, the ones that belong to family $\mathcal{F}$, because the set of awards vectors is the intersection of a rectangle with a hyperplane. Secondly, the core of the utopia game of an inclusion-wise maximal set of the family $\mathcal{F}$ is a regular simplex and, thus, its centroid is easy to obtain. To be able to generalize this method for convex games we face two challenges. First, utopia games have to be properly defined so that the imputation set admits a decomposition by their cores. Secondly, some pieces of the decomposition must be simple enough so that its centroid is known.

Acknowledgements This work was supported by grants PID2019-106281GB-I00 and PID2021- 124030NB-C33 that are funded by MCIN/AEI/10.13039/501100011033/ and by "ERDF A way of making Europe"/EU, and by grant ED481A 2021/325 funded by Programa de axudas á etapa predoutoral da Xunta de Galicia, Consellería de Educación, Universidade e Formación Profesional.

Funding Open Access funding provided thanks to the CRUE-CSIC agreement with Springer Nature. Funding for open access charge: Universidade de Vigo/CISUG.

[7] A game $v \in G^N$ is reasonable stable if $C(v) = R(v)$.

Appendix

A The core of the utopia games

Proposition A.1 *Let* $(E, d) \in C_+^N$, $v \in G^N$ *the coalitional game associated with* $(E, d) \in C^N$, $T \in \mathcal{P}$, $(\tilde{E}_T, \tilde{d}_T) \in C^N$ *the* T*-utopia claims problem,* $a_T \in G^N$ *the* T*-utopia additive game, and* $v_T \in G^N$ *the* T*-utopia game. We have that:*

1. v_T *is a convex game,* $v_T(N) = E$, $v_T(T) = E - v(N\backslash T)$, $I(v_T) = a_T + I(\tilde{v}_T)$, $C(v_T) = a_T + C(\tilde{v}_T)$, $\text{Vol}(I(v_T)) = \text{Vol}(I(\tilde{v}_T))$, $\text{Vol}(C(v_T)) = \text{Vol}(C(\tilde{v}_T))$, *and* $\mu(v_T) = a_T + \mu(\tilde{v}_T)$.
2. $v_T(S) \geq v(S)$ *if* $S \supset T$.
3. *If* $d(T) \geq E$ *then* $v_T = \tilde{v}_T$ *and* $\text{Vol}(C(v_T)) = 0$.
4. *If* $d(T) < E$ *then* $\tilde{v}_T$ *is a zero-normalized game and*
$$v_T(S) = \begin{cases} d(T) + v(S \cap (N\backslash T)) & \text{if } T \subset S \\ d(S \cap T) + \sum\limits_{l \in S \cap (N\backslash T)} v(l) & \text{otherwise.} \end{cases}$$
If, in addition, $|T| = n - 1$ *then* $v_T = a_T$ *is an additive games and its core is a singleton,* $C(v_T) = \{a_T\}$.
5. $v_\emptyset = v$ *and* $\tilde{v}_\emptyset$ *is the zero-normalization of* v.

Proof Note that $\tilde{v}_T \in G^N$ is a convex game, because it is the coalitional game associated with a claims problem. By definition, v_T and $\tilde{v}_T$ are strategically equivalent, so v_T is also a convex game. Since $v_T = a_T + \tilde{v}_T$, the equalities of the first statement are straightforward.

Let $S \in 2^N$ such that $S \supset T$. If $d(T) \geq E$ then $v_T(S) = \tilde{v}_T(S) = \max\{0, E - d((N\backslash S) \cap T)\} = \max\{0, E - d(N\backslash(S \cup (N\backslash T))\} = E = v(N)$. Since $v \in G^N$ is the game associated with (E, d) we conclude that $v_T(S) = v(N) \geq v(S)$. If $d(T) < E$, since $v(N\backslash T) = E - d(T) = v(N) - d(T)$, we have that $v_T(S) = v(S \cap (N\backslash T)) + d(T) = v(S \cap (N\backslash T)) + v(N) - v(N\backslash T) \geq v(S)$, where the last inequality holds by convexity of v. Therefore, the second statement is true.

Assume that $d(T) \geq E$. By Definition 3.2, $v_T = \tilde{v}_T$ so $C(v_T) = X(\tilde{E}_T, \tilde{d}_T)$. But, according to Proposition 3.1, $X(\tilde{E}_T, \tilde{d}_T)$ is not full dimensional because $\tilde{d}_T(N\backslash T) = 0$. That proves the third item.

Let $T \in \mathcal{P}$ such that $d(T) < E$. Then, we claim that, for all $S \in 2^N$,

$$\tilde{v}_T(S) = \max\left\{0,\ \tilde{E}_T - \tilde{d}_T(N\backslash S)\right\} = \begin{cases} v(S \cap (N\backslash T)) - \sum\limits_{\ell \in S \cap (N\backslash T)} v(\ell) & \text{if } T \subset S \\ 0 & \text{otherwise} \end{cases}. \quad (2)$$

Indeed, if there is $i \in T \cap (N\backslash S)$ then $\tilde{v}_T(S) = 0$ because $\tilde{d}_T(N\backslash S) = \tilde{E}_T + \tilde{d}_T(N\backslash(S \cup \{i\})) \geq \tilde{E}_T$. On the other hand, if $T \subset S$ then $N\backslash S \subset N\backslash T$, so:

$$\tilde{v}_T(S) = \max\left\{0,\ \tilde{E}_T - \tilde{d}_T(N\backslash S)\right\} = \max\left\{0, E - d(T) - \sum_{\ell \in N\backslash T} v(\ell) - \sum_{\ell \in N\backslash S} (d_\ell - v(\ell))\right\}$$
$$= \max\left\{0, E - d((N\backslash S) \cup T)) - \sum_{\ell \in S \cap (N\backslash T)} v(\ell)\right\}. \quad (3)$$

We distinguish two cases:

Case 1. If $E \leq d((N\backslash S)\cup T)$ then, from (3), $\tilde{v}_T(S) = 0$. By definition, $v(S \cap (N\backslash T)) = \max\{0, E - d((N\backslash S) \cup T)\} = 0$. Also, for each $\ell \in S \cap (N\backslash T)$ we have that $v(\ell) = \max\{0, E - d(N\backslash\{\ell\})\} = 0$. Therefore, $\tilde{v}_T(S) = v(S \cap (N\backslash T)) - \sum_{\ell \in S\cap(N\backslash T)} v(\ell) = 0$.

Case 2. If $E > d((N\backslash S) \cup T)$ then $v(S \cap (N\backslash T)) = E - d\big((N\backslash S) \cup T\big) > 0$. Since v is convex $v(S \cap (N\backslash T)) \geq \sum_{\ell \in S\cap(N\backslash T)} v(\ell)$. Therefore, by equality (3), we have $\tilde{v}_T(S) = E - d((N\backslash S) \cup T)) - \sum_{\ell \in S\cap(N\backslash T)} v(\ell)$ and, again, $\tilde{v}_T(S) = v(S \cap (N\backslash T)) - \sum_{\ell \in S\cap(N\backslash T)} v(\ell)$. We have just proved that equality (2) holds. In particular, $\tilde{v}_T(i) = 0$ for all $i \in N$, so $\tilde{v}_T$ is a zero-normalized game. Now, for all $S \in 2^N$, $a_T(S) = d(S \cap T) + \sum_{l \in S\cap(N\backslash T)} v(l)$ and $v_T = a_T + \tilde{v}_T$, so

$$v_T(S) = \begin{cases} v(S \cap (N\backslash T)) + d(T) & \text{if } T \subset S \\ d(S \cap T) + \sum_{l \in S\cap(N\backslash T)} v(l) & \text{otherwise} \end{cases} \cdot \tag{4}$$

In particular, let $T = N\backslash\{j\}$ for some $j \in N$ such that $d(T) < E$. Then $a_T(j) = v(j) = E - d(T)$ and $a_T(k) = d_k$ for $k \in T$. Then $v_T(N) = E = a_T(N)$ and if $S \in 2^N$, with $S \neq N$, $v_T(S) = a_T(S)$, because, by (3), $\tilde{v}_T(S) = 0$. Therefore, in fact, $v_T \in G^N$ coincides with the additive game defined by the vector $a_T \in \mathbb{R}^N$, so $C(v_T) = \{a_T\}$.[8] Finally, the last statement is straightforward. □

Proposition A.2 *Let* $(E, d) \in C_+^N$, $T \in \mathcal{P}$, *and* $v_T \in G^N$ *the* T*-utopia game. Then,* $\text{Vol}(C(v_T)) > 0$ *if and only if* $T \in \mathcal{F}$.

Proof Let $v \in G^N$ be the coalitional game associated with $(E, d) \in C_+^N$ and $\tilde{v}_T \in G^N$ the coalitional game associated with the T-utopia claims problem $(\tilde{E}_T, \tilde{d}_T) \in C^N$. By Proposition A.1, $\text{Vol}(C(v_T)) = \text{Vol}(C(\tilde{v}_T))$. We will show that $\text{Vol}(C(\tilde{v}_T)) > 0$ if and only if $T \in \mathcal{F}$. First, we prove sufficiency. Assume that $|T| \leq n-2$ and $d(T) < E$. Let us check that none of the conditions of Proposition 3.1 are satisfied by $(\tilde{E}_T, \tilde{d}_T) \in C^N$. Assume that $\tilde{E}_T = 0$ so $\sum_{\ell \in N\backslash T} v(\ell) = E - d(T) > 0$. Now, $N\backslash T$ has al least 2 players, since $|T| \leq n-2$. Denote $A = \{\ell \in N\backslash T : v(\ell) > 0\}$ and $B = \{j \in N\backslash T : v(j) = 0\}$. So, $N = A \cup B \cup T$ and $|A| \geq 1$ because $\sum_{\ell \in N\backslash T} v(\ell) > 0$. But,

$$\begin{aligned} E - d(T) &= \sum_{\ell \in N\backslash T} v(\ell) = \sum_{\ell \in A} v(\ell) = \sum_{\ell \in A} \big(E - d(N\backslash\{\ell\})\big) \\ &= |A|E - (|A| - 1)d(A) - |A|(d(T) + d(B)) \\ &= E - d(T) + (|A| - 1)\big(E - d(N)\big) - d(B). \end{aligned}$$

Therefore, $(|A| - 1)\big(E - d(N)\big) = d(B)$. So either $|A| = 1$ and $d(B) = 0$ in which case $|B| \geq 1$ and $d_i = 0$ for all $i \in B$; or $|A| > 1$ and $E \geq d(N)$. Since both situations lead to a contradiction, we conclude that, in fact, $\tilde{E}_T > 0$. Next, we show that $\tilde{d}_T(N) = |T|\tilde{E}_T + \tilde{d}_T(N\backslash T) > \tilde{E}_T$. Whenever $|T| > 1$ or $|T| = 0$ this property is obvious, and for $|T| = 1$ is a direct consequence of the fact that $v(j) = m_j(E, d) < d_j$ for all $j \in N$, since otherwise $E = d(N)$. For the same reason $(\tilde{d}_T)_i > 0$ for all $i \in N$.

[8] Observe that $a_T \in \mathbb{R}^N$ is a marginal worth vector of game $v \in G^N$.

To prove necessity, observe that if $d(T) \geq E$ then according to Proposition A.1, $\mathrm{Vol}(C(\tilde{v}_T)) = 0$. On the other hand, if $T = N \backslash \{i\}$ such that $d(T) < E$ then by Proposition A.1, $C(v_T) = \{a_T\}$. ◻

Proposition A.3 *Let $(E, d) \in C_+^N$, $T \in \mathcal{F}$, and $v_T \in G^N$ the T-utopia game. Then $I(v_T) \subset I(E, d)$ and $C(v_T) = \big\{x \in I(v_T) : x_i \leq v_T(N) - v_T(N \backslash \{i\})$ for all $i \in N \backslash T\big\}$.*

Proof Let $v \in G^N$ be the coalitional game associated with $(E, d) \in C_+^N$, $a_T \in G^N$ the T-utopia additive game, and $\tilde{v}_T \in G^N$ the coalitional game associated with the T-utopia claims problem $(\tilde{E}_T, \tilde{d}_T) \in C_+^N$. The imputation set $I(v_T)$ is the regular simplex spanned by the n points $b^i = a_T + \tilde{E}_T e^i \in \mathbb{R}^n$, $i \in N$. Therefore, to see that $I(v_T) \subset I(v)$ it suffices to prove that $b^i \in I(v)$ for all $i \in N$. So, let $i \in N$. Clearly, $\sum\limits_{j \in N} b_j^i = \sum\limits_{j \in T} d_j + \sum\limits_{j \in N \backslash T} v(j) + \tilde{E}_T = E$ so $b^i \in H(v)$. Besides, if $j \neq i$, $b_j^i = a_T(j) \geq v(j)$. But, if $i \in T$, then $b_i^i = \tilde{E}_T + d_i \geq d_i \geq v(i)$. Analogously, if $i \notin T$, then $b_i^i = \tilde{E}_T + v(i) \geq v(i)$.

On the other hand, since $C(v_T) = a_T + C(\tilde{v}_T) = a_T + \big\{\tilde{x} \in I(\tilde{v}_T) : \tilde{x}_i \leq \min\{\tilde{E}_T, (\tilde{d}_T)_i\}$ for all $i \in N\big\}$ and $\min\{\tilde{E}_T, (\tilde{d}_T)_i\} = \tilde{E}_T$ for all $i \in T$, the lower bound constraints for the players in T in the former representation of $C(v_T)$ are redundant. The result follows directly. ◻

B The imputation set and the utopia games

Proposition B.1 *Let $(E, d) \in C_+^N$, $T \in \mathcal{F}$, and $v_T \in G^N$ the T-utopia game. If T is a maximal element of $\mathcal{F}$ then $C(v_T) = I(v_T)$.*

Proof Let $v \in G^N$ and $\tilde{v}_T \in G^N$ be the coalitional games associated with the claims problem $(E, d) \in C_+^N$ and the T-utopia claims problem $(\tilde{E}_T, \tilde{d}_T) \in C_+^N$ respectively. If $T \in \mathcal{F}$ is maximal then $d(T) < E$ and $d(T) + d_j \geq E$ whenever $j \notin T$. We know that v_T is strategically equivalent to $\tilde{v}_T$. So according to Proposition 3.1, we just have to prove that $(\tilde{d}_T)_k \geq \tilde{E}_T$ for all $k \in N$. Indeed, $(\tilde{d}_T)_i = \tilde{E}_T$ if $i \in T$. But, if $j \notin T$, we have

$$
\begin{aligned}
(\tilde{d}_T)_j - \tilde{E}_T &= (d_j - v(j)) - \Big(E - d(T) - \sum_{\ell \in N \backslash T} v(\ell)\Big) = \big(d(T) + d_j - E\big) + \sum_{\ell \in N \backslash (T \cup \{j\})} v(\ell) \\
&\geq \big(d(T) + d_j - E\big) \geq 0
\end{aligned}
$$

because T is maximal. ◻

Lemma B.2 *Let $(E, d) \in C_+^N$, $T, R \in \mathcal{F}$ such that $R \subset N \backslash T$, and $v_T \in G^N$ the T-utopia game. Then $E - v_T(N \backslash R) \leq d(R)$.*

Proof Since $R \in \mathcal{F}$, $d(R) = E - v(N \backslash R)$. Besides, $N \backslash R \supset T$, and then by Proposition A.1, $v_T(N \backslash R) \geq v(N \backslash R)$. ◻

Lemma B.3 *Let $(E, d) \in C_+^N$, $T, R \in \mathcal{F}$ such that $T \neq R$, and $v_T, v_R \in G^N$ the T-utopia and R-utopia games, respectively. Then* $\mathrm{Vol}\big(C(v_T) \cap C(v_R)\big) = 0$.

Proof Let $v \in G^N$ be the coalitional game associated with $(E, d) \in C_+^N$. We show that there is a hyperplane separating $C(v_T)$ and $C(v_R)$. We distinguish three cases.
Case 1: Assume that $T = \emptyset$. Then $v_T = v$ and the hyperplane $x(R) = d(R)$ separates $C(v)$ and $C(v_R)$. Indeed, if $y \in C(v)$ then $y(R) = E - y(N \backslash R) \leq E - v(N \backslash R) = d(R)$. On the other hand, if $z \in C(v_R)$ then $z(R) \geq v_R(R) = d(R)$.

Case 2: Assume that $T, R \neq \emptyset$ and $R \cap T = \emptyset$. The hyperplane $x(T) = d(T)$ separates $C(v_R)$ and $C(v_T)$. Certainly, if $y \in C(v_T)$ then $y(T) \geq v_T(T) = d(T)$. On the other hand, if $z \in C(v_R)$ then $z(T) = E - z(N \backslash T) \leq E - v_R(N \backslash T) \leq d(T)$, where the last inequality holds by Lemma B.2.
Case 3: Assume that $R \cap T \neq \emptyset$ and that $|R| \geq |T|$. Then $S = R \cap (N \backslash T) \neq \emptyset$. Let $a_R \in G^N$ the R-utopia additive game, and $\tilde{v}_R \in G^N$ the coalitional game associated with the R-utopia claims problem $(\tilde{E}_R, \tilde{d}_T) \in C^N$. Trivially, $a_R(S) = d(S)$ and $\tilde{v}_R(S) = 0$. The hyperplane $x(S) = d(S)$ separates $C(v_R)$ and $C(v_T)$. Indeed, if $y \in C(v_R)$ then $y(S) \geq v_R(S) = a_R(S) + \tilde{v}_R(S) = d(S)$. On the other hand, let $z \in C(v_T)$. Since $S \subset N \backslash T$, we can apply Lemma B.2, so $z(S) = E - z(N \backslash S) \leq E - v_T(N \backslash S) \leq d(S)$. $\square$

Proof of Theorem 4.1

Let $v \in G^N$ be the coalitional game associated with $(E, d) \in C^N_+$. First, observe that if $\mathcal{F} = \{\emptyset\}$ then, according to Proposition 3.1, $I(v) = C(v)$. So, assume that $\mathcal{F}$ contains at least one non-empty coalition. Then, for all $T \in \mathcal{F}$, by Proposition A.3, we have that $C(v_T) \subset I(v_T) \subset I(v)$. Therefore, $I(v) \supset \bigcup_{T \in \mathcal{F}} C(v_T)$. To show that $I(v) \subset \bigcup_{T \in \mathcal{F}} C(v_T)$ it suffices to prove that if $x \in I(v) \backslash C(v)$ then there exists $T \in \mathcal{F}$ such that $x \in C(v_T)$. Take $T = \{i \in N : x_i > d_i\}$. First of all, since $x \in I(v)$ then $x_j \leq E$ for all $j \in N$. But if $x \notin C(v)$, there exists $i \in N$ for which $x_i > \min\{E, d_i\}$ and then $x_i > d_i$. Therefore, $|T| \geq 1$. Now, we show that $|T| \geq n - 1$ leads to a contradiction. Indeed, if $|T| = n$ then $E = x(N) > d(N)$. If $|T| = n - 1$ then $T = N \backslash \{j\}$ for some $j \in N$. Consequently, $x_j = E - x(N \backslash \{j\}) < E - d(N \backslash \{j\}) \leq v(j)$ and $x \notin I(v)$. Finally, $T \in \mathcal{F}$ because $d(T) < x(T) \leq x(N) = E$.

Let $v_T \in G^N$ be the T-utopia game. According to Proposition A.3, $x \in C(v_T)$ if $x \in I(v_T)$ and for any $i \notin T, x_i \leq v_T(N) - v_T(N \backslash \{i\})$. First we check that $x \in I(v_T)$. Clearly, if $i \notin T$, $v_T(i) = v(i) \leq x_i$ because $x \in I(v)$. But if $i \in T$, $x_i > d_i = v_T(i)$. It remains to be proved that if $i \notin T$ then $x_i \leq v_T(N) - v_T(N \backslash \{i\}) = \min\Big\{d_i, E - d(T) - \sum_{\ell \in N \backslash (T \cup \{i\})} v(\ell)\Big\}$. But, if $i \notin T$ then $x_i \leq d_i$. Now, the fact that $x \in I(v)$ implies $x_i = E - x(T) - \sum_{\ell \in N \backslash (T \cup \{i\})} x_\ell \leq E - d(T) - \sum_{\ell \in N \backslash (T \cup \{i\})} v(\ell)$.

The fact that if $T, R \in \mathcal{F}$, $T \neq R$, then $\mathrm{Vol}\big(C(v_T) \cap C(v_R)\big) = 0$, follows at once by Lemma B.3. $\square$

Proof of Theorem 4.2

Let $v \in G^N$ be the coalitional game associated with $(E, d) \in C^N_+$, $a_T \in G^N$ the T-utopia additive game, and $\tilde{v}_T \in G^N$ the coalitional game associated with the T-utopia claims problem $(\tilde{E}_T, \tilde{d}_T) \in C^N_+$. By Proposition A.1 and Theorem 4.1,

$$I(v_T) = a_T + I(\tilde{v}_T) = \bigcup_{R \in \tilde{\mathcal{F}}} C\big(a_T + (\tilde{v}_T)_R\big) \tag{5}$$

where $\tilde{\mathcal{F}} = \{R \in \mathcal{P}\colon \tilde{d}_T(R) < \tilde{E}_T, |R| \leq n - 2\}$ and $(\tilde{v}_T)_R \in G^N$ is the R-utopia game associated with the T-claims problem $(\tilde{E}_T, \tilde{d}_T) \in C^N_+$. Take $R \in \tilde{\mathcal{F}}$. First, observe that $R \subset N \backslash T$ (otherwise $(\tilde{d}_T)_i = \tilde{E}_T$ for each $i \in T \cap R$ and $\tilde{d}_T(R) \geq \tilde{E}_T$). Besides,

$\tilde{d}_T(R) = d(R) - \sum_{\ell \in R} v(\ell) < E - d(T) - \sum_{\ell \in N \setminus T} v(\ell) = \tilde{E}_T$. So, $d(R) + d(T) < E - \sum_{\ell \in N \setminus T} v(\ell) + \sum_{\ell \in R} v(\ell) < E$, and then, $T \cup R \in \mathcal{F}$ whenever $|T \cup R| \leq n - 2$.[9] Now,

$$\begin{aligned} \tilde{E}_T - \tilde{d}_T(R) &= E - d(T) - \sum_{\ell \in N \setminus T} v(\ell) - d(R) + \sum_{j \in R} v(j) \\ &= E - d(T \cup R) - \sum_{\ell \in N \setminus (T \cup R)} v(\ell) = \tilde{E}_{T \cup R} \end{aligned}$$

where the last equality holds since $T \cup R \in \mathcal{F}$. Let $a_{T \cup R} \in G^N$ the $(T \cup R)$-utopia additive game, and $\tilde{v}_{T \cup R} \in G^N$ the coalitional game associated with the $(T \cup R)$-utopia claims problem $(\tilde{E}_{T \cup R}, \tilde{d}_{T \cup R}) \in C_+^N$. By definition:

$$(a_{T \cup R})_i = \begin{cases} d_i & \text{if } i \in T \cup R \\ v(i) & \text{if } i \in N \setminus (T \cup R) \end{cases}, \quad (\tilde{d}_{T \cup R})_i = \begin{cases} \tilde{E}_{T \cup R} & \text{if } i \in T \cup R \\ d_i - v(i) & \text{if } i \in N \setminus (T \cup R) \end{cases}.$$

On the other hand, associated with the T-utopia claims problem $(\tilde{E}_T, \tilde{d}_T) \in C_+^N$ we have the R-utopia game $(\tilde{v}_T)_R \in G^N$. Denote by $a_{TR} \in G^N$ the R-utopia additive game and by $\tilde{v}_{TR} \in G^N$ the coalitional game associated with the claims problem $\left(\tilde{E}_{TR}, \tilde{d}_{TR}\right) \in C_+^N$ such that $(\tilde{v}_T)_R = a_{TR} + \tilde{v}_{TR}$. From Proposition A.1 we know that $\tilde{v}_T \in G^N$ is a zero-normalized game, so $\tilde{E}_{TR} = \tilde{E}_T - \tilde{d}_T(R) = \tilde{E}_{T \cup R}$. Moreover,

$$(a_{TR})_i = \begin{cases} d_i - v(i) & \text{if } i \in R \\ 0 & \text{if } i \in N \setminus R \end{cases}, \quad (\tilde{d}_{TR})_i = \begin{cases} \tilde{E}_{T \cup R} & \text{if } i \in R \\ \tilde{E}_T & \text{if } i \in T \\ d_i - v(i) & \text{if } i \in N \setminus (T \cup R) \end{cases}.$$

It is easy to check that $a_{T \cup R} = a_T + a_{TR}$. Since $(\tilde{d}_{TR})_j = (\tilde{d}_{T \cup R})_j$ for all $j \in N \setminus T$ and $(\tilde{d}_{TR})_i = \tilde{E}_T \geq \tilde{E}_{T \cup R} = (\tilde{d}_{T \cup R})_i$ for all $i \in T$ we conclude that $\tilde{v}_{TR} = \tilde{v}_{T \cup R}$. Therefore, $v_{T \cup R} = a_{T \cup R} + \tilde{v}_{T \cup R} = a_T + a_{TR} + \tilde{v}_{TR} = a_T + (\tilde{v}_T)_R$. Finally, from (5), we have $I(v_T) = \bigcup_{R \in \tilde{\mathcal{F}}} C\left(a_T + (\tilde{v}_T)_R\right) = \bigcup_{R \in \tilde{\mathcal{F}}} C\left(v_{T \cup R}\right) = \bigcup_{S \in \mathcal{F}_T} C(v_S)$. □

C The algorithm

Proposition C.1 *Let $(E, d) \in C_+^N$, $v \in G^N$ the associated coalitional game, $T \in \mathcal{F}$, $v_T \in G^N$ the T-utopia game, $a_T \in G^N$ the T-utopia additive game, and $\tilde{v}_T \in G^N$ the coalitional game associated with the T-utopia claims problem $(\tilde{E}_T, \tilde{d}_T) \in C_+^N$. Let $\tilde{E}_T^* > 0$ and $a_T^* \in \mathbb{R}^N$, defined by backward recurrence as:*

$$\tilde{E}_T^* = \begin{cases} \tilde{E}_T & \textit{if } |T| = c \\ \frac{p_T^I}{p_T} \tilde{E}_T - \sum\limits_{\substack{S \in \mathcal{F}_T \\ S \neq T}} \frac{p_S}{p_T} \tilde{E}_S^* & \textit{otherwise} \end{cases}, \quad a_T^* = \begin{cases} a_T & \textit{if } |T| = c \\ \frac{p_T^I}{p_T} a_T - \sum\limits_{\substack{S \in \mathcal{F}_T \\ S \neq T}} \frac{p_S}{p_T} a_S^* & \textit{otherwise} \end{cases}.$$

Let $(\tilde{E}_T^*, \tilde{d}_T^*) \in C^N$ be the claims problem such that $(\tilde{d}_T^*)_i = \tilde{E}_T^*$ for all $i \in N$, $\tilde{v}_T^* \in G^N$ the associated coalitional game, and $v_T^* = a_T^* + \tilde{v}_T^* \in G^N$. Then:

[9] If $|T \cup R| \geq n - 2$, then the core of the $(T \cup R)$-utopia game is not full dimensional.

1. $p_T = (\tilde{E}_T)^{n-1} - \sum_{\substack{S\in\mathcal{F}_T \\ S\neq T}} p_S$.
2. $\tilde{E}^*_T = \tilde{E}_T + \sum_{\substack{S\in\mathcal{F}_T \\ |S|=|T|+1}} \frac{p_S}{p_T}\big(\tilde{E}_T - \tilde{E}_S\big)$ *and* $\tilde{E}^*_T \geq \tilde{E}_T \geq \tilde{E}_S > 0$, *for all* $S \in \mathcal{F}_T$.
3. $a^*_T(i) = d_i$ *if* $i \in T$ *and* $a^*_T(i) = v(i) - \frac{p_S}{p_T}(d_i - v(i))$ *if* $i \notin T$, $S = T \cup \{i\}$.
4. $I(\tilde{v}^*_T) = C(\tilde{v}^*_T)$; $\mu_i(\tilde{v}^*_T) = \frac{\tilde{E}^*_T}{n}$, *for all* $i \in N$; *and* $\mu_i(v_T) = \mu_i(v^*_T) = a^*_T(i) + \frac{\tilde{E}^*_T}{n}$, *for all* $i \in N$.

Proof The first statement follows from the properties of the imputation set $I(v_T)$ and equality (1). Doing some algebraic manipulations, it can be proved, by backward induction on the cardinality of T, the equality of the second statement. Clearly, $\tilde{E}_T \geq \tilde{E}_S > 0$, for all $S \in \mathcal{F}_T$. That, in turn, implies $\tilde{E}^*_T \geq \tilde{E}_T$. Analogously, it can be proved that $a^*_T(i) = a_T(i) + \sum_{\substack{S\in\mathcal{F}_T \\ |S|=|T|+1}} \frac{p_S}{p_T}\big(\tilde{E}_T - \tilde{E}_S\big)$ for all $i \in N$. The expressions in the third statement are now straightforward. If T is a maximal coalition of $\mathcal{F}$ then $\tilde{E}^*_T = \tilde{E}_T$ and $C(v_T) = I(v_T)$. Now, let us proceed by backward recurrence on the cardinality of those coalitions $T \in \mathcal{F}$ such that there exists $S \in \mathcal{F}_T$, $S \neq T$. Then, in every step, from the equalities in (1), we have,

$$p^I_T \mu(a_T + \tilde{v}^I_T) = p_T \mu(v_T) + \sum_{\substack{S\in\mathcal{F}_T \\ S\neq T}} p_S \mu(a^*_S + \tilde{v}^*_S).$$

Then, $\mu(v_T) = \mu\Big(\big(\frac{p^I_T}{p_T} a_T - \sum_{\substack{S\in\mathcal{F}_T \\ S\neq T}} \frac{p_S}{p_T} a^*_S\big) + \big(\frac{p^I_T}{p_T}\tilde{v}^I_T - \sum_{\substack{S\in\mathcal{F}_T \\ S\neq T}} \frac{p_S}{p_T}\tilde{v}^*_S\big)\Big) = \mu(a^*_T + \tilde{v}^*_T) = a^*_T + \mu(\tilde{v}^*_T)$. Obviously, $a^*_T \in G^N$ is an additive game, $\tilde{v}^*_T(N) = \tilde{E}^*_T$, and $\tilde{v}^*_T(S) = 0$ if $S \neq N$. □

Proof of Theorem 5.2

Let $v \in G^N$ be the coalitional game associated with $(E, d) \in C^N_+$. Since $E \leq \min\{\frac{1}{2}d(N), d(N\backslash\{n\})\}$, we conclude that v is a zero-normalized game. We know, from Proposition C.1, that when $T = \emptyset$ there is $E^* = E + \sum_{i\in\mathcal{I}} \frac{p_i}{p} d_i > 0$ and a vector $a^* \in \mathbb{R}^n$ such that $\mu_i(v) = a^*_i + \frac{E^*}{n}$ for all $i \in N$. Moreover, $v(i) = 0$ for $i \in N$ implies $a^*(i) = -\frac{p_i}{p} d_i$ if $i \in \mathcal{I}$ and $a^*(i) = 0$ otherwise. The result is now straightforward. □

D The rules

- Proportional rule (PRO): For each $(E, d) \in C^N$ and each $i \in N$, $\text{PRO}_i(E, d) = \frac{d_i}{d(N)} E$ if $d(N) \neq 0$ and $\text{PRO}_i(E, 0) = 0$.
- Constrained equal awards rule (CEA): For each $(E, d) \in C^N$ and each $i \in N$, $\text{CEA}_i(E, d) = \min\{\alpha, d_i\}$, where $\alpha \geq 0$ is chosen such that $E = \sum_{i\in N} \text{CEA}_i(E, d)$.
- Talmud rule (T): For each $(E, d) \in C^N$ and each $i \in N$,

$$\text{T}_i(E, d) = \begin{cases} \text{CEA}_i(E, \frac{d}{2}) & \text{if } E \leq \frac{1}{2}d(N) \\ d_i - \text{CEA}_i(d(N) - E, \frac{d}{2}) & \text{if } E \geq \frac{1}{2}d(N) \end{cases}.$$

- Random arrival rule (RA): For each $(E, d) \in C^N$ and each $i \in N$,

$$\mathrm{RA}_i(E, d) = \frac{1}{|N|!} \sum_{\pi \in \Pi^N} \min\big\{d_i, \max\{0, E - d(P_\pi(i))\}\big\},$$

where Π^N is the set of strict orders on N and $P_\pi(i) = \{j \in N : \pi(j) < \pi(i)\}$ for $\pi \in \Pi^N$.

References

Aumann, R. J., & Maschler, M. (1985). Game theoretic analysis of a bankruptcy problem from the Talmud. *Journal of Economic Theory, 36*, 195–213.

Curiel, I. J., Maschler, M., & Tijs, S. H. (1987). Bankruptcy games. *Zeitschrift für Operations Research, 31*, A143–A159.

Dietzenbacher, B. (2018). Bankruptcy games with nontransferable utility. *Mathematical Social Sciences, 92*, 16–21.

Driessen, T. (1986). k-convex n-person games and their cores. *Zeitschrift für Operations Research, 30*, A49–A64.

Duro Moreno, J., Giménez-Gómez, J., & Vilella Bach, C. (2020). The allocation of CO_2 emissions as a claims problem. *Energy Economics, 86*, 104652.

Estévez-Fernández, A., Fiestras-Janeiro, M. G., Mosquera, M. A., & Sánchez-Rodríguez, E. (2012). A bankruptcy approach to the core cover. *Mathematical Methods of Operations Research, 76*, 195–213.

Gerard-Varet, L. A., & Zamir, S. (1987). Remarks on the reasonable set of outcomes in a general coalition function form game. *International Journal of Game Theory, 16*, 123–143.

Giménez-Gómez, J. M., Teixidó-Figueras, J., & Vilella, C. (2016). The global carbon budget: A conflicting claims problem. *Climatic Change, 136*, 693–703.

Gong, D., Dietzenbacher, B., & Peters, H. (2022). Two-bound core games and the nucleolus. *Annals of Operations Research*. https://doi.org/10.1007/s10479-022-04949-0

González-Díaz, J., Mirás Calvo, M. A., Quinteiro Sandomingo, C., & Sánchez Rodríguez, E. (2016). Airport games: The core and its center. *Mathematical Social Sciences, 82*, 105–115.

González-Díaz, J., & Sánchez-Rodríguez, E. (2007). A natural selection from the core of a TU game: The core-center. *International Journal of Game Theory, 36*, 27–46.

González-Díaz, J., & Sánchez-Rodríguez, E. (2008). Cores of convex and strictly convex games. *Games and Economic Behavior, 62*, 100–105.

Heo, E., & Lee, J. (2022). Allocating CO_2 emissions: A dynamic claims problem. *Review of Economic Design*. https://doi.org/10.1007/s10058-021-00286-z

Lasserre, J. B. (1983). An analytical expression and an algorithm for the volume of a convex polyhedron in $\mathbb{R}^n$. *Journal of Optimization Theory and Applications, 39*, 363–377.

Mirás Calvo, M. A., Núñez Lugilde, I., Quinteiro Sandomingo, C., & Sánchez Rodríguez, E. (2022). Deviation from proportionality and Lorenz-domination for claims problems. *Review of Economic Design*. https://doi.org/10.1007/s10058-022-00300-y

Mirás Calvo, M. A., Núñez Lugilde, I., Quinteiro Sandomingo, C., & Sánchez Rodríguez, E. (2023). An operational toolbox for solving conflicting claims problems. *Decision Analytics Journal, 6*, 100160.

Mirás Calvo, M. A., Quinteiro Sandomingo, C., & Sánchez Rodríguez, E. (2020). The boundary of the core of a balanced game: Face games. *International Journal of Game Theory, 49*, 579–599.

Mirás Calvo, M. A., Quinteiro Sandomingo, C., & Sánchez Rodríguez, E. (2022). The average-of-awards rule for claims problems. *Social Choice and Welfare, 59*, 863–888.

Muto, S., Nakayama, M., Potters, J., & Tijs, S. (1988). On big boss games. *The Economic Studies Quarterly, 39*, 303–321.

Núñez Lugilde, I., Mirás Calvo, M.A., Quinteiro Sandomingo, C., & Sánchez Rodríguez, E. (2023). *ClaimsProblems: Analysis of Conflicting Claims*, R package version 0.2.1.

O'Neill, B. (1982). A problem of rights arbitration from the Talmud. *Mathematical Social Sciences, 2*, 345–371.

Potters, J., Poos, R., Tijs, S., & Muto, S. (1989). Clan games. *Games and Economic Behaviour, 1*, 275–293.

Quant, M., Borm, P., Reijnierse, H., & van Velzen, B. (2005). The core cover in relation to the nucleolus and the Weber set. *International Journal of Game Theory, 33*, 491–503.

Thomson, W. (2019). *How to divide when there isn't enough. From Aristotle, the Talmud, and Maimonides to the axiomatics of resource allocation*, Cambridge University Press.

Tijs, S. H., & Lipperts, F. A. S. (1982). The hypercube and the core cover of n-person cooperative games. *Cahiers du Centre d'Études de Researche Opérationelle, 24*, 27–37.
United Nations Environment Program. (2019). *Emissions Gap Report 2019*. UNEP.