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ARL Evaluation of a DEWMA Control Chart for Autocorrelated Data: A Case Study on Prices of Major Industrial Commodities

Yadpirun Supharakonsakun [1], Yupaporn Areepong [2*]

[1] *Faculty of Science and Technology, Phetchabun Rajabhat University, Phetchabun, 67000 Thailand.*

[2] *Faculty of Applied Science, King Mongkut's University of Technology Bangkok, Bangkok,10800 Thailand.*

Abstract

The double exponentially weighted moving average (DEWMA) control chart, an extension of the EWMA control chart, is a useful statistical process control tool for detecting small shift sizes in the mean of processes with either independent or autocorrelated observations. In this study, we derived explicit formulas to compute the average run length (ARL) for a moving average of order q (MA(q)) process with exponential white noise running on a DEWMA control chart and verified their accuracy by comparison with the numerical integral equation (NIE) method. The results for both were in good agreement with the actual ARL. To investigate the efficiency of the proposed procedure on the DEWMA control chart, a performance comparison between it and the standard and modified EWMA control charts was also conducted to determine which provided the smallest out-of-control ARL value for several scenarios involving MA(q) processes. It was found that the DEWMA control chart provided the lowest out-of-control ARL for all cases of varying the exponential smoothing parameter and shift size values. To illustrate the efficacy of the proposed methodology, the presented approach was applied to datasets of the prices of several major industrial commodities in Thailand. The findings show that the DEWMA procedure performed well in almost all of the scenarios tested.

Keywords:

Explicit Formulas;
Moving Average Process;
Autocorrelated Observation;
Statistical Process Control;
Average Run Length.

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1- Introduction

Statistical process control (SPC) is very important for verifying the quality of a product and also monitoring and improving the process for its manufacturing. One of the tools used to achieve this is the control chart. They have been applied in various fields to monitor a process and detect changes therein. Well-known ones include the Shewhart, cumulative sum (CUSUM) [1], and exponentially weighted moving average (EWMA) [2] control charts. The Shewhart control chart is good for detecting large changes in the process mean, whereas the CUSUM and EWMA control charts are better at detecting small-to-moderate shifts. The double EWMA (DEWMA) control chart is an extension of the EWMA control scheme carried out by using exponential smoothing parameters [3, 4]. The DEWMA control chart is good for detecting small sustained shifts in the mean of a process with normally distributed observations and performs better than the EWMA control chart in detecting small shifts in the process mean ranging from 0.1 to 0.5 of the process standard deviation [5].

Comparing its performance with that of the standard EWMA control chart can be evaluated based on the zero-stat performance [6]. It can be said that the DEWMA control procedure with the larger exponential smoothing parameter values ($\lambda > 0.05$) performs better than the EWMA control chart in detecting very small sustained shifts of the process mean. Besides, the DEWMA control chart was used to monitor Poisson data. The simulation results show that the

* **CONTACT**: yupaporn.a@sci.kmutnb.ac.th

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DEWMA control chart is more sensitive to small downward process mean shifts than the EWMA control chart [7]. The modified EWMA (MEWMA) control chart is highly effective at detecting small and abrupt changes in the mean of processes comprising independent normally distributed or autocorrelated observations. Various experimental studies on processes involving non-normal distributions [10–12] and real data [13–15] indicate that the MEWMA control chart is more effective at detecting changes in the process mean than the EWMA control chart.

Time-series data are habitually autocorrelated, which can be serially correlated over a long period of discretely timed observations. The degree of correlation of the same variable over two successive time intervals can be measured by how much the lagged version of the observations of the variable is related to the original observations. For various situations, the analysis of autocorrelation helps to determine repeating periodic patterns in the form of a mathematical representation of the degree of similarity between the original and lagged versions over a long time period. Autocorrelation can provide information about short-term trends to support the prediction of future points with a short holding period. The autoregressive moving average (ARMA) model [16–18] is an amalgamation of autoregressive (AR) [19–21] and moving average (MA) [22–24] models.

Control charts have been employed for monitoring and detecting processes in which the observations are independently normally distributed [25–27], non-normally distributed [11, 28, 29], or autocorrelated [30–32]. The measurement of a control chart's performance can be undertaken to ensure that it is appropriate for a particular process, for which the average run length (ARL) is the most often used evaluation method. It represents the average number of observations until an out-of-control signal occurs. Monte Carlo simulation [33, 34] and the numerical integral equation (NIE) method [35–37] are the most often used schemes to compute the ARL. Moreover, the exact ARL has been employed in several studies [38–42]. From the outcomes of comparative studies, the EWMA control chart is more powerful than the CUSUM control chart for monitoring and detecting small and abrupt changes in the mean of a process involving autocorrelated observations [43, 44], while the MEWMA control chart is more effective than either of them in this scenario [14, 18, 20, 31, 32]. However, deriving the ARL for an autocorrelated process with exponential white noise running on a DEWMA control chart using explicit formulas has not previously been reported. Thus, this became the focus of the present study. A comparison of explicit formula-derived ARLs of MA(q) processes with exponential white noise running on DEWMA, MEWMA, and EWMA control charts is also presented.

2- The Characteristic of Control Chart Investigation

2-1- EWMA Control Chart

The EWMA control chart was the first in the quality control literature proposed by Roberts (1959) [2]. It has been widely applied in statistical process control (SPC) due to their performance is more powerful than exist charts in continually monitors and detects small changes in the process mean. The past and current observations are considered to create the control statistic with their weighted average. The EWMA control statistic can be written by the recursive equation as,

$$Z_i = \lambda X_i + (1-\lambda)Z_{i-1}, \quad i = 0, 1, 2, \dots, n. \tag{1}$$

where λ is an exponential smoothing parameter, which is $0 < \lambda < 1$. The initial value is defined by $Z_0 = X_0$ with the target value μ_0, X_t is the process with mean μ and variance σ^2.

The variance of $Z_i$is $\sigma^2_{Z_t} = \sigma^2\left(\frac{\lambda}{2-\lambda}\right)(1-(1-\lambda)^{2t})$. If t gets large, the term $(1-\lambda)^{2t}$ converge to 0. Therefore, the general upper control limit (UCL), center line (CL) and lower control limit (LCL) to detect the sequence Z_i are given by

$$\begin{aligned} UCL &= \mu_0 + L_1\sigma\sqrt{\tfrac{\lambda}{2-\lambda}}, \\ CL &= \mu_0, \\ LCL &= \mu_0 - L_1\sigma\sqrt{\tfrac{\lambda}{2-\lambda}}. \end{aligned} \tag{2}$$

where μ_0 is the target mean, σ is the process standard deviation, and L_1 is the appropriate control width limit.

2-2- Double EWMA Control Chart

A DEWMA control chart was the initially introduced by Brown (1962). He indicated a different context to forecast future time series observations. Later, Shama & Shamma (1992) developed and evaluated the idea of using the increase of the sensitivity of the EWMA control chart to smaller shifts in the process mean via a double exponential weighting of moving averages implementation. The recursive control statistic of the DEWMA chart is defined as [45]:

$$Z_i = \lambda_2 X_i + (1-\lambda_2) Z_{i-1},$$
$$W_i = \lambda_1 Z_i + (1-\lambda_1) W_{i-1}, i = 1,2,\ldots, \tag{3}$$

where λ_1 and λ_2 is an exponential smoothing parameter, which is $0 < \lambda_1, \lambda_2 \leq 1$. The initial value is defined by $Z_0 = W_0 = X_0$ with the target value μ_0, X_i is the process with mean μ and variance σ^2.

The variance of $W_i$is $\sigma_{W_t}^2 = \frac{\lambda_1^2\lambda_2^2}{(\lambda_1-\lambda_2)^2}\sigma^2\left(\frac{(1-\lambda_2)^2[1-(1-\lambda_2)^{2t}]}{1-(1-\lambda_2)^2} + \frac{(1-\lambda_1)^2[1-(1-\lambda_1)^{2t}]}{1-(1-\lambda_1)^2} - 2\frac{(1-\lambda_1)(1-\lambda_2)[1-\{(1-\lambda_1)(1-\lambda_2)\}^t]}{1-(1-\lambda_1)(1-\lambda_2)}\right)$.

For large values of t, the asymptotic variance becomes to $\sigma_{W_t}^2 = \frac{\lambda_1^2\lambda_2^2}{(\lambda_1-\lambda_2)^2}\sigma^2\left[\frac{(1-\lambda_2)^2}{1-(1-\lambda_2)^2} + \frac{(1-\lambda_1)^2}{1-(1-\lambda_1)^2} - 2\frac{(1-\lambda_1)(1-\lambda_2)}{1-(1-\lambda_1)(1-\lambda_2)}\right]$. Therefore, the general upper control limit (UCL), center line (CL) and lower control limit (LCL) to detect the sequence W_i are given by:

$$UCL = \mu_0 + L_2\sigma\sqrt{\frac{\lambda_1^2\lambda_2^2}{(\lambda_1-\lambda_2)^2}\left[\frac{(1-\lambda_2)^2}{1-(1-\lambda_2)^2} + \frac{(1-\lambda_1)^2}{1-(1-\lambda_1)^2} - 2\frac{(1-\lambda_1)(1-\lambda_2)}{1-(1-\lambda_1)(1-\lambda_2)}\right]},$$
$$CL = \mu_0, \tag{4}$$
$$LCL = \mu_0 - L_2\sigma\sqrt{\frac{\lambda_1^2\lambda_2^2}{(\lambda_1-\lambda_2)^2}\left[\frac{(1-\lambda_2)^2}{1-(1-\lambda_2)^2} + \frac{(1-\lambda_1)^2}{1-(1-\lambda_1)^2} - 2\frac{(1-\lambda_1)(1-\lambda_2)}{1-(1-\lambda_1)(1-\lambda_2)}\right]}.$$

where μ_0 is the target mean, σ is the process standard deviation, and L_2 is the optimal control width limit.

3- Derivation of Average Run Length

3-1- Explicit Formula

There are many methods to evaluate the ARL, one of them is explicit formula which is provide the exact ARL values. Various studies have been implemented under different situations of their works both model is based on assumption and violate process.

Autocorrelation is a characteristic of data which shows the degree of similarity between values of the same variables over successive time intervals where the basic assumption of instance independence underlines most conventional models is infringed. Multitudinous processes of time series modeling have been employed in numerous applications.

The observation of this study is the moving average in the general order note that by MA(q) process. It can be described as follows:

$$X_i = \mu + \varepsilon_i - \theta_1\varepsilon_{i-1} - \theta_2\varepsilon_{i-2} - \ldots - \theta_q\varepsilon_{i-q}, \tag{5}$$

where μ is the mean of the process, $\varepsilon_t$is a white noise which is assumed to be the exponential distribution, θ_t is a coefficient which is $|\theta_i| < 1; i = 1,2,3, \ldots q$.

Therefore, the Double EWMA statistic for the MA(q) process can be written as:

$$W_i = \lambda_1\lambda_2[\mu + \varepsilon_i - \theta_1\varepsilon_{i-1} - \theta_2\varepsilon_{i-2} - \ldots - \theta_q\varepsilon_{i-q}] + \lambda_1(1-\lambda_2)Z_{i-1} + (1-\lambda_1)W_{i-1}, \tag{6}$$

where $i = 1, 2, 3, \ldots, n$. We use one side of the control limit (i.e., $LCL = 0$and $UCL = u$). Then obtain:

$$W_1 = \lambda_1\lambda_2[\mu + \varepsilon_1 - \theta_1\varepsilon_0 - \ldots - \theta_q\varepsilon_{1-q}] + \lambda_1(1-\lambda_2)Z_0 + (1-\lambda_1)W_0 \tag{7}$$

If X_1 causes the out-of-control stat for W_1 with the starting value $W_0 = \omega$, then:

$\lambda_1\lambda_2[\mu + \varepsilon_1 - \theta_1\varepsilon_0 - \ldots - \theta_q\varepsilon_{1-q}] + \lambda_1(1-\lambda_2)Z_0 + (1-\lambda_1)\omega > u$ or

$\lambda_1\lambda_2[\mu + \varepsilon_1 - \theta_1\varepsilon_0 - \ldots - \theta_q\varepsilon_{1-q}] + \lambda_1(1-\lambda_2)Z_0 + (1-\lambda_1)\omega < 0$

If X_1 causes the in-control stat for W_1, then:

$0 < \lambda_1\lambda_2[\mu + \varepsilon_1 - \theta_1\varepsilon_0 - \ldots - \theta_q\varepsilon_{1-q}] + \lambda_1(1-\lambda_2)Z_0 + (1-\lambda_1)\omega < u$.

If can be written in the form as:

$$-\frac{\lambda_1\lambda_2[\mu - \theta_1\varepsilon_0 - \ldots - \theta_q\varepsilon_{i-q}] + \lambda_1(1-\lambda_2)Z_0 + (1-\lambda_1)\omega}{\lambda_1\lambda_2} \leq \varepsilon_1 \leq u - \frac{\lambda_1\lambda_2[\mu - \theta_1\varepsilon_0 - \ldots - \theta_q\varepsilon_{i-q}] + \lambda_1(1-\lambda_2)Z_0 + (1-\lambda_1)\omega}{\lambda_1\lambda_2}.$$

The probability that ε_1 satisfies the bounds mentioned above for probability distribution function ε_1 is derived in the form:

$$\left[-\frac{\lambda_1\lambda_2[\mu-\theta_1\varepsilon_0-\ldots-\theta_q\varepsilon_{i-q}]+\lambda_1(1-\lambda_2)Z_0+(1-\lambda_1)\omega}{\lambda_1\lambda_2} \le \varepsilon_1 \le u-\frac{\lambda_1\lambda_2[\mu-\theta_1\varepsilon_0-\ldots-\theta_q\varepsilon_{i-q}]+\lambda_1(1-\lambda_2)Z_0+(1-\lambda_1)\omega}{\lambda_1\lambda_2}\right] =$$
$$\int_{-\frac{\lambda_1\lambda_2[\mu-\theta_1\varepsilon_0-\ldots-\theta_q\varepsilon_{i-q}]+\lambda_1(1-\lambda_2)Z_0+(1-\lambda_1)\omega}{\lambda_1\lambda_2}}^{u-\frac{\lambda_1\lambda_2[\mu-\theta_1\varepsilon_0-\ldots-\theta_q\varepsilon_{i-q}]+\lambda_1(1-\lambda_2)Z_0+(1-\lambda_1)\omega}{\lambda_1\lambda_2}} f(y)dy$$

According to the method of Champ and Rigdon [46], the ARL of Double EWMA control chart for the MA(q) model can be written in the form of the integral equation as:

$$ARL = 1+\int_{-\frac{\lambda_1\lambda_2[\mu-\theta_1\varepsilon_0-\ldots-\theta_q\varepsilon_{i-q}]+\lambda_1(1-\lambda_2)Z_0+(1-\lambda_1)\omega}{\lambda_1\lambda_2}}^{u-\frac{\lambda_1\lambda_2[\mu-\theta_1\varepsilon_0-\ldots-\theta_q\varepsilon_{i-q}]+\lambda_1(1-\lambda_2)Z_0+(1-\lambda_1)\omega}{\lambda_1\lambda_2}} L\left[\lambda_1\lambda_2\left[y+\mu-\theta_1\varepsilon_0-\ldots-\theta_q\varepsilon_{1-q}\right]+\lambda_1(1-\lambda_2)Z_0+(1-\lambda_1)\omega\right]f(y)dy. \tag{8}$$

Changing the integral variable, we obtain:

$$ARL = 1+\frac{1}{\lambda_1\lambda_2}\int_0^u L(\omega) f\left[\frac{k-\lambda_1\lambda_2[\mu-\theta_1\varepsilon_0-\theta_2\varepsilon_2-\ldots-\theta_q\varepsilon_{i-q}]-\lambda_1(1-\lambda_2)Z_0-(1-\lambda_1)\omega}{\lambda_1\lambda_2}\right]d\omega \tag{9}$$

In this study, $\varepsilon_i$is defined to be the exponentially distributed with parameter α. Therefore, a Fredholm integral equation of the second kind for the ARL can be written as:

$$ARL = 1+e^{\frac{\lambda_1\lambda_2[\mu-\theta_1\varepsilon_0-\theta_2\varepsilon_2-\ldots-\theta_q\varepsilon_{i-q}]+\lambda_1(1-\lambda_2)Z_0+(1-\lambda_1)\omega}{\alpha\lambda_1\lambda_2}}\frac{1}{\alpha\lambda_1\lambda_2}\int_0^u L(\omega)\cdot e^{-\frac{\omega}{\alpha\lambda_1\lambda_2}}d\omega = 1+G(\omega)A \tag{10}$$

where $G(\omega) = e^{\frac{\lambda_1\lambda_2[\mu-\theta_1\varepsilon_0-\theta_2\varepsilon_2-\ldots-\theta_q\varepsilon_{i-q}]+\lambda_1(1-\lambda_2)Z_0+(1-\lambda_1)\omega}{\alpha\lambda_1\lambda_2}}$ and;

$$\begin{aligned}
A &= \frac{1}{\alpha\lambda_1\lambda_2}\int_0^u L(\omega)\cdot e^{-\frac{\omega}{\alpha\lambda_1\lambda_2}}d\omega \\
&= \frac{1}{\alpha\lambda_1\lambda_2}\int_0^u [1+G(\omega)A]\cdot e^{-\frac{\omega}{\alpha\lambda_1\lambda_2}}d\omega \\
&= \frac{1}{\alpha\lambda_1\lambda_2}\int_0^u e^{-\frac{\omega}{\alpha\lambda_1\lambda_2}}d\omega+\frac{A}{\alpha\lambda_1\lambda_2}\int_0^u e^{\frac{\lambda_1\lambda_2[\mu-\theta_1\varepsilon_0-\theta_2\varepsilon_2-\ldots-\theta_q\varepsilon_{i-q}]+\lambda_1(1-\lambda_2)Z_0+(1-\lambda_1)\omega}{\alpha\lambda_1\lambda_2}}\cdot e^{-\frac{\omega}{\alpha\lambda_1\lambda_2}}d\omega \\
&= -\alpha\lambda_1\lambda_2\left[e^{-\frac{u}{\alpha\lambda_1\lambda_2}}-1\right]-Ae^{\frac{\frac{\lambda_1\lambda_2[\mu-\theta_1\varepsilon_0-\theta_2\varepsilon_2-\ldots-\theta_q\varepsilon_{i-q}]+\lambda_1(1-\lambda_2)Z_0+(1-\lambda_1)\omega}{\alpha\lambda_1\lambda_2}}{\lambda_1}}\left[e^{-\frac{u}{\alpha\lambda_2}}-1\right]
\end{aligned}$$

$$A = \frac{-\alpha\lambda_1\lambda_2\left[e^{-\frac{u}{\alpha\lambda_1\lambda_2}}-1\right]}{1+e^{\frac{\frac{\lambda_1\lambda_2[\mu-\theta_1\varepsilon_0-\theta_2\varepsilon_2-\ldots-\theta_q\varepsilon_{i-q}]+\lambda_1(1-\lambda_2)Z_0}{\alpha\lambda_1\lambda_2}}{\lambda_1}}\left[e^{-\frac{u}{\alpha\lambda_2}}-1\right]} \tag{11}$$

Finally, the equation 10 is replaced by equation 11. The ARL of the Double EWMA control chart for the MA(q) process with exponential white noise is provided by deriving a Fredholm integral equation of the second kind as follows:

It can be written in the form:

$$ARL = 1-\frac{\lambda_1 e^{\frac{(1-\lambda_1)\omega}{\alpha\lambda_1\lambda_2}}\left[e^{-\frac{u}{\alpha\lambda_1\lambda_2}}-1\right]}{1+e^{-\frac{\lambda_1\lambda_2[\mu-\theta_1\varepsilon_0-\theta_2\varepsilon_2-\ldots-\theta_q\varepsilon_{i-q}]+\lambda_1(1-\lambda_2)Z_0}{\alpha\lambda_1\lambda_2}}+\left[e^{-\frac{u}{\alpha\lambda_2}}-1\right]} \tag{12}$$

when ε_0, ω and Z_0 are the starting value of the process for MA(q) model. The in-control process of *ARL* corresponds to *ARL*$_0$ when $\alpha_0 = 1$. An out-of-control process when $\alpha_1 > 1$, provide the *ARL*$_1$ values.

3-2- Numerical Integral Equation

The numerical integral equation (NIE) is a method of estimation the ARL approach. In this study, it is used to compare the accuracy of ARL obtain by explicit formulas method. According to the integral equation in (9), the Gauss-Legendre quadrature rule approach is approximated the integral by finite sum of areas of rectangles with base $\frac{u}{m}$ and heights chosen as values of f midpoints of the one-side interval which divide $[0,u]$ into a partition from 0 to u. It can be derived the ARL calculation of the integral equation as follows.

$$ARL(a_t) = 1 + \frac{1}{\lambda_1\lambda_2}\sum_{j=1}^{m} w_j\, ARL(a_j) f\left[\frac{a_j - \lambda_1\lambda_2[\mu - \theta_1\varepsilon_0 - \theta_2\varepsilon_2 - \ldots - \theta_q\varepsilon_{i-q}] - \lambda_1(1-\lambda_2)Z_0 - (1-\lambda_1)\omega}{\lambda_1\lambda_2}\right] \tag{13}$$

where $j = 1,2,3,\ldots,m$, $a_j = \frac{u}{m}\left(j - \frac{1}{2}\right)$ and $w_j = \frac{u}{m}$

In order to compute the ARL values, the relation of a matrix form can be rewritten in a form as:

$$ARL_{m\times 1} = 1_{m\times 1} + R_{m\times m}ARL_{m\times 1} \text{ or } ARL_{m\times 1} = \left(I_{m\times m} - R_{m\times m}\right)^{-1} 1_{m\times 1} \tag{14}$$

where $ARL_{m\times 1} = \begin{bmatrix} ARL(a_1) \\ \vdots \\ ARL(a_1) \end{bmatrix}$, $1_{m\times 1} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$, $I_{m\times m}$ is a identity matrix and $R_{m\times m} = \frac{1}{\lambda_1\lambda_2}\begin{bmatrix} w_1 f_{11} & \cdots & w_m f_{1m} \\ w_1 f_{11} & \cdots & w_m f_{2m} \\ \vdots & & \vdots \\ w_1 f_{11} & \cdots & w_m f_{mm} \end{bmatrix}$, when $f_{ij} = f\left(a_j - \lambda_1\lambda_2[\mu - \theta_1\varepsilon_0 - \theta_2\varepsilon_2 - \ldots - \theta_q\varepsilon_{i-q}] - \lambda_1(1-\lambda_2)Z_0 - (1-\lambda_1)\omega\right)$.

Hence, the approximation of *ARL* by using NIE method for the Double EWMA control chart of MA(q) process is expressed in the form as follows:

$$A\tilde{R}L = 1 + \frac{1}{\lambda_1\lambda_2}\sum_{j=1}^{m} w_j\, ARL(a_j) f\left[\frac{a_j - \lambda_1\lambda_2[\mu - \theta_1\varepsilon_0 - \theta_2\varepsilon_2 - \ldots - \theta_q\varepsilon_{i-q}] - \lambda_1(1-\lambda_2)Z_0 - (1-\lambda_1)\omega}{\lambda_1\lambda_2}\right] \tag{15}$$

3-3- Existence and Uniqueness of ARL Demonstrate

Here, Banach's Fixed-point Theorem is used to show the uniquely exists solution of the integral equation for explicit formula. In this section, let Tbe an operation in the class of all continuous functions expressed by:

$$T(ARL) = 1 + \frac{e^{\frac{\lambda_1\lambda_2[\mu - \theta_1\varepsilon_0 - \theta_2\varepsilon_2 - \ldots - \theta_q\varepsilon_{i-q}] + \lambda_1(1-\lambda_2)Z_0 + (1-\lambda_1)\omega}{\alpha\lambda_1\lambda_2}}}{\alpha\lambda_1\lambda_2}\int_0^u L(\omega)\cdot e^{-\frac{\omega}{\alpha\lambda_1\lambda_2}} d\omega\,. \tag{16}$$

As stated by Banach's Fixed-point Theorem, if an operator T is a contraction, then the fixed-point equation $T(ARL) = ARL$ has a unique solution. To prove that Equation 16 exists and has a unique solution, the following theorem can be claimed.

Banach's Fixed-point Theorem:

Let (X, r)be a complete metric space and $T: X \to X$be a contraction mapping with contraction constant $0 \le p < 1$ such that $\|T(A_1) - T(A_2)\| \le p\|A_1 - A_2\|$ for all $A_1, A_2 \in X$. Then there exists a unique $A(\cdot) \in X$such that $T(ARL) = ARL$, i.e., a unique fixed-point in X [47].

Proof: To show that T in (16) is a contraction mapping for $A_1, A_2 \in [0,u]$ by show that $\|T(A_1) - T(A_2)\| \le p\|A_1 - A_2\|$ for all $A_1, A_2 \in C[0,u]$ with $0 \le p < 1$ under the norm $\|A\|_\infty = sup_{a\in(0,u)}|ARL|$. From Equations 10 and 16:

$$\|T(A_1) - T(A_2)\|_\infty = \sup_{a\in(0,u)}\left|\frac{G(\omega)}{\alpha}\int_0^u \left(A_1(\omega) - (A_2(\omega)\right)e^{-\frac{\omega}{\alpha\lambda_1\lambda_2}} d\omega\right|$$

$$\le \sup_{a\in(0,u)}\left|\|A_1 - A_2\|_\infty G(\omega)\left[e^{-\frac{u}{\alpha\lambda_1\lambda_2}} - 1\right]\right|$$

$$= \|A_1 - A_2\|_\infty \sup_{a\in(0,u)}|G(\omega)|\left|e^{-\frac{u}{\alpha\lambda_1\lambda_2}} - 1\right|.$$

$$\le p\|A_1 - A_2\|_\infty,$$

where $p = \sup_{a\in(0,u)}|G(\omega)|\left|e^{-\frac{u}{\alpha\lambda_1\lambda_2}} - 1\right|$ and $G(\omega) = e^{\frac{\lambda_1\lambda_2[\mu - \theta_1\varepsilon_0 - \theta_2\varepsilon_2 - \ldots - \theta_q\varepsilon_{i-q}] + \lambda_1(1-\lambda_2)Z_0 + (1-\lambda_1)\omega}{\alpha\lambda_1\lambda_2}}$; $0 \le p < 1$.

From Banach's Fixed-point Theorem, we get the following result approved to existence and uniqueness of a solution of the ARL for MA(q) process on the DEWMA control chart.

4- Experimental Results

In this section, the results for evaluating the *ARL* obtained by explicit formulas and NIE method which is accurate method (viewed as the most accurate method) are presented in the term of the absolute percentage relative error (APRE) (a measure of the exactness of the ARL), which is given by:

$$APRE(\%) = \frac{|ARL - A\tilde{R}L|}{ARL}\times 100 \tag{17}$$

where $ARL$and $A\tilde{R}L$ are exact and approximation of ARLs derived by explicit formulas and NIE method, respectively, as provided by MATHEMATICA. The experimental results are reported in Tables 1 to 3.

Table 1. ARLs for MA(1) of Double EWMA using explicit formula and NIE for $\mu = 3$, $\lambda_1 = 0.05$, $\lambda_2 = 0.05$, ARL₀=370 and 500

θ	δ	ARL$_0$=370			ARL$_0$=500		
		Explicit	NIE	APRE	Explicit	NIE	APRE
$\theta = -0.1$*	0.0000	370.50714959	370.57748949	1.898E-04	501.71693916	500.86635583	1.695E-03
	0.0001	200.75916840	200.82436152	3.247E-04	233.81960666	233.69232691	5.444E-04
	0.0003	105.02826510	104.99266205	3.390E-04	113.28697135	113.28454771	2.139E-05
	0.0005	71.19431570	71.18983590	6.292E-05	74.87846828	74.88961468	1.489E-04
	0.0010	39.59828081	39.59553886	6.924E-05	40.69999650	40.70120578	2.971E-05
	0.0050	9.08265365	9.08274598	1.016E-05	9.13463185	9.13471371	8.962E-06
	0.0100	4.87669125	4.87670629	3.085E-06	4.89009036	4.89004937	8.382E-06
	0.0300	2.04768806	2.04768908	4.979E-07	2.04920213	2.04920019	9.480E-07
	0.0500	1.50321171	1.50321081	5.975E-07	1.50374268	1.50374342	4.899E-07
	0.1000	1.14131875	1.14131880	3.829E-08	1.14143235	1.14143233	1.936E-08
$\theta = -0.6$ **	0.0000	370.78227233	370.94202310	4.308E-04	500.49245685	499.96209703	1.060E-03
	0.0001	202.88130761	202.95987236	3.872E-04	236.32143590	236.24179079	3.370E-04
	0.0003	106.69575092	106.68597687	9.161E-05	115.17721328	115.17720354	8.461E-08
	0.0005	72.47063486	72.47463475	5.519E-05	76.26912000	76.27772115	1.128E-04
	0.0010	40.37691517	40.37831222	3.460E-05	41.51668365	41.51879403	5.083E-05
	0.0050	9.26940425	9.26955445	1.620E-05	9.32335069	9.32333683	1.486E-06
	0.0100	4.97242527	4.97242324	4.083E-07	4.98626511	4.98624029	4.978E-06
	0.0300	2.07965732	2.07965773	1.975E-07	2.08122678	2.08122517	7.714E-07
	0.0500	1.52166260	1.52166304	2.857E-07	1.52221692	1.52221696	3.160E-08
	0.1000	1.14886943	1.14886940	3.012E-08	1.14898865	1.14898872	5.405E-08
$\theta = 0.7$***	0.0000	370.75635234	369.89600209	2.321E-03	502.03825198	500.28123519	3.500E-03
	0.0001	197.23178122	197.47134292	1.215E-03	229.01754145	229.29959142	1.232E-03
	0.0003	102.39383177	102.38531269	8.320E-05	110.30588760	110.28484007	1.908E-04
	0.0005	69.25975195	69.22347006	5.239E-04	72.69192484	72.72898292	5.098E-04
	0.0010	38.39381265	38.40403407	2.662E-04	39.43774963	39.44643909	2.203E-04
	0.0050	8.79978972	8.79986268	8.291E-06	8.84880732	8.84862371	2.075E-05
	0.0100	4.73197650	4.73188934	1.842E-05	4.74436952	4.74439827	6.060E-06
	0.0300	1.99944084	1.99943265	4.094E-06	2.00084288	2.00084544	1.282E-06
	0.0500	1.47545493	1.47545485	5.246E-08	1.47595114	1.47595063	3.443E-07
	0.1000	1.13011547	1.13011534	1.083E-07	1.13021935	1.13021924	1.004E-07

*$u_1 = 1.26726 \times 10^{-12}$ for ARL$_0$=370 and $u_2 = 7.69171 \times 10^{-13}$ for ARL$_0$=500.

**$u_1 = 1.26726 \times 10^{-12}$ for ARL$_0$=370 and$u_2 = 1.268144 \times 10^{-12}$ for ARL$_0$=500.

*** $u_1 = 3.45366 \times 10^{-13}$for ARL$_0$=370 and $u_2 = 3.4561 \times 10^{-13}$ for ARL$_0$=500.

Table 2. ARLs for MA(2) of Double EWMA using explicit formula and NIE for $\mu = 3$,$\lambda_1 = 0.1$, $\lambda_2 = 0.1$, ARL₀=370 and 500

θ_i	δ	ARL$_0$=370			ARL$_0$=500		
		Explicit	NIE	APRE	Explicit	NIE	APRE
$\theta_1 = -0.4$ $\theta_2 = 0.6$*	0.0000	370.24317768	370.24317777	2.614E-10	500.19993709	500.19992300	2.816E-08
	0.0001	248.98102637	248.98102537	4.037E-09	301.60736291	301.60735801	1.623E-08
	0.0005	108.01120836	108.01120806	2.730E-09	116.80625423	116.80625319	8.879E-09
	0.0010	63.42636995	63.42636975	3.071E-09	66.34036343	66.34036357	2.201E-09
	0.0050	15.12797985	15.12797986	6.478E-10	15.27987449	15.27987451	1.086E-09
	0.0100	8.00471427	8.00471427	3.748E-11	8.04436256	8.04436257	2.735E-10
	0.0300	3.11217326	3.11217326	1.285E-10	3.11680540	3.11680540	6.417E-11
	0.0500	2.13351334	2.13351334	0.000E+00	2.13521632	2.13521632	0.000E+00
	0.1000	1.42665805	1.42665805	0.000E+00	1.42708648	1.42708648	0.000E+00

$\theta_1 = -0.3$ $\theta_2 = -0.7$**	0.0000	370.20283796	370.20283585	5.708E-09	500.12681627	500.12681136	9.813E-09
	0.0001	256.59731214	256.59731270	2.191E-09	312.86473878	312.86473574	9.702E-09
	0.0005	115.41620981	115.41621012	2.692E-09	125.52152553	125.52152525	2.226E-09
	0.0010	68.55863878	68.55863882	6.768E-10	71.98031312	71.98031320	1.086E-09
	0.0050	16.52590941	16.52590941	4.720E-10	16.70844045	16.70844045	5.985E-12
	0.0100	8.73605009	8.73605009	1.717E-10	8.78386334	8.78386333	1.252E-10
	0.0300	3.36496653	3.36496653	0.000E+00	3.37057535	3.37057535	2.967E-11
	0.0500	2.28588039	2.28588039	0.000E+00	2.28795048	2.28795048	0.000E+00
	0.1000	1.50044485	1.50044485	0.000E+00	1.50097338	1.50097338	0.000E+00
$\theta_1 = 0.2$ $\theta_2 = 0.8$***	0.0000	370.56603665	370.56601974	4.563E-08	500.10984438	500.10983336	2.203E-08
	0.0001	244.27859336	244.27859654	1.303E-08	294.49437014	294.49436891	4.188E-09
	0.0005	103.60575138	103.60575341	1.964E-08	111.63413223	111.63413429	1.842E-08
	0.0010	60.42199845	60.42199798	7.805E-09	63.04789082	63.04789115	5.305E-09
	0.0050	14.32458864	14.32458862	9.843E-10	14.45963107	14.45963104	1.743E-09
	0.0100	7.58563562	7.58563562	3.955E-10	7.62081257	7.62081257	4.593E-10
	0.0300	2.96781544	2.96781544	2.696E-10	2.97191445	2.97191445	1.009E-10
	0.0500	2.04679837	2.04679837	0.000E+00	2.04830122	2.04830122	4.882E-11
	0.1000	1.38520315	1.38520315	0.000E+00	1.38557724	1.38557724	0.000E+00

* $u_1 = 5.01687 \times 10^{-8}$for ARL$_0$=370 and $u_2 = 5.0204 \times 10^{-8}$ for ARL$_0$=500.

** $u_1 = 1.66566 \times 10^{-7}$for ARL$_0$=370 and $u_2 = 1.666832 \times 10^{-7}$ for ARL$_0$=500.

*** $u_1 = 2.25423 \times 10^{-8}$for ARL$_0$=370 and $u_2 = 2.25581 \times 10^{-8}$ for ARL$_0$=500.

Table 3. ARLs for MA(3) of Double EWMA using explicit formula and NIE for $\mu = 3$, $\lambda_1 = 0.05$, $\lambda_2 = 0.1$, ARL$_0$=370 and 500

θ_i	δ	ARL$_0$=370			ARL$_0$=500		
		Explicit	NIE	APRE	Explicit	NIE	APRE
$\theta_1 = -0.3$ $\theta_2 = 0.5$ $\theta_3 = -0.7$*	0.0000	370.03177426	370.03176210	3.286E-08	500.09716678	500.09716176	1.004E-08
	0.0001	253.27995775	252.27996135	3.948E-03	308.05070039	307.05069931	3.246E-03
	0.0005	112.19315180	112.19315271	8.176E-09	121.73272685	121.73272741	4.553E-09
	0.0010	66.31585426	66.31585389	5.644E-09	69.51582408	69.51582362	6.548E-09
	0.0050	15.91185692	15.91185690	1.244E-09	16.08085596	16.08085595	3.731E-10
	0.0100	8.41452935	8.41452936	6.655E-10	8.45873035	8.45873034	5.084E-10
	0.0300	3.25371247	3.25371247	6.147E-11	3.25888867	3.25888867	9.206E-11
	0.0500	2.21875231	2.21875231	4.507E-11	2.22065970	2.22065970	4.503E-11
	0.1000	1.46780365	1.46780365	0.000E+00	1.46828773	1.46828773	0.000E+00
$\theta_1 = -0.4$ $\theta_2 = -0.6$ $\theta_3 = -0.8$**	0.0000	370.38904503	370.38904278	6.057E-09	500.04199070	499.04199411	2.000E-03
	0.0001	262.04573909	261.04574060	3.816E-03	320.83371689	319.83371786	3.117E-03
	0.0005	120.96899225	120.96899217	7.002E-10	132.09242254	132.09242271	1.333E-09
	0.0010	72.47974926	72.47974925	1.269E-10	76.30841266	76.30841261	6.395E-10
	0.0050	17.61694426	17.61694426	1.703E-10	17.82485079	17.82485079	2.020E-10
	0.0100	9.30874076	9.30874076	1.074E-10	9.36334882	9.36334882	1.175E-10
	0.0300	3.56363956	3.56363956	0.000E+00	3.57006379	3.57006379	0.000E+00
	0.0500	2.40600670	2.40600670	4.156E-11	2.40838353	2.40838353	4.152E-11
	0.1000	1.55930848	1.55930848	0.000E+00	1.55992088	1.55992088	0.000E+00
$\theta_1 = 0.2$ $\theta_2 = 0.4$ $\theta_3 = 0.7$***	0.0000	370.03065404	370.03062745	7.188E-08	500.25466313	500.25461175	1.027E-07
	0.0001	242.27983943	241.27981673	4.128E-03	291.97363681	290.97359760	3.425E-03
	0.0005	101.99589225	101.99589834	5.975E-08	109.81976529	109.81977019	4.462E-08
	0.0010	59.35138828	59.35138612	3.630E-08	61.89923888	61.89923653	3.804E-08
	0.0050	14.04475356	14.04475359	2.563E-09	14.17519108	14.17519102	4.480E-09
	0.0100	7.44012309	7.44012311	1.828E-09	7.47407699	7.47407696	3.586E-09
	0.0300	2.91784125	2.91784125	3.427E-11	2.92179400	2.92179400	2.396E-10
	0.0500	2.01684440	2.01684440	9.916E-11	2.01829209	2.01829209	4.955E-11
	0.1000	1.37099281	1.37099281	0.000E+00	1.37135170	1.37135170	7.292E-11

* $u_1 = 5.05136 \times 10^{-8}$for ARL$_0$=370 and $u_2 = 5.05492 \times 10^{-8}$ for ARL$_0$=500.

** $u_1 = 1.8535 \times 10^{-7}$for ARL$_0$=370 and $u_2 = 1.854801 \times 10^{-7}$ for ARL$_0$=500.

*** $u_1 = 8.34984 \times 10^{-9}$for ARL$_0$=370 and $u_2 = 8.35573 \times 10^{-9}$ for ARL$_0$=500.

The methodology of our process of computation the ARL values by using the explicit formulas and NIE method running on the double EWMA control chart for the MA(q) process with exponential white noise is presented in Figure 1.

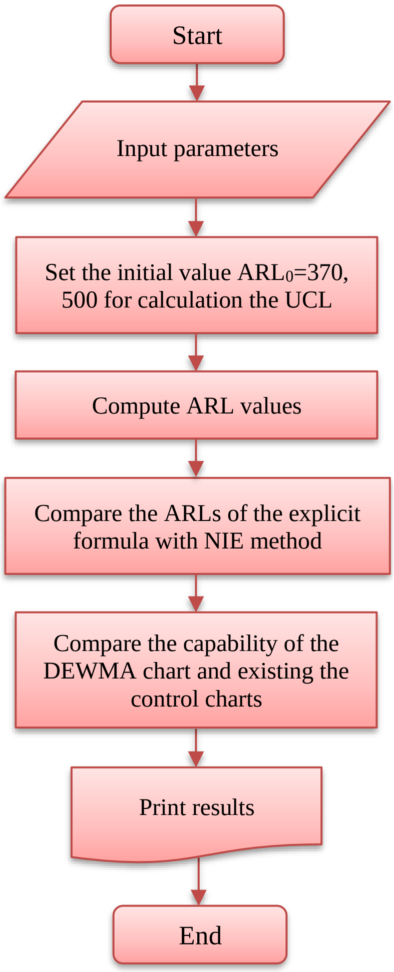


Figure 1. The diagram of the research methodology

Table 1 shows the ARL values for an MA(1) process with exponential white noise running on the DEWMA control chart when $\mu = 3; \theta = -0.1, -0.6$, or 0.7; $\lambda_1 = 0.05$; and $\lambda_2 = 0.05$ for ARL$_0$ = 370 or 500 .The numerical results were obtained after setting in-control process parameter $\alpha_0 = 1$and out-of-control process parameter $\alpha_1 > 1$) i.e., $\alpha_1 = (1 + \delta)\alpha_0$ (for shift size δ = 0.0001, 0.0003, 0.0005, 0.001, 0.005, 0.01, 0.03, 0.05, 0.1, or 0.5 and where u_1 and u_2 are the upper control limits)UCLs(for ARL$_0$ = 370 and 500, respectively .The results using the two methods indicate that they are in excellent agreement since the APRE values were less than 3.5×10^{-3} .$\lambda_1 = \lambda_2 = 0.05$, as is usually recommended for the EWMA control chart, was used for computing the ARL .The ARL results show that the DEWMA control chart is very sensitive for detecting changes in the process mean when the shift size was very small for $\lambda_1 = \lambda_2 = 0.1$.

The ARL results for MA(2) and MA(3) processes, δ = 0.0001, 0.0005, 0.001, 0.005, 0.01, 0.03, 0.05, 0.1, or 0.5 and ARL$_0$ =370 or 500 are given in Tables 2 and 3, respectively .The other settings for the simulation were $\mu = 3; \theta_1 = -0.1$, 0.2, or –0.3;$\theta_2 = 0.6$, –0.7, or 0.8; $\lambda_1 = 0.05$; and $\lambda_2 = 0.05$for the MA(2) process and $\mu = 3; \theta_1 = -0.3, -0.4$, or 0.2; $\theta_2 = 0.5$, –0.6, or 0.4;$\theta_3 = -0.7$, –0.8, or 0.7; $\lambda_1 = 0.05$; and $\lambda_2 = 0.05$for the MA(3) process .Moreover, u_1 and u_2 are the UCLs for ARL$_0$ = 370 and 500, respectively. The results in Tables 2 and 3 indicate that the exact ARL provided by the explicit formulas and the estimated ARLs provided by the NIE method are in good agreement, as indicated by the APRE results being close to zero for all of the cases studied. In addition, the ARL$_1$ values were sensitive to small changes in the process mean. Consequently, the exact ARL values provided by the explicit formulas can be utilized to efficiently and rapidly detect changes in the mean of an MA(q) process with exponential white noise running on a DEWMA control chart.

Tables 4 to 6 report the ARLs for MA(1), MA(2), and MA(3) processes running on the DEWMA, EWMA, and MEWMA control charts, respectively .For these experiments, δ = 0.0001, 0.0005, 0.001, 0.005, 0.01, 0.03, 0.05, 0.1, or 0.5 .Meanwhile, $\lambda_1 = \lambda_2 = \lambda = 0.05$ for the MA(1) model and $\lambda_1 = \lambda_2 = \lambda = 0.1$ for the MA(2) and MA(3) models . Moreover, h_1 and h_2 are the UCLs for the EWMA control chart, b_1 and b_2 are the UCLs for the MEWMA control chart, and u_1 and u_2 are the UCLs for the DEWMA control chart for $ARL_0 = 370$ and 500, respectively. The experimental results show that the ARL values provided by the DEWMA control chart for the MA(1) process with $\lambda_1 = \lambda_2 = 0.05$ were much lower than the others for both $ARL_0 = 370$ and 500, which was also the case for the MA(2) and MA(3) processes, indicating that it could detect changes more quickly and more sensitively than the other two control charts . These results are graphically presented in Figures 2 to 4.

Table 4. Comparison ARLs for MA(1) of EWMA, modified EWMA and Double EWMA using explicit formula for $\mu = 2$, $\lambda_1 = 0.05$, $\lambda_2 = 0.05$, ARL_0=370 and 500

θ	δ	ARL_0=370			ARL_0=500		
		EWMA	MEWMA	DEWMA	EWMA	MEWMA	DEWMA
$\theta = -0.2$*	0.000	370.420516	370.394639	370.557944	500.151500	500.433051	500.362162
	0.0001	369.763596	356.678265	205.290702	499.263887	475.726593	239.663500
	0.0005	367.148846	310.659489	74.045414	495.730904	397.270443	78.035169
	0.001	363.909285	267.513368	41.348261	491.353694	329.367460	42.554722
	0.005	339.108224	126.699476	9.504206	457.843145	139.103249	9.561361
	0.010	310.713835	76.405097	5.092886	419.477385	80.761050	5.107554
	0.030	220.874463	29.523459	2.120001	298.088755	30.158783	2.121667
	0.050	159.087077	18.314113	1.545020	214.603235	18.558721	1.545612
	0.100	73.919331	9.452589	1.158538	99.526762	9.518188	1.158666
	0.500	1.910715	2.304331	1.000648	2.230536	2.307695	1.000649
$\theta = 0.5$**	0.000	370.493400	370.491438	370.496940	500.241764	500.297837	500.248510
	0.0001	369.810536	359.714488	202.350449	499.319111	480.847729	235.929507
	0.0005	367.092993	322.223992	72.205613	495.647297	416.137091	76.006726
	0.001	363.727139	285.085666	40.217627	491.099517	356.217654	41.357351
	0.005	337.999245	148.346904	9.231227	456.337224	165.563359	9.285131
	0.010	308.635388	92.768414	4.952901	416.662193	99.226597	4.966724
	0.030	216.479441	37.189052	2.073137	292.145505	38.188258	2.074709
	0.050	153.934489	23.308205	1.517899	207.637767	23.697520	1.518452
	0.100	69.437034	12.147268	1.147323	93.468856	12.252214	1.147442
	0.500	1.721326	2.909056	1.000513	1.974622	2.914368	1.000514

*$h_1 = 9.34 \times 10^{-7}$ for ARL_0=370 and $h_2 = 1.262 \times 10^{-6}$ for ARL_0=500 on EWMA chart,
$b_1 = 3.01952 \times 10^{-1}$ for ARL_0=370 and $b_2 = 3.02414 \times 10^{-1}$ for ARL_0=500 on modified EWMA chart,
$u_1 = 2.30909 \times 10^{-12}$ for ARL_0=370 and $u_2 = 2.31071 \times 10^{-12}$ for ARL_0=500 on double EWMA chart.
**$h_1 = 4.639 \times 10^{-7}$ for ARL_0=370 and $h_2 = 6.268 \times 10^{-7}$ for ARL_0=500 on EWMA chart,
$b_1 = 6.12996 \times 10^{-1}$ for ARL_0=370 and $b_2 = 6.13825 \times 10^{-1}$ for ARL_0=500 on modified EWMA chart,
$u_1 = 1.14666 \times 10^{-12}$ for ARL_0=370 and $u_2 = 1.147467 \times 10^{-12}$ for ARL_0=500 on double EWMA chart.

Table 5. Comparison ARLs for MA(2) of EWMA, modified EWMA and Double EWMA using explicit formula for $\mu = 2$, $\lambda_1 = 0.1$, $\lambda_2 = 0.1$, ARL_0=370 and 500

θ_i	δ	ARL_0=370			ARL_0=500		
		EWMA	MEWMA	DEWMA	EWMA	MEWMA	DEWMA
$\theta_1 = -0.6$ $\theta_2 = -0.3$*	0.000	370.442007	370.688469	370.282277	500.280903	500.404149	500.721193
	0.0001	370.206255	352.379302	262.677817	499.986803	467.617568	322.143935
	0.0005	369.265201	294.245238	121.687138	498.812727	370.512782	133.011189
	0.001	368.093261	243.939769	72.996928	497.350352	294.156968	76.902712
	0.005	358.890000	103.028774	17.763595	485.856911	111.059955	17.976172
	0.010	347.803269	59.834537	9.385927	471.988577	62.467032	9.441781
	0.030	307.678169	22.381213	3.590480	421.571801	22.743925	3.597053
	0.050	273.407974	13.801259	2.422260	378.197982	13.939484	2.424692
	0.100	207.225284	7.124820	1.567313	293.410598	7.161775	1.567941
	0.500	42.853698	1.861519	1.023479	70.935699	1.863427	1.023496

$\theta_1 = 0.4$ $\theta_2 = 0.5$**	0.0000	370.476061	370.431514	370.915331	500.211410	500.336744	500.134946
	0.0001	370.150120	360.755176	251.154930	499.772316	482.841359	304.327726
	0.0005	368.849837	326.630368	109.831066	498.020624	423.599669	118.866553
	0.0010	367.232290	292.099374	64.65599	495.841495	367.282108	67.664410
	0.0050	354.597872	158.330489	15.455183	478.819582	178.076451	15.612815
	0.0100	339.540527	100.768796	8.175341	458.530799	108.413348	8.216519
	0.0300	286.588631	41.214183	3.170985	387.157890	42.433449	3.175800
	0.0500	243.391335	26.011894	2.168898	328.902333	26.490290	2.170670
	0.1000	165.856433	13.679155	1.443691	224.253470	13.808689	1.444139
	0.5000	19.073215	3.319012	1.012751	25.601274	3.325528	1.012760

*$h_1 = 1.456 \times 10^{-1}$ for ARL_0=370 and $h_2 = 7.33 \times 10^{-1}$ for ARL_0=500 on EWMA chart,

$b_1 = 1.5039 \times 10^{-1}$ for ARL_0=370 and $b_2 = 1.50616 \times 10^{-1}$ for ARL_0=500 on modified EWMA chart,

$u_1 = 4.09687 \times 10^{-7}$ for ARL_0=370 and $u_2 = 4.09976 \times 10^{-7}$ for ARL_0=500 on double EWMA chart.

**$h_1 = 1.4 \times 10^{-2}$ for ARL_0=370 and $h_2 = 1.905 \times 10^{-2}$ for ARL_0=500 on EWMA chart,

$b_1 = 9.4434 \times 10^{-1}$ for ARL_0=370 and $b_2 = 9.4544 \times 10^{-1}$ for ARL_0=500 on modified EWMA chart,

$u_1 = 6.7721 \times 10^{-8}$ for ARL_0=370 and $u_2 = 6.77683 \times 10^{-8}$ for ARL_0=500 on double EWMA chart.

Table 6. Comparison ARLs for MA(3) of EWMA, modified EWMA and Double EWMA using explicit formula for $\mu = 2$, $\lambda_1 = 0.1$, $\lambda_2 = 0.1$, ARL_0=370 and 500

θ_i	δ	ARL_0=370			ARL_0=500		
		EWMA	**MEWMA**	**DEWMA**	**EWMA**	**MEWMA**	**DEWMA**
$\theta_1 = -0.5$ $\theta_2 = -0.3$ $\theta_3 = -0.4$*	0.0000	370.232951	370.295033	370.279493	500.242481	500.277843	500.667609
	0.0001	370.028234	350.570618	264.754200	500.025456	464.949528	325.254239
	0.0005	369.210927	288.994227	123.928903	499.159021	362.540635	135.692587
	0.0010	368.192789	236.964955	74.606546	498.079716	284.271431	78.691402
	0.0050	360.185433	97.098726	18.219698	489.592603	104.230015	18.443676
	0.0100	350.510577	55.873910	9.626005	479.341725	58.173296	9.684921
	0.0300	315.212099	20.728554	3.673991	441.993163	21.041206	3.680932
	0.0500	284.670425	12.756672	2.472855	409.788070	12.875514	2.475427
	0.1000	224.413910	6.578182	1.592285	346.933562	6.609904	1.592950
	0.5000	60.260919	1.749892	1.026013	268.238244	1.751513	1.026032
$\theta_1 = 0.4$ $\theta_2 = 0.7$ $\theta_3 = 0.6$**	0.0000	370.511926	370.321510	370.910954	500.262991	500.618332	500.510253
	0.0001	370.153608	366.614425	246.226216	499.778772	493.865419	297.251211
	0.0005	368.724487	352.494340	105.252142	497.847496	468.572821	113.537613
	0.0010	366.947373	336.292549	61.527071	495.445956	440.362147	64.249208
	0.0050	353.094062	245.649075	14.615966	476.725024	296.872576	14.756613
	0.0100	336.649208	183.425695	7.737339	454.502051	210.532824	7.774002
	0.0300	279.422272	90.305107	3.019985	377.168576	96.397731	3.024261
	0.0500	233.514924	59.393075	2.078103	315.133089	61.959689	2.079673
	0.1000	153.265639	31.500715	1.400116	206.694881	32.200648	1.400508
	0.5000	14.524592	6.374223	1.009737	19.255481	6.398828	1.009744

*$h_1 = 3.73 \times 10^{-1}$ for ARL_0=370 and $h_2 = 1.773$ for ARL_0=500 on EWMA chart,

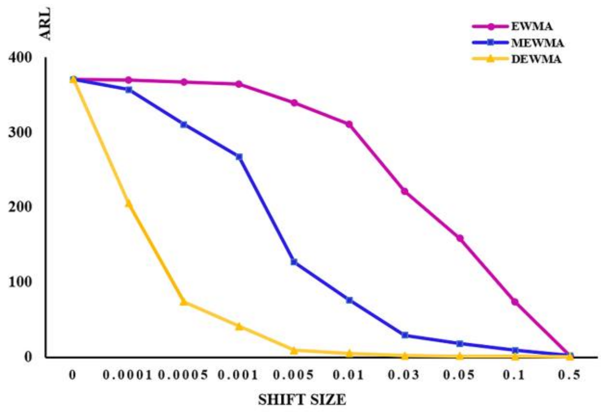
$b_1 = 1.11203 \times 10^{-1}$ for ARL_0=370 and $b_2 = 1.11373 \times 10^{-1}$ for ARL_0=500 on modified EWMA chart,

$u_1 = 5.5302 \times 10^{-7}$ for ARL_0=370 and $u_2 = 5.5341 \times 10^{-7}$ for ARL_0=500 on double EWMA chart.

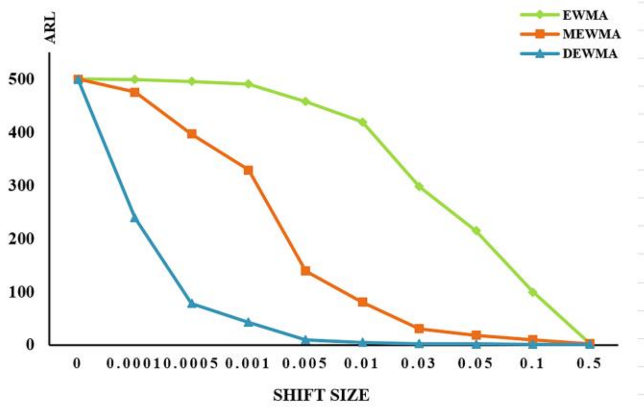
**$h_1 = 6.06 \times 10^{-3}$ for ARL_0=370 and $h_2 = 8.14 \times 10^{-3}$ for ARL_0=500 on EWMA chart,

$b_1 = 2.2279$ for ARL_0=370 and $b_2 = 2.22977$ for ARL_0=500 on modified EWMA chart,

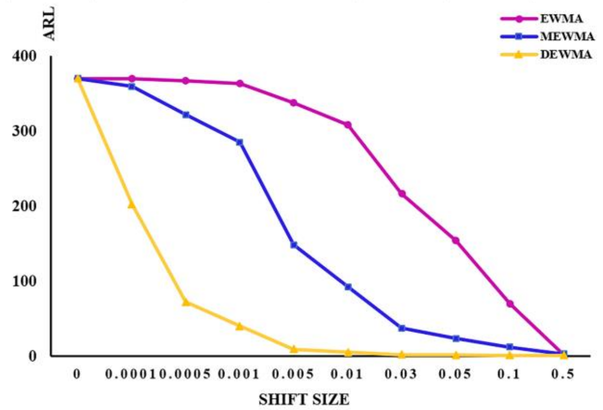
$u_1 = 3.0429 \times 10^{-8}$ for ARL_0=370 and $u_2 = 3.04503 \times 10^{-8}$ for ARL_0=500 on double EWMA chart.



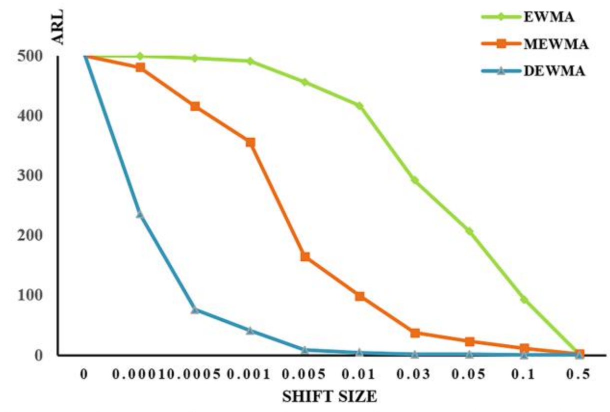
(a) ARL for $\theta = -0.2$ given ARL_0=370



(b) ARL for $\theta = -0.2$ given ARL_0=500

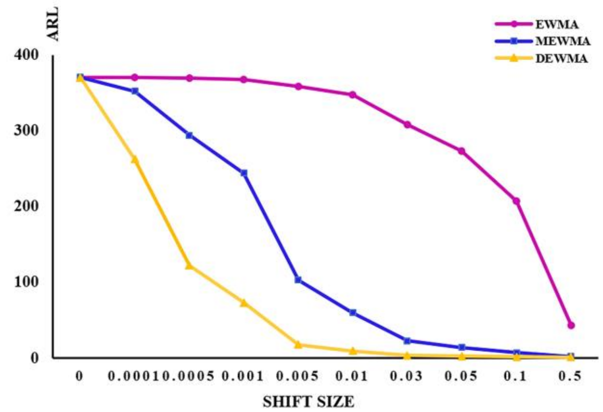


(c) ARL for $\theta = 0.5$ given ARL_0=370

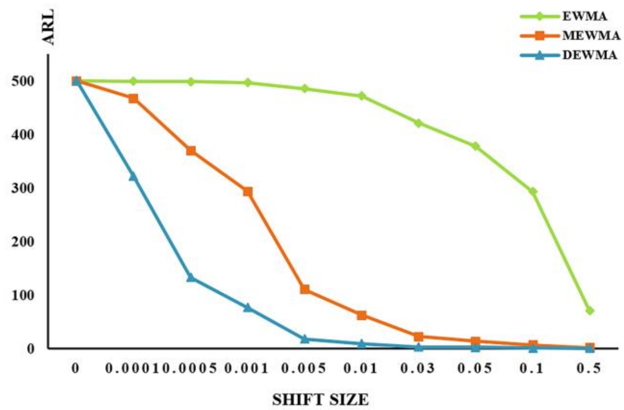


(d) ARL for $\theta = 0.5$ given ARL_0=500

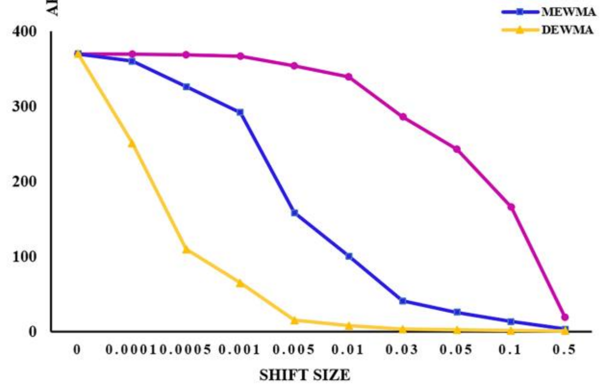
Figure 2. The ARL comparison of the EWMA, modified EWMA and double EWMA charts for MA(1) process



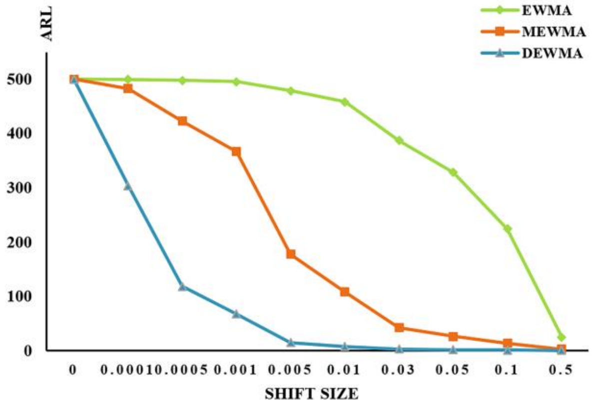
(a) ARL for $\theta_1 = -0.6, \theta_2 = -0.3$ given ARL_0=370



(b) ARL for $\theta_1 = -0.6, \theta_2 = -0.3$ given ARL_0=500

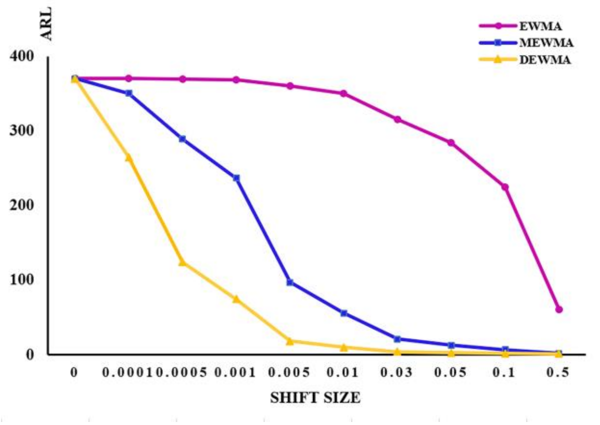


(c) ARL for $\theta_1 = 0.4, \theta_2 = 0.5$ given ARL_0=370

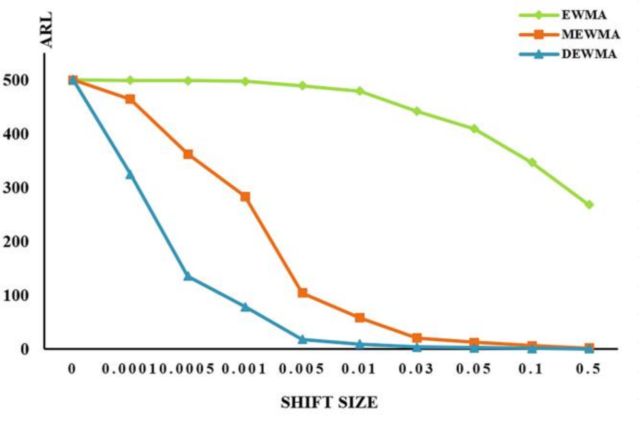


(d) ARL for $\theta_1 = 0.4, \theta_2 = 0.5$ given ARL_0=500

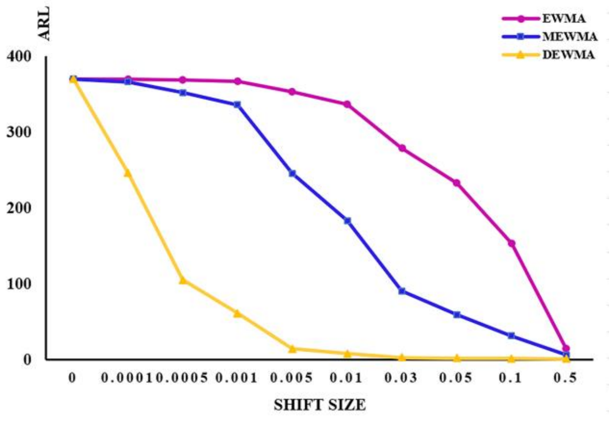
Figure 3. The ARL comparison of the EWMA, modified EWMA and double EWMA charts for MA(2) process



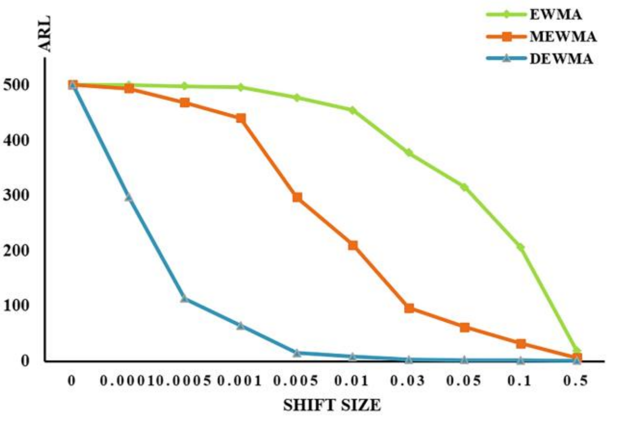
(a) ARL for $\theta_1 = -0.5, \theta_2 = -0.3, \theta_3 = -0.4$ given ARL_0=370



(b) ARL for $\theta_1 = -0.5, \theta_2 = -0.3, \theta_3 = -0.4$ given ARL_0=500



(c) ARL for $\theta_1 = 0.4, \theta_2 = 0.7, \theta_3 = 0.6$ given ARL_0=370



(d) ARL for $\theta_1 = 0.4, \theta_2 = 0.7, \theta_3 = 0.6$ given ARL_0=500

Figure 4. The ARL comparison of the EWMA, modified EWMA and double EWMA charts for MA(3) process

5- Application of Real Data

It is well-known that commodity prices are subject to volatility over time, which depends on demand .When the prices of energy products including oil and natural gas rise, the prices of agricultural and other essential products also rise due to increased transportation costs .Thus, commodity prices play a major role in the economy .Thus, using a control chart to anticipate future market changes provides an interesting opportunity.

The real datasets used in this study comprise the price data for major industrial commodities (diesel, gasoline 95, and NGV (natural gas for vehicles) prices; unit: Baht) in Thailand. The Box-Jenkins technique was used to determine whether the observations were autocorrelated, while the t-statistic was used to assess whether the time-series data follow an MA(q) process. Dataset 1 comprises the prices of diesel collected every three months from the third quarter of 2011 to the fourth quarter of 2022. The 39 observations were proved to be autocorrelated and suitable for the MA(1) model written as follows:

$X_i = 26.947 - 0.649\varepsilon_{i-1} + \varepsilon_i$, where $\varepsilon_i \sim Exp(1.5563)$

Dataset 2 comprises the prices of gasoline 95 from the fourth quarter of 2013 to the fourth quarter of 2022. The 29 observations were proved to be autocorrelated and suitable for an MA(2) model expressed as follows:

$X_i = 34.405 - 0.852\varepsilon_{i-1} - 0.389\varepsilon_{i-2} + \varepsilon_i$, where $\varepsilon_i \sim Exp(1.4950)$

Dataset 3 comprises the prices of NGV from the third quarter of 2011 to the fourth quarter of 2022. The 39 observations were proved to be autocorrelated and suitable for an MA(3) model expressed as follows:

$X_i = 13.093 - 0.842\varepsilon_{i-1} - 0.747\varepsilon_{i-2} - 0.753\varepsilon_{i-3} + \varepsilon_i$, where $\varepsilon_i \sim Exp(0.6354)$.

The ARLs on the MA(1), MA(2), and MA(3) processes comprising diesel, gasoline 95, and NGV prices running on a DEWMA control chart for $\lambda_1 = \lambda_2 = \lambda = 0.1$are reported in Tables 7 to 9, respectively. The performance evaluation of the proposed scheme was also conducted to assess the ARLs of the processes derived by using explicit formulas running on EWMA [30] and MEWMA [23, 31] control charts, where h_1 and h_2 are the UCLs for the EWMA control chart, b_1 and b_2 are the UCLs for the MEWMA control chart, and u_1 and u_2 are the UCLs for the DEWMA control chart for ARL_0 = 370 and 500, respectively.

Table 7. Comparison ARLs for MA(1) for the diesel price of EWMA, modified EWMA and Double EWMA using explicit formula for $\mu = 26.947, \theta = -0.649$, $\lambda_1 = \lambda_2 = \lambda = 0.1$ when $\alpha_0 = 1.5563$ given ARL_0=370 and 500

δ	ARL_0=370			ARL_0=500		
	EWMA ($h_1 = 7.58 \times 10^{-9}$)	MEWMA ($b_1 = 6.0865 \times 10^{-8}$)	DEWMA ($u_1 = 9.3492 \times 10^{-11}$)	EWMA ($h_2 = 1.025 \times 10^{-8}$)	MEWMA ($b_2 = 6.0937 \times 10^{-8}$)	DEWMA ($u_2 = 9.35576 \times 10^{-11}$)
0.0000	370.203716	370.716740	370.639850	500.253042	500.217084	500.450172
0.0001	369.640784	295.317482	251.634933	499.491821	372.029380	305.324007
0.0005	367.398327	162.887955	110.369658	496.459476	183.773548	119.542412
0.0010	364.615998	97.437801	65.025627	492.697092	112.624145	68.084105
0.0050	343.163528	26.615320	15.537699	463.688143	27.622638	15.697846
0.0100	318.254521	14.171646	8.206145	430.005120	14.310240	8.247845,
0.0300	254.600096	5.024696	3.164868	319.555180	5.040943	3.169683
0.0500	236.575442	3.166451	2.155496	239.263742	3.172360	2.157246
0.1000	88.847375	1.815094	1.426273	119.790975	1.816539	1.426700
0.5000	2.106560	1.017543	1.007627	2.496338	1.017564	1.007633

Table 8. Comparison ARLs for MA(2) for the gasoline 95 price of EWMA, modified EWMA and Double EWMA using explicit formula for $\mu = 34.405$, $\theta_1 = -0.852$, $\theta_2 = -0.389$, $\lambda_1 = \lambda_2 = \lambda = 0.1$ when$\alpha_0 = 1.4950$ given ARL_0=370 and 500

δ	ARL_0=370			ARL_0=500		
	EWMA ($h_1 = 1.518 \times 10^{-9}$)	MEWMA ($b_1 = 1.3274 \times 10^{-10}$)	DEWMA ($u_1 = 1.42052 \times 10^{-12}$)	EWMA ($h_2 = 2.052 \times 10^{-9}$)	MEWMA ($b_2 = 1.329 \times 10^{-10}$)	DEWMA ($u_2 = 1.42152 \times 10^{-12}$)
0.0000	370.312651	370.622974	370.503145	500.228959	500.298538	500.034988
0.0001	369.687810	274.171177	232.190478	499.384312	339.199810	277.107458
0.0005	367.199808	135.482679	93.471109	496.021086	149.299782	99.953362
0.0010	364.115217	82.946390	53.680486	491.851400	88.045176	55.775642
0.0050	340.423239	20.341692	12.560368	459.825089	20.626109	12.663342
0.0100	313.124381	10.554741	6.660192	422.923072	10.631371	6.686891
0.0300	225.394810	3.763853	2.638217	304.332115	3.772591	2.641262
0.0500	163.680166	2.412408	1.842751	220.907577	2.415516	1.843842
0.1000	76.362943	1.463881	1.282056	102.874018	1.464595	1.282310
0.5000	1.624144	1.003332	1.002307	1.843704	1.003336	1.002308

Table 9. Comparison ARLs for MA(3) for the NGV price of EWMA, modified EWMA and Double EWMA using explicit formula for $\mu = 13.093$, $\theta_1 = -0.842$, $\theta_2 = -0.747$, $\theta_3 = -0.753$,$\lambda_1 = \lambda_2 = \lambda = 0.1$ when$\alpha_0 = 0.6354$ given ARL_0=370 and 500

δ	ARL_0=370			ARL_0=500		
	EWMA($h_1 = 3.016 \times 10^{-7}$)	MEWMA($b_1 = 7.9087 \times 10^{-11}$)	DEWMA($u_1 = 3.03148 \times 10^{-11}$)	EWMA($h_2 = 4.077 \times 10^{-7}$)	MEWMA($b_2 = 7.9287 \times 10^{-11}$)	DEWMA($u_2 = 4.15585 \times 10^{-11}$)
0.0000	370.287689	370.386144	370.380825	500.199155	500.083732	500.521507
0.0001	369.175418	267.416281	170.697861	498.695599	329.179753	195.721913
0.0005	364.763097	126.075293	54.398606	492.731070	138.315989	57.310969
0.0010	359.329419	75.657213	32.536879	485.385883	79.979176	30.639940
0.0050	318.944221	17.539143	6.882687	430.793607	17.776736	6.884161
0.0100	275.358308	8.683398	3.397220	371.874636	8.745227	3.755962
0.0300	156.438698	2.737418	2.151639	211.120380	2.744051	1.687787
0.0500	91.953642	1.689314	1.205840	123.950169	1.691421	1.305511
0.1000	27.969293	1.120333	1.050986	37.456806	1.120649	1.071243
0.5000	1.069228	1.000048	1.000101	1.093582	1.000048	1.000139

The ARL results for the real processes in Tables 7 to 9 are in good agreement with those from the experimental study. The DEWMA control chart was exceptionally good at detecting shifts in the process mean for any magnitude of shift size for all of the scenarios tested except for $\delta = 0.5$ with the MA(3) process (NGV price), for which the MEWMA control chart provides the smallest ARL_1.

6- Conclusion

Herein, the computation of the ARL of an MA(q) process with exponential white noise running on a DEWMA control chart using explicit formulas is presented .For comparison, the NIE method was applied to approximate the ARL under the control limit of the explicit formulas to check the accuracy of the explicit formulas method .In the experimental study, the parameters for the control chart and observations were varied to cover as many cases as possible .The experimental results provided by using both methods were in excellent agreement with only small APREs.

In the comparative study of the capability of the control charts, the EWMA and MEWMA control charts were compared with the DEWMA control in terms of effectiveness by using the explicit formulas to derive their ARLs for MA(q) processes with exponential white noise .Both the experimental study and application using real datasets of the prices of major industrial commodities in Thailand provided similar ARL results .Moreover, the proposed procedure for deriving the ARL of MA(q) processes with white noise running on a DEWMA control chart was superior to the others. The desired ARL_1 values of the DEWMA control chart detected changes in the process mean more quickly for all magnitudes of shift size studied. That is to say, the DEWMA control chart is appropriate for monitoring and detecting very small shifts in the process mean of MA(q) processes with exponential white noise. In addition, the comparison of control charts' efficiencies revealed that the DEWMA control chart performed the best in almost all cases except for the MA(3) model for the NGV price data, with which the MEWMA control chart yielded the smallest ARL for a process mean shift size of 0.5. The findings from the ARL evaluation using explicit formulas indicate that the DEWMA control chart performed well for most of the investigated scenarios. However, we also discovered that the EWMA and MEWMA control charts performed well when the exponential smoothing parameter was 0.05 and - 0.1, respectively, and are thus good alternatives for detecting shifts in the mean of MA(q) processes with exponential white noise.

7- Declarations

7-1- Author Contributions

Conceptualization, Y.S. and Y.A.; methodology, Y.S.; software, Y.S.; validation, Y.S. and Y.A.; formal analysis, Y.S..; investigation, Y.A.; resources, Y.A.; data curation, Y.A.; writing—original draft preparation, Y.S.; writing—review and editing, Y.A. and Y.S.; visualization, Y.S.; supervision, Y.A.; project administration, Y.A.; funding acquisition, Y.A. All authors have read and agreed to the published version of the manuscript.

7-2- Data Availability Statement

The dataset to be prices of major industrial commodities in Thailand can be found here: https://app.bot.or.th/BTWS_STAT/statistics/BOTWEBSTAT.aspx?reportID=90&language=ENG.

7-3- Funding

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7-4- Institutional Review Board Statement

Not applicable.

7-5- Informed Consent Statement

Not applicable.

7-6- Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this manuscript. In addition, the ethical issues, including plagiarism, informed consent, misconduct, data fabrication and/or falsification, double publication and/or submission, and redundancies have been completely observed by the authors.

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