Available online at www.ijournalse.org

Emerging Science Journal

(ISSN: 2610-9182)

Vol. 7, No. 4, August, 2023

On the Locating Rainbow Connection Number of Trees and Regular Bipartite Graphs

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Abstract

Locating the rainbow connection number of graphs is a new mathematical concept that combines the concepts of the rainbow vertex coloring and the partition dimension. In this research, we determine the lower and upper bounds of the locating rainbow connection number of a graph and provide the characterization of graphs with the locating rainbow connection number equal to its upper and lower bounds to restrict the upper and lower bounds of the locating rainbow connection number of a graph. We also found the locating rainbow connection number of trees and regular bipartite graphs. The method used in this study is a deductive method that begins with a literature study related to relevant previous research concepts and results, making hypotheses, conducting proofs, and drawing conclusions. This research concludes that only path graphs with orders 2, 3, 4, and complete graphs have a locating rainbow connection number equal to 2 and the order of graph G, respectively. We also showed that the locating rainbow connection number of bipartite regular graphs is in the range of $r - \left\lfloor \frac{n}{4} \right\rfloor + 2$ to $\frac{n}{2} + 1$, and the locating rainbow connection number of a tree is determined based on the maximum number of pendants or the maximum number of internal vertices.

Keywords:

Bipartite Regular Graph;

Locating Rainbow Connection Number;

Rainbow Code;

Rainbow Vertex Connection Number;

Rainbow Vertex Path; Tree.

Article History:

Received:	23	January	2023
Revised:	20	June	2023
Accepted:	04	July	2023
Available online:	12	July	2023

1- Introduction

Graph theory is a necessary field in discrete mathematics. One of the interesting concepts in graph theory is graph coloring. Graph coloring has become increasingly popular since almost all problems inconceivable to any discipline can be solved by using graph models. The graph coloring problem field in discrete mathematics, such as vertex coloring, is an NP-complete problem. Besides that, optimization of vertex coloring is an NP-hard problem. For example, the concept of chromatic coloring [1] is one of the solutions to optimizing the time to finish all the schedules without conflicts using the discretization algorithm.

In recent years, various concepts of graph coloring have been developed, and the rainbow connection number of a graph, which was first shown in 2008 [2], is one of them. This concept was inspired by secret communications between government agencies to secure the distribution of classified information after the terrorist attacks on September 11, 2001, which resulted in an observation made by Ericksen [3]. The deadly attack that occurred significantly impacted the

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DOI: http://dx.doi.org/10.28991/ESJ-2023-07-04-016

security of state data. The incident sparked the realization that intelligence agencies and law enforcement authorities were unable to interact with one another through their usual routes. The technology system is a separate entity, and shared access is prohibited. In other words, the authorities have no way of cross-checking information from one organization to another. Meanwhile, data related to national security must be protected, and procedures must also allow access by authorized parties. This problem can be overcome by establishing communication lines between agencies that can be mediated by other agencies and that require passwords and firewalls large enough to prevent network breakers or hackers but not too large to control [4]. This problem raises the question, "What is the minimum number of passwords or keywords required for some paths to share passwords differently during communication between two institutions?" This problem can be simulated by using rainbow coloring. This concept has been studied widely with a variety of graph operations (e.g., [5–10]).

Motivated by the theory above, in 2010, Krivelevich & Yuster first studied a new concept in graph coloring, i.e., the rainbow vertex connection number of a graph [11]. This concept has undergone significant development. Many researchers apply it to various classes of graphs, such as pencil graphs [12], connected graphs [13], star fan graphs [14], and some special graphs [15]. Chen et al. [16, 17] provided the rainbow vertex connection of a graph and its complexity and showed that it was an NP-hard problem. The NP-hard problem is at least as complex as the NP problem but can be much more difficult or complex. Solving the NP-hard problem will lead to the discovery of an algorithm with a polynomial running time for all NP problems. Hence, research related to the NP-hard problem is very interesting. Therefore, a new topic that combines the concepts of rainbow vertex coloring and partition dimensions [18] was introduced in 2021, and the concept is called locating rainbow connection numbers [19].

Let G be a simple, connected, and undirected graph with order n. For $k \in N$, the color of vertices of G is known as a *rainbow vertex k-coloring* if there is a function $\alpha: V(G) \longrightarrow \{1,2, \ldots, k\}$ so that for each distinct pair of vertices $x, y \in V(G)$ there is *rainbow vertex $x - y$-path* whose internal vertices are assigned a different color. The *rainbow vertex connection number* of G, which is given by the equation $rvc(G) = min\{k: k \in N\}$, such that G has a rainbow vertex k-coloring. For $i \in \{1, 2, \ldots, k\}$, let R_i be the set of vertices that have the color i and $\Pi = \{R_1, R_2, \ldots, R_k\}$ be an ordered partition of $V(G)$. Writing $rc_\Pi(v) = \big(d(v, R_1), d(v, R_2), \ldots, d(v, R_k)\big)$, where $d(v, R_i) = \min\{\, d(v, y): y \in R_i\}$ for every $i \in \{1,2, \ldots, k\}$. Further, we call $rc_\Pi(v)$ as the *rainbow code* of v of G with respect to Π. If $rc_\Pi(v_j) \neq rc_\Pi(v_l)$ for distinct $j, l \in \{1,2, \ldots, n\}$ then α is called a *locating rainbow k-coloring* of G. The *locating rainbow connection number* of G is denoted by the $rvcl(G) = min\{k: k \in N\}$ so that G has a locating rainbow k-coloring. Every locating rainbow k-coloring of G is a rainbow vertex coloring of G, therefore we get $rvc(G) \leq rvcl(G)$. Based on an easy observation by Yuster and Krivelevich [11], $rvc(G)$ is more than or equal to the diameter minus one. Therefore, the following logical conclusion holds true.

Corollary 1.1. *If G is a simple connected graph and $diam(G)$ denotes a diameter of G, then $rvcl(G) \geq diam(G) - 1$.*

One of the applications of the locating rainbow connection number is for building security systems (see Figure 1). Suppose that G represents a system for placing biometric scanning devices at the doors of every room in a building. The edges and vertices of graph G represent the hallways that connect the doors and the doors of all rooms, respectively. The color of the vertex represents the type of biometric scanner.

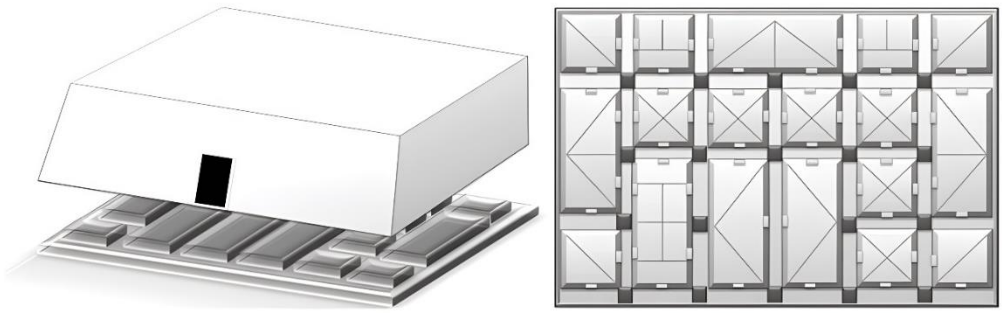


Figure 1. A building plan with several secret rooms

Biometric scanning devices can be categorized into several types, such as facial patterns, irises, fingerprints, voices, and palms or fingers, with each type utilizing different biometric authentication methods that measure and analyze unique characteristics inherent to an individual to confirm their identity.

In a placement system for building security, it may be tempting to install the same type of biometric scanner at every door. However, if a hacker manages to compromise or damage one of the scanners, they can easily access all the rooms in the building. Therefore, the most effective way to ensure high security is to equip each door with a different type of biometric scanning device. However, this approach can be expensive, especially if there are only a limited number of biometric identifiers available. To address this issue, we aim to minimize the number of scanning device types required by using the theory of rainbow vertex connection. The minimum number of device types needed can be expressed as the rainbow vertex connection number of a graph.

To improve the security system, it is essential to take further steps. One possible measure is to assign a unique code to each door based on its type of biometric scanning device. This way, if access to certain rooms is compromised or damaged, the building owner can easily detect which rooms are affected. This approach can be implemented using the concept of partition dimensions. For a visual representation, refer to Figure 2. In this figure, the vertices of G correspond to the doors where the scanning device is installed, and each vertex's color indicates the type of scanning device used. For instance, the green color represents the type 1 scanning device, so the door equipped with this device is included in the R_1 set. Similarly, the light blue color represents the type 2 scanning device, and doors fitted with this device belong to the R_2set, and so on for the other device types.

Suppose the biometric scanner at the door of room v_2 is damaged. The installation system that uses the concept of the locating rainbow connection number offers two advantages. First, it is easy to detect room v_2 since each room has a unique door code. In this case, a signal will send a unique code (1,0,1,2,2,1,3,4). Second, after identifying the damaged room, the building owner can immediately determine the closest rooms to v_2 and increase their security system by using the unique code of v_2. Based on that unique code, the security system of the rooms in sets R_1, R_3, and R_6 must be improved because these sets contain the rooms closest to v_2. Therefore, these rooms can be the first to be affected by the damage to room v_2. For the same reason, the rooms that require security system improvements are those in sets R_4 and R_5, followed by those in R_7 and R_8 (refer to Figure 2).

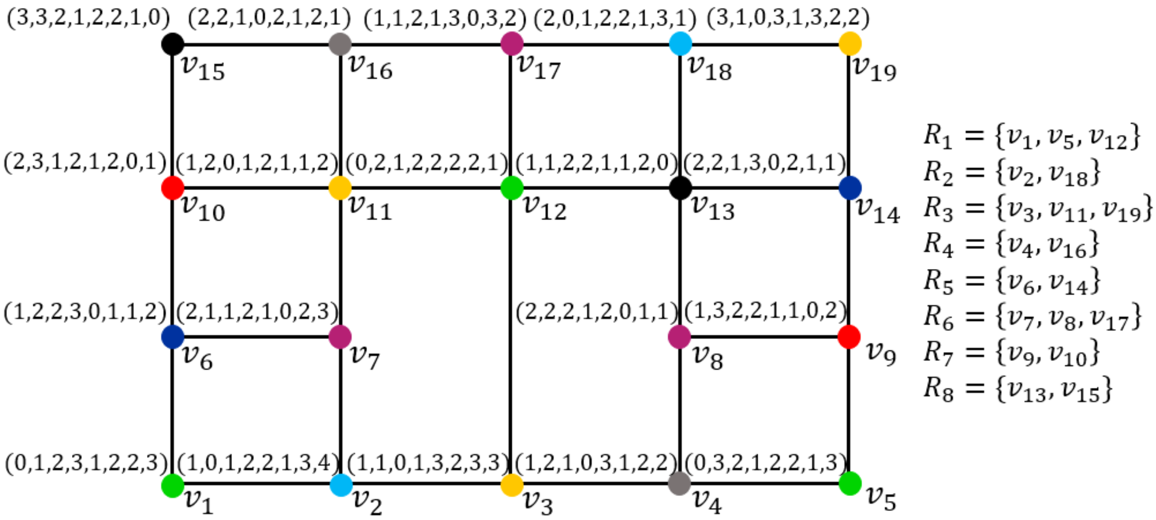


Figure 2. A representation of a building plan in a graph

Taking into consideration the benefits of applying the concept of locating rainbow coloring, we are interested in developing it further by applying it to various classes of graphs, thereby enabling a wider range of applications for the concept of locating rainbow coloring.

In Bustan et al. [19], lower and upper bounds for $rvcl(G)$ were established, with the smallest value being 2 and the largest being the order of a graph G. Consequently, this study identified graphs with locating rainbow connection numbers equal to 2 and the order of a graph G, resulting in a new range of values for the locating rainbow connection number for restricting the upper and lower bounds of $rvcl(G)$ to help other researchers determine the $rvcl(G)$.

Moreover, in addition to providing characterizations, we determined the locating rainbow connection number for a large class of graphs. Additionally, in Bustan et al. [19], the locating rainbow connection number was already determined for various simple graph classes, including stars and paths, which are both included in trees. Furthermore, since every tree is a bipartite graph, this study also presents the locating rainbow connection number for trees and other bipartite graph classes.

2- Research Methodology

This study uses the deductive method, which begins with a review of existing literature on the theory of locating rainbow coloring, rainbow vertex coloring, partition dimension, and the characteristics of the graphs studied in this paper, such as complete graphs, graphs containing cycles, trees, and regular bipartite graphs. Based on the information gathered through the literature review, we identified several statements and formulated them as hypotheses regarding the location of the rainbow connection number, along with the necessary proof methods to establish them.

The proof process is divided into two stages: proving the lower bound and the upper bound. For the lower bound, it is known that the value of the locating rainbow connection number of a graph is never less than the diameter minus one, the maximum number of leaves adjacent to a vertex, or the number of cut vertices. If the lower bound is trivial, i.e., if it satisfies one of the properties, then the proof is straightforward. Otherwise, the proof is established through contradiction or contrapositive. For the upper bound, it is known that the maximum value of the locating rainbow connection number of a graph is equal to the order of the graph. If it is less than the order, the proof is established through a vertex coloring construction, where the given coloring not only shows that there is always a rainbow path between any two vertices but also that all vertices have different rainbow codes.

After going through the proof process, if the given hypothesis is proven to be true, it will be concluded as a lemma, theorem, or corollary. However, if the hypothesis is proven false, the research will be repeated by reconsidering the hypothesis and the proof methods used.

The flowchart for this study is presented in Figure 3.

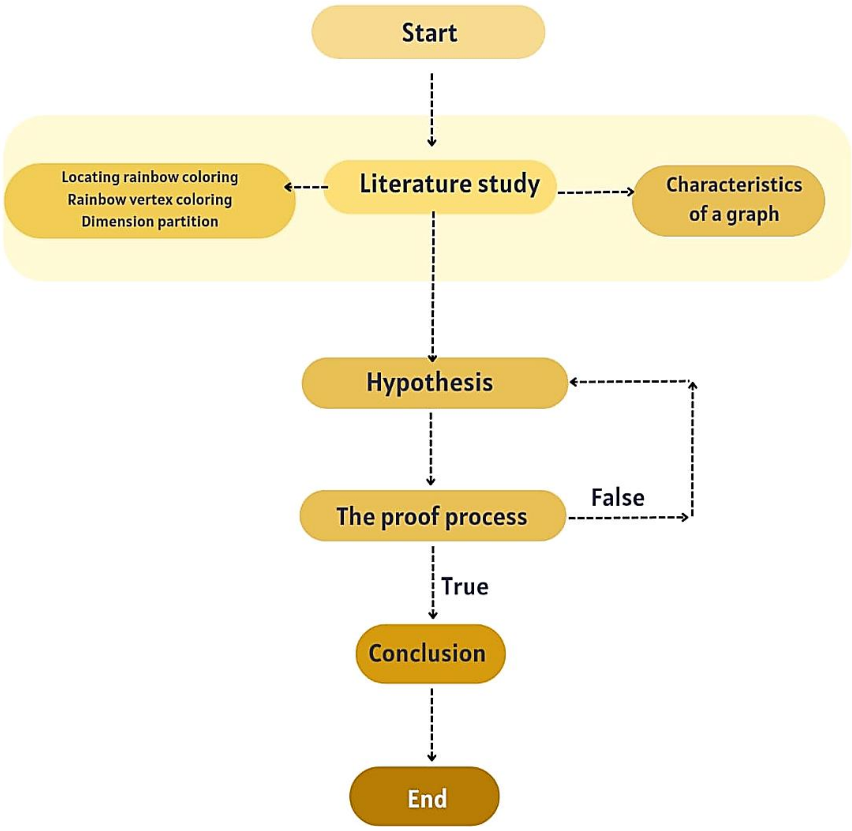


Figure 3. Research process flowchart

3- Main Results

All results in this paper are given in this section. In particular, in Subsections 3-1 and 3-2, we provide a characterization of graphs by locating rainbow connection numbers that are equal to 2 or their order. In Subsection 3-3 we determine locating rainbow connection number of trees, and in Subsections 3-4, we consider some regular bipartite graphs to calculate the locating rainbow connection number for those graphs. For simplification, denote $\{n \in N \mid a \leq n \leq b\}$ by $[a, b]$.

3-1- Graphs with Locating Rainbow Connection Number 2

In [19, Theorem 2.2], one of the results shows the locating rainbow connection number of paths. In the main theorem of this subsection, we show that a path P_n for $n \in [2,4]$ is the only graph with $rvcl(G) = 2$. The next step is to make sure that Lemma 3.1, which will be applied in the proof of Theorem 3.1, is correct.

Lemma 3.1. *If G is a simple connected graph of order $n \geq 3$ which contains a cycle, then $rvcl(G) \geq 3$*

Proof. Suppose that $rvcl(G)$ is less than or equals 2 and by [19, Lemma 2.1], we get $rvcl(G) = 2$. Since $rvcl(G) \geq rvc(G)$, based on Corollary 1.1, we have $diam(G) \leq 3$. Let $C = u, v, \ldots, t, u$ be the shortest cycle contained in G. Since $rvcl(G) = 2$, we consider two cases.

First, suppose all vertices in C have the same color, say color 1, if $G \cong C$, by [19, Lemma 2.1], we have a contradiction. Conversely, then there exists $x \notin C$ and without loss of generality, let $ux \in E(G)$. The cyclegraph has an order of at least three, so that if $diam(G) \leq 3$, then color 1 can only be used a maximum of three times $(0,1), (0,2), (0,3)$, consequently $c(x) = 2$. If $xt \in E(G)$ or $xv \in E(G)$, therefore $rc_\Pi(t) = rc_\Pi(u) = (0,1)$ or $rc_\Pi(x) = rc_\Pi(u) = (0,1)$. If $xt \notin E(G)$ or $xv \notin E(G)$, thus $rc_\Pi(t) = rc_\Pi(v) = (0,2)$, a contradiction.

Second, suppose all vertices in C are assigned with colors 1 and 2. We consider two subcases, the odd cycle, and the even cycle. Since we have 2 colors, based on [19, Theorem 2.4] for an odd cycle, there are at least two vertices with the same colors and rainbow codes, which is a contradiction.

For an even cycle, there are two different vertices, w and s, so that $ws \in E(G)$ in C, $c(w) = 1$ and $c(s) = 2$. Since it is an even cycle, we have at least two other vertices in C, so there are $wy \in E(G)$ and $st \in E(G)$. To make all vertices have different rainbow codes, $c(y) = 1, c(t) = 2$. Consider yt, if $yt \in E(G)$, then $rc_\Pi(y) = rc_\Pi(w)$, a contradiction. Conversely, if $yt \notin E(G)$ by [19, Theorem 2.4], we only have a maximum of two other vertices, which are uncolored.

Since G contains an even cycle, there are only three possible conditions: one vertex in G and not in C, two vertices in G and not in C, and two vertices in C. For one vertex in G but not in C or two vertices in G but not in C, we get $rc_\Pi(t) = rc_\Pi(s) = (1,0)$.

For two vertices in C, there are at least $rz \in E(G)$ in C, where $ry \in E(G)$ and $zr \in E(G)$. To get $rc_\Pi(r) \neq rc_\Pi(s)$, it must be $c(r) = 1$ and $rc_\Pi(z) = rc_\Pi(w) = (0,1)$. Therefore, there isa contradiction; thus, $rvcl(G) \geq 3$.

Furthermore, to prove Theorem 3.1, the adjacency properties of the vertex in the graph will be used. Consider that $|N(v)|$ is the number of vertices adjacent to the vertex v, for $v \in V(G)$.

Theorem 3.1. *Let G be a connected graph with order $n \in \{2,3,4\}$. Then $rvcl(G) = 2$ if and only if G is isomorphic to a path of order n.*

Proof. Let G be a path graph of order n. Based on [19, Theorem 2.2], we get $rvcl(P_n) = 2$ for $n \in [2,4]$. Conversely, suppose $rvcl(G) = 2$. Based on Lemma 3.1, Corollary 1.1, and [19, Lemma 2.2], we have G as a tree with $l \leq 2$ and $diam(G) \leq 3$. This means that the graph has a maximum of three edges with a maximal order of 4, and a vertex that is adjacent maximally to two pendants. Hence, the graph with $rvcl(G) = 2$ are P_2, P_3, and P_4 as shown in Figure 4.

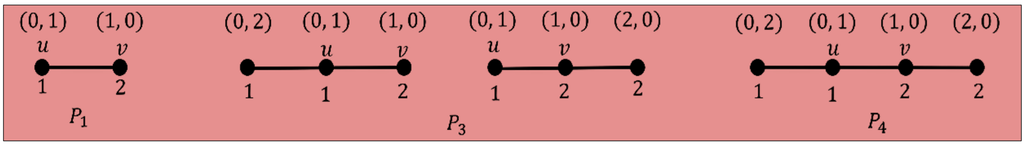


Figure 4. The locating rainbow 2-coloring of P_2, P_3, and P_4

3-2- Graphs with Locating Rainbow Connection Number n

In Bustan et al. [19], we know that $rvcl(K_n) = n$, where K_n is a complete graph of order n. Next, we show that complete graphs are the only graph classes with $rvcl(G) = n$. We use the cut vertex properties of the graph in order to show Theorem 3.1. A vertex $v \in V(G)$ is said to be a *cut vertex* if $G - v$ make graph G disconnected.

Theorem 3.2. *Suppose G is a connected graph of order $n \geq 3$. Then $rvcl(G) = n$ if and only if G is isomorphic to complete graphs.*

Proof. Suppose that G is not a complete graph. Then G has two vertices, u and v of G so that $uv \notin E(G)$. Consider the shortest path $P = u, w, \ldots, v$ of G. We have two cases.

First, u and w are not cut vertices. We make $c(u) = c(w) = 1$, and color $n-1, n-2, \ldots, 2$ to other vertices, differently. As u and w are not cut vertices, for every pair of vertices connected by a path containing u and w, there exists an alternative path that does not contain w and u, which is obviously a rainbow vertex path. Since all vertices besides other than vertices u and w have different colors, any path connecting every two vertices in G and not containing vertices u and w is the rainbow vertex path.

Furthermore, each vertex with a different color must have a different rainbow code. We only consider vertices u and w, with $c(u) = c(w) = 1$. Since $P = u, w, \ldots, v$ is the shortest path connecting vertices u and v; consequently, $d(w, v) < d(u, v)$, so $rc_\Pi(w) \neq rc_\Pi(u)$.

Second, u and/or w are cut vertices. Without loss of generality, let u be a cut vertex. Since u is a cut vertex, there is at least one vertex y other than vertex w adjacent to vertex u so that $G - \{u\}$ is disconnected and puts y and w on different components (see Figure 5. for illustration). Next, see the U component, which is the subgraph component that contains y in $G - \{u\}$. Choose a vertex in U, say vertex z, which is the farthest vertex from u when in G.

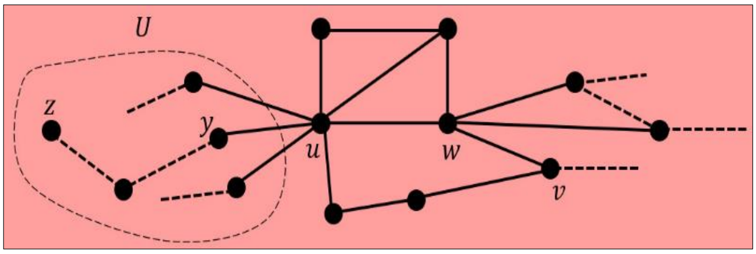


Figure 5. Graph G with cut vertex u

Next, give color 1 for the vertices u and z, $c(u) = c(z) = 1$, and colors $2,3, \ldots, n$ for the other vertices in G differently. Since z and w are in two components, $zw \notin E(G)$. Thus, $d(w,u) = 1$, $d(w,z) > 1$. Therefore, $rc_{\Pi}(z) \neq rc_{\Pi}(w)$.

The rainbow vertex coloring of two cases produces the rainbow vertex coloring of G. Observe that each vertex has a unique rainbow code. Therefore, $rvcl(G) \leq n - 1$, which is a contradiction. Conversely, according to [19, Theorem 2.1], $rvcl(G) = n$, where G is a complete graph.

3-3- Locating Rainbow Connection Number of Trees

In this subsection, we determine the locating rainbow connection number of bipartite graphs, particularly for trees. However, we provide both lower and upper bounds for the locating rainbow connection number of bipartite graphs in general. Let a and b be positive integers, and $n = a + b$. A graph G with order n is called a *bipartite graph*, denoted by $B_{a,b}$ if there exist independent subsets U, V with $|U| = a$ and $|V| = b$, such that every edge of the graph connects one vertex in U to at least one vertex in V. Note that every path is a bipartite graph. According to Theorem 3.1, graphs with locating rainbow connection number 2 are P_n for $n = 2,3,4$. Thus, based on [19, Theorem 2.1], Corollary 3.2 is true.

Corollary 3.2. *For $a \geq 1$ and $b \geq 2$, let $B_{a,b}$ be a bipartite graph of order $a + b$, where $B_{a,b} \neq P_3$ and $B_{a,b} \neq P_4$. Then* $3 \leq rvcl(B_{a,b}) \leq a + b - 1$.

A star is a bipartite graph $B_{1,n-1}$. In [15, Theorem 2.2], we find the locating rainbow connection number of stars, which have a locating rainbow connection number equal to the upper bound of Corollary 3.2. It is known that a bipartite graph has no odd-length cycles. A *tree* is a connected acyclic graph. Therefore, every tree is a bipartite graph. Next, in Theorem 3.3, we show the locating rainbow connection number of trees using internal vertices and pendants.

Theorem 3.3. *Let T be a tree graph with k internal vertices, and l be the maximum number of pendants adjacent to a vertex in T. Then:*

$$rvcl(T) = \begin{cases} l, & k < l \\ k, & k \geq l \end{cases}$$

Proof. Let p_i be an internal vertex of T, and $p_{i,j}$ is the j-pendant that is adjacent to the internal vertex p_i for $j \in [1, l]$ and $i \in [1, k]$. See Figure 6 for an illustration.

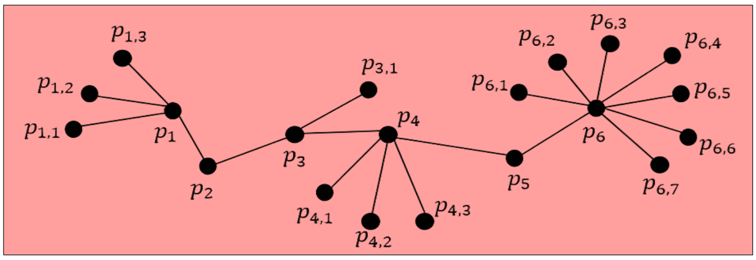


Figure 6. Tree T with $k = 6$ and $l = 7$

First, we prove the lower bound of the locating rainbow connection number of trees. For $k < l$, based on [15, lemma 2.2], we get $rvcl(T) \geq l$. For $k \geq l$, suppose $rvcl(T) \leq k - 1$. As a result, there are two internal vertices, u and v, so that $c(u) = c(v)$. Furthermore, since u and v are not pendants, $d(u) = d(v) \geq 2$. Therefore, there are s and t vertices, where $su \in E(T)$ and $vt \in E(T)$, so that $P = s, u, \ldots v, t$ in T. Since every path on T is unique, there is no rainbow vertex path connecting vertices s and vertex t, which is a contradiction. Thus, $rvcl(T) \geq k$.

Next, we show $rvcl(T) \leq l$ for $k < l$ and $rvcl(T) \leq k$ for $k \geq l$ by defining a vertex coloring $c: V(T) \leftrightarrow [1, max\{k, l\}]$ as follows:

$$c(p_i) = i, \text{ for } i \in [1, l] \tag{1}$$

$c(p_{i,j}) = j$, for $j \in [1, l]$ and $i \in [1, k]$

Since all internal vertices are distinct colors, for every two vertices in T, there is always a rainbow vertex path connecting the vertices. Furthermore, we show that the rainbow codes for each vertex are distinct. From the vertex coloring above, we have the following.

- $c(p_{i,j}) \neq c(p_{i,q})$ for $i \in [1, k]$, and for distinct $j, q \in [1, m]$.
- $c(p_i) \neq c(p_j)$ for $i, j \in [1, k]$.
- $d(p_{i,j}, R_i) = 1$ and $d(p_{q,l}, R_i) > 1$ for distinct $i, q \in [1, k]$, and $j, l \in [1, m]$.
- $c(p_{i,j}) = c(p_i)$ for $i = j$, $d(p_{i,j}, R_{i+1}) = 2$ and $d(p_i, R_{i+1}) = 1$ for $i \in [1, k-1]$ or $d(p_{i,j}, R_{i-1}) = 2$ and $d(p_i, R_{i-1}) = 1$ for $i = k$.
- $c(p_{i,j}) = c(p_a)$ for $i \neq a$, but $|N(p_{i,j})| = 1$ and $|N(p_a)| \geq 2$ where there are at least two vertices $z, w \in N(p_a)$ so that $c(z) \neq c(w) \neq c(p_a)$.

Thus, the rainbow codes for each vertex are distinct. The conditions above apply to $k < l$ and $k \geq l$. Therefore, $rvcl(T) = k$ for $k \geq l$ and $rvcl(T) = l$ for $k < l$.

3-4- Locating Rainbow Connection Number of Regular Bipartite Graphs

A bipartite graph with partite sets U and V with $|U| = |V|$ is called *balanced*. Let $r \in N$ and $r \geq 1$. A bipartite graph whose vertex has a degree r is called a *r-regular bipartite graph*. Note that all r-regular bipartite graphs are balanced. Let $V(G) = \left\{u_i \mid i \in \left[1, \frac{n}{2}\right]\right\} \cup \left\{v_j \mid j \in \left[1, \frac{n}{2}\right]\right\}$, such that $E(G) = \left\{u_i v_j \mid i \in \left[1, \frac{n}{2}\right], j = (i+k) \bmod \frac{n}{2}, k \in [0, r-1]\right\}$. In Theorem 3.4, we show the sharp lower and upper bounds for $rvcl(G)$, where G is r-regular bipartite graphs. We first verify some lemmas to help prove Theorem 3.4.

Lemma 3.2. *Let G be an r-regular bipartite graph with $|G| = n \geq 4$, n be even, and $diam(G)$ denote the diameter of G. If $r \geq \left\lceil \frac{n}{4} \right\rceil + 1$, then $diam(G) = 3$.*

Proof. Consider $u_i u_j$ and $v_i v_j$ for distinct $i, j \in \left[1, \frac{n}{2}\right]$. We claim that the set of neighbors of any two vertices in the same partition set is not empty. Suppose $N(u_i) \cap N(u_j) = \emptyset$. Then there are at least two vertices u_j and u_i so that the two vertices do not have the same neighbors in V. In other words, $|N(u_i)| + |N(u_j)| \leq \frac{n}{2}$. Meanwhile, by condition of r, we have $r \geq \left\lceil \frac{n}{4} \right\rceil + 1$. It means $|N(u_i)| \geq \left\lceil \frac{n}{4} \right\rceil + 1$ and $|N(u_j)| \geq \left\lceil \frac{n}{4} \right\rceil + 1$. Thus, it takes at least $\left(\left\lceil \frac{n}{4} \right\rceil + 1\right) + \left(\left\lceil \frac{n}{4} \right\rceil + 1\right) = 2\left(\left\lceil \frac{n}{4} \right\rceil\right) + 2$ vertices in V, which is contradictory to $|N(u_i)| + |N(u_j)| \leq \frac{n}{2}$. Therefore, $N(u_i) \cap N(u_j) \neq \emptyset$. The same logic applies to obtaining $N(v_i) \cap N(v_j) \neq \emptyset$. Thus $d(u_i, u_j) = d(v_i, v_j) = 2$ for distinct $i, j \in \left[1, \frac{n}{2}\right]$. Next, consider the distance between u_i and v_j, for $i, j \in \left[1, \frac{n}{2}\right]$. We have $u_i v_j \in E(G), j = (i+k) mod \frac{n}{2}$ for $k \in [0, r-1]$. For $k \in \left[r, \frac{n}{2}\right]$, we obtain $u_i v_j \notin E(G)$, but since $r \geq \left\lceil \frac{n}{4} \right\rceil + 1$, we have u_i, v_i, u_j, v_j. Therefore, we always get a path with a maximum length of three, so $diam(G) = 3$.

Lemma 3.3. *Let G be an r-regular bipartite graph with $|G| = n \geq 4$, n be even, R_a be a set of vertices with color a, and $r \geq \left\lceil \frac{n}{4} \right\rceil + 1$. For any distinct $i, j \in \left[1, \frac{n}{2}\right]$.*

1. *If $d(u_i, R_a) = 2$, then $d(u_j, R_a) \neq 3$.*
2. *If $d(u_i, R_a) = 3$, then $d(u_j, R_a) \neq 2$.*

Proof.

1. Given $d(u_i, R_a) = 2$, there is at least a vertex u_j for $i \neq j$, so that $c(u_j) = a$. Based on Lemma 2.2 for $i \neq j$ we have the intersection of sets $N(u_i)$ and $N(u_j)$ that is not empty. Therefore, $d(u_j, R_a) \leq 2$.
2. Given $d(u_i, R_a) = 3$, there is no vertex $u_j \in U$ for $i \neq j$, such that $c(u_j) \neq a$. Therefore, $d(u_i, R_a) \neq 2$.

□

Lemma 3.4. *Let G be an r-regular bipartite graph with $|G| = n \geq 4$, n be even, R_a be a set of vertices with color a, and $r \geq \left\lceil \frac{n}{4} \right\rceil + 1$. If $d(u_i, R_a) = 1$ for some $i \in \left[1, \frac{n}{2}\right]$, then the maximum number of other vertices in U with a distance greater than one from R_a is $\frac{n}{2} - r$.*

Proof. Since $d(u_i, R_a) = 1$, there exists at least one vertex $v \in V$ such that $vu_i \in E(G)$ and $c(v) = a$. Each vertex $v \in V$ is adjacent to r vertices $u \in U$. Therefore, the minimum number of vertices in U which has a distance of one from R_a is r, and the maximum number of other vertices in U with a distance greater than one from R_a is $\frac{n}{2} - r$.

□

Lemma 3.5. *Let G be an r-regular bipartite graph with $|G| = n \geq 4$. Let $r \geq \left\lfloor \frac{n}{4} \right\rfloor + 1$, n be even, and R_j be a set of vertices with color j. Then*

1. $d(u, R_j) \neq 3$ for each $j \in \left[1, r - \left\lfloor \frac{n}{4} \right\rfloor + 1\right]$.
2. $d(u, R_j) = 2$ For only one $k \in \left[1, r - \left\lfloor \frac{n}{4} \right\rfloor + 1\right]$., there exists an exact one j index, such that $d(u, R_j) = 2$.

Proof.

1. By Lemma 3.2, $d(v, R_j) \in \{0,1,2,3\}$. Suppose there is a vertex u, such that $d(u, R_j) = 3$. Without loss of generality, let $d\left(u_1, R_{r-\left\lfloor \frac{n}{4} \right\rfloor+1}\right) = 3$. Then there is at least a vertex $y \in V$ with $u_1 y \notin E(G)$ such that $c(y) = r - \left\lfloor \frac{n}{4} \right\rfloor + 1$ and $c(x) \neq r - \left\lfloor \frac{n}{4} \right\rfloor + 1$, for every $x \in (V(G) - \{y\})$. Next, based on Lemma 2.3, $c(u_i) \neq r - \left\lfloor \frac{n}{4} \right\rfloor + 1$ for $i \in \left[1, \frac{n}{2}\right]$. Since $r - \left\lfloor \frac{n}{4} \right\rfloor$ colors can be used to color $v \in V$ with $u_1 v \in E(G)$, at least one color is used as much as $\frac{n}{2} - r + 1$ times. Without loss of generality, suppose the color is 2. Since $r \geq \left\lfloor \frac{n}{4} \right\rfloor + 2$, the color 2 is used at least twice. Furthermore, from the coloring, at least r vertices on U have a distance 1 to $R_{r-\left\lfloor \frac{n}{4} \right\rfloor+1}$ and R_2. Therefore, it takes at least r different rainbow codes. If color 2 is used exactly twice, then as many $r - \left\lfloor \frac{n}{4} \right\rfloor - 1$ colors can be assigned to those vertices. Meanwhile, if color 2 is used more than twice, then only $r - \left\lfloor \frac{n}{4} \right\rfloor - 2$ can be assigned to those vertices. However, based on Lemma 3.3 and Lemma 3.4, less than r of different rainbow codes could be formed. We get a contradiction. Thus, $d(u, R_j) \neq 3$ for $j \in \left[1, r - \left\lfloor \frac{n}{4} \right\rfloor + 1\right]$.
2. Without loss of generality, suppose that there exists a vertex u_1 such that $d(u_1, R_2) = 2$ and $d(u_1, R_3) = 2$. Thus, at least two vertices, u_a and u_b are in U, so that $c(u_a) = 2$ and $c(u_b) = 3$. Since u_1 is adjacent to r vertices in V, at least $\frac{n}{2} - r + 1$ vertices in $v \in V$ with $u_1 v \in E(G)$ have the same color. Furthermore, a contradiction is obtained in the same way as the prior proof. Therefore, for every $j \in \left[1, r - \left\lfloor \frac{n}{4} \right\rfloor + 1\right]$, there exists exactly one j index, such that $d(u, R_k) = 2$.

Theorem 3.4. *Let G be an r-regular bipartite graph with $|G| = n \geq 4$. If $r \geq 2$, and n is even, then $r - \left\lfloor \frac{n}{4} \right\rfloor + 2 \leq rvcl(G) \leq \frac{n}{2} + 1$.*

Proof. Suppose that $rvcl(G) = r - \left\lfloor \frac{n}{4} \right\rfloor + 1$ and G is an r-regular bipartite graph, partitioned into two sets U and V, where $|U| = |V| = \frac{n}{2}$ and $r \leq |U|$. Furthermore, the proof is divided into two cases as follows.

1. $r = \frac{n}{2}$.

Since $rvcl(G) < r - \left\lfloor \frac{n}{4} \right\rfloor + 1$, there exist two vertices $u_1, u_2 \in U$ such that $c(u_1) = c(u_2)$. Note that each vertex $u \in U$ is adjacent to r vertices of $V(G)$;since $N(u_1) = N(u_2)$, we have $rc_\Pi(u_1) = rc_\Pi(u_2)$, which is a contradiction. Hence, $rvcl(G) \geq r - \left\lfloor \frac{n}{4} \right\rfloor + 2$

2. $r < \frac{n}{2}$.

We consider two sub-cases as follows:

a. $r - \left\lfloor \frac{n}{4} \right\rfloor + 1 \leq 2$.

Based on Corollary 3.2, we have a contradiction. Thus, $rvcl(G) \geq r - \left\lfloor \frac{n}{4} \right\rfloor + 2$.

b. $r - \left\lfloor \frac{n}{4} \right\rfloor + 1 \geq 3$.

We also divide this subcase into two conditions, for $n > r \times diam^{r-1}$ and $n \leq r \times diam^{r-1}$. By [19, Theorem 2.4], we have a contradiction for $n > r \times diam^{r-1}$. Therefore, $rvcl(G) \geq r - \left\lfloor \frac{n}{4} \right\rfloor + 2$. Next, we will prove the same for $n \leq r \times diam^{r-1}$. The proof will be shown by eliminating the rainbow code generated from locating rainbow coloring on the graph with $r - \left\lfloor \frac{n}{4} \right\rfloor + 1$ colors. Since $r \geq \left\lfloor \frac{n}{4} \right\rfloor + 2$, based on Lemma 3.5, the number of distinct rainbow codes formed is $\left(r - \left\lfloor \frac{n}{4} \right\rfloor + 1\right)^2$. If $n \geq \left(r - \left\lfloor \frac{n}{4} \right\rfloor + 1\right)^2 + 1$, there are at least two vertices with the same rainbow code. We have a contradiction. But if $n \leq \left(r - \left\lfloor \frac{n}{4} \right\rfloor + 1\right)^2$, then consider the following conditions.

1) Let c and s be two positive integers, c be a color assigned to the vertices of G, and s be the number of vertices with a distance of more than one from R_c. If we get s only from one set U or V, then based on Lemma 3.4, we have $s \leq \frac{n}{2} - r$ in G, and the maximum number of distinct rainbow codes formed is $\left(\left(r - \left\lfloor\frac{n}{4}\right\rfloor + 1\right)^2\right) - \left(\left(r - \left\lfloor\frac{n}{4}\right\rfloor + 1\right)\left(2r - \left\lfloor\frac{n}{4}\right\rfloor - \frac{n}{2}\right)\right)$.

2) Suppose we get s from sets U and V. Based on Lemma 3.4, we have $s \leq 2\left(\frac{n}{2} - r\right)$ in G. If t is a number of colors c with $t \geq 1$, then $s + t \leq 2\left(\frac{n}{2} - r + 1\right)$.

Based on conditions 1) and 2), at least two vertices have the same rainbow code, which is a contradiction. Thus, $rvcl(G) \geq r - \left\lfloor\frac{n}{4}\right\rfloor + 2$.

Based on the first and second subcases, we obtain $rvcl(G) \geq r - \left\lfloor\frac{n}{4}\right\rfloor + 2$. Furthermore, we show that $rvcl(G) \leq \frac{n}{2} + 1$ by defining a rainbow vertex coloring $c: V(G) \longrightarrow \left[1, \frac{n}{2} + 1\right]$ as follows:

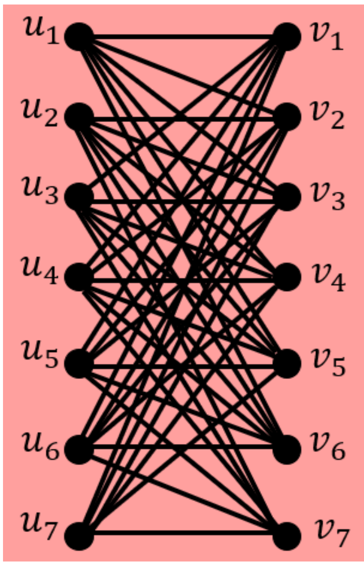
$$c(u_i) = i, \text{ for } i \in \left[1, \frac{n}{2}\right] \tag{2}$$

$$c(v_j) = \begin{cases} \left\lceil\frac{n}{4}\right\rceil + 1, & \text{for } j = 1 \\ \frac{n}{2} + 1, & \text{for } j = 2 \\ j - \left\lfloor\frac{n}{4}\right\rfloor, & \text{for } j \in \left[\left\lfloor\frac{n}{4}\right\rfloor + 1, \frac{n}{2}\right] \\ j + \left(\left\lceil\frac{n}{4}\right\rceil - 1\right), & otherwise \end{cases} \tag{3}$$

Based on the coloring above, we have $c(u_i) \neq c(u_k)$ for distinct $i, k \in \left[1, \frac{n}{2}\right]$ and $c(vj) \neq c(u_l)$ for distinct $j, l \in \left[1, \frac{n}{2}\right]$. We also have all vertices in sets: $\{u_i \,|\, i \in \left[1, \left\lceil\frac{n}{4}\right\rceil\right]\} \cup \{v_j \,|\, j \in \left[1, \left\lfloor\frac{n}{4}\right\rfloor\right]\}$ and $\{u_k \,|\, k \in \left[\left\lceil\frac{n}{4}\right\rceil + 1, \frac{n}{2}\right]\} \cup \{v_l \,|\, l \in \left[\left\lfloor\frac{n}{4}\right\rfloor + 1, \frac{n}{2}\right]\}$ assigned distinct colors. Therefore, there exists a rainbow vertex path between any two vertices. Next, we show that each vertex has different rainbow codes. Based on the coloring above, we have the following.

- Each color has been used a maximum of two times, once in U and again in V.
- Color $\frac{n}{2}$ is only used at the vertex in U.
- Color $\frac{n}{2} + 1$ is only used at vertex in V.
- $d(u_i, u_k)$ and $d(v_j, v_l)$ are even, whereas $d(u_i, v_j)$ and $d(v_j, u_i)$ are odd.

Thus, the rainbow codes for each vertex are distinct. Therefore, we get a rainbow vertex coloring of G at most $\frac{n}{2} + 1$ colors, and all vertices have different color codes. Figure 7 shows the possible rainbow codes of a 6-regular graph with $\frac{n}{2} = 7$, without entry 3, and containing only one entry 2.



R_1	R_2	R_3	R_4
0	1	1	1
0	1	1	2
0	1	2	1
0	2	1	1
1	0	1	1
1	0	1	2
1	0	2	1
2	0	1	1

R_1	R_2	R_3	R_4
1	1	0	1
1	1	0	2
1	2	0	1
2	1	0	1
1	1	1	0
1	1	2	0
1	2	1	0
2	1	1	0

Figure 7. Rainbow codes of a 6-regular graph with $\frac{n}{2} = 7$, without entry 3, and contain only one entry 2.

One of the regular bipartite graphs is the complete bipartite graph $K_{,aa}$. Thus, the next theorem determines the locating rainbow connection number of the complete bipartite graph $K_{,aa}$. Not only that, but we also determined the locating rainbow connection number for all complete bipartite graphs in our findings. A *complete bipartite graph* is a bipartite graph whose every two vertices, $u \in U$ and $v \in V$, are connected by an edge in $E(G)$. A complete bipartite graph with partitions of size $|U| = a$ and $|V| = b$ is denoted by $K_{a,b}$. We now show the locating rainbow connection of the complete bipartite graph.

Theorem 3.5 *Let $K_{a,b}$ be a complete bipartite graph with $1 \le a \le b$. Then*

$$rvcl(K_{a,b}) = \begin{cases} a, & for\ a > b \\ a+1, & for\ a = b \end{cases}$$

Proof.

1. $a > b$.

Suppose $rvcl(K_{a,b}) = a - 1$. Consequently, there exist two vertices $u_1, u_2 \in U$ such that $c(u_1) = c(u_2)$. Since $d(u_1, v_i) = d(u_2, v_i)$ for $i \in [1, b]$ and $d(u_1, u_i) = d(u_2, u_i) = 2$ for $i \in [3, a]$, we obtain $rc_\Pi(u_1) = rc_\Pi(u_2)$, which is a contradiction. Hence $rvcl(K_{a,b}) \ge a$. Furthermore, to show $rvcl(K_{a,b}) \le a$, we define a vertex coloring $c: V(K_{a,b}) \longrightarrow [1, a]$ as follows:

$$\begin{aligned} &c(u_i) = i, \text{ for } i \in [1, a]; \\ &c(v_j) = j, \text{ for } j \in [1, b] \end{aligned} \tag{4}$$

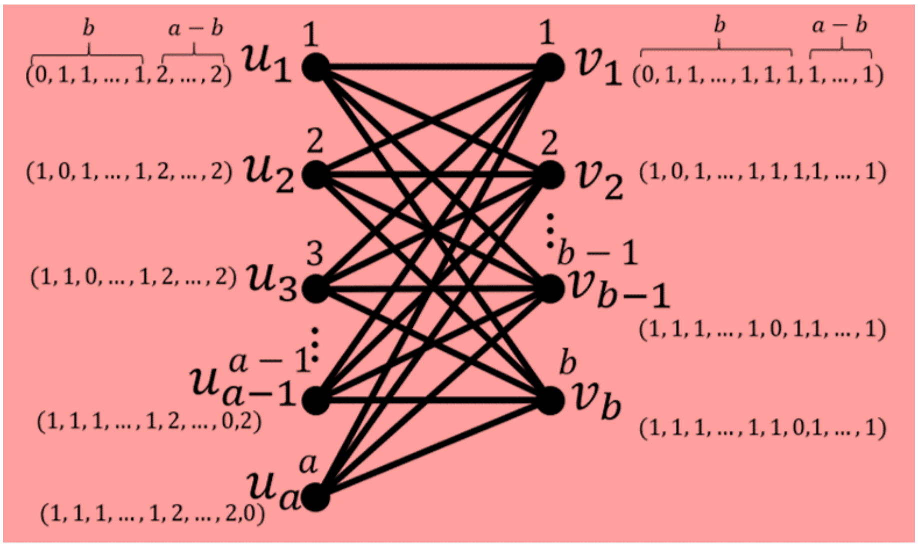


Figure 8. Rainbow code of $K_{a,b}$ for $a > b$

In fact, the diameter of a complete bipartite graph $K_{a,b}$ is 2, and by coloring graph $K_{a,b}$ with a colors, we can identify a rainbow vertex path connecting any two vertices on graph $K_{a,b}$. Additionally, by the vertex coloring above, for $i = j$ we get $c(u_i) = c(v_j)$. Because $a > b$, so $c(v_j) \ne a$. Therefore, $d(u_i, R_a) = 2$ for $i \in [1, a-1]$ and $d(v_j, R_a) = 1$ for $j \in [1, b]$, such that each vertex of $K_{a,b}$ has a unique rainbow code. Thus, $rvcl(K_{a,b}) = a$.

2. $a = b$

Since the distances between every two vertices in U or V and other vertices in $K_{a,a}$ are the same, every vertex in the same partition will be given a distinct colors. Consequently, there are two vertices, u and v, where $c(u) = c(v)$, so $rc_\Pi(u) = rc_\Pi(v)$, and we obtain a contradiction. Thus, $rvcl(K_{a,a}) \ge a + 1$. Next, we show $rvcl(K_{a,a}) \le a + 1$ by defining a vertex coloring $c: V(K_{a,a}) \longrightarrow [1, a+1]$ as follows:

$$\begin{aligned} &c(u_i) = i, \text{ for } i \in [1, a]; \\ &c(v_j) = j + 1, \text{ for } j \in [1, a] \end{aligned} \tag{5}$$

Using the same logic as in the first case of this proof, we always find a rainbow vertex path between any two vertices on graph $K_{a,a}$. Furthermore, by the vertex coloring above, we have $c(u_i) = c(v_j)$ for $i = j + 1$, $i \in [2, a]$ and $j \in [1, a-1]$. However, $d(u_i, R_{a+1}) = 1$ and $d(v_j, R_{a+1}) = 2$, so $rc_\Pi(u_i) \ne rc_\Pi(v_i)$. Thus, $rvcl(K_{a,a}) = a + 1$.

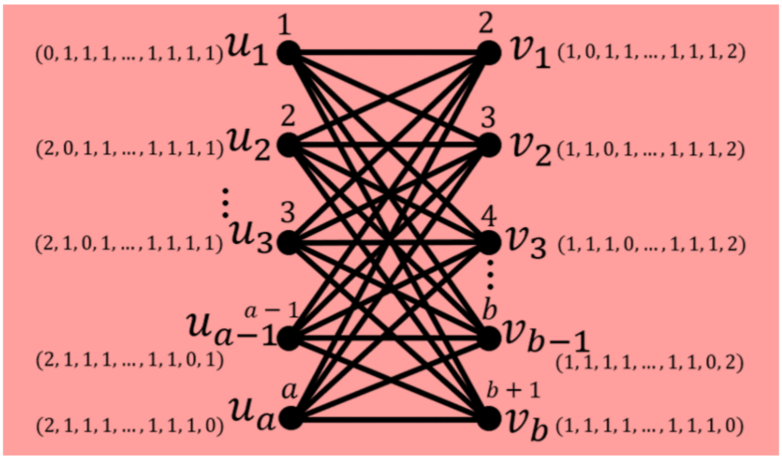


Figure 9. **Rainbow code of $K_{a,b}$ for $a = b$**

$rvcl(K_{a,a})$ is equal to the upper bound of the r-regular bipartite graph. Next, we show a class of bipartite graphs whose locating rainbow connection number is equal to the lower bound of the locating rainbow vertex connection number for regular bipartite graphs. The graph is formed by deleting perfect matching in regular complete bipartite graphs, denoted by $K_{\frac{n}{2},\frac{n}{2}} - M$. A *perfect matching* in G is an independent edge set $E(G)$, such that every vertex in the vertex set $V(G)$ is adjacent to exactly one edge in M. The result from Theorem 3.6 shows a significant difference in the locating rainbow connection number of a graph before and after removing some edges on the graph.

Theorem 3.6. *If $K_{\frac{n}{2},\frac{n}{2}}$ be a complete bipartite graph with $n \geq 6$, n be even, and M be a matching in $K_{\frac{n}{2},\frac{n}{2}}$, then* $rvcl\left(K_{\frac{n}{2},\frac{n}{2}} - M\right) = \left\lceil\frac{n}{4}\right\rceil + 1$.

Proof. Based on Theorem 3.4 we have $rvcl\left(K_{\frac{n}{2},\frac{n}{2}}\right) - M \geq \left\lceil\frac{n}{4}\right\rceil + 1$. Next, we show that $rvcl\left(K_{\frac{n}{2},\frac{n}{2}}\right) - M \leq \left\lceil\frac{n}{4}\right\rceil + 1$

In Figure 10, we have $rvcl\left(K_{\frac{n}{2},\frac{n}{2}}\right) - M \leq \left\lceil\frac{n}{4}\right\rceil + 1$ for $n = 6,8$. For $n \geq 10$, we define a rainbow vertex coloring $c: V(G) \longrightarrow \left[1, \left\lceil\frac{n}{4}\right\rceil + 1\right]$ as follows $\left(i, j \in \left[1, \frac{n}{2}\right]\right)$:

$$c(u_i) = \begin{cases} 1, & \text{for } i \in \left[1, \left\lceil\frac{n}{4}\right\rceil + 1\right]; \\ i - \left\lceil\frac{n}{4}\right\rceil, & \text{for otherwise.} \end{cases} \tag{6}$$

$$c(v_j) = \begin{cases} j + 1, & \text{for } j \in \left[1, \left\lceil\frac{n}{4}\right\rceil\right]; \\ 1, & \text{for otherwise.} \end{cases} \tag{7}$$

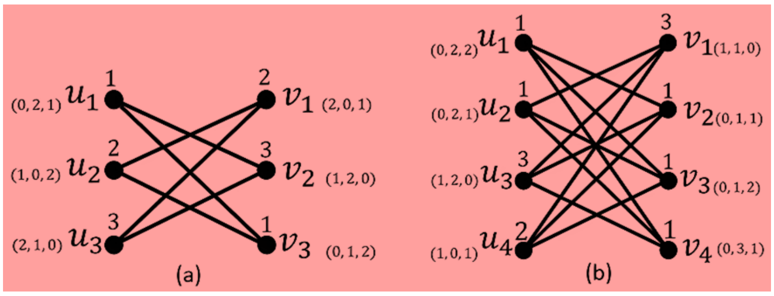


Figure 10. **Rainbow codes of (a) $K_{3,3} - M$, (b) $K_{4,4} - M$**

By Lemma 3.2, we have $diam\left(K_{\frac{n}{2},\frac{n}{2}}\right) = 3$; thus, a rainbow vertex path can be seen between any two vertices with a distance of 1 or 2. Each vertex of U is adjacent with $\frac{n}{2} - 1$ vertices of V, and we also have a rainbow vertex path between u_i and v_j for $i \neq j$ (see Figure 11 for an illustration). Next, we show that each vertex of $K_{\frac{n}{2},\frac{n}{2}} - M$ has distinct rainbow codes. From the coloring above, we have the following.

- Color $\left\lceil\frac{n}{4}\right\rceil + 1$ is only used once for $v_{\left\lceil\frac{n}{4}\right\rceil}$.
- $c(u_i) = c(v_j) = 1$ for $i \in \left[1, \left\lceil\frac{n}{4}\right\rceil + 1\right], j \in \left[\left\lceil\frac{n}{4}\right\rceil + 1, \frac{n}{2}\right]$.
  $d\left(u_i, R_{\left\lceil\frac{n}{4}\right\rceil+1}\right) = 1$, for $i \neq \left\lceil\frac{n}{4}\right\rceil$.

$d\left(v_j, R_{\left\lceil\frac{n}{4}\right\rceil+1}\right) = 2.$

$d\left(u_{\left\lceil\frac{n}{4}\right\rceil}, R_{\left\lceil\frac{n}{4}\right\rceil+1}\right) = 3.$

- $c(u_i) = c\left(v_{i-\left(\left\lceil\frac{n}{4}\right\rceil+1\right)}\right)$, for $i \in \left[\left\lceil\frac{n}{4}\right\rceil + 1, \frac{n}{2}\right]$.

  $d\left(u_i, R_{\left\lceil\frac{n}{4}\right\rceil+1}\right) = 1.$

  $d\left(v_{i-\left(\left\lceil\frac{n}{4}\right\rceil+1\right)}, R_{\left\lceil\frac{n}{4}\right\rceil+1}\right) \neq 1$

Thus, we obtain a rainbow vertex coloring of $K_{\frac{n}{2},\frac{n}{2}} - for\, \frac{n}{2} \geq$ with $\left\lceil\frac{n}{4}\right\rceil + 1$ colors, and all vertices have distinct rainbow codes.

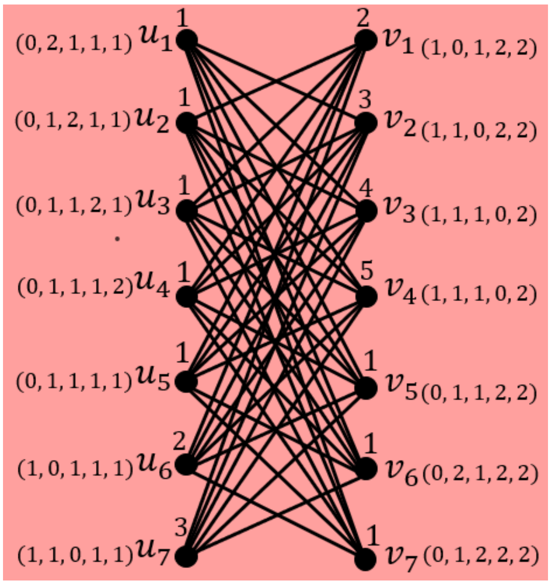


Figure 11. Rainbow codes of $K_{7,7} - M$

4- Conclusion

We determined the lower and upper bounds of the locating rainbow connection number of a graph and provided the characterization of graphs with the locating rainbow connection number equal to its upper and lower bounds, where only path graphs with orders 2, 3, 4 and complete graphs have a locating rainbow connection number equal to 2 and the order of graph G, respectively. These results can facilitate further research in determining the upper and lower bounds of $rvcl(G)$ for any connected graph G. Additionally, the main result finds a lemma that states that for any graph G that contains cycles, $rvcl(G)$ must be greater than or equal to 3. These results certainly serve as guidelines for determining the lower bound of $rvcl(G)$ for any graph G that contains cycles.

Locating the rainbow connection number of a tree is determined based on the maximum number of pendants or internal vertices. A tree is a class of graphs that are classified into other small graph classes, such as path, caterpillar, star, and double star. Thus, by determining the locating rainbow connection number of trees, locating rainbow connection numbers for other tree graph classes can be determined, which will make it easier for future researchers to identify locating rainbow connection numbers from various types of tree graph classesNext, we determined the range of values for the locating rainbow connection number of bipartite regular graphs. In addition, we defined regular bipartite graphs whose values are equal to the upper and lower bounds. This, of course, minimizes the possibility of value assumptions that appear to be related to the locating rainbow connection number of a bipartite regular graph.

5- Declarations

5-1- Author Contributions

Conceptualization, A.W.B., A.N.M.S., and P.E.P.; methodology, A.W.B., A.N.M.S., and P.E.P.; validation, A.N.M.S. and P.E.P.; writing—review and editing, A.W.B., A.N.M.S., P.E.P., and Z.Y.A.; writing—original draft preparation, A.W.B.; visualization, A.W.B.; supervision, A.N.M.S. and P.E.P. All authors have read and agreed to the published version of the manuscript.

5-2- Data Availability Statement

Data sharing is not applicable to this article.

5-3- Funding

This research was supported financially by LPDP from Ministry of Finance Indonesia and Riset ITB Program from Institut Teknologi Bandung.

5-4- Acknowledgements

This research was supported financially by LPDP from Ministry of Finance Indonesia and Riset ITB Program from Institut Teknologi Bandung.

5-5- Institutional Review Board Statement

Not applicable.

5-6- Informed Consent Statement

Not applicable.

5-7- Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this manuscript. In addition, the ethical issues, including plagiarism, informed consent, misconduct, data fabrication and/or falsification, double publication and/or submission, and redundancies have been completely observed by the authors.

6- References

[1] Mahmoudi, S., & Lotfi, S. (2015). Modified cuckoo optimization algorithm (MCOA) to solve graph coloring problem. Applied Soft Computing, 33, 48–64. doi:10.1016/j.asoc.2015.04.020.

[2] Chartrand, G., Johns, G. L., McKeon, K. A., & Zhang, P. (2008). Rainbow connection in graphs. Mathematica Bohemica, 133(1), 85–98. doi:10.21136/mb.2008.133947.

[3] Ericksen, A. B. (2007). A matter of security. Graduating Engineer & Computer Careers, 24, 28.

[4] Li, X., Shi, Y., & Sun, Y. (2013). Rainbow Connections of Graphs: A Survey. Graphs and Combinatorics, 29(1), 1–38. doi:10.1007/s00373-012-1243-2.

[5] Fitriani, D., & Salman, A. N. M. (2016). Rainbow connection number of amalgamation of some graphs. AKCE International Journal of Graphs and Combinatorics, 13(1), 90–99. doi:10.1016/j.akcej.2016.03.004.

[6] Li, H., & Ma, Y. (2017). Rainbow connection number and graph operations. Discrete Applied Mathematics, 230, 91–99. doi:10.1016/j.dam.2017.06.004.

[7] Susanti, B. H., Salman, A. N. M., & Simanjuntak, R. (2020). The rainbow 2-connectivity of Cartesian products of 2-connected graphs and paths. Electronic Journal of Graph Theory and Applications, 8(1), 145–156. doi:10.5614/EJGTA.2020.8.1.11.

[8] Umbara, R. F., Salman, A. N. M., & Putri, P. E. (2023). On the inverse graph of a finite group and its rainbow connection number. Electronic Journal of Graph Theory and Applications, 11(1), 135–147. doi:10.5614/ejgta.2023.11.1.11.

[9] Fitriani, D., Salman, A. N. M., & Awanis, Z. Y. (2022). Rainbow connection number of comb product of graphs. Electronic Journal of Graph Theory and Applications, 10(2), 461–473. doi:10.5614/ejgta.2022.10.2.9.

[10] Hasan, M. A., Wulandari, R. Y., & Salman, A. N. M. (2022). Rainbow Connection Number of Shackle Graphs. Proceedings of the International Conference on Mathematics, Geometry, Statistics, and Computation (IC-MaGeStiC 2021), Jember, Indonesia. doi:10.2991/acsr.k.220202.013.

[11] Krivelevich, M., & Yuster, R. (2010). The rainbow connection of a graph is (at most) reciprocal to its minimum degree. Journal of Graph Theory, 63(3), 185–191. doi:10.1002/jgt.20418.

[12] Simamora, D. N. S., & Salman, A. N. M. (2015). The Rainbow (Vertex) Connection Number of Pencil Graphs. Procedia Computer Science, 74, 138–142. doi:10.1016/j.procs.2015.12.089.

[13] Li, X., & Liu, S. (2014). Tight upper bound of the rainbow vertex-connection number for 2-connected graphs. Discrete Applied Mathematics, 173, 62–69. doi:10.1016/j.dam.2014.04.002.

[14] Bustan, A. W., & Salman, A. N. M. (2018). The Rainbow Vertex-Connection Number of Star Fan Graphs. CAUCHY: Journal of Pure Mathematics and Applications, 5(3), 112–116. doi:10.18860/ca.v5i3.5516.

[15] Ma, Y., Xue, Y., & Zhang, X. (2023). Proper (Strong) Rainbow Connection and Proper (Strong) Rainbow Vertex Connection of Some Special Graphs. Journal of Interconnection Networks, 2250006. doi:10.1142/s0219265922500062.

[16] Chen, L., Li, X., & Shi, Y. (2011). The complexity of determining the rainbow vertex-connection of a graph. Theoretical Computer Science, 412(35), 4531–4535. doi:10.1016/j.tcs.2011.04.032.

[17] Chen, L., Huo, B., & Ma, Y. (2016). Hardness results for total rainbow connection of graphs. Discussiones Mathematicae - Graph Theory, 36(2), 355–362. doi:10.7151/dmgt.1856.

[18] Chartrand, G., Salehi, E., & Zhang, P. (2000). The partition dimension of a graph. Aequationes Mathematicae, 59(1–2), 45–54. doi:10.1007/PL00000127.

[19] Bustan, A. W., Salman, A. N. M., & Putri, P. E. (2021). On the Locating Rainbow Connection Number of a Graph. Journal of Physics: Conference Series, 1764(1), 12057. doi:10.1088/1742-6596/1764/1/012057.