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Interval Estimation of the Dependence Parameter in Bivariate Clayton Copulas

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Abstract

In various disciplines, discerning dependencies between variables remains a crucial undertaking. While correlation measures like Pearson, Spearman, and Kendall provide insight into the degree of two-variable relationships, they fall short of revealing the intricate structure of dependencies between these variables. The Clayton copula, known for its flexible attributes, becomes instrumental in unveiling this dependency structure. This paper aims to advance knowledge by providing an explicit formula for creating Wald confidence intervals (CIs) for the dependence parameter in a bivariate Clayton copula, along with a mathematical derivation of the observed Fisher information. In comparison, we also propose likelihood CIs, whose performance we examine in simulation studies using both coverage probability and average length of CIs as performance indicators. Our findings reveal that in scenarios characterized by small sample sizes, likelihood-based CIs, despite their slightly more complex computational requirements, outperform Wald CIs, yielding a coverage probability more proximate to the nominal confidence level of 0.95. However, in situations involving large samples and a dependence parameter distant from zero, both Wald and likelihood-based CIs demonstrate comparable utility. For real-world data applications, the daily closing prices of two cryptocurrencies are analyzed using the proposed CIs.

Keywords:

Bivariate Clayton;

Dependence;

Wald Interval;

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1- Introduction

In recent years, the advent of big data has resulted in an increase in research on both individual variables and the joint distributions of multiple variables. Applying the Copula model to real-world data has proven useful for numerous researchers in a variety of disciplines. Notably, Di Clemente and Romano [1] have outlined a number of statistical procedures for calibrating copula functions to financial market data, providing methods for identifying the function that most closely corresponds to the available financial data. Simultaneously, Stulajter [2] demonstrated that, in the context of portfolio design, copula functions permit the separation of modeling dependence aspects and modeling marginal distributions of financial assets.

Naifar [3] employed a copula to investigate the intricate structure of dependency between credit default swaps and jump risk, whereas Darabi & Baghban [4] applied a copula for portfolio optimization. Further, in the specialized field of hydrological analysis, specifically drought research, Abdi et al. [5], Evkaya et al. [6], and Fan et al. [7] utilized copulas to explore the relationships between various variables. Zhang [8] delved into the relationship between crude oil prices, petroleum prices, and tanker freight rates in 2018 through copula theory. Meanwhile, Bhatti & Do [9] researched the evolution of copula models and their applications in energy, fuel cells, forestry, and environmental sciences. In the sphere

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of agricultural insurance, Rusyda et al. [10] used a copula to simulate the dependency structure between crop yield and multicrop pricing in Indonesia. Despite these significant strides, the description and prediction of the characteristics of these multivariate distributions remain formidable tasks due to their inherent complexity.

Indeed, Pearson's correlation, due to its computational ease and interpretability, has been widely used not just in statistics but in the natural and social sciences [11]. However, this measurement only evaluates the linear relationship between two variables. This issue can be mitigated with the application of copula [12]. Copula offers a powerful alternative for modeling and estimating multivariate distributions. Grounded in Sklar's theorem from 1959 [13], a copula is a function that merges univariate marginals to create a multivariate distribution. It provides a broad class of marginal distributions and a variety of dependence structures [14], and these components can be modeled separately [15], which explains its appeal in multiple disciplines. Copulas can be categorized into three major classes: elliptical, extreme-value, and Archimedean [16]. Elliptical copulas are comprised of copulas with elliptical distributions; extreme-value copulas correspond to multivariate extreme-value distributions; and Archimedean copulas are derived from the generator function. The Archimedean class, particularly the Clayton copula, is especially attractive due to its simple construction, mathematical tractability [17], and stochastic properties of its elements [18]. Its capacity for managing strong left-tail dependence [17] has found important applications in economics and finance [11]. It also comes into play in shared frailty models; indeed, these models predate the general theories of copula [19].

Estimation procedures for the dependence parameter of the copula can be divided into three kinds of inference approaches: parametric, semi-parametric, and nonparametric. Parametric approaches include maximum likelihood estimators (MLE) and inference functions for margins (IFM) [20], with IFM offering a highly efficient alternative to MLE estimation [21]. The semi-parametric method utilizes a nonparametric approach, such as empirical distribution functions or their scaled versions, to estimate univariate marginal distributions before applying the maximization of the contribution to the log-likelihood function for the estimation of the copula parameter [22]. Various studies have compared the robustness of the estimation method under differing scenarios, such as univariate margin misspecification [23] and the performance of point estimation [21, 24, 25].

However, discourse on the confidence interval (CI) for the dependence parameter of the copula, which plays a pivotal role in statistical inference and multivariate modeling, remains scarce. Genest et al. [22] and Kojadinovic & Yan [24] proposed a variance estimator for the semiparametric estimator. Hofert et al. [26] pointed out that obtaining the expected Fisher information in terms of the score function for constructing Wald CIs becomes challenging in high-dimensional situations with known margins. To circumvent this problem, the likelihood-based confidence interval approach has been considered, but both approaches come with complicated and intractable formulas. Moreover, the construction of confidence intervals, such as the asymptotic CIs based on the second derivative of the likelihood function [26], typically depends on numerical approaches rather than direct calculations. Although this might be more convenient in practice, the results only provide approximate values. Therefore, this paper aims to offer a formula to construct CIs for the dependence parameter of a bivariate Clayton copula using the Wald method. Additionally, the results will be compared with likelihood-based CIs. Both of these approaches leverage the likelihood function, a valuable tool for objective reasoning with data, especially when handling uncertainty due to the limited information contained within the data.

The remainder of this paper is structured as follows: the first section defines the bivariate copula and introduces the Clayton copula. Section 3 presents two CIs for the dependence parameter of the bivariate Clayton copula. Section 4 focuses on Monte Carlo simulation studies, while Section 5 applies these studies to real-world datasets. Section 6 discusses the findings of this study with the previous studies, and the final section summarizes the conclusions drawn from the paper.

2- Background

Let X_1 and X_2 be two real-value random variables on a common probability space $(\Omega, \mathcal{F}, \mathrm{P})$ with distribution functions $F_1(x_1) = P(X_1 \leq x_1)$ and $F_2(x_2) = P(X_2 \leq x_2)$, respectively, and a joint distribution function $F(x_1, x_2) = P(X_1 \leq x_1, X_2 \leq x_2)$. Then, if the marginal distributions of the marginals are known, the quantile transformation or the probability integral transformation will apply to transform a random variable with distribution function F into a standard uniform random variable. Meanwhile, the marginals are unknown, and the empirical distribution function is suggested for the transformation [22].

A copula is a multivariate distribution function with standard uniform univariate margins, or $U(0,1)$. The first concept regarding copula theory was stated by Abe Sklar in 1959, called Sklar's theorem. The theorem states that for a bivariate distribution function F with margins F_1 and F_2, there exists a copula C such that

$$F(x_1, x_2) = C\{F_1(x_1), F_2(x_2)\} = C\{u, v\},$$

where $x_1, x_2 \in \mathbb{R}$. In the case where F_1 and F_2 are continuous, copula C is uniquely defined on $[0,1]^2$. This explains why the majority of statistical applications of copula involve the modeling of continuous random vectors, that is, random vectors with continuous marginal distribution functions [16].

Archimedean copula is a popular class that is not computed using Sklar's theorem but depends on the generator function. In general, the bivariate form of the Archimedean copula is

$$C(u, v) = \varphi^{-1}\big(\varphi(u) + \varphi(v)\big), \tag{1}$$

where $0 \leq u \leq 1$, $0 \leq v \leq 1$, and φ is a generator function. A function C of the form in (1) is an Archimedean copula if and only if its generator is a convex decreasing function from $(0,1)$ to $(0, \infty)$ such that $\varphi(1) = 0$ [15].

Clayton, or the Mardia-Takahasi-Clayton-Cook-Johnson copula, is an Archimedean family member. The first indirect application of joint-life models resulted from Clayton's study of bivariate life tables of fathers and offspring [27]. As shown in Figure 1, the Clayton copula captures positive lower-tail dependence but less association in upper-tail dependence [28]. The Clayton copula generator function is $\varphi(t) = (1+t)^{-\frac{1}{t}}$ where $t \geq 0$, and its inverse is $\varphi^{-1}(t) = t^{-\theta} - 1$. This function is present in Laplace transformation (LT) families that implement the required properties of the Archimedean generator. The resulting bivariate Clayton copula function with a dependence parameter (θ) is defined as

$$C(u, v) = \left(u^{-\theta} + v^{-\theta} - 1\right)^{-\frac{1}{\theta}}, \tag{2}$$

where $0 \leq u, v < 1$ and $0 \leq \theta < \infty$. In addition, the Clayton copula density function can be expressed as

$$c(u, v) = \frac{\partial C(u,v)}{\partial u \partial v} = (\theta + 1)(uv)^{-\theta-1}\left(u^{-\theta} + v^{-\theta} - 1\right)^{-2-\frac{1}{\theta}}. \tag{3}$$

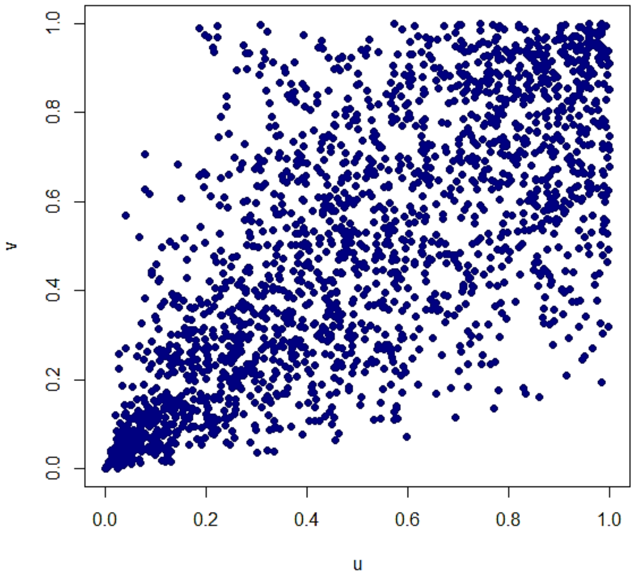
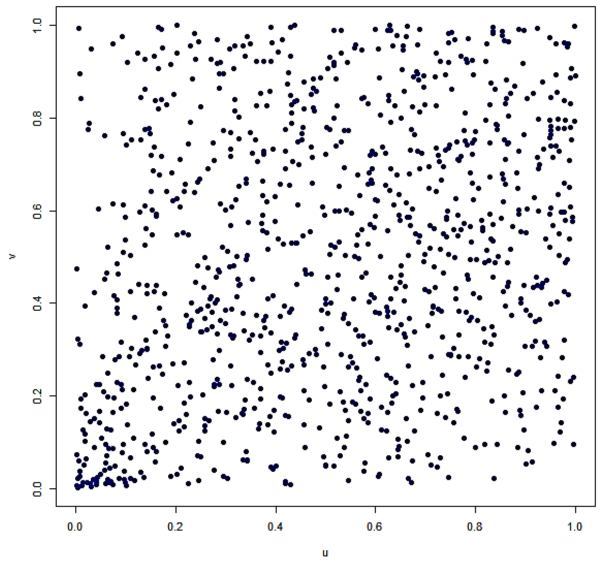
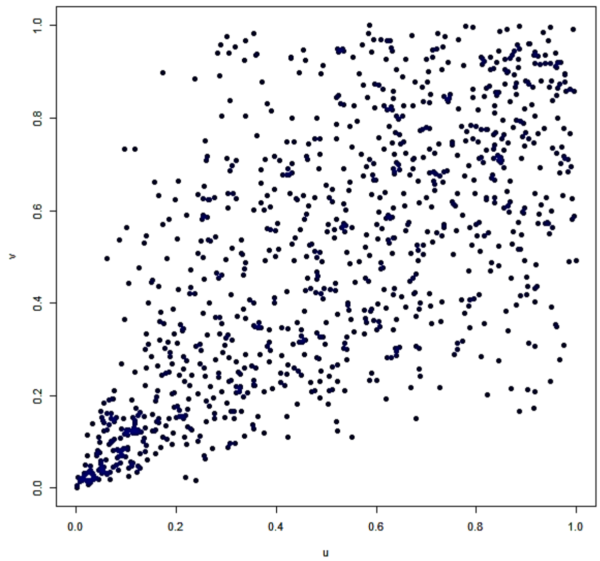


Figure 1. Scatter plot of 2,000 random samples from Clayton copula at θ = 2

The LT families can be parameterized such that the dependence of the copula increases as the value of the dependence parameter (θ) increases, as depicted in Figure 2.





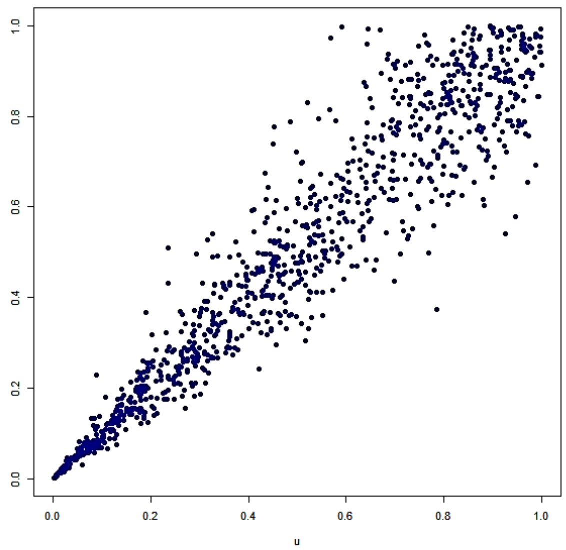


Figure 2. Scatter plot of 1,000 random samples from Clayton copula at θ = 0.5, 2, and 8

Copula $C(u, v)$ is invariant with respect to the one-to-one transformation of the marginal variables X_1 and X_2. In this respect, it is analogous to the invariance of Kendall's Tau rank correlation coefficient, a well-known measure of dependence [11, 14], defined as:

$$\tau(X_1, X_2) = 4\int_{[0,1]^2} C(u, v)dC(u, v) - 1 = \frac{\theta}{\theta+2}.$$

A number of publications, such as Genest et al. [22] and Kojadinovic & Yan [24], have focused on the semiparametric inference of dependence parameters in copula. Hofert et al. [26] demonstrated the procedure for constructing CIs for the copula parameter under known margins. Consider the observed Fisher information, which is defined in Hofert's study as:

$$J(\theta; \boldsymbol{u}_1, \cdots, \boldsymbol{u}_n) = \sum_{i=1}^{n} -\frac{d^2}{d\theta} l(\theta; \boldsymbol{u}_i).$$

Under regularity conditions, the Fisher information satisfies:

$$I(\theta) = J(\theta; \boldsymbol{U}) = \mathbb{E}\left\{\sum_{i=1}^{n} -\frac{d^2}{d\theta} l(\theta; \boldsymbol{U})\right\}.$$

From this and definition of the Fisher information, the following choices for $\widehat{I(\theta)}$ are:

$$I(\hat{\theta}_n) = \mathbb{E}\{s_{\hat{\theta}_n}(\boldsymbol{U}) s_{\hat{\theta}_n}(\boldsymbol{U})^T\} \text{ and}$$

$$I^{(2)}(\hat{\theta}_n) = \frac{1}{n}\sum_{i=1}^{n} -\frac{d^2}{d\theta} l(\theta; \boldsymbol{u})$$

and the score function for the Clayton Family is:

$$s_\theta(\boldsymbol{U}) = \sum_{k=0}^{d-1} \frac{k}{\theta k+1} - \sum_{j=1}^{d} \log u^{(j)} + \frac{1}{\theta^2}\log\{1 + t_\theta(\boldsymbol{u})\} - (d + 1/\theta)\frac{t_\theta(\boldsymbol{u})}{1+t_\theta(\boldsymbol{u})},$$

where $t_\theta(\boldsymbol{u}) = \varphi^{-1}(u_1) + \cdots + \varphi^{-1}(u_d)$.

The observed Fisher information of Clayton copula is computed by:

$$I^{(2)}(\hat{\theta}_n) = \frac{1}{n}\sum_{i=1}^{n} -\left\{-\sum_{k=0}^{d-1}\left(\frac{k}{\theta k+1}\right)^2 + \frac{2}{\theta^2}\left[\frac{t'_\theta(\boldsymbol{u})}{1+t_\theta(\boldsymbol{u})} - \frac{1}{\theta}\ln\{1 + t_\theta(\boldsymbol{u})\}\right] + (d + 1/\theta)\left[\left\{\frac{t'_\theta(\boldsymbol{u})}{1+t_\theta(\boldsymbol{u})}\right\}^2 - \frac{\sum_{j=1}^{d}(\ln u_j)^2 u_j^{-\theta}}{1+t_\theta(\boldsymbol{u})}\right]\right\},$$

where $t'_\theta(\boldsymbol{u}) = \sum_{j=1}^{d}(\ln u_j)u_j^{-\theta}$. From the aforementioned theoretical background, Wald CIs can be constructed and will be presented in the next section.

3- Confidence Intervals

In this section, the key findings will be presented, including a mathematical proof of Wald and likelihood-based intervals. By definition, a $100(1 - \alpha)\%$ confidence interval for θ will satisfy the following property:

$$P[\theta_L(T_n) \leq \theta \leq \theta_U(T_n)] = 1 - \alpha,$$

where $1-\alpha$ is a confidence coefficient, the statistics $\theta_L(T_n)$ and $\theta_U(T_n)$ are the limits of the CI, and $\theta_L(T_n) \leq \theta_U(T_n)$ [29]. For the case of bivariate Clayton copulas, it is started with likelihood functions, which X_1 and X_2 are continuous random variables with respective densities f_1 and f_2, and distribution functions F_1 and F_2. Let $u = F_1(x_1)$ and $v = F_2(x_2)$, then the likelihood function of the Clayton copula is

$$L(\theta; x_1, x_2) = c(u,v)\prod_{i=1}^{2} f_i(x_i) = (\theta+1)(uv)^{-\theta-1}\left(u^{-\theta}+v^{-\theta}-1\right)^{-2-\frac{1}{\theta}}\prod_{i=1}^{2} f_i(x_i). \tag{4}$$

In cases where marginal distributions F_1 and F_2 are unknown, they are replaced by rescaled versions of their empirical distribution, $F_{n,j}(x) = \frac{1}{n+1}\sum_{i=1}^{n} 1\left(X_{ij} \leq x\right)$ [16]. The log-likelihood function will have a form of

$$l(\theta) = \sum_{j=1}^{n} \log c\left(u^{(j)}, v^{(j)}\right) + \sum_{i=1}^{2}\sum_{j=1}^{n} \log f_i\left(x_i^{(j)}\right) = \log(\theta+1) + \sum_{j=1}^{n}(-\theta-1)\log\left(u^{(j)}v^{(j)}\right) + \sum_{j=1}^{n}\left(-2-\frac{1}{\theta}\right)\log\left({u^{(j)}}^{-\theta} + {v^{(j)}}^{-\theta} - 1\right) + \sum_{i=1}^{2}\sum_{j=1}^{n}\log f_i\left(x_i^{(j)}\right), \tag{5}$$

where $i = 1,2$ or the number of random variables and $j = 1,2,\dots,n$ or the number of observations.

3-1- Wald Confidence Intervals

Wald intervals, which may also be referred to as MLE-based symmetric confidence intervals, are the most well-known intervals [30, 31]. This interval is determined using the Wald statistic for testing the null hypothesis $H_0: \theta = \theta_0$ against the alternative hypothesis $H_1: \theta = \theta_1$, where $\theta_1$is a specified value. Under the null hypothesis, the following are the two test statistics, both of which have asymptotically normal asymptotic distributions:

$$\sqrt{I\left(\hat{\theta}_{ML}\right)}\left(\hat{\theta}_{ML}-\theta\right) \overset{a}{\sim} N(0,1) \text{ and } \sqrt{J\left(\hat{\theta}_{ML}\right)}\left(\hat{\theta}_{ML}-\theta\right) \overset{a}{\sim} N(0,1),$$

where $I\left(\hat{\theta}_{ML}\right)$ and $J\left(\hat{\theta}_{ML}\right)$ are the observed Fisher information (FI) and expected FI. The observed FI is the second derivative of the log-likelihood. Here, the FI formula is derived:

$$\begin{aligned}
I(\theta) &= \frac{\partial l(\theta)}{\partial^2\theta} \\
&= \frac{\partial}{\partial^2\theta}\left(\log(\theta+1) + \log u^{-\theta-1} + \log v^{-\theta-1} + \left(-2-\frac{1}{\theta}\right)\log\left(u^{-\theta}+v^{-\theta}-1\right)\right) + \frac{\partial}{\partial^2\theta}\sum_{i=1}^{2}\sum_{j=1}^{n}\log f_i\left(x_i^{(j)}\right) \\
&= \frac{\partial}{\partial\theta}\left(\frac{\partial}{\partial\theta}\log(\theta+1) + \log u^{-\theta-1} + \log v^{-\theta-1} + \log\left(u^{-\theta}+v^{-\theta}-1\right)^{-2-\frac{1}{\theta}}\right) \\
&= \frac{\partial}{\partial\theta}\left(\frac{1}{1+\theta} - \log u - \log v + \left[\frac{1}{\theta^2}\log\left(u^{-\theta}+v^{-\theta}-1\right) + \frac{\left(-2-\frac{1}{\theta}\right)\left(-u^{-\theta}\log u - v^{-\theta}\log v\right)}{u^{-\theta}+v^{-\theta}-1}\right]\right) \\
&= \frac{\partial}{\partial\theta}\left(\frac{1}{1+\theta}\right) + \frac{\partial}{\partial\theta}\left(-\log u - \log v\right) + \frac{\partial}{\partial\theta}\left(\frac{1}{\theta^2}\log\left(u^{-\theta}+v^{-\theta}-1\right) - \frac{\left(-2-\frac{1}{\theta}\right)\left(-u^{-\theta}\log u - v^{-\theta}\log v\right)}{u^{-\theta}+v^{-\theta}-1}\right) \\
&= -\frac{1}{(\theta+1)^2} + \frac{\partial}{\partial\theta}\left(\frac{1}{\theta^2}\log\left(u^{-\theta}+v^{-\theta}-1\right) - \frac{\left(-2-\frac{1}{\theta}\right)\left(-u^{-\theta}\log u - v^{-\theta}\log v\right)}{\left(u^{-\theta}+v^{-\theta}-1\right)}\right) \\
&= -\frac{1}{(\theta+1)^2} + \left[\frac{1}{\left(u^{-\theta}+v^{-\theta}-1\right)^2}\left(-\left(-2-\frac{1}{\theta}\right)\left(-u^{-\theta}\log u - v^{-\theta}\log v\right)^2 + \left(u^{-\theta}+v^{-\theta}-1\right)\left(\frac{-u^{-\theta}\log u - v^{-\theta}\log v}{\theta^2} + \right.\right.\right. \\
&\quad \left.\left.\left.\left(-2-\frac{1}{\theta}\right)\left(u^{-\theta}\log^2 u + v^{-\theta}\log^2 v\right)\right)\right)\right] - \frac{\partial}{\partial\theta}\left(\frac{1}{\theta^2}\log\left(u^{-\theta}+v^{-\theta}-1\right)\right) \\
&= -\frac{1}{(1+\theta)^2} + \left[\frac{1}{\left(u^{-\theta}+v^{-\theta}-1\right)^2}\left(-\left(-2-\frac{1}{\theta}\right)\left(-u^{-\theta}\log u - v^{-\theta}\log v\right)^2 + \left(u^{-\theta}+v^{-\theta}-1\right)\left(\frac{-u^{-\theta}\log u - v^{-\theta}\log v}{\theta^2} + \right.\right.\right. \\
&\quad \left.\left.\left.\left(-2-\frac{1}{\theta}\right)\left(u^{-\theta}\log^2 u + v^{-\theta}\log^2 v\right)\right)\right)\right] + \frac{-u^{-\theta}\log u - v^{-\theta}\log v}{\theta^2\left(u^{-\theta}+v^{-\theta}-1\right)} - \frac{2\log\left(u^{-\theta}+v^{-\theta}+1\right)}{\theta^3}
\end{aligned}$$

resulting in:

$$I(\theta) = -\sum_{j=1}^{n}\left(-\left({v^{(j)}}^{\theta} - {u^{(j)}}^{\theta}\left({v^{(j)}}^{\theta}-1\right)\right)^2\left(\theta^3 + 2(\theta+1)^2\log\left({u^{(j)}}^{-\theta}+{v^{(j)}}^{-\theta}-1\right)\right) + \theta^2(\theta+1)^2(2\theta+1){u^{(j)}}^{\theta}{v^{(j)}}^{\theta}\left({v^{(j)}}^{\theta}-1\right)\log\left(u^{(j)}\right)^2 + \theta^2(\theta+1)^2(2\theta+1)\left({u^{(j)}}^{\theta} - 1\right){u^{(j)}}^{\theta}{v^{(j)}}^{\theta}\log\left(v^{(j)}\right)^2 + 2\theta(\theta+1)^2{u^{(j)}}^{\theta}\log\left(v^{(j)}\right)\left({u^{(j)}}^{\theta}\left({v^{(j)}}^{\theta}-1\right) - {v^{(j)}}^{\theta}\right) + 2\theta(\theta + \right. \tag{6}$$

$$1)^2 v^{(j)^{\theta}} \log\left(u^{(j)}\right) \left(u^{(j)^{\theta}} \left(v^{(j)^{\theta}} - 1\right) + \theta(2\theta + 1) u^{(j)^{\theta}} \log\left(v^{(j)}\right) - v^{(j)^{\theta}}\right)\right) / \left(\theta^3 (\theta + 1)^2 \left(v^{(j)^{\theta}} - u^{(j)^{\theta}} \left(v^{(j)^{\theta}} - 1\right)\right)^2\right)\right).$$

Consequently, a $(1-\alpha)100\%$ Wald CI for θ that is evaluated at $(\hat{\theta}_{ML})$ will be

$$\hat{\theta}_{ML} \pm z_{1-\frac{\alpha}{2}} \sqrt{I^{-1}(\hat{\theta}_{ML})}, \tag{7}$$

where $\hat{\theta}_{ML} = \arg\max \sum_{j=1}^{n} \log c\left(u^{(j)}, v^{(j)}\right)$ is the maximum likelihood estimators which satisfies the asymptotic properties [15] and $I^{-1}(\hat{\theta}_{ML}) = 1/I(\hat{\theta}_{ML})$. In this paper, the maximum likelihood estimates were obtained from R package bbmle version 1.0.25. Although the FI formula in Equation 6 appears complex, the Wald CI shown in Equation 7 provides an explicit formula rather than an approximation, as do numerical methods.

3-2- Likelihood-based Confidence Intervals

Fisher [32] proposed the likelihood CI, which can be derived from the likelihood function without requiring the likelihood function's derivatives. A likelihood-based CI of θ will be

$$\left\{\theta \middle| \frac{L(\theta)}{\max L(\theta)} \geq c\right\} = \left\{\theta \middle| \frac{L(\theta)}{L(\hat{\theta}_{ML})} \geq c\right\}.$$

The constant c can be found from the Wilk's likelihood ratio statistic which is defined as

$$W = -2\log\frac{L(\theta)}{L(\hat{\theta}_{ML})} \sim \chi_1^2.$$

Thus, a $(1-\alpha)100\%$ likelihood-based CI of θ using the likelihood-based method will be

$$\left\{\theta \middle| \frac{L(\theta)}{L(\hat{\theta}_{ML})} \geq \exp\left(-\frac{1}{2}\chi^2_{1,(1-\alpha)}\right)\right\}, \tag{8}$$

where $\chi^2_{1,(1-\alpha)}$ is the $(1-\alpha)$th quantile of the chi-squared distribution with degree freedom of one. In other words, let

$$g(\theta) = \frac{L(\theta)}{L(\hat{\theta}_{ML})} = \frac{\exp(l(\theta))}{\exp(l(\hat{\theta}_{ML}))},$$

where $l(\theta)$is presented in Equation 5. The $(1-\alpha)100\%$ CI is (θ_L, θ_U), where $\theta_L = g^{-1}(c)$, $\theta_U = g^{-1}(c)$, $\theta_L < \theta_U$, and $c = \exp\left(-\frac{1}{2}\chi^2_{1,\,(1-\alpha)}\right)$. To find θ_L and θ_U, a numerical method can be used for solving equations $\theta_L = g^{-1}(c)$ and $\theta_U = g^{-1}(c)$. In this paper, function approxfun in R package stats version 4.2.3 was utilized to find the solutions. Figure 3 shows examples of functions $g(\theta)$ with lower and upper limits of CIs when 1,000 random samples are from Clayton copula at θ = 0.5, 2, and 8. Note that the log-likelihood function can be approximated by the quadratic function as follows:

$$\log\frac{L(\theta)}{L(\hat{\theta})} \approx -\frac{1}{2} I(\hat{\theta})(\theta - \hat{\theta})^2. \tag{9}$$

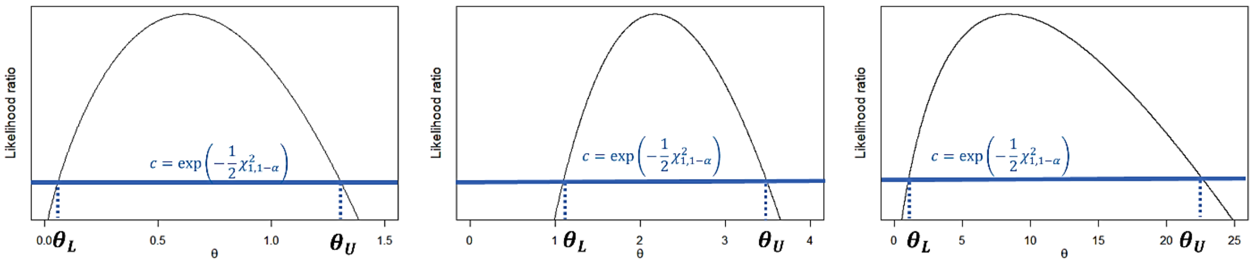


Figure 3. Likelihood-based confidence intervals when θ = 0.5 (left), 2 (middle), and 8 (right) and n = 1,000

Observations from Figure 3 also indicate that the symmetry of the likelihood ratio functions is not guaranteed, even when operating with large sample sizes (n = 1,000). It becomes evident that the symmetry of these functions is not solely contingent on the sample size but is also significantly influenced by the dependence parameter.

4- Monte Carlo Simulation Studies

For the Monte Carlo simulation studies, Clayton sample data is generated using the rcopula function from R package copula version 1.1-2. The data sets display a range of characteristics, with the Clayton dependence parameter (θ) set to values of 0.22, 0.5, 0.86, 2, 4.67, and 18. These correspond to τ values of 0.1, 0.2, 0.3, 0.5, 0.7, and 0.9. Furthermore, the sample sizes (n) tested in this study include 10, 20, 30, 50, 100, and 500, representing small to large sample sizes. In order to evaluate the reliability and precision of the CIs, the coverage probability (CP) and average length (AL) are estimated via 1,000 Monte Carlo simulation repetitions. CP represents the proportion of times, among the 1,000 repetitions, when the true parameter falls within the computed CI. The AL is determined by averaging the lengths of the CIs computed across all repetitions, and the formula is defined as:

$$AL = \frac{\sum_{i=1}^{1000}(U_i - L_i)}{1000}$$

This application of Monte Carlo simulations enables a comprehensive analysis of how CIs behave under varying dependence parameters and sample sizes. This method provides an in-depth overview of the effects these variables have on the CP and AL, thereby demonstrating the effectiveness of interval estimation for the dependence parameter in bivariate Clayton copulas. The most efficient CI technique yielded a CP near or greater than 0.95 and the shortest AL. Figure 4 illustrates the Monte Carlo simulations process.

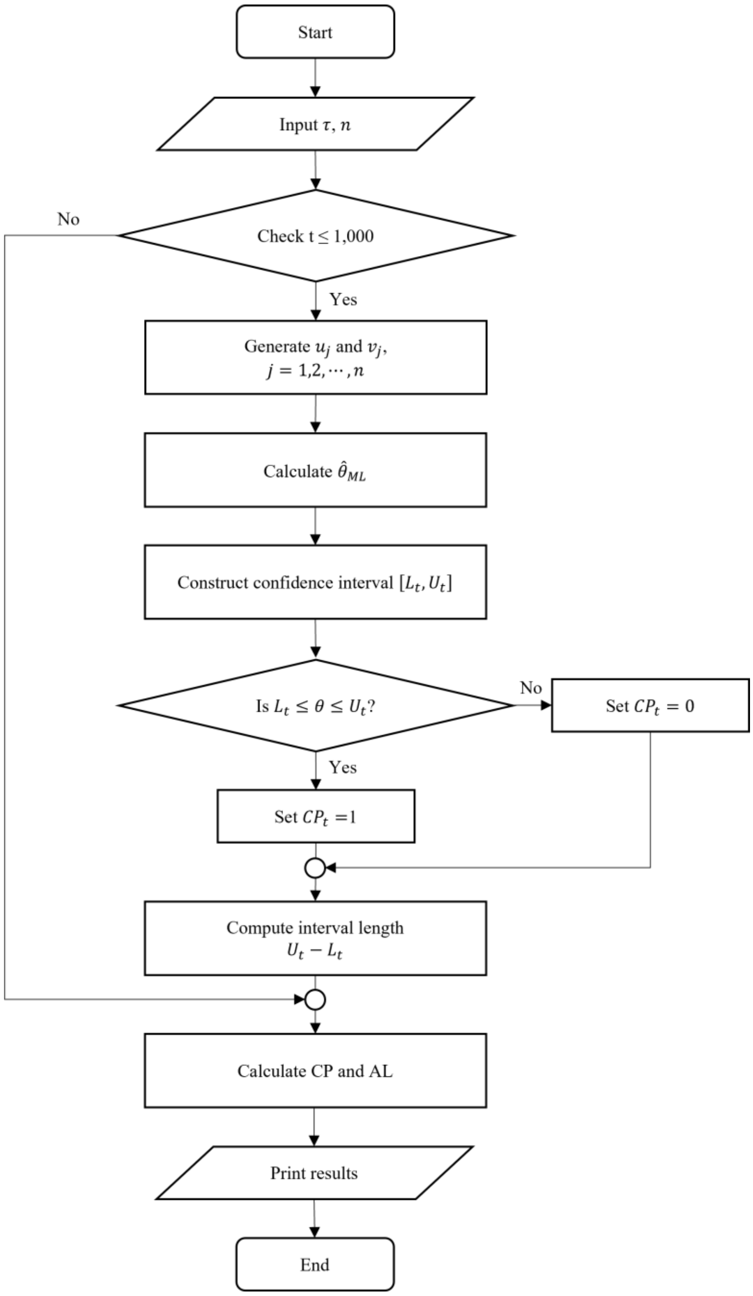


Figure 4. **Monte Carlo simulation flowchart**

In Figure 5, the performance is examined through the CPs, and it is found that when θ is greater than 0.86, the Wald CI performs as well as the likelihood-based CI for most sample sizes (n = 20, 30, 50, 100 and 500); the CPs are all close to 0.95. When θ has a low value, such as 0.22 or 0.5, and the sample size is small (less than 30), the likelihood-based CI outperforms the Wald CI. Figure 6 used the same information as Figure 5, but highlights the impact of varying values of the dependence parameters by using dependence as the x-axis and maintaining a fixed sample size. This visualization reveals that when the sample size is exceptionally small (n = 10), the CP is significantly below the stated confidence. Moreover, as the degree of dependence increases, the Wald CI becomes increasingly closer to the nominal coverage probability of 0.95. In contrast, the likelihood CIs appear to maintain robustness across all simulated scenarios. Regardless of the degree of dependence, likelihood CIs consistently perform well, emphasizing their superiority.

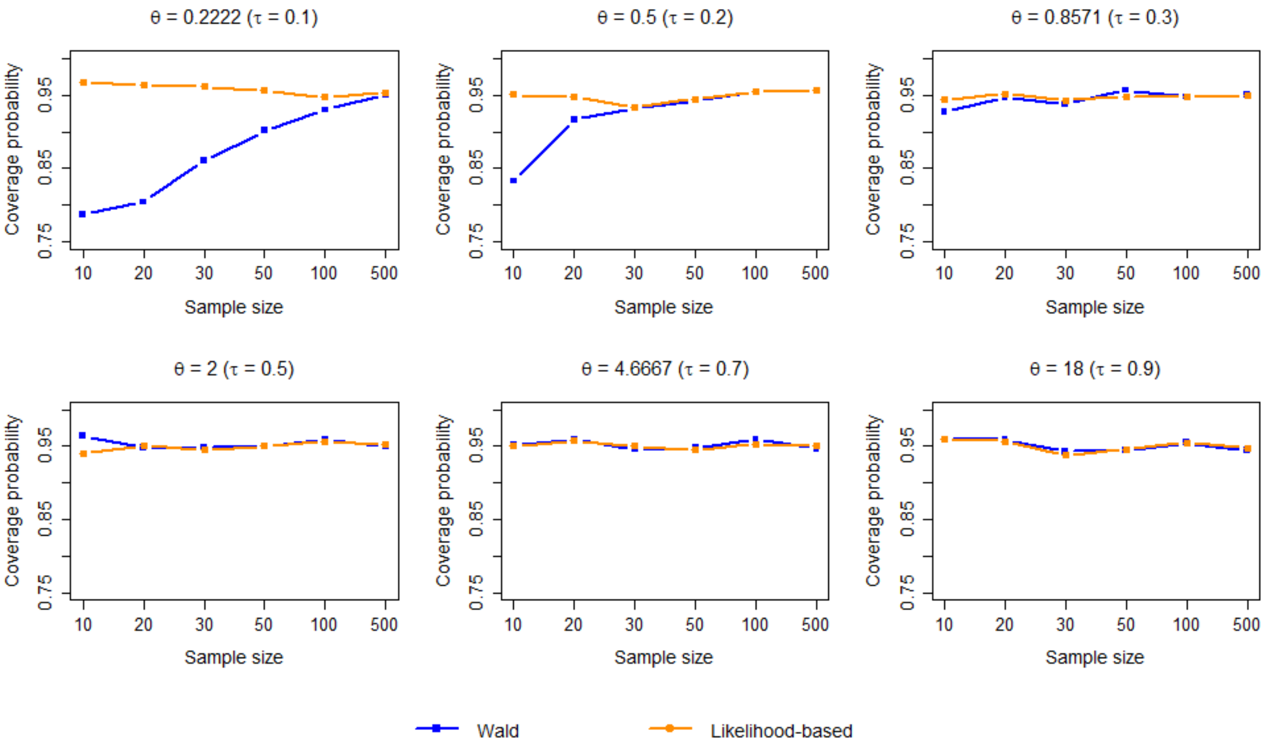


Figure 5. **Plots of coverage probabilities of Wald and likelihood intervals obtained under the bivariate Clayton using sample sizes as the x-axis**

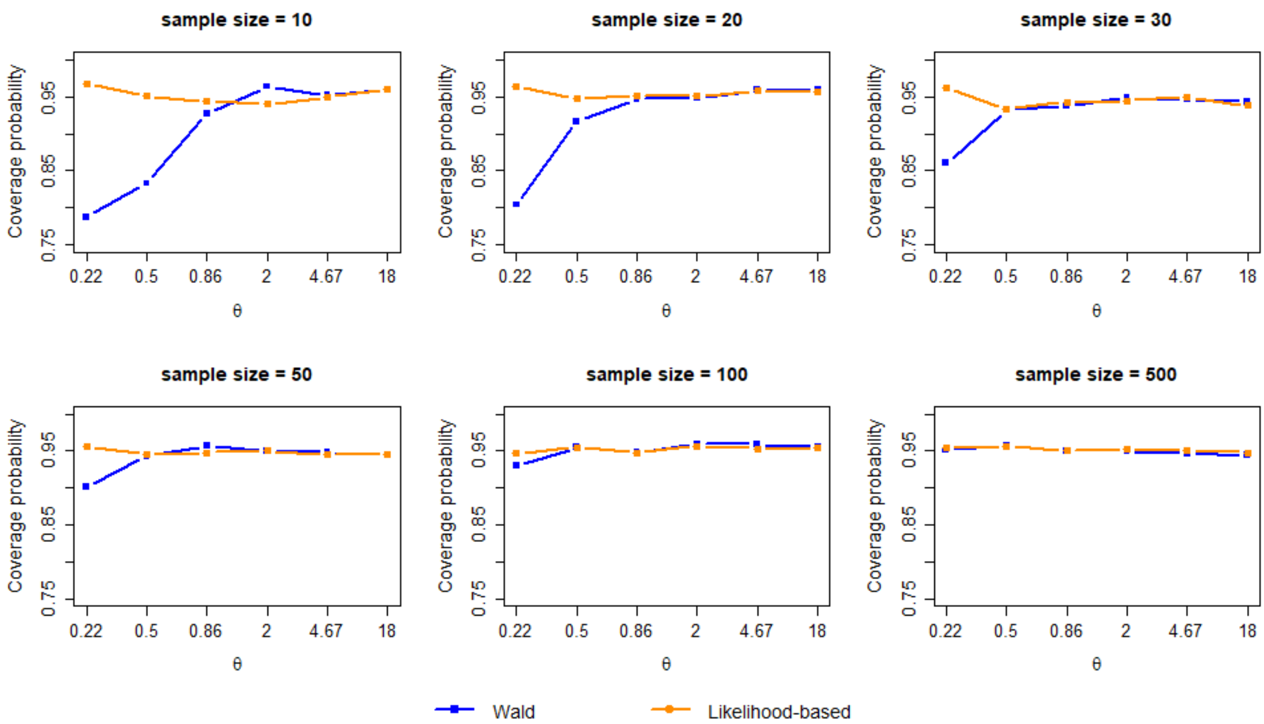


Figure 6. **Plots of coverage probabilities of Wald and likelihood intervals obtained under the bivariate Clayton using dependence parameters as the x-axis**

In addition to the criterion of CPs, the average length (AL) serves as another metric to assess the performance of CIs. Given equivalent CPs, a CI boasting a shorter AL is typically more desirable. As summarized in Table 1, an increase in sample size accompanied by a fixed θ value results in a decreasing range of ALs. For example, with $\theta = 2$ and a sample size of 30, the AL of the likelihood-based CI diminishes from 1.9254 to 0.4621 as the sample size escalates to 500. Further, given a fixed sample size, larger theta values correspond with increased AL values across all intervals. For instance, at a sample size of 50, the ALs of the Wald CI for θ values of 0.5, 2, and 18 are 0.8048, 1.4817, and 8.5612, respectively. Generally, the Wald CI exhibits a slightly shorter AL in comparison to the likelihood-based CI, but its CP can be far from 0.95 in some cases.

Table 1. The average length of Wald and likelihood intervals under the bivariate Clayton

θ	Method	Sample Size					
		10	20	30	50	100	500
0.22	Wald	1.1606	0.8033	0.7006	0.5812	0.4428	0.2205
	LL	1.3253	0.9002	0.7510	0.6071	0.4483	0.2199
0.5	Wald	1.5219	1.1974	1.0002	0.8048	0.5926	0.2632
	LL	1.6619	1.2264	1.0020	0.8052	0.5889	0.2629
0.86	Wald	2.1780	1.5421	1.2621	1.0089	0.6923	0.3083
	LL	2.2075	1.5527	1.2674	0.9963	0.6914	0.3082
2	Wald	3.5333	2.4055	1.9396	1.4817	1.0466	0.4611
	LL	3.4478	2.4060	1.9254	1.4785	1.0438	0.4621
4.67	Wald	6.2976	4.3656	3.4705	2.6413	1.8772	0.8259
	LL	6.2377	4.3348	3.4703	2.6251	1.8645	0.8292
18	Wald	20.4983	14.1017	11.0606	8.5612	6.0535	2.6755
	LL	20.1457	14.0274	10.9592	8.5646	6.0476	2.6856

LL: Likelihood-based confidence interval

5- Application

The effective implementation of CI calculations is illustrated using the daily closing prices of two prominent cryptocurrencies, Bitcoin (BTC) and Litecoin (LTC), throughout 2021. Data for these 365 observations, presented in Figure 7, was sourced from the Coin Metrices, accessible online at *https://charts.coinmetrics.io/network-data/*. An understanding of the dependence between BTC and LTC is critical for several reasons. Investors and market analysts can benefit from knowing the correlation between these two assets for portfolio diversification and risk management purposes. This is essential in the rapidly changing and often volatile cryptocurrency market, where risk-return trade-offs are stark.

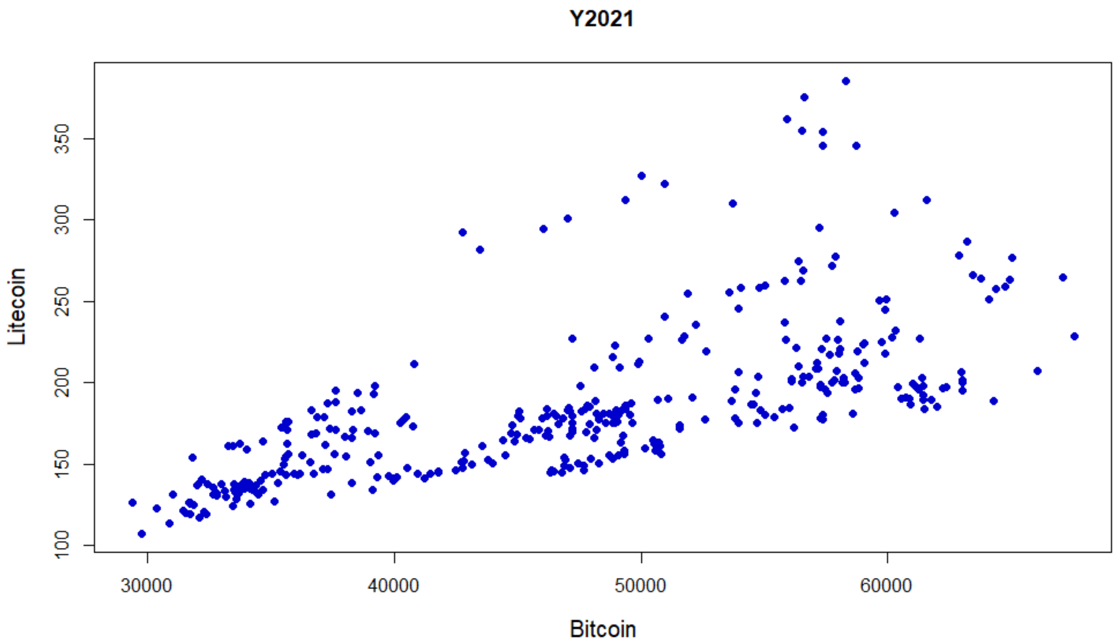


Figure 7. Scatter plot of 365 observations of BTC and LTC per USD-denominated closing price in log-scale in 2021

However, simply relying on Pearson's correlation may not provide a comprehensive understanding of their interrelationship. Pearson's correlation assumes a linear relationship and a similar scale among variables, which might not be the case in real-world scenarios, such as with the price movements of cryptocurrencies, which can be non-linear. Here, the marginal distributions of BTC and LTC are unknown, so the empirical distribution is applied to rescale the observations into a standard uniform. The plot of transformed observations is shown in Figure 8.

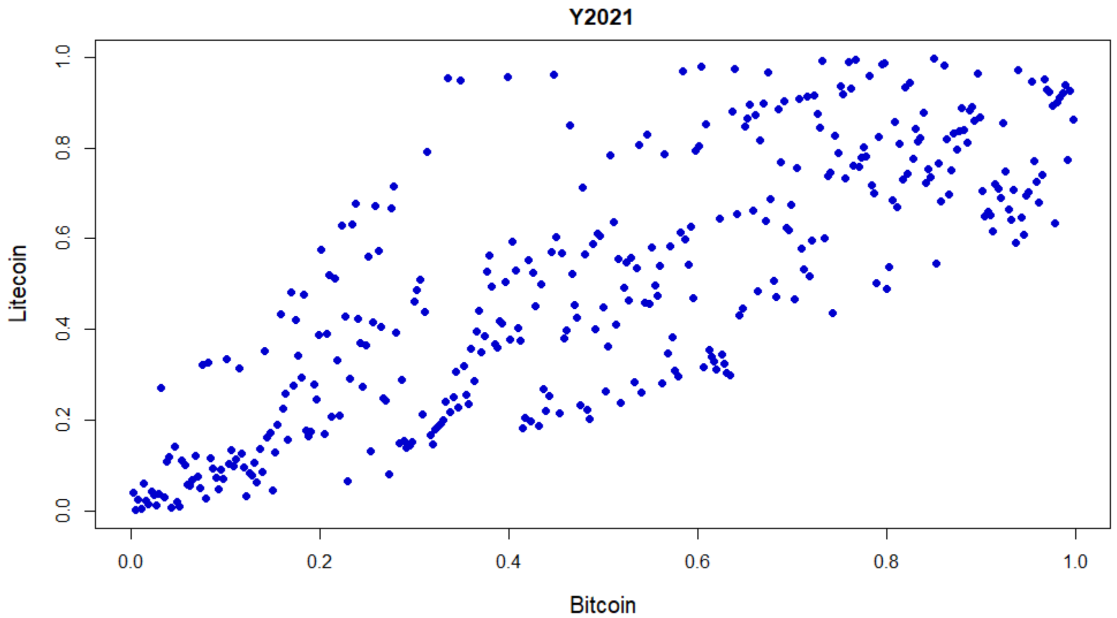


Figure 8. Scatter plot of 365 transformed observations of BTC and LTC per USD-denominated closing price in log-scale in 2021

In this particular case, the maximum likelihood estimate of θ ($\hat{\theta}_{ML}$) comes out to be 2.45, accompanied by an estimated variance of 0.0266, derived from the observed Fisher information. Applying both the Wald and likelihood-based methods, the resulting 95% CIs for the dependence parameter fall within (2.1335, 2.7730) and (2.1415, 2.7812) respectively. As seen in these outcomes, the interval length of the Wald method is marginally shorter, at 0.6395, compared to the likelihood-based method, which stands at 0.6397. The closest scenario in simulation studies is the case with $\theta = 2$ and a sample size of 500 which gives the AL of 0.4611 for the Wald CI and 0.4621 for the likelihood CI. This reinforces the conclusion drawn from the simulation study about the Wald method providing a slightly more concise interval.

6- Discussion

Confidence intervals serve as a fundamental statistical tool, offering a range of plausible values for true parameters. Within the scope of Archimedean copulas, Hofert *et al*. [26] introduced a method for constructing CIs for the dependence parameter when marginals are known. Their work hinged on the properties of maximum likelihood estimators and the score function, leading to the derivation of CIs in a general form that could be applied to any member of the Archimedean copulas. However, the inherent complexity of the general form made it difficult to use practically, thus making an explicit formula desirable. Furthermore, the likelihood CIs remained unexplored in the literature.

Our research significantly contributes to this field by focusing on CI of the dependence parameter of the bivariate Clayton copula, an Archimedean copula. In contrast to the previous approach, the observed FI of the bivariate Clayton copula has been mathematically derived in an explicit, therefore calculable form and is available here.

In discussing the results of the simulation studies, it is observed that Wald CIs tend to underperform compared to their likelihood-based counterparts, particularly in scenarios involving small sample sizes and low dependence parameter values. This discrepancy can be attributed to the fact that Wald CIs are formulated based on a normal distribution approximation, with the Wald statistic being a quadratic approximation of the likelihood ratio statistics, as detailed in Equation 9. Consider Figure 9, which highlights that with a sample size of 10, the shape of the likelihood ratio significantly deviates from the quadratic approximation, particularly at θ values that greatly differ from the MLE. On the contrary, with a sample size of 500, the likelihood ratio and quadratic function lines converge closely. Therefore, researchers, prior to the selection of Wald CIs, should take a preliminary step to examine the contours of both the likelihood ratio and quadratic functions. Without such information, likelihood CIs are recommended over Wald CIs.

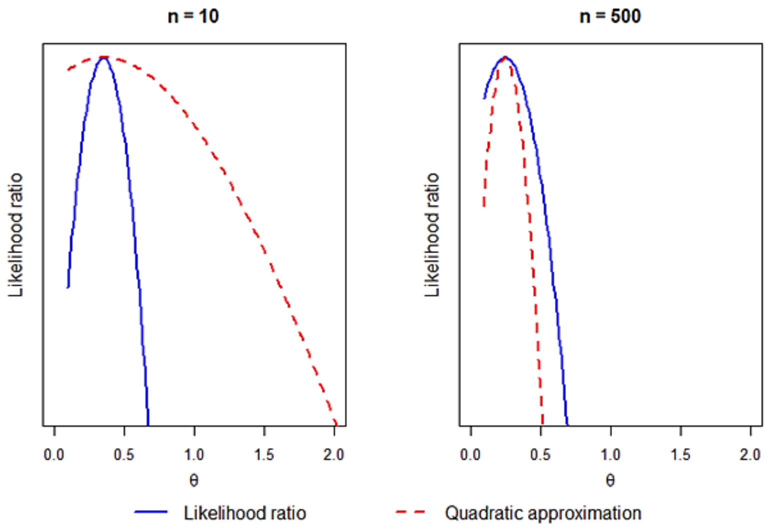


Figure 9. **Poor (left) and good (right) quadratic approximation (dotted) of the likelihood function (solid)**

7- Conclusion

This study aimed to construct an observed FI formula, which is used for the Wald CI for the dependence parameter of a bivariate Clayton copula. This research's contribution lies in offering an explicit formula for a CI for Clayton copula parameter, a widely-used approach despite the lack of a clear formula for a CI until now. Using Monte Carlo simulations, the performance of the Wald CI was compared to that of the likelihood-based CI based on the criteria of coverage probability and average CI length. In situations with a small sample size, likelihood-based CIs outperform Wald CIs by providing a coverage probability that is closer to the nominal confidence level of 0.95. Nonetheless, for large samples and when the dependence parameter, θ, is not close to zero, both the Wald and likelihood-based confidence intervals are recommended equally.

As a continuation of this research, future studies could explore higher-dimensional Clayton dependency structures. This would allow the estimation of the dependence parameter for Clayton copula across multiple variables simultaneously. In turn, it could broaden the understanding of various types of dependencies among various variables, contributing further to the field of copula studies.

8- Declarations

8-1- Author Contributions

Conceptualization, P.S.; methodology, P.S. and U.K.; software, U.K.; validation, P.S.; formal analysis, U.K.; investigation, P.S.; resources, P.S. and U.K.; data curation, U.K.; writing—original draft preparation, U.K.; writing—review and editing, P.S.; visualization, U.K.; supervision, P.S.; funding acquisition, P.S. and U.K. All authors have read and agreed to the published version of the manuscript.

8-2- Data Availability Statement

Publicly available datasets were analyzed in this study. This data can be found here: https://charts.coinmetrics.io/network-data/.

8-3- Funding and Acknowledgements

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8-4- Institutional Review Board Statement

Not applicable.

8-5- Informed Consent Statement

Not applicable.

8-6- Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this manuscript. In addition, the ethical issues, including plagiarism, informed consent, misconduct, data fabrication and/or falsification, double publication and/or submission, and redundancies have been completely observed by the authors.

9- References

[1] Di Clemente, A., & Romano, C. (2021). Calibrating and Simulating Copula Functions in Financial Applications. Frontiers in Applied Mathematics and Statistics, 7. doi:10.3389/fams.2021.642210.

[2] Štulajter, F. (2009). Comparison of different copula assumptions and their application in portfolio construction. Ekonomická Revue - Central European Review of Economic Issues, 12(4), 191–204. doi:10.7327/cerei.2009.12.04.

[3] Naifar, N. (2011). Modelling dependence structure with Archimedean copulas and applications to the iTraxx CDS index. Journal of Computational and Applied Mathematics, 235(8), 2459–2466. doi:10.1016/j.cam.2010.10.047.

[4] Darabi, R., & Baghban, M. (2018). Application of Clayton Copula in Portfolio Optimization and its Comparison with Markowitz Mean-Variance Analysis. Advances in Mathematical Finance & Applications, 3(1), 33–51. doi:10.22034/AMFA.2018.539133.

[5] Abdi, A., Hassanzadeh, Y., Talatahari, S., Fakheri-Fard, A., & Mirabbasi, R. (2017). Parameter estimation of copula functions using an optimization-based method. Theoretical and Applied Climatology, 129(1–2), 21–32. doi:10.1007/s00704-016-1757-2.

[6] Evkaya, O., Yozgatligil, C., & Sevtap Selcuk-Kestel, A. (2019). Drought analysis using copula approach: a case study for Turkey. Communications in Statistics: Case Studies, Data Analysis and Applications, 5(3), 243–260. doi:10.1080/23737484.2019.1635923.

[7] Fan, L., Wang, H., Wang, C., Lai, W., & Zhao, Y. (2017). Exploration of Use of Copulas in Analysing the Relationship between Precipitation and Meteorological Drought in Beijing, China. Advances in Meteorology, 2017, 1–11. doi:10.1155/2017/4650284.

[8] Zhang, Y. (2018). Investigating dependencies among oil price and tanker market variables by copula-based multivariate models. Energy, 161, 435–446. doi:10.1016/j.energy.2018.07.165.

[9] Bhatti, M. I., & Do, H. Q. (2019). Recent development in copula and its applications to the energy, forestry and environmental sciences. International Journal of Hydrogen Energy, 44(36), 19453–19473. doi:10.1016/j.ijhydene.2019.06.015.

[10] Rusyda, H. A., Noviyanti, L., Soleh, A. Z., Chadidjah, A., & Indrayatna, F. (2021). The design of multiple crop insurance in Indonesia based on revenue risk using the copula model approach. Journal of Applied Statistics, 48(13–15), 2920–2930. doi:10.1080/02664763.2021.1897089.

[11] Tjøstheim, D., Otneim, H., & Støve, B. (2022). Statistical Dependence: Beyond Pearson's ρ. Statistical Science, 37(1), 90–109. doi:10.1214/21-STS823.

[12] Schweizer, B., & Wolff, E. F. (1981). On nonparametric measures of dependence for random variables. The annals of statistics, 9(4), 879-885. doi:10.1214/aos/1176345528.

[13] Nelsen, R. B. (2006). An Introduction to Copulas. Springer, New York, United States. doi:10.1007/0-387-28678-0.

[14] Frees, E. W., & Valdez, E. A. (1998). Understanding relationships using copulas. North American Actuarial Journal, 2(1), 1–25. doi:10.1080/10920277.1998.10595667.

[15] Joe, H. (2014). Dependence Modeling with Copulas. Chapman and Hall/CRC, New York, United States. doi:10.1201/b17116.

[16] Hofert, M., Kojadinovic, I., Mächler, M., & Yan, J. (2018). Elements of Copula Modeling with R. Springer, Cham, Switzerland. doi:10.1007/978-3-319-89635-9.

[17] Atique, F., & Attoh-Okine, N. (2018). Copula parameter estimation using Bayesian inference for pipe data analysis. Canadian Journal of Civil Engineering, 45(1), 61–70. doi:10.1139/cjce-2017-0084.

[18] Genest, C., & Rivest, L.-P. (1993). Statistical Inference Procedures for Bivariate Archimedean Copulas. Journal of the American Statistical Association, 88(423), 1034. doi:10.2307/2290796.

[19] Shemyakin, A., & Kniazev, A. (2017). Introduction to Bayesian Estimation and Copula Models of Dependence. John Wiley & Sons, Hoboken, United States. doi:10.1002/9781118959046.

[20] Choroś, B., Ibragimov, R., Permiakova, E. (2010). Copula Estimation. Copula Theory and Its Applications. Lecture Notes in Statistics, vol 198. Springer, Berlin, Germany. doi:10.1007/978-3-642-12465-5_3.

[21] Kim, G., Silvapulle, M. J., & Silvapulle, P. (2007). Comparison of semiparametric and parametric methods for estimating copulas. Computational Statistics & Data Analysis, 51(6), 2836–2850. doi:10.1016/j.csda.2006.10.009.

[22] Genest, C., Ghoudi, K., & Rivest, L. P. (1995). A Semiparametric Estimation Procedure of Dependence Parameters in Multivariate Families of Distributions. Biometrika, 82(3), 543. doi:10.2307/2337532.

[23] Ko, V., & Hjort, N. L. (2019). Model robust inference with two-stage maximum likelihood estimation for copulas. Journal of Multivariate Analysis, 171, 362–381. doi:10.1016/j.jmva.2019.01.004.

[24] Kojadinovic, I., & Yan, J. (2010). Comparison of three semiparametric methods for estimating dependence parameters in copula models. Insurance: Mathematics and Economics, 47(1), 52–63. doi:10.1016/j.insmatheco.2010.03.008.

[25] Alipour, M., La Puma, I., Picotte, J., Shamsaei, K., Rowell, E., Watts, A., ... & Taciroglu, E. (2023). A multimodal data fusion and deep learning framework for large-scale wildfire surface fuel mapping. Fire, 6(2), 36. doi:10.3390/fire6020036.

[26] Hofert, M., Mächler, M., & McNeil, A. J. (2012). Likelihood inference for Archimedean copulas in high dimensions under known margins. Journal of Multivariate Analysis, 110, 133–150. doi:10.1016/j.jmva.2012.02.019.

[27] Clayton, D. G. (1978). A model for association in bivariate life tables and its application in epidemiological studies of familial tendency in chronic disease incidence. Biometrika, 65(1), 141–151. doi:10.2307/2335289.

[28] Mai, J.-F., & Scherer, M. (2014). Financial Engineering with Copulas Explained. Palgrave Macmillan, London, United Kingdom. doi:10.1057/9781137346315.

[29] Rohde, C A. (2014). Introductory Statistical Inference with the Likelihood Function. Springer, Cham, Switzerland. doi:10.1007/978-3-319-10461-4.

[30] Pawitan, Y. (2001). In all likelihood: statistical modelling and inference using likelihood. Oxford University Press, Oxford, United Kingdom.

[31] Sodanin, K., Niwitpong, S. A., & Niwitpong, S. (2022). Estimating Simultaneous Confidence Intervals for Multiple Contrasts of Means of Normal Distribution with Known Coefficients of Variation. Emerging Science Journal, 6(4), 705-716. doi:10.28991/ESJ-2022-06-04-04.

[32] Fisher, R. A. (1956). Statistical methods and scientific inference. Hafner Publishing Company, Pennsylvania, United States.