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Recovering Coefficients of Second-Order Hyperbolic and Plate Equations via Finite Measurements on the Boundary

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Abstract

In this dissertation, we consider the inverse problem for a second-order hyperbolic equation of recovering $n + 3$ unknown coefficients defined on an open bounded domain with a smooth enough boundary. We also consider the inverse problem of recovering an unknown coefficient on the Euler-Bernoulli plate equation on a lower-order term again defined on an open bounded domain with a smooth enough boundary. For the second-order hyperbolic equation, we show that we can uniquely and (Lipschitz) stably recover all these coefficients from only using half of the corresponding boundary measurements of their solutions, and for the plate equation, we show that we can uniquely and stably recover the coefficient by using two measurements on the boundary. The proofs for solving both inverse problems are based on a post-Carleman estimate strategy developed by Isakov in [19], continuous observability inequalities, and regularity theory.

Dedication

This work is dedicated to my parents, my father, Claude (Randy) Randall Scruggs, and my mother, Martha Elizabeth-Green Scruggs. Ever since childhood, you both believed I could accomplish anything I set my mind to. Thank you both for supporting me over the years, showing me unconditional love, and encouraging me when I doubted myself. I am fortunate to have you two as my parents.

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I want to thank my entire family for their patience, kindness, and love during my time as a graduate student. To my parents: I am very grateful for all of your support over the years. To my sister Elizabeth: Thank you for your support and allowing me to play copious amounts of Pokémon during my visits. I am always happy to share my love of the series with you.

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Out of all my friends outside of Clemson, I want to thank my dear friend Brooke Watkins. From being weird undergrads to being (kind-of) well-adjusted adults, I am glad our friendship has remained strong over the last decade. Thank you for all the assurance, kindness, compassion, and patience you constantly gave me. Your friendship has helped me so much, more than you can imagine. I am glad to see the person you have become, and I wish you nothing but the best for your future.

To finish my acknowledgments, I want to thank two people who, without them, completing graduate school would not be possible. First, I want to thank Dr. Lynne Cory, my therapist, during my first few years at Clemson. I reached the lowest point in my mental health during my early 20s, and I thought I was going insane for all the self-deprecating thoughts and thoughts of self-harm I was experiencing. Your sessions helped me learn that I was not going insane but was confused, scared, and simply needed help. Thank you for your patience and kindness during the darkest period of my life. My therapy sessions with you have helped me accept and cope with my thought process, and they encouraged me to help others going through similar experiences. I cannot stress how invaluable your commitment to my mental health and well-being was toward my success in graduate school.

Finally, thank you to Shelton Pinson. In March 2016, I decided there was no point in living, and I planned to end everything. Before I left BCM that night, you convinced me to stay for that evening's worship service, and during the service, I decided to continue living and seek help. Your act of kindness saved my life, and I would not have obtained my doctorate nor met most of the

people listed in the acknowledgments without your intervention. I cannot thank you enough, and you will forever have my respect and gratitude.

Table of Contents

Chapter 1

Introduction

The primary focus of this dissertation is to solve the inverse problem of recovering coefficients from second-order hyperbolic partial differential equations (PDEs) and the Euler-Bernoulli plate equations under hinged boundary conditions. Our approach utilizes a sharp Carleman estimate for second-order hyperbolic equations [31, 37, 46] and a Carleman estimate for the plate equation derived using similar techniques as in [1, 47]. To use the Carleman estimates, we first convert the original inverse problems to inverse source problems and prove uniqueness and stability for the system of second-order hyperbolic equations and the plate equation. An crucial result used in the proofs is a "post-Carleman estimate" technique from [19, Theorem 8.2.2]. This result creates a strategy to prove uniqueness for the inverse source problems. To prove stability, we use corresponding continuous observability inequalities from [27, 40, 41, 42, 46]. In this chapter, we provide a brief overview of Carleman estimates as they pertain to inverse problem for systems of PDEs. We also provide the underlying assumptions used for each inverse problem.

1.1 Carleman Estimates

The origin of Carleman estimates and their namesake is from the mathematician Carleman in 1939. In [9], he formed a technique to prove uniqueness to the Cauchy problem in two variables. Years later, Hörmander generalized Carleman's method to work on a more general class of differential operators [15]. The general representation is the following:

$$\sum_{|\alpha|<m} \tau^{2(m-|\alpha|-1)} \int \left|D^{\alpha} u e^{2\tau\varphi}\right| \, dx \leq C \int |P(x,D) u e^{2\tau\varphi}|^2 \, dx, \; u \in C_0^{\infty} \tag{1.1}$$

where φ is a weight function, $P(x, D)$ is a partial differential operator, α is multi-index notation, and τ is a sufficiently large parameter. Hörmander used (1.1) to prove the *Unique Continuation Property*: given u as a solution to the PDE $P(x, D)u = 0$ on a bounded domain $\Omega \subset \mathbb{R}^n$ and $u = 0$ for some $\varphi(x) > 0$, where the function $\varphi : \Omega \to \mathbb{R}$ defines a smooth hypersurface in Ω, then this implies that $u = 0$ on a neighborhood of $\varphi = 0$.

The main setback of early results on Carleman estimates, however, is that they were only applicable when the solutions to the PDE system were compactly supported. By this assumption, early Carleman estimates did not contain boundary terms, which limits their usefulness in applications to control theory and inverse problems. To exacerbate the issue, homogenizing the Cauchy data produced a term on the right-hand side of the estimate with norms of boundary traces a half derivative higher than the norm of u on the left-hand side of the estimate [30]. Hence, early Carleman estimates failed in provided decent results when applied to boundary value problems. Thus, the need to develop improved Carleman-type inequalities that produce good results for solutions of boundary value problems increased.

The improvement to Carleman estimates that include boundary terms is credited to two sources. The first source is credited to Daniel Tataru [45] at the University of California, Berkeley. He proposed extending the main Carleman estimate to general pseudo-differential operators. To establish this, specific geometric properties must be met with the domain, including a surface that must be pseudo-convex. Tataru's work was inspired by Lasiecka and Triggiani [34] where they developed a sharp Carleman estimate for second-order hyperbolic equations by using the multiplier method, which refers to multiplying the governing PDE by differential multipliers. These multipliers differ depending on the PDE they were used on. The other method of providing useful Carleman estimates for boundary value problems is credited to Lavrentiev, Romanov, and Shishatski [39]. They first establish an initial pointwise Carleman estimate with the resulting integral form aiding them in producing boundary terms. Lasiecka, Triggiani, and Zhang extended their work in [37] where they established a pointwise inequality for general second-order hyperbolic equations. They applied similar techniques in the papers [34, 36, 38].

1.2 Geometric Assumptions

In this section we present the main geometrical assumptions necessary to establish the Carleman estimates used throughout this paper. These assumptions are well known, and they can be found in [4, 5, 7, 22, 26, 29, 30, 37, 41, 42, 43, 46] and the numerous sources cited within.

1.2.1 Second-Order System of Hyperbolic Equations

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be an open bounded domain with smooth enough boundary $\Gamma = \partial\Omega = \overline{\Gamma_0 \cup \Gamma_1}$, where $\Gamma_0 \cap \Gamma_1 = \emptyset$. We refer Γ_1 as the *observed* part of the boundary where boundary measurements are taken, and Γ_0 as the *unobserved* part of the boundary,, which is left alone during boundary measurements. We consider the following general second-order hyperbolic equation for $w = w(x,t)$ defined on $Q = \Omega \times [-T, T]$ with initial conditions $\{w_0, w_1\}$ and Dirichlet boundary condition $w|_\Sigma = h$ on $\Sigma = \Gamma \times [-T, T]$:

$$\begin{cases} w_{tt} - c^2(x)\Delta w + q_1(x)w_t + q_0(x)w + \mathbf{q}(x) \cdot \nabla w = 0 & \text{in } Q \\ w\,(x, 0) = w_0(x);\ \ w_t\,(x, 0) = w_1(x) & \text{in } \Omega \\ (x, t) = h(x, t) & \text{in } \Sigma. \end{cases} \tag{1.2}$$

Here $q_1 \in L^\infty(\Omega)$, $q_0 \in L^\infty(\Omega)$, and $\mathbf{q} \in (L^\infty(\Omega))^n$ are the damping, potential, and gradient coefficients, respectively, and wave speed $c(x)$ satisfies

$$c \in \mathcal{C} = \{c \in C^1(\Omega) : c_0^{-1} \leq c(x) \leq c_0,\ \text{for some } c_0 > 0\}.$$

Because the wave speed is on the principal part of (1.2), our geometrical assumptions are made with respect to Riemannian geometry so we can define a metric to make the principle part constant. The definitions and some identities are given in Appendix B of this paper. Given the triplet $\{\Omega, \Gamma_0, \Gamma_1\}$, we make the following assumptions on the unobserved part Γ_0:

(A.1) There exists a strictly convex function $d : \overline{\Omega} \to \mathbb{R}$ in the metric $g = c^{-2}(x)dx^2$, and of class $C^3(\overline{\Omega})$, such that the following two properties hold true (through translation and rescaling if necessary):

(i) The normal derivative of d on the unobserved part Γ_0 of the boundary is non-positive.

Namely,

$$\frac{\partial d}{\partial \nu} = \langle Dd(x), \nu(x) \rangle \leq 0, \quad \forall x \in \Gamma_0,$$

where $Dd = \nabla_g d$ is the gradient vector field on Ω with respect to g, and $\langle X, Y \rangle = g(X, Y)$ for all $X, Y \in M_x$, where M_x is the tangent space at $x \in \Omega$.

(ii) The Hessian of d, denoted as $D^2 d(X, X)$, is strictly positive definite,

$$D^2 d(X, X) = \langle D_X(Dd), X \rangle_g \geq 2|X|_g^2, \ \forall X \in M_x, \ \min_{x \in \overline{\Omega}} d(x) = m_0 > 0$$

where D_X is the covariant derivative of a vector field with respect to X.

(A.2) $d(x)$ has no critical point on $\overline{\Omega}$. In other words,

$$\inf_{x \in \overline{\Omega}} |Dd| > 0, \text{ so that we may take } \inf_{x \in \overline{\Omega}} \frac{|Dd|^2}{d} > 4.$$

Remark 1.2.1. The geometrical assumptions above permit the construction of a vector field that enables a pseudo-convex function necessary for allowing a Carleman estimate containing no lower-order terms for the general second-order equation (1.1). These assumptions were first formulated in [37] under the framework of a Euclidean metric, with [46] employing them under the more general Riemannian framework. The reader can find examples and illustrations of large general classes of domains $\{\Omega, \Gamma_1, \Gamma_0\}$ satisfying (A.1) and (A.2) in [46, Appendix B]. One canonical example is to take $d(x) = |x - x_0|^2$, with x_0 being a point outside $\overline{\Omega}$, if the wave speed c satisfies

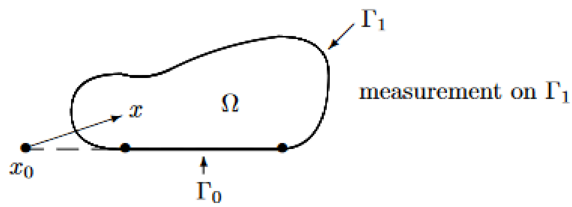
$$\left| \frac{\nabla c(x) \cdot (x - x_0)}{2c(x)} \right| \leq r_c < 1 \text{ for some } r_c \in (0, 1). \tag{1.3}$$

More details are provided in [7, Theorem 6.1], [17, Theorem 1], and [19, Theorem 3.2.1]. Condition (1.3) on c may be improved to the following condition given in [22, Theorem 2.5.1]:

$$\nabla c^{-2}(x) \cdot (x - x_0) \geq 0, \ x \in \overline{\Omega} \tag{1.4}$$

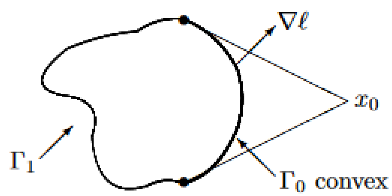
To illustrate the geometric examples and their importance, we will present a few examples to the geometrical examples (A.1) and (A.2) where c is constant. More details can be found in [30, 37, 46].

Example 1.2.2. Here let the dimension of Ω be greater than or equal to two. Here the unobserved portion of the boundary Γ_0 is flat.

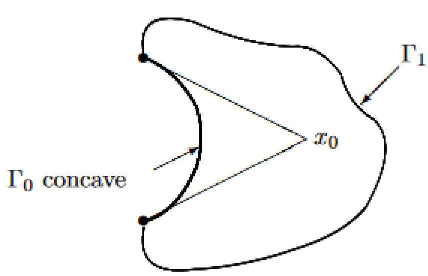


For any point x_0 in the hyperplane of Γ_0, then $d(x) = \|x - x_0\|^2$ and $h(x) = \nabla d(x) = 2(x - x_0)$.

Example 1.2.3. Again let Ω be a domain with dimension greater than or equal to two, but now let Γ_0 be convex, subtended by a common point x_0. The specific $d(x)$ can be found in [37, Theorem A.4.1]. Under this setting, $\Gamma_0 = \ell(x) =$ level set and $(x - x_0) \cdot \nabla \ell(x) \leq 0$ on Γ_0.



Example 1.2.4. Now let Γ_0 be concave, subtended by a common point x_0. Again, the specific $d(x)$ can be found in [37, Theorem A.4.1].



Example 1.2.5. Fix the dimension of Ω to 2, and let Γ_0 be neither convex or concave. Γ_0 can be described by the graph

$$y = \begin{cases} f_1(x) & x_0 < x < x_1,\ y \geq 0; \\ f_2(x) & x_2 < x < x_1,\ y < 0, \end{cases}$$

Here, f_1 and f_2 are logarithmic concave on $x_0 < x < x_1$, i.e. $\sin(x) + 1$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$; $\cos(x) + 1$, $0 < x < \pi$.

The function $d(x)$ can be found in [37, Equation (A.2.7)].

1.2.2 Plate Equation

As before, let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be an open bounded domain with smooth enough boundary $\Gamma = \partial\Omega = \overline{\Gamma_0 \cup \Gamma_1}$, where $\Gamma_0 \cap \Gamma_1 = \emptyset$. In this subsection, we consider the geometrical assumptions for the following plate equation for $w = w(x,t)$ defined on Q with initial conditions $\{w_0, w_1\}$ and hinged boundary conditions h_1 and h_2 on Σ:

$$\begin{cases} w_{tt} + \Delta^2 w + q(x)w = 0 & \text{in } Q \\ w(x,0) = w_0(x)\ w_t(x,0) = w_1(x) & \text{in } \Omega \\ w(x,t) = h_1(x,t),\ \Delta w(x,t) = h_2(x,t) & \text{on } \Sigma \end{cases} \tag{1.5}$$

with $q \in L^\infty(\Omega)$. The geometric assumptions for problem (1.5) are similar to the geometrical assumptions for (1.2); however, since the coefficient of interest is not on the principal part of the equation, our assumptions are stated in the Euclidean framework $\mathbb{R}^n$:

(A.1') There exists a strictly convex function $d : \overline{\Omega} \to \mathbb{R}$ of class $C^3(\overline{\Omega})$, such that $h \equiv \nabla d$ for every $x \in \Omega$ (h is radial) and the following properties hold:

(i) We have $\nabla d \cdot \nu = h \cdot \nu \leq 0$ for all $x \in \Gamma_0$ in the Dirichlet B.C. case. If (1.5) has Neumann B.C., our assumption changes to $h \cdot \nu = 0$ for all $x \in \Gamma_0$.

(ii) The Hessian matrix of $d(x)$ (the Jacobian matrix of $h(x)$) is strictly positive definite on $\overline{\Omega}$: there exists constant $\rho > 0$ such that for all $x \in \overline{\Omega}$

$$\mathcal{H}_d(x) = J_h(x) = \begin{pmatrix} d_{x_1x_1} & \cdots & d_{x_1x_n} \\ \vdots & \ddots & \vdots \\ d_{x_nx_1} & \cdots & d_{x_nx_n} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial h_1}{\partial x_1} & \cdots & \dfrac{\partial h_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial h_n}{\partial h_1} & \cdots & \dfrac{\partial h_n}{\partial x_n} \end{pmatrix} \geq \rho I \tag{1.6}$$

(A.2') $d(x)$ has no critical points within $\overline{\Omega}$:

$$\inf_{x \in \Omega} |h(x)| = \inf_{x \in \Omega} |\nabla d(x)| = p > 0. \tag{1.7}$$

1.3 Weight Functions

The reason for assuming the geometrical assumptions (A.1) and (A.2) for (1.2) and (A.1') and (A.2') for (1.5) is because it allows us to construct a pseudo-convex (weight) function necessary to obtain Carleman estimates for our equations. We list the corresponding weight functions and their properties in this section. The weight functions below are standard, and they can be found in [13, 37, 38, 41, 42, 46], and the papers cited within these works.

1.3.1 Second Order Hyperbolic Equation

Having chosen, on the strength of geometrical assumption (A.1) for (1.2), a strictly convex function $d(x)$, we can define the function $\varphi(x,t) : \Omega \times \mathbb{R} \to \mathbb{R}$ of class C^3 by setting

$$\varphi(x,t) = d(x) - \alpha t^2, \quad x \in \Omega, \; t \in [-T, T], \tag{1.8}$$

where $T > T_0$. The threshold time T_0 is set by

$$T_0^2 \equiv 4 \max_{x \in \overline{\Omega}} d(x). \tag{1.9}$$

This definition is due to the assumption in (A.1) from [37] where it is assumed that $c = 1$. Notice that T_0 is affected by the scaling of $d(x)$. Thus, a smaller threshold time permits a smaller (forces a larger) final time T. We also assume that $d(x) > 0$, $x \in \Omega$ since otherwise we can translate $d(x)$ so that positivity over the domain is satisfied. So from (1.9), we have the existence of $\delta > 0$, fixed, that satisfies

$$T^2 > 4 \max_{x \in \overline{\Omega}} d(x) + 4\delta. \tag{1.10}$$

Now let $\alpha \in (0,1)$ be selected as follows: for $T > T_0$, for the δ fixed satisfying (1.10), there exists a constant $\alpha \in (0,1)$, such that

$$\alpha T^2 > 4 \max_{x \in \overline{\Omega}} d(x) + 4\delta. \tag{1.11}$$

From our definition of the weight function (1.8) and related definitions(1.9)-(1.11), φ has the following properties:

(a) For the constant $\delta > 0$ fixed above, we have

$$\varphi(x, -T) = \varphi(x, T) \leq \max_{x \in \overline{\Omega}} d(x) - \alpha T^2 \leq -\delta \text{ uniformly in } x \in \Omega; \tag{1.12}$$

and

$$\varphi(x,t) \leq \varphi(x, 0), \quad \text{for any } t \in [-T, T] \text{ and any } x \in \Omega. \tag{1.13}$$

(b) There are t_0 and t_1, with $-T < t_0 < 0 < t_1 < T$, say, chosen symmetrically about 0, such that

$$\min_{x \in \overline{\Omega}, t \in [t_0, t_1]} \varphi(x,t) \geq \sigma, \quad \text{where } 0 < \sigma < m_0 = \min_{x \in \overline{\Omega}} d(x). \tag{1.14}$$

Moreover, let $Q(\sigma)$ be the subset of $Q = \Omega \times [-T, T]$ defined by

$$Q(\sigma) = \{(x,t) : \varphi(x,t) \geq \sigma > 0, x \in \Omega, -T \leq t \leq T\}, \tag{1.15}$$

Then we have

$$\Omega \times [t_0, t_1] \subset Q(\sigma) \subset Q. \tag{1.16}$$

The region $Q(\sigma)$ in (1.15) will be relevant when proving Theorem 3.1.3 in Chapter 3.

Remark 1.3.1. Property (1.14) is only required for the Carleman estimate for (1.2). For the Carleman estimate (1.5), it upholds the assumption $w(x, -T) = w(x, T) = 0$, which circumvents the need of $Q(\sigma)$. See [30] for details.

1.3.2 Plate Equation

A key ingredient for the Carleman estimate of the plate equation is to use the Carleman estimate for the Schrödinger equation. Hence, our weight function for the plate equation is made to be compatible with the Schrödinger equation. Having chosen a convex function $d(x)$ under the assumption (A.1'), we define our weight function as follows:

$$\varphi(x,t) = d(x) - \beta t^2;\ x \in \Omega,\ t \in [-T,T]. \tag{1.17}$$

Unlike the weight function defined in (1.8), we let $T > 0$ be arbitrary since we do not need to satisfy finite propagation speed. Likewise, we choose the constant $\beta = \beta_T$ large enough to such that

$$\beta T^2 > 4 \max_{x\in\overline{\Omega}} d(x). \tag{1.18}$$

From (1.18), we fix a small $\delta > 0$ such that

$$\beta T^2 > 4 \max_{x\in\overline{\Omega}} d(x) + 4\delta. \tag{1.19}$$

This specific weight function has the following properties:

(a) For the constant δ fixed in (1.18), we have the following:

$$\varphi(x,-T) = \varphi(x,T) = d(x) - \beta T^2 \le -\delta \text{ uniformly in } x \in \Omega; \tag{1.20}$$

and

$$\varphi(x,t) \le \varphi(x,0) \text{ for all } t \in [-T,T] \text{ and } x \in \Omega. \tag{1.21}$$

(b) There are t_0 and t_1, with $-T < t_0 < 0 < t_1 < T$, such that

$$\min_{x\in\overline{\Omega};[t_0,t_1]} \varphi(x,t) \ge -\frac{\delta}{2} \tag{1.22}$$

since $\varphi(x,0) = d(x) \ge 0$ for all $x \in \Omega$.

1.4 Controllability and Observability

On an evolution system modeled by a PDE, (exact) controllability is defined as the existence of a control function that drives the system, within some time T, from an initial state to a desired final state [44]. For a hyperbolic system, this can be summarized by steering any initial condition to 0 at the target time T through the use of a control function [30]. The control function may act on either the boundary (entire or a portion) or a region contained within an open bounded domain.

A standard reference in the controllability of PDE systems is Lions' paper [40], where he used the *Hilbert Uniqueness Method* (HUM) to establish exact boundary controllability for both hyperbolic and Petrowsky-type systems using both Dirichlet and Neumann boundary control functions. To summarize, HUM transforms the exact controllability problem into an observability problem for the dual of the PDE system. Lasiecka and Triggiani also demonstrated the exact controllability of second order hyperbolic equations under Dirichlet and Neumann boundary controls using the relationship between controllability of the original system and observability of its dual system in [32]. In this paper, they focused on the surjectivity of the "control-to-solution" operator that maps the boundary control to the final state of the solution under a target space [30].

The observability problem for the dual system refers to establishing an observability inequality on an energy term. This inequality can be interpreted as the initial energy being "observed" through a suitable boundary trace of the solution to the dual system, which is homogeneous on the boundary in the same boundary condition. For interior control, the observed data occurs in the specified region within the domain. Observability inequalities for both hyperbolic equations and plate equations have traditionally been established via the moment method in one dimension [27, 44] and the multiplier method for general dimensions, see [13, 32, 40, 49, 50] and the references within. Recently, Green, Liu, and Mitkovski in [11] extended the moment method to general dimensions for the viscoelastic wave equation. However, these methods are not strong enough to accommodate PDE systems with lower-order terms or variable coefficients. These issues were solved by using Carleman estimates on these system, which yields sharper inequalities that include boundary terms. See for example [7, 30, 34, 35, 37, 38] and the numerous sources cited within.

1.5 Inverse Problems

Inverse problems can be phrased as finding the cause of an event given knowledge of their effects. The field has been motivated by practical applications in various areas of science and engineering such as geophysical explorations, biomedical imaging, weather predictions, mine detection, and civil engineering [48]. In the context of a PDE system, solving the inverse problem usually means recovering coefficients, either a single or multiple coefficients, of the system over some measurement taken in a region either within the domain or on the boundary (observed part). Specific applications for inverse problems of hyperbolic systems of PDEs include electromagnetic, acoustics, and elastic waves [30], while applications to the inverse problem of plate equation include the elastic bending of thin plates, determination of magnitude of contact force produced during impact of objects, and determining the time-history of wind pressure on exposed surfaces [20].

Both inverse problems in this dissertation were inspired by the multidimensional inverse problem for second-order hyperbolic equations where one measurement is taken on the boundary. This problem was pioneered by Bukhgeim and Kilbanov in [8], which was one of the first papers to use Carleman estimates to solve the inverse problem. This was further expanded on by Klibanov in [21]. The development in the field has improved the process for determining the uniqueness of coefficients, establishing a procedure when working with second-order hyperbolic equations or parabolic equations [30].

The typical method of solving the inverse problem involves the use of appropriate Carleman estimates for the underlying system. Most papers determining uniqueness and stability of coefficients of PDE systems typically use one of the two primary techniques. Imanuvilov and Yamamoto in [16] used Carleman estimates directly to show the stability of recovering the coefficients of the wave equation. They focused on stability since it implies uniqueness of recovering the coefficient. The downside of this process is that there must be increased restrictions on the unknown coefficient of the system. The unknown coefficient is typically denoted as q, which represents either the damping or potential coefficient. In particular, along with the typical requirement that $q \in L^{\infty}(\Omega)$, their technique also requires q to be in an admissible set that imposed more regularity [30]. A second approach developed by Isakov, found in [19, Theorem 8.2.2], uses a post-Carleman technique to first demonstrate that q is uniquely recoverable, and afterwards uses the observability inequality for the system to demonstrate stability separately. More details about the inverse problems with a single measurement formulation can be found in [7, 18, 22, 24, 25, 43] and the numerous references within the works.

The main focus of this thesis is to solve the inverse problems of the second-order hyperbolic equation (1.1) and the plate equation (1.5). Our approach follows Isakov's approach of first proving that our unknown coefficients are uniquely recoverable, and then we demonstrate stability using the resulting observability inequalities. We organize the paper as follows: Chapter 2 discusses the Carleman estimates used in this paper. Chapter 3 solves the inverse problem for the second-order hyperbolic equation, while Chapter 4 solves an inverse problem for the plate equation. Appendix A of this dissertation includes a pointwise estimate for a Carleman estimate of the Schrödinger equation, which is used to derive the Carleman estimate for the plate equation, and Appendix B includes some definitions and properties of Riemannian geometry.

Chapter 2

Carleman Estimates for Riemannian Wave Equation and Plate Equations

In this chapter, we discuss the Carleman estimates used throughout this paper to solve the inverse problems for (1.2) and (1.5). We begin this chapter with the Carleman estimates for second-order hyperbolic equations. As mentioned in Remark 1.2.1, the Carleman estimates for second-order hyperbolic equations were first formulated in [37] under an Euclidean setting and later formulated under the Riemannian manifold setting in [46]. Because one of the coefficients we are interested in recovering is the wave speed c and it is on the principle part of the equation, we discuss the Carleman estimate for second-order hyperbolic equation in the setting of Riemannian geometry. It is straightforward to convert the Carleman estimate to a Euclidean setting if we were not interested in recovering wave speed. Definitions for the terms used can be found in Appendix B.

For the plate equation, we follow a similar strategy to [1] and [47] to obtain a Carleman estimate by using Carleman estimates of Schrödinger operators. As mentioned previously, since the coefficient we are interested in lies on the lower-order terms and not on the principle part of the plate equation, our estimate remains in terms of Euclidean geometry. We also discuss the continuous observability inequalities related to the Carleman estimates.

2.1 Carleman Estimates for Second-Order Hyperbolic Equation

Consider a Riemannian metric $g(\cdot,\cdot) = \langle \cdot,\cdot \rangle$ and squared norm $|X|^2 = g(X,X)$ on a smooth finite-dimensional manifold M. Define Ω as an open bounded, connected set of M with smooth boundary $\Gamma = \overline{\Gamma_0 \cup \Gamma_1}$, where $\Gamma_0 \cap \Gamma_1 = \emptyset$. Let ν be the unit outward normal field with boundary Γ. Denote Δ_g as the Laplace-Beltrami operator on manifold M and D as Levi-Civita connection on M.

Under these setting, we consider the following second-order hyperbolic equation with energy level terms defined on $Q = \Omega \times [-T,T]$ for some $T > 0$:

$$w_{tt}(x,t) - \Delta_g w(x,t) + F(w) = G(x,t), \ (x,t) \in Q, \tag{2.1}$$

where the forcing term $G \in L^2(Q)$ and the energy level differential term $F(w)$ is given by

$$F(w) = \langle \mathbf{P}(x,t), Dw \rangle + P_1(x,t) w_t + P_0(x,t) w, \tag{2.2}$$

where P_0 and P_1 are functions on Q and $\mathbf{P}(x,t)$ is a vector field on M for $t \in [-T,T]$. Further, we assume that there exists a constant $C > 0$ such that

$$|F(w)| \leq C[w^2 + w_t^2 + |Dw|^2], \ \forall (x,t) \in Q.$$

Now consider solutions $w(x,t)$ in the class

$$\begin{cases} w \in H^{1,1}(Q) = L^2(-T,T;H^1(\Omega)) \cap H^1(-T,T;L^2(\Omega)); \\ w_t, \frac{\partial w}{\partial \nu} \in L^2(-T,T;L^2(\Gamma)). \end{cases} \tag{2.3}$$

We now state the corresponding Carleman estimate for (2.1). This Carleman estimate was proven by Triggiani and Yao in [46].

Theorem 2.1.1. *Consider* (2.1) *and solutions in the class listed in* (2.3). *Then with the geometric assumptions (A.1) and (A.2), on* Ω, *the following family of estimates hold true, where* $\beta > 0$ *being* *a suitable constant, for all $\tau > 0$ sufficiently large and $\epsilon > 0$ small:*

$$BT(w) + 2\int_Q e^{2\tau\varphi}|G|^2\,dQ + Ce^{2\tau\sigma}\int_Q w^2\,dQ + C\tau^3 e^{-2\tau\delta}[E_w(-T) + E_w(T)]$$
$$\geq C_{1,\tau}\int_Q e^{2\tau\varphi}[w_t^2 + |Dw|^2]\,dQ + C_{2,\tau}\int_{Q(\sigma)} e^{2\tau\varphi}w^2\,dxdt \quad (2.4)$$

where

$$\begin{aligned} C_{1,\tau} &= \tau\epsilon(1-\alpha) - 2C, \\ C_{2,\tau} &= 2\tau^3\beta + \mathcal{O}(\tau^2) - 2C. \end{aligned} \quad (2.5)$$

In (2.4), $\delta,\ \sigma > 0$ are the constants from Section 1.3, φ is the weight function defined in (1.8), and C denotes a generic positive constant that may depend on T and d but independent of τ. The energy function $E_w(t)$ is defined as

$$E_w(t) = \int_\Omega \left[w^2(x,t) + w_t^2(x,t) + |Dw(x,t)|^2\right]\,d\Omega. \quad (2.6)$$

The boundary terms, denoted as $BT(w)$, can be explicitly calculated as

$$\begin{aligned} BT(w) &= 2\tau\int_\Sigma e^{2\tau\varphi}\left(w_t^2 - |Dw|^2\right)\langle Dd,\nu\rangle\,d\Sigma \\ &+ 4\tau\int_\Sigma e^{2\tau\varphi}\langle Dd,Dw\rangle\langle Dw,\nu\rangle\,d\Sigma + 8\alpha\tau\int_\Sigma e^{2\tau\varphi}tw_t\langle Dw,\nu\rangle\,d\Sigma \\ &+ 4\tau^2\int_\Sigma e^{2\tau\varphi}\left[|Dd|^2 - 4\alpha^2t^2 + \frac{\Delta d - \alpha - 1}{2\tau}\right]w\langle Dw,\nu\rangle\,d\Sigma \\ &+ 2\tau\int_\Sigma e^{2\tau\varphi}\left[2\tau^2\left(|Dd|^2 - 4\alpha^2t^2\right) + \tau(3\alpha+1)\right]w^2\langle Dd,\nu\rangle\,d\Sigma. \end{aligned}$$

Remark 2.1.1. If we have $w|_{\Gamma\times[-T,T]} = 0$ and $\dfrac{\partial w}{\partial\nu} = \langle Dw,\nu\rangle = 0$ on $\Gamma_1\times[-T,T]$, then the boundary terms may be simplified to

$$BT(w) = 2\tau\int_{\Sigma_0} e^{2\tau\varphi}|Dw|^2\langle Dd,\nu\rangle\,dxdt \leq 0, \quad (2.7)$$

where $\Sigma_0 = \Gamma_0\times[-T,T]$. The last inequality comes from assumption (A.1).

As a corollary of the Carleman estimate, we have the following continuous observability inequality for (2.1)

$$C_T E_w(0) \leq \int_{\Sigma_1} \left(\frac{\partial w}{\partial \nu}\right)^2 d\Gamma dt + \|G\|^2_{L^2(Q)} \tag{2.8}$$

with homogeneous Dirichlet boundary condition $w|_\Sigma = 0$ and $\Sigma_1 = \Gamma_1 \times [-T, T]$. The way to interpret (2.8) is that if the hyperbolic equation (2.1) has homogeneous Dirichlet boundary conditions, nonhomogeneous forcing term $G \in L^2(Q)$, and Neumann boundary trace $\frac{\partial w}{\partial \nu} \in L^2(\Sigma_1)$, then the initial conditions $\{w(\cdot, 0), w_t(\cdot, 0)\}$ must lie in the space $H^1_0(\Omega) \times L^2(\Omega)$. This fact is used in Chapter 3 when proving stability for the system.

For constant coefficient wave equations, inequality (2.8) can be proved using the multiplier method, first showed by Ho in [14]. As mentioned in Section 1.4, observability for variable coefficients in hyperbolic equations with lower-order terms is typically done with Carleman estimates pioneered in [23]. More work on the continuous observability inequality for hyperbolic equations can be found in [22, Theorem 2.7.1], [24, Theorem 2.4.1], [37, 46], and the sources cited in Section 1.4.

To remain compatible with our Dirichlet boundary conditions, the following interior and boundary regularity results for the solution w in (2.1) must hold true: For $\gamma \geq 0$ (not necessarily an integer), if the given data satisfy the following regularity assumptions

$$\begin{cases} G \in L^1(0, T; H^\gamma(\Omega)),\ \partial_t^{(\gamma)} G \in L^1(0, T; L^2(\Omega)), \\ w_0 \in H^{\gamma+1}(\Omega),\ w_1 \in H^\gamma(\Omega),\ h \in H^{\gamma+1}(\Sigma) \end{cases} \tag{2.9}$$

with all compatibility conditions (trace coincidence). Then we have the following regularity for the solution w:

$$w \in C([0, T]; H^{\gamma+1}(\Omega)),\ \partial_t^{(\gamma+1)} w \in C([0, T]; L^2(\Omega));\ \frac{\partial w}{\partial \nu} \in H^\gamma(\Sigma). \tag{2.10}$$

See [31] for further details.

2.2 Carleman Estimates for Plate Equation

Consider the following version of the plate equation:

$$\begin{cases} w_{tt} + \Delta^2 w = q(x)w + f(x,t) & \text{in } Q \\ w(x,0) = w_0(x),\ w_t(x,0) = w_1(x) & \text{in } \Omega \\ w = h_1(x,t),\ \Delta w = h_2(x,t) & \text{in } \Sigma \end{cases} \tag{2.11}$$

with $f \in L^2(Q)$ as the forcing term. The goal is to construct a Carleman estimate for (2.11) to solve the inverse problem of recovering the coefficient $q \in L^\infty(\Omega)$. Lasiecka and Triggiani derived Carleman estimates for the plate equation in [13, 35] via the multiplier method; more specifically, they multiplied (2.11) with the following operators:

$$e^{\tau\varphi(x,t)} Dd(\Delta_g w),\ \operatorname{div}(e^{\tau\varphi} Dd)\, w. \tag{2.12}$$

However, these Carleman estimates have two main issues when solving the inverse problem that the Carleman estimate (2.4) does not have. The first is that there exist some constants on the larger side of the inequality that depend on the parameter τ. The second issue is that the parameter τ appears in the denominator of the smaller side of the inequality, which causes issues in solving the inverse problem for sufficiently large τ. It is worth noting that the Carleman estimates presented in these works are sufficient enough to prove observability and controllability of the plate equation, which were the authors' goals in the papers cited.

One paper that inspired our approach of obtaining the Carleman estimate for the plate equation is [1]. In this paper, the author obtains a Carleman estimate for (2.11) by first decomposing it as the product of two Schrödinger operators. They then proved Carleman estimates for each Schrödinger operator from the decomposition and combined the two to obtain a Carleman estimate for the plate equation. Their weight function is different from the one in (1.8), which is defined as

$$\varphi(x,t) = g(t)\left(e^{\lambda\psi(x)} - 2e^{\lambda\Phi}\right),\ \Phi = \|\psi\|_{L^\infty(\Omega)} \tag{2.13}$$

with

$$g(t) = \frac{1}{t(T-t)},\ t \in [0,T]. \tag{2.14}$$

and $\psi(x)$ being a function such that $\nabla\psi(x) \neq 0$ for all $x \in \Omega$. The author chose this weight function because they were interested in proving observability of the heat equation coupled with the plate equation, so the weight function (2.13) was designed so that it approaches $-\infty$ at time 0.

Another paper that inspired our approach of obtaining a Carleman estimate for (2.11) is [47], which solved the inverse source problem for the following plate equation:

$$\begin{cases} y_{tt}(x,t) + \Delta^2 y(x,t) = a_0(x,t)y(x,t) + \mu(t)f(x), & (x,t) \in \Omega \times (0,T) \\ y(x,0) = y_t(x,0) = 0, & x \in \Omega \\ y(x,t) = \Delta y(x,t) = 0, & (x,t) \in \Sigma = \partial\Omega \times (0,T) \end{cases} \tag{2.15}$$

where

$$\mu \in C^3[0,T];\ \ \min_{t\in[0,T]} |\mu(t)| > 0;\ a_0 \in W^{1,\infty}\left(0,T; L^\infty(\Omega)\right) \tag{2.16}$$

In this paper, the author also used a Carleman estimate for the Schrödinger equation to obtain an observability inequality for the following plate equation with memory:

$$\begin{cases} \phi_{tt} + \Delta^2\phi = b_0(x,t)\phi + b_1(x,t)\phi_t + b_2(x,t)\displaystyle\int_0^s \phi(x,s)\,ds & \text{in } Q \\ \phi(x,0) = 0,\ \phi_t(x,0) = \phi_1(x) & \text{in } \Omega \\ \phi = \Delta\phi = 0 & \text{on } \Sigma \end{cases} \tag{2.17}$$

where $b_i \in L^\infty\left(\Omega \times (0,T)\right)$ for $i = 0,1,2$. The author then used the observability inequality of (2.17) to uniquely determine the source term f in (2.15) using Neumann boundary trace data of y and Δy on a suitable subset of the boundary for an arbitrary small observation time using the transformation $\phi = \frac{\partial}{\partial t}\left(\frac{y}{\mu}\right)$. This work was extended in [2], where the authors proved identifiability for the source term $\sum_{j=1}^N \lambda_j(t)\delta_{\xi_j}$, where λ_j are unknown functions of time and $\xi_j \in \Omega$ are unknown for each $j = 1,\ldots,N$. Numerical methods for recovering $f(x,t)$ in (2.11) were discussed in [12].

Using a similar strategy as Albano and Wang, we derive a Carleman estimate for the plate equation (2.11) using two Schrödinger operators. We will assume the geometrical assumptions (A.1') and (A.2') and define the weight function as in (1.8). Define the energy term $\mathbb{E}_w(t)$ as

$$\mathbb{E}_w(t) = \int_\Omega \left[|\nabla w(t)|^2 + |w(t)|^2 \right] \, dt = \|w(t)\|^2_{H^1(\Omega)}. \tag{2.18}$$

Before we proceed to the Carleman estimate for the plate equation, we first list the corresponding Carleman estimates for the two Schrödinger operators used. The first estimate can be found in [13, 38], and the second estimate is obtained in Appendix A.

Theorem 2.2.1. *Let $T > 0$ be arbitrary and β be a constant defined in* (1.18). *Let $d(x) \in C^3(\overline{\Omega})$ be the nonnegative, real, strictly convex function satisfying assumptions (A.1') and (A.2'). Define $\varphi(x,t)$ by* (1.17), *and let w be a solution to*

$$iw + \Delta w = f, \tag{2.19}$$

where $w \in H^{2,2}(Q) \equiv L^2(-T,T;H^2(\Omega)) \cap H^2(-T,T;L^2(\Omega))$ so that

$$\left.\frac{\partial w}{\partial w}\right|_\Gamma \in L^2(-T,T;H^{\frac{1}{2}}(\Gamma)); \;\; w_t \in L^2(-T,T;H^1(\Omega)); \;\; w_t|_\Gamma \in L^2(-T,T;H^{\frac{1}{2}}(\Gamma)). \tag{2.20}$$

Then for all τ sufficiently large, we have the following estimate holds:

$$\begin{aligned} &BT_1(w) + 4\int_Q e^{2\tau\varphi}|f|^2 \, dQ \\ &\qquad \geq \tilde{C}_{1,\tau} \int_Q e^{2\tau\varphi}|\nabla w|^2 \, dQ + \tilde{C}_{2,\tau} \int_Q e^{2\tau\varphi}|w|^2 \, dQ - C_{d,T}\tau e^{-2\tau\delta} \left[\mathbb{E}(T) + \mathbb{E}(-T)\right] \\ &\qquad \geq m \int_Q \left[|\nabla w|^2 + |w|^2 \right] \, dQ - C\tau e^{-2\tau\delta} \left[\mathbb{E}(T) + \mathbb{E}(-T)\right] \end{aligned} \tag{2.21}$$

where

$$\tilde{C}_{1,\tau} = 2\tau\rho - \frac{1}{2} - 4C \tag{2.22}$$

$$\tilde{C}_{2,\tau} = 4\tau^3\rho p^2 + \mathcal{O}(\tau^2) - 4C \tag{2.23}$$

$$m = \min\{\tilde{C}_{1,\tau}, \tilde{C}_{2,\tau}\} \nearrow \infty \text{ as } \tau \to \infty, \tag{2.24}$$

where ρ, p, $\delta > 0$ are defined by (1.6), (1.7), (1.20), *and C is a generic positive constant depending on d and T but not on τ.*

Setting $h \equiv \nabla d$, the boundary term $BT_1(w)$ are given as follows, where $\xi = \text{Re }(w)$ and $\eta = \text{Im }(w)$:

$$\begin{aligned} BT_1(w) = 2\tau \int_\Sigma e^{2\tau\varphi}\left[2\tau^2|h|^2 + \Phi\right]|w|^2 h\cdot\nu\, d\Sigma - 2\beta\tau\int_\Sigma e^{2\tau\varphi} t\left[\eta\frac{\partial\xi}{\partial\nu} - \xi\frac{\partial\eta}{\partial\nu}\right] d\Sigma \\ - 2\tau\int_\Sigma e^{2\tau\varphi}[\xi_t\eta - \xi\eta_t] h\cdot\nu\, d\Sigma + \int_\Sigma e^{2\tau\varphi}[2\tau^2|h|^2 - \tau\Delta d]\left[\overline{w}\frac{\partial w}{\partial\nu} + w\frac{\partial\overline{w}}{\partial\nu}\right] d\Sigma \\ + 2\tau\int_\Sigma e^{2\tau\varphi} h\cdot\left[\nabla\overline{w}\frac{\partial w}{\partial\nu} + \nabla w\frac{\partial\overline{w}}{\partial\nu}\right] d\Sigma - 2\tau\int_\Sigma e^{2\tau\varphi}|\nabla w|^2 h\cdot\nu\, d\Sigma, \end{aligned} \tag{2.25}$$

where the function Φ may be taken to satisfy either $\Phi \equiv 0$ or else $\Phi \equiv \tau\Delta d$.

Theorem 2.2.2. *Let w be a solution to the following Schrödinger equation:*

$$iw_t - \Delta w = u. \tag{2.26}$$

Then under the same conditions as Theorem 2.2.1, for all sufficiently large τ, we have the following estimate holds:

$$\begin{aligned} &BT_1(w) + 4\int_Q e^{2\tau\varphi}|u|^2\, dQ \\ &\quad\geq \tilde{C}_{3,\tau}\int_Q e^{2\tau\varphi}|\nabla w|^2\, dQ + \tilde{C}_{4,\tau}\int_Q e^{2\tau\varphi}|w|^2\, dQ - C_{d,T}\tau e^{-2\tau\delta}\left[\mathbb{E}(T) + \mathbb{E}(-T)\right] \\ &\quad\geq \tilde{m}\int_Q \left[|\nabla w|^2 + |w|^2\right] dQ - C\tau e^{-2\tau\delta}\left[\mathbb{E}(T) + \mathbb{E}(-T)\right] \end{aligned} \tag{2.27}$$

where

$$\tilde{C}_{3,\tau} = \left(2\tau\rho - \frac{1}{2}\right)a - 4C \tag{2.28}$$

$$\tilde{C}_{4,\tau} = 4\tau^3\rho p^2(2 - k + 2c) + \mathcal{O}(\tau^2) - 2C \tag{2.29}$$

$$\tilde{m} = \min\{\tilde{C}_{3,\tau}, \tilde{C}_{4,\tau}\}. \tag{2.30}$$

The constant a is such that $1 - 4\Delta d > a$, and the boundary term $BT_1(w)$ is defined as in (2.25). This and the $2 - k - 2c$ term in (2.29) were originally specified in [37].

Using the Carleman estimates (2.21) and (2.27), we obtain the following Carleman estimate for the plate equation (2.11).

Theorem 2.2.3. *Assume (A.1') and (A.2') and* $f \in L^2(\Omega)$*. Let* w *be a solution to the plate equation* (2.11)*. Further assume that*

$$\begin{cases} \{w, w_t\} \in L^2\left((-T,T); H^3(\Omega) \times H^1(\Omega)\right); \\ \dfrac{\partial w}{\partial \nu},\ \dfrac{\partial \Delta w}{\partial \nu} \in L^2(\Sigma) \end{cases}$$

Then for all sufficiently large τ*, the following estimate holds:*

$$\begin{aligned} &BT_1^*(w) + C\int_Q e^{2\tau\varphi}|f|^2\,dQ \\ &\quad \geq \tau\int_Q e^{2\tau\varphi}\left[|\nabla w|^2 + |\nabla w_t|^2 + |\nabla \Delta w|^2\right]\,dQ + \tau^3\int_Q e^{2\tau\varphi}\left[|w|^2 + |w_t|^2 + |\Delta w|^2\right]\,dQ \\ &\quad - C_{d,T}\tau e^{-2\tau\delta}\left[\mathbb{E}_{-iw_t+\Delta w}(T) + \mathbb{E}_{-iw_t+\Delta w}(-T) + \mathbb{E}_w(T) + \mathbb{E}_w(-T)\right] \end{aligned} \tag{2.31}$$

where the boundary terms are defined as follows:

$$BT_1^*(w) = BT_1(-iw_t + \Delta w) + BT_1(w). \tag{2.32}$$

Note that $C_{1,\tau}$ and $C_{3,\tau}$ are similar to τ, and $C_{2,\tau}$ and $C_{4,\tau}$ are similar to τ^3. For simplifying our work, we replace these constants with τ and τ^3.

Proof. Define u as follows:

$$u = -iw_t + \Delta w. \tag{2.33}$$

Notice that $iu_t = w_{tt} + i\Delta w_t$ and $\Delta u = -i\Delta w_t + \Delta^2 w$, so we readily have $iu_t + \Delta u = w_{tt} + \Delta^2 w = qw + f$. Applying Theorem 2.2.1 on $iu_t + \Delta u$ yields the following:

$$\begin{aligned} &C\int_Q e^{2\tau\varphi}|w|^2\,dQ + C\int_Q e^{2\tau\varphi}|f|^2\,dQ + BT_1(u) \\ &\quad \geq \tau\int_Q e^{2\tau\varphi}|\nabla u|^2\,dQ + \tau^3\int_Q e^{2\tau\phi}|u|^2\,dQ - \tau C e^{-2\tau\delta}\left[\mathbb{E}_u(T) + \mathbb{E}_u(-T)\right] \end{aligned} \tag{2.34}$$

Similarly, we apply Theorem 2.2.2 to (2.33) and obtain the following:

$$\begin{aligned} &C\int_Q e^{2\tau\varphi}|u|^2\,dQ + BT_1(w) \\ &\geq \tau\int_Q e^{2\tau\varphi}|\nabla w|^2\,dQ + \tau^3\int_Q e^{2\tau\varphi}|w|^2\,dQ - \tau C_{d,T}e^{-2\tau\delta}\left[\mathbb{E}_w(T)+\mathbb{E}_w(-T)\right] \end{aligned} \tag{2.35}$$

We now add (2.34) and (2.35) together to obtain the following:

$$\begin{aligned} &C\int_Q e^{2\tau\varphi}|u|^2\,dQ + C\int_Q e^{2\tau\varphi}|w|^2\,dQ + C\int_Q e^{2\tau\varphi}|f|^2\,dQ + BT_1(w) + BT_1(u) \\ &\geq \tau\int_Q e^{2\tau\varphi}\left[|\nabla u|^2+|\nabla w|^2\right]\,dQ + \tau^3\int_Q e^{2\tau\varphi}\left[|u|^2+|w|^2\right]\,dQ \\ &- \tau C_{d,T}e^{-2\tau\delta}\left[\mathbb{E}_w(T)+\mathbb{E}_u(T)+\mathbb{E}_w(-T)+\mathbb{E}_u(-T)\right] \end{aligned} \tag{2.36}$$

Note that we can absorb the $C\int_Q e^{2\tau\varphi}|u|^2\,dQ$ and $C\int_Q e^{2\tau\varphi}|w|^2\,dQ$ terms on the LHS of (2.36) with the $\tau^3\int_Q e^{2\tau\varphi}|u|^2\,dQ$ and $\tau^3\int_Q e^{2\tau\varphi}|w|^2\,dQ$ terms on the RHS of (2.36), respectively. Doing this gives us the following:

$$\begin{aligned} &C\int_Q e^{2\tau\varphi}|f|^2\,dQ + BT_1(w) + BT_1(u) \\ &\geq \tau\int_Q e^{2\tau\varphi}\left[|\nabla u|^2+|\nabla w|^2\right]\,dQ + \tau^3\int_Q e^{2\tau\varphi}\left[|u|^2+|w|^2\right]\,dQ \\ &- \tau C_{d,T}e^{-2\tau\delta}\left[\mathbb{E}_w(T)\mathbb{E}_u(T)+\mathbb{E}_w(-T)+\mathbb{E}_u(-T)\right] \end{aligned} \tag{2.37}$$

Converting the $u-$ terms in (2.37) to terms of w, we have the following:

$$\begin{aligned} \int_Q e^{2\tau\varphi}|u|^2\,dQ &= \int_Q e^{2\tau\varphi}\left[|w_t|^2+|\Delta w|^2\right]\,dQ \\ \int_Q e^{2\tau\varphi}|\nabla u|^2\,dQ &= \int_Q e^{2\tau\varphi}\left[|\nabla w_t|^2+|\nabla\Delta w|^2\right]\,dQ \end{aligned} \tag{2.38}$$

Applying (2.38) to (2.37) yields the desired result. □

Remark 2.2.4. As with Remark 2.1.1, if we have $w|_{\Gamma\times[-T,T]} = \Delta w|_{\Gamma\times[-T,T]} = 0$ and $\left.\dfrac{\partial w}{\partial\nu}\right|_{\Gamma_1\times[-T,T]} =$

$\left.\dfrac{\partial \Delta w}{\partial \nu}\right|_{\Gamma_1\times[-T,T]} = 0$, the boundary terms simplify to (recall that $h \equiv \nabla d$)

$$BT_1^*(w) = 2\tau \int_{\Sigma_0} \left[\left|\frac{\partial w}{\partial \nu}\right|^2 + \left|\frac{\partial w_t}{\partial \nu}\right|^2 + \left|\frac{\partial \Delta w}{\partial \nu}\right|^2 \right] h \cdot \nu \, dxdt \leq 0, \tag{2.39}$$

where the last inequality comes from assumption (A.1')

From our Carleman estimate, we obtain the following continuous observability inequality.

Corollary 2.2.5. *Let w be a solution to* (2.11)*. Then under assumptions (A.1') and (A.2'), there exists a constant $C > 0$ such that*

$$CE'(0) \leq \int_{\Sigma_1} \left[\left(\frac{\partial \Delta w}{\partial \nu}\right)^2 + \left(\frac{\partial w_t}{\partial \nu}\right)^2 \right] d\Gamma dt + \|f\|^2_{L^2(Q)} \tag{2.40}$$

where

$$E'(t) = \int_\Omega \left[|\nabla \Delta w(t)|^2 + |\nabla w_t(t)|^2 \right] dx \tag{2.41}$$

with homogeneous Dirichlet boundary conditions $w|_\Sigma = \Delta w|_\Sigma = 0$.

Proof. Again, let $u = -iw_t + \Delta w$. Under the Dirichlet boundary conditions, the boundary terms $BT_1(u)$ and $BT_1(w)$ simplifies to the following after using assumption (A.1') and [38, Theorem 8.2]:

$$\begin{aligned} BT_1(u) &= 2\tau \int_\Sigma e^{2\tau\varphi} \left|\frac{\partial u}{\partial \nu}\right|^2 h \cdot \nu \, d\Sigma \leq 2\tau \int_{\Sigma_1} e^{2\tau\varphi} \left[\left|\frac{\partial \Delta w}{\partial \nu}\right|^2 + \left|\frac{\partial w_t}{\partial \nu}\right|^2 \right] d\Sigma_1 \\ BT_1(w) &= 2\tau \int_\Sigma e^{2\tau\varphi} \left|\frac{\partial w}{\partial \nu}\right|^2 h \cdot \nu \, d\Sigma \leq 2\tau \int_{\Sigma_1} e^{2\tau\varphi} \left|\frac{\partial w}{\partial \nu}\right|^2 d\Sigma_1. \end{aligned} \tag{2.42}$$

Via the properties of the pseudo-convex function φ, assumptions (A.1') and (A.2'), and since $w(x,0) = 0$ in Ω, (2.42) can be rewritten as follows:

$$BT_1^*(w) \leq \tau e^{C\tau} \int_{\Sigma_1} \left[\left|\frac{\partial \Delta w}{\partial \nu}\right|^2 + \left|\frac{\partial w_t}{\partial \nu}\right|^2 \right] d\Sigma_1, \tag{2.43}$$

where C is a constant that depends on d. Apply (2.43) to (2.31) then, after dropping unnecessary terms (assuming $f \equiv 0$), gives us the following:

$$\tau \int_Q e^{2\tau\varphi} \left[|\nabla w|^2 + |\nabla \Delta w|^2 \right] dQ \leq \tau e^{C\tau} \int_{\Sigma_1} \left[\left|\frac{\partial \Delta w}{\partial \nu}\right|^2 + \left|\frac{\partial w_t}{\partial \nu}\right|^2 \right] d\Sigma_1. \tag{2.44}$$

Now we bound the LHS of (2.44). Notice that $\varphi(x,0) \geq R := \min_{x\in\overline{\Omega}} d(x)$ for all $x \in \Omega$. Thus, there exists $\epsilon \in (0,1)$ such that $\varphi(x,t) \geq \frac{R}{2}$. Define $\tilde{T} = \epsilon T$ and $Q_0 = \Omega \times [-\tilde{T}, \tilde{T}]$. Since $Q_0 \subset Q$, we have

$$\begin{aligned}\tau \int_Q e^{2\tau\varphi} \left[|\nabla w|^2 + |\nabla \Delta w|^2\right] dQ &\geq \tau \int_{Q_0} e^{2\tau\varphi} \left[|\nabla w|^2 + |\nabla \Delta w|^2\right] dQ_0 \\ &\geq \tau e^{R\tau} \int_{Q_0} [|\nabla w|^2 + |\nabla \Delta w|^2]\, dQ_0.\end{aligned}$$

Apply this to (2.44) and using Lemma 3.2 in [47], we have the following:

$$C\tau e^{R\tau} E'(0) \leq C\tau e^{R\tau} E'(t) \leq \tau e^{C\tau} \int_{\Sigma_1} \left[\left|\frac{\partial \Delta w}{\partial \nu}\right|^2 + \left|\frac{\partial w_t}{\partial \nu}\right|^2\right] d\Sigma_1. \tag{2.45}$$

Taking τ large enough yields the desired result. $\square$

As with the other observability inequality (2.8), we can interpret (2.40) as follows: if the plate equation has homogeneous Dirichlet boundary conditions, nonhomogeneous forcing term $f \in L^2(Q)$, and Neumann boundary traces $\dfrac{\partial w}{\partial \nu}, \dfrac{\partial \Delta w}{\partial \nu} \in L^2(\Sigma_1)$, then the initial conditions $\{w(\cdot,0), w_t(\cdot,0)\}$ must lie in the space $H^3(\Omega) \times H^1(\Omega)$. We will use this in Chapter 4 when proving stability.

The observability inequality (2.40) can be proven using the multiplier method, as shown by Lasiecka and Triggiani in [35] and Lions in [40], and by using the Carleman estimate for Schrödinger equation as done by Lasiecka, Triggiani, and Zhang in [38] and Wang in [47]. Our observability inequality remains consistent with those derived in the literature. More work on the continuous observability inequality can be found in these works and the references cited within.

To remain compatible with the boundary conditions, we assume the following interior and boundary regularity results for solution w of (2.11): If the given data satisfies the following regularity assumptions:

$$\begin{cases} f \in L^1(0,T; H_0^1(\Omega)),\ \partial_t f \in L^1(0,T; L^2(\Omega)), \\ \\ \{w_0, w_1\} \in X,\ \{h_1, h_2\} \in L^2(\Sigma) \times H^2(\Sigma), \end{cases} \tag{2.46}$$

where

$$X = H^{-1}(\Omega) \times V \tag{2.47}$$

$$V = \left\{v \in H^3(\Omega) : v|_\Gamma = \Delta v|_\Gamma = 0\right\}, \tag{2.48}$$

then we have the following:

$$\{w, w_t\} \in C\left([0,T]; X\right), \ \left\{\frac{\partial w}{\partial \nu}, \ \frac{\partial \Delta w}{\partial \nu}\right\} \in L^2(\Sigma) \tag{2.49}$$

See [32, 40, 47] for details.

Chapter 3

Inverse Problem for Second-Order Hyperbolic Equations

In this chapter, we provide the main theorems and proofs for the uniqueness and stability of the inverse problem of recovering the coefficients of the second-order hyperbolic equation (1.2), which is restated below:

$$\begin{cases} w_{tt} - c^2(x)\Delta w + q_1(x)w_t + q_0(x)w + \mathbf{q}(x)\cdot\nabla w = 0 & \text{in } Q \\ w\,(x,0) = w_0(x);\ \ w_t\,(x,0) = w_1(x) & \text{in } \Omega \\ w(x,t) = h(x,t) & \text{in } \Sigma. \end{cases} \tag{3.1}$$

Recall that we assume $q_1, q_0 \in L^\infty(\Omega)$, $\mathbf{q} \in (L^\infty(\Omega))^n$, and the wave speed $c(x)$ satisfies

$$c \in \mathcal{C} = \{c \in C^1(\Omega) : c_0^{-1} \leq c(x) \leq c_0,\ \text{for some } c_0 > 0\}.$$

For the inverse problem of (3.1), we are recovering $n+3$ unknown functions: n from the gradient coefficient $\mathbf{q}$ and 3 from recovering the damping, potential, and wave speed coefficients q_1, q_0, and c, respectively. Since we have $n+3$ unknown coefficients, one would expect to solve the inverse problem by making $n+3$ measurements on the boundary. In the following theorems, we show that it is possible to solve the inverse problem using only *half* of the measurements; more explicitly, by appropriately choosing $\lfloor \frac{n+4}{2} \rfloor$ pairs of initial conditions $\{w_0, w_1\}$ and a boundary condition h, we can uniquely and stably recover all unknown coefficients at once from the corresponding Neumann boundary measurements of their solutions.

Previous works that addressed similar results were [6] and [7, Chapter 9]. In these works, the authors considered recovering the coefficient matrix of the principle part in an anisotropic hyperbolic equation from a finite set of measurements on a portion of the boundary. What differs from the results below is that the number of unknown functions is $\frac{n(n+1)}{2}$ and the number of measurements is $\frac{n(n+3)}{2}$.

3.1 Statement of Theorems

We state the uniqueness and stability theorems associated with solving the inverse problem of (3.1). The proofs of the results will be given in the subsequent sections.

Theorem 3.1.1. *Under the geometrical assumptions (A.1) and (A.2), let*

$$T > T_0 = 2\sqrt{\max_{x\in\overline{\Omega}} d(x)}. \tag{3.2}$$

Suppose the initial and boundary conditions are in the following function spaces

$$\{w_0, w_1\} \in H^{\gamma+1}(\Omega) \times H^{\gamma}(\Omega),\ \ h \in H^{\gamma+1}(\Sigma),\ \ \textit{where } \gamma > \frac{n}{2} + 4 \tag{3.3}$$

along with all compatibility conditions (trace coincidence) which make sense. Let $w^{(i)}(c, q_1, q_0, \mathbf{q})$ and $w^{(i)}(\tilde{c}, p_1, p_0, \mathbf{p})$ be the corresponding solutions of equation (3.1) with different coefficients $\{c, q_1, q_0, \mathbf{q}\}$ and $\{\tilde{c}, p_1, p_0, \mathbf{p}\}$, as well as the initial and boundary conditions $\{w_0^{(i)}, w_1^{(i)}, h\}$, $i = 1, \cdots, m+2$. In addition, depending on the dimension n of the space, we assume the following positivity condition: There exists $r_0 > 0$ such that

Case I: If n is odd, i.e., $n = 2m+1$ for some $m \in \mathbb{N}$, then we choose $m+2$ pairs of initial conditions $\{w_0^{(i)}, w_1^{(i)}\}$, $i = 1, \ldots, m+2$, and a boundary condition h so satisfying (3.3) and

$$|\det W(x)| \geq r_0,\ \ a.e.\ x \in \Omega \tag{3.4}$$

where $W(x)$ is the $(n+3)\times(n+3)$ matrix defined by

$$W(x) = \begin{bmatrix} w_0^{(1)}(x) & w_1^{(1)}(x) & \partial_{x_1} w_0^{(1)}(x) & \cdots & \partial_{x_n} w_0^{(1)}(x) & \Delta w_0^{(1)}(x) \\ w_1^{(1)}(x) & w_{tt}^{(1)}(x,0) & \partial_{x_1} w_1^{(1)}(x) & \cdots & \partial_{x_n} w_1^{(1)}(x) & \Delta w_1^{(1)}(x) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ w_0^{(m+2)}(x) & w_1^{(m+2)}(x) & \partial_{x_1} w_0^{(m+2)}(x) & \cdots & \partial_{x_n} w_0^{(m+2)}(x) & \Delta w_0^{(m+2)}(x) \\ w_1^{(m+2)}(x) & w_{tt}^{(m+2)}(x,0) & \partial_{x_1} w_1^{(m+2)}(x) & \cdots & \partial_{x_n} w_1^{(m+2)}(x) & \Delta w_1^{(m+2)}(x) \end{bmatrix} \tag{3.5}$$

Case II: If n is even, i.e., $n = 2m$ for some $m \in \mathbb{N}$, then we choose $m+2$ pairs of initial conditions $\{w_0^{(i)}, w_1^{(i)}\}$, $i = 1, \ldots, m+2$, and a boundary condition h so that they satisfy (3.3) and

$$|\det \widetilde{W}(x)| \geq r_0, \;\; a.e. \; x \in \Omega \tag{3.6}$$

where $\widetilde{W}(x)$ is the $(n+3)\times(n+3)$ matrix defined by

$$\widetilde{W}(x) = \begin{bmatrix} w_0^{(1)}(x) & w_1^{(1)}(x) & \partial_{x_1} w_0^{(1)}(x) & \cdots & \partial_{x_n} w_0^{(1)}(x) & \Delta w_0^{(1)}(x) \\ w_1^{(1)}(x) & w_{tt}^{(1)}(x,0) & \partial_{x_1} w_1^{(1)}(x) & \cdots & \partial_{x_n} w_1^{(1)}(x) & \Delta w_1^{(1)}(x) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ w_0^{(m+1)}(x) & w_1^{(m+1)}(x) & \partial_{x_1} w_0^{(m+1)}(x) & \cdots & \partial_{x_n} w_0^{(m+1)}(x) & \Delta w_0^{(m+1)}(x) \\ w_1^{(m+1)}(x) & w_{tt}^{(m+1)}(x,0) & \partial_{x_1} w_1^{(m+1)}(x) & \cdots & \partial_{x_n} w_1^{(m+1)}(x) & \Delta w_1^{(m+1)}(x) \\ w_0^{(m+2)}(x) & w_1^{(m+2)}(x) & \partial_{x_1} w_0^{(m+2)}(x) & \cdots & \partial_{x_n} w_0^{(m+2)}(x) & \Delta w_0^{(m+2)}(x) \end{bmatrix} \tag{3.7}$$

If we have the same Neumann boundary traces over the observed part Γ_1 of the boundary and over the time interval $[-T, T]$, i.e., for $i = 1, \cdots, m+2$,

$$\frac{\partial w^{(i)}(c, q_1, q_0, \mathbf{q})}{\partial \nu}(x,t) = \frac{\partial w^{(i)}(\tilde{c}, p_1, p_0, \mathbf{p})}{\partial \nu}(x,t), \;\; (x,t) \in \Gamma_1 \times [-T, T], \tag{3.8}$$

then we must have that all the coefficients coincide,

$$c(x) = \tilde{c}(x), \;\; q_1(x) = p_1(x), \;\; q_0(x) = p_0(x), \;\; \mathbf{q}(x) = \mathbf{p}(x) \;\;\; a.e. \; x \in \Omega. \tag{3.9}$$

Remark 3.1.1. The matrices $W(x)$ and $\widetilde{W}(x)$ defined in (3.5) and (3.7) both contain the $w_{tt}^{(i)}(x,0)$ terms, $i=1,\ldots,m+2$, in the second column, which contains unknown coefficients due to the initial conditions of (3.1). However, if we choose the first pair of initial conditions $\{w_0^{(1)}, w_1^{(1)}\} = \{0,1\}$ while having the other $m+1$ pairs of initial conditions satisfy that the $(n+1)\times(n+1)$ sub-matrices on the lower right corner of $W(x)$ and $\widetilde{W}(x)$ has determinants that are bounded away from 0, then we prevent this issue from affecting the positivitity assumptions.

After proving the above uniqueness theorem, we may also get the following Lipschitz stability result for recovering all coefficients $\{c, q_1, q_0, \mathbf{q}\}$ from the corresponding finite sets of boundary measurements. The full theorem is stated below.

Theorem 3.1.2. *Under the assumptions in Theorem 3.1.1, again let $w^{(i)}(c, q_1, q_0, \mathbf{q})$ and $w^{(i)}(\tilde{c}, p_1, p_0, \mathbf{p})$ denote the corresponding solutions of equation (3.1) with coefficients $\{c, q_1, q_0, \mathbf{q}\}$ and $\{\tilde{c}, p_1, p_0, \mathbf{p}\}$, as well as the initial and boundary conditions $\{w_0^{(i)}, w_1^{(i)}, h\}$, $i=1,\cdots,m+2$ (either n is odd or even). Then there exists $C>0$ depends on Ω, T, Γ_1, c, q_1, q_0, $\mathbf{q}$, $w_0^{(i)}$, $w_1^{(i)}$, h such that*

$$\begin{aligned} & \|c^2-\tilde{c}^2\|_{L^2(\Omega)}^2 + \|q_1-p_1\|_{L^2(\Omega)}^2 + \|q_0-p_0\|_{L^2(\Omega)}^2 + \|\mathbf{q}-\mathbf{p}\|_{\mathbf{L}^2(\Omega)}^2 \\ \leq \quad & C\sum_{i=1}^{m+2} \left\| \frac{\partial w_{tt}^{(i)}(c,q_1,q_0,\mathbf{q})}{\partial\nu} - \frac{\partial w_{tt}^{(i)}(\tilde{c},p_1,p_0,\mathbf{p})}{\partial\nu} \right\|_{L^2(\Sigma_1)}^2, \end{aligned} \tag{3.10}$$

for all such coefficients $c, \tilde{c}, q_1, q_0, p_1, p_0 \in H_0^1(\Omega)$, $\mathbf{q}, \mathbf{p} \in \left(H_0^1(\Omega)\right)^n$, where $\|\cdot\|_{\mathbf{L}^2(\Omega)}$ is defined as

$$\|\mathbf{r}\|_{\mathbf{L}^2(\Omega)} = \left(\int_\Omega \sum_{i=1}^n |r_i(x)|^2\,dx\right)^{\frac{1}{2}}, \ \textit{for} \ \ \mathbf{r}(x) = (r_1(x),\cdots,r_n(x)).$$

The first step to solve the inverse problem is to first convert this into an inverse source problem. To accomplish this, let

$$\begin{aligned} & f_2(x) = c^2(x) - \tilde{c}^2(x), \quad f_1(x) = p_1(x) - q_1(x), \\ & f_0(x) = p_0(x) - q_0(x), \quad \mathbf{f}(x) = \mathbf{p}(x) - \mathbf{q}(x); \\ & u(x,t) = w(c,q_1,q_0,\mathbf{q}) - w(\tilde{c},p_1,p_0,\mathbf{p}), \ R(x,t) = w(\tilde{c},p_1,p_0,\mathbf{p})(x,t), \end{aligned} \tag{3.11}$$

then $u = u(x,t)$ is readily seen to satisfy the following homogeneous mixed problem

$$\begin{cases} u_{tt} - c^2(x)\Delta u + q_1(x)u_t + q_0(x)u + \mathbf{q}(x)\cdot\nabla u = S(x,t) & \text{in } Q \\ u\,(x,0) = u_t\,(x,0) = 0 & \text{in } \Omega \\ u(x,t) = 0 & \text{in } \Sigma, \end{cases} \tag{3.12}$$

where

$$S(x,t) = f_0(x)R(x,t) + f_1(x)R_t(x,t) + \mathbf{f}(x)\cdot\nabla R(x,t) + f_2(x)\Delta R(x,t). \tag{3.13}$$

In this setting, we assume that $c \in \mathcal{C}$, $q_0,\ q_1 \in L^\infty(\Omega)$ and $\mathbf{q} \in (L^\infty(\Omega))^n$ are given and fixed, and we assume that $R = R(x,t)$ is a given function that can be suitably chosen. On the other hand, the source coefficients $f_0, f_1, f_2 \in L^2(\Omega)$ and $\mathbf{f} \in \left(L^2(\Omega)\right)^n$ are assumed to be unknown. This transforms the inverse problem of (3.1) to the following inverse source problem for (3.12): determine f_0, f_1, f_2 and $\mathbf{f}$ from the Neumann boundary measurements of u over the observed part Γ_1 of the boundary and over a sufficiently long time interval $[-T,T]$.

Because our new inverse problem is now an inverse source problem, corresponding to Theorems 3.1.1 and 3.1.2, we prove the following uniqueness and stability theorems:

Theorem 3.1.3. *Under geometrical assumptions (A.1) and (A.2) and let T satisfy (3.2). Depending on the dimension n, we assume the following regularity and positivity conditions:*

Case I: If n is odd, i.e., $n = 2m+1$ for some $m \in \mathbb{N}$, then we choose $m+2$ functions $R^{(1)}$, $\cdots$, $R^{(m+2)}$ such that they satisfy

$$R^{(i)}, R_t^{(i)}, R_{tt}^{(i)}, R_{ttt}^{(i)} \in W^{2,\infty}(Q),\ \ i = 1,\cdots,m+2 \tag{3.14}$$

and there exists $r_0 > 0$ such that

$$|\det U(x)| \geq r_0,\ \ a.e.\ x \in \Omega \tag{3.15}$$

where $U(x)$ is the $(n+3)\times(n+3)$ matrix defined by

$$U(x) = \begin{bmatrix} R^{(1)}(x,0) & R_t^{(1)}(x,0) & \partial_{x_1}R^{(1)}(x,0) & \cdots & \partial_{x_n}R^{(1)}(x,0) & \Delta R^{(1)}(x,0) \\ R_t^{(1)}(x,0) & R_{tt}^{(1)}(x,0) & \partial_{x_1}R_t^{(1)}(x,0) & \cdots & \partial_{x_n}R_t^{(1)}(x,0) & \Delta R_t^{(1)}(x,0) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ R^{(m+2)}(x,0) & R_t^{(m+2)}(x,0) & \partial_{x_1}R^{(m+2)}(x,0) & \cdots & \partial_{x_n}R^{(m+2)}(x,0) & \Delta R^{(m+2)}(x,0) \\ R_t^{(m+2)}(x,0) & R_{tt}^{(m+2)}(x,0) & \partial_{x_1}R_t^{(m+2)}(x,0) & \cdots & \partial_{x_n}R_t^{(m+2)}(x,0) & \Delta R_t^{(m+2)}(x,0) \end{bmatrix} \tag{3.16}$$

Case II: If n is even, i.e., $n = 2m$ for some $m \in \mathbb{N}$, then we choose $m+2$ functions $R^{(1)}$, $\cdots$, $R^{(m+2)}$ such that they satisfy (3.14) and there exists $r_0 > 0$ such that

$$|\det \widetilde{U}(x)| \geq r_0, \ \ a.e.\ x \in \Omega \tag{3.17}$$

where $\widetilde{U}(x)$ is the $(n+3)\times(n+3)$ matrix defined by

$$\widetilde{U}(x) = \begin{bmatrix} R^{(1)}(x,0) & R_t^{(1)}(x,0) & \partial_{x_1}R^{(1)}(x,0) & \cdots & \partial_{x_n}R^{(1)}(x,0) & \Delta R^{(1)}(x,0) \\ R_t^{(1)}(x,0) & R_{tt}^{(1)}(x,0) & \partial_{x_1}R_t^{(1)}(x,0) & \cdots & \partial_{x_n}R_t^{(1)}(x,0) & \Delta R_t^{(1)}(x,0) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ R^{(m+1)}(x,0) & R_t^{(m+1)}(x,0) & \partial_{x_1}R^{(m+1)}(x,0) & \cdots & \partial_{x_n}R^{(m+1)}(x,0) & \Delta R^{(m+1)}(x,0) \\ R_t^{(m+1)}(x,0) & R_{tt}^{(m+1)}(x,0) & \partial_{x_1}R_t^{(m+1)}(x,0) & \cdots & \partial_{x_n}R_t^{(m+1)}(x,0) & \Delta R_t^{(m+1)}(x,0) \\ R^{(m+2)}(x,0) & R_t^{(m+2)}(x,0) & \partial_{x_1}R^{(m+2)}(x,0) & \cdots & \partial_{x_n}R^{(m+2)}(x,0) & \Delta R^{(m+2)}(x,0) \end{bmatrix} \tag{3.18}$$

Let $u^{(i)}(f_0, f_1, f_2, \mathbf{f})$ be the solutions of equation (3.12) with the functions $R^{(i)}$, $i = 1, \cdots, m+2$. If

$$\frac{\partial u^{(i)}(f_0, f_1, f_2, \mathbf{f})}{\partial \nu}(x,t) = 0, \ \ (x,t) \in \Gamma_1 \times [-T, T], \ \ i = 1, \cdots, m+2, \tag{3.19}$$

then we must have

$$f_0(x) = f_1(x) = f_2(x) = \mathbf{f}(x) = 0, \qquad a.e.\ x \in \Omega. \tag{3.20}$$

Theorem 3.1.4. *Under the assumptions in Theorem 3.1.3, again let $u^{(i)}(f_0, f_1, f_2, \mathbf{f})$ denote the solutions of equation (3.12) with the functions $R^{(i)}$, $i = 1, \cdots, m+2$ (either n is odd or even). Then there exists $C > 0$ depends on Ω, T, Γ_1, c, q_1, q_0, $\mathbf{q}$, $w_0^{(i)}$, $w_1^{(i)}$, h such that*

$$\|f_0\|^2_{L^2(\Omega)} + \|f_1\|^2_{L^2(\Omega)} + \|f_2\|^2_{L^2(\Omega)} + \|\mathbf{f}\|^2_{\mathbf{L}^2(\Omega)} \leq C \sum_{i=1}^{m+2} \left\| \frac{\partial u_{tt}^{(i)}(f_0, f_1, f_2, \mathbf{f})}{\partial \nu} \right\|^2_{L^2(\Sigma_1)} \tag{3.21}$$

for all $f_0, f_1, f_2 \in H_0^1(\Omega)$ and $\mathbf{f} \in \left(H_0^1(\Omega)\right)^n$.

It is straightfoward to see that Theorems 3.1.1 and 3.1.2 are immediate corollaries of Theorems 3.1.3 and 3.1.4, respectively. Therefore, we will shift our focus to prove Theorems 3.1.3 and 3.1.4, and then use them to prove Theorems 3.1.1 and 3.1.2.

3.2 Uniqueness of Inverse Source Problem: Proof of Theorem 3.1.3

We now prove Theorem 3.1.3 to show uniqueness for the inverse source problem of (3.12). We will divide this proof into various steps so that the proof is clear. For convenience, we use C to denote a generic positive constant which may depend on Ω, T, c, q_1, q_0, $\mathbf{q}$, r_0, $w^{(i)}$, $u^{(i)}$, $R^{(i)}$, $i = 1, \cdots, m+2$, but independent of the free large parameter τ appearing in the Carleman estimate.

Step 1: Consider the case when n is odd, i.e. $n = 2m+1$, $m \in \mathbb{N}$. Corresponding with the choice of $R^{(i)}$, $i = 1, \cdots, m+2$, we have $m+2$ equations of the form (3.12) with solutions $u^{(i)} = u^{(i)}(x,t)$ that satisfy

$$\begin{cases} u_{tt}^{(i)} - c^2(x)\Delta u^{(i)} + q_1(x)u_t^{(i)} + q_0(x)u^{(i)} + \mathbf{q}(x)\cdot\nabla u^{(i)} = S^{(i)}(x,t) & \text{in } Q \\ u^{(i)}(x,0) = u_t^{(i)}(x,0) = 0 & \text{in } \Omega \\ u^{(i)}|_{\Gamma\times[-T,T]} = 0, \quad \dfrac{\partial u^{(i)}}{\partial \nu}|_{\Gamma_1\times[-T,T]} = 0 & \text{in } \Sigma, \Sigma_1, \end{cases} \tag{3.22}$$

where

$$S^{(i)}(x,t) = f_0(x)R^{(i)}(x,t) + f_1(x)R_t^{(i)}(x,t) + \mathbf{f}(x)\cdot\nabla R^{(i)}(x,t) + f_2(x)\Delta R^{(i)}(x,t), \tag{3.23}$$

for $i = 1, \ldots, m+2$. In other words, $S^{(i)}(x,t)$ is in (3.13) with R being replaced by $R^{(i)}$.

Since $c \in \mathcal{C}$, $q_1, q_0 \in L^\infty(\Omega)$ and $\mathbf{q} \in (L^\infty(\Omega))^n$, equation (3.22) can be written as a Riemannian wave equation with respect to the metric $g = c^{-2}(x)dx^2$, modulo lower-order terms

$$u_{tt}^{(i)} - \Delta_g u^{(i)} + \text{“lower-order terms”} = S^{(i)}(x,t).$$

By the regularity assumption (3.14), we have that $S^{(i)} \in L^2(Q)$ for each $i = 1, \ldots, m+2$, and by the Cauchy–Schwartz inequality we have

$$|S^{(i)}(x,t)|^2 \leq C \left(|f_0(x)|^2 + |f_1(x)|^2 + |\mathbf{f}(x)|^2 + |f_2(x)|^2 \right).$$

Thus, can apply the Carleman estimate (2.4) for solution $u^{(i)}$ in the class (2.3) and get the following inequality for sufficiently large τ:

$$\begin{aligned} \tau \int_Q e^{2\tau\varphi}[(u_t^{(i)})^2 + |Du^{(i)}|^2]dQ + \tau^3 \int_{Q(\sigma)} e^{2\tau\varphi}(u^{(i)})^2 dxdt \\ \leq C \int_Q e^{2\tau\varphi} \left(|f_0(x)|^2 + |f_1(x)|^2 + |\mathbf{f}(x)|^2 + |f_2(x)|^2 \right) dQ + Ce^{2\tau\sigma}. \end{aligned} \tag{3.24}$$

Note here we have dropped the unnecessary terms in the Carleman estimate (2.4) as well as the boundary terms $BT(u^{(i)})$ since the homogeneous boundary data $u^{(i)}|_{\Gamma\times[-T,T]} = \frac{\partial u^{(i)}}{\partial \nu}|_{\Gamma_1\times[-T,T]} = 0$ implies that $BT(u^{(i)}) \leq 0$, as mentioned in Remark 2.1.1.

Step 2: We now differentiate the $u^{(i)}$-system (3.22) with respect to time t. This results in the following $u_t^{(i)}$-system:

$$\begin{cases} (u_t^{(i)})_{tt} - c^2(x)\Delta(u_t^{(i)}) + q_1(x)(u_t^{(i)})_t + q_0(x)(u_t^{(i)}) + \mathbf{q}(x)\cdot\nabla(u_t^{(i)}) = S_t^{(i)}(x,t) & \text{in } Q \\ (u_t^{(i)})(x,0) = 0, \ (u_t^{(i)})_t(x,0) = S^{(i)}(x,0) & \text{in } \Omega \\ (u_t^{(i)})|_{\Gamma\times[-T,T]} = 0, \quad \dfrac{\partial u_t^{(i)}}{\partial\nu}|_{\Gamma_1\times[-T,T]} = 0 & \text{in } \Sigma, \Sigma_1. \end{cases} \tag{3.25}$$

By our regularity assumptions (3.14), we have $S_t^{(i)} \in L^2(Q)$, and by Cauchy–Schwartz inequality, we obtain the following:

$$|S_t^{(i)}(x,t)|^2 \leq C \left(|f_0(x)|^2 + |f_1(x)|^2 + |\mathbf{f}(x)|^2 + |f_2(x)|^2 \right).$$

In addition, $BT(u_t^{(i)}) \leq 0$ by Remark 2.1.1 since $u_t^{(i)}|_{\Gamma\times[-T,T]} = \frac{\partial u_t^{(i)}}{\partial \nu}|_{\Gamma_1\times[-T,T]} = 0$.

Thus, similar to (3.24) we can apply Carleman estimate (2.4) for solutions $u_t^{(i)}$ and get the following inequality for sufficiently large τ:

$$\begin{aligned}\tau\int_Q e^{2\tau\varphi}[(u_{tt}^{(i)})^2 + |Du_t^{(i)}|^2]dQ + \tau^3\int_{Q(\sigma)} e^{2\tau\varphi}(u_t^{(i)})^2 dxdt \\ \leq C\int_Q e^{2\tau\varphi}\left(|f_0(x)|^2 + |f_1(x)|^2 + |\mathbf{f}(x)|^2 + |f_2(x)|^2\right)dQ + Ce^{2\tau\sigma}.\end{aligned} \tag{3.26}$$

Step 3: Repeating this process, we differentiate (3.22) in t two more times, and we get the corresponding $u_{tt}^{(i)}$ and $u_{ttt}^{(i)}$-systems:

$$\begin{cases}(u_{tt}^{(i)})_{tt} - c^2(x)\Delta(u_{tt}^{(i)}) + q_1(x)(u_{tt}^{(i)})_t + q_0(x)(u_{tt}^{(i)}) + \mathbf{q}(x)\cdot\nabla(u_{tt}^{(i)}) = S_{tt}^{(i)}(x,t) \\ (u_{tt}^{(i)})(x,0) = S^{(i)}(x,0),\ \ (u_{tt}^{(i)})_t(x,0) = S_t^{(i)}(x,0) - q_1(x)S^{(i)}(x,0) \\ (u_{tt}^{(i)})|_{\Gamma\times[-T,T]} = 0, \quad \dfrac{\partial u_{tt}^{(i)}}{\partial\nu}|_{\Gamma_1\times[-T,T]} = 0\end{cases} \tag{3.27}$$

$$\begin{cases}(u_{ttt}^{(i)})_{tt} - c^2(x)\Delta(u_{ttt}^{(i)}) + q_1(x)(u_{ttt}^{(i)})_t + q_0(x)(u_{ttt}^{(i)}) + \mathbf{q}(x)\cdot\nabla(u_{ttt}^{(i)}) = S_{ttt}^{(i)}(x,t) \\ (u_{ttt}^{(i)})(x,0) = S_t^{(i)}(x,0) - q_1(x)S^{(i)}(x,0) \\ (u_{ttt}^{(i)})_t(x,0) = S_{tt}^{(i)}(x,0) + c^2\Delta S^{(i)}(x,0) - q_1 S_t^{(i)}(x,0) - q_0 S^{(i)}(x,0) - \mathbf{q}\cdot\nabla S^{(i)}(x,0) \\ (u_{ttt}^{(i)})|_{\Gamma\times[-T,T]} = 0, \quad \dfrac{\partial u_{ttt}^{(i)}}{\partial\nu}|_{\Gamma_1\times[-T,T]} = 0.\end{cases} \tag{3.28}$$

Again by (3.14), Cauchy–Schwartz inequality and the homogeneous Dirichlet and Neumann boundary data, we can apply Carleman estimate (2.4) to the corresponding $u_{tt}^{(i)}$, $u_{ttt}^{(i)}$-systems above and get the following inequalities that are similar to (3.24) and (3.26), for τ sufficiently large

$$\begin{aligned}\tau\int_Q e^{2\tau\varphi}[(u_{ttt}^{(i)})^2 + |Du_{tt}^{(i)}|^2]dQ + \tau^3\int_{Q(\sigma)} e^{2\tau\varphi}(u_{tt}^{(i)})^2 dxdt \\ \leq C\int_Q e^{2\tau\varphi}\left(|f_0(x)|^2 + |f_1(x)|^2 + |\mathbf{f}(x)|^2 + |f_2(x)|^2\right)dQ + Ce^{2\tau\sigma}.\end{aligned} \tag{3.29}$$

$$\begin{aligned}&\tau \int_Q e^{2\tau\varphi}[(u^{(i)}_{tttt})^2 + |Du^{(i)}_{ttt}|^2]dQ + \tau^3 \int_{Q(\sigma)} e^{2\tau\varphi}(u^{(i)}_{ttt})^2 dxdt \\ &\qquad\qquad \le C \int_Q e^{2\tau\varphi}\left(|f_0(x)|^2 + |f_1(x)|^2 + |\mathbf{f}(x)|^2 + |f_2(x)|^2\right) dQ + Ce^{2\tau\sigma}.\end{aligned} \tag{3.30}$$

Adding (3.24), (3.26), (3.29), and (3.30) yields the overall inequality

$$\begin{aligned}&\tau \int_Q e^{2\tau\varphi}\left[(u^{(i)}_{tttt})^2 + (u^{(i)}_{ttt})^2 + (u^{(i)}_{tt})^2 + (u^{(i)}_{t})^2 + |Du^{(i)}_{ttt}|^2 + |Du^{(i)}_{tt}|^2 + |Du^{(i)}_{t}|^2 + |Du^{(i)}|^2\right] dQ \\ &+ \tau^3 \int_{Q(\sigma)} e^{2\tau\varphi}\left[(u^{(i)}_{ttt})^2 + (u^{(i)}_{tt})^2 + (u^{(i)}_{t})^2 + (u^{(i)})^2\right] dxdt \\ &\le C \int_Q e^{2\tau\varphi}\left(|f_0(x)|^2 + |f_1(x)|^2 + |\mathbf{f}(x)|^2 + |f_2(x)|^2\right) dQ + Ce^{2\tau\sigma}.\end{aligned} \tag{3.31}$$

Step 4: We now analyze the integral term on the right-hand side of (3.31). By estimating the $u^{(i)}$-equation in (3.22) and $u^{(i)}_t$-equation in (3.25) at time $t = 0$, we get

$$\begin{cases} u^{(i)}_{tt}(x,0) = S^{(i)}(x,0) \\ u^{(i)}_{ttt}(x,0) = S^{(i)}_t(x,0) - q_1(x)S^{(i)}(x,0). \end{cases} \tag{3.32}$$

Since (3.32) holds for any $i = 1, \ldots, m+2$, putting the $m+2$ systems together gives us the following $(n+3) \times (n+3)$ linear system

$$\left[u^{(1)}_{tt}(x,0), u^{(1)}_{ttt}(x,0), \cdots, u^{(m+2)}_{tt}(x,0), u^{(m+2)}_{ttt}(x,0)\right]^T = U_{q_1}(x)\left[f_0(x), f_1(x), \mathbf{f}(x), f_2(x)\right]^T \tag{3.33}$$

where $U_{q_1}(x)$ is the $(n+3)\times(n+3)$ coefficient matrix:

$$U_{q_1}(x)=\begin{bmatrix} R^{(1)}(x,0) & R_t^{(1)}(x,0) & \partial_{x_1}R^{(1)}(x,0) & \cdots & \partial_{x_n}R^{(1)}(x,0) & \Delta R^{(1)}(x,0)\\ \tilde{a}^{(1)}(x) & \tilde{b}^{(1)}(x) & \tilde{m}_1^{(1)}(x) & \cdots & \tilde{m}_n^{(1)}(x) & \tilde{\ell}^{(1)}(x)\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ R^{(m+2)}(x,0) & R_t^{(m+2)}(x,0) & \partial_{x_1}R^{(m+2)}(x,0) & \cdots & \partial_{x_n}R^{(m+2)}(x,0) & \Delta R^{(m+2)}(x,0)\\ \tilde{a}^{(m+2)}(x) & \tilde{b}^{(m+2)}(x) & \tilde{m}_1^{(m+2)}(x) & \cdots & \tilde{m}_n^{(m+2)}(x) & \tilde{\ell}^{(m+2)}(x)\end{bmatrix} \tag{3.34}$$

with

$$\begin{aligned} &\tilde{a}^{(i)}(x)=R_t^{(i)}(x,0)-q_1(x)R^{(i)}(x,0),\quad \tilde{b}^{(i)}(x)=R_{tt}^{(i)}(x,0)-q_1(x)R_t^{(i)}(x,0),\\ &\tilde{m}_k^{(i)}(x)=\partial_{x_k}R_t^{(i)}(x,0)-q_1(x)\partial_{x_k}R^{(i)}(x),\quad \tilde{\ell}^{(i)}(x,0)=\Delta R_t^{(i)}(x,0)-q_1(x)\Delta R^{(i)}(x,0). \end{aligned} \tag{3.35}$$

Notice that by doing elementary row operations, specifically, adding q_1 multiplied by an odd row to the subsequent even row, the matrix $U_{q_1}(x)$ and $U(x)$ as defined in (3.16) have the same determinant. Thus the positivity assumption (3.15) implies that we may invert $U_{q_1}(x)$ in (3.34) to obtain

$$\begin{aligned} |f_0(x)|^2+|f_1(x)|^2+|f_2(x)|^2+|\mathbf{f}(x)|^2 &\le C\sum_{i=1}^{m+2}\left(|u_{tt}^{(i)}(x,0)|^2+|u_{ttt}^{(i)}(x,0)|^2\right)\\ &= C\left(|\mathbf{u}_{tt}(x,0)|^2+|\mathbf{u}_{ttt}(x,0)|^2\right) \end{aligned} \tag{3.36}$$

where we denote $\mathbf{u}(x,t)=\left(u^{(1)}(x,t),u^{(2)}(x,t),\cdots,u^{(m+2)}(x,t)\right)$.

Step 5: By using properties of the pseudo-convex function $\varphi(x)$ as defined in (1.8), we have the following estimate of the right-hand side of (3.31):

$$\begin{aligned} &\int_Q e^{2\tau\varphi(x,t)}\left(|f_0(x)|^2+|f_1(x)|^2+|\mathbf{f}(x)|^2+|f_2(x)|^2\right)dQ \qquad (3.37)\\ \le\ & C\int_\Omega e^{2\tau\varphi(x,0)}\left(|\mathbf{u}_{tt}(x,0)|^2+|\mathbf{u}_{ttt}(x,0)|^2\right)dx\\ \le\ & C\left(\int_\Omega\int_{-T}^{0}\frac{d}{ds}[e^{2\tau\varphi(x,s)}\left(|\mathbf{u}_{tt}(x,s)|^2+|\mathbf{u}_{ttt}(x,s)|^2\right)]ds\,dx \end{aligned}$$

$$
\begin{aligned}
&\quad + \int_\Omega e^{2\tau\varphi(x,-T)} \left(|\mathbf{u}_{tt}(x,-T)|^2 + |\mathbf{u}_{ttt}(x,-T)|^2\right) dx\Bigg) \\
\leq &\quad C\Bigg(\tau \int_\Omega \int_{-T}^{0} e^{2\tau\varphi(x,s)} \left(|\mathbf{u}_{tt}(x,s)|^2 + |\mathbf{u}_{ttt}(x,s)|^2\right)]ds\, dx \\
&\quad + 2\int_\Omega \int_{-T}^{0} e^{2\tau\varphi(x,s)} \left(\mathbf{u}_{tt}\cdot\mathbf{u}_{ttt} + \mathbf{u}_{ttt}\cdot\mathbf{u}_{tttt}\right)]ds\, dx \\
&\quad + \int_\Omega e^{2\tau\varphi(x,-T)} \left(|\mathbf{u}_{tt}(x,-T)|^2 + |\mathbf{u}_{ttt}(x,-T)|^2\right) dx\Bigg) \\
\leq &\quad C\left(\tau \int_Q e^{2\tau\varphi}|\mathbf{u}_{tt}|^2 dQ + \tau\int_Q e^{2\tau\varphi}|\mathbf{u}_{ttt}|^2 dQ + \int_Q e^{2\tau\varphi}|\mathbf{u}_{tttt}|^2 dQ\right).
\end{aligned}
$$

Since (3.31) holds for all $i = 1, \cdots m+2$, summing over i in (3.31) and dropping the non-negative gradient terms on the left-hand side, we can apply (3.37) to (3.31) and get that for τ sufficiently large

$$
\begin{aligned}
&\quad \tau \int_Q e^{2\tau\varphi}\left(|\mathbf{u}_{tttt}|^2 + |\mathbf{u}_{ttt}|^2 + |\mathbf{u}_{tt}|^2 + |\mathbf{u}_t|^2\right) dQ \\
+ &\quad \tau^3 \int_{Q(\sigma)} e^{2\tau\varphi}\left(|\mathbf{u}_{ttt}|^2 + |\mathbf{u}_{tt}|^2 + |\mathbf{u}_t|^2 + |\mathbf{u}|^2\right) dx\, dt \\
\leq &\quad C\tau \int_Q e^{2\tau\varphi}(|\mathbf{u}_{tt}|^2 + |\mathbf{u}_{ttt}|^2) dQ + C\int_Q e^{2\tau\varphi}|\mathbf{u}_{tttt}|^2 dQ + Ce^{2\tau\sigma}.
\end{aligned}
\tag{3.38}
$$

Step 6: Since $e^{2\tau\varphi} < e^{2\tau\sigma}$ on $Q\backslash Q(\sigma)$ from the definition of $Q(\sigma)$ (1.15), we have the following

$$
\begin{aligned}
&\quad \int_Q e^{2\tau\varphi}\left(|\mathbf{u}_{tt}|^2 + |\mathbf{u}_{ttt}|^2\right) dQ \\
= &\quad \int_{Q(\sigma)} e^{2\tau\varphi}\left(|\mathbf{u}_{tt}|^2 + |\mathbf{u}_{ttt}|^2\right) dx\, dt + \int_{Q\backslash Q(\sigma)} e^{2\tau\varphi}\left(|\mathbf{u}_{tt}|^2 + |\mathbf{u}_{ttt}|^2\right) dx\, dt \\
\leq &\quad \int_{Q(\sigma)} e^{2\tau\varphi}\left(|\mathbf{u}_{tt}|^2 + |\mathbf{u}_{ttt}|^2\right) dx\, dt + \; e^{2\tau\sigma}\int_{Q\backslash Q(\sigma)} \left(|\mathbf{u}_{tt}|^2 + |\mathbf{u}_{ttt}|^2\right) dx\, dt
\end{aligned}
$$

This transforms (3.38) into

$$
\begin{aligned}
& \tau \int_Q e^{2\tau\varphi} \left(|\mathbf{u}_{tttt}|^2 + |\mathbf{u}_{ttt}|^2 + |\mathbf{u}_{tt}|^2 + |\mathbf{u}_t|^2\right) dQ \\
+ \quad & \tau^3 \int_{Q(\sigma)} e^{2\tau\varphi} \left(|\mathbf{u}_{ttt}|^2 + |\mathbf{u}_{tt}|^2 + |\mathbf{u}_t|^2 + |\mathbf{u}|^2\right) dx\, dt \\
\leq \quad & C\tau \int_{Q(\sigma)} e^{2\tau\varphi} \left(|\mathbf{u}_{tt}|^2 + |\mathbf{u}_{ttt}|^2\right) dx\, dt + C \int_Q e^{2\tau\varphi} |\mathbf{u}_{tttt}|^2 dQ + Ce^{2\tau\sigma}.
\end{aligned}
\tag{3.39}
$$

Step 7: The first and second terms on the right-hand side (RHS) of (3.39) can be absorbed by the corresponding terms on the left-hand side (LHS) when τ is taken large enough. This yields the following estimate for sufficiently large τ:

$$
\tau^3 \int_{Q(\sigma)} e^{2\tau\varphi} \left(|\mathbf{u}_{ttt}|^2 + |\mathbf{u}_{tt}|^2 + |\mathbf{u}_t|^2 + |\mathbf{u}|^2\right) dx\, dt \leq C\tau e^{2\tau\sigma}.
$$

Since $\varphi(x,t) \geq \sigma$ on $Q(\sigma)$, we get

$$
\tau^2 \int_{Q(\sigma)} |\mathbf{u}_{ttt}|^2 + |\mathbf{u}_{tt}|^2 + |\mathbf{u}_t|^2 + |\mathbf{u}|^2 \ dx\, dt \leq C.
$$

Since $\tau > 0$ in a free large parameter and the constants C do not depend on τ, the above inequality implies we must have $\mathbf{u} = \mathbf{0}$ a.e. on $Q(\sigma)$.

Step 8: From (1.16) the subspace $Q(\sigma)$ satisfies the property $\Omega \times [t_0, t_1] \subset Q(\sigma) \subset Q$ with $t_0 < 0 < t_1$, therefore by evaluating the $\mathbf{u}$ and $\mathbf{u}_t$-systems of equations at $t = 0$, we get the $(n+3) \times (n+3)$ linear system by (3.33):

$$
U_{q_1}(x)[f_0(x), f_1(x), \mathbf{f}(x), f_2(x)]^T = \mathbf{0}, \ \textit{a.e. } x \in \Omega.
$$

As the coefficient matrix $U_{q_1}(x)$ is invertible from assumption (3.15), we must have

$$
f_0(x) = f_1(x) = f_2(x) = \mathbf{f}(x) = 0, \ \textit{a.e. } x \in \Omega,
$$

as desired.

Step 9: For the case when n is even, i.e., $n = 2m$, $m \in \mathbb{N}$, the above proof can be repeated with obvious adjustments. The only difference worth mentioning is that since $n = 2m$ is even, the linear system (3.33) contains an odd number $(n+3)$ of equations. Therefore we only need $m+1$ pairs of equations from (3.32) plus one more equation from $u_{tt}^{(m+2)}(x,0)$. Doing this yields the matrix $\widetilde{U}_{q_1}(x)$, where

$$\widetilde{U}_{q_1}(x) = \begin{bmatrix} R^{(1)}(x,0) & R_t^{(1)}(x,0) & \partial_{x_1}R^{(1)}(x,0) & \cdots & \partial_{x_n}R^{(1)}(x,0) & \Delta R^{(1)}(x,0) \\ \tilde{a}^{(1)}(x) & \tilde{b}^{(1)}(x) & \tilde{m}_1^{(1)}(x) & \cdots & \tilde{m}_n^{(1)}(x) & \tilde{\ell}^{(1)}(x) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ R^{(m+1)}(x,0) & R_t^{(m+1)}(x,0) & \partial_{x_1}R^{(m+1)}(x,0) & \cdots & \partial_{x_n}R^{(m+1)}(x,0) & \Delta R^{(m+1)}(x,0) \\ \tilde{a}^{(m+1)}(x) & \tilde{b}^{(m+1)}(x) & \tilde{m}_1^{(m+1)}(x) & \cdots & \tilde{m}_n^{(m+1)}(x) & \tilde{\ell}^{(m+1)}(x) \\ R^{(m+2)}(x,0) & R_t^{(m+2)}(x,0) & \partial_{x_1}R^{(m+2)}(x,0) & \cdots & \partial_{x_n}R^{(m+2)}(x,0) & \Delta R^{(m+2)}(x,0) \end{bmatrix} \tag{3.40}$$

with $\tilde{a}^{(i)}$, $\tilde{b}^{(i)}$, $\tilde{m}_k^{(i)}$ and $\tilde{\ell}^{(i)}$ defined as in (3.35). Again, since elementary row operations does not change the determinant, $\widetilde{U}_{q_1}(x)$ will have the same determinant as the matrix $\widetilde{U}(x)$, thus giving us the positivity assumption (3.17). This completes the proof of Theorem 3.1.3.

3.3 Uniqueness of Inverse Problem: Proof of Theorem 3.1.1

As in (3.11), for $x \in \Omega$, $t \in [-T,T]$, set

$$\begin{aligned} & f_2(x) = c^2(x) - \tilde{c}^2(x), \quad f_1(x) = p_1(x) - q_1(x), \\ & f_0(x) = p_0(x) - q_0(x), \quad \mathbf{f}(x) = \mathbf{p}(x) - \mathbf{q}(x); \\ & u(x,t) = w(c,q_1,q_0,\mathbf{q}) - w(\tilde{c},p_1,p_0,\mathbf{p}),\ R(x,t) = w(\tilde{c},p_1,p_0,\mathbf{p})(x,t), \end{aligned} \tag{3.41}$$

Then $u(x,t)$ solves (3.12). By (3.41), $f_0, f_1 \in L^\infty(\Omega) \subset L^2(\Omega)$, $f_2(x) \in L^2(\Omega)$, $\mathbf{f} \in (L^\infty(\Omega))^n \subset (L^2(\Omega))^n$. Moreover, by assumption (3.3), we have (3.14).

We also have that positivity assumptions (3.4) and (3.6) imply the positivity assumptions (3.15) and (3.17). In addition, by assumption (3.8) that the traces coincide:

$$\frac{\partial w^{(i)}(c,q_1,q_0,\mathbf{q})}{\partial \nu}(x,t) = \frac{\partial w^{(i)}(\tilde{c},p_1,p_0,\mathbf{p})}{\partial \nu}(x,t),\ (x,t) \in \Gamma_1 \times [-T,T], \tag{3.42}$$

implies via (3.41) that we have

$$\frac{\partial u^{(i)}(f_0, f_1, f_2, \mathbf{f})}{\partial \nu}(x,t) = 0, \ (x,t) \in \Gamma_1 \times [-T,T], \ i = 1, \cdots, m+2. \tag{3.43}$$

Therefore, Theorem 3.1.3 applies, and we have (3.20), which yields our desired relation (3.9).

3.4 Stability of Inverse Source Problem: Proof of Theorem 3.1.4

Now we prove Theorem 3.1.4 to show stability of the inverse source problem (3.12). As with the proof of Theorem 3.1.3, we will break the proof into various steps to make the argument clear. The proof for the cases whether n is even or odd is essentially the same; the only difference is in the choices of $R^{(i)}$, $i = 1, 2, \ldots, m+2$. Hence, without loss of generality, let n be odd.

Step 1: We return to inequality (3.36) from Section 3.2. Integrating over Ω yields

$$\|f_0\|^2_{L^2(\Omega)} + \|f_1\|^2_{L^2(\Omega)} + \|f_2\|^2_{L^2(\Omega)} + \|\mathbf{f}\|^2_{\mathbf{L}^2(\Omega)} \leq C \sum_{i=1}^{m+2} \left(\|u^{(i)}_{tt}(\cdot, 0)\|^2_{L^2(\Omega)} + \|u^{(i)}_{ttt}(\cdot, 0)\|^2_{L^2(\Omega)} \right) \tag{3.44}$$

For each i, $1 \leq i \leq m+2$, (3.44) suggests we return to the $u^{(i)}_{tt}$-system:

$$\begin{cases} (u^{(i)}_{tt})_{tt} - c^2(x)\Delta(u^{(i)}_{tt}) + q_1(x)(u^{(i)}_{tt})_t + q_0(x)(u^{(i)}_{tt}) + \mathbf{q}(x) \cdot \nabla(u^{(i)}_{tt}) = S^{(i)}_{tt}(x,t) \\ (u^{(i)}_{tt})(x,0) = S^{(i)}(x,0), \ (u^{(i)}_{tt})_t(x,0) = S^{(i)}_t(x,0) - q_1(x)S^{(i)}(x,0) \\ (u^{(i)}_{tt})|_{\Gamma \times [-T,T]} = 0 \end{cases} \tag{3.45}$$

Again we assume

$$c \in \mathcal{C}, \ q_0, q_1, q_2 \in L^\infty(\Omega), \ \mathbf{q} \in \left(L^\infty(\Omega)\right)^n, f_0, f_1, f_2 \in H^1_0(\Omega), \ \mathbf{f} \in \left(H^1_0(\Omega)\right)^n \tag{3.46}$$

and $R^{(i)}$ satisfies (3.14) and (3.15) (or (3.17) if n is even).

Step 2: By linearity of the solution $u^{(i)}_{tt}$, we split $u^{(i)}_{tt}$ into two systems, $u^{(i)}_{tt} = y^{(i)} + z^{(i)}$, where

$y^{(i)} = y^{(i)}(x,t)$ satisfies the homogeneous forcing term and nonhomogeneous initial conditions

$$\begin{cases} y_{tt}^{(i)} - c^2(x)\Delta y^{(i)} + q_1(x)y_t^{(i)} + q_0(x)y^{(i)} + \mathbf{q}(x)\cdot\nabla y^{(i)} = 0 & \text{in } Q \\ y^{(i)}(x,0) = u_{tt}^{(i)}(x,0) = S^{(i)}(x,0) & \text{in } \Omega \\ y_t^{(i)}(x,0) = (u_{tt}^{(i)})_t(x,0) = S_t^{(i)}(x,0) - q_1(x)S^{(i)}(x,0) & \text{in } \Omega \\ y^{(i)}|_{\Gamma\times[-T,T]} = 0 & \text{in } \Sigma \end{cases} \tag{3.47}$$

and $z^{(i)} = z^{(i)}(x,t)$ has the nonhomogeneous forcing term and homogeneous initial conditions

$$\begin{cases} z_{tt}^{(i)} - c^2(x)\Delta z^{(i)} + q_1(x)z_t^{(i)} + q_0(x)z^{(i)} + \mathbf{q}(x)\cdot\nabla z^{(i)} = S_{tt}^{(i)}(x,t) & \text{in } Q \\ z^{(i)}(x,0) = z_t^{(i)}(x,0) = 0 & \text{in } \Omega \\ z^{(i)}|_{\Gamma\times[-T,T]} = 0 & \text{in } \Sigma. \end{cases} \tag{3.48}$$

For the $y^{(i)}$-system, note by assumptions (3.46) and (3.14) we have that $S^{(i)}(\cdot,0) \in H_0^1(\Omega)$ and $S_t^{(i)}(\cdot,0) - q_1(\cdot)S^{(i)}(\cdot,0) \in L^2(\Omega)$.

Step 3: From this, we may apply the continuous observability inequality (2.8) (with $g = c^{-2}(x)dx^2$) to get

$$\|y^{(i)}(\cdot,0)\|_{H_0^1(\Omega)}^2 + \|y_t^{(i)}(\cdot,0)\|_{L^2(\Omega)}^2 = \|u_{tt}^{(i)}(\cdot,0)\|_{H_0^1(\Omega)}^2 + \|u_{ttt}^{(i)}(\cdot,0)\|_{L^2(\Omega)}^2 \le C\left\|\frac{\partial y^{(i)}}{\partial\nu}\right\|_{L^2(\Sigma_1)}^2. \tag{3.49}$$

Sum the above inequality over i, using (3.44) and the decomposition $u_{tt}^{(i)} = y^{(i)} + z^{(i)}$, and applying

Poincaré's inequality, we have

$$\begin{aligned}
&\|f_0\|^2_{L^2(\Omega)} + \|f_1\|^2_{L^2(\Omega)} + \|f_2\|^2_{L^2(\Omega)} + \|\mathbf{f}\|^2_{\mathbf{L}^2(\Omega)} \qquad (3.50)\\
\leq\ & C \sum_{i=1}^{m+2} \left(\|u^{(i)}_{ttt}(\cdot,0)\|^2_{H^1_0(\Omega)} + \|u^{(i)}_{ttt}(\cdot,0)\|^2_{L^2(\Omega)} \right)\\
\leq\ & C \sum_{i=1}^{m+2} \left\| \frac{\partial y^{(i)}}{\partial \nu} \right\|^2_{L^2(\Sigma_1)}\\
=\ & C \sum_{i=1}^{m+2} \left\| \frac{\partial u^{(i)}_{tt}}{\partial \nu} - \frac{\partial z^{(i)}}{\partial \nu} \right\|^2_{L^2(\Sigma_1)}\\
\leq\ & C \sum_{i=1}^{m+2} \left\| \frac{\partial u^{(i)}_{tt}}{\partial \nu} \right\|^2_{L^2(\Sigma_1)} + C \sum_{i=1}^{m+2} \left\| \frac{\partial z^{(i)}}{\partial \nu} \right\|^2_{L^2(\Sigma_1)}.
\end{aligned}$$

Step 4: Notice from the previous step that the right-hand side of (3.50) is our desired stability estimate (3.21) for Theorem 3.1.4 polluted by the $z^{(i)}$ terms. To obtain our desired result, we will show that the polluted terms can be absorbed by relying on a compactness-uniqueness argument. The uniqueness portion of this argument relies on Theorem 3.1.3.

To accomplish this, we first prove a related result regarding the $z^{(i)}-$system (3.48). In particular, the lemma shows the polluted terms in (3.50) are compact operators for each $i = 1, \ldots, m+2$.

Lemma 3.4.1. *For each $i = 1, \cdots, m+2$, the operator define by*

$$\begin{aligned}
\mathcal{K}_i : L^2(\Omega) \times L^2(\Omega) \times L^2(\Omega) \times \left(L^2(\Omega)\right)^n \quad &\to \quad L^2(\Sigma_1) \qquad (3.51)\\
(f_0, f_1, f_2, \mathbf{f}) \quad &\mapsto \quad \left.\frac{\partial z^{(i)}}{\partial \nu}\right|_{\Sigma_1},
\end{aligned}$$

is a compact operator.

Proof. By assumptions (3.46) and (3.14), we have that $S^{(i)}_{tt} \in H^1(Q)$. Using the regularity result (2.10), we have that

$$S^{(i)}_{tt} \in H^1(Q) \ \Rightarrow\ \frac{\partial z^{(i)}}{\partial \nu} \in H^1(\Sigma_1) \ \text{ continuously.}$$

This then implies the map $(f_0, f_1, f_2, \mathbf{f}) \mapsto \mathcal{K}_i(f_0, f_1, f_2, \mathbf{f}) \in H^1(\Sigma_1)$ is continuous and hence $(f_0, f_1, f_2, \mathbf{f}) \mapsto \mathcal{K}_i(f_0, f_1, f_2, \mathbf{f}) \in L^2(\Sigma_1)$ is compact. $\square$

Step 5: Thanks to Lemma 3.4.1, we may drop the $z^{(i)}$ terms in (3.50) to get the desired stability estimate (3.21) using a compact-uniqueness argument. This direction is inspired by [41] and [42].

Suppose, to the contrary, that the stability estimate (3.21) does not hold. Then there exist sequences $\{f_0^k\}$, $\{f_1^k\}$, $\{f_2^k\}$ and $\{\mathbf{f}^k\}$, with $f_0^k, f_1^k, f_2^k \in H_0^1(\Omega)$ and $\mathbf{f}^k \in \left(H_0^1(\Omega)\right)^n$, $\forall k \in \mathbb{N}$, such that

$$\left\|f_0^k\right\|^2_{L^2(\Omega)} + \left\|f_1^k\right\|^2_{L^2(\Omega)} + \left\|f_2^k\right\|^2_{L^2(\Omega)} + \left\|\mathbf{f}^k\right\|^2_{\mathbf{L}^2(\Omega)} = 1 \tag{3.52}$$

$$\lim_{k\to\infty} \sum_{i=1}^{m+2} \left\| \frac{\partial u_{tt}^{(i)}(f_0^k, f_1^k, f_2^k, \mathbf{f}^k)}{\partial \nu} \right\|_{L^2(\Sigma_1)} = 0 \tag{3.53}$$

where $u^{(i)}(f_0^k, f_1^k, f_2^k, \mathbf{f}^k)$ solves the system (3.22) with $f_0 = f_0^k$, $f_1 = f_1^k$, $f_2 = f_2^k$ and $\mathbf{f} = \mathbf{f}^k$.

From (3.52), there exist subsequences, still denoted as $\{f_0^k\}$, $\{f_1^k\}$, $\{f_2^k\}$ and $\{\mathbf{f}^k\}$, such that

$$f_i{}^k \rightharpoonup f_i^* \text{ and } \mathbf{f}^k \rightharpoonup \mathbf{f}^* \text{ weakly for some } f_i^* \in L^2(\Omega) \text{ and } \mathbf{f}^* \in \left(L^2(\Omega)\right)^n, i = 0, 1, 2. \tag{3.54}$$

Moreover, since the operators $\mathcal{K}_i$, $i = 1, \cdots, m+2$, are compact by Lemma 3.4.1, we also have the strong convergence below ([3, Theorem 3.2.3], [28, Theorem 8.1-7]):

$$\lim_{k,l\to\infty} \left\|\mathcal{K}_i(f_0^k, f_1^k, f_2^k, \mathbf{f}^k) - \mathcal{K}_i(f_0^l, f_1^l, f_2^l, \mathbf{f}^l)\right\|_{L^2(\Sigma_1)} = 0,\ \forall i = 1, \cdots, m+2. \tag{3.55}$$

Step 6: Since the map $(f_0, f_1, f_2, \mathbf{f}) \mapsto u^{(i)}(f_0, f_1, f_2, \mathbf{f})$ is linear, we have from (3.50) that

$$\begin{aligned}
& \left\|f_0^k - f_0^l\right\|^2_{L^2(\Omega)} + \left\|f_1^k - f_1^l\right\|^2_{L^2(\Omega)} + \left\|f_2^k - f_2^l\right\|^2_{L^2(\Omega)} + \left\|\mathbf{f}^k - \mathbf{f}^l\right\|^2_{\mathbf{L}^2(\Omega)} \\
\leq\ & C \sum_{i=1}^{m+2} \left\| \frac{\partial u_{tt}^{(i)}(f_0^k, f_1^k, f_2^k, \mathbf{f}^k)}{\partial \nu} - \frac{\partial u_{tt}^{(i)}(f_0^l, f_1^l, f_2^l, \mathbf{f}^l)}{\partial \nu} \right\|^2_{L^2(\Sigma_1)} \\
& + C \sum_{i=1}^{m+2} \left\|\mathcal{K}_i(f_0^k, f_1^k, f_2^k, \mathbf{f}^k) - \mathcal{K}_i(f_0^l, f_1^l, f_2^l, \mathbf{f}^l)\right\|^2_{L^2(\Sigma_1)} \\
\leq\ & C \sum_{i=1}^{m+2} \left\| \frac{\partial u_{tt}^{(i)}(f_0^k, f_1^k, f_2^k, \mathbf{f}^k)}{\partial \nu} \right\|^2_{L^2(\Sigma_1)} + C \sum_{i=1}^{m+2} \left\| \frac{\partial u_{tt}^{(i)}(f_0^l, f_1^l, f_2^l, \mathbf{f}^l)}{\partial \nu} \right\|^2_{L^2(\Sigma_1)} \\
& + C \sum_{i=1}^{m+2} \left\|\mathcal{K}_i(f_0^k, f_1^k, f_2^k, \mathbf{f}^k) - \mathcal{K}_i(f_0^l, f_1^l, f_2^l, \mathbf{f}^l)\right\|^2_{L^2(\Sigma_1)}
\end{aligned}$$

Therefore, by (3.53) and (3.55) we get

$$\lim_{k,l\to\infty} \left\|f_i^k - f_i^l\right\|_{L^2(\Omega)} = \lim_{k,l\to\infty} \left\|\mathbf{f}^k - \mathbf{f}^l\right\|_{\mathbf{L}^2(\Omega)} = 0,\ i = 0,1,2.$$

Namely, $\{f_0^k\}$, $\{f_1^k\}$, $\{f_2^k\}$ are Cauchy sequences in $L^2(\Omega)$ and $\{\mathbf{f}_k\}$ is a Cauchy sequence in $(L^2(\Omega))^n$. By uniqueness of limits and (3.54), we must have $\{f_i^k\}$ converges to f_i^* strongly, $i = 0,1,2$, and $\{\mathbf{f}^k\}$ converges to $\mathbf{f}^*$ strongly. Hence we have from (3.52)

$$\|f_0^*\|^2_{L^2(\Omega)} + \|f_1^*\|^2_{L^2(\Omega)} + \|f_2^*\|^2_{L^2(\Omega)} + \|\mathbf{f}^*\|^2_{\mathbf{L}^2(\Omega)} = 1. \tag{3.56}$$

Step 7: We now return to the $u_{tt}^{(i)}$-system (3.45). By the regularity theory (2.10), we have that the map $(f_0, f_1, f_2, \mathbf{f}) \mapsto \frac{\partial u_{tt}^{(i)}(f_0,f_1,f_2,\mathbf{f})}{\partial\nu} \in L^2(\Sigma)$ is continuous and hence

$$\left\|\frac{\partial u_{tt}^{(i)}(f_0, f_1, f_2, \mathbf{f})}{\partial\nu}\right\|^2_{L^2(\Sigma)} \le C\left(\|f_0\|^2_{L^2(\Omega)} + \|f_1\|^2_{L^2(\Omega)} + \|f_2\|^2_{L^2(\Omega)} + \|\mathbf{f}\|^2_{\mathbf{L}^2(\Omega)}\right). \tag{3.57}$$

Since the map $(f_0, f_1, f_2, \mathbf{f}) \mapsto u_{tt}^{(i)}(f_0, f_1, f_2, \mathbf{f})|_\Sigma$ is linear, then it follows from that we have

$$\begin{aligned} &\left\|\frac{\partial u_{tt}^{(i)}(f_0^k, f_1^k, f_2^k, \mathbf{f}^k)}{\partial\nu} - \frac{\partial u_{tt}^{(i)}(f_0^*, f_1^*, f_2^*, \mathbf{f}^*)}{\partial\nu}\right\|^2_{L^2(\Sigma_1)} \\ \le\ & C\left(\|f_0^k - f_0^*\|^2_{L^2(\Omega)} + \|f_1^k - f_1^*\|^2_{L^2(\Omega)} + \|f_2^k - f_2^*\|^2_{L^2(\Omega)} + \|\mathbf{f}^k - \mathbf{f}^*\|^2_{\mathbf{L}^2(\Omega)}\right). \end{aligned} \tag{3.58}$$

Because $f_i^k \to f_i^*$, $i = 0,1,2$ and $\mathbf{f}^k \to \mathbf{f}^*$ strongly, (3.58) implies

$$\lim_{k\to\infty}\left\|\frac{\partial u_{tt}^{(i)}(f_0^k, f_1^k, f_2^k, \mathbf{f}^k)}{\partial\nu} - \frac{\partial u_{tt}^{(i)}(f_0^*, f_1^*, f_2^*, \mathbf{f}^*)}{\partial\nu}\right\|_{L^2(\Sigma_1)} = 0$$

and hence $\frac{\partial u_{tt}^{(i)}(f_0^*,f_1^*,f_2^*,\mathbf{f}^*)}{\partial\nu} = 0$ in $L^2(\Sigma_1)$ in view of (3.53). In other words, $\frac{\partial u_t^{(i)}(f_0^*,f_1^*,f_2^*,\mathbf{f}^*)}{\partial\nu}$ is constant in $t \in [-T, T]$.

Step 8: We now claim that $\frac{\partial u_t^{(i)}(f_0^*,f_1^*,f_2^*,\mathbf{f}^*)}{\partial\nu} = 0$ on Σ_1. To see this, consider now the $u_t^{(i)}(f_0^k, f_1^k, f_2^k, \mathbf{f}^k)$-

system

$$
\begin{cases}
(u_t^{(i)})_{tt} - c^2(x)\Delta(u_t^{(i)}) + q_1(x)(u_t^{(i)})_t + q_0(x)(u_t^{(i)}) + \mathbf{q}(x)\cdot\nabla u_t^{(i)} = (S_k^{(i)})_t(x,t) & \text{in } Q \\
(u_t^{(i)})(x,0) = 0,\ (u_t^{(i)})_t(x,0) = S_k^{(i)}(x,0) & \text{in } \Omega \\
u_t^{(i)}|_{\Gamma\times[-T,T]} = 0 & \text{in } \Sigma
\end{cases}
\tag{3.59}
$$

where for $i = 1,\cdots,m+2$, $R^{(i)} = R^{(i)}(x,t)$ and

$$
S_k^{(i)}(x,t) = f_0^k(x)R^{(i)} + f_1^k(x)R_t^{(i)} + \mathbf{f}^k(x)\cdot\nabla R^{(i)} + f_2^k(x)\Delta R^{(i)}.
$$

The regularity theory (2.10) and trace theory imply

$$
\begin{aligned}
& \left\| u_t^{(i)}(f_0^k, f_1^k, f_2^k, \mathbf{f}^k) - u_t^{(i)}(f_0^*, f_1^*, f_2^*, \mathbf{f}^*) \right\|^2_{C([0,T];H_0^1(\Omega))} \\
\leq \quad & C\left(\|f_0^k - f_0^*\|^2_{L^2(\Omega)} + \|f_1^k - f_1^*\|^2_{L^2(\Omega)} + \|f_2^k - f_2^*\|^2_{L^2(\Omega)} + \|\mathbf{f}^k - \mathbf{f}^*\|^2_{\mathbf{L}^2(\Omega)} \right) \\
& \left\| u_t^{(i)}(f_0^k, f_1^k, f_2^k, \mathbf{f}^k) - u_t^{(i)}(f_0^*, f_1^*, f_2^*, \mathbf{f}^*) \right\|^2_{C([0,T];H^{\frac{1}{2}}(\Sigma)} \\
\leq \quad & C\left(\|f_0^k - f_0^*\|^2_{L^2(\Omega)} + \|f_1^k - f_1^*\|^2_{L^2(\Omega)} + \|f_2^k - f_2^*\|^2_{L^2(\Omega)} + \|\mathbf{f}^k - \mathbf{f}^*\|^2_{\mathbf{L}^2(\Omega)} \right).
\end{aligned}
$$

By the initial conditions in (3.59), $u_t^{(i)}(f_0^k, f_1^k, f_2^k, \mathbf{f}^k)(x,0) = 0$ for all $k \in \mathbb{N}$. Combining this, the inequalities above, and the strong convergence $f_i^k \to f_i^*$ for $i = 0,1,2$, and $\mathbf{f}^k \to \mathbf{f}^*$, letting $k \to \infty$ yields $u_t^{(i)}(f_0^*, f_1^*, f_2^*, \mathbf{f}^*)(x,0) = 0$ in Ω and $u_t^{(i)}(f_0^*, f_1^*, f_2^*, \mathbf{f}^*)|_\Sigma = 0$. Hence $\frac{\partial u_t^{(i)}(f_0^*,f_1^*,f_2^*,\mathbf{f}^*)}{\partial\nu}(x,0) = 0$ on Σ. Since we know $\frac{\partial u_t^{(i)}(f_0^*,f_1^*,f_2^*,\mathbf{f}^*)}{\partial\nu}$ is a constant in t, we must have $\frac{\partial u_t^{(i)}(f_0^*,f_1^*,f_2^*,\mathbf{f}^*)}{\partial\nu} = 0$ on Σ_1, as desired.

The conclusion above implies $\frac{\partial u^{(i)}(f_0^*,f_1^*,f_2^*,\mathbf{f}^*)}{\partial\nu}$ is also a constant in t. By repeating the same argument, this time using the regularity theory for the $u^{(i)}(f_0^k, f_1^k, f_2^k, \mathbf{f}^k)$-system and taking limit $k \to \infty$, we finally get $\frac{\partial u^{(i)}(f_0^*,f_1^*,f_2^*,\mathbf{f}^*)}{\partial\nu} = 0$ on Σ_1.

Step 9: From our previous step, we have that $u^{(i)}(f_0^*, f_1^*, f_2^*, \mathbf{f}^*)$ satisfies the following system

$$\begin{cases} u_{tt}^{(i)} - c^2(x)\Delta u^{(i)} + q_1(x)u_t^{(i)} + q_0(x)u^{(i)} + \mathbf{q}(x) \cdot \nabla u^{(i)} = S_*^{(i)}(x,t) & \text{in } \; Q \\ u^{(i)}(x,0) = u_t^{(i)}(x,0) = 0 & \text{in } \; \Omega \\ u^{(i)}|_{\Gamma \times [-T,T]} = 0, \quad \left.\dfrac{\partial u^{(i)}}{\partial \nu}\right|_{\Gamma_1 \times [-T,T]} = 0 & \text{in } \; \Sigma, \Sigma_1 \end{cases} \tag{3.60}$$

with

$$S_*^{(i)}(x,t) = f_0^*(x)R^{(i)} + f_1^*(x)R_t^{(i)} + \mathbf{f}^*(x) \cdot \nabla R^{(i)} + f_2^*(x)\Delta R^{(i)}, i = 1, \cdots, m+2.$$

By the uniqueness result Theorem 3.1.3, this implies $f_0^* = f_1^* = f_2^* = \mathbf{f}^* = 0$, which contradicts with (3.56). Therefore we must be able to drop the $z^{(i)}$ terms in (3.50), completing the proof of Theorem 3.1.4.

3.5 Stability of Inverse Problem: Proof of Theorem 3.1.2

By the regularity theory (2.10) the assumption (3.3) on the initial and boundary conditions $\{w_0^{(i)}, w_1^{(i)}, h\}$ implies the solutions $w^{(i)}$, $i = 1, \cdots, m+2$, satisfy

$$\{w^{(i)}, w_t^{(i)}, w_{tt}^{(i)}, w_{ttt}^{(i)}\} \in C\left([-T,T]; H^{\gamma+1}(\Omega) \times H^{\gamma}(\Omega) \times H^{\gamma-1}(\Omega) \times H^{\gamma-2}(\Omega)\right).$$

As $\gamma > \frac{n}{2} + 4$, we have the following embedding $H^{\gamma-2}(\Omega) \hookrightarrow W^{2,\infty}(\Omega)$ and hence the regularity assumption (3.3) implies the corresponding regularity assumption (3.14) for the inverse source problem. Therefore, we have Theorem 3.1.2 as an immediate consequence of Theorem 3.1.4.

Chapter 4

Inverse Problem for the Plate Equation

In this chapter, we solve the inverse problem for recovering $q \in L^\infty(\Omega)$ for the plate equation (1.5) using finite measurements on the boundary. We restate it below for convenience:

$$\begin{cases} w_{tt} + \Delta^2 w + q(x)w = 0 & \text{in } Q \\ w(x,0) = w_0(x)\ w_t(x,0) = w_1(x) & \text{in } \Omega \\ w(x,t) = h_1(x,t),\ \Delta w(x,t) = h_2(x,t) & \text{on } \Sigma \end{cases} \tag{4.1}$$

As in Chapter 3, we will solve the inverse problem following a post-Carleman estimate approach. In particular, we will show that we can recover the coefficient q using two measurements on the boundary. The two measurements is needed due the the boundary terms $BT_1^*(w)$ in (2.32) and Remark 2.2.4.

The major contribution of our inverse problem of recovering q in (4.1) is that the recovery is on a lower-order term. Most inverse problems of the plate equation are inverse source problems because of the decomposition into two Schrödinger equations being on the principal part of the equation. Our decomposition allows us to incorporate lower-order terms with our Carleman estimates, which allows us to solve the inverse problem on lower-order terms.

Similar to the previous chapter, we start by listing our major theorems followed by giving the proofs in the subsequent sections.

4.1 Statement of Theorems

We now state the main theorems of uniqueness of solving the inverse problem associated with (4.1).

Theorem 4.1.1. *Assume the geometrical assumptions (A.1') and (A.2') are satisfied, and let* $T > 0$*. Suppose the initial and boundary conditions satisfy the following:*

$$\{w_0, w_1\} \in H^{\gamma+3}(\Omega) \times H^{\gamma+1}(\Omega),\ \{h_1, h_2\} \in H^{\gamma+3}(\Sigma),\ \textit{for } \gamma > \frac{n}{2} + 3, \tag{4.2}$$

along with all compatability conditions. Moreover, assume that the initial condition has the following positivity condition:

$$|w_0(x)| \geq r_0 > 0, \tag{4.3}$$

for some constant $r_0 > 0$ *and* $x \in \overline{\Omega}$*. Let* $w(q)$ *and* $w(p)$ *be solutions of equation* (4.1) *with coefficients* q *and* p*, respectively. If we have the same Neumann boundary traces over* Γ_1 *and the time interval:*

$$\begin{aligned} \frac{\partial w(q)}{\partial \nu} &= \frac{\partial w(p)}{\partial \nu} \\ \frac{\partial \Delta w(q)}{\partial \nu} &= \frac{\partial \Delta w(p)}{\partial \nu}, \end{aligned} \tag{4.4}$$

then the coefficients coincide:

$$q(x) = p(x)\ \textit{a.e } x \in \Omega. \tag{4.5}$$

Just as before, after proving Theorem 4.1.1 giving us uniqueness, we may also obtain the following stability result for recovering q from a single measurement on the boundary.

Theorem 4.1.2. *Under the assumptions listed in Theorem 4.1.1, let* $w(q)$ *and* $w(p)$ *denote the solutions of equation* (4.1) *with coefficients* q *and* p *with initial and boundary conditions* $\{w_0, w_1, h_1, h_2\}$*. Then there exists* $C > 0$ *that depends on* Ω*,* T*,* Γ_1*,* q*,* w_0*,* w_1*,* h_1*,* h_2 *such that*

$$\|q - p\|^2_{L^2(\Omega)} \leq \left\| \frac{\partial w_{tt}(q)}{\partial \nu} - \frac{\partial w_{tt}(p)}{\partial \nu} \right\|^2_{L^2(\Sigma_1)} + \left\| \frac{\partial \Delta w_t(q)}{\partial \nu} - \frac{\partial \Delta w_t(p)}{\partial \nu} \right\|^2_{L^2(\Sigma_1)} \tag{4.6}$$

for all $p, q \in H^3(\Omega) \cap H_0^1(\Omega)$.

Our first step is to convert convert our inverse problem into an inverse source problem. Let

$$\begin{aligned} f(x) &= q(x) - p(x) \\ u(x,t) &= w(q)(x,t) - w(p)(x,t) \\ R(x,t) &= w(p)(x,t), \end{aligned} \tag{4.7}$$

then $u = u(x,t)$ satisfies the following homogeneous mixed problem

$$\begin{cases} u_{tt} + \Delta^2 u = q(x)u + f(x)R(x,t) & \text{in } Q \\ u(x,0) = u_t(x,0) = 0 & \text{in } \Omega \\ u(x,t) = \Delta u(x,t) = 0, & \text{on } \Sigma. \end{cases} \tag{4.8}$$

Assume $q \in L^\infty(\Omega)$ is given and fixed and $R = R(x,t)$ is a given function suitably chosen, and the source coefficient $f \in L^2(\Omega)$ is assumed to be unknown. This transforms the inverse problem of (4.1) into the following inverse source problem for (4.8): determine f from Neumann boundary measurements of u over the observed part Γ_1 of the boundary and over the time interval $[-T, T]$.

Converting our inverse problem into an inverse source problem yields the following theorems for the inverse source problem.

Theorem 4.1.3. *Assume (A.1') and (A.2'), and let* $T > 0$. *We also assume the following regularity conditions:*

$$q \in L^\infty(\Omega);\ \ R, R_t, R_{tt} \in W^{2,\infty}(Q). \tag{4.9}$$

Further, suppose we have the following positivity condition:

$$|R(x,0)| \geq r_0 > 0 \tag{4.10}$$

for some constant $r_0 > 0$ *and* $x \in \overline{\Omega}$. *Let* $u(f)$ *be the solution to* (4.8). *If*

$$\frac{\partial u(f)}{\partial \nu} = \frac{\partial \Delta u(f)}{\partial \nu} = 0,\ \ (x,t) \in \Gamma_1 \times [-T, T], \tag{4.11}$$

then we must have $f(x) = 0$ *a.e.* $x \in \Omega$.

Theorem 4.1.4. *Under the assumptions in Theorem 4.1.3, let* $u(f)$ *denote the solutions of* (4.8) *with function* R. *Then there exists* $C > 0$ *that depends on* Ω, T, Γ_1, q, w_0, w_1, h_1, *and* h_2 *such that*

$$\|f\|^2_{L^2(\Omega)} \le C \left\| \frac{\partial u_{tt}(f)}{\partial \nu} \right\|^2_{L^2(\Sigma_1)} + \left\| \frac{\partial \Delta u_t(f)}{\partial \nu} \right\|^2_{L^2(\Sigma_1)} \tag{4.12}$$

for $f \in H^3(\Omega) \cap H^1_0(\Omega)$.

Since Theorems 4.1.1 and 4.1.2 are immediate from Theorems 4.1.3 and 4.1.4, we first prove the theorems for the inverse source problem and then prove the theorems to the original inverse problem.

4.2 Uniqueness of Inverse Source Problem: Proof of Theorem 4.1.3

In this section, we show uniqueness for the inverse source problem of (4.8). As with the proof of Theorem 3.1.3, we divide the proof into various steps to make the proof clear. For the remainder of this chapter, C will denote a generic constant that may depend on Ω, T, β, q, w, u, and R, but remains independent of the parameter τ.

Step 1: We return to (4.8) under the assumptions given in the theorem statement of Theorem 4.1.3. Adding our measurement on the boundary gives us the u-system below

$$\begin{cases} u_{tt} + \Delta^2 u = q(x)u + f(x)R(x,t) & \text{in } Q \\ u(x,0) = u_t(x,0) = 0 & \text{in } \Omega \\ u(x,t) = \Delta u(x,t) = 0, & \text{on } \Sigma \\ \left.\dfrac{\partial u}{\partial \nu}\right|_{\Sigma_1} = \left.\dfrac{\partial \Delta u}{\partial \nu}\right|_{\Sigma_1} = 0, & \text{on } \Sigma_1. \end{cases} \tag{4.13}$$

Because of regularity assumptions (4.9), we can apply the Carleman estimate (2.31) for the solution

(4.13) and get the following for sufficiently large τ, after dropping unnecessary terms:

$$\tau^3 \int_Q e^{2\tau\varphi} \left[|u|^2 + |u_t|^2 + |\Delta u|^2\right] dQ + \tau \int_Q e^{2\tau\varphi} \left[|\nabla u|^2 + |\nabla u_t|^2 + |\nabla \Delta u|^2\right] dQ \le C \int_Q e^{2\tau\varphi} |fR|^2 \, dQ \tag{4.14}$$

Here we dropped the unnecessary terms in the Carleman estimate (2.31). We also dropped the boundary terms $BT_1^*(u)$ since the homogeneous boundary data

$$u|_\Sigma = \Delta u|_\Sigma = \left.\frac{\partial u}{\partial \nu}\right|_{\Sigma_1} = \left.\frac{\partial \Delta u}{\partial \nu}\right|_{\Sigma_1} = 0 \tag{4.15}$$

implies $BT_1^*(u) \le 0$ (see Remark 2.2.4). Also, since $R \in W^{2,\infty}(Q)$, we have that $fR \in L^2(Q)$. Thus, we have the following:

$$|fR|^2 \le \tilde{C}|f|^2$$

where $\tilde{C} = \|R\|_{L^\infty(Q)}$. Apply this to (4.14) yields

$$\tau^3 \int_Q e^{2\tau\varphi} \left[|u|^2 + |u_t|^2 + |\Delta u|^2\right] dQ + \tau \int_Q e^{2\tau\varphi} \left[|\nabla u|^2 + |\nabla u_t|^2 + |\nabla \Delta u|^2\right] dQ \le C \int_Q e^{2\tau\varphi} |f|^2 \, dQ, \tag{4.16}$$

where we combined $\tilde{C}$ with the generic constant C originally in front of the integral on the RHS of (4.14).

Step 2: We now differentiate the $u-$system (4.13) with respect to time to get the following u_t-system:

$$\begin{cases} (u_t)_{tt} + \Delta^2(u_t) = q(x)(u_t) + f(x)R_t(x,t) & \text{in } Q \\ (u_t)(x,0) = 0; \ (u_t)_t(x,0) = f(x)R(x,0) & \text{in } \Omega \\ (u_t)(x,t) = \Delta(u_t)(x,t) = 0, & \text{on } \Sigma \\ \left.\dfrac{\partial(u_t)}{\partial \nu}\right|_{\Sigma_1} = \left.\dfrac{\partial \Delta(u_t)}{\partial \nu}\right|_{\Sigma_1} = 0, & \text{on } \Sigma_1. \end{cases} \tag{4.17}$$

Notice that $f(x)R_t(x,t) \in L^2(Q)$ since $R_t \in W^{2,\infty}(Q)$, so we have

$$|f(x)R_t(x,t)|^2 \le \tilde{C'}|f(x)|^2$$

where $\tilde{C'} = \|R_t\|_{L^\infty(Q)}$. We can again apply the Carleman estimate (2.31) for solution u_t in (4.17) to get the following for sufficiently large τ:

$$\tau^3 \int_Q e^{2\tau\varphi} \left[|u_t|^2 + |u_{tt}|^2 + |\Delta u_t|^2\right] dQ + \tau \int_Q e^{2\tau\varphi} \left[|\nabla u_t|^2 + |\nabla u_{tt}|^2 + |\nabla \Delta u_t|^2\right] dQ \leq C \int_Q e^{2\tau\varphi} |f|^2 \, dQ \tag{4.18}$$

Since the boundary terms and boundary data are zero, $BT^*(u_t) \leq 0$. Again, we absorb $\tilde{C}'$ with the C in the RHS of (4.18).

Step 3: Repeating this process, we differentiate (4.17) with respect to time to get the following u_{tt}-system:

$$\begin{cases} (u_{tt})_{tt} + \Delta^2(u_{tt}) = q((u_{tt}) + f(x)R_{tt}(x,t) & \text{in } Q \\ (u_{tt})(x,0) = f(x)R(x,0);\ (u_{tt})_t(x,0) = f(x)R_t(x,0) & \text{in } \Omega \\ (u_{tt})(x,t) = \Delta(u_{tt})(x,t) = 0, & \text{on } \Sigma \\ \left.\dfrac{\partial(u_{tt})}{\partial\nu}\right|_{\Sigma_1} = \left.\dfrac{\partial\Delta(u_{tt})}{\partial\nu}\right|_{\Sigma_1} = 0, & \text{on } \Sigma_1. \end{cases} \tag{4.19}$$

As before, we can apply the Carleman estimate (2.31) for solution u_{tt} and get the following for sufficiently large τ:

$$\tau^3 \int_Q e^{2\tau\varphi} \left[|u_{tt}|^2 + |u_{ttt}|^2 + |\Delta u_{tt}|^2\right] dQ + \tau \int_Q e^{2\tau\varphi} \left[|\nabla u_{tt}|^2 + |\nabla u_{ttt}|^2 + |\nabla \Delta u_{tt}|^2\right] dQ \leq C \int_Q e^{2\tau\varphi} |f|^2 \, dQ \tag{4.20}$$

Adding inequalities (4.16), (4.18), and (4.20) yields the overall estimate:

$$\begin{aligned} &\tau^3 \int_Q e^{2\tau\varphi} \left[|u|^2 + |u_t|^2 + |u_{tt}|^2 + |u_{ttt}|^2 + |\Delta u|^2 + |\Delta u_t|^2 + |\Delta u_{tt}|^2\right] dQ \\ &+ \tau \int_Q e^{2\tau\varphi} \left[|\nabla u|^2 + |\nabla u_t|^2 + |\nabla u_{tt}|^2 + |\nabla u_{ttt}|^2 + |\nabla \Delta u|^2 + |\nabla \Delta u_t|^2 + |\nabla \Delta u_{tt}|^2\right] dQ \\ &\leq C \int_Q e^{2\tau\varphi} |f|^2 \, dQ \end{aligned} \tag{4.21}$$

Step 4: We now analyze the integral term on the right-hand side of (4.21). We return to the $u-$system (4.13) and evaluate u at the initial time $t = 0$. Doing this gives us the following after applying the positivity assumption (4.10):

$$|u_{tt}(x,0)+\underbrace{\Delta^2 u(x,0)}_{=0}|=|\underbrace{q(x)u(x,0)}_{=0}+f(x)R(x,0)|=|f(x)|\;|R(x,0)|\geq r_0|f(x)| \tag{4.22}$$

for $r_0>0$. Thus, we readily obtain the following estimate for f:

$$|f(x)|\leq \frac{1}{r_0}|u_{tt}(x,0)|,\;\; x\in\overline{\Omega}. \tag{4.23}$$

Step 5: Using (4.23) and properties of the weight function φ defined in (1.17), we obtain the following:

$$\begin{aligned}
\int_Q e^{2\tau\varphi}|f|^2\,dQ &\leq \frac{1}{r_0^2}\int_Q e^{2\tau\varphi}|u_{tt}(x,0)|^2\,dQ \\
&\leq \frac{1}{r_0^2}\int_Q e^{2\tau\varphi(x,0)}|u_{tt}(x,0)|^2\,dQ \\
&= \frac{2T}{r_0^2}\int_\Omega e^{2\tau\varphi(x,0)}|u_{tt}(x,0)|^2\,dx \\
&= \frac{2T}{r_0^2}\int_\Omega\int_{-T}^0 \frac{d}{ds}\left(e^{2\tau\varphi(x,s)}|u_{tt}(x,s)|^2\right)\,dtdx+\frac{2T}{r_0^2}\int_\Omega e^{2\tau\varphi(x,-T)}|u_{tt}(x,-T)|^2\,dx \\
&= \tau\frac{8\beta T}{r_0^2}\int_\Omega\int_{-T}^0(-s)e^{2\tau\varphi(x,s)}|u_{tt}(x,s)|^2\,dxdt+\frac{4T}{r_0^2}\int_\Omega\int_{-T}^0 e^{2\tau\varphi(x,s)}(|u_{tt}(x,s)|\;|u_{ttt}(x,s)|)\,dsdx \\
&+\frac{2T}{r_0^2}\int_\Omega e^{2\tau\varphi(x,-T)}|u_{tt}(x,-T)|^2\,dx \\
&\leq \tau\frac{8\beta T^2}{r_0^2}\int_\Omega\int_{-T}^0 e^{2\tau\varphi(x,s)}|u_{tt}(x,s)|^2\,dsdx+\frac{4T}{r_0^2}\int_\Omega\int_{-T}^0 e^{2\tau\varphi(x,s)}(|u_{tt}(x,s)|\;|u_{ttt}(x,s)|)\,dsdx \\
&+\frac{2T}{r_0^2}\int_\Omega|u_{tt}(x,-T)|^2\,dx \qquad (4.24)\\
&\leq \tau\frac{8\beta T^2}{r_0^2}\int_\Omega\int_{-T}^0 e^{2\tau\varphi(x,s)}|u_{tt}(x,s)|^2\,dsdx+\frac{2T}{r_0^2}\int_\Omega\int_{-T}^0 e^{2\tau\varphi(x,s)}\left[|u_{tt}(x,s)|^2+|u_{ttt}(x,s)|^2\right]\,dsdx \\
&+\frac{2T}{r_0^2}\int_\Omega|u_{tt}(x,-T)|^2\,dx \\
&\leq C\left((\tau+1)\int_Q e^{2\tau\varphi}|u_{tt}|^2\,dQ+\int_Q e^{2\tau\varphi}|u_{ttt}|^2\,dQ+\int_\Omega|u_{tt}(x,-T)|^2\,dx\right) \qquad (4.25)
\end{aligned}$$

where C is a constant that depends on β, r_0, and T but independent of τ. We obtain inequality

(4.24) by recalling $\varphi(x,-T) \leq -\delta$ from (1.20), which makes $e^{2\tau\varphi(x,-T)} \leq 1$.

Step 6: We substitute (4.25) for the integral term of (4.21) and obtain the following:

$$\begin{aligned}
&\tau^3 \int_Q e^{2\tau\varphi} \left[|u|^2 + |u_t|^2 + |u_{tt}|^2 + |u_{ttt}|^2 + |\Delta u|^2 + |\Delta u_t|^2 + |\Delta u_{tt}|^2\right] dQ \\
&+ \tau \int_Q e^{2\tau\varphi} \left[|\nabla u|^2 + |\nabla u_t|^2 + |\nabla u_{tt}|^2 + |\nabla u_{ttt}|^2 + |\nabla \Delta u|^2 + |\nabla \Delta u_t|^2 + |\nabla \Delta u_{tt}|^2\right] dQ \qquad (4.26) \\
&\leq C \left((\tau + 1) \int_Q e^{2\tau\varphi} |u_{tt}|^2 \, dQ + \int_Q e^{2\tau\varphi} |u_{ttt}|^2 \, dQ + k_u \right)
\end{aligned}$$

where we set

$$k_u = \int_\Omega |u_{tt}(x,-T)|^2 \, dx. \qquad (4.27)$$

The RHS term $C\,(\tau+1) \int_Q e^{2\tau\varphi} |u_{tt}|^2$ of (4.26) absorbed by the LHS term $\tau^3 \int_Q e^{2\tau\varphi} |u_{tt}|^2$ when τ is large enough. Likewise, the RHS term $C \int_Q e^{2\tau\varphi} |u_{ttt}|^2$ can be absorbed by the LHS term $\tau^3 \int_Q e^{2\tau\varphi} |u_{ttt}|^2$ when τ is large enough. This simplifies (4.26) into the following:

$$\begin{aligned}
&\tau^3 \int_Q e^{2\tau\varphi} \left[|u|^2 + |u_t|^2 + |u_{tt}|^2 + |u_{ttt}|^2 + |\Delta u|^2 + |\Delta u_t|^2 + |\Delta u_{tt}|^2\right] dQ \\
&+ \tau \int_Q e^{2\tau\varphi} \left[|\nabla u|^2 + |\nabla u_t|^2 + |\nabla u_{tt}|^2 + |\nabla u_{ttt}|^2 + |\nabla \Delta u|^2 + |\nabla \Delta u_t|^2 + |\nabla \Delta u_{tt}|^2\right] dQ \qquad (4.28) \\
&\leq C k_u
\end{aligned}$$

Step 7: We can simplify inequality (4.28) into the following inequality by dropping the second positive term on the LHS of (4.28):

$$\tau^3 \int_Q e^{2\tau\varphi} \left[|u|^2 + |u_t|^2 + |u_{tt}|^2 + |u_{ttt}|^2 + |\Delta u|^2 + |\Delta u_t|^2 + |\Delta u_{tt}|^2\right] dQ \leq C k_u \qquad (4.29)$$

where the constants C and k_u are independent of τ. Since τ is an arbitrary large parameter, letting $\tau \to \infty$ shows that $u = 0$ on Q since k_u does not depend on τ. Returning to (4.23) implies

$$|f(x)| \leq \frac{1}{r_0} |u_{tt}(x,0)| = 0 \implies f(x) = 0 \text{ a.e } x \in \overline{\Omega}.$$

as desired. This completes the proof of Theorem 4.1.3.

4.3 Uniqueness of Inverse Problem: Proof of Theorem 4.1.1

The proof of Theorem 4.1.1 reduces to Theorem 4.1.3 after slight transformations. For any $(x,t) \in Q$, set

$$u(x,t) = w(q)(x,t) - w(p)(x,t);\; R(x,t) = w(p)(x,t);\; f(x) = q(x) - p(x). \tag{4.30}$$

After the transformation (4.30), u solves (4.8). Moreover, by the regularity assumption on the initial conditions (4.2), we readily have (4.9).

It is obvious that the positivity assumption (4.3) implies the positivity assumption (4.10). Moreover, since $q, p \in L^\infty(\Omega)$, we have that $f(x) \in L^\infty(\Omega) \subset L^2(\Omega)$, and by assumption (4.2) on the I.C. $\{w_0, w_1\}$ and B.C $\{h_1, h_2\}$, we have the regularity properties

$$R, R_t, R_{tt} \in W^{2,\infty}(Q). \tag{4.31}$$

Furthermore, assumption (4.4) that

$$\begin{aligned} \frac{\partial w(q)}{\partial \nu} &= \frac{\partial w(p)}{\partial \nu} \\ \frac{\partial \Delta w(q)}{\partial \nu} &= \frac{\partial \Delta w(p)}{\partial \nu}, \end{aligned} \tag{4.32}$$

implies via (4.30) that we have

$$\begin{aligned} \frac{\partial u(f)}{\partial \nu} &= 0 \\ \frac{\partial \Delta u(f)}{\partial \nu} &= 0, \end{aligned} \tag{4.33}$$

for any $(x,t) \in \Sigma_1 = \Gamma_1 \times [-T, T]$. Therefore, Theorem 4.1.3 applies, and we conclude $f(x) = q(x) - p(x) = 0$, or $q(x) = p(x)$ a.e. $x \in \Omega$. This finishes the proof of Theorem 4.1.1.

4.4 Stability of Inverse Source Problem: Proof of Theorem 4.1.4

We now prove the stability of the inverse source problem (4.8). The proof has a similar strategy to the proof of Theorem 3.1.4 where we perform a compact-uniqueness argument.

Step 1: Let $u(f)$ denote the solution to (4.13) with initial data

$$q \in L^\infty(\Omega),\ \ R, R_t, R_{tt} \in L^\infty(Q),\ \ |R(x,0)| \geq r_0 > 0$$

with unknown term $f \in H^3(\Omega) \cap H_0^1(\Omega)$. We return to (4.23). We integrate over Ω to obtain the following inequality

$$\|f\|^2_{L^2(\Omega)} \leq C\|u_{tt}(x,0)\|^2_{L^2(\Omega)}. \tag{4.34}$$

This suggest that we return to the u_t-system, which we rewrite below for convenience:

$$\begin{cases} (u_t)_{tt} + \Delta^2(u_t) = q(x)(u_t) + f(x)R_t(x,t) & \text{in } Q \\ (u_t)(x,0) = 0;\ \ (u_t)_t(x,0) = f(x)R(x,0) & \text{in } \Omega \\ u_t(x,t) = \Delta u_t(x,t) = 0, & \text{on } \Sigma \end{cases} \tag{4.35}$$

Step 2: By the linearity of the system, we split it into $u_t = y + z$, where $y = y(x,t)$ has homogeneous forcing term

$$\begin{cases} y_{tt} + \Delta^2 y = q(x)y & \text{in } Q \\ y(x,0) = u_t(x,0) & \text{in } \Omega \\ y_t(x,0) = f(x)R(x,0) & \text{in } \Omega \\ y = \Delta y = 0 & \text{on } \Sigma \end{cases} \tag{4.36}$$

and $z = z(x,t)$ has homogeneous initial conditions

$$\begin{cases} z_{tt} + \Delta^2 z = q(x)z + f(x)R_t(x,t) & \text{in } Q \\ z(x,0) = z_t(x,0) = 0 & \text{in } \Omega \\ z = \Delta z = 0 & \text{on } \Sigma. \end{cases} \tag{4.37}$$

Step 3: For the y-system, since $f \in H^3(\Omega) \cap H_0^1(\Omega)$ and by the regularity assumptions on R, we have that $f(x)R(x,0) \in H^3(\Omega) \cap H_0^1(\Omega)$. Thus, we may apply the observability inequality (2.40) to the y-system (4.36) after applying Poincaré's inequality:

$$\|u_{tt}(x,0)\|^2_{L^2(\Omega)} = \|y_t(x,0)\|^2_{L^2(\Omega)} \leq \|\nabla y_t(x,0)\|^2_{L^2(\Omega)} + \|\nabla \Delta y(x,0)\|^2_{L^2(\Omega)} \leq \left\|\frac{\partial y_t}{\partial \nu}\right\|^2_{L^2(\Sigma_1)} + \left\|\frac{\partial \Delta y}{\partial \nu}\right\|^2_{L^2(\Sigma_1)} \tag{4.38}$$

Using the observability inequality and Poincaré's inequality as in (4.38) yield the following inequalities:

$$\begin{aligned} \|f\|^2_{L^2(\Omega)} &\leq C\|u_{tt}(x,0)\|^2_{L^2(\Omega)} \\ &\leq C\left(\|y_t(x,0)\|^2_{L^2(\Omega)} + \|\Delta y(x,0)\|^2_{L^2(\Omega)}\right) \\ &\leq C\left(\|\nabla y_t(x,0)\|^2_{L^2(\Omega)} + \|\nabla \Delta y(x,0)\|^2_{L^2(\Omega)}\right) \\ &\leq C\left(\left\|\frac{\partial y_t}{\partial \nu}\right\|^2_{L^2(\Sigma_1)} + \left\|\frac{\partial \Delta y}{\partial \nu}\right\|^2_{L^2(\Sigma_1)}\right) \\ &= C\left(\left\|\frac{\partial u_{tt}}{\partial \nu} - \frac{\partial z_t}{\partial \nu}\right\|^2_{L^2(\Sigma_1)} + \left\|\frac{\partial \Delta u_t}{\partial \nu} - \frac{\partial \Delta z}{\partial \nu}\right\|^2_{L^2(\Sigma_1)}\right) \\ &\leq C\left(\left\|\frac{\partial \Delta u_t}{\partial \nu}\right\|^2_{L^2(\Sigma_1)} + \left\|\frac{\partial u_{tt}}{\partial \nu}\right\|^2_{L^2(\Sigma_1)} + \left\|\frac{\partial \Delta z}{\partial \nu}\right\|^2_{L^2(\Sigma_1)} + \left\|\frac{\partial z_t}{\partial \nu}\right\|^2_{L^2(\Sigma_1)}\right) \end{aligned} \tag{4.39}$$

Step 4: Notice that (4.39) our desired stability estimate polluted by the $z-$terms. We will now use a compactness-uniqueness argument to absorb the polluted terms. We first proceed by proving the polluted $z-$ terms are compact operators.

Lemma 4.4.1. *The two operators defined by*

$$\begin{aligned} G_1 : L^2(\Omega) &\to L^2(\Sigma_1) \\ f &\mapsto \left.\frac{\partial z_t}{\partial \nu}\right|_{\Sigma_1} \end{aligned} \tag{4.40}$$

$$\begin{aligned} G_2 : L^2(\Omega) &\to L^2(\Sigma_1) \\ f &\mapsto \left.\frac{\partial \Delta z}{\partial \nu}\right|_{\Sigma_1} \end{aligned} \tag{4.41}$$

are compact operators.

Proof. Assumptions $f \in H^3(\Omega) \cap H_0^1(\Omega)$ and $R \in L^\infty(Q)$ imply that $f(x)R_t(x,t) \in H^3(Q) \cap H^1(Q)$, so

$$f(x)R(x,t) \in H^3(Q) \cap H^1(Q) \implies \frac{\partial z}{\partial \nu} \in H^1(\Sigma_1) \text{ continuously.}$$

Thus, $f \mapsto G_1(f) \in H^1(\Sigma_1)$ is continuous, which implies that G_1 is compact. The proof showing that G_2 is compact is similar. $\square$

Step 5: We will now show that we may drop the z-terms in (4.39) to obtain our desired stability estimate. Suppose, to the contrary, that the stability estimate does not hold. Then there exists a sequence $\{f_k\}$, $f_k \in L^2(\Omega)$ for all $k \in \mathbb{N}$ such that

$$\begin{aligned} \|f_k\|^2_{L^2(\Omega)} &= 1 \\ \lim_{k\to\infty} \left(\left\| \frac{\partial \Delta u_t(f_k)}{\partial \nu} \right\|^2_{L^2(\Sigma_1)} + \left\| \frac{\partial u_{tt}(f_k)}{\partial \nu} \right\|^2_{L^2(\Sigma_1)} \right) &= 0. \end{aligned} \tag{4.42}$$

Then there exists a subsequence, still denoted as $\{f_k\}$ such that

$$f_k \rightharpoonup f_0 \text{ weakly to some } f_0 \in L^2(\Omega). \tag{4.43}$$

Since G_1 and G_2 are compact operators by Lemma 4.4.1, we have the following strong convergence

([3, Theorem 3.2.3], [28, Theorem 8.1-7]):

$$\begin{aligned} &\lim_{k,l\to\infty} \|G_1(f_k) - G_1(f_l)\|^2_{L^2(\Sigma_1)} = 0 \\ &\lim_{k,l\to\infty} \|G_2(f_k) - G_2(f_l)\|^2_{L^2(\Sigma_1)} = 0 \end{aligned} \tag{4.44}$$

Step 6: Since the map $f \mapsto u(f)$ is linear, we have from (4.39) that

$$\begin{aligned} \|f_k - f_l\|^2_{L^2(\Omega} &\leq C\left(\left\|\frac{\partial u_{tt}(f_k)}{\partial\nu} - \frac{\partial u_{tt}(f_l)}{\partial\nu}\right\|^2_{L^2(\Sigma_1)}\right) + C\left(\left\|\frac{\partial \Delta u_t(f_k)}{\partial\nu} - \frac{\partial \Delta u_t(f_l)}{\partial\nu}\right\|^2_{L^2(\Sigma_1}\right) \\ &\quad + C\|G_1(f_k) - G_1(f_l)\|^2_{L^2(\Sigma_1)} + C\|G_2(f_k) - G_2(f_l)\|^2_{L^2(\Sigma_1} \\ &\leq C\left\|\frac{\partial u_{tt}(f_k)}{\partial\nu}\right\|^2_{L^2(\Sigma_1)} + C\left\|\frac{\partial u_{tt}(f_l)}{\partial\nu}\right\|^2_{L^2(\Sigma_1)} + C\left\|\frac{\partial \Delta u_t(f_k)}{\partial\nu}\right\|_{L^2\Sigma_1{}^2} + \left\|\frac{\partial \Delta u_t(f_l)}{\partial\nu}\right\|^2_{L^2(\Sigma_1} \\ &\quad + C\|G_1(f_k) - G_1(f_l)\|^2_{L^2(\Sigma_1)} + C\|G_2(f_k) - G_2(f_l)\|^2_{L^2(\Sigma_1}. \end{aligned}$$

Hence, by (4.42) and (4.44), letting $k, l \to \infty$ gives us the following:

$$\lim_{k,l\to\infty} \|f_k - f_l\|^2_{L^2(\Omega)} = 0.$$

Thus, $\{f_k\}$ is a Cauchy sequence in $L^2(\Omega)$. By the uniqueness of the limit, we must have

$$\lim_{k\to\infty} \|f_k - f_0\|_{L^2(\Omega)} = 0, \tag{4.45}$$

so we also have

$$\|f_0\|^2_{L^2(\Omega)} = 1. \tag{4.46}$$

Step 7: By the regularity assumption, we have that $f \mapsto G_1(f) \in L^2(\Sigma_1)$ is continuous. Hence,

$$\left\|\frac{\partial u_{tt}(f)}{\partial\nu}\right\|^2_{L^2(\Sigma)} \leq C\|f\|^2_{L^2(\Omega)}.$$

Since the map $f \mapsto u_{tt}(f)$ is linear, we have

$$\left\|\frac{\partial u_{tt}(f_k)}{\partial\nu} - \frac{\partial u_{tt}(f_0)}{\partial\nu}\right\|^2_{L^2(\Sigma)} \leq C\|f_k - f_0\|^2_{L^2(\Omega)}. \tag{4.47}$$

Thus, by (4.45), we have that

$$\lim_{k\to\infty}\left\|\frac{\partial u_{tt}(f_k)}{\partial\nu}-\frac{\partial u_{tt}(f_0)}{\partial\nu}\right\|_{L^2(\Sigma)}=0$$

and hence $\frac{\partial u_{tt}(f_0)}{\partial\nu}=0$ in $L^2(\Sigma_1)$. This means that $\frac{\partial u_t(f_0)}{\partial\nu}$ is constant with respect to $t\in[-T,T]$.

Step 8: Similarly, by the regularity assumption, we have that $f\mapsto G_2(f)\in L^2(\Sigma_1)$ is continuous. Hence,

$$\left\|\frac{\partial\Delta u_t(f)}{\partial\nu}\right\|^2_{L^2(\Sigma)}\le C\|f\|^2_{L^2(\Omega)}.$$

Since the map $f\mapsto\Delta u_t(f)$ is linear, we have

$$\left\|\frac{\partial\Delta u_t(f_k)}{\partial\nu}-\frac{\partial\Delta u_t(f_0)}{\partial\nu}\right\|^2_{L^2(\Sigma)}\le C\|f_k-f_0\|^2_{L^2(\Omega)}. \tag{4.48}$$

Thus, by (4.45), we have again that

$$\lim_{k\to\infty}\left\|\frac{\partial\Delta u_t(f_k)}{\partial\nu}-\frac{\partial\Delta u_t(f_0)}{\partial\nu}\right\|_{L^2(\Sigma)}=0$$

and hence $\frac{\partial\Delta u_t(f_0)}{\partial\nu}=0$ in $L^2(\Sigma_1)$, meaning that $\frac{\partial\Delta u(f_0)}{\partial\nu}$ is constant with respect to $t\in[-T,T]$.

Step 9: Now we show that $\frac{\partial u_t(f_0)}{\partial\nu}=\frac{\partial\Delta u(f_0)}{\partial\nu}=0$ on Σ_1. Consider the original u-system with the source term f replaced with f_k:

$$\begin{cases} u_{tt}+\Delta^2u=q(x)u+f_k(x)R(x,t) & \text{in } Q\\ u(x,0)=u_t(x,0)=0 & \text{in } \Omega\\ u(x,t)=\Delta u(x,t)=0, & \text{on } \Sigma. \end{cases} \tag{4.49}$$

By the regularity theory (2.49) and trace theory, we have

$$\begin{aligned} \|u_t(f_k)-u_t(f_0)\|_{C[0,T];H^3(\Omega)} &\le C\|f_k-f_0\|^2_{L^2(\Omega)}\\ \|\Delta u(f_k)-\Delta u(f_0)\|_{C[0,T];H^1(\Omega)} &\le C\|f_k-f_0\|^2_{L^2(\Omega)} \end{aligned} \tag{4.50}$$

Since $u_t(f_k)(x,0)=0$, by strong convergence of $\{f_k\}$ and by linearity, we must have $u_t(f_0)(x,0)=$

$\Delta u(f_0)(x,0) = 0$ in Ω and $u_t(f_0)|_\Sigma = \Delta u(f_0)|_\Sigma = 0$. Hence, we have that $\frac{\partial u_t(f)}{\partial \nu}(x,0) = \frac{\partial \Delta u(f_0)}{\partial \nu} = 0$ on Σ_1. However, since $\frac{\partial u_t(f)}{\partial \nu}$ and $\frac{\partial \Delta u(f_0)}{\partial \nu}$ are constant with respect to t, we must have that $\frac{\partial u_t(f_0)}{\partial \nu} = \frac{\partial \Delta u(f_0)}{\partial \nu} = 0$ on Σ_1. Also notice that since $u(f_k)(x,0) = 0$ in Ω and $f_k \to f_0$, we also have that $\frac{\partial u(f_0)}{\partial \nu} = 0$ on Σ_1 by a similar argument.

Step 10: Since we have $\frac{\partial u(f_0)}{\partial \nu} = \frac{\partial \Delta u(f_0)}{\partial \nu} = 0$ on Σ_1, $u(f_0)$ satisfies the following:

$$\begin{cases} u_{tt} + \Delta^2 u = q(x)u + f_0(x)R(x,t) & \text{in } Q \\ u(x,0) = u_t(x,0) = 0 & \text{in } \Omega \\ u(x,t) = \Delta u(x,t) = 0 & \text{on } \Sigma \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial \Delta u}{\partial \nu} = 0 & \text{on } \Sigma_1. \end{cases} \tag{4.51}$$

By the uniqueness result in Theorem 4.1.3, we must have that $f_0 = 0$, which contradicts (4.46). Therefore, we have that the z-terms in (4.39) must be dropped. This completes the proof of Theorem 4.1.4.

4.5 Stability of Inverse Problem: Proof of Theorem 4.1.2

The result is an immediate consequence of Theorem 4.1.4. By the regularity assumption (2.49) on the initial and boundary conditions $\{w_0, w_1, h_1, h_2\}$ (4.2) imply that the solutions must satisfy

$$\{w, w_t, w_{tt}\} \in C\left([-T,T]; H^{\gamma+3}(\Omega) \times H^{\gamma+1}(\Omega) \times H^{\gamma-1}(\Omega)\right).$$

As $\gamma > \frac{n}{2} + 3$, we have the embedding $H^{\gamma-1}(\Omega) \hookrightarrow W^{2,\infty}(\Omega)$, so the regularity assumption (4.2) implies the corresponding regularity assumption (4.9). Therefore, we obtain Theorem 4.1.2 from Theorem 4.1.4.

Chapter 5

Conclusions

To summarize our results, we have shown that we can solve the inverse problem of a second-order hyperbolic system (1.2) of recovering $n+3$ unknown coefficients using half of the measurements on the boundary. We also showed that we can solve the inverse problem of recovering an unknown coefficient q from the plate equation (1.5) on the lower-order term via two measurements on the boundary.

Below we leave a few remarks about each inverse problem that could lead to future research.

(1) For the second-order hyperbolic equations, one could also recover the $n+3$ unknown coefficients by choosing $n+3$ initial conditions $\{w_0, w_1\}$ and boundary condition h with their corresponding boundary measurements. The positivity assumption will become $|\det(W')| \geq r_0 > 0$, where W' is the following $(n+3) \times (n+3)$ matrix:

$$W' = \begin{bmatrix} w_0^{(1)}(x) & w_1^{(1)}(x) & \partial_{x_1} w_0^{(1)}(x) & \cdots & \partial_{x_n} w_0^{(1)}(x) & \Delta w_0^{(1)}(x) \\ w_0^{(2)}(x) & w_1^{(2)}(x) & \partial_{x_1} w_0^{(2)}(x) & \cdots & \partial_{x_n} w_0^{(2)}(x) & \Delta w_0^{(2)}(x) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ w_0^{(n+3)}(x) & w_1^{(n+3)}(x) & \partial_{x_1} w_0^{(n+3)}(x) & \cdots & \partial_{x_n} w_0^{(n+3)}(x) & \Delta w_0^{(n+3)}(x) \end{bmatrix} \tag{5.1}$$

The main difference in proving this inverse problem compared to using only half of the measurements is that we only need to differentiate the $u-$equation (3.12) twice with respect to t instead of three times. Since we reduce the number of times we need to differentiate, we also get a better stability estimate, which is provided below:

$$\|c^2-\tilde{c}^2\|^2_{L^2(\Omega)}+\|q_1-p_1\|^2_{L^2(\Omega)}+\|q_0-p_0\|^2_{L^2(\Omega)}+\|\mathbf{q}-\mathbf{p}\|^2_{\mathbf{L}^2(\Omega)}$$
$$\leq \quad C\sum_{i=1}^{m+2}\left\|\frac{\partial w_t^{(i)}(c,q_1,q_0,\mathbf{q})}{\partial\nu}-\frac{\partial w_t^{(i)}(\tilde{c},p_1,p_0,\mathbf{p})}{\partial\nu}\right\|^2_{L^2(\Sigma_1)}, \tag{5.2}$$

(2) Using a similar strategy to the proofs in Chapter 4, it is possible to recover more unknown coefficients of the plate equation on lower-order terms ($w_t, \Delta w, \Delta w_t, \nabla\Delta w$, etc.) under hinged boundary conditions. The first step to accomplish this is to adapt the Carleman estimate (2.31) for the lower-order terms with each of the desired coefficients. It is unknown whether this requires more measurements on the boundary to successfully solve this inverse problem.

(3) The inverse problem for the plate equation (1.5) was conducted with *hinged* boundary conditions. However, for the plate equation, one can also consider *clamped* boundary conditions:

$$\begin{cases} w_{tt}+\Delta^2 w+q(x)w=0; & \text{in } Q \\ w(x,0)=w_0,\ w_t(x,0)=w_1; & \text{in } \Omega \\ w=h_1,\ \dfrac{\partial w}{\partial\nu}=h_2; & \text{on } \Sigma \end{cases} \tag{5.3}$$

Currently, it is unknown how working with clamped boundary conditions will affect the inverse problem of recovering the unknown coefficient q. Based on the boundary term $BT_1^*(w)$ in (2.32), we suspect that our measurements on the boundary should be the Dirichlet and Neumann boundary traces of Δw, $\Delta w|_{\Sigma_1}$, $\left.\dfrac{\partial\Delta w}{\partial\nu}\right|_{\Sigma_1}$ to solve this inverse problem.

(4) For both inverse problems, we used the time interval $[-T,T]$ and used $t=0$ as the initial time. For the plate equation, we could also work on the time interval $[0,T]$ as well. The main difference is that we would have to incorporate an even extension from $\Omega\times[0,T]$ to $\Omega\times[-T,T]$ via the change of variables $t\to t-T$.

For the second-order hyperbolic equation (1.2), we cannot perform the even extension due to the damping coefficient term $q_1(x)w_t$. If we were not recovering the damping coefficient, then we can also perform the even extension mentioned above. However, even if we were not recovering the damping coefficient, the current time interval allows us to recover all coefficients in the second-order hyperbolic problem with fewer choices of initial conditions and fewer measurements on the boundary. This is because we can use both equations in (3.32) simultaneously instead of only one equation if we initially assume the time interval $[0, T]$.

(5) The inverse problem for second-order hyperbolic equation can also be set up by assuming Neumann boundary condition $\frac{\partial w}{\partial \nu}$ on Σ by making measurements of Dirichlet boundary traces of w on Σ_1. This requires a more demanding geometrical assumption on the unobserved portion of the boundary Γ_0 by assuming $\frac{\partial d}{\partial \nu} = \langle Dd, \nu \rangle = 0$ on Γ_0. In addition, more regularity theory for second-order hyperbolic equation with nonhomogeneous Neumann boundary condition will also need to be invoked, see [33, 35]. Besides that, the main ideas for solving the inverse problems remain the same.

Appendices

Appendix A

Carleman type Estimates for the Schrödinger Equation

In this appendix, we focus on the Carleman-type estimates for the Schrödinger equation in (2.26) and provide a proof for Theorem 2.2.2. We first establish an initial pointwise inequality and then restate the inequality after specializations are made.

A.1 Initial Pointwise Inequality

We first derive the following pointwise estimate for the Schrödinger equation $iw_t - \Delta w = f$.

Lemma A.1.1. *Let*

$$\begin{gathered} w(x,t) \in C^2(\mathbb{R}^n_x \times \mathbb{R}_t; \mathbb{C});\;\; \ell(x,t) \in C^3(\mathbb{R}^n_x \times \mathbb{R}_t, \mathbb{R}); \\ \Psi(x,t),\;\; \Phi(x,t) \in C^1(\mathbb{R}^n_x \times \mathbb{R}_t, \mathbb{R}) \end{gathered} \tag{A.1.1}$$

Further, let

$$\ell_{tx_j} \equiv 0;\;\; \theta(x,t) = e^{\ell(x,t)};\;\; v(x,t) = \theta(x,t) w(x,t). \tag{A.1.2}$$

Then we have the following pointwise inequality:

$$\theta^2\left(1+\frac{1}{\epsilon}\right)|iw_t-\Delta w|^2-\frac{\partial M}{\partial t}-div\,V$$
$$\geq\Bigg[-2(\Psi+\Delta\ell)(|\nabla\ell|^2-\Delta\ell)+4\sum_{j,k}\ell_{x_k}\ell_{x_j}\ell_{x_jx_k}-2\nabla\Phi\cdot\nabla\ell-2\nabla\ell\cdot\nabla\Delta\ell-(\Phi^2+\Psi^2)$$
$$-2\Phi\Delta\ell+\ell_{tt}-4\nabla\Psi\cdot\nabla\ell+2\nabla\ell\cdot\nabla(\Psi-\Delta\ell)-\frac{n}{\epsilon}\sum_{j,k}\ell^2_{x_jx_jx_k}-\epsilon(\Psi-\Delta\ell)^2-2|\Psi-\Delta\ell||\nabla\ell|^2\Bigg]|v|^2$$
$$+2\left\{\sum_j\nabla\ell_{x_j}(v_{x_j}\nabla\overline{v}+\overline{v}_{x_j}\nabla v)+(\Psi+\Delta\ell)|\nabla v|^2\right\}-2\theta^2\left((\Psi-\Delta\ell)+|\Psi-\Delta\ell|\right)|\nabla w|^2-\epsilon|\nabla v|^2 \tag{A.1.3}$$

where

$$M=-\theta^2\left(2\sum_j(\ell_{x_j}(\xi_{x_j}\eta-\xi\eta_{x_j}))-\ell_t|w|^2\right) \tag{A.1.4}$$

$$V_j=V_j(w)=2\theta^2\Bigg(-[2|\nabla\ell|^2-\Delta\ell+\Phi-\Psi]\ell_{x_j}|w|^2-[\ell_t(\xi_{x_j}\eta-\xi\eta_{x_j})-\ell_{x_j}(\xi_t\eta-\xi\eta_t)]$$
$$+\frac{1}{2}\left(2|\nabla\ell|^2+\Delta\ell\right)(w_{x_j}\overline{w}+\overline{w}_{x_j}w)-\left[\sum_k[\ell_{x_k}(w_{x_j}\overline{w}_{x_k}+w_{x_k}\overline{w}_{x_j})-\ell_{x_j}|\nabla w|^2]\right]\Bigg\}\Bigg) \tag{A.1.5}$$

Proof. The proof of Lemma A.1.1 is long, so we break the it into multiple steps to help make the presentation clear.

Step 1: Based on our definitions in (A.1.2), we immediately obtain the following identities:

$$\theta_t=\theta\ell_t;\ v_t=\theta_tw+\theta w_t\ (\text{or }\theta w_t=v_t-\ell_tv) \tag{A.1.6a}$$

$$\theta w_{x_jx_j}=v_{x_jx_j}-2\ell_{x_j}v_{x_j}+(\ell^2_{x_j}-\ell_{x_jx_j})v,\ j=1,\dots,n \tag{A.1.6b}$$

$$\theta\Delta w=\Delta v-2\nabla\ell\cdot\nabla v+(|\nabla\ell|^2-\Delta\ell)v \tag{A.1.6c}$$

Multiplying the principle part by the exponential weight, squaring, and by (A.1.6a)-(A.1.6c), we have

$$
\begin{aligned}
|\theta(iw_t - \Delta w)|^2 &= |i(v_t - \ell_t v) - \Delta v + 2\nabla\ell \cdot \nabla v - (|\nabla\ell|^2 - \Delta\ell)v + \Phi v - \Phi v + \Psi v - \Psi v|^2 \\
&= |(iv_t - \Delta v - (|\nabla\ell|^2 - \Delta\ell)v - \Psi v) - (-2\nabla\ell \cdot \nabla v + \Phi v + i\ell_t v) + (\Phi + \Psi)v|^2 \\
&= |I_1 - I_2 + I_3|^2 \\
&= |I_1|^2 + |I_2|^2 + |I_3|^2 - (I_2\overline{I_2} + \overline{I_1}I_2) - (I_3\overline{I_2} + \overline{I_3}I_2) + (I_3\overline{I_1} + \overline{I_3}I_1)
\end{aligned}
\tag{A.1.7}
$$

where

$$
\begin{aligned}
I_1 &\equiv iv_t - \Delta v - (|\nabla\ell|^2 - \Delta\ell)v - \Psi v \\
I_2 &\equiv -2\nabla\ell \cdot \nabla v + \Phi v + i\ell_t v \\
I_3 &\equiv (\Phi + \Psi)v
\end{aligned}
\tag{A.1.8}
$$

Remark A.1.2. The reason behind introducing Φ and Ψ will be shown in later sections once specializations are made.

After dropping the $|I_1|^2 + |I_2|^2$ term in (A.1.7), we have

$$
|\theta(iw_t - \Delta w)|^2 \geq |I_3|^2 - (I_1\overline{I_2} + \overline{I_1}I_2) - (I_3\overline{I_2} + \overline{I_3}I_2) + (I_3\overline{I_1} + \overline{I_3}I_1) \tag{A.1.9}
$$

Step 2: The goal now is to expand the cross-terms of (A.1.9). Let us start with the $\overline{I_1}I_3 + I_1\overline{I_3}$ cross-term. Expanding this cross-term yields

$$
\begin{aligned}
I_3\overline{I_1} + \overline{I_3}I_1 &= \overline{I_1}(\Phi + \Psi)v + I_1(\Phi + \Psi)\overline{v} \\
&= \Phi(I_1\overline{v} + \overline{I_1}v) + \Psi(I_1\overline{v} + \overline{I_1}v) \\
&= \Phi(I_1\overline{v} + \overline{I_1}v) + (-i\overline{v}_t - \Delta\overline{v} - (|\nabla\ell|^2 - \Delta\ell)\overline{v} - \Psi\overline{v})\Psi v + (iv_t - \Delta v - (|\nabla\ell|^2 - \Delta\ell)v - \Psi v)\Psi\overline{v} \\
&= \Phi(I_1\overline{v} + \overline{I_1}v) + i\Psi(\overline{v}v_t - \overline{v}_t v) - \Psi(v\Delta\overline{v} + \overline{v}\Delta v) - 2\Psi(|\nabla\ell|^2 - \Delta\ell + \Psi)|v|^2.
\end{aligned}
\tag{A.1.10}
$$

Notice by direct computation we have the following identity:

$$\begin{aligned}\sum_j [\Psi(v_{x_j}\overline{v}+\overline{v}_{x_j}v)]_{x_j} &= \sum_j \Psi(v_{x_jx_j}\overline{v}+v_{x_j}\overline{v}_{x_j}+\overline{v}_{x_jx_j}v+\overline{v}_{x_j}v_{x_j})+\sum_j \Psi_{x_j}(v_{x_j}\overline{v}+\overline{v}_{x_j}v)\\ &= \Psi[\overline{v}\Delta v+v\Delta\overline{v}]+2\Psi|\nabla v|^2+\nabla\Psi\cdot[\nabla v\overline{v}+\nabla\overline{v}v].\end{aligned}$$

Apply this to the third term of (A.1.10) yields

$$\begin{aligned}I_3\overline{I_1}+\overline{I_3}I_1 = \Phi(I_1\overline{v}+\overline{I_1}v)+i\Psi(\overline{v}v_t-\overline{v}_tv)-\sum_j[\Psi(v_{x_j}\overline{v}+\overline{v}_{x_j}v)]_{x_j}\\ +2\Psi|\nabla v|^2+\nabla\Psi\cdot[\nabla v\overline{v}+\nabla\overline{v}v]-2\Psi(|\nabla\ell|^2-\Delta\ell+\Psi)|v|^2.\end{aligned} \tag{A.1.11}$$

Step 3: We now expand the $I_2\overline{I}_3+\overline{I}_2I_3$ cross-term. The work is provided below:

$$\begin{aligned}I_2\overline{I_3}+\overline{I_2}I_3 &= (-2\nabla\ell\cdot\nabla v+\Phi v+i\ell_tv)(\Phi+\Psi)\overline{v}+(-2\nabla\ell\cdot\nabla\overline{v}+\Phi\overline{v}-i\ell_t\overline{v})(\Phi+\Psi)v\\ &= -2(\Phi+\Psi)[\nabla\ell\cdot\nabla v\overline{v}+\nabla\ell\cdot\nabla\overline{v}v]+2\Phi(\Phi+\Psi)|v|^2\\ &= -2(\Phi+\Psi)\nabla\ell\cdot\nabla(|v|^2)+2\Phi(\Phi+\Psi)|v|^2,\end{aligned} \tag{A.1.12}$$

after canceling the $i\ell_t(\Phi+\Psi)|v|^2$ terms. By direct computation, the next identity is easily verified:

$$-2\sum_j[(\Phi+\Psi)\ell_{x_j}|v|^2]_{x_j} = -2\nabla(\Phi+\Psi)\cdot\nabla\ell|v|^2-2(\Phi+\Psi)\Delta\ell|v|^2-2(\Phi+\Psi)\nabla\ell\cdot\nabla(|v|^2). \tag{A.1.13}$$

Substitute (A.1.13) to the first term on the right-hand side of (A.1.12) yields

$$\begin{aligned}I_2\overline{I_3}+\overline{I_2}I_3 &= -2\sum_j[(\Phi+\Psi)\ell_{x_j}|v|^2]_{x_j}+2\nabla(\Phi+\Psi)\cdot\nabla\ell|v|^2+2(\Phi+\Psi)\Delta\ell|v|^2+2\Phi(\Phi+\Psi)|v|^2\\ &= -2\sum_j[(\Phi+\Psi)\ell_{x_j}|v|^2]_{x_j}+2(\Phi+\Psi)(\Phi+\Delta\ell)|v|^2+2[\nabla(\Phi+\Psi)\cdot\nabla\ell]|v|^2.\end{aligned} \tag{A.1.14}$$

Step 4: We now proceed to expand the $I_1\overline{I_2}+\overline{I_1}I_2$ cross-terms from (A.1.9). We claim that this cross-term has the following expansion:

$$I_1\overline{I_2}+\overline{I_1}I_2=\Phi(I_1\overline{v}+\overline{I_1}v)-i\left\{2\left(\nabla\ell\cdot\nabla\overline{v}v\right)_t-2\sum_j\left(\ell_{x_j}\overline{v}_tv\right)_{x_j}+2\Delta\ell(v\overline{v}_t)+\sum_j[\ell_t(v\overline{v}_{x_j}-\overline{v}v_{x_j})]_{x_j}\right\}$$

$$+(\ell_t|v|^2)_t-\ell_{tt}|v|^2+(C)$$

$$(C)=(B)-2\sum_{j,k}[\ell_{x_kx_j}(v_{x_j}\overline{v}_{x_k}-\overline{v}_{x_j}v_{x_k})]+2\Delta\ell|\nabla v|^2$$

$$-2\sum_j\left(\sum_k[2\ell_{x_k}\ell_{x_jx_k}-\ell_{x_kx_kx_j}]-\Psi_{x_j}\right)\ell_{x_j}|v|^2-2[|\nabla\ell|^2-\Delta\ell+\Psi]\Delta\ell|v|^2$$

$$(B)=2\sum_j\left\{\sum_k[\ell_{x_k}(v_{x_j}\overline{v}_{x_k}+\overline{v}_{x_j}v_{x_k})-\ell_{x_j}|v_{x_k}|^2]+[|\nabla\ell|^2-\Delta\ell+\Psi]\ell_{x_j}|v|^2\right\}_{x_j} \tag{A.1.15}$$

Proof of (A.1.15): Establishing (A.1.15) requires more work than the other cross-terms of (A.1.9), so we will break the proof into multiple steps.

(i) First, we obtain the following via direct computation:

$$I_1\overline{I_2}+\overline{I_1}I_2=I_1(-2\nabla\ell\cdot\nabla\overline{v}+\Phi\overline{v}-i\ell_t\overline{v})+\overline{I_1}(-2\nabla\ell\cdot\nabla v+\Phi v+i\ell_tv)$$

$$=\Phi(I_1\overline{v}+\overline{I_1}v)+(iv_t-\Delta v-(|\nabla\ell|^2-\Delta\ell)v-\Psi v)(-2\nabla\ell\cdot\nabla\overline{v}-i\ell_t\overline{v})$$

$$+(-i\overline{v}_t-\Delta\overline{v}-(|\nabla\ell|^2-\Delta\ell)\overline{v}-\Psi\overline{v})(-2\nabla\ell\cdot\nabla v+i\ell_tv)$$

$$=\Phi(I_1\overline{v}+\overline{I_1}v)-2iv_t\nabla\ell\cdot\nabla\overline{v}+iv_t(-i\ell_t\overline{v})+2i\overline{v}_t\nabla\ell\cdot\nabla v+(-i\overline{v}_t)(i\ell_tv)$$

$$+[-\Delta v-(|\nabla\ell|^2-\Delta\ell)v-\Psi v](-2\nabla\ell\cdot\nabla\overline{v})+[-\Delta v-(|\nabla\ell|^2-\Delta\ell)v-\Psi v](-i\ell_t\overline{v})$$

$$+[-\Delta\overline{v}-(|\nabla\ell|^2-\Delta\ell)\overline{v}-\Psi\overline{v}](-2\nabla\ell\cdot\nabla v)+[-\Delta\overline{v}-(|\nabla\ell|^2-\Delta\ell)\overline{v}-\Psi\overline{v}](i\ell_tv)$$

$$=\Phi(I_1\overline{v}+\overline{I_1}v)-2i\nabla\ell\cdot(\nabla\overline{v}v_t-\nabla v\overline{v}_t)+\ell_t\left[v_t\overline{v}+\overline{v}_tv\right]+2\nabla\ell\cdot[\Delta\overline{v}\nabla v+\Delta v\nabla\overline{v}]$$

$$+2\nabla\ell\cdot(\nabla\overline{v}v+\nabla v\overline{v})\left[|\nabla\ell|^2-\Delta\ell+\Psi\right]-i\ell_t[\Delta\overline{v}v-\Delta v\overline{v}] \tag{A.1.16}$$

after a cancellation of the $i\ell_t(|\nabla\ell|^2-\Delta\ell)|v|^2$ and $i\ell_t\Psi|v|^2$ terms.

(ii) Next, after adding and subtracting terms, we have the following identities:

$$\overline{v}_{x_j} v_t - v_{x_j} \overline{v}_t = (v\overline{v}_{x_j})_t - (v_t v)_{x_j} \tag{A.1.17a}$$

$$\nabla \ell \cdot [\nabla \overline{v} v_t - \nabla v \overline{v}_t] = \sum_j \ell_{x_j} \left[(v\overline{v}_{x_j})_t - (v_t v)_{x_j} \right] \tag{A.1.17b}$$

$$v_{x_k x_k} \overline{v}_{x_j} + \overline{v}_{x_k x_k} v_{x_j} = (v_{x_k} \overline{v}_{x_j} + \overline{v}_{x_k} v_{x_j})_{x_k} - \left(|v_{x_k}|^2\right)_{x_j} \tag{A.1.18a}$$

$$\nabla \ell \cdot [\nabla \overline{v} \Delta v + \nabla v \Delta \overline{v}] = \sum_{j,k} \ell_{x_j} \left[(v_{x_k} \overline{v}_{x_j} + \overline{v}_{x_k} v_{x_j})_{x_k} - \left(|v_{x_k}|^2\right)_{x_j} \right] \tag{A.1.18b}$$

$$v \overline{v}_{x_j x_j} - \overline{v} v_{x_j x_j} = (v \overline{v}_{x_j} - \overline{v} v_{x_j})_{x_j} \tag{A.1.19a}$$

$$v \Delta \overline{v} - \overline{v} \Delta v = \sum_j (v \overline{v}_{x_j} - \overline{v} v_{x_j})_{x_j} \tag{A.1.19b}$$

Substitute (A.1.17)-(A.1.19) to the second, fourth, and sixth terms of (A.1.16), respectively, to obtain

$$\begin{aligned} I_1 \overline{I_2} + \overline{I_1} I_2 = {} & \Phi(I_1 \overline{v} + \overline{I_1} v) - 2i \sum_j \left\{ \ell_{x_j} [(v\overline{v}_{x_j})_t - (v_t v)_{x_j}] \right\} + \ell_t (|v|^2)_t \\ & + 2 \sum_{j,k} \ell_{x_j} [(v_{x_k} \overline{v}_{x_j} + \overline{v}_{x_k} v_{x_j})_{x_k}] - 2 \nabla \ell \cdot \nabla (|\nabla v|^2) \\ & + 2 \nabla \ell \cdot \nabla (|v|^2) [|\nabla \ell|^2 - \Delta \ell + \Psi] - i \ell_t \sum_j (v \overline{v}_{x_j} - \overline{v} v_{x_j})_{x_j}. \end{aligned} \tag{A.1.20}$$

(iii) Since $\ell_{t x_j} = 0$ for $j = 1, \ldots, n$, we also have the following identities:

$$\left(\ell_{x_j} v \overline{v}_{x_j}\right)_t - \left(\ell_{x_j} \overline{v}_t v\right)_{x_j} + \ell_{x_j x_j} v \overline{v}_t = \ell_{x_j} \left[(v\overline{v}_{x_j})_t - (\overline{v}_t v)_{x_j} \right] \tag{A.1.21a}$$

$$\sum_j \left\{ \left(\ell_{x_j} v \overline{v}_{x_j}\right)_t - \left(\ell_{x_j} \overline{v}_t v\right)_{x_j} \right\} + \Delta \ell v v_t = \sum_j \left\{ \ell_{x_j} [(v\overline{v}_{x_j})_t - (\overline{v}_t v)_{x_j}] \right\} \tag{A.1.21b}$$

$$\ell_t (v \overline{v}_{x_j} - \overline{v} v_{x_j})_{x_j} = [\ell_t (v \overline{v}_{x_j} - \overline{v} v_{x_j})]_{x_j} \tag{A.1.22a}$$

$$\sum_j \ell_t (v \overline{v}_{x_j} - \overline{v} v_{x_j})_{x_j} = \sum_j [\ell_t (v \overline{v}_{x_j} - \overline{v} v_{x_j})]_{x_j} \tag{A.1.22b}$$

After substituting (A.1.21)-(A.1.22) to the second and last terms of the right-hand side of

(A.1.20), we obtain the following:

$$
\begin{aligned}
I_1\overline{I_2}+\overline{I_1}I_2 &= \Phi(I_1\overline{v}+\overline{I_1}v)-2i\sum_j\{\left(\ell_{x_j}v\overline{v}_{x_j}\right)_t-\left(\ell_{x_j}\overline{v}_t v\right)_{x_j}\}-2i\Delta\ell v v_t+\ell_t(|v|^2)_t\\
&\quad -i\sum_j[\ell_t(v\overline{v}_{x_j}-\overline{v}v_{x_j})]_{x_j}+(A)\\
(A) &= 2\sum_{j,k}\ell_{x_j}[(v_{x_k}\overline{v}_{x_j}+\overline{v}_{x_k}v_{x_j})_{x_k}]-2\nabla\ell\cdot\nabla(|\nabla v|^2)+2\nabla\ell\cdot\nabla(|v|^2)[|\nabla\ell|^2-\Delta\ell+\Psi]
\end{aligned}
\tag{A.1.23}
$$

Recalling the definition of (B) in (A.1.15), we can relate it to (A) in the following manner:

$$
\begin{aligned}
(B) &= 2\sum_j\left\{\sum_k[\ell_{x_k}(v_{x_j}\overline{v}_{x_k}+\overline{v}_{x_j}v_{x_k})-\ell_{x_j}|v_{x_k}|^2]+[|\nabla\ell|^2-\Delta\ell+\Psi]\ell_{x_j}|v|^2\right\}_{x_j}\\
&= (A)+2\sum_{j,k}[\ell_{x_kx_j}(v_{x_j}\overline{v}_{x_k}+\overline{v}_{x_j}v_{x_k})]-2\Delta\ell|\nabla v|^2\\
&\quad +2\sum_j[|\nabla\ell|^2-\Delta\ell+\Psi]_{x_j}\ell_{x_j}|v|^2+2[|\nabla\ell|^2-\Delta\ell+\Psi]\Delta\ell|v|^2
\end{aligned}
\tag{A.1.24}
$$

Substituting (A.1.24) to the right-hand side of (A.1.23) yields

$$
\begin{aligned}
I_1\overline{I_2}+\overline{I_1}I_2 &= \Phi(I_1\overline{v}+\overline{I_1}v)-2i\sum_j\{\left(\ell_{x_j}v\overline{v}_{x_j}\right)_t-\left(\ell_{x_j}\overline{v}_t v\right)_{x_j}\}-2i\Delta\ell v v_t+\ell_t(|v|^2)_t\\
&\quad -i\sum_j[\ell_t(v\overline{v}_{x_j}-\overline{v}v_{x_j})]_{x_j}+(B)-2\sum_{j,k}[\ell_{x_kx_j}(v_{x_j}\overline{v}_{x_k}+\overline{v}_{x_j}v_{x_k})]\\
&\quad +2\Delta\ell|\nabla v|^2-2\sum_j[|\nabla\ell|^2-\Delta\ell+\Psi]_{x_j}\ell_{x_j}|v|^2-2[|\nabla\ell|^2-\Delta\ell+\Psi]\Delta\ell|v|^2
\end{aligned}
\tag{A.1.25}
$$

Hence, using the following identities

$$
\ell_t(|v|^2)_t=(\ell_t|v|^2)_t-\ell_{tt}|v|^2;\;\; [|\nabla\ell|^2-\Delta-\Psi]_{x_j}=\sum_k[2\ell_{x_k}\ell_{x_jx_k}-\ell_{x_kx_kx_j}-\Psi_{x_j}]
$$

gives us (A.1.15). ◇

Step 5: Now that we have expanded the cross-terms of (A.1.9), we will establish the following:

$$
\begin{aligned}
|\theta(iw_t-\Delta w)|^2 &\geq |I_3|^2-(I_1\overline{I_2}+\overline{I_1}I_2)-(I_3\overline{I_2}+\overline{I_3}I_2)+(I_3\overline{I_1}+\overline{I_3}I_1)\\
&= X_1+X_2+X_3+X_4
\end{aligned}
\tag{A.1.26}
$$

where

$$X_1 = \Big[-2(\Psi + \Delta\ell)(|\nabla\ell|^2 - \Delta\ell) + 4\sum_{j,k} \ell_{x_k}\ell_{x_j}\ell_{x_jx_k} - 2\nabla\Phi\cdot\nabla\ell$$
$$- 2\sum_{j,k}\ell_{x_j}\ell_{x_jx_kx_k} - \Phi^2 - \Psi^2 - 2\Phi\Delta\ell + \ell_{tt} + 4\nabla\Psi\cdot\nabla\ell\Big]|v|^2 \tag{A.1.27}$$
$$+ 2\left\{\sum_{j,k}\ell_{x_jx_k}(v_{x_j}\overline{v}_{x_k} + \overline{v}_{x_j}v_{x_k}) + (\Psi + \Delta\ell)|\nabla v|^2\right\}$$

$$X_2 = -\frac{\partial}{\partial t}(\ell_t|v|^2) + 2\sum_j \frac{\partial}{\partial x_j}\Big\{[2\Psi - |\nabla\ell|^2 + \Delta\ell - \Phi]\ell_{x_j}|v|^2 - \frac{\Psi}{2}(v_{x_j}\overline{v} + \overline{v}_{x_j}v)$$
$$- \nabla\ell\cdot(\nabla\overline{v}v_j + \nabla v\overline{v}_{x_j}) + \ell_{x_j}|\nabla v|^2\Big\} \tag{A.1.28}$$

$$X_3 = \nabla\Psi\cdot(\nabla v\overline{v} + \nabla\overline{v}v) + i(\Psi - \Delta\ell)(v_t\overline{v} - \overline{v}_tv) \tag{A.1.29}$$

$$X_4 = i\sum_j\left\{\frac{\partial}{\partial t}[2\ell_{x_j}v\overline{v}_{x_j} + \ell_{x_jx_j}|v|^2] + \frac{\partial}{\partial x_j}[\ell_t(v\overline{v}_{x_j} - \overline{v}v_{x_j}) - 2\ell_{x_j}\overline{v}_tv]\right\} \tag{A.1.30}$$

Indeed, substituting (A.1.11), (A.1.14), and (A.1.15) into (A.1.9) yields

$$|\theta(iw_t - \Delta w)|^2 \geq |I_3|^2 - (I_1\overline{I_2} + \overline{I_1}I_2) - (I_3\overline{I_2} + \overline{I_3}I_2) + (I_3\overline{I_1} + \overline{I_3}I_1)$$
$$= |(\Phi + \Psi)v|^2 - \Bigg\{\Bigg\{\Phi(I_1\overline{v} + \overline{I_1}v) - i\Big\{2\left(\nabla\ell\cdot\nabla\overline{v}v\right)_t - 2\sum_j\left(\ell_{x_j}\overline{v}_tv\right)_{x_j} + 2\Delta\ell v\overline{v}_t$$
$$+ \sum_j[\ell_t(v\overline{v}_{x_j} - \overline{v}v_{x_j})]_{x_j}\Big\} + (\ell_t|v|^2)_t - \ell_{tt}|v|^2$$
$$+ 2\sum_j\left\{\sum_k[\ell_{x_k}(v_{x_j}\overline{v}_{x_k} + \overline{v}_{x_j}v_{x_k}) - \ell_{x_j}|v_{x_k}|^2] + [|\nabla\ell|^2 - \Delta\ell + \Psi]\ell_{x_j}|v|^2\right\}_{x_j}$$
$$- 2\sum_{j,k}[\ell_{x_kx_j}(v_{x_j}\overline{v}_{x_k} - \overline{v}_{x_j}v_{x_k})] + 2\Delta\ell|\nabla v|^2 - 2[|\nabla\ell|^2 - \Delta\ell + \Psi]\Delta\ell|v|^2$$
$$- 2\Big\{\sum_j\left(\sum_k[2\ell_{x_k}\ell_{x_jx_k} - \ell_{x_kx_kx_j}]\right)\ell_{x_j} - \nabla\Psi\cdot\nabla\ell\Big\}|v|^2\Bigg\}\Bigg\}$$
$$- \Bigg[\Bigg[-2\sum_j[(\Phi + \Psi)\ell_{x_j}|v|^2]_{x_j} + 2(\Phi + \Psi)(\Phi + \Delta\ell)|v|^2 + 2[\nabla(\Phi + \Psi)\cdot\nabla\ell]|v|^2\Bigg]\Bigg]$$

$$+\Bigg(\Bigg(\Phi(I_1\overline{v}+\overline{I_1}v)+i\Psi(\overline{v}v_t-\overline{v}_tv)-\sum_j[\Psi(v_{x_j}\overline{v}+\overline{v}_{x_j}v)]_{x_j}$$

$$+2\Psi|\nabla v|^2+\nabla\Psi\cdot[\nabla v\overline{v}+\nabla\overline{v}v]-2\Psi(|\nabla\ell|^2-\Delta\ell+\Psi)|v|^2\Bigg)\Bigg)$$

$$=|(\Phi+\Psi)v|^2-\Bigg\{\Bigg\{\cancel{\Phi(I_1\overline{v}+\overline{I_1}v)}-i\Bigg\{2\left(\nabla\ell\cdot\nabla\overline{v}v\right)_t-2\sum_j\left(\ell_{x_j}\overline{v}_tv\right)_{x_j}+2\Delta\ell v\overline{v}_t$$

$$+\sum_j[\ell_t(v\overline{v}_{x_j}-\overline{v}v_{x_j})]_{x_j}\Bigg\}+(\ell_t|v|^2)_t-\ell_{tt}|v|^2$$

$$+2\sum_j\left\{\sum_k[\ell_{x_k}(v_{x_j}\overline{v}_{x_k}+\overline{v}_{x_j}v_{x_k})-\ell_{x_j}|v_{x_k}|^2]+[|\nabla\ell|^2-\Delta\ell]\ell_{x_j}|v|^2\right\}_{x_j}-2\sum_j\{\Psi\ell_{x_j}|v|^2\}_{x_j}$$

$$-2\sum_{j,k}[\ell_{x_kx_j}(v_{x_j}\overline{v}_{x_k}-\overline{v}_{x_j}v_{x_k})]+2\Delta\ell|\nabla v|^2-2[|\nabla\ell|^2-\Delta\ell]\Delta\ell|v|^2-\cancel{2\Psi\Delta\ell|v|^2}$$

$$-2\Bigg\{\sum_j\left(\sum_k[2\ell_{x_k}\ell_{x_jx_k}-\ell_{x_kx_kx_j}]\right)\ell_{x_j}-\nabla\Psi\cdot\nabla\ell\Bigg\}|v|^2\Bigg\}\Bigg\}$$

$$-\Bigg[\Bigg[-2\sum_j[(\Phi+\mathbf{\Psi})\ell_{\mathbf{x_j}}|\mathbf{v}|^{\mathbf{2}}]_{\mathbf{x_j}}+2[\Phi^2+\Psi^2+\Psi\Phi+\Delta\ell\Psi+\nabla\Phi\cdot\nabla\ell+\nabla\mathbf{\Psi}\cdot\nabla\ell]|\mathbf{v}|^{\mathbf{2}}\Bigg]\Bigg]$$

$$+\Bigg(\Bigg(\cancel{\Phi(I_1\overline{v}+\overline{I_1}v)}+i\Psi(\overline{v}v_t-\overline{v}_tv)-\sum_j[\Psi(v_{x_j}\overline{v}+\overline{v}_{x_j}v)]_{x_j}$$

$$+2\Psi|\nabla v|^2+\nabla\Psi\cdot[\nabla v\overline{v}+\nabla\overline{v}v]-2\Psi[|\nabla\ell|^2+\Psi]|v|^2-\cancel{2\Psi\Delta\ell|v|^2}\Bigg)\Bigg).\qquad\text{(A.1.31)}$$

In (A.1.31), $\{\{\ \}\}$ refers to (A.1.15), $[[\ ]]$ refers to (A.1.14), and $((\ ))$ refers to (A.1.11). From (A.1.31), we obtain X_4 by collecting the terms pre-multiplied by i and using the identities

$$(v\overline{v}_t)=(|v|^2)_t-(v_t\overline{v}-\overline{v}_tv),\ \ell_{x_jx_j}(|v|^2)_t=(\ell_{x_jx_j}|v|^2)_t$$

by invoking $\ell_{tx_j}\equiv 0$. Likewise, if we collect the other terms, we obtain $X_1+X_2+X_3$ in (A.1.27)-(A.1.29). Thus, we obtain (A.1.26).

Step 6: Our goal now is to convert (A.1.26) from v to w. To this extent, we use the following identities:

$$\begin{aligned} &v = \theta w,\ \theta = e^{\ell},\ \theta_t = \theta \ell_t,\ v_t = \theta[w_t + \ell_t w],\ \theta_{x_j} = \theta \ell_{x_j} \\ &v_{x_j} = \theta(w_{x_j} + \ell_{x_j} w),\ |\nabla v|^2 = \theta^2 \sum_j [w_{x_j} + \ell_{x_j} w]^2 \end{aligned} \tag{A.1.32}$$

We begin with X_4. First, rewrite X_4 as

$$X_4 = i \sum_j \mu_j = i \sum_j \left\{ \frac{\partial}{\partial t}[2\ell_{x_j} v \overline{v}_{x_j} + \ell_{x_j x_j} |v|^2] + \frac{\partial}{\partial x_j}[\ell_t (v \overline{v}_{x_j} - \overline{v} v_{x_j}) - 2\ell_{x_j} \overline{v}_t v] \right\} \tag{A.1.33}$$

We then obtain the following by using identities (A.1.32):

$$\begin{aligned} \mu_j &\equiv 2\left((\ell_{x_j} v \overline{v}_{x_j})_t - (\ell_{x_j} \overline{v}_t v)_{x_j}\right) + \left(\ell_{x_j x_j} |v|^2\right)_t + \left(\ell_t (v \overline{v}_{x_j} - \overline{v} v_{x_j})\right)_{x_j} \\ &= 2\left((\theta^2 \ell_{x_j}^2 |w|^2 + \theta^2 \ell_{x_j} w \overline{w}_{x_j})_t - (\theta^2 \ell_{x_j} \ell_t |w|^2 + \theta^2 \ell_{x_j} w \overline{w}_t)_{x_j}\right) + (\theta^2 \ell_{x_j x_j} |w|^2)_t + \left(\theta^2 \ell_t (w \overline{w}_{x_j} - \overline{w} w_{x_j})\right)_{x_j} \end{aligned} \tag{A.1.34}$$

Hence, we can rewrite X_4 as follows:

$$X_4 = -\frac{\partial}{\partial t}\left\{ 2 \sum_j (\theta^2 \ell_j (\xi_{x_j} \eta - \xi \eta_{x_j})) \right\} - \sum_j \frac{\partial}{\partial x_j} \{2\theta^2 [\ell_t (\xi_{x_j} \eta - \xi \eta_{x_j}) - \ell_{x_j} (\xi_t \eta - \xi \eta_t)]\} \tag{A.1.35}$$

Step 7: Now we rewrite X_3 in terms of w. First, we consider the last term of X_3 in (A.1.29) and prove the following identity:

$$\begin{aligned} i(\Psi - \Delta\ell)(v_t \overline{v} - \overline{v}_t v) &= \theta^2 (\Psi - \Delta\ell)\left((\mathcal{P}w)\overline{w} + w(\overline{\mathcal{P}w}\right) - 2\theta^2 (\Psi - \Delta\ell)|\nabla w|^2 \\ &\quad + \theta^2 (\Psi - \Delta\ell) \sum_k (w_{x_k} \overline{w} + \overline{w}_{x_k} w)_{x_k} \end{aligned} \tag{A.1.36}$$

where $\mathcal{P} = iw_t - \Delta w$. To prove (A.1.36), direct computation gives us

$$i(\Psi - \Delta\ell)(v_t\overline{v} - \overline{v}_t v) = i(\Psi - \Delta\ell)\left(\theta(w_t + \ell_t w)(\theta\overline{w}) - \theta(\overline{w}_t + \ell_t\overline{w})(\theta w)\right)$$
$$= \theta^2(\Psi - \Delta\ell)(iw_t\overline{w} + \overline{(iw_t)}w) \tag{A.1.37}$$

$$iw_t\overline{w} + \overline{(iw_t)}w = (\mathcal{P}w)\overline{w} + (\overline{\mathcal{P}w})w + (\Delta w\overline{w} + \Delta\overline{w}w) \tag{A.1.38}$$

$$\sum_k (w_{x_k}\overline{w} + \overline{w}_{x_k}w)_{x_k} = \Delta w\overline{w} + \Delta\overline{w}w + 2|\nabla w|^2 \tag{A.1.39}$$

Notice that (A.1.38) follows directly from the definition of $\mathcal{P}$. Substituting (A.1.38) and (A.1.39) into (A.1.37) gives us (A.1.36), as desired .

Step 8: For the last term in (A.1.36), we have

$$\begin{aligned}\theta^2(\Psi - \Delta\ell)\sum_k (w_{x_k}\overline{w} + \overline{w}_{x_k}w)_{x_k} &= \sum_k \left(\theta^2(\Psi - \Delta\ell)(w_{x_k}\overline{w} + \overline{w}_{x)k}w)\right)_{x_k} \\ - 2\theta^2(\Psi - \Delta\ell)\sum_k \ell_{x_k}(w_{x_k}\overline{w} + \overline{w}_{x_k}w) &- \sum_k \theta^2(\Psi_{x_k} - \Delta\ell_{x_k})(w_{x_k}\overline{w} + \overline{w}_{x_k}w).\end{aligned} \tag{A.1.40}$$

For the second term on the RHS of (A.1.40), since ℓ is real-valued, we have the following identity from [38]:

$$\begin{aligned}\sum_k \left(w_{x_k}(\overline{\ell_{x_k}w}) + \overline{w}_{x_k}(\ell_{x_k}w)\right) &= \sum_k 2\mathrm{Re}(w_{x_k}\overline{(\ell_{x_k}w)}) \geq -\sum_k (|w_{x_k}|^2 + \ell_{x_k}^2|w|^2) \\ &= -\left(|\nabla w|^2 + |\nabla\ell|^2|w|^2\right).\end{aligned} \tag{A.1.41}$$

Apply (A.1.41) to the second term on the right-hand side of (A.1.40) gives us the following:

$$\begin{aligned}i(\Psi - \Delta\ell)(v_t\overline{v} - \overline{v}_t v) \geq\ & \theta^2(\Psi - \Delta\ell)\left((\mathcal{P}w)\overline{w} + w(\overline{\mathcal{P}w})\right) + 2\theta^2\left((\Psi - \Delta\ell)|\nabla w|^2 + |\Psi - \Delta\ell||\nabla w|^2\right) \\ &+ \sum_k \left(\theta^2(\Psi - \Delta\ell)(w_{x_k}\overline{w} + \overline{w}_{x_k}w)\right)_{x_k} - 2\theta^2|\Psi - \Delta\ell|\ |\nabla\ell|^2|w|^2 \\ &- \sum_k \left(\theta^2(\Psi_{x_k} - \Delta\ell_{x_k})(w_{x_k}\overline{w} + \overline{w}_{x_k}w)\right).\end{aligned} \tag{A.1.42}$$

Step 9: In this step, we establish the following estimate for X_3:

$$\begin{aligned} X_3 \geq\ & \theta^2(\Psi - \Delta\ell)\left((\mathcal{P}w)\overline{w} + w(\overline{\mathcal{P}w})\right) + 2\theta^2\left((\Psi - \Delta\ell)|\nabla w|^2 + |\Psi - \Delta\ell||\nabla w|^2\right) \\ & + \sum_k \left(\theta^2\Psi(w_{x_k}\overline{w} + \overline{w}_{x_k}w)\right)_{x_k} + \sum_k \left(\theta^2\Delta\ell(w_{x_k}\overline{w} + \overline{w}_{x_k}w)\right)_{x_k} \\ & + \sum_k [\Delta\ell_{x_k}(v_{x_k}\overline{v} + \overline{v}_{x_k}v)] + 2\sum_k [(\Psi_{x_k} - \Delta\ell_{x_k})\ell_{x_k})]\,|v|^2 - 2\theta^2|\Psi - \Delta\ell||\nabla\ell|^2|w|^2 \end{aligned} \tag{A.1.43}$$

To show (A.1.43), we use (A.1.42), leaving most of the identity unchanged. The exception is the last term of (A.1.42), where we use the following identity from [38]:

$$\sum_k (\Psi_{x_k} - \Delta\ell_{x_k})(v_{x_k}\overline{v} + \overline{v}_{x_k}v - 2\ell_{x_k}|v|^2) = \sum_k \theta^2[(\Psi_{x_k} - \Delta\ell_{x_k})(w_{x_k}\overline{w} + \overline{w}_{x_k}w)]. \tag{A.1.44}$$

Identity (A.1.44) follows by direct computation, using (A.1.32). After substituting (A.1.44) into (A.1.42) and recalling the definition of X_3 (A.1.29), we get (A.1.43).

Step 10: Now we rewrite X_2 from (A.1.28) in terms of w. We claim

$$\begin{aligned} X_2 =\ & -\frac{\partial}{\partial t}(\ell_t\theta^2|w|^2) + 2\sum_j \frac{\partial}{\partial x_j}\Bigg\{ -\theta^2[2|\nabla\ell|^2 - \Delta\ell + \Phi - \Psi]\ell_{x_j}|w|^2 + \theta^2\left(|\nabla\ell|^2 - \frac{\Psi}{2}\right)(w_{x_j}\overline{w} + \overline{w}_{x_j}w) \\ & - \theta^2\left[\sum_k [\ell_{x_k}(w_{x_j}\overline{w}_{x_k} + w_{x_k}\overline{w}_{x_j}) - \ell_{x_j}|\nabla w|^2]\right]\Bigg\}. \end{aligned} \tag{A.1.45}$$

To prove (A.1.45), we use the following identities from [38], which are immediate by direct computations.

$$v_{x_j} + \overline{v} + \overline{v}_{x_j}v = \theta^2[2\ell_{x_j}|w|^2 + w_{x_j}\overline{w} + \overline{w}_{x_j}w]; \tag{A.1.46}$$

$$\begin{aligned} v_{x_j}\overline{v}_{x_k} + \overline{v}_{x_j}v_{x_k} = \theta^2\Big[& 2\ell_{x_j}\ell_{x_k}|w|^2 + \ell_{x_j}(w\overline{w}_{x_k} + \overline{w}w_{x_k}) \\ & + \ell_{x_k}(w_{x_j}\overline{w} + \overline{w}_{x_j}w) + (w_{x_j}\overline{w}_{x_k} + \overline{w}_{x_j}w_{x_k})\Big]; \end{aligned} \tag{A.1.47}$$

$$|v_{x_k}|^2 = \theta^2[\ell_{x_k}^2|w|^2 + \ell_{x_k}(w\overline{w}_{x_k} + \overline{w}w_{x_k}) + |w_{x_k}|^2] \tag{A.1.48}$$

Substitute (A.1.46)-(A.1.48) into (A.1.28), and we readily obtain (A.1.45).

Step 11: We return to (A.1.26) and apply (A.1.45) for X_2, (A.1.43) for X_3, and (A.1.35) for X_4, leaving X_1 untouched. Notice that the term

$$-\sum_j \frac{\partial}{\partial x_j}\left\{\theta^2\Psi(w_{x_j}\overline{w}+\overline{w}_{x_j}w)\right\}$$

in (A.1.45) will cancel the term of opposite sign in (A.1.43). Afterwards, we obtain the following:

$$\begin{aligned}\theta^2|iw_t-\Delta w|^2 \geq X_1 + \Bigg(\Bigg(&-\frac{\partial}{\partial t}(\ell_t\theta^2|w|^2)+2\sum_j\frac{\partial}{\partial x_j}\Bigg\{-\theta^2[2|\nabla\ell|^2-\Delta\ell+\Phi-\Psi]\ell_{x_j}|w|^2\\
&+\theta^2|\nabla\ell|^2(w_{x_j}\overline{w}+\overline{w}_{x_j}w)-\theta^2\left[\sum_k[\ell_{x_k}(w_{x_j}\overline{w}_{x_k}+w_{x_k}\overline{w}_{x_j})-\ell_{x_j}|\nabla w|^2]\right]\Bigg\}\Bigg)\Bigg)\\
&+\Bigg[\Bigg[\theta^2(\Psi-\Delta\ell)\left((\mathcal{P}w)\overline{w}+w(\overline{\mathcal{P}w})\right)+2\theta^2\left((\Psi-\Delta\ell)|\nabla w|^2+|\Psi-\Delta\ell||\nabla w|^2\right)\\
&+\sum_k\left(\theta^2\Delta\ell(w_{x_k}\overline{w}+\overline{w}_{x_k}w)\right)_{x_k}+\sum_k[\Delta\ell_{x_k}\left(v_{x_k}\overline{v}+\overline{v}_{x_k}v\right)]\\
&+2\sum_k[(\Psi_{x_k}-\Delta\ell_{x_k})\ell_{x_k})]\,|v|^2-2\theta^2|\Psi-\Delta\ell||\nabla\ell|^2|w|^2\Bigg]\Bigg]\\
&+\Bigg\{\Bigg\{-\frac{\partial}{\partial t}\Bigg\{2\sum_j(\theta^2\ell_j(\xi_{x_j}\eta-\xi\eta_{x_j}))\Bigg\}-\sum_j\frac{\partial}{\partial x_j}\{2\theta^2[\ell_t(\xi_{x_j}\eta-\xi\eta_{x_j})-\ell_{x_j}(\xi_t\eta-\xi\eta_t)]\}\Bigg\}\Bigg\}\end{aligned} \tag{A.1.49}$$

To finish the proof, for any $\epsilon>0$, we use the following inequalities from [38]:

$$(\Psi-\Delta\ell)\left[(\mathcal{P}w)\overline{w}+(\overline{\mathcal{P}w})w\right]\geq-\epsilon|\Psi-\Delta\ell|^2|w|^2-\frac{1}{\epsilon}|\mathcal{P}w|^2 \tag{A.1.50}$$

$$\sum_j\Delta\ell_{x_j}\left(v_{x_j}\overline{v}+\overline{v}_{x_j}v\right)\geq-\epsilon|\nabla v|^2-\frac{1}{\epsilon}\sum_j|\Delta\ell_{x_j}|^2|v|^2 \tag{A.1.51}$$

Inequalities (A.1.50) and (A.1.51) are applications of the inequality $|a\overline{b}+\overline{a}b|=2\mathrm{Re}(a\overline{b})\geq-\epsilon|a|^2-\frac{1}{\epsilon}|b|^2$. Thus, applying them to the first and fourth terms in the [[ ]] group of (A.1.49), recalling the terms X_1 in (A.1.27), to get

$$\begin{aligned}\theta^2\left(1+\frac{1}{\epsilon}\right)|iw_t-\Delta w|^2\geq\Bigg[&-2(\Psi+\Delta\ell)(|\nabla\ell|^2-\Delta\ell)+4\sum_{j,k}\ell_{x_k}\ell_{x_j}\ell_{x_jx_k}-2\nabla\Phi\cdot\nabla\ell\\
&-2\nabla\ell\cdot\nabla\Delta\ell-(\Phi^2+\Psi^2)-2\Phi\Delta\ell+\ell_{tt}+4\nabla\Psi\cdot\nabla\ell\Bigg]|v|^2\end{aligned}$$

$$
\begin{aligned}
&+2\left\{\sum_j \nabla\ell_{x_j}(v_{x_j}\nabla\overline{v}+\overline{v}_{x_j}\nabla v)+(\Psi+\Delta\ell)|\nabla v|^2\right\}\\
&+\Bigg(\Bigg(-\frac{\partial}{\partial t}(\ell_t\theta^2|w|^2)+2\sum_j\frac{\partial}{\partial x_j}\Big\{-\theta^2[2|\nabla\ell|^2-\Delta\ell+\Phi-\Psi]\ell_{x_j}|w|^2\\
&+\theta^2|\nabla\ell|^2(w_{x_j}\overline{w}+\overline{w}_{x_j}w)-\theta^2\left[\sum_k[\ell_{x_k}(w_{x_j}\overline{w}_{x_k}+w_{x_k}\overline{w}_{x_j})-\ell_{x_j}|\nabla w|^2]\right]\Big\}\Bigg)\Bigg)\\
&+\Bigg[\Bigg[-\epsilon\theta^2|\Psi-\Delta\ell|^2+2\theta^2\left((\Psi-\Delta\ell)|\nabla w|^2+|\Psi-\Delta\ell||\nabla w|^2\right)\\
&+\sum_k\left(\theta^2\Delta\ell(w_{x_k}\overline{w}+\overline{w}_{x_k}w)\right)_{x_k}-\epsilon|\nabla v|^2-\frac{n}{\epsilon}\sum_{j,k}\ell^2_{x_jx_jx_k}|v|^2\\
&+2\sum_k\left[(\Psi_{x_k}-\Delta\ell_{x_k})\ell_{x_k}\right]|v|^2+2\theta^2|\Psi-\Delta\ell||\nabla\ell|^2|w|^2\Bigg]\Bigg]\\
&+\left\{\left\{-\frac{\partial}{\partial t}\left\{2\sum_j(\theta^2\ell_j(\xi_{x_j}\eta-\xi\eta_{x_j}))\right\}-\sum_j\frac{\partial}{\partial x_j}\{2\theta^2[\ell_t(\xi_{x_j}\eta-\xi\eta_{x_j})-\ell_{x_j}(\xi_t\eta-\xi\eta_t)]\}\right\}\right\}
\end{aligned}
\tag{A.1.52}
$$

Finally, recalling M and V from (A.1.4) and (A.1.5), we can rewrite (A.1.52) as

$$
\begin{aligned}
&\theta^2\left(1+\frac{1}{\epsilon}\right)|iw_t-\Delta w|^2-\frac{\partial M}{\partial t}-\operatorname{div}V\\
&\qquad\geq\Bigg[-2(\Psi+\Delta\ell)(|\nabla\ell|^2-\Delta\ell)+4\sum_{j,k}\ell_{x_k}\ell_{x_j}\ell_{x_jx_k}-2\nabla\Phi\cdot\nabla\ell-2\nabla\ell\cdot\nabla\Delta\ell-(\Phi^2+\Psi^2)\\
&\qquad-2\Phi\Delta\ell+\ell_{tt}+4\nabla\Psi\cdot\nabla\ell+2\nabla\ell\cdot\nabla(\Psi-\Delta\ell)-\frac{n}{\epsilon}\sum_{j,k}\ell^2_{x_jx_jx_k}-\epsilon(\Psi-\Delta\ell)^2-2|\Psi-\Delta\ell||\nabla\ell|^2\Bigg]|v|^2\\
&\qquad+2\left\{\sum_j\nabla\ell_{x_j}(v_{x_j}\nabla\overline{v}+\overline{v}_{x_j}\nabla v)+(\Psi+\Delta\ell)|\nabla v|^2\right\}+2\theta^2\left((\Psi-\Delta\ell)+|\Psi-\Delta\ell|\right)|\nabla w|^2-\epsilon|\nabla v|^2
\end{aligned}
\tag{A.1.53}
$$

which is our desired inequality (A.1.3). □

A.2 Pointwise Inequality After Specializations

In this section, we convert (A.1.53) for specific choices of $\ell(x,t)$, $\Psi(x,t)$, and $\Phi(x,t)$. These specifications are made to allow us to obtain our desired Carleman estimate.

Theorem A.2.1. *Let $w\in C^2(\mathbb{R}^n_x\times\mathbb{R}_t;\mathbb{C})$ and $d(x)\in C^3(\mathbb{R}^n_x;\mathbb{R})$. Let $\tau>0$ be a parameter, and* *we also make the following selections for $\ell(x,t)$, $\Psi(x,t)$, and $\Phi(x,t)$:*

$$\ell(x,t) := \tau\left[d(x) - \beta\left(t - \frac{T}{2}\right)^2\right] \equiv \tau\varphi(x,t) \in C^3(\mathbb{R}^n_x \times \mathbb{R}_t; \mathbb{R}) \tag{A.2.1}$$

$$\Psi(x,t) := -\Delta\ell(x,t) \in C^1(\mathbb{R}^n_x \times \mathbb{R}_t; \mathbb{R}) \tag{A.2.2}$$

$$\Phi(x,t) := \Delta\ell(x,t) \ \text{or}\ \Phi(x,t) := 0 \tag{A.2.3}$$

(a) With the above choices,we get the following (set $h \equiv \nabla d$)

$$\begin{aligned}
&\ell_{x_j} = \tau d_{x_j};\ |\nabla\ell|^2 = \tau^2|\nabla d|^2;\ \ell_{x_k x_j} = \tau d_{x_k x_j};\ \ell_{x_j x_j x_k} = \tau d_{x_j x_j x_k};\ \sum_{j,k} \ell^2_{x_j x_j x_k} = \tau^2 \sum_{j,k} d^2_{x_j x_j x_k} \\
&\quad 4\mathcal{H}_\ell \nabla\ell \cdot \nabla\ell = 4\sum_{j,k} \ell_{x_j x_k}\ell_{x_j}\ell_{x_k} = 4\tau^3 \sum_{j,k} d_{x_k x_j} d_{x_j} d_{x_k} = 4\tau^3 \mathcal{H}_d \nabla d \cdot \nabla d \\
&\quad 2\mathcal{H}_\ell[\nabla v \cdot \nabla\overline{v} + \nabla\overline{v} \cdot \nabla v] = 2\sum_{j,k} \ell_{x_j x_k}(v_{x_j}\overline{v}_{x_k} + v_{x_k}\overline{v}_{x_j}) = 2\tau[\mathcal{H}_d \nabla v \cdot \nabla\overline{v} + \mathcal{H}_d \nabla\overline{v} \cdot \nabla v] \\
&\nabla\ell = \tau\nabla d;\ \Delta\ell = \tau\Delta d;\ \ell_t = -2\beta\tau\left(t - \frac{T}{2}\right); \ell_{tt} = -2\beta\tau;\ \ell_{tx_j} = 0; \\
&\quad -(\Psi^2 + \Phi^2) - 2\Phi\Delta\ell = -(\Phi + \Delta\ell)^2
\end{aligned} \tag{A.2.4}$$

Further, we have the following identity:

$$2\nabla[\Phi + \Delta\ell] \cdot \nabla\ell + (\Phi + \Delta\ell)^2 = \begin{cases} 4\tau^2[\nabla\Delta d \cdot \nabla d + (\Delta d)^2], & \Phi = \Delta\ell \\ \tau^2[2\nabla\Delta d \cdot \nabla d + (\Delta d)^2], & \Phi = 0 \end{cases} \tag{A.2.5}$$

via equation (A.2.3).

(b) Using (A.2.1)-(A.2.4)*, our estimate* (A.1.53) *becomes*

$$\theta^2\left(1+\frac{1}{\epsilon}\right)|iw_t-\Delta w|^2-\frac{\partial M}{\partial t}-div\ V$$
$$\geq\Bigg[4\tau^3\mathcal{H}_d\nabla d\cdot\nabla d-(2\nabla(\Phi+\Delta\ell)\cdot\nabla\ell+(\Phi+\Delta\ell)^2)-2\beta\tau$$
$$-\frac{n}{\epsilon}\tau^2\sum_{j,k}d^2_{x_jx_jx_k}-4\epsilon\tau^2(\Delta d)^2+4\tau^3\Delta d|\nabla d|^2\Bigg]|v|^2+2\tau[\mathcal{H}_d\nabla v\cdot\nabla\overline{v}+\mathcal{H}_d\nabla\overline{v}\cdot\nabla v]-\epsilon|\nabla v|^2 \tag{A.2.6}$$

(b_1) If we assume $\Phi=\Delta\ell$*,*

$$\theta^2\left(1+\frac{1}{\epsilon}\right)|iw_t-\Delta w|^2-\frac{\partial M}{\partial t}-div\ V$$
$$\geq\Bigg[4\tau^3\mathcal{H}_d\nabla d\cdot\nabla d-4\tau^2[\nabla\Delta d\cdot\nabla d+(\Delta d)^2]-2\beta\tau-\frac{n}{\epsilon}\tau^2\sum_{j,k}d^2_{x_jx_jx_k}\quad.$$
$$-4\epsilon\tau^2(\Delta d)^2+4\tau^3\Delta d|\nabla d|^2\Bigg]|v|^2+2\tau[\mathcal{H}_d\nabla v\cdot\nabla\overline{v}+\mathcal{H}_d\nabla\overline{v}\cdot\nabla v]-\epsilon|\nabla v|^2 \tag{A.2.7}$$

(b_2) If we assume $\Phi=0$*,*

$$\theta^2\left(1+\frac{1}{\epsilon}\right)|iw_t-\Delta w|^2-\frac{\partial M}{\partial t}-div\ V$$
$$\geq\Bigg[4\tau^3\mathcal{H}_d\nabla d\cdot\nabla d-\tau^2[2\nabla\Delta d\cdot\nabla d+(\Delta d)^2]-2\beta\tau-\frac{n}{\epsilon}\tau^2\sum_{j,k}d^2_{x_jx_jx_k}$$
$$-4\epsilon\tau^2(\Delta d)^2+4\tau^3\Delta d|\nabla d|^2\Bigg]|v|^2+2\tau[\mathcal{H}_d\nabla v\cdot\nabla\overline{v}+\mathcal{H}_d\nabla\overline{v}\cdot\nabla v]-\epsilon|\nabla v|^2 \tag{A.2.8}$$

(c) We can combine (A.2.7) *and* (A.2.8) *as*

$$\theta^2\left(1+\frac{1}{\epsilon}\right)|iw_t-\Delta w|^2-\frac{\partial M}{\partial t}-div\ V$$
$$\geq\Big[4\tau^3\mathcal{H}_d\nabla d\cdot\nabla d+\mathcal{O}(\tau^2)+4\tau^3\Delta d|\nabla d|^2\Big]|v|^2+2\tau[\mathcal{H}_d\nabla v\cdot\nabla\overline{v}+\mathcal{H}_d\nabla\overline{v}\cdot\nabla v]-\epsilon|\nabla v|^2 \tag{A.2.9}$$

where $\mathcal{O}$ depends on $d, n, \beta,$ and ϵ.

(d) Using assumptions (A.1)-(A.2), (A.2.9) *becomes*

$$\begin{aligned}\theta^2\left(1+\frac{1}{\epsilon}\right)|iw_t - \Delta w|^2 - \frac{\partial M}{\partial t} - div\ V \\ \geq \left[4\tau^3\rho p^2 + \mathcal{O}(\tau^2) + 4\tau^3 p^2 \Delta d\right]|v|^2 + (4\tau\rho - \epsilon)|\nabla v|^2\end{aligned} \tag{A.2.10}$$

Proof. The proof is a direct verification based on (A.2.1)-(A.2.3). The reasoning behind the specific choices of (A.2.1)-(A.2.3) are similar to the choices made in [38]. □

Remark A.2.2. Notice that the function φ above is similar to the function φ in (1.17). The $t - \frac{T}{2}$ term is on the time interval $[0, T]$. Doing a transformation $t \to t - \frac{T}{2}$ returns to the pseudo-convex function used previously.

With Theorem A.2.1, we can now obtain our pointwise estimate used to get the Carleman estimate for the Schrödinger equation.

Corollary A.2.3. *Let $d(x) \in C^3(\mathbb{R}^n_x)$ satisfy the geometric assumptions (A.1) and (A.2) in Chapter 1. Let ℓ, Ψ, and Φ be defined as in* (A.2.1)*,* (A.2.2)*, and* (A.2.3)*.*

(i) Inequality (A.2.9) *becomes the following, for any $\epsilon > 0$ and sufficiently large τ:*

$$\begin{aligned}\theta^2\left(1+\frac{1}{\epsilon}\right)|iw_t - \Delta w|^2 - \frac{\partial M}{\partial t} - div\ V \\ \geq \left[4\tau^3\rho p^2 + \mathcal{O}(\tau^2) + 4\tau^3 p^2 \Delta d\right]|v|^2 + (4\tau\rho - \epsilon)(\theta^2|\nabla w|^2 - 2\tau^2|\nabla d|^2\theta^2|w|^2) \\ \geq \delta_0\left(2\tau\rho - \frac{\epsilon}{2} - 4\Delta d\right)\theta^2|\nabla w|^2 + \left(4\tau^3\rho p^2(1-\delta_0+\Delta d) + \mathcal{O}(\tau^2)\right)\theta^2|w|^2\end{aligned} \tag{A.2.11}$$

for some $0 < \delta_0 < 1$. Further, from [37], we can bound the second Δd on the right-hand side *of* (A.2.11) *to get the following:*

$$
\begin{aligned}
\theta^2\left(1+\frac{1}{\epsilon}\right)&|iw_t-\Delta w|^2-\frac{\partial M}{\partial t}-div\ V\\
&\geq\left[4\tau^3\rho p^2+\mathcal{O}(\tau^2)+4\tau^3p^2\Delta d\right]|v|^2+(4\tau\rho-\epsilon)(\theta^2|\nabla w|^2-2\tau^2|\nabla d|^2\theta^2|w|^2)\\
&\geq\delta_0\left(2\tau\rho-\frac{\epsilon}{2}-4\Delta d\right)\theta^2|\nabla w|^2+\left(4\tau^3\rho p^2(2-k+2\beta-\delta_0)+\mathcal{O}(\tau^2)\right)\theta^2|w|^2
\end{aligned}
\tag{A.2.12}
$$

for some $0<k<1$.

(ii) On the boundary, we have the following identity, where Φ *is left uncommitted:*

$$
\begin{aligned}
\int_\Omega div(V)\,dx=&-2\int_\Gamma\theta^2(2\tau^2|h|^2+\Phi)\tau|w|^2h\cdot\nu\,d\Gamma+2\beta\int_\Gamma\theta^2\tau\left(t-\frac{T}{2}\right)\left[\eta\frac{\partial\xi}{\partial\nu}-\xi\frac{\partial\eta}{\partial\nu}\right]d\Gamma\\
&+2\int_\Gamma\theta^2\left(\xi_t\eta-\xi\eta_t\right)\tau h\cdot\nu\,d\Gamma-\int_\Gamma\theta^2\left[2\tau^2|h|^2+\tau\Delta d\right]\left(\overline{w}\frac{\partial w}{\partial\nu}+w\frac{\partial\overline{w}}{\partial\nu}\right)d\Gamma\\
&-2\int_\Gamma\tau h\cdot\left[\nabla\overline{w}\frac{\partial w}{\partial\nu}+\nabla w\frac{\partial\overline{w}}{\partial\nu}\right]d\Gamma+2\int_\Gamma\theta^2|\nabla w|^2h\cdot\nu\,d\Gamma
\end{aligned}
\tag{A.2.13}
$$

(iii) For the M *term, we have the following:*

$$
\begin{aligned}
\left|\int_Q\frac{\partial M}{\partial t}\,dQ\right|&=\left[\int_\Omega M\,dx\right]_0^T\\
&\leq\tau C_{d,T}\left[\int_\Omega e^{2\tau\varphi}[|\nabla w|^2+|w|^2]\,dx\right]_0^T\\
&\leq C_{d,T}\tau e^{-2\tau\delta}[\mathbb{E}(T)-\mathbb{E}(0)].
\end{aligned}
\tag{A.2.14}
$$

Proof. (i) The first inequality in (A.2.11) follows from (A.2.9) as a direct consequence of the geometric assumptions (A.1) and (A.2). The second inequality in (A.2.11) is obtained by the following inequality from [37]:

$$
2|\nabla v|^2\geq\theta^2|\nabla w|^2=2\tau^2|\nabla d|^2|v|^2=\theta^2|\nabla w|^2-2\tau^2|\nabla d|^2\theta^2|w|^2. \tag{A.2.15}
$$

(ii) Equation (A.2.13) is immediate from the Divergence Theorem and the definition of V in (A.1.5).

(iii) Recalling (A.1.4),

$$\begin{aligned}
\left|\int_Q \frac{\partial M}{\partial t}\right| &= \left|\left[\int_\Omega M\right]_0^T\right| \\
&= \left|\left[\int_\Omega -\theta^2[2\tau(\nabla d\cdot\nabla\xi)\eta - 2\tau(\nabla d\cdot\nabla\eta)\xi + 2\beta\tau\left(t-\frac{T}{2}\right)|w|^2]\right]_0^T\right| \\
&= \left|\left[\int_\Omega \theta^2[2\tau(\nabla d\cdot\nabla\xi)\eta - 2\tau(\nabla d\cdot\nabla\eta)\xi + 2\beta\tau\left(t-\frac{T}{2}\right)|w|^2]\right]_0^T\right| \\
&= \left|\left[\int_\Omega \theta^2[2\tau\nabla d\cdot(\eta\nabla\xi - \xi\nabla\eta) + 2\beta\tau\left(t-\frac{T}{2}\right)|w|^2]\right]_0^T\right| \\
&\le \tau C_{d,T}\left|\left[\int_\Omega e^{2\tau\varphi}[|\nabla w|\ |w| + |w|^2]\right]_0^T\right| \\
&\le \tau C_{d,T}\left|\left[\int_\Omega e^{2\tau\varphi}[|\nabla w|^2 + |w|^2]\right]_0^T\right|
\end{aligned}$$

The next to last line above comes from the observation that $\xi = \mathrm{Re}(w)$ and $\eta = \mathrm{Im}(w)$. Recalling property (1.20) and $\mathbb{E}(t)$ defined as

$$\mathbb{E}_w(t) = \int_\Omega [|w(t)|^2 + |\nabla w(t)|^2]\,dx = \|w(t)\|^2_{H^1(\Omega)}$$

we obtain (A.2.14).

$\square$

Now we have everything set to obtain the Carleman estimate in Theorem 2.2.2. Integrate (A.2.12) over Q and using the Schrödinger equation along with equations (A.2.13) and (A.2.14).

Notice that $BT_1(w)$ defined in (2.25) is equal to (A.2.13). This gives us the following:

$$\begin{aligned}
&\left(1+\frac{1}{\epsilon}\right)\int_Q \theta^2|f|^2\,dQ + \tau C_{d,T}e^{-2\tau\delta}[\mathbb{E}(T)+\mathbb{E}(0)] + BT_1(w) \\
&\qquad\geq \left(1+\frac{1}{\epsilon}\right)\int_Q |iw_t-\Delta w|^2 - \left[\int_\Omega M\,dx\right]_0^T - \operatorname{div} V \\
&\qquad\geq \delta_0\left(2\tau\rho - \frac{\epsilon}{2} - 4\Delta d\right)\int_Q \theta^2|\nabla w|^2\,dQ \\
&\qquad+ \left(4\tau^3\rho p^2(2-k+2\beta-\delta_0)+\mathcal{O}(\tau^2)\right)\int_Q \theta^2|w|^2\,dQ - \tau C_{d,T}e^{-2\tau\delta}[\mathbb{E}(T)+\mathbb{E}(0)],
\end{aligned} \tag{A.2.16}$$

establishing the desired inequality in Theorem 2.2.1 by taking $\epsilon = 1$.

Appendix B

Terminology and Properties of Riemannian Geometry

In this appendix, we provide some definitions and properties of Riemannian geometry referenced in this paper. These definitions are found in various works [7, 10, 35, 49] and the references cited within, which we collected here for convenience.

B.1 Definitions

Definition B.1.1. Let M be a manifold. A *Riemannian metric* g on M is defined as a map which associates to any vector fields X and Y on M a function $g(X, Y)$ on M such that the following properties hold:

1. $g(X_1 + X_2, Y) = g(X_1, Y) + g(X_2, Y)$
2. $g(X, Y_1 + Y_2) = g(X, Y_1) + g(X, Y_2)$
3. $g(fX, Y) = fg(X, Y)$
4. $g(X, Y) = g(Y, X)$

for all real-valued functions f and vector fields X, X_1, X_2, Y, Y_1, Y_2, and $g(X, X) > 0$ when $X \neq 0$.

A *Riemannian manifold* (M,g) is a manifold M equipped with the Riemannian metric g.

In terms of local coordinates, g is given by a positive definite and symmetric matrix function g defined by

$$g_{jk} = g(\partial_j, \partial_k)$$

Definition B.1.2. For $x \in M$, the inner product $\langle X, Y \rangle$ and the norm $|X|$ for $X, Y \in M_x$ are defined by

$$g(X,Y) = \langle X,Y \rangle_g = \langle X,Y \rangle := \sum_{j,k=1}^{n} g_{jk}\alpha_j\beta_k \tag{B.1.1}$$

$$|X|_g = |X| := \langle X, X \rangle_g^{1/2} \tag{B.1.2}$$

for $X = \sum_{i=1}^{n} \alpha_i \dfrac{\partial}{\partial x_i}$ and $Y = \sum_{i=1}^{n} \beta_i \dfrac{\partial}{\partial x_i}$

where, for each $x \in M$, M_x denotes the tangent space of M at x. Notice that the coefficients $g_{ij}(x)$ of g form a symmetric and positive definite for any $x \in M$. The inverse matrix of $(g_{ij}(x))$ is denoted by $(g^{ij}(x))$.

Definition B.1.3. Let f be a C^1 scalar function on manifold M and let X be a vector field on M. The *Levi-Civita connection*, denoted by D, is defined as follows: If f is a scalar C^1-function on M and X is a vector field on M, then the continuous linear functional $X(f)$ (in this context, it defines the derivative of f in the direction of X) is given by (via Riesz Representation Theorem)

$$X(f) = \langle Df, X \rangle = \langle \nabla f, X \rangle \tag{B.1.3}$$

where ∇f is the gradient of f. Also, if H, X, and Y are vector fields on M, then DH denotes the *covariant differential* of H. This determines a bilinear form on $M_x \times M_x$, for each $x \in M$, defined by

$$DH(X,Y) = \langle D_Y H, X \rangle, \;\; X, Y \in M_x, \;\; x \in M \tag{B.1.4}$$

where $D_Y H$ is the *covariant derivative* of H with respect to Y. The equation of calculating the covariant derivative is provided below:

$$D_Y H = \sum_{k=1}^{n} Y(h_k)\frac{\partial}{\partial x_k} + \sum_{k,i=1}^{n} h_k \beta_i D_{\partial/\partial x_i}\left(\frac{\partial}{\partial x_k}\right) \tag{B.1.5}$$

where

$$\begin{aligned} Y(h_k) &= \langle \nabla h_k, X \rangle_g \\ D_{\partial/\partial x_i}\left(\frac{\partial}{\partial x_k}\right) &= \sum_{l=1}^{n} \Gamma_{ik}^{l} \frac{\partial}{\partial x_l} \end{aligned} \tag{B.1.6}$$

with Γ_{ik}^{l} being the connection coefficients of the connection D,

$$\Gamma_{ik}^{l}(x) = \frac{1}{2}\sum_{p=1}^{n} g^{kp}(x)\left(\frac{\partial g_{ji}}{\partial x_i} + \frac{\partial g_{ip}}{\partial x_j} - \frac{\partial g_{ik}}{\partial x_p}\right),\ 1 \le i, j, k \le n. \tag{B.1.7}$$

Definition B.1.4. Given vector field $X = \sum_{i=1}^{n} \alpha_i \frac{\partial}{\partial x_i}$ and metric g, define the *divergence* of vector field X as follows:

$$\mathrm{div}_g X = \frac{1}{\sqrt{\det\, g}} \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\sqrt{\det\, g}\ \alpha_i\right) = \sum_{i=1}^{n} \left[D_{\partial/\partial x_i} X\right]_i \tag{B.1.8}$$

The *Laplace-Beltrami* operator is given by

$$\Delta_g f = \mathrm{div}_g(Df). \tag{B.1.9}$$

Similar to Euclidean space, if f and h are C^2 functions, we have the following identity:

$$\mathrm{div}\,(fDh) = f\Delta_g h + \langle Df, Dh\rangle.$$

If f is a C^2 scalar function on M, then its *Hessian* $D^2 f(\cdot,\cdot)$ with respect to metric g is defined by

$$D^2 f(X,Y) = \langle D_Y(Df), X\rangle. \tag{B.1.10}$$

Moreover, if ν and ρ denote the unit normal and tangential vectors, respectively along the boundary $\partial\Omega$ of $\Omega \subset M$, then

$$\begin{aligned} \frac{\partial\ \cdot}{\partial \nu} &= \langle D\ \cdot, \nu\rangle \\ \frac{\partial\ \cdot}{\partial \rho} &= \langle D\ \cdot, \rho\rangle \end{aligned} \tag{B.1.11}$$

B.2 Properties of Riemannian Geometry

In this section, we list some properties based on the definitions given in the previous section. These are well-known results; hence, the proofs are omitted. The proofs can be found in the references listed with each result.

Proposition B.2.1. *([36, 49, 50]) For any function f and vector field H on manifold M, the following identity holds on each $x \in M$:*

$$\langle Df, D(H(f))\rangle = DH(Df, Df) + \frac{1}{2}\left(div_g(|Df|^2 H) - |Df|^2 div_g H\right) \tag{B.2.12}$$

Proposition B.2.2. *([7]) Let $u \in C^1(M)$ satisfying $u|_{\partial M} = 0$and set $\partial_N u = (\nabla u \cdot \nu)$. The following properties hold:*

$$\begin{aligned} \partial_N u &= \frac{1}{(g^{-1}\nu \cdot \nu)}\frac{\partial u}{\partial \nu};\ \nabla_g u = (\partial_N u) g^{-1}\nu \\ |\nabla_g u|^2 = \langle \nabla_g u, \nabla_g u\rangle &= \frac{1}{g^{-1}\nu \cdot \nu}\left(\frac{\partial u}{\partial \nu}\right)^2 \\ \langle \nabla_g u, \nabla_g \psi\rangle &= \frac{1}{(g^{-1}\nu \cdot \nu)}\frac{\partial u}{\partial \nu}\ \frac{\partial \psi}{\partial \nu}\ \text{ on } \partial M \text{ for } u, \psi \in C^1(M) \text{ such that } u|_{\partial M} = 0 \\ \langle H, \nabla_g u\rangle &= \frac{1}{(g^{-1}\nu \cdot \nu)}(H \cdot \nu)\frac{\partial u}{\partial \nu}\ \text{ on } \partial M \text{ for a vector field } H. \end{aligned} \tag{B.2.13}$$

Proposition B.2.3. *Let $A(x)$ and $G(x)$ denote the $n \times n$ matrices with coordinates (g^{ik}) and (g_{ik}), respectively for $x \in \mathbb{R}^n$ (i.e $G(x) = [A(x)]^{-1}$). Let $f, h \in C^1(\overline{\Omega})$ and H, X be vector fields. Then*

$$\begin{aligned} &\langle H(x), A(x)X(x)\rangle_g = H(x) \cdot X(x), \\ &Df(x) = A(x)\nabla f,\ \text{ where } \nabla f \text{ denotes the regular Euclidean gradient,} \\ &\langle Df, Dh\rangle_g = \langle A(x)\nabla f, Dh\rangle_g = \nabla f Dh = \nabla f^T \cdot A(x)\nabla h \\ &D^2 f(X, X) = \sum_{i,j=1}^{n} \alpha_i \left(\sum_{l=1}^{n} \frac{\partial f_l}{\partial x_i} g_{lj} + \sum_{k,l=1}^{n} f_k g_{lj}\Gamma^l_{ik}\right)\alpha_j, \end{aligned} \tag{B.2.14}$$

where f_l is the lth coordinate of Df.

References

[1] P. Albano, *Carleman estimates for the euler-bernoulli plate operator*, Electronic Journal of Differential Equations, 2000 (2000), pp. 1–13.

[2] C. Alves, A. L. Silvestre, T. Takahashi, and M. Tucsnak, *Solving inverse source problems using observability. applications to the euler–bernoulli plate equation*, SIAM Journal on Control and Optimization, 48 (2009), pp. 1632–1659.

[3] A. V. Balakrishnan, *Applied Functional Analysis*, vol. 3, Springer Science & Business Media, 2012.

[4] M. Bellassoued, *Uniqueness and stability in determining the speed of propagation of second-order hyperbolic equation with variable coefficients*, Applicable Analysis, 83 (2004), pp. 983–1014.

[5] M. Bellassoued, D. Jellali, and M. Yamamoto, *Lipschitz stability for a hyperbolic inverse problem by finite local boundary data*, Applicable Analysis, 85 (2006), pp. 1219–1243.

[6] ——, *Lipschitz stability in an inverse problem for a hyperbolic equation with a finite set of boundary data*, Applicable Analysis, 87 (2008), pp. 1105–1119.

[7] M. Bellassoued and M. Yamamoto, *Carleman estimates and applications to inverse problems for hyperbolic systems*, vol. 8, Springer, 2017.

[8] A. L. Bukhgeim and M. V. Klibanov, *Global uniqueness of a class of multidimensional inverse problems*, in Doklady Akademii Nauk, vol. 260, Russian Academy of Sciences, 1981, pp. 269–272.

[9] T. Carleman, *Sur un problème d'unicité pour les systèmes d'équations aux dérivées partielles à deux variables indépendantes*, Almqvist & Wiksell, 1939.

[10] M. P. Do Carmo and J. Flaherty Francis, *Riemannian geometry*, vol. 6, Springer, 1992.

[11] W. Green, S. Liu, and M. Mitkovski, *Boundary observability for the viscoelastic wave equation*, SIAM Journal on Control and Optimization, 57 (2019), pp. 1629–1645.

[12] Z.-j. Gu and Y.-j. Tan, *Fundamental solution method for inverse source problem of plate equation*, Applied Mathematics and Mechanics, 33 (2012), pp. 1513–1532.

[13] R. Gulliver, W. Littman, I. Lasiecka, and R. Triggiani, *The case for differential geometry in the control of single and coupled PDEs: the structural acoustic chamber*, Springer, 2004.

[14] L. F. Ho, *Observabilité frontière de l'équation des ondes*, Comptes rendus de l'Académie des sciences. Série 1, Mathématique, 302 (1986), pp. 443–446.

[15] L. Hörmander, *Linear partial differential operators*, vol. 116, Springer, 2013.

[16] O. Y. Imanuvilov and M. Yamamoto, *Global uniqueness and stability in determining coefficients of wave equations*, Communications in Partial Differential Equations, 26 (2001), pp. 1409–1425.

[17] O. Y. Imanuvilov and M. Yamamoto, *Determination of a coefficient in an acoustic equation with a single measurement*, Inverse Problems, 19 (2003), pp. 157–171.

[18] V. Isakov, *Inverse source problems*, no. 34, American Mathematical Soc., 1990.

[19] ——, *Inverse Problems for Partial Differential Equations*, vol. 127, Springer International Publishing, 2017.

[20] A. Kawano and A. Morassi, *Uniqueness in the determination of loads in multi-span beams and plates*, European Journal of Applied Mathematics, 30 (2019), pp. 176–195.

[21] M. Klibanov, *Carleman estimates and inverse problems*, Inverse Problems, 8 (1992), pp. 575–96.

[22] M. V. Klibanov and J. Li, *Inverse problems and carleman estimates: global uniqueness, global convergence and experimental data*, vol. 63, Walter de Gruyter GmbH & Co KG, 2021.

[23] M. V. Klibanov and J. Malinsky, *Newton-kantorovich method for three-dimensional potential inverse scattering problem and stability of the hyperbolic cauchy problem with time-dependent data*, Inverse problems, 7 (1991), pp. 577–595.

[24] M. V. Klibanov and A. A. Timonov, *Carleman estimates for coefficient inverse problems and numerical applications*, de Gruyter, 2004.

[25] ——, *Carleman estimates for coefficient inverse problems and numerical applications*, in Carleman Estimates for Coefficient Inverse Problems and Numerical Applications, de Gruyter, 2012.

[26] M. V. Klibanov and M. Yamamoto, *Lipschitz stability of an inverse problem for an acoustic equation*, Applicable Analysis, 85 (2006), pp. 515–538.

[27] V. Komornik and P. Loreti, *Fourier series in control theory*, Springer, 2005.

[28] E. Kreyszig, *Introductory functional analysis with applications*, vol. 17, John Wiley & Sons, 1991.

[29] J. Kurz, S. Liu, and P. Pei, *Recovering density for the mindlin–timoshenko system by means of a single boundary measurement*, Applicable Analysis, 101 (2022), pp. 1682–1698.

[30] J. A. Kurz, *Exact Controllability and Inverse Problem for the Mindlin-Timoshenko System*, PhD thesis, Clemson University, 2021.

[31] I. Lasiecka, J.-L. Lions, and R. Triggiani, *Non homogeneous boundary value problems for second order hyperbolic operators*, Journal de Mathématiques pures et Appliquées, 65 (1986), pp. 149–192.

[32] I. Lasiecka and R. Triggiani, *Exact controllability of the euler–bernoulli equation with controls in the dirichlet and neumann boundary conditions: a nonconservative case*, SIAM journal on control and optimization, 27 (1989), pp. 330–373.

[33] I. Lasiecka and R. Triggiani, *Sharp regularity theory for second order hyperbolic equations of neumann type*, Annali di Matematica Pura ed Applicata, 157 (1990), pp. 285–367.

[34] ——, *Carleman estimates and exact boundary controllability for a system of coupled, nonconservative second-order hyperbolic equations*, in Partial differential equation methods in control and shape analysis, CRC Press, 1997, pp. 231–260.

[35] I. Lasiecka, R. Triggiani, and P. Yao, *Carleman estimates for a plate equation on a riemann manifold with energy level terms*, in Analysis and Applications—ISAAC 2001, Springer, 2003, pp. 199–236.

[36] I. Lasiecka, R. Triggiani, and P.-F. Yao, *Inverse/observability estimates for second-order hyperbolic equations with variable coefficients*, Journal of Mathematical Analysis and Applications, 235 (1999), pp. 13–57.

[37] I. Lasiecka, R. Triggiani, and X. Zhang, *Nonconservative wave equations with unobserved neumann bc: global uniqueness and observability in one shot*, Contemporary Mathematics, 268 (2000), pp. 227–326.

[38] ——, *Global uniqueness, observability, and stabilization of nonconservative schrödinger equations via pointwise carleman estimates*, Journal of Inverse Ill-Posed Problems, 11 (2003), pp. 1–96.

[39] M. M. Lavrent_ev, V. G. Romanov, and S. P. Shishatski_, *Ill-posed problems of mathematical physics and analysis*, vol. 64, American Mathematical Soc., 1986.

[40] J.-L. Lions, *Exact controllability, stabilization and perturbations for distributed systems*, SIAM review, 30 (1988), pp. 1–68.

[41] S. Liu and R. Triggiani, *Global uniqueness and stability in determining the damping and potential coefficients of an inverse hyperbolic problem*, Nonlinear Analysis: Real World Applications, 12 (2011), pp. 1562–1590.

[42] ——, *Global uniqueness and stability in determining the damping coefficient of an inverse hyperbolic problem with non-homogeneous dirichlet bc through an additional localized neumann boundary trace*, Applicable Analysis, 91 (2012), pp. 1551–1581.

[43] S. Liu and R. Triggiani, *Boundary control and boundary inverse theory for non-homogeneous second-order hyperbolic equations: a common carleman estimates approach*, HCDTE Lecture notes, AIMS Book Series on Applied Mathematics, 6 (2013), pp. 227–343.

[44] S. Micu and E. Zuazua, *An introduction to the controllability of partial differential equations*, Quelques questions de théorie du contrôle. Sari, T., ed., Collection Travaux en Cours Hermann, to appear, (2004).

[45] D. Tataru, *Boundary controllability for conservative pdes*, Applied Mathematics and Optimization, 31 (1995), pp. 257–295.

[46] R. Triggiani et al., *Carleman estimates with no lower-order terms for general riemann wave equations. global uniqueness and observability in one shot*, Applied Mathematics & Optimization, 46 (2002), pp. 331–375.

[47] Y. Wang, *Global uniqueness and stability for an inverse plate problem*, Journal of Optimization Theory and Applications, 132 (2007), pp. 161–173.

[48] F. Yaman, V. G. Yakhno, and R. Potthast, *Recent theory and applications on inverse problems*, 2013.

[49] P.-F. Yao, *On the observability inequalities for exact controllability of wave equations with variable coefficients*, SIAM Journal on Control and Optimization, 37 (1999), pp. 1568–1599.

[50] P.-F. Yao, *Observability inequalities for the euler-bernoulli plate with variable coefficients*, Contemporary Mathematics, 268 (2000), pp. 383–406.