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Asymptotic Cones of Quadratically Defined Sets and Their Applications to QCQPs

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Abstract

Quadratically constrained quadratic programs (QCQPs) are a set of optimization problems defined by a quadratic objective function and quadratic constraints. QCQPs cover a diverse set of problems, but the nonconvexity and unboundedness of quadratic constraints lead to difficulties in globally solving a QCQP. This thesis covers properties of unbounded quadratic constraints via a description of the asymptotic cone of a set defined by a single quadratic constraint. A description of the asymptotic cone is provided, including properties such as retractiveness and horizon directions.

Using the characterization of the asymptotic cone, we generalize existing results for bounded quadratically defined regions with non-intersecting constraints. The newer result provides a sufficient condition for when the intersection of the lifted convex hulls of quadratically defined sets equals the lifted convex hull of the intersection. This document goes further by expanding the non-intersecting property to cover affine linear constraints.

The Frank-Wolfe theorem provides conditions for when a problem defined by a quadratic objective function over affine linear constraints has an optimal solution. Over time, this theorem has been extended to cover cases involving convex quadratic constraints. We discuss more current results through the lens of the asymptotic cone of a quadratically defined set. This discussion expands current results and provides a sufficient condition for when a QCQP with one quadratic constraint with an indefinite Hessian has an optimal solution.

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Chapter 1

Introduction

This dissertation focuses on asymptotic cones and their applications to solving quadratically constrained quadratic programs, also known as QCQPs. A QCQP takes the general form as follows:

$$\begin{aligned} \inf \quad & f(x) = x^T A_0 x + 2a_0^T x \\ s.t. \quad & g_i(x) = x^T A_i x + 2a_i^T x + \alpha_i \leq 0, \ i = 1, \dots, m, \end{aligned} \tag{QCQP}$$

where $A_i \in \mathcal{S}^n$ are $n \times n$ real symmetric matrices, $a_i \in \mathbb{R}^n$ are real vectors for all $i = 0, \dots, m$, and $\alpha_i \in \mathbb{R}$ are scalars for all $i = 1, \dots, m$. For ease of notation, the feasible region will be denoted as follows:

$$\mathcal{F} := \{ \, x \mid g_i(x) \leq 0, \ i = 1, \dots, m \, \} .$$

This general problem structure connects linear programs ($A_i = 0, \ i = 0, \dots, m$) to nonlinear programs. The general QCQP covers a diverse set of problems and applications such as the facility location problem, the Max-Cut problem, the trust region method, binary programming, polynomial programming, production planning, and the pooling problem. With this level of diversity, one can expect that solving an arbitrary QCQP will be NP-Hard in general. This difficulty can arise from several problems, such as nonconvexity or nonlinearity. In fact, the potential unboundedness of quadratic constraints yields an extra level of difficulty occurs when determining if the QCQP has an optimal solution.

In general, (QCQP) is not convex (requires A_i to be positive semidefinite for $i = 0, 1, \dots, m$).

This lack of convexity poses difficulty for finding a global solution. One approach to this is convexifying (QCQP). Similar to other nonconvex problem types like integer programming, the initial step is to find a convex relaxation and then apply valid inequalities. For (QCQP), this can be handled using conic programming. An example of this is semidefinite programming. Many semidefinite relaxations and reformulations have been discussed such as using the cone of nonnegative functions and matrix decompositions to transform (QCQP) to a linear conic programming problems [31].

The primary question that is addressed in this document is how the unboundedness of a quadratic feasible region affects the existence of an optimal solution as well as the convexification of a QCQP. Over a convex feasible region, the recession cone provides an understanding of how the set behaves as it tends from the origin. Extending this to a nonconvex QCQP, this notion is extended to asymptotic directions. Asymptotic directions are used to describe the end behavior of sets and functions [17, 20, 28] as well as the horizon limits of sequences of lower level sets [5, 29].

In several cases, the exact semidefinite representable relaxations for $\mathcal{F}$ are straightforward. For instance, when $\mathcal{F}$ is defined by a single inequality, a single equality, and an interval-bounded inequality, then the relatively simple Shor relaxation provides an exact, convex reformulation to (QCQP). Other semidefinite representable cases include when $\mathcal{F}$ is defined as a low dimensional polyhedron or when $\mathcal{F}$ is defined by a convex quadratic constraint with multiple non-intersecting constraints. Several of these results will be discussed in Chapter 2 and for more recent results on the convexification of quadratically defined sets, refer to [9, 12, 14, 25, 30, 33].

The difficulty of globally solving (QCQP) begs the question of if an optimal solution exists. While this is a straightforward question with linear programs, or in terms of the Weierstrass theorem, solving over a compact feasible region. However, the general unboundedness and lack of convexity may cause the optimal solution of (QCQP) to not exist. There is a wide variety of methods used to determine optimality conditions of (QCQP) such as conic programming to verify KKT solutions [21] and duality theory [40]. A more in-depth review of this topic is presented in Chapter 6.

1.1 Outline and Overview

This document highlights asymptotic cones and several applications of them in terms of QCQPs. Chapter 2 discusses conic relaxations and convexification of (QCQP) and provides a nonexhaustive list of results related to this topic. Chapter 3 provides an introduction to asymptotic cones and directions along with definitions and results. The chapter concludes with an analysis of these definitions in regards a set defined by a single quadratic constraint. Chapter 4 describes the convexification of sets that have an underlying set of non-intersecting constraints. This work relies on a complete description of the asymptotic cones of sets defined by a single quadratic equality and a partial understanding of the recession cone $\overline{\mathcal{C}}(\mathcal{F})$. The proof generalizes existing results for bounded quadratically defined sets with non-intersecting constraints and provides a sufficient condition for when the lifted closed convex hull of the intersection equals the intersection of the lifted closed convex hulls.

Chapter 5 extends the results of Chapter 4. Chapter 4 focuses on quadratic non-intersecting sets and the proofs of the results show a gap when extended to affine linear constraints. This chapter contains two sections. Section one explores the definition of non-intersecting in terms of an affine linear constraint and the effect it has on the asymptotic cone of the intersection as well as the recession cone of the lifted closed convex hull of the intersection. Section two explores whether the intermediate step of homogenizing the original space can extend the results of Chapter 4.

Chapter 6 changes the focus of the document to instead consider whether an optimal solution to (QCQP) exists when (QCQP) is defined by a constraint with an indefinite Hessian. While asymptotic cones play an important role in results related to this question, this document poses that the "center" of constraints could also play a vital role. Using asymptotic cones, this chapter extends a result of Tam and Nghi [32] and provides conditions that coincide with the results of Bertsekas and Tseng [5].

1.2 Notation

For a nonempty set $S \subseteq \mathbb{R}^n$, denote its boundary, interior, and closure by $\text{bd}(S)$, $\text{int}(S)$, and $\overline{S}$, respectively. The cone generated by S is denoted by $\text{cone}(S)$, and the conic hull of S is represented as $\text{cone}\,\text{conv}(S)$. Their closures are represented as $\overline{\text{cone}}(S)$ and $\overline{\text{cone}}\,\text{conv}(S)$, respectively. The cardinality of S is denoted by $|S|$ and $\pm S$ is the set $S \cup (-S)$. When S is convex, the recession cone of S is denoted by $\text{Rec}(S)$, and the set of extreme points of S is denoted by $\text{ext}(S)$. For vectors x and matrices X, $x \geq 0$ and $X \geq 0$ implies that x and X have only non-negative entries.

In proving some of the results in Chapter 2, copositive optimization is used. This can arise when $x \geq 0$ is a constraint defining $\mathcal{F}$. When lifting this set into the matrix space, there are relations to the set of completely positive matrices (CP). A matrix X is completely positive if and only if $X = \sum_k x_i x_i^T$ for some $x_i \geq 0$, $i = 1, \ldots, k$. A completely positive matrix is always doubly nonnegative. A matrix X is doubly nonnegative if and only if $X \succeq 0$ and $X \geq 0$.

Note: While results in Chapters 4 and 5 are focused over the set $\{ (x, xx^T) \mid x \in S \}$ for a given set S, other results can be found for the set $\left\{ \binom{1}{x}\binom{1}{x}^T \,\middle|\, x \in S \right\}$. Some results in Chapter 2 will use the alternative notation.

Chapter 2

Literature on Convex Hull Results

Let $\mathcal{F}$ be a nonempty closed set. The majority of this document is interested in the structure of the lifted closed convex hull

$$\overline{\mathcal{C}}(\mathcal{F}) := \overline{\text{conv}}\,\{\,(x, xx^T) \mid x \in \mathcal{F}\,\}\,.$$

The set $\overline{\mathcal{C}}(\mathcal{F})$ is related to optimization problems with quadratic objectives. Specifically, defining $Q \in \mathcal{S}^n$, where $\mathcal{S}^n$ is the set of $n \times n$ real symmetric matrices, and $q \in \mathbb{R}^n$, a quadratic function of x over $\mathcal{F}$ can be formulated as an optimization problem of the form:

$$\begin{aligned} v(q, \mathcal{F}) := \inf_{x} &\quad x^T Q x + 2q^T x \\ s.t. &\quad x \in \mathcal{F}. \end{aligned} \tag{2.1}$$

Introducing a new variable $X = xx^T$, the objective function of (2.1) can be linearized as $Q \bullet X + 2q^T x$ where $Q \bullet X := \text{Trace}(Q^T X)$ is the Frobenius inner product of Q and X. Due to the linearity of the objective function, the next step is to convexify the feasible region and obtain the following convex formulation (e.g. [17, 21]):

$$\begin{aligned} v(q, \overline{\mathcal{C}}(\mathcal{F})) = \inf_{(x,X)} &\quad Q \bullet X + 2q^T x \\ s.t. &\quad (x, X) \in \overline{\mathcal{C}}(\mathcal{F}). \end{aligned} \tag{2.2}$$

To see that $v(q, \mathcal{F}) = v(q, \overline{\mathcal{C}}(\mathcal{F}))$, consider the following proposition.

Proposition 1. *For all quadratic functions of the form $q(x) = x^T Qx + 2q^T x$, the following holds $v(q, \mathcal{F}) = v(q, \overline{\mathcal{C}}(\mathcal{F}))$.*

Proof. The proof will consider two cases of the set $\mathcal{C}(\mathcal{F}) = \text{conv}\left\{ (x, xx^T) \mid x \in \mathcal{F} \right\}$ since $v(q, \mathcal{C}(\mathcal{F})) = v(q, \overline{\mathcal{C}}(\mathcal{F}))$. Case one is when $v(q, \mathcal{F}) = -\infty$ (the case of $v(q, \mathcal{C}(\mathcal{F})) = -\infty$ follows the same approach), and case two is when $v(q, \mathcal{F})$ is bounded.

First, let $v(q, \mathcal{F}) = \infty$. Then there exists a sequence $\{x_i\}_{i \geq 0} \subseteq \mathcal{F}$ such that $q(x_i) = x_i^T Q x_i + 2q^T x_i \to \infty$ as $x_i \to \infty$. Otherwise, there exists $M \in \mathbb{R}$ such that $q(x) \geq M$ for all $x \in \mathcal{F}$. In this case, for any $(x, X) \in \mathcal{C}(\mathcal{F})$,

$$Q \bullet X + 2q^T x = \sum_i \lambda_i (Q \bullet (x_i x_i^T) + 2q^T x_i) \geq M,$$

where $\lambda_i \geq 0$ and $\sum_i \lambda_i = 1$.

Now suppose that $v(q, \mathcal{F})$ is attainable for some $\hat{x} \in \mathcal{F}$. Then $(\hat{x}, \hat{x}\hat{x}^T) \in \mathcal{C}(\mathcal{F})$ where $Q \bullet (\hat{x}\hat{x}^T) + 2q^T \hat{x} = v(q, \mathcal{F})$. If $v(q, \mathcal{C}(\mathcal{F})) < v(q, \mathcal{F})$, then there exists $(x, X) \in \mathcal{C}(\mathcal{F})$ such that $v(q, \mathcal{C}(\mathcal{F})) \leq Q \bullet X + 2q^T x < v(q, \mathcal{F})$. Decomposing (x, X) into a convex combination of elements in $\mathcal{C}(\mathcal{F})$ results in $(\bar{x}, \bar{x}\bar{x}^T) \in \mathcal{C}(\mathcal{F})$ where $\bar{x} \in \mathcal{F}$ and

$$Q \bullet (\bar{x}\bar{x}^T) + 2q^T \bar{x} = \bar{x}^T Q \bar{x} + 2q^T \bar{x} + q^T \bar{x} = q(\bar{x}) < q(\hat{x}) = v(q, \mathcal{F}).$$

Hence a contradiction since $\hat{x}$ is the optimal solution. Therefore, $v(q, \mathcal{F}) = v(q, \mathcal{C}(\mathcal{F}))$, and $v(q, \mathcal{C}(\mathcal{F}))$ is attained at $(\hat{x}, \hat{x}\hat{x}^T)$. □

Due to the lack of explicit expressions for $\overline{\mathcal{C}}(\mathcal{F})$, (2.2) is computationally intractable. Therefore, even a partial understanding of $\overline{\mathcal{C}}(\mathcal{F})$ is desired, as valid inequalities of $\overline{\mathcal{C}}(\mathcal{F})$ can help tighten the lower bounds of the convex relaxation of (2.1).

In this chapter, results relating to the lifted closed convex hull of different classes of quadratic sets will be discussed. This class of problems is represented in (QCQP) with feasible region

$$\mathcal{F} = \left\{ x \mid x^T A_i x + 2a_i^T x + \alpha_i \leq 0,\ i = 1, \ldots, m \right\}.$$

As mentioned earlier, calculating $\overline{\mathcal{C}}(\mathcal{F})$ is a difficult task but the structure of $\mathcal{F}$ can provide insight in the forms of valid cuts and inequalities. In general these cuts are applied to the Shor relaxation of $\mathcal{F}$, denoted below.

$$\mathcal{S}(\mathcal{F}) = \left\{ (x, X) \;\middle|\; A_i \bullet X + 2a_i^T x + \alpha \leq 0,\ i = 1, \ldots, m \right\},$$

The rest of the chapter serves as a survey of lifted convex hull results. If any of the following structures can be found in a quadratically defined set under the assumptions in Chapter 4, they can be applied to more convex sets, increasing the utility of these results. Before that, let's look at a one-dimensional set to have a graphical understanding of $\overline{\mathcal{C}}(\mathcal{F})$. Consider $\mathcal{F}$ in

$$\mathcal{F}_{int} = \{ x \in \mathbb{R} \mid -2 \leq x \leq 1 \}. \tag{INT}$$

This set can be shifted and scaled to be an interval of any size. Optimizing a quadratic objective function over (INT) yields the same value as optimizing a linear function over $\overline{\mathcal{C}}(\mathcal{F}_{int}) = \overline{\text{conv}}\,\{ (x, x^2) \mid x \in \mathcal{F}_{int} \}$. Graphically, this is presented in Figure 2.1. This lifted set $\overline{\mathcal{C}}(\mathcal{F}_{int})$ can be defined using the convex inequalities $x^2 \leq X$ and $X \leq 2 - x$. It is importantly crucial to note that for an optimal $(\hat{x}, \hat{X}) \in \overline{\mathcal{C}}(\mathcal{F}_{int})$, $\hat{x}$ might not be optimal for the original problem. This arises because the set of optimal solutions over $\overline{\mathcal{C}}(\mathcal{F}_{int})$ is the convex hull of optimal solutions in the set $\{ (x, x^2) \mid x \in \mathcal{F}_{int} \}$ [7]. Figure 2.1 illustrates this, where the boundary of $\overline{\mathcal{C}}(\mathcal{F})$ is either an extreme point (x, xx^T) or the convex combination of the points $(-2, 4)$ and $(1, 1)$. One sufficient condition for $\hat{x}$ to be an optimal solution to the original QCQP is to require $X = x^2$, which is equivalent to the nonconvex rank-one condition.

Figure 2.1, while in low dimensions, gives insights to the geometrical properties that need to be considered. For example, in Figure 2.1, $\mathcal{F}_{int}$ is a bounded set, and thus $\mathcal{C}(\mathcal{F}_{int}) = \overline{\mathcal{C}}(\mathcal{F}_{int})$ is compact. This is not the case for when $\mathcal{F}$ is unbounded, say $\mathcal{F} = \{ x \in \mathbb{R} \mid x \leq 1 \}$. Consider Figure 2.2. The left plot is $\mathcal{C}(\mathcal{F})$ with $\text{Rec}(\mathcal{C}(\mathcal{F})) = \{0\}$. Taking the closure of $\mathcal{C}(\mathcal{F})$, $\overline{\mathcal{C}}(\mathcal{F})$, the recession cone is non-empty and broadens the tools that can be used to analyze this set.

Section 2.1 gives a brief introduction to the S-Procedure [36] and a few derivations of the Shor relaxation, $\mathcal{S}(\mathcal{F})$. Section 2.2 presents several results where, for a given set $\mathcal{F}$, the Shor relaxation $\mathcal{S}(\mathcal{F})$ provides an exact relaxation. There is a brief overview about sets defined by one inequality, followed by a section about the generalized trust region subproblem, and then a small dive

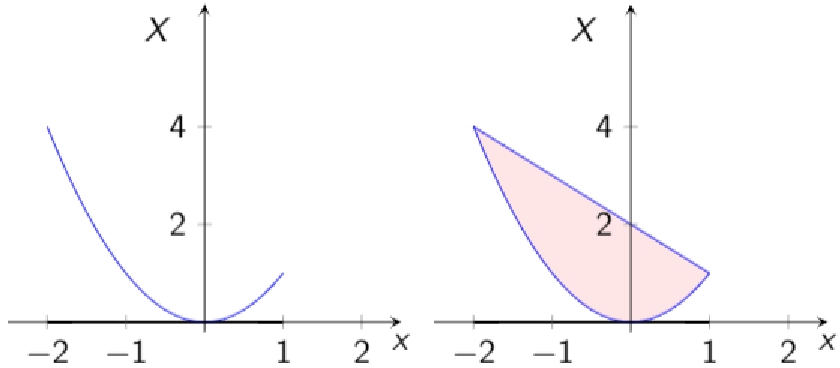


Figure 2.1: Convexification of the lifted set $\{ (x, x^2) \mid x \in \mathcal{F}_{int} \}$ for $\mathcal{F} = \{ x \mid -2 \leq x \leq 1 \}$. The left plot depicts the lifted set, and the right plot depicts the convex hull.

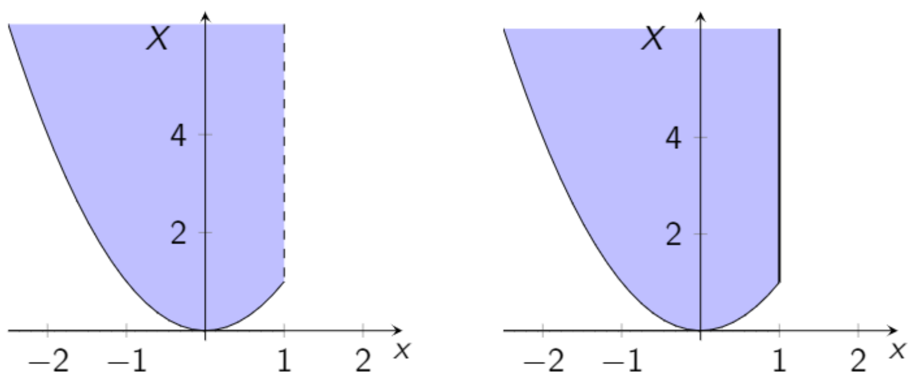


Figure 2.2: Two plots of $\mathcal{C}(\mathcal{F})$ (left) and $\overline{\mathcal{C}}(\mathcal{F})$ (right) for $\mathcal{F} = \{ x \in \mathbb{R} \mid x \leq 1 \}$. The thickened boundary on the right plot represents what is added when taking the closure.

into rank-one sets and non-intersecting constraints. Section 2.3 introduces results from copositive programming where the lifted closed convex hull is not fully defined by the Shor relaxation. Most results will focus on small dimensional polyhedron. Section 2.4 explores the two cases of bounded sets with “non-intersecting” constraints.

2.1 Preliminaries

Lemma 1 (S-Lemma [36]). *Let $A, B \in \mathcal{S}^n$, $a, b \in \mathbb{R}^n$, $\alpha, \beta \in \mathbb{R}$, and suppose there exists an $\hat{x}$ with*

$$\hat{x}^T B \hat{x} + 2b^T \hat{x} + \beta < 0.$$

Then there exists an $x \in \mathbb{R}^n$ *satisfying*

$$x^T Ax + 2a^T x + \alpha < 0, \qquad x^T Bx + 2b^T x + \beta \leq 0$$

if and only if there exists no λ *such that*

$$\lambda \geq 0, \quad \begin{pmatrix} \alpha & a^T \\ a & A \end{pmatrix} + \lambda \begin{pmatrix} \beta & b^T \\ b & B \end{pmatrix} \succeq 0.$$

This theorem of alternatives has been used in many comparisons of quadratic sets and is used to derive results relating objective functions to sets defined by one constraint. Lemma 1, under convexity assumptions and Slater's condition, has been combined with Farkas' lemma to help find hidden convexities for globally solving non-convex non-quadratic problems [39].

Given (QCQP), solving the lifted problem over the Shor relaxation provides a lower bound on the optimal solution. This paper briefly introduces how to show that the Shor relaxation is a valid relaxation of (QCQP). The first way is by lifting (QCQP) into a higher dimensional space

$$\begin{aligned} v_R \coloneqq \min_{(x,X)} \quad & Q \bullet X + 2q^T x \\ s.t. \quad & A_i \bullet X + 2a_i^T x + \alpha \leq 0 \\ & X = xx^T. \end{aligned} \tag{2.3}$$

Here, (2.3) has a linear objective over linear constraints except for the nonconvex rank-one constraint $X = xx^T$. Relaxing the rank one constraint to $X \succeq xx^T$, (2.3) is now solving a linear objective function over the Shor relaxation $\mathcal{S}(\mathcal{F})$. The second approach uses the Lagrangian dual. While not as direct, it provides a deeper analysis into the structure of the problem with the Lagrangian dual of (QCQP). The Lagrangian function is

$$L(x, \lambda) = x^T \left(Q + \sum_{i \in I} \lambda_i A_i \right) x + 2 \left(q + \sum_{i \in I} \lambda_i a_i \right)^T x + \sum_{i \in I} \lambda_i \alpha_i,$$

where $\lambda_i \geq 0$. Using the Schur complement, the dual problem of (QCQP) can be expressed as the semi-definite dual in the form of

$$\begin{aligned} \max \quad & \gamma \\ s.t. \quad & \begin{pmatrix} \sum_{i\in I} \lambda_i \alpha_i - \gamma & q^T + \sum_{i\in I} \lambda_i a_i^T \\ q + \sum_{i\in I} \lambda_i a_i & Q + \sum_{i\in I} A_i \end{pmatrix} \succeq 0 \\ & \lambda_i \geq 0, \quad i \in I. \end{aligned}$$

For the QCQP and its Lagrangian dual to have the same optimal value, again Slater's condition must be satisfied. Taking the dual of the Lagrangian dual (in its SDP form) gives the following formulation

$$\begin{aligned} v \coloneqq \min_{(x,X)} \quad & Q \bullet X + 2q^T x \\ s.t. \quad & A_i \bullet X + 2a_i^T x + \alpha_i \leq 0, \; i \in I \\ & X \succeq xx^T, \end{aligned}$$

which is equivalent to

$$\begin{aligned} v \coloneqq \min_{(x,X)} \quad & Q \bullet X + 2q^T x \\ s.t. \quad & (x, X) \in \mathcal{S}(\mathcal{F}). \end{aligned} \tag{2.4}$$

It is important to note that for the Langrangian dual and the SDP relaxation to have the same optimal value, Slater's condition must hold. This method is considered in Section 2.2.1 for determining when the Shor relaxation is tight.

2.2 When the Shor Relaxation is Exact

The first result to be mentioned is that of Sturm and Zhang [31]. This paper analyzes the structure of the cone of non-negative quadratic functions over a set, defined as follows for a set $\mathcal{F}$:

$$D = \left\{ U \in \mathcal{S}^{n+1} \;\middle|\; \begin{pmatrix} 1 \\ x \end{pmatrix}^T U \begin{pmatrix} 1 \\ x \end{pmatrix} \geq 0, \; \forall x \in \mathcal{F} \right\}. \tag{2.5}$$

The dual cone of D, D^*, is defined as $D^* = \overline{\text{cone}}(Z)$ where

$$Z = \left\{ Y \in \mathcal{S}^{n+1} \;\middle|\; Y = \begin{pmatrix} 1 \\ x \end{pmatrix} \begin{pmatrix} 1 \\ x \end{pmatrix}^T,\ x \in \mathcal{F} \right\}. \tag{2.6}$$

In relation to the Shor relaxation, $\mathcal{S}(\mathcal{F})$, D^* can be rewritten as

$$D^* = \overline{\text{cone}}(Z) = \text{cone}(\mathcal{S}(\mathcal{F})),$$

which geometrically implies that $\mathcal{S}(\mathcal{F})$ is an affine slice of a closed convex cone. This result, used in papers, can be presented in a different way. Interested in the two trust-region subproblem ($\mathcal{F}_{TTS}$), [38] shows that finding an exact relaxation can be accomplished by finding the family of all quadratic functions that are non-negative over $\mathcal{F}_{TTS}$. In fact, the optimal value over $\mathcal{F}_{TTS}$ is equal to $v(\mathcal{T})$ where

$$\mathcal{T} := \{ (R, r, \rho) \in \mathcal{S}^n \times \mathbb{R}^n \times \mathbb{R} \mid x^T Rx + 2r^T x + \rho \geq 0 \;\; \forall x \in \mathcal{F}_{TTS} \}.$$

In regards to [31], they provide a generalization of Lemma 1 and proves that the Shor relaxation of a set defined by a single quadratic inequality constraint is tight. They use the fact that $X \in \mathcal{S}^n_+$ is a positive semi-definite (psd) matrix of rank r if and only if, there exists $x_i \in \mathbb{R}^n$, $i = 1, \ldots, r$, such that

$$X = \sum_{i=1}^{r} x_i x_i^T. \tag{2.7}$$

In other words, a psd matrix of rank r has a rank-one decomposition of r vectors. This results in the following lemma.

Lemma 2 ([31])**.** *Let $X \in \mathcal{S}^n$ be a positive semidefinite matrix of rank r and $Q \in \mathcal{S}^n$ be a given matrix. Then, $G \bullet X \leq 0$ if and only if there exists $x_i \in \mathbb{R}^n$, $i = 1, \ldots, r$, such that*

$$X = \sum_{i=1}^{r} x_i x_i^T \quad \text{and} \quad x_i^T Q x_i \leq 0 \quad \text{for all } i = 1, \ldots, r.$$

Using this and the homogenized cone of $\mathcal{F} = \{ x \in \mathbb{R}^n \mid x^T Ax + 2a^T x + \alpha \leq 0 \}$, they prove the following result.

Theorem 1 (Single Quadratic Inequality, [31])**.** *Let $\mathcal{F} = \{ x \in \mathbb{R}^n \mid x^T Ax + 2a^T x + \alpha \leq 0 \}$ be a* *closed, nonempty set. Then*

$$\overline{\text{conv}}\,\{\,(x, xx^T) \mid x \in \mathcal{F}\,\} = \{\,(x, X) \in \mathbb{R}^n \times \mathcal{S}^n \mid A \bullet X + 2a^T x + \alpha \leq 0,\ X \succeq xx^T\,\}\,. \tag{2.8}$$

That is, $\overline{\mathcal{C}}(\mathcal{F}) = \mathcal{S}(\mathcal{F})$.

This is not the only result where the Shor relaxation provides an exact reformulation. While working with the interval bounded generalized trust region subproblem (GTRS), Pong and Wolkowicz [27] determined when the SDP relaxation is tight.

2.2.1 Generalized Trust Region Subproblem

The results pertaining to the generalized trust region subproblem (GTRS) relate the QCQP to its SDP representation through the consideration of its Lagrangian dual. A more generalized version of this approach was presented in Section 2.1. Here, [27, 35] explore the conditions needed for the SDP relaxation of (GTRS) to be exact.

$$\begin{aligned} v_{GTRS} = \inf_{x} \quad & x^T Q x - 2q^T x \\ s.t. \quad & \ell \leq x^T A x + 2a^T x \leq u \end{aligned} \tag{GTRS}$$

This is a problem with two constraints but, since the constraints both relate to the same A and a, they can be expressed as a single interval bounded constraint. A similar structure can be seen in the relaxed SDP presented in (SDP-GTRS).

$$\begin{aligned} v_* = \inf_{(x,X)} \quad & Q \bullet X - 2q^T x \\ s.t. \quad & \ell \leq A \bullet X + 2a^T x \leq u \\ & X \succeq xx^T. \end{aligned} \tag{SDP-GTRS}$$

In showing the exactness of this relaxation, the authors start with the dual of GTRS

$$\begin{aligned} d_{GTRS} = \sup_{\lambda} \quad & h(\lambda) + \ell\lambda_+ - u\lambda_- \\ s.t. \quad & Q - \lambda A \succeq 0, \end{aligned} \tag{D-GTRS}$$

for $\lambda_+ = \max\{\lambda, 0\}$, $\lambda_- = \min\{\lambda, 0\}$, and

$$h(\lambda) = \inf_x x^T(Q-\lambda A)x - 2(q-\lambda a)^T x$$
$$= \begin{cases} -(q-\lambda a)^T(Q-\lambda A)^\dagger(q-\lambda a) & \text{if } q-\lambda a \in \text{Range}(Q-\lambda A) \\ & \text{and } Q-\lambda A \succeq 0, \\ -\inf & \text{otherwise,} \end{cases}$$

where $(Q-\lambda A)^\dagger$ is the pseudo inverse of $Q-\lambda A$. This problem is then shown to have the exact solution of the dual of (SDP-GTRS), stated below:

$$\begin{aligned} d_* = \sup_{\lambda,\gamma} \quad & \ell\lambda_+ - u\lambda_- - \gamma \\ s.t. \quad & \begin{pmatrix} \gamma & -(q-\lambda a)^T \\ (q-\lambda a) & Q-\lambda A \end{pmatrix} \succeq 0. \end{aligned} \tag{SDD-GTRS}$$

The semidefinite dual of the GTRS, (SDD-GTRS), is used to show that (D-GTRS) is the dual of (SDP-GTRS) under the assumptions listed in Assumption 1. There are multiple citations for this set of assumptions, but the later two reduce the necessary list of conditions.

Assumption 1 (GTRS Assumptions, [27, 34, 35])**.**

1. *$A \neq 0$*
2. *The following relative interior constraint qualification (RICQ) holds:*

$$\ell < A \bullet \hat{X} + 2a^T\hat{x} < u \quad \textit{for some } \hat{X} \succ \hat{x}\hat{x}^T.$$

3. *(GTRS) is bounded below.*

Under these assumptions, we have the following theorem,

Theorem 2 (Interval Bounded Quadratic Inequality, [27])**.** *Suppose that Assumption 1 holds. Then the following holds for GTRS:*

1. *The optimal values of GTRS and its SDP relaxation are equal, that is*

$$v_{GTRS} = v_*. \tag{2.9}$$

2. *Strong duality holds for GTRS, i.e.* $v_{GTRS} = d_{GTRS}$ *and the dual value* d_{GTRS} *is attained. Moreover,*

$$d_{GTRS} = d_* = v_* = v_{GTRS}.$$

In other words, the Shor relaxation of a given set defined by a single interval bounded quadratic constraint is exact (i.e. $\mathcal{S}(\mathcal{F}) = \overline{\mathcal{C}}(\mathcal{F})$) under mild assumptions. The path followed by Pong and Wolkowicz can be seen in Figure 2.3. In this figure, it is seen that Slater's condition is required to show equality between the primal and its dual for both the original and lifted space. In regards to what happens when $\ell = u$, a more strict "double-sided" Slater's condition is necessary.

Assumption 2 (Double-Sided Slater's Condition, [35])**.** *Given a function* $f : \mathbb{R}^n \to \mathbb{R}$*,* $f(x)$ *yields both positive and negative values. That is, there exists* $x_1, x_2 \in \mathbb{R}^n$ *such that* $f(x_1) < 0 < f(x_2)$*.*

With this assumption and the S-Lemma with Equality, [34, 35] prove the following theorem.

Theorem 3 (Single Quadratic Equality Constraint, [34])**.** *For a set* $\mathcal{F} = \{x \in \mathbb{R}^n \mid x^T Ax + 2a^T x + \alpha = 0\}$*, under Assumption 2 with* $A \neq 0$*,* $\overline{\mathcal{C}}(\mathcal{F}) = \mathcal{S}(\mathcal{F})$*. That is, the Shor relaxation of* $\mathcal{F}$ *is exact.*

Similar to [27], [34] also follows the path given in Figure 2.3.

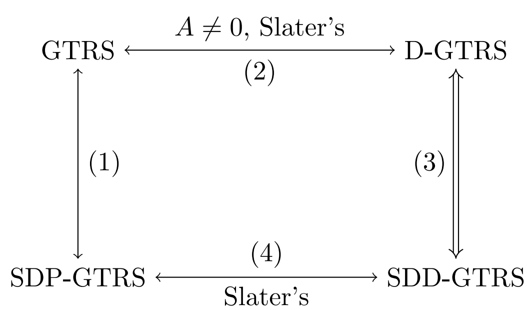


Figure 2.3: A flowchart showing the approach used to prove SDP exactness. The method used in [27, 34] traverses arcs (2),(3) and (4).

The results so far have shown that for a set defined by single quadratic constraint under mild assumptions, the lifted convex hull can be represented by the Shor relaxation of that set. In considering sets defined by multiple constraints, more assumptions are required for $\overline{\mathcal{C}}(\mathcal{F}) = \mathcal{S}(\mathcal{F})$.

2.2.2 Rank-One Generated Cones

This section focuses on the work of Argue et. al [3] concentrating on rank-one generated (ROG) sets. That is, for a set $S \subset \mathbb{R}^n$, S is ROG if

$$S = \overline{\text{conv}}(S \cap \{\, xx^T \mid x \in \mathbb{R}^n \,\}).$$

Equivalently, a set S is ROG if and only if it is equal to the closed convex hull of its rank-one matrices. Thus, for a closed convex cone $S \subseteq \mathcal{S}^n$, S is ROG. Also, if $X \in S$ where $\text{rank}(X) = 1$, then X is an extreme ray of S. The importance behind the ROG property is if the feasible region of (ROG-SDP) is ROG, then the SDP relaxation is exact. That is, the feasible reason of (ROG-SDP) is ROG if and only if there exists rank-one solutions in the feasible region that approach the optimal value of any arbitrary objective function [3].

For this section, the quadratic constraints will be labeled differently for ease of presentation. Given a quadratic constraint $y^T A_i y + 2a_i^T y + \alpha_i \leq 0$ where $y \in \mathbb{R}^{n-1}$, it is reexpressed as $x^T M_i x$ where $x = \begin{pmatrix} 1 \\ y \end{pmatrix}$ and $M_i = \begin{pmatrix} 1 & a_i^T \\ a_i & A_i \end{pmatrix}$. (QCQP) is rewritten as

$$\begin{aligned} \inf_{x \in \mathbb{R}^n} \quad & x^T \begin{pmatrix} 0 & q^T \\ q & Q \end{pmatrix} x \\ s.t. \quad & x^T M_i x \leq 0, \quad i \in I \\ & x_1^2 = 1, \end{aligned} \tag{ROG-QCQP}$$

with the SDP relaxation

$$\begin{aligned} \inf_{X \in \mathcal{S}^n} \quad & \begin{pmatrix} 0 & q^T \\ q & Q \end{pmatrix} \bullet X \\ s.t. \quad & M_i \bullet X \leq 0, \quad i \in I \\ & X_{1,1} = 1. \end{aligned} \tag{ROG-SDP}$$

With ROG cones, there is an interest in the linear matrix inequalities (LMI) $M_i \bullet X \leq 0$. Denote the set $\mathcal{M} \subseteq \mathcal{S}^n$ as the collection of M_i's, $M_i \in \mathcal{M}$ for $i \in I$. The interest of this paper [3] is to determine if $\mathcal{S}(\mathcal{M})$ is a ROG cone, where $\mathcal{S}(\mathcal{M})$ is defined as

$$\mathcal{S}(\mathcal{M}) \coloneqq \{\, X \in \mathcal{S}^n_+ \mid M \bullet X \leq 0,\ \ \forall M \in \mathcal{M} \,\}\,.$$

The following lemma states the exactness of an SDP with a ROG feasible region.

Lemma 3 ([3])**.** *Let $\mathcal{M} \subseteq \mathcal{S}^n$. If $\mathcal{S}(\mathcal{M})$ is ROG, then*

$$\begin{aligned} \inf_{x\in\mathbb{R}^n} \quad & x^T \begin{pmatrix} 0 & q^T \\ q & Q \end{pmatrix} x \\ s.t. \quad & x^T M_i x \leq 0, \quad i \in I \\ & x^T B x = 1 \end{aligned}$$

has the same optimal solution as

$$\begin{aligned} \inf_{X\in\mathcal{S}^n} \quad & \begin{pmatrix} 0 & q^T \\ q & Q \end{pmatrix} \bullet X \\ s.t. \quad & M_i \bullet X \leq 0, \quad i \in I \\ & B \bullet X = 1 \\ & X \succeq 0 \end{aligned}$$

for all B, $\begin{pmatrix} 0 & q^T \\ q & Q \end{pmatrix} \in \mathcal{S}^n$ for which the optimum SDP objective value is bounded from below. In particular, this equality holds whenever the SDP feasible domain is bounded.

Lemma 3 generalizes the constraints $x_1^2 = 1$ and $X_{1,1} = 1$ in (ROG-QCQP) and (ROG-SDP), respectively. Also, the lemma has the assumption that the SDP objective value is bounded from below. The importance of this assumption is shown with Example 1.

Example 1. *For matrix M, define $Sym(M) = (M + M^T)/2$, and let e_i be a vector where the only* *nonzero term is 1 in the ith entry. Let $n = 2$ and $\mathcal{M} = \{\, Sym(e_1e_2^T),\ -Sym(e_1e_2^T)\,\}$ so that*

$$\begin{aligned}
\mathcal{S}(\mathcal{M}) &= \left\{ \begin{pmatrix} x_1^2 & 0 \\ 0 & x_2^2 \end{pmatrix} \;\middle|\; x \in \mathbb{R}^2 \right\} \\
&= \operatorname{conv}\left(\left\{ \begin{pmatrix} x_1 \\ 0 \end{pmatrix}\begin{pmatrix} x_1 \\ 0 \end{pmatrix}^T \;\middle|\; x \in \mathbb{R}^2 \right\} \right) \\
&\qquad \cup \quad \operatorname{conv}\left(\left\{ \begin{pmatrix} 0 \\ x_2 \end{pmatrix}\begin{pmatrix} 0 \\ x_2 \end{pmatrix}^T \;\middle|\; x \in \mathbb{R}^2 \right\} \right).
\end{aligned}$$

The representation on the right shows that $\mathcal{S}(\mathcal{M})$ is ROG. On the other hand, setting $B = e_1e_1^T$ and $Q = -e_2e_2^T$,

$$\inf_{x\in\mathbb{R}^2} \left\{ x^TQx \;\middle|\; \begin{array}{l} xx^T \in \mathcal{S}(\mathcal{M}) \\ x^TBx = 1 \end{array} \right\} = \inf_{x\in\mathbb{R}^2} \left\{ x^TQx \;\middle|\; \begin{array}{l} x_1x_2 = 0 \\ x_1^2 = 1 \end{array} \right\} = 0,$$

which is not equal to

$$\inf_{X\in\mathcal{S}^2} \left\{ Q \bullet X \;\middle|\; \begin{array}{l} X \in \mathcal{S}(\mathcal{M}) \\ B \bullet X = 1 \end{array} \right\} = \inf_{x\in\mathbb{R}^2} \left\{ -x_2^2 \;\middle|\; x_1^2 = 0 \right\} = -\infty.$$

This discussion will now check if $\mathcal{S}(\mathcal{M})$ is ROG and highlight some operations that preserve the ROG property.

Lemma 4 ([3])**.** *For any $\mathcal{M} \subseteq \mathcal{S}^n$, the following are equivalent:*

1. *$\mathcal{S}(\mathcal{M})$ is ROG.*

2. *Every face of $\mathcal{S}(\mathcal{M})$ is ROG.*

3. *$\mathcal{S}(\mathcal{M}) \cap \mathcal{T}(\mathcal{M}')$ is ROG for every $\mathcal{M}' \subseteq \mathcal{M}$ where*

$$\mathcal{T}(\mathcal{M}) \coloneqq \{\, X \in \mathcal{S}_+^n \mid M \bullet X = 0,\ \ \forall M \in \mathcal{M} \,\}.$$

As a byproduct of item 3 of Lemma 4, if $\mathcal{S}(\mathcal{M})$ is ROG, then so is $\mathcal{T}(\mathcal{M})$. With Lemma 4 and the nullspace, $\mathrm{Null}(\mathcal{M})$, the writers recover the result from [31] with the following lemma.

Lemma 5 (One Inequality Constraint ROG [3]). *Consider any $M \in \mathcal{S}^n$ and let $\mathcal{M} = \{M\}$. Then $\mathcal{S}(\mathcal{M})$ is ROG.*

The operation of interest is the intersection of constraints, or in the sense of this paper, the union of M_i's.

Lemma 6 ([3]). *Let $\mathcal{M} \subset \mathcal{S}^n$ be a finite union of compact sets $\mathcal{M} = \cap_{i=1}^{k} \mathcal{M}_i$. Further, suppose that for all nonzero $X \in \mathcal{S}_+^n$ and $i = 1, \ldots, k$, if $M_i \bullet X = 0$ for some $M_j \in \mathcal{M}_i$, then $M \bullet X < 0$ for all $M \in \mathcal{M} \setminus \mathcal{M}_i$. Then $\mathcal{S}(\mathcal{M})$ is ROG if and only if $\mathcal{S}(\mathcal{M}_i)$ is ROG for all $i = 1, \ldots, k$.*

In terms of quadratic constraints, suppose $\mathcal{F} = \{\, x \mid x^T A_i x + 2a_i^T + \alpha_i \leq 0,\ i \in I \,\}$. Then if there exists $\hat{x}$ such that $\hat{x}^T A_j \hat{x} + 2a_j^T \hat{x} + \alpha_j = 0$ for some $j \in I$, then $\hat{x}^T A_k \hat{x} + 2a_k^T \hat{x} + \alpha_k < 0$ for all $k \in I \setminus j$. Visually, this can be seen in Figure 2.4 where the boundaries of the constraints do not intersect. This is formalized in the following theorem.

Theorem 4 (Non-Interacting Constraints, [3]). *Consider $\mathcal{F} \in \mathbb{R}^n$ to be a quadratically defined set such that $\mathcal{F} = \{\, x \mid x^T A_i x + 2a_i^T + \alpha_i \leq 0,\ i \in I \,\}$. Then $\overline{\mathcal{C}}(\mathcal{F}) = \mathcal{S}(\mathcal{F})$, where $\mathcal{S}(\mathcal{F})$ is the Shor relaxation of $\mathcal{F}$, if for all nonzero $x \in \mathbb{R}^n$, if $x^T A_i + 2a_i^T x + \alpha_i = 0$ for some $i \in I$, then $x^T A_j x + 2a_j^T x + \alpha_j < 0$ for all $j \in I \setminus i$.*

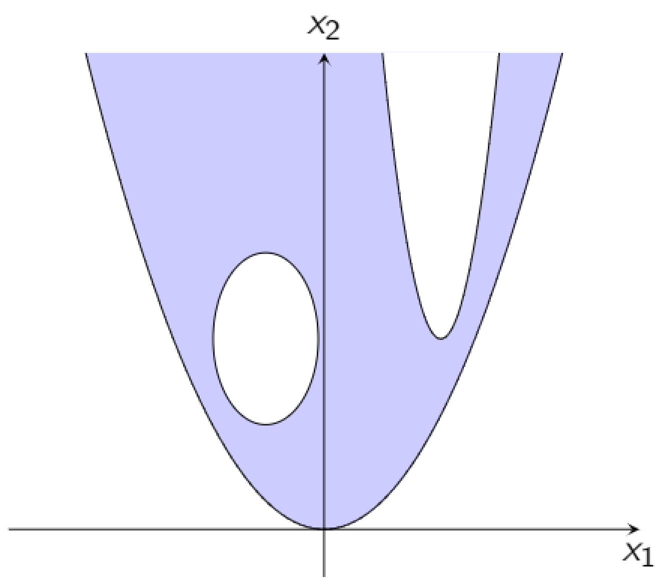


Figure 2.4: Non-interacting constraints

2.3 Other Reformulations of $\overline{\mathcal{C}}(\mathcal{F})$

As expected, not every quadratically defined set $\mathcal{F}$ has an exact Shor relaxation. The results in this section, for the most part, come from different techniques to create new valid inequalities through reformulation linearization technique (RLT) and through analyzing the geometry of a set and its lifted space.

2.3.1 Low Dimensional Polyhedral Sets

Solving general nonconvex quadratic programs over polyhedral sets, i.e.

$$\mathcal{F}_P = \{ x \in \mathbb{R}^n \mid Ax = b,\ x \geq 0 \}, \tag{POLY}$$

can be difficult. In convexifying the problem, one can use copositive optimization to create families of copositive cuts. Copositive optimization is linear optimization over the convex cone of copositive matrices. The dual of this linear problem is a linear optimization problem of the cone of completely positive matrices. One benefit of finding these copositive cuts is that the cuts have no dependence on the objective or the values of A or b but instead on the constraint $x \geq 0$ [7]. This approach extends beyond polyhedrons to sets with binary constraints [6], ellipsoids [7], etc.

This section provides several results using copositive optimization with a focus on polyhedral sets of dimension $n \leq 4$. First, let's look at a one-dimensional interval set.

Consider the standard simplex

$$\mathcal{F}_{Simp} = \{ x \in \mathbb{R}^n_+ \mid e^T x = 1 \}, \tag{SIMP}$$

where $n \leq 4$ and $e \in \mathbb{R}^n$ is the vector with each component being 1. Lifting (SIMP) into the semi-definite matrix space, there is the following formulation

$$\{ (x, X) \mid E \bullet X = 1,\ x \geq 0,\ X \succeq xx^T,\ X \in CP \}, \tag{SDP-SIMP}$$

where $E = ee^T$ and CP is the set of completely positive matrices, i.e. $X \in CP$ if and only if $X = \sum_{i=1}^k x_i x_i^T,\ x_i \geq 0,\ i = 1, \ldots, k$. As expected, solving over CP is difficult. However, there is a known relation between CP and the doubly non-negative cone DNN. For all dimensions n, CP $\subseteq DNN$ with equality for $n \leq 4$ [24], as seen in Lemma 7.

Lemma 7 ([24]). *Let $n \leq 4$, and let $\mathcal{S}^n$ be the space of $n \times n$ symmetric matrices. If $Z \in \mathcal{S}^n$ is positive semidefinite with nonnegative entries, then there exists a collection of n-dimensional nonnegative vectors $\{z_i\}$ such that $Z = \sum_i z_i z_i^T$.*

This leads to the following result.

Theorem 5 (Standard Simplex, $n \leq 4$. [2]). *For all $n \leq 4$, if $\mathcal{F} = \{\, x \in \mathbb{R}^n \mid e^T x = 1,\ x \geq 0 \,\}$ then $\overline{\mathcal{C}}(\mathcal{F})$ is defined by $\overline{\mathcal{C}}(\mathcal{F}) = \{\, (x, X) \mid E \bullet X = 1,\ x \geq 0,\ X \succeq xx^T,\ X \geq 0 \,\}$.*

This section wraps up with convex hull results for triangles (tetrahedrons) and quadrilaterals. That is,

$$\mathcal{F}_{tri} = \{\, x \in \mathbb{R}^2 \mid Ax \leq b \,\}, \quad A \in \mathbb{R}^{3\times 2} \tag{TRI}$$

and

$$\mathcal{F}_{quad} = \{\, x \in \mathbb{R}^2 \mid Ax \leq b \,\}, \quad A \in \mathbb{R}^{4\times 2}, \tag{QUAD}$$

where both sets are bounded polygons with nonempty interior. For both sets, define $C = (b, -A)$ such that the inequality $Cy \geq 0$ defines the polyhedral cone in $\mathbb{R}^3$, $\mathbb{R}^4$ respectively. The analysis of both polygons follow the same idea by considering the set

$$\mathcal{K} = \overline{\text{conv}}\,\{\, yy^T \mid Cy \geq 0 \,\},$$

where $\mathcal{K}$ is the closed conic hull of matrices yy^T with $Cy \geq 0$. So $\mathcal{K}$ is the closure of all possible sums of such matrices yy^T. After an analysis of this set with Lemma 7, there is Lemma 8.

Lemma 8. *Regarding* (TRI) *and* (QUAD), *$\mathcal{K} := \overline{\text{conv}}\,\{\, yy^T \mid Cy \geq 0 \,\} = \{\, Y \succeq 0 \mid CYC^T \geq 0 \,\}$, where $C := (b, -A)$.*

Lemma 8 leads to the exact formulation of the lifted closed convex hull $\overline{\mathcal{C}}(\mathcal{F}_{tri})$ and $\overline{\mathcal{C}}(\mathcal{F}_{quad})$. Before stating this result, recall that

$$Y(x, X) := \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix}.$$

Theorem 6 (Convex Hull of Triangles, Tetrahedron, and Quadrilaterals [7])**.** *Regarding* (TRI) *and* (QUAD)*, it holds that*

$$\begin{aligned}\overline{\mathcal{C}}(\mathcal{F}) &:= \overline{\text{conv}}\,\{\,(x, xx^T) \mid x \in \mathcal{F}\,\} \\ &= \{\,(x, X) \mid Y(x, X) \in \mathcal{K}\,\} \\ &= \left\{ (x, X) \;\middle|\; \begin{array}{c} Y = Y(x, X) \\ Y \succeq 0, \;\; CYC^T \geq 0 \end{array} \right\}\end{aligned}$$

where $C := (b, -A)$.

Theorem 6 also applies to tetrahedrons, as this extension comes from $n \leq 4$.

2.4 Non-Intersecting Constraints

In this section, the focus is on *non-intersecting* constraints. This term will be explored in different contexts for the two subsections.

2.4.1 Ball Constraint with Non-Intersecting Linear Constraints

Before presenting the results for a set defined by a ball constraint with non-intersecting linear constraints, there will be a re-derivation of the exactness when $\mathcal{F}$ is defined by a single constraint. This focus is on an ellipsoidal set, (ELL). In this formulation, the set is defined by the unit ball inequality $||x|| \leq 1$. Recall that this result was already presented in Theorem 1 for an arbitrary set defined by one quadratic inequality. This follows the approach given by [7] with the focus on ellipsoids ($A \succ 0$). This analysis is sufficient since, by affine transformations, any ellipsoid can be expressed as (ELL).

$$\mathcal{F}_{ell} = \{\, x \in \mathbb{R}^n \mid \|x\| \leq 1 \,\}\,. \tag{ELL}$$

This set is equivalent to the feasible region of the Trust Region Subproblem (TRS). Similar with the previous results, the interest lies in determining the exact formulation of $\overline{\mathcal{C}}(\mathcal{F}_{ell})$. Displaying this result requires the introduction of the second-order cone in R^{n+1}:

$$\begin{aligned}\mathcal{L} &= \{\, y \in \mathbb{R}^{n+1} \mid \sqrt{y_2^2 + y_3^2 + \cdots + y_{n+1}^2} \le y_1 \,\} \\ &= \{\, y \in \mathbb{R}^{n+1} \mid y_2^2 + y_3^2 + \cdots + y_{n+1}^2 \le y_1^2 \,\} \\ &= \{\, y \in \mathbb{R}^{n+1} \mid y_1 \ge 0,\ y^T L y \ge 0 \,\}\end{aligned}$$

where L is a diagonal matrix with entries $(1, -1, \ldots, -1)$. Similar to Theorem 6, the defined convex hull of matrices yy^T, where $y \in \mathcal{L}$, is

$$\mathcal{K} \coloneqq \overline{\text{conv}}\, \{\, yy^T \mid x \in \mathcal{L} \,\} = \{\, Y \succeq 0 \mid \mathcal{L} \bullet Y \ge 0 \,\}.$$

Using the set $\mathcal{K}$, the results of [31] (a single quadratic inequality), are recovered for bounded sets with Theorem 1.

Theorem 7 (Ellipsoids, [7])**.** *Regarding* (ELL)*, it holds that*

$$\begin{aligned}\overline{\mathcal{C}}(\mathcal{F}_{ell}) &\coloneqq \overline{\text{conv}}\, \{\, (x, xx^T) \mid x \in \mathcal{F} \,\} \\ &= \{\, (x, X) \mid Y(x, X) \in \mathcal{K} \,\} \\ &= \{\, (x, X) \mid Y = Y(x, X),\ Y \ge 0,\ L \bullet Y \ge 0 \,\}.\end{aligned}$$

Or in terms of (x, X)*, there is the alternate representation*

$$\overline{\mathcal{C}}(\mathcal{F}_{ell}) \coloneqq \{\, (x, X) \mid X \succeq xx^T,\ \ I_n \bullet X \le 1 \,\} = \mathcal{S}(\mathcal{F}_{ell}),$$

where I_n *is the* $n \times n$ *identity matrix.*

Recall that for two quadratic constraints $q_1(x) \le 0,\ q_2(x) \le 0$ to be non-interacting, then if there exists $\hat{x}$ such that $q_1(\hat{x}) = 0$, then $q_2(\hat{x}) < 0$. This can be relaxed to non-intersecting if there exists $\hat{x}$ such that $q_1(\hat{x}) = 0$, then $q_2(\hat{x}) \le 0$. Geometrically, this implies that the boundaries of both constraints intersect at a given point.

The focus now shifts to the feasible region defining the extended trust region subproblem (eTRS), which is the trust region intersected by m linear inequality constraints, defined below as

$$\mathcal{F}_m := \left\{ x \in \mathbb{R}^n \;\middle|\; \begin{array}{c} ||x|| \leq 1 \\ a_i^T x \leq \alpha_i, \;\; i = 1, \dots, m \end{array} \right\}. \tag{F_m}$$

When $m = 1$, there are the results of [7, 31]. Adding a single linear constraint with $m = 1$ and denoting the SDP relaxation as R_1, the relaxation requires a second-order cone (SOC) constraint and has presentation:

$$R_1 := \left\{ x \in \mathbb{R}^n \;\middle|\; \begin{array}{c} I_n \bullet X \leq 1, \;\; X \succeq xx^T \\ ||\alpha_1 x - X a_1|| \leq \alpha_1 - a_1^T x \end{array} \right\}. \tag{R_1}$$

The SOC constraint $||\alpha_1 x - X a_1|| \leq \alpha_1 - a_1^T x$ is constructed by the second-order cone reformulation-linearization technique (SOC-RLT) in combination with the constraints $||x|| \leq 1$ and $a_i^T x \leq \alpha_i$ [8]. This is also shown to be an exact representation in [31]. Since the relaxation becomes more complicated with extra constraints, it can be assumed that the generalization will contain more constraints. For $m = 2$, [8, 41] derived the relaxation

$$R_2 := \left\{ x \in \mathbb{R}^n \;\middle|\; \begin{array}{c} I_n \bullet X \leq 1, \;\; X \succeq xx^T \\ ||\alpha_1 x - X a_1|| \leq \alpha_1 - a_1^T x, \;\; i = 1, 2 \\ \alpha_1 \alpha_2 - \alpha_2 a_1^T x - \alpha_1 a_2^T x + a_1^T X a_2 \geq 0 \end{array} \right\}. \tag{R_2}$$

This result can be further generalized in [11] under an additional assumption. This assumption is that the linear constraints are non-intersecting inside of the ball region.

Theorem 8 (Ball with Non-Intersecting Linear Constraints, [11])**.** *In regards to $\mathcal{F}_m$ in (F_m), if for all $i < j$, there exists no $x \in \mathcal{F}_m$ such that $a_i^T = \alpha_i$ and $a_j^T x = \alpha_j$, then*

$$\overline{\mathcal{C}}(\mathcal{F}_m) := \left\{ (x, X) \in \mathbb{R}^n \times \mathcal{S}^n \;\middle|\; \begin{array}{c} I_n \bullet X \leq 1, \;\; X \succeq xx^T \\ ||\alpha_1 x - X a_1|| \leq \alpha_1 - a_1^T x, \;\; i = 1, \dots, m \\ \alpha_i \alpha_j - \alpha_j a_i^T x - \alpha_i a_j^T x + a_i^T X a_j \geq 0, \;\; 1 \leq i < j \leq m \end{array} \right\}.$$

2.4.2 Bounded Quadratic Sets with Bounded Hollows

The previous section referred to non-intersecting constraints as linear constraints that can only intersect outside of the ellipsoid. In [37], non-intersecting constraints are contained inside of the larger set. The results of this paper consider a complicated, bounded, quadratically defined set $\mathcal{G}$ that can be decomposed into two sets $\mathcal{F}$ and $\mathcal{H}$ such that

$$\mathcal{F} := \left\{ x \in \mathbb{R}^n \mid x^T A_i x + 2a_i^T x + \alpha_i \leq 0,\ i = 1, \dots m \right\},$$

and

$$\mathcal{H} := \left\{ x \in \mathbb{R}^n \mid x^T W_i x + 2w_i^T x + \omega_i \geq 0,\ i = 1, \dots m \right\}$$

where $\mathcal{F}$ is a bounded, full-dimensional set and $\mathcal{H}$ is defined by positive matrices $W_i \in \mathcal{S}^n$, $w_i \in \mathbb{R}^n$, and $\omega_i \in \mathbb{R}$, and $\mathcal{G} = \mathcal{F} \cap \mathcal{H}$. One way of expressing this is to say that "$\mathcal{H}$ induces non-intersecting hollows in $\mathcal{F}$". Since this work is generalized in Chapter 4, only the result of this paper is presented.

Theorem 9 (Bounded Quadratic Regions with Non-Intersecting Hollows, [37]). *If $\mathcal{H}$ induces non-intersecting hollows in $\mathcal{F}$, then $\overline{\mathcal{C}}(\mathcal{G}) = \overline{\mathcal{C}}(\mathcal{F}) \cap \mathcal{H}$.*

Chapter 3

Asymptotic Cones

This chapter focuses on describing the end behavior of unbounded, nonconvex sets. For a nonempty set $\mathcal{F} \subseteq \mathbb{R}^n$, the asymptotic cone of $\mathcal{F}$ is defined as

$$\mathcal{F}_\infty := \left\{ d \in \mathbb{R}^n \;\middle|\; \exists\, t_k \to \infty, \{x_k\}_{k \geq 0} \subseteq \mathcal{F} \text{ such that } \lim_{k \to \infty} \frac{x_k}{t_k} = d \right\}.$$

We define the nonzero elements of $\mathcal{F}_\infty$ as asymptotic directions. These directions appear in many applications, such as defining cosmic compactness in [29], convexifying sets defined by quadratic functions [15, 20], and determining the existence of optimal solutions of QCQPs [5, 32]. Throughout this document, a more direct definition of asymptotic directions will be used. Setting the sequence $t_k = \|x_k\|$, the following expression for $\mathcal{F}_\infty$ can be used.

Proposition 2 (Normalized Set [4])**.** *Let $\mathcal{F} \subseteq \mathbb{R}^n$ be nonempty. Then $\mathcal{F}_\infty = \{ \lambda d \mid \lambda \geq 0,\ d \in \mathcal{F}_N \}$ where*

$$\mathcal{F}_N := \left\{ d \in \mathbb{R}^n \;\middle|\; \exists \{x_k\} \subseteq \mathcal{F},\ \|x_k\| \to \infty \textit{ with } \lim_{k \to \infty} \frac{x_k}{\|x_k\|} = d \right\}.$$

Using Proposition 2, it is easier to see that the asymptotic cone $\mathcal{F}_\infty$ provides insight about the end behavior at infinity. The generalization of the recession cone of $\mathcal{F}$ can be considered as $\mathcal{F}_\infty$. In particular, when $\mathcal{F}$ is convex, $\mathcal{F}_\infty$ is the recession cone of $\mathcal{F}$. For a more detailed discussion on $\mathcal{F}_\infty$ and the behavior of an asymptotic function through its epigraph, the reader is referred to [4]. However, the following lemma includes properties of $\mathcal{F}_\infty$ that will be utilized and expanded upon in this document.

Lemma 9 ([4])**.** *Let $\mathcal{F} \subseteq \mathbb{R}^n$ be nonempty. Then:*

1. *$\mathcal{F}_\infty$ is a closed cone;*

2. *$(\operatorname{cl} \mathcal{F})_\infty = \mathcal{F}_\infty$;*

3. *if $\mathcal{F}$ is a cone, then $\mathcal{F}_\infty = \operatorname{cl} \mathcal{F}$;*

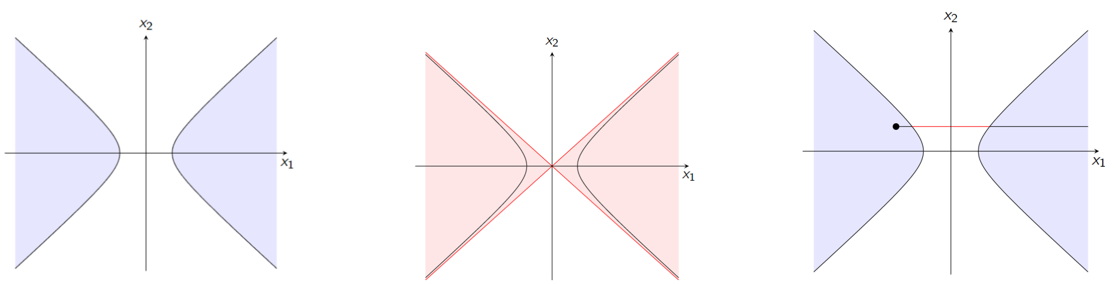
4. *$\mathcal{F}$ is bounded if and only if $\mathcal{F}_\infty = \{0\}$;*

5. *for any $\emptyset \neq T \subseteq \mathcal{F}$, $T_\infty \subseteq \mathcal{F}_\infty$;*

6. *for any $T \subseteq \mathbb{R}^n$ such that $\mathcal{F} \cap T \neq \emptyset$, $(\mathcal{F} \cap T)_\infty \subseteq \mathcal{F}_\infty \cap T_\infty$;*

7. *if S is a closed convex set that contains no line, then $\mathcal{F} = \operatorname{conv}(\operatorname{ext}(\mathcal{F})) + \mathcal{F}_\infty$.*

The asymptotic cone can have properties that relate to those of the recession cone. Consider a convex set $\mathcal{F}$ with recession direction d and point $x \in \mathcal{F}$. Since d is a recession direction, for all $\lambda \geq 0$, $x + \lambda d \in \mathcal{F}$. For a nonconvex set $\mathcal{F}$, the direction d can carry $x + \lambda d$ out of $\mathcal{F}$ for large enough lambda. However, there may be a positive interval of λ, say (λ_1, λ_2) such that for $x \in \mathcal{F}$ and asymptotic direction d, then $x + \lambda d \notin \mathcal{F}$ for $\lambda \in (\lambda_1, \lambda_2)$. This phenomena can be seen in Figure 3.1. In subfigure 3.1c, one can see that as they follow the $d = (1, 0)$ direction, the red interval is the set of $\lambda's$ such that $x + \lambda d \notin \mathcal{F}$. These directions are called horizon directions described in the following definition.



(a) $\mathcal{F}$ (b) $\mathcal{F}_\infty$ (c) Horizon Direction Example

Figure 3.1: Three graphs describing properties of a quadratically defined set $\mathcal{F}$. 3.1a is a graphical image of $\mathcal{F}$. 3.1b is the asymptotic cone of $\mathcal{F}$ presented over $\mathcal{F}$. 3.1c is an example of a local horizon direction of $\mathcal{F}$ where the red interval is the interval with $x + \lambda d \notin \mathcal{F}$.

Definition 1 (Horizon Directions [5])**.** *Given a closed set $\mathcal{F} \subseteq \mathbb{R}^n$, an asymptotic direction $d \in \mathcal{F}_\infty$ is a horizon direction with respect to a set S, if, for every $x \in S$, there exists a scalar $\bar{\mu} \geq 0$ such* *that $x + \mu d \in \mathcal{F}$ for all $\mu \geq \bar{\mu}$. We say that d is a global horizon direction if $S = \mathbb{R}^n$, and it is a local horizon direction if $S = \mathcal{F}$.*

The last definition to be discussed is the idea of retractiveness. This property is used in the discussion of the existence of an optimal solution for a given problem by guaranteeing that an intersection of retractive nested sequence of nonempty closed sets has nonempty intersection [5].

Definition 2 (Retractive Directions [5])**.** *Given a closed set $\mathcal{F} \subseteq \mathbb{R}^n$, an asymptotic direction $d \in \mathcal{F}_\infty$ is a retractive direction if, for every corresponding asymptotic sequence $\{ x_k \}_k \subseteq \mathcal{F}$, there exists an integer $\bar{k}$ such that*

$$x_k - d \in \mathcal{F}, \quad \forall k \geq \bar{k}.$$

A set $\mathcal{F}$ is called retractive if all its asymptotic directions are retractive.

There exists a weaker definition for retractiveness: d is retractive if for every corresponding asymptotic sequence $\{ x_k \}$, there exists a bounded sequence of positive scalars $\{ \gamma_k \}$ and $\bar{k} \geq 0$ such that $x_k - \gamma_k d \in \mathcal{F}$ for all $k \geq \bar{k}$. While this definition is present in literature [32], it does not work well when intersecting multiple sets [5]. As such, Definition 2 will be the preferred choice for this document. This property of retractiveness is used primarily in determining the existence of optimal solutions [5, 32], and the benefits of retractive directions will be discussed in Chapter 6.

3.1 Sets Defined by One Quadratic Constraint

It is well known that when $\mathcal{F}$ is a polyhedron, $\mathcal{F}_\infty$ (i.e. Rec($\mathcal{F}$)) can be explicitly defined. To the author's knowledge, there are no current results about the general characterization of $\mathcal{F}_\infty$ for $\mathcal{F}$ defined by multiple quadratic constraints. However, Dickinson et al [15] provided the following characterization for $\mathcal{F}$ defined by a single quadratic constraint.

Proposition 3 ([15])**.** *For $\mathcal{F} = \{ x \in \mathbb{R}^n \mid x^T A x + 2a^T x + \alpha \leq 0 \} \neq \emptyset$, we have*

$$\mathcal{F}_\infty = \begin{cases} \{ d \in \mathbb{R}^n \mid d^T A d \leq 0 \}, & \text{if } A \not\succeq 0 \\ \{ d \in \mathbb{R}^n \mid d^T A d \leq 0,\ a^T d \leq 0 \}, & \text{if } A \succeq 0. \end{cases}$$

As can be expected, the asymptotic cone depends on the definiteness of the Hessian A with an included linear term for $A \succeq 0$. Chapter 4 works with the boundary of a quadratically defined set and requires the asymptotic cone of a set defined by an equality constraint. Similar to the positive semidefinite case of Proposition 3, the linear term of $\mathcal{F}$ has an impact on the asymptotic cone of the boundary.

Lemma 10 ([20]). *Suppose that* $\mathcal{F} = \{ x \in \mathbb{R}^n \mid x^T A x + 2a^T x + \alpha = 0 \}$ *is nonempty and* A *has a positive eigenvalue. If* d *satisfies* $d^T A d = 0$ *and* $a^T d < 0$*, then* $d \in S_\infty$.

Proof. Let v be any vector such that $v^T A v > 0$. Consider the following bivariate quadratic function

$$\begin{aligned} f(k, \Delta) &:= (kd + \Delta v)^T A (kd + \Delta v) + 2a^T (kd + \Delta v) + \alpha \\ &= (v^T A v)\Delta^2 + 2(kd^T A v + a^T v)\Delta + (2ka^T d + \alpha). \end{aligned}$$

Since $a^T d < 0$, there exists $K \in \mathbb{R}$ such that $f(k, 0) = 2ka^T d + \alpha < 0$ for all $k \geq K$. For each $k \geq K$, let $\delta_k := a^T v + kd^T A v$ and

$$\Delta_k := \begin{cases} \left(-\delta_k + \sqrt{\delta_k^2 - (v^T A v)(2ka^T d + \alpha)}\right) / \left(v^T A v\right), & \text{if } \delta_k \geq 0 \\ \left(-\delta_k - \sqrt{\delta_k^2 - (v^T A v)(2ka^T d + \alpha)}\right) / \left(v^T A v\right), & \text{if } \delta_k < 0. \end{cases} \tag{3.1}$$

Then $f(k, \Delta_k) = 0$ and $\lim_{k\to\infty} \Delta_k / k = 0$. This implies that $d \in \mathcal{F}_\infty$ as $kd + \Delta_k v \in S$ and $\lim_{k\to\infty}(kd + \Delta_k v)/k = d$. (See Figure 3.2 for the three cases of the level set $\{ (k, \Delta) \mid f(k, \Delta) = 0 \}$.)

□

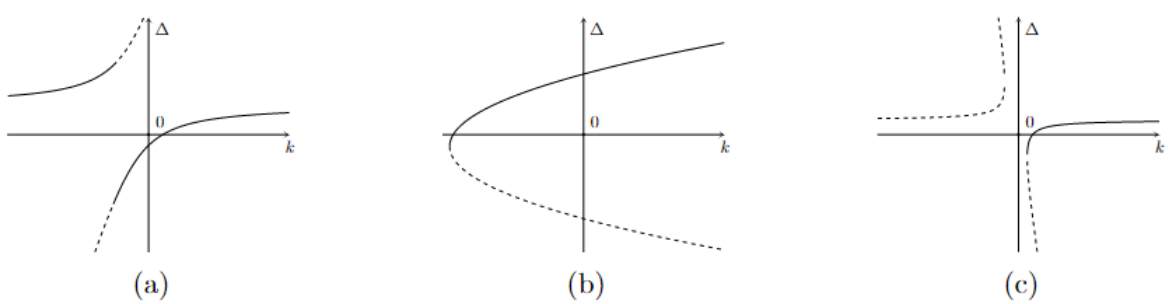


Figure 3.2: Plots of possible level sets $\{ (k, \Delta) \mid f(k, \Delta) = 0 \}$ in the proof of Lemma 10. The solid segments represent how Δ_k is defined in (3.1)

.

Lemma 11 ([20]). *Suppose* $S = \{ x \in \mathbb{R}^n \mid x^T A x + 2a^T x + \alpha = 0 \}$ *is nonempty. If* d *satisfies* $d^T A d = 0$ *and* $a^T d = 0$*, then* $\pm d \in S_\infty$.

Proof. The proof is straightforward when $A = 0$. Now suppose that $A \neq 0$, and v is a vector in S. Similar to the proof of Lemma 10, let

$$\begin{aligned} f(k, \Delta) :&= (kd + \Delta v)^T A (kd + \Delta v) + 2a^T (kd + \Delta v) + \alpha \\ &= (v^T A v)\Delta^2 + 2(kd^T A v + a^T v)\Delta + \alpha. \end{aligned}$$

We consider two cases. First, if $d^T A v = 0$, then $f(k, 1) = v^T A v + 2a^T v + \alpha = 0$ for all $k \in \mathbb{R}$. Therefore, $d \in S_\infty$ as $kd + v \in S$ and $\lim_{k\to\infty}(kd + v)/k = d$. Second, if $d^T A v \neq 0$, then $|kd^T A v + a^T v| \to \infty$ as $k \to \infty$. For sufficiently large k, if $v^T A v \neq 0$, we set Δ_k as defined in (3.1) (with $a^T d = 0$). If $v^T A v = 0$, we set $\Delta_k := -\alpha/(2kd^T A v + 2a^T v)$. With either definition of Δ_k, $kd + \Delta_k \in S$ and $\lim_{k\to\infty} \Delta_k = 0$. Therefore, $d \in S_\infty$. Replacing d with $-d$ in the above proof, we have $-d \in S_\infty$.

□

Lemmas 10 and 11 add an additional level of focus on the linear term $a \in \mathbb{R}^n$. In the positive semidefinite and negative semidefinite cases, the linear term helps determine the shape of the feasible region. In Figure 3.3, A is positive semidefinite and there are two cases. In (a), $a \notin \text{Range}(A)$ and as such, $d \in (\text{bd}\,\mathcal{F})_\infty$ does not guarantee that $-d \in (\text{bd}\,\mathcal{F})_\infty$. Likewise, in (b), $a \in \text{Range}(A)$ and $\mathcal{F}$ contains a lineality space. This effect when A is semidefinite is shown in the following proposition.

Proposition 4 ([20])**.** *For* $\mathcal{F} = \{\, x \in \mathbb{R}^n \mid x^T A x + 2a^T x + \alpha = 0 \,\} \neq \emptyset$*, we have*

$$\mathcal{F}_\infty = \begin{cases} \{\, d \in \mathbb{R}^n \mid d^T A d = 0,\ a^T d \leq 0 \,\}, & \textit{if } A \succeq 0 \textit{ and } A \neq 0 \\ \{\, d \in \mathbb{R}^n \mid d^T A d = 0,\ a^T d \geq 0 \,\}, & \textit{if } A \preceq 0 \textit{ and } A \neq 0 \\ \{\, d \in \mathbb{R}^n \mid d^T A d = 0 \,\}, & \textit{if } A \textit{ is indefinite} \\ \{\, d \in \mathbb{R}^n \mid a^T d = 0 \,\}, & \textit{if } A = 0. \end{cases}$$

Proof. For the forward containment, note that

$$\begin{aligned}
S_\infty &= \big(\{\, x \in \mathbb{R}^n \mid x^T A x + 2a^T x + \alpha \le 0 \,\} \cap \{\, x \in \mathbb{R}^n \mid x^T A x + 2a^T x + \alpha \ge 0 \,\}\big)_\infty \\
&\subseteq \{\, x \in \mathbb{R}^n \mid x^T A x + 2a^T x + \alpha \le 0 \,\}_\infty \cap \{\, x \in \mathbb{R}^n \mid x^T A x + 2a^T x + \alpha \ge 0 \,\}_\infty \\
&= \begin{cases}
\{\, d \in \mathbb{R}^n \mid d^T A d = 0,\ a^T d \le 0 \,\}, & \text{if } A \succeq 0 \text{ and } A \ne 0 \\
\{\, d \in \mathbb{R}^n \mid d^T A d = 0,\ a^T d \ge 0 \,\}, & \text{if } A \preceq 0 \text{ and } A \ne 0 \\
\{\, d \in \mathbb{R}^n \mid d^T A d = 0 \,\}, & \text{if } A \text{ is indefinite} \\
\{\, d \in \mathbb{R}^n \mid a^T d = 0 \,\}, & \text{if } A = 0,
\end{cases}
\end{aligned}$$

where the last equality comes from Proposition 3.

For the reverse containment, note that A has a positive eigenvalue in the first and third cases, and $-A$ has a positive eigenvalue in the second and third cases. Then the reverse containment is a direct consequence of Lemmas 10 and 11. □

With formal definitions describing the geometry of the asymptotic cone of $\mathcal{F}$, the horizon and retractive directions of $\mathcal{F}$ can be understood. Since the definitions relied on the definiteness of the Hessian A, it is expected that additional properties of $\mathcal{F}_\infty$ may also rely on A.

3.1.1 Horizon Directions and Retractive Direcions

Recall that for $d \in \mathcal{F}_\infty$ to be a local (global) horizon direction, starting at any point $x \in \mathcal{F}$ ($x \in \mathbb{R}^n$), there exists $\bar{\lambda} \ge 0$ such that $x + \lambda d \in \mathcal{F}$, for all $\lambda \ge \bar{\lambda}$, and for d to be a retractive direction, for all corresponding asymptotic sequences $\{x_k\}_k \subseteq \mathcal{F}$, $x_k - d \in \mathcal{F}$ for all $k \ge \bar{k} \ge 0$.

Similar to partitioning $\mathcal{F}_\infty$ by the definiteness of the Hessian A, a similar approach applies to both horizon and retractive directions. Before considering $A \ne 0$, if $A = 0$, then $\mathcal{F}$ is a halfspace, which is a retractive set [5]. In fact, if $\mathcal{F} = \{\, x \mid a^T x + \alpha \le 0 \,\}$, then $\mathcal{F}_\infty = \{\, x \mid a^T x \le 0 \,\}$. With this in mind, $\{\, x \mid a^T x < 0 \,\}$ is the set of global horizon directions while $\{\, x \mid a^T x = 0 \,\}$ is the set of local horizon directions. Extending this to $A \ne 0$, we have the following propositions.

Proposition 5 ($A \succeq 0,\ A \ne 0$)**.** *Let* $\mathcal{F} = \{\, x \mid x^T A x + 2a^T x + \alpha \le 0 \,\}$ *be a nonempty set such that* $A \succeq 0$ *nonzero. Then*

1. $(\mathcal{F}_\infty)_= := \{\, d \ne 0 \mid Ad = 0,\ a^T d = 0 \,\}$ *is the set of retractive local horizon directions.*

2. $(\mathcal{F}_\infty)_< := \{\, d \neq 0 \mid Ad = 0,\ a^T d < 0 \,\}$ *is the set of non-retractive, global horizon directions.*

Proof. Let $d \in \mathcal{F}_\infty$ with corresponding asymptotic sequence $\{\, x_k \,\} \subseteq \mathcal{F}$, $\gamma > 0$, and note that

$$f(x + \gamma d) = x^T Ax + 2a^T x + \alpha + 2\gamma a^T d.$$

If d is a retractive direction then for asymptotic sequences $\{\, x_k \,\} \subseteq \mathcal{F}$, $f(x_k - d) = f(x_k) - 2a^T d \leq 0$ for $k \geq \bar{k}$. By Propositions 3 and 4, $\mathcal{F}_\infty = \mathrm{bd}(\mathcal{F})_\infty$ for $A \succeq 0$. Then there exists $\{\, \hat{x}_k \,\} \subseteq \mathrm{bd}(\mathcal{F}) \subseteq \mathcal{F}$ such that $\{\, \hat{x}_k \,\}$ is a corresponding asymptotic sequence to d. Then for all k, $f(\hat{x}_k - d) = f(\hat{x}_k) - 2a^T d = -2a^T d \leq 0$ for all $k \geq \hat{k}$. Therefore, $a^T d = 0$ and $d \in (\mathcal{F}_\infty)_=$.

If d is a global horizon direction then for all $x \notin \mathcal{F}$, there exists $\bar{\gamma} \geq 0$ such that for all $\gamma \geq \bar{\gamma}$,

$$f(x + \gamma d) = \underbrace{f(x)}_{>0} + 2\gamma a^T d \leq 0.$$

Therefore, $a^T d < 0$ and $d \in (\mathcal{F}_\infty)_<$. If d is a local horizon direction that is not global then for all $x \in \mathcal{F}$, there exists $\hat{\gamma} \geq 0$ such that for all $\gamma \geq \hat{\gamma}$,

$$f(x + \gamma d) = \underbrace{f(x)}_{\leq 0} + 2\gamma a^T d \leq 0.$$

Therefore, $a^T d = 0$ and $d \in (\mathcal{F}_\infty)_=$.

If $d \in (\mathcal{F}_\infty)_=$, then for all $\gamma > 0$, $f(x + \gamma d) = f(x)$. This implies that d is a local horizon direction and not a global horizon direction since $f(x + \gamma d) \leq 0$ if and only if $f(x) \leq 0$. To see that d is retractive, $f(x_k - \gamma d) = f(x_k) \leq 0$.

If $d \in (\mathcal{F}_\infty)_<$, then for all $\gamma > 0$, $f(x + \gamma d) < f(x)$. Therefore, for any $x \in \mathbb{R}^n$, there exists $\gamma^* > 0$ such that $f(x + \gamma d) \leq 0$ for all $\gamma \geq \gamma^*$. To see that d is not retractive, note that

$$(\mathcal{F}_\infty)_< \subseteq \left\{\, d \;\middle|\; d^T Ad = 0,\ a^T d \leq 0 \,\right\} = \left(\left\{\, x \;\middle|\; x^T Ax + 2a^T x + \alpha = 0 \,\right\}\right)_\infty.$$

Therefore, for any $d \in (\mathcal{F}_\infty)_<$, d is an asymptotic direction of $\left\{\, x \;\middle|\; x^T Ax + 2a^T x + \alpha = 0 \,\right\}$. That is, there is a sequence $\{x_k\}$ such that $x_k^T Ax_k + 2a^T x_k + \alpha = 0$ for every k, $\lim_{k\to\infty} \|x_k\| = \infty$, and

$\lim_{k\to\infty} \frac{x_k}{\|x_k\|} = d$. We show that $f(x_k - d) > 0$ for all k. In fact, since $f(x_k) = 0$,

$$\begin{aligned} f(x_k - d) &= x_k^T A x_k - 2x_k^T \underbrace{Ad}_{=0} + d^T \underbrace{Ad}_{=0} + 2a^T x_k - 2a^T d + a \\ &= -2a^T d > 0. \end{aligned}$$

Therefore, d is not retractive. $\square$

The two cases in Proposition 5 give geometric insight to what makes a direction retractive. Consider the cases with Figure 3.3. In (b), the case describes when $a \in \text{Range}(A)$ and the boundary of the quadratically defined set has linear features. Since the asymptotic directions are linear and are defined by the set and its boundary, all directions are local directions and $x_k - d \in \mathcal{F}$ for all k. However, the case in (a), covers when the boundary of $\mathcal{F}$ is asymptotically sublinear. Then for every sequence $\{x_k\}$ along the boundary of $\mathcal{F}$, $x_k - d \notin \mathcal{F}$ for all k. Therefore, all asymptotic directions of this case are global horizon directions but not retractive.

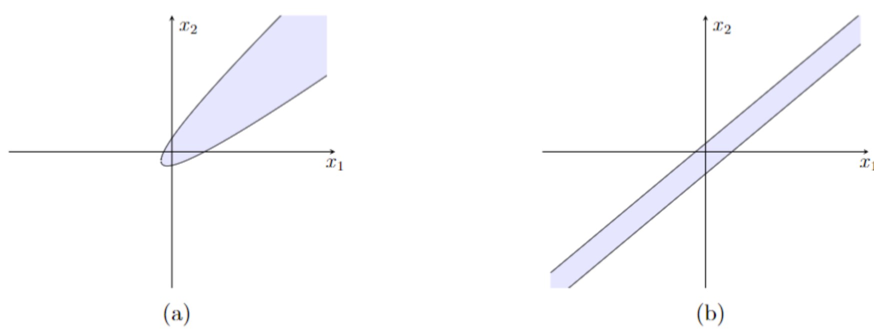


Figure 3.3: The two 2D cases when A is positive semidefinite. In (a), $a \notin \text{Range}(A)$ and the boundary of $\mathcal{F}$ has a curved behavior. In (b), $a \in \text{Range}(A)$ and as a result, the boundary of $\mathcal{F}$ is linear.

In terms of semidefinite sets, the positive semidefinite case is more interesting than that of the negative semidefinite. When $A \preceq 0$ is nonzero, Proposition 3 states that $\mathcal{F}_\infty = \left\{ x \mid x^T A x \leq 0 \right\} = \mathbb{R}^n$. Even though the asymptotic cone contains all real vectors, $\mathcal{F}$ does not cover $\mathbb{R}^n$, and the following proposition provides a geometric intuition to how the set behaves.

Proposition 6 ($A \preceq 0,\ A \neq 0$)**.** *Let $\mathcal{F} = \{ x \mid x^T A x + 2a^T x + \alpha \leq 0 \}$ be a nonempty set such that $A \preceq 0$ nonzero. Then $\mathcal{F}$ is retractive [5] and*

1. $(\mathcal{F}_\infty)_1 := \left\{ d \neq 0 \,\middle|\, d^T A d < 0 \right\} \cup \left\{ d \neq 0 \,\middle|\, Ad = 0, \quad a^T d < 0 \right\}$ *is the set of global horizon directions*

2. $(\mathcal{F}_\infty)_2 := \left\{ d \neq 0 \,\middle|\, Ad = 0, \quad a^T d = 0 \right\}$ *is the set of local horizon directions that are not global.*

3. $(\mathcal{F}_\infty)_3 := \left\{ d \neq 0 \,\middle|\, Ad = 0, \quad a^T d > 0 \right\}$ *is the set of directions that are not local horizon directions.*

Proof. Since the complement of $\mathcal{F}$ is convex, then by [5], $\mathcal{F}$ is retractive.

Let $d \in \mathcal{F}_\infty$, $\gamma > 0$, and note that

$$f(x + \gamma d) = x^T A x + 2a^T x + \alpha + 2\gamma a^T d + \gamma^2 d^T A d.$$

If d is a global horizon direction. Then for all $x \notin \mathcal{F}$ and $\lambda \geq 0$,

$$f(x + \gamma d) = \underbrace{f(x)}_{>0} + 2\gamma a^T d + \gamma^2 d^T A d \leq 0,$$

for all $\gamma \geq \bar{\gamma} \geq 0$. Since $f(x) > 0$, then

1. if $d^T A d < 0$, for large enough γ, $f(x + \gamma d) \leq 0$ and $d \in (\mathcal{F}_\infty)_1$;

2. if $d^T A d = 0$, $a^T d < 0$ and $d \in (\mathcal{F}_\infty)_1$.

If d is a local horizon direction that is not global, then for all $x \in \mathcal{F}$,

$$f(x + \gamma d) = \underbrace{f(x)}_{\leq 0} + 2\gamma a^T d + \gamma^2 d^T A d \leq 0,$$

for all $\gamma \geq \hat{\gamma} \geq 0$. Then, since d is not a global horizon direction, $d^T A d = 0$, $a^T d = 0$ and $d \in (\mathcal{F}_\infty)_2$.

Let d be a direction that is not a local horizon direction, then d can not be an element of $(\mathcal{F}_\infty)_1$ or $(\mathcal{F}_\infty)_2$. Therefore, since $d \in \mathcal{F}_\infty$, $d \in (\mathcal{F}_\infty)_3$.

If $f(x) > 0$, then $x \notin \mathcal{F}$. For d to be a global horizon direction, one needs $\gamma(2a^T d + \gamma d^T A d) < 0$ for all $\gamma \geq \gamma^* > 0$. If $d \in (\mathcal{F}_\infty)_1$, then the inequality holds.

If $f(x) \leq 0$, then $x \in \mathcal{F}$. For d to be a local horizon direction, one needs $\gamma(2a^T d + \gamma d^T A d) \leq 0$ for all $\gamma \geq \gamma^* > 0$. While this is satisfied for $d \in (\mathcal{F}_\infty)_1$, $d \in (\mathcal{F}_\infty)_2$ also satisfies the inequality.

Let $d \in (\mathcal{F}_\infty)_3$, then $\gamma(2a^T d + \gamma d^T A d) > 0$ since $\gamma > 0$. Therefore, $(\mathcal{F}_\infty)_3$ is the set of directions that are not local horizon directions. □

Overall, determining if a direction is a horizon direction is not a difficult task as seen above. In fact, based off of the analysis above, one would imagine determining retractiveness to be straightforward. This changes slightly when the Hessian, A, is indefinite. Unlike the semi-definite cases, having an indefinite Hessian A decomposes the analysis to both the interior and boundary of the asymptotic cone $\mathcal{F}_\infty$. Before beginning this analysis, it is best to note that the interior of $\mathcal{F}_\infty$, $\text{int}(\mathcal{F}_\infty)$, is not the asymptotic cone of $\text{int}(\mathcal{F})$. In fact, since $\mathcal{F}_\infty$ is a closed cone, $\mathcal{F}_\infty$ is not only the asymptotic cone of $\mathcal{F}$ but also the asymptotic cone of the interior, $\text{int}(\mathcal{F})$.

Proposition 7. *Let* $\mathcal{F} = \left\{ x \in \mathbb{R}^n \;\middle|\; x^T A x + 2a^T x + \alpha \leq 0 \right\}$. *Then* $\text{int}(\mathcal{F}_\infty)$ *is the set of retractive, global horizon directions.*

Proof. Let $d \in D_<$ with corresponding asymptotic sequence $\{ x_k \}$, $\|x_k\| \to \infty$, and $x \in \mathbb{R}^n$. Then

$$f(x + \gamma d) = f(x) + \gamma^2 d^T A d + 2\gamma (Ax + a)^T d.$$

Since $d^T A d < 0$, there exists γ^* such that $\gamma^2 d^T A d < 2\gamma(Ax + a)^T d$ for all $\gamma \geq \gamma^*$. Therefore, d is a global horizon direction.

For d to be retractive, then $f(x_k - d) \leq 0$ for $k \geq k^*$. Note that

$$\begin{aligned}\lim_{k\to\infty} \frac{f(x_k - d)}{\|x_k\|^2} &= \lim_{k\to\infty} \frac{x_k^T A x_k + 2a^T x_k + \alpha - d^T A d - 2(Ax_k + a)^T d}{\|x_k\|^2} \\ &= d^T A_k d < 0.\end{aligned}$$

Therefore, there must exists $k^* > 0$ such that $f(x_k - d) < 0$ for all $k \geq k^*$ and d is retractive. ☐

Proposition 7 can be visualized in Figure 3.4. In this figure, $\mathcal{F} = \left\{ x \in \mathbb{R}^2 \;\middle|\; x^T A x + \alpha \leq 0 \right\}$. Both (a) and (c) describe the possible shapes of $\mathcal{F} \subseteq \mathbb{R}^2$ when α is negative (a) or positive (c) with the shared asymptotic cone in (b). In terms of Proposition 7, following any path along a direction d from the interior of $\mathcal{F}_\infty$ in (b), you will either stay in $\mathcal{F}$ or enter $\mathcal{F}$ and never leave for both (a) and (c). This occurs because the boundary of $\mathcal{F}$ converges to the boundary of $\mathcal{F}_\infty$ in both cases, and $d \in \text{int}(\mathcal{F}_\infty)$ will cross that threshold if the path starts outside of $\mathcal{F}_\infty$. Likewise, $d \in \text{bd}(\mathcal{F}_\infty)$ is not even a local horizon direction. Consider any point in the left side of (a), say x where $x_1 < 0$. Then choose $d \in \text{bd}(\mathcal{F}_\infty)$ such that $d > 0$. At some value $\gamma \geq 0$, $x + \gamma d$ will leave $\mathcal{F}$ and since it travels parallel to $\text{bd}(\mathcal{F}_\infty)$, $x + \hat{\gamma} d \notin \mathcal{F}$ for all $\hat{\gamma} \geq \gamma$.

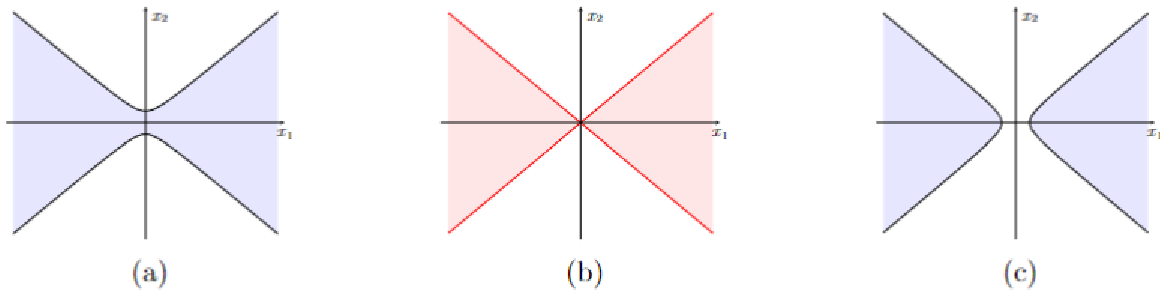


Figure 3.4: Graphs depicting the sets: (a) $\mathcal{F}_1 = \left\{ x \;\middle|\; -x_1^2 + x_2^2 - 1 \leq 0 \right\}$, (c) $\mathcal{F}_2 = \left\{ x \;\middle|\; -x_1^2 + x_2^2 + 1 \leq 0 \right\}$, and (b) their shared asymptotic cone.

Next, to discuss the retractiveness of the boundary of the asymptotic cone first requires simplifying the set in a similar way to Figure 3.4. Consider $\mathcal{F} = \left\{ x \;\middle|\; x^T A x + \alpha \leq 0 \right\}$ such that there is no linear term $a \in \mathbb{R}^n$. In this form, $\mathcal{F}$ is similar to its asymptotic cone $\mathcal{F}_\infty = \left\{ x \;\middle|\; x^T A x \leq 0 \right\}$ with a minor transformation to accommodate α. Upon quick observation, the α term may have impact in determining the retractiveness for the set $\mathcal{F}$. However, this does not extend to higher dimensions. Consider the following propositions.

Proposition 8. *Let $\mathcal{F} = \left\{ x \;\middle|\; x^T A x + \alpha \leq 0 \right\}$, where $A \in \mathcal{S}^n$ is an invertible, indefinite, diagonal matrix and $n \geq 3$. For any nonzero $d \in \text{bd}(\mathcal{F}_\infty)$, d is not retractive.*

Proof. For d to not be retractive, it suffices to show that there exists a sequence $\{ x_k \}$ corresponding to d such that $x_k - d \notin \mathcal{F}$ for every large enough k. That is, $(x_k - d)^T A (x_k - d) + \alpha > 0$ for $k \geq k'$.

Since A is an indefinite, diagonal matrix with P nonnegative entries and $N = n - P$ negative entries, without loss of generality, assume that the first P entries of A are nonnegative with the first entry being positive. That is, $A = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_P, -\lambda_{(P+1)}, \ldots, -\lambda_{(P+N)})$ where $\lambda_i \geq 0$ and $\lambda_1 > 0$.

Denote $d = (d_1, d_2, \ldots, d_n)^T$ and then consider $x_k \in \text{bd}(\mathcal{F})$ such that

$$x_k = \begin{pmatrix} kd_1 + \frac{\mu}{k} \\ kd_2 \\ \vdots \\ kd_P \\ \gamma_k d_{P+1} \\ \vdots \\ \gamma_k d_{P+N} \end{pmatrix}, \tag{3.2}$$

where

$$\gamma_k = k\sqrt{\frac{\alpha + 2\mu\lambda_1 d_1 + \lambda_1 \frac{\mu^2}{k^2}}{k^2 \sum_{j=1}^{N} \lambda_{P+j} d_{P+j}^2} + 1}.$$

Setting $c_k = \alpha + 2\mu\lambda_1 d_1 + \lambda_1 \frac{\mu^2}{k^2}$, for γ_k to exists choose $\mu, \bar{k} \in \mathbb{R}$ such that $c_k > 0$ for all $k \geq \bar{k}$. It is easy to verify that $x_k \in \text{bd}(\mathcal{F})$. However, to see that $\{ x_k \}$ is an asymptotic sequence of d, note that

$$\begin{aligned}
\gamma_k &= \sqrt{\frac{c_k + k^2 \sum_{i=1}^{P} \lambda_i d_i^2}{\sum_{j=1}^{N} \lambda_{P+j} d_{P+j}^2}} \\
&= \sqrt{\frac{c_k}{\sum_{j=1}^{N} \lambda_{P+j} d_{P+j}^2} + \frac{k^2 \sum_{j=1}^{N} \lambda_{P+j} d_{P+j}^2}{\sum_{j=1}^{N} \lambda_{P+j} d_{P+j}^2}} \\
&= k\sqrt{\frac{c_k}{k^2 \sum_{j=1}^{N} \lambda_{P+j} d_{P+j}^2} + 1},
\end{aligned} \tag{3.3}$$

where the second equality comes from the fact that $d^T A d = 0$ and

$$\begin{aligned}
\|x_k\|^2 &= k^2 \sum_{i=1}^{P} d_i^2 + \frac{c_k + k^2 \sum_{i=1}^{P} \lambda_i d_i^2}{\sum_{j=1}^{N} \lambda_{P+j} d_{P+j}^2} \sum_{j=1}^{N} d_{P+j}^2 \\
&= k^2 \sum_{i=1}^{P} d_i^2 + \frac{c_k + k^2 \sum_{j=1}^{N} \lambda_{P+j} d_{P+j}^2}{\sum_{j=1}^{N} \lambda_{P+j(P+j)} d_{P+j}^2} \sum_{j=1}^{N} d_{P+j}^2 \\
&= k^2 \sum_{i=1}^{n} d_i^2 + \frac{c_k}{\sum_{j=1}^{N} \lambda_{P+j} d_{P+j}^2} \sum_{j=1}^{N} d_{P+j}^2,
\end{aligned}$$

where $\gamma_k \to k$ and $\|x_k\| \to k\|d\|$ as $k \to \infty$. Then $\|x_k\| \to \infty$ and $x_k\|x_k\|^{-1} \to d\|d\|^{-1}$ and $\{ x_k \}$ is an asymptotic sequence converging to d.

Then,

$$\begin{aligned}
x_k^T Ad &= k\sum_{i=1}^{P} \lambda_i d_i^2 - \sum_{j=1}^{N} \lambda_{P+j} d_{P+j}^2 \sqrt{\frac{c_k + k^2 \sum_{i=1}^{P} \lambda_i d_i^2}{\sum_{j=1}^{N} \lambda_{P+j} d_{P+j}^2}} \\
&= \frac{k^2 \left(\sum_{i=1}^{P} \lambda_i d_i^2\right)^2 - \left(\sum_{j=1}^{N} \lambda_{P+j} d_{P+j}^2\right)^2 \frac{c_k + k^2 \sum_{i=1}^{P} \lambda_i d_i^2}{\sum_{j=1}^{N} \lambda_{P+j} d_{P+j}^2}}{k\sum_{i=1}^{P} \lambda_i d_i^2 + \gamma_k \sum_{j=1}^{N} \lambda_{P+j} d_{P+j}^2} \\
&= \frac{k^2 \left(\sum_{i=1}^{P} \lambda_i d_i^2\right)^2 - k^2 \left(\sum_{i=1}^{P} \lambda_i d_i^2\right) \sum_{j=1}^{N} \lambda_{P+j} d_{P+j}^2 - c_k \sum_{j=1}^{N} \lambda_{P+j} d_{P+j}^2}{k\sum_{i=1}^{P} \lambda_i d_i^2 + \gamma_k \sum_{j=1}^{N} \lambda_{P+j} d_{P+j}^2} \\
&= \frac{k^2 \sum_{i=1}^{P} \lambda_i d_i^2 \left(\sum_{i=1}^{P} \lambda_i d_i^2 - \sum_{j=1}^{N} \lambda_{P+j} d_{P+j}^2\right) - c_k \sum_{j=1}^{N} \lambda_{P+j} d_{P+j}^2}{k\sum_{i=1}^{P} \lambda_i d_i^2 + \gamma_k \sum_{j=1}^{N} \lambda_{P+j} d_{P+j}^2} \\
&= \frac{k^2 (d^T Ad) \sum_{i=1}^{P} \lambda_i d_i^2 - c_k \sum_{j=1}^{N} \lambda_{P+j} d_{P+j}^2}{k\sum_{i=1}^{P} \lambda_i d_i^2 + \gamma_k \sum_{j=1}^{N} \lambda_{P+j} d_{P+j}^2} \\
&= \frac{-c_k \sum_{j=1}^{N} \lambda_{P+j} d_{P+j}^2}{k\sum_{i=1}^{P} \lambda_i d_i^2 + \gamma_k \sum_{j=1}^{N} \lambda_{P+j} d_{P+j}^2} \\
&< 0,
\end{aligned}$$

where the final inequality comes from $c_k > 0$ for all $k \geq \hat{k}$. Therefore, for all k, $x_k^T Ad < 0$. As a result, for sequence $\{ x_k \}$ defined in (3.2) and γ_k defined in (3.3), $\{ x_k \}$ is a converging sequence of d and

$$(x_k - d)^T A(x_k - d) + \alpha = x_k^T Ax_k + \alpha + d^T Ad - 2x_k^T Ad > 0.$$

That is, d is not retractive. □

When $n = 2$, there will be at least one case when $\mathcal{F}$ is retractive. In order to describe the entire set $\mathcal{F}$ as retractive, consider the following proposition from Bertsekas and Tseng [5] followed by an additional result.

Proposition 9 ([5]). *Let $\mathcal{F}$ be a nonempty, closed set. Then the following hold:*

1. *If $\mathcal{F}$ is the complement of an open convex set, then $\mathcal{F}$ is retractive.*

2. *If $\mathcal{F}$ is the union or cross product of retractive sets, then $\mathcal{F}$ is retractive.*

3. *If $\mathcal{F}$ is the nonempty intersection of retractive sets, then $\mathcal{F}$ is retractive.*

Proposition 10. *Let* $\mathcal{F} = \left\{ x \mid x^T A x + \alpha \leq 0 \right\}$*, where* $A \in \mathcal{S}^n$ *is an indefinite, diagonal matrix,* $n = 2$*, and nonzero* $d \in \operatorname{bd}(\mathcal{F}_\infty)$*. Then the following hold:*

1. *If* $\alpha > 0$*, then* d *is not retractive.*

2. *If* $\alpha \leq 0$*, then* d *is retractive.*

Proof. Let $d \in \operatorname{bd}(\mathcal{F}_\infty)$, that is, $d^T A d = 0$. If $\alpha > 0$, then consider the sequence

$$x_k = \begin{pmatrix} kd_1 \\ kd_2\sqrt{\frac{\alpha}{k^2 A_{22} d_2^2} + 1} \end{pmatrix}.$$

From the same process of Proposition 8, it can be seen that $x_k^T A d < 0$ and d is not retractive.

If $\alpha = 0$, then $\mathcal{F} = \mathcal{F}_\infty$ and can be decomposed into two, 2D polyhedral cones. Since each of the two cones are the intersection of halfspaces, by Proposition 9, they are retractive. Also, by Proposition 9, since $\mathcal{F}$ is the union of two retractive sets, $\mathcal{F}$ is retractive .

If $\alpha < 0$, $\mathcal{F} \subseteq \mathcal{F}_\infty$ we follow the same approach as when $\alpha = 0$. Consider the closure of the complement of $\mathcal{F}$, $\operatorname{cl} \mathcal{F}^C = \left\{ x \mid x^T A x - \alpha \leq 0 \right\}$. As in Figure 3.5, there exists $c \in \mathbb{R}^2$ such that $\mathcal{F}^C$ can be decomposed into two sets,

$$\hat{\mathcal{F}}_1 = \left\{ x \mid c^T x \leq 0 \right\} \cap \operatorname{cl} \mathcal{F}^C, \quad \hat{\mathcal{F}}_2 = \left\{ x \mid c^T x > 0 \right\} \cap \operatorname{cl} \mathcal{F}^C,$$

where both $\hat{\mathcal{F}}_1$ and $\hat{\mathcal{F}}_2$ are convex sets that are symmetric over $\{ x \mid c^T x = 0 \}$ [42], and by Proposition 9, $\operatorname{cl} \hat{\mathcal{F}}_1^C$ and $\operatorname{cl} \hat{\mathcal{F}}_2^C$ are retractive. By Proposition 9, it suffices to show that $\mathcal{F} = \operatorname{cl} \hat{\mathcal{F}}_1^C \cap \operatorname{cl} \hat{\mathcal{F}}_2^C$.

$$\begin{aligned} \operatorname{cl} \hat{\mathcal{F}}_1^C \cap \operatorname{cl} \hat{\mathcal{F}}_2^C &= \left\{ x \mid x^T A x - \alpha \leq 0,\ c^T x \leq 0 \right\}^C \cap \left\{ x \mid x^T A x - \alpha \leq 0,\ c^T x > 0 \right\}^C \\ &= \mathcal{F} \cup (\mathcal{F} \cap \left\{ x \mid c^T x > 0 \right\}) \cup (\mathcal{F} \cap \left\{ x \mid c^T x \leq 0 \right\}) \\ &= \mathcal{F}. \end{aligned}$$

Therefore, since the intersection of closed, retractive sets are retractive, $\mathcal{F}$ is retractive.

□

Both Propositions 8 and 10 cover cases when A is an indefinite, diagonal matrix with no linear term. When A is not diagonal, consider Corollary 1 below. Extending this to include linear

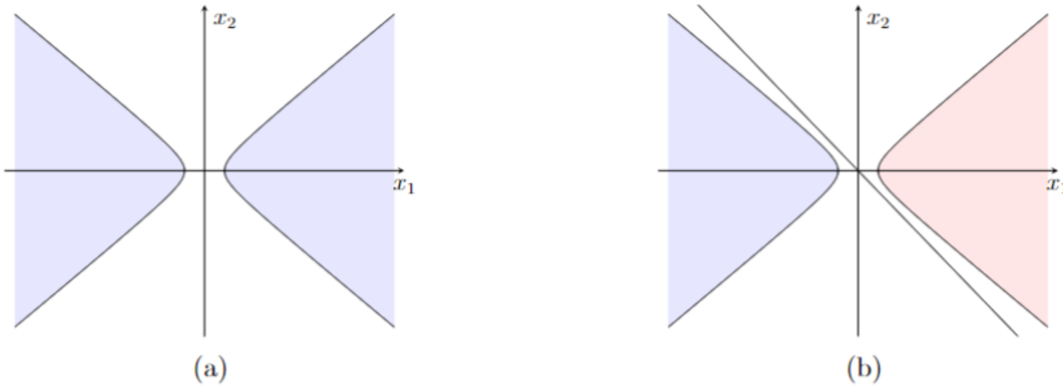


Figure 3.5: In the 2D case when A is indefinite and $\alpha > 0$, there exists a vector such that $\mathcal{F}$ is symmetric over that vector. In fact, the partitioning in (b) decomposes $\mathcal{F}$ into two convex sets.

terms will be a two part process. First consider $a \in \text{Range}(A)$ in Proposition 11. Geometrically, this is translating a set $\mathcal{F} = \left\{ x \mid x^T Ax + \alpha \leq 0 \right\}$ by $-c$.

Corollary 1. *Let $\mathcal{F} = \left\{ x \mid x^T Ax + \alpha \leq 0 \right\}$, where $A \in \mathcal{S}^n$ is an invertible, indefinite matrix and $n \geq 3$. For any nonzero $d \in \text{bd}(\mathcal{F})_\infty$, d is not retractive.*

Proof. Similar to Proposition 8, it suffices to find a sequence $\{ x_k \}$ corresponding to $d \in \text{bd}(\mathcal{F})_\infty$ such that $x_k - d \notin \mathcal{F}$ for every large enough k. Since A is a symmetric matrix, there exists an orthogonal matrix U and diagonal matrix D such that $A = U^T DU$. Without loss of generality, assume that the first P entries of D are nonnegative and the remaining $N = n - P$ entries are negative. Since $d \in \text{bd}(\mathcal{F}_\infty)$,

$$d^T Ad = 0 \quad \Rightarrow \quad d^T U^T DUd = 0 \quad \Rightarrow \quad (Ud)^T D(Ud) = 0,$$

and $(Ud) \in \text{bd}(\left\{ x \mid x^T Dx + \alpha \leq 0 \right\}_\infty)$. Denote $y = Ud = (y_1, y_2, \ldots, y_n)^T$ and then consider

$x_k \in \mathrm{bd}(\{\, x \mid x^T D x + \alpha \le 0 \,\})$ such that

$$x_k = \begin{pmatrix} ky_1 + \frac{\mu}{k} \\ ky_2 \\ \vdots \\ ky_P \\ \gamma_k y_{P+1} \\ \vdots \\ \gamma_k y_{P+N} \end{pmatrix}. \tag{3.4}$$

With this sequence, the proof follows similarly to that of Proposition 8. □

Proposition 11. *Let $\mathcal{F} = \{\, x \mid (x+c)^T A(x+c) + \alpha \le 0 \,\}$, where $A \in \mathcal{S}^n$ is an invertible, indefinite matrix, $c \in \mathbb{R}^n$, and $n \ge 3$. For any nonzero $d \in \mathrm{bd}(\mathcal{F}_\infty)$, d is not retractive.*

Proof. Let $\mathcal{F}' := \{\, y \mid y^T A y + \alpha \le 0 \,\}$. By Corollary 1, for any $d \in \mathcal{F}_\infty = (\mathcal{F}')_\infty$, there exists $\{y_k\} \subseteq \mathcal{F}'$ and $k' > 0$ such that $y_k - d \notin \mathcal{F}'$ for all $k > k'$, $\|y_k\| \to \infty$, and $(y_k/\|y_k\|) \to (d/\|d\|)$ as $k \to \infty$. Let $\{x_k\} := \{y_k\} - c$. It is clear that $\{x_k\} \subseteq \mathcal{F}$ and $x_k - d \notin \mathcal{F}$ for all $k > k'$. To show that d is not retractive, it suffices to show that $\|x_k\| \to \infty$ and $(x_k/\|x_k\|) \to (d/\|d\|)$ as $k \to \infty$. Since $\|x_k\| \ge \|y_k\| - \|c\|$,

$$\frac{\|y_k\| - \|c\|}{\|y_k\|} \le \frac{\|x_k\|}{\|y_k\|} \le \frac{\|y_k\| + \|c\|}{\|y_k\|},$$

and $\|y_k\| \to \infty$ as $k \to \infty$, we have $\|x_k\| \to \infty$ and $\|x_k\|/\|y_k\| \to 1$ as $k \to \infty$. Therefore,

$$\frac{x_k}{\|x_k\|} = \frac{y_k - c}{\|y_k\|} \cdot \frac{\|y_k\|}{\|x_k\|} = \left(\frac{y_k}{\|y_k\|} - \frac{c}{\|y_k\|} \right) \frac{\|y_k\|}{\|x_k\|} \to \left(\frac{d}{\|d\|} - 0 \right) \cdot 1 = \frac{d}{\|d\|} \quad \text{as } k \to \infty.$$

□

Corollary 2. *Let $\mathcal{F} = \{\, x \mid x^T A x + 2a^T x + \alpha \le 0 \,\}$, where $A \in \mathcal{S}^n$ is an invertible, indefinite matrix with $a_i \in \mathbb{R}^n$, and $n \ge 3$. For any nonzero $d \in \mathrm{bd}(\mathcal{F}_\infty)$, d is not retractive.*

After this analysis, for the general quadratically defined feasible region, the boundary of that set is most likely not retractive. This consideration is explored in Chapter 6 where retractiveness can be used to determine the existence of an optimal solution.

3.2 Non-Intersecting Constraints

While this section has focused on the properties of a set defined by one quadratic constraint, there is an increased difficulty when adding addtional constraints. If given a nonempty intersection of sets, then the asymptotic cone of the intersection is a subset of the intersection of asymptotic cones. One instance of this being equality is when the sets are convex. This subsection explores the case where $\mathcal{F}_2$ induces a non-intersecting constraint in $\mathcal{F}_1$. That is, $\text{bd}(\mathcal{F}_2) \subseteq \mathcal{F}_1$. The non-intersecting property is examined more in Chapter 4 in terms of convexifying sets in the lifted matrix space $\mathcal{S}^{n+1}$. In terms of asymptotic cones, this property leads to the following result.

Proposition 12. *Let $\mathcal{F}_1 \subseteq \mathbb{R}^n$ be a closed, nonempty set and let $\mathcal{F}_2 = \left\{ x \;\middle|\; x^T Ax + 2a^T x + \alpha \le 0 \right\}$ where $\text{bd}(\mathcal{F}_2) \subseteq \mathcal{F}_1$. Then*

$$(\mathcal{F}_1 \cap \mathcal{F}_2)_\infty = (\mathcal{F}_1)_\infty \cap (\mathcal{F}_2)_\infty.$$

Proof. The forward containment follows from Proposition 9. For the reverse containment, let $d \in (\mathcal{F}_1)_\infty \cap (\mathcal{F}_2)_\infty$. Then there exists a sequence $\{ x_k \}_k \subseteq \mathcal{F}_1$ such that $\|x_k\| \to \infty$ and $x_k \|x_k\|^{-1} \to d$. Since $d \in (\mathcal{F}_2)_\infty$, $d^T W d \le 0$. These two cases are discussed below.

If $d \in \text{int}(\mathcal{F}_2)_\infty$, then there exists some $\bar{k} \in \mathbb{Z}_+$ such that for all $k \ge \bar{k}$, $x_k^T A x_k + 2a^T x_k + \alpha < 0$. Therefore there exists a subsequence $\{ x_k \}_{k \ge \bar{k}} \subseteq \mathcal{F}_2$ such that $\|x_k\| \to \infty$ and $x_k \|x_k\|^{-1} \to d$ thus $\{ x_k \}_{k \ge \bar{x}} \subseteq \mathcal{F}_1 \cap \mathcal{F}_2$. Hence $d \in (\mathcal{F}_1 \cap \mathcal{F}_2)_\infty$.

If $d \in \text{bd}(\mathcal{F}_2)_\infty$, since $\mathcal{F}_2$ induces a non-intersecting constraint in $\mathcal{F}_1$, there exists $\{ x_k \}_k \subseteq \text{bd}(\mathcal{F}_2) \subseteq \mathcal{F}_1$. Therefore $\{ x_k \}_k \subseteq \mathcal{F}_1 \cap \mathcal{F}_2$ and $d \in (\mathcal{F}_1 \cap \mathcal{F}_2)_\infty$. $\square$

If $\mathcal{F}_1$ is defined by a single quadratic constraint, Proposition 12 can be checked by using the S-Procedure [36]. Otherwise, if $\mathcal{F}_1$ is defined by multiple quadratic constraints, say $\mathcal{F}_1 = \{ x \mid x^T A_i x + 2a_i^T x + \alpha_i \le 0,\ i = 1, \ldots, m \}$ and $\mathcal{F}_2 = \{ x \mid x^T W x + 2w^T x + \omega \le 0 \}$, this property holds if an only if

$$\max \left\{ x^T A_i x + 2a_i^T x + \alpha_i \le 0 \;\middle|\; x^T W x + 2w^T x + \omega \le 0 \right\}.$$

This procedure is discussed more in Chapter 4 addition of multiple non-intersecting constraints and the impact of the Proposition 12 when $\mathcal{F}_2$ is defined by an affine linear constraint is examined in Chapter 5.

Chapter 4

Convex Hull Results on Quadratic Programs with Non-Intersecting Constraints

Chapter 2 introduced the idea of the convexification of a nonconvex set $\mathcal{F}$, denoted $\overline{\mathcal{C}}(\mathcal{F})$. While finding this higher dimensional set can be computationally difficult in general, $\mathcal{F}$ can have structural properties that provide easy to generate inequalities. In fact, there could be non-trivial constraints with certain properties that, when added to $\mathcal{F}$, do not add extra levels of difficulty to finding $\overline{\mathcal{C}}(\mathcal{F})$. This chapter focuses on adding an additional constraint to $\mathcal{F}$, $\mathcal{H}$, that has a non-intersecting boundary with $\mathcal{F}$.

4.1 Introduction

To explore the lifted closed convex hull for more sets, we take a small step and consider a closed set $\mathcal{G} \in \mathbb{R}^n$ resulting by adding one more constraint to an $\mathcal{F}$ with known $\overline{\mathcal{C}}(\mathcal{F})$. Specifically, suppose that $\mathcal{G} := \mathcal{F} \cap \mathcal{H}$ where $\mathcal{H}$ is defined by a single inequality, we hope to derive $\overline{\mathcal{C}}(\mathcal{G})$ based on $\overline{\mathcal{C}}(\mathcal{F})$ and the simple structure of $\mathcal{H}$. If this is successful, we can repeat the process to generate more convex hull results in the lifted space.

We are interested in the relation between $\overline{\mathcal{C}}(\mathcal{G})$, $\overline{\mathcal{C}}(\mathcal{F})$ and $\overline{\mathcal{C}}(\mathcal{H})$. By definition, it is clear that

$$\overline{\mathcal{C}}(\mathcal{G}) = \overline{\mathcal{C}}(\mathcal{F} \cap \mathcal{H}) \subseteq \overline{\mathcal{C}}(\mathcal{F}) \cap \overline{\mathcal{C}}(\mathcal{H}).$$

On the other hand, $\overline{\mathcal{C}}(\mathcal{G})$ can be a proper subset of $\overline{\mathcal{C}}(\mathcal{F}) \cap \overline{\mathcal{C}}(\mathcal{H})$ in general. Such an example can be found even when $\mathcal{F}$ and $\mathcal{H}$ are as simple as two intersecting ellipsoids [8].

In this paper, we propose a sufficient condition for $\overline{\mathcal{C}}(\mathcal{G}) = \overline{\mathcal{C}}(\mathcal{F}) \cap \overline{\mathcal{C}}(\mathcal{H})$. Specifically, we show that the equation holds when $\mathcal{H}$ is defined by a non-intersecting quadratic inequality with nonzero Hessian. For the rest of the paper, unless stated otherwise, we focus on sets with the following structure:

- $\mathcal{F}$ is a nonempty closed set in $\mathbb{R}^n$;
- $\mathcal{H} := \{ x \in \mathbb{R}^n \mid x^T W x + 2w^T x + \omega \leq 0 \}$ is a nonempty proper subset of $\mathbb{R}^n$, where $W \in \mathcal{S}^n$, $w \in \mathbb{R}^n$, $\omega \in \mathbb{R}$;
- $\mathcal{G} = \mathcal{F} \cap \mathcal{H}$.

Note that although the study is motivated by and primarily applied to quadratically defined sets, our approach does not rely on the quadratic structure of $\mathcal{F}$ or $\mathcal{G}$. Moreover, we omit the trivial cases when $\mathcal{H} = \emptyset$ or $\mathbb{R}^n$ in the discussion. We show in the paper that $\overline{\mathcal{C}}(\mathcal{G}) = \overline{\mathcal{C}}(\mathcal{F}) \cap \overline{\mathcal{C}}(\mathcal{H})$ under the following assumptions.

Assumption 3 (nonzero)**.** *$W \neq 0$.*

The nonzero assumption assures that $\mathcal{H}$ is not linearly defined. The necessity of the assumption is demonstrated by Example 2 in Section 4.2.

Assumption 4 (non-intersecting)**.**

$$x^T W x + 2w^T x + \omega = 0 \quad \Longrightarrow \quad x \in \mathcal{F}. \tag{4.1}$$

Geometrically, the non-intersecting assumption is satisfied if and only if the boundary of $\mathcal{H}$ is contained in $\mathcal{F}$ (and equivalently contained in $\mathcal{G}$).

When $\mathcal{F}$ is quadratically defined , the non-intersecting assumption (4.1) holds if and only if

$$\sup \{ x^T A_i x + 2a_i^T x + \alpha_i \mid h(x) = 0 \} \leq 0 \quad \forall\, i \in I. \tag{4.2}$$

where $h(x) := x^T W x + 2w^T x + \omega$. If there exist $\hat{x}$ and $\bar{x}$ such that $h(\hat{x}) < 0 < h(\bar{x})$, then the optimization problem in (4.2) enjoys exact Shor relaxations due to the S-Lemma with equality [35]. Therefore, the non-intersecting assumption (4.1) can be checked by solving semidefinite programs. On the other hand, if $h(x) \geq 0$ for all $x \in \mathbb{R}^n$ and $\mathcal{H} \neq \emptyset$, then $W \succeq 0$ and $w \in \text{Range}(W)$. In this case, $h(x) = (x+b)^T W (x+b)^T + \beta$ for some $b \in \mathbb{R}^n$ and $\beta \in \mathbb{R}$. We observe that $\beta \geq 0$ since $h(x) \geq 0$ for all $x \in \mathbb{R}^n$, and that $\beta \leq 0$ since $\mathcal{H} \neq \emptyset$. Therefore, the constraint $h(x) = 0$ is equivalent to the affine equality constraint $W^{1/2}(x + b) = 0$. With suitable substitution, the optimization problem in (4.2) can be transformed to unconstrained quadratic problems and solved easily.

Concepts similar to Assumption 4 have been mentioned in [37] and [3]. We restate those concepts here to avoid possible confusion. In [37], two linear constraints are called "non-intersecting" if the hyperplanes defined by the constraints do not intersect *inside the unit ball*. In [3], "non-interacting" constraints are explained as that if any of the constraints is active at a certain point x, then all the other constraints are satisfied *strictly* at x.

Under Assumptions 3 and 4, we show in this paper that $\overline{\mathcal{C}}(\mathcal{G}) = \overline{\mathcal{C}}(\mathcal{F}) \cap \mathcal{L}(\mathcal{H})$ (Theorem 10), where

$$\mathcal{L}(\mathcal{H}) := \left\{ (x, X) \;\middle|\; W \bullet X + 2w^T x + \omega \leq 0 \right\}.$$

Since $\mathcal{H}$ is defined by a single quadratic inequality, it is known that $\overline{\mathcal{C}}(\mathcal{H}) = \mathcal{S}(\mathcal{H})$. We then have $\overline{\mathcal{C}}(\mathcal{G}) = \overline{\mathcal{C}}(\mathcal{F}) \cap \mathcal{L}(\mathcal{H}) = \overline{\mathcal{C}}(\mathcal{F}) \cap \mathcal{S}(\mathcal{H}) = \overline{\mathcal{C}}(\mathcal{F}) \cap \overline{\mathcal{C}}(\mathcal{H})$. Our proof approach is motivated by the prior work on bounded quadratic programs with hollows [37]. When $\mathcal{F}$ is bounded, so is $\mathcal{G}$, and $\overline{\mathcal{C}}(\mathcal{G})$ is reduced to

$$\mathcal{C}(\mathcal{G}) := \text{conv} \left\{ (x, xx^T) \;\middle|\; x \in \mathcal{G} \right\}.$$

When $\mathcal{F}$ is bounded and quadratically defined, it is shown in [37] that $\mathcal{C}(\mathcal{G}) = \mathcal{C}(\mathcal{F}) \cap \mathcal{L}(\mathcal{H})$ under the non-intersecting assumption. (The non-intersecting assumption implies the nonzero assumption in the bounded case.) In this paper, we generalize the result to allow general unbounded closed $\mathcal{F}$. This generalization allows more intriguing applications. We provide four quadratically defined[1]

We remark here that the proofs in [37] do not generalize to the unbounded case directly. In particular, two proofs are provided in [37]. The first one relies on discussing the locations of optimal solutions of $\min \{ x^T Q x + 2c^T x \mid x \in \mathcal{F} \}$, while the second considers the extreme points of $\mathcal{C}(\mathcal{F}) \cap$

[1]The examples are all quadratically defined because little is known about $\overline{\mathcal{C}}(\mathcal{F})$ when $\mathcal{F}$ is non-quadratic. examples in Section 4.3.

$\mathcal{L}(\mathcal{H})$. For the first proof, when $\mathcal{F}$ is compact, an optimal solution of $\min\{\, x^TQx + 2c^Tx \mid x \in \mathcal{F}\,\}$ always exists due to the Weierstrass extreme value theorem. However, in the unbounded case, an optimal solution may be unattainable even if the optimal value is finite. For the second, when $\mathcal{F}$ is bounded, $\mathcal{C}(\mathcal{G})$ is closed and is generated by its extreme points, which are in the form of (x, xx^T). When $\mathcal{F}$ is unbounded, $\mathcal{C}(\mathcal{G})$ is not necessarily closed, and $\overline{\mathcal{C}}(\mathcal{G})$ is generated by both its extreme points and its extreme directions. However, characterizing the extreme directions of $\overline{\mathcal{C}}(\mathcal{G})$ seems not to be an easy task.

To overcome the difficulty, we tailor a technical lemma by Dickinson et al. [15] to build a connection between $\overline{\mathcal{C}}(\mathcal{G})$ and $\mathcal{C}(\mathcal{G})$. As the connection is related to the asymptotic cone of $\mathcal{G}$, we use basic properties of the asymptotic cones and utilize their properties that are listed in Chapter 3. In Section 4.2, we use the connection to build the proof of $\overline{\mathcal{C}}(\mathcal{G}) = \overline{\mathcal{C}}(\mathcal{F}) \cap \mathcal{L}(\mathcal{H})$, for which two nontrivial pieces of the claim are considered sequentially. To show the necessity of the assumptions, a counterexample is provided after the proof. Three corollaries follow the main result with a generalization of the non-intersecting concept to allow multiple constraints in $\mathcal{H}$. In Section 4.3, we provide four examples where the theory can be applied. The paper is concluded in Section 4.4.

4.2 The closed convex hull result

In this section, we prove $\overline{\mathcal{C}}(\mathcal{G}) = \overline{\mathcal{C}}(\mathcal{F}) \cap \mathcal{L}(\mathcal{H})$ under Assumptions 3 and 4. We start from a simple observation. Let $T := \{\, Y \in \mathcal{S}^{n+1} \mid Y_{11} = 1\,\}$. By definition, it is easy to check that

$$\operatorname{conv}\left\{ \begin{pmatrix}1\\x\end{pmatrix}\begin{pmatrix}1\\x\end{pmatrix}^T \,\middle|\, x \in S \right\} = \operatorname{cone}\operatorname{conv}\left\{ \begin{pmatrix}1\\x\end{pmatrix}\begin{pmatrix}1\\x\end{pmatrix}^T \,\middle|\, x \in S \right\} \cap T,$$

where Y_{11} is the top left element of Y. We observe that the same statement also holds for the corresponding closures.

Lemma 12. *Let $S \subseteq \mathbb{R}^n$ be a nonempty closed set. Then*

$$\overline{\operatorname{conv}}\left\{ \begin{pmatrix}1\\x\end{pmatrix}\begin{pmatrix}1\\x\end{pmatrix}^T \,\middle|\, x \in S \right\} = \overline{\operatorname{cone}}\operatorname{conv}\left\{ \begin{pmatrix}1\\x\end{pmatrix}\begin{pmatrix}1\\x\end{pmatrix}^T \,\middle|\, x \in S \right\} \cap T.$$

Proof. The forward containment "$\subseteq$" is straightforward. Now let Y be a matrix of the set on the right side of the equation. Then $Y_{11} = 1$, and there exists a sequence $\{Y_m\}_m$ such that $Y_m \to Y$ as

$m \to \infty$ and

$$Y_m = \sum_{i=1}^{k_m} \lambda_{m_i} \begin{pmatrix} 1 \\ x_{m_i} \end{pmatrix} \begin{pmatrix} 1 \\ x_{m_i} \end{pmatrix}^T$$

for some $k_m \geq 0$, λ_{m_i} and $x_{m_i} \in S$. In particular, $\lambda_m := \sum_{i=1}^{k_m} \lambda_{m_i} \to 1$ as $m \to \infty$. Then $\tilde{Y}_m := Y_m/\lambda_m \in \operatorname{conv}\{ \binom{1}{x}\binom{1}{x}^T \mid x \in S \}$ and $\tilde{Y}_m \to Y$ as $m \to \infty$. Therefore, $Y \in \overline{\operatorname{conv}}\{ \binom{1}{x}\binom{1}{x}^T \mid x \in S \}$.

□

The following lemma from [15] is crucial to characterize the closed convex hull.

Lemma 13 ([15])**.** *Let $S \subseteq \mathbb{R}^n$ be a nonempty closed set. Then*

$$\overline{\operatorname{cone}} \operatorname{conv} \{ yy^T \mid y \in \{1\} \times S \} = \operatorname{conv} \{ yy^T \mid y \in \operatorname{cone}(\{1\} \times S) \cup (\{0\} \times S_\infty) \}.$$

Interpreting Lemma 13 by rewriting the equation in equivalent forms, we have the following lemma to characterize the difference between the convex hull $\mathcal{C}(\mathcal{F})$ and its closure $\overline{\mathcal{C}}(\mathcal{F})$.

Lemma 14. *Let $S \subseteq \mathbb{R}^n$ be a nonempty closed set. Then*

$$\overline{\mathcal{C}}(S) = \mathcal{C}(S) + \operatorname{conv} \{ (0, dd^T) \mid d \in S_\infty \}.$$

Proof. By Lemma 13,

$$\begin{aligned}
&\overline{\operatorname{cone}} \operatorname{conv} \left\{ \begin{pmatrix} 1 \\ x \end{pmatrix} \begin{pmatrix} 1 \\ x \end{pmatrix}^T \middle| \, x \in S \right\} \\
&= \operatorname{conv} \left\{ yy^T \;\middle|\; y \in \operatorname{cone}(\{ 1 \} \times S) \cup (\{ 0 \} \times S_\infty) \right\} \\
&= \operatorname{conv} \left\{ yy^T \;\middle|\; y \in \operatorname{cone}(\{ 1 \} \times S) \right\} + \operatorname{conv} \left\{ yy^T \;\middle|\; y \in \{ 0 \} \times S_\infty \right\} \\
&= \operatorname{cone} \operatorname{conv} \left\{ \begin{pmatrix} 1 \\ x \end{pmatrix} \begin{pmatrix} 1 \\ x \end{pmatrix}^T \middle| \, x \in S \right\} + \operatorname{conv} \left\{ \begin{pmatrix} 0 \\ d \end{pmatrix} \begin{pmatrix} 0 \\ d \end{pmatrix}^T \middle| \, d \in S_\infty \right\}.
\end{aligned}$$

Intersecting both sides of the equation with T, we have

$$\begin{aligned}
\overline{\operatorname{conv}} \left\{ \begin{pmatrix} 1 \\ x \end{pmatrix} \begin{pmatrix} 1 \\ x \end{pmatrix}^T \middle| \, x \in S \right\} = \operatorname{conv} &\left\{ \begin{pmatrix} 1 \\ x \end{pmatrix} \begin{pmatrix} 1 \\ x \end{pmatrix}^T \middle| \, x \in S \right\} \\
&+ \operatorname{conv} \left\{ \begin{pmatrix} 0 \\ d \end{pmatrix} \begin{pmatrix} 0 \\ d \end{pmatrix}^T \middle| \, d \in S_\infty \right\}
\end{aligned}$$

by Lemma 12. Dropping the first component of the matrices, which is fixed to 1, the above equation is equivalent to $\overline{\mathcal{C}}(S) = \mathcal{C}(S) + \text{conv}\{(0, dd^T) \,|\, d \in S_\infty\}$. $\square$

In fact, Lemma 14 helps build a connection between $\text{Rec}(\overline{\mathcal{C}}(S))$ and S_∞. Applying the description of the asymptotic cone in Chapter 3 to $\text{bd}(\mathcal{H}) = \{\, x \in \mathbb{R}^n \mid x^T W x + 2w^T x + \omega = 0 \,\}$, we have the following key observation.

Proposition 13. *If $d^T W d = 0$, then $(0, dd^T) \in \text{Rec}(\overline{\mathcal{C}}(\mathcal{G}))$.*

Proof. Since $W \neq 0$ by Assumption 3, Proposition 4 indicates that $\{d \in \mathbb{R}^n \,|\, d^T W d = 0\} = \pm(\text{bd}(\mathcal{H}))_\infty$. By Assumption 4, $\text{bd}(\mathcal{H})$ is contained in $\mathcal{G}$. Consequently, $\pm(\text{bd}(\mathcal{H}))_\infty \subseteq \pm\mathcal{G}_\infty$ by Lemma 9. Therefore, for any $(x, X) \in \overline{\mathcal{C}}(\mathcal{G})$, $\lambda \geq 0$ and $d \in \mathbb{R}^n$ such that $d^T W d = 0$,

$$\begin{aligned}(x, X) + \lambda(0, dd^T) &\in \overline{\mathcal{C}}(\mathcal{G}) + \text{conv}\,\{\, (0, dd^T) \mid d \in \pm\,\mathcal{G}_\infty \,\} \\ &= \overline{\mathcal{C}}(\mathcal{G}) + \text{conv}\,\{\, (0, dd^T) \mid d \in \mathcal{G}_\infty \,\} \\ &= \mathcal{C}(\mathcal{G}) + \text{conv}\,\{\, (0, dd^T) \mid d \in \mathcal{G}_\infty \,\} = \overline{\mathcal{C}}(\mathcal{G}),\end{aligned}$$

where the second equation holds because of Lemma 14. That is, $(0, dd^T) \in \text{Rec}(\overline{\mathcal{C}}(\mathcal{G}))$.

$\square$

With the help of Assumption 3 and Proposition 4, Proposition 13 establishes a connection between $\{d \in \mathbb{R}^n \,|\, d^T W d = 0\}$ and $\text{bd}(\mathcal{H})$. Without Assumption 3, $\{d \in \mathbb{R}^n \,|\, d^T W d = 0\}$ is equal to $\mathbb{R}^n$ and provides no information about $\mathcal{H}$. In addition, the proof of Proposition 13 is the only place where Assumption 3 and Proposition 4 are explicitly used in the proof of the main result (Theorem 10).

As another technical lemma for the main proof, we restate the famous rank-1 decomposition by Sturm and Zhang.

Lemma 15 ([31])**.** *Let V be a symmetric matrix, and suppose $Y \succeq 0$ with $V \bullet Y = 0$ and $\text{rank}(Y) = r$. Then there exists a rank-1 decomposition $Y = \sum_{i=1}^r y^i (y^i)^T$ such that $y^i \neq 0$ and $(y^i)^T V y^i = 0$ for all $i = 1, \ldots, r$.*

The proof of our main theorem is constructed by the following two propositions. In the first proposition, we show that $\overline{\mathcal{C}}(\mathcal{F}) \cap \text{bd}(\mathcal{L}(\mathcal{H})) \subseteq \overline{\mathcal{C}}(\mathcal{G})$ by proving a more general statement.

Proposition 14. *If $X \succeq xx^T$ and $W \bullet X + 2w^T x + \omega = 0$, then $(x, X) \in \overline{\mathcal{C}}(\mathcal{G})$.*

Proof. Since $\begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \succeq 0$ and $\begin{pmatrix} \omega & w^T \\ w & W \end{pmatrix} \bullet \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} = 0$, by Lemma 15, there exist nonzero $y^j = \begin{pmatrix} z_0^j \\ z^j \end{pmatrix} \in \mathbb{R}^{1+n}$ for $j = 1, \ldots, r$ such that

$$0 = \begin{pmatrix} z_0^j \\ z^j \end{pmatrix}^T \begin{pmatrix} \omega & w^T \\ w & W \end{pmatrix} \begin{pmatrix} z_0^j \\ z^j \end{pmatrix} = (z^j)^T W z^j + 2 z_0^j w^T z^j + \omega (z_0^j)^2 \tag{4.3}$$

and

$$\begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} = \sum_{i=1}^{r} y^i (y^i)^T = \sum_{j \in J} (z_0^j)^2 \begin{pmatrix} 1 \\ x^j \end{pmatrix} \begin{pmatrix} 1 \\ x^j \end{pmatrix}^T + \sum_{j \notin J} \begin{pmatrix} 0 \\ z^j \end{pmatrix} \begin{pmatrix} 0 \\ z^j \end{pmatrix}^T,$$

where $J \coloneqq \{\, j \mid y_1^j \neq 0 \,\}$ and $x^j = z^j / (z_0^j)$ for $j \in J$. Equivalently, $\sum_{j \in J} (z_0^j)^2 = 1$ and

$$(x, X) = \sum_{j \in J} (z_0^j)^2 (x^j, x^j (x^j)^T) + \sum_{j \notin J} (0, z^j (z^j)^T).$$

By (4.3), $(z^j)^T W z^j = 0$ for $j \notin J$. Therefore, Proposition 13 indicates that $\sum_{j \notin J} (0, z^j (z^j)^T) \in \text{Rec}(\overline{\mathcal{C}}(\mathcal{G}))$. Also by (4.3), $x^j \in \text{bd}(\mathcal{H}) \subseteq \mathcal{G}$ for all $j \in J$. Therefore, $(x, X) \in \mathcal{C}(\mathcal{G}) + \text{Rec}(\overline{\mathcal{C}}(\mathcal{G})) \subseteq \overline{\mathcal{C}}(\mathcal{G})$. □

We remark here that when W is (positive or negative) definite, $|J| = r$ and the proof of Proposition 14 reduces to the alternative proof of Corollary 1 in [37]. When W is not definite, the term $\sum_{j \notin j} (0, z^j (z^j)^T)$ is related to $\text{Rec}(\overline{\mathcal{C}}(\mathcal{G}))$ in our proof by the prior discussion on the asymptotic cones, which helps generalize the result.

Leveraging Proposition 14, we show in the following proposition that $\mathcal{C}(\mathcal{F}) \cap \text{int}(\mathcal{L}(\mathcal{H})) \subseteq \overline{\mathcal{C}}(\mathcal{G})$. In [37], this case is trivial as it suffices to consider the extreme points of $\mathcal{C}(\mathcal{F}) \cap \mathcal{L}(\mathcal{H})$ when $\mathcal{F}$ is compact. Here, we need to adopt a different approach due to the unboundedness of $\mathcal{F}$. We consider an arbitrary point (x, X) in $\mathcal{C}(\mathcal{F}) \cap \text{int}(\mathcal{L}(\mathcal{H}))$ and decompose it into "rank-1" points in $\mathcal{C}(\mathcal{F})$. If all the "rank-1" points are in $\mathcal{L}(\mathcal{H})$, then (x, X) is in $\mathcal{C}(\mathcal{G})$; if some "rank-1" point is not in $\mathcal{L}(\mathcal{H})$, we construct a convex combination of the point and (x, X) which is in $\text{bd}(\mathcal{L}(\mathcal{H}))$, and consider the convex combination instead.

Proposition 15. *If $(x, X) \in \mathcal{C}(\mathcal{F})$ and $W \bullet X + 2 w^T x + \omega < 0$, then $(x, X) \in \overline{\mathcal{C}}(\mathcal{G})$.*

Proof. Since $(x, X) \in \mathcal{C}(\mathcal{F})$, there exist $x^j \in \mathcal{F}$, $\mu_j > 0$ for $j = 1, \ldots, p$, such that $\sum \mu_j = 1$ and

$$(x, X) = \sum_{j=1}^{p} \mu_j (x^j, x^j (x^j)^T).$$

Let $J := \{ j \mid (x^j)^T W x^j + 2w^T x^j + \omega \leq 0 \}$. If $|J| = p$, then $x^j \in \mathcal{F} \cap \mathcal{H} = \mathcal{G}$ for $j = 1, \ldots, p$, and therefore $(x, X) \in \mathcal{C}(\mathcal{G}) \subseteq \overline{\mathcal{C}}(\mathcal{G})$. If $|J| < p$, then for each $j \notin J$, we have $W \bullet (x^j (x^j)^T) + 2w^T x^j + \omega > 0$. Since the hyperplane $\{ (x, X) \mid W \bullet X + 2w^T x + \omega = 0 \}$ separates $(x^j, x^j (x^j)^T)$ and (x, X), there exists $\gamma_j \in (0, 1)$ such that

$$(\hat{x}^j, \hat{X}^j) := \gamma_j (x, X) + (1 - \gamma_j)(x^j, x^j (x^j)^T)$$

satisfies $W \bullet \hat{X}^j + w^T \hat{x}^j + \omega = 0$. Moreover, as a convex combination of two points (x, X) and $(x^j, x^j (x^j)^T)$ in $\mathcal{C}(\mathcal{F})$, $(\hat{x}^j, \hat{X}^j)$ is in $\mathcal{C}(\mathcal{F})$. Since $\hat{X}^j \succeq \hat{x}^j (\hat{x}^j)^T$, Proposition 14 indicates that $(\hat{x}^j, \hat{X}^j) \in \overline{\mathcal{C}}(\mathcal{G})$. By the definition of $(\hat{x}^j, \hat{X}^j)$,

$$\begin{aligned}
(x, X) &= \sum_{j \in J} \mu_j (x^j, x^j (x^j)^T) + \sum_{j \notin J} \mu_j (x^j, x^j (x^j)^T) \\
&= \sum_{j \in J} \mu_j (x^j, x^j (x^j)^T) + \sum_{j \notin J} \mu_j \left(\frac{1}{1 - \gamma_j} (\hat{x}^j, \hat{X}^j) - \frac{\gamma_j}{1 - \gamma_j} (x, X) \right).
\end{aligned}$$

Let $\sigma := 1 + \sum_{j \notin J} \frac{\mu_j \gamma_j}{1 - \gamma_j}$, then

$$(x, X) = \sum_{j \in J} \frac{\mu_j}{\sigma} (x^j, x^j (x^j)^T) + \sum_{j \notin J} \frac{\mu_j}{\sigma (1 - \gamma_j)} (\hat{x}^j, \hat{X}^j),$$

which is a convex combination of points in $\overline{\mathcal{C}}(\mathcal{G})$. Therefore, $(x, X) \in \overline{\mathcal{C}}(\mathcal{G})$. □

Using a continuity argument, Proposition 15 can be generalized to $\overline{\mathcal{C}}(\mathcal{F}) \cap \operatorname{int}(\mathcal{L}(\mathcal{H})) \subseteq \overline{\mathcal{C}}(\mathcal{G})$.

Corollary 3. *If $(x, X) \in \overline{\mathcal{C}}(\mathcal{F})$ and $W \bullet X + 2w^T x + \omega < 0$, then $(x, X) \in \overline{\mathcal{C}}(\mathcal{G})$.*

Proof. If $(x, X) \in \overline{\mathcal{C}}(\mathcal{F})$, there exists a sequence $\{ (x^t, X^t) \}_t \subseteq \mathcal{C}(\mathcal{F})$ such that $(x^t, X^t) \to (x, X)$ as $t \to \infty$. Since $W \bullet X + 2w^T x + \omega < 0$, for sufficiently large t, $W \bullet X^t + 2w^T x^t + \omega < 0$. By Proposition 15, $(x^t, X^t) \in \overline{\mathcal{C}}(\mathcal{G})$. The proof is completed by taking $t \to \infty$. □

Summarizing the above, we state the main theorem of this section as follows.

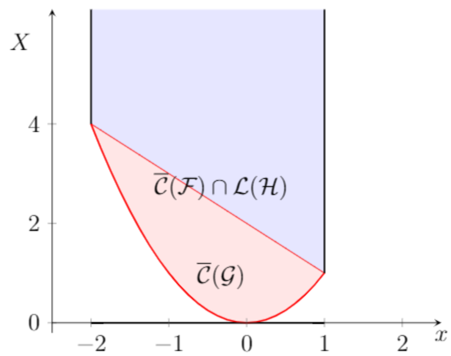


Figure 4.1: In Example 2, $\overline{\mathcal{C}}(\mathcal{G})$ is a bounded set while $\overline{\mathcal{C}}(\mathcal{F}) \cap \mathcal{L}(\mathcal{H})$ is unbounded.

Theorem 10. *Under Assumptions 3 and 4,* $\overline{\mathcal{C}}(\mathcal{G}) = \overline{\mathcal{C}}(\mathcal{F}) \cap \mathcal{L}(\mathcal{H})$.

Proof. The forward direction "$\subseteq$" is easy since $\overline{\mathcal{C}}(\mathcal{G}) \subseteq \overline{\mathcal{C}}(\mathcal{F}) \cap \overline{\mathcal{C}}(\mathcal{H}) \subseteq \overline{\mathcal{C}}(\mathcal{F}) \cap \mathcal{L}(\mathcal{H})$. The other direction is given by combining Proposition 14 and Corollary 3. □

The non-intersecting assumption (Assumption 4) is essential in Theorem 10. We refer the readers to [37] for counterexamples when $W \succ 0$ and the non-intersecting assumption is missing. The following example shows that the nonzero assumption (Assumption 3) cannot be dropped.

Example 2. *Let* $\mathcal{F} = \{ x \in \mathbb{R} \mid -x - 2 \leq 0 \} = [-2, \infty)$, $\mathcal{H} = \{x \in \mathbb{R} \mid -x + 1 \geq 0\} = (-\infty, 1]$, *and* $\mathcal{G} := \mathcal{F} \cap \mathcal{H} = [-2, 1]$. *Obviously, the non-intersecting assumption is satisfied as* $\mathrm{bd}(\mathcal{H}) = \{1\} \subseteq \mathcal{F}$. *However,*

$$\overline{\mathcal{C}}(\mathcal{G}) = \{ (x, X) \in \mathbb{R}^2 \mid X \leq 2 - x, X \geq x^2 \},$$

which is a proper subset of

$$\overline{\mathcal{C}}(\mathcal{F}) \cap \mathcal{L}(\mathcal{H}) = \{ (x, X) \in \mathbb{R}^2 \mid -2 \leq x \leq 1, X \geq x^2 \}.$$

We conclude this section with three corollaries of Theorem 10, which extend our main result to sets defined by multiple quadratic constraints. Let $\mathcal{H}' := \{ x \in \mathbb{R}^n \mid x^T W_k x + 2w_k^T x + \omega_k \leq 0, \ k \in K \}$ be a nonempty proper subset of $\mathbb{R}^n$, where $W_k \in \mathcal{S}^n$, $w_k \in \mathbb{R}^n$, $\omega_k \in \mathbb{R}$, and $K = \{1, \ldots, \ell\}$. Correspondingly, we define

$$\mathcal{L}(\mathcal{H}') := \left\{ (x, X) \;\middle|\; W_k \bullet X + 2w_k^T x + \omega_k \leq 0, \ \forall\, k \in K \right\}.$$

The first corollary is a direct extension of Theorem 10.

Corollary 4. *For a nonempty closed set* $\mathcal{F} \subseteq \mathbb{R}^n$ *and* $\mathcal{G} = \mathcal{F} \cap \mathcal{H}'$, $\overline{\mathcal{C}}(\mathcal{G}) = \overline{\mathcal{C}}(\mathcal{F}) \cap \mathcal{L}(\mathcal{H}')$ *under the following assumptions.*

Assumption 5. $W_k \neq 0$ *for all* $k \in K$.

Assumption 6. *For all* $k \in K$,

$$x^T W_k x + 2w_k^T x + \omega_k = 0 \quad \Longrightarrow \quad \begin{cases} x \in \mathcal{F}, \\ x^T W_j x + 2w_j^T x + \omega_j \leq 0, \quad \forall\, j \in K \setminus \{k\}. \end{cases}$$

Proof. Let $\mathcal{H}_k := \{\, x \in \mathbb{R}^n \mid x^T W_k x + 2w_k^T x + \omega_k \leq 0 \,\}$ for each $k \in K$. We have $\mathcal{H}' = \bigcap_{k \in K} \mathcal{H}_k$. When $\ell = 1$, the statement is reduced to Theorem 10. For $\ell \geq 2$, the corollary can be proved by repeatedly applying Theorem 10 to $\mathcal{H}_k$ and $\mathcal{F} \cap \mathcal{H}_1 \cap \cdots \cap \mathcal{H}_{k-1}$.

□

The second corollary shows that $\overline{\mathcal{C}}(\mathcal{G}) = \mathcal{S}(\mathcal{G})$ when $\mathcal{G}$ is defined by non-intersecting quadratic constraints with nonzero Hessians. Special cases and variants of the corollary can be spotted in the literature. To name a few: the non-binding constraints in [41], the generalized trust region subproblem in [27], and the non-interacting constraints in [3].

Corollary 5. *Let* $\mathcal{G} = \{\, x \in \mathbb{R}^n \mid x^T W_k x + 2w_k^T x + \omega_k \leq 0,\ k \in K \,\}$ *be a set defined by non-intersecting quadratic inequalities with nonzero Hessian matrices. That is, for all* $k \in K$, $W_k \neq 0$ *and*

$$x^T W_k x + 2w_k^T x + \omega_k = 0 \quad \Longrightarrow \quad x^T W_j x + 2w_j^T x + \omega_j \leq 0 \ \ \forall\, j \in K \setminus \{k\}.$$

Then, $\overline{\mathcal{C}}(\mathcal{G}) = \mathcal{S}(\mathcal{G}) = \{\, (x, X) \mid W_k \bullet X + 2w_k^T x + \omega_k \leq 0,\ k \in K,\ X \succeq xx^T \,\}$.

Proof. Note that $\mathcal{G}$ can be decomposed as $\mathcal{G} = \mathcal{F} \cap \mathcal{H}'$, where $\mathcal{F} = \mathbb{R}^n$ and $\mathcal{H}' = \mathcal{G}$. Since Assumptions 5 and 6 are satisfied, Corollary 4 implies

$$\begin{aligned} \overline{\mathcal{C}}(\mathcal{G}) &= \overline{\mathcal{C}}(\mathbb{R}^n) \cap \mathcal{L}(\mathcal{G}) \\ &= \{\, (x, X) \mid W_k \bullet X + 2w_k^T x + \omega_k \leq 0,\ k \in K,\ X \succeq xx^T \,\} \\ &= \mathcal{S}(\mathcal{G}). \end{aligned}$$

□

The last corollary can be interpreted as a sufficient condition for $\overline{\mathcal{C}}(\mathcal{F}\cap\mathcal{H}')=\overline{\mathcal{C}}(\mathcal{F})\cap\overline{\mathcal{C}}(\mathcal{H}')$. Note that $\overline{\mathcal{C}}(\mathcal{F})\cap\mathcal{L}(\mathcal{H}')=\overline{\mathcal{C}}(\mathcal{F})\cap\mathcal{S}(\mathcal{H}')$. By Theorem 10 and Corollary 5, we have the following statement.

Corollary 6. *For a nonempty closed set $\mathcal{F}\subseteq\mathbb{R}^n$, $\overline{\mathcal{C}}(\mathcal{F}\cap\mathcal{H}')=\overline{\mathcal{C}}(\mathcal{F})\cap\overline{\mathcal{C}}(\mathcal{H}')$ under Assumptions 5 and 6.*

4.3 Examples

In this section, we provide four examples to show how the theory in Section 4.2 can be applied to derive new convex hull results in the lifted space. The first example is a toy example, which is depicted in Figure 4.2.

Example 3. *Let $\mathcal{F}:=\{\,x\in\mathbb{R}^2\mid x_1\geq 0, x_2\geq 0\,\}$ be the nonnegative quadrant, $\mathcal{H}:=\{x\in\mathbb{R}^2\mid -(x_1-x_2)^2+2x_2-1\leq 0\}$, and $\mathcal{G}:=\mathcal{F}\cap\mathcal{H}$. It is known that $\overline{\mathcal{C}}(\mathcal{F})$ is the doubly nonnegative cone [2], that is,*

$$\overline{\mathcal{C}}(\mathcal{F})=\{\,(x,X)\mid X\succeq xx^T,\ X\geq 0,\ x\geq 0\,\}.$$

Since $\operatorname{bd}(\mathcal{H})\subseteq\mathcal{G}$, the non-intersecting assumption (Assumption 4) is satisfied. Therefore, Theorem 10 indicates that

$$\overline{\mathcal{C}}(\mathcal{G})=\overline{\mathcal{C}}(\mathcal{F})\cap\mathcal{L}(\mathcal{H})=\left\{(x,X)\;\middle|\;\begin{array}{l}-(X_{11}+X_{22})+2X_{12}+2x_2-1\leq 0,\\ X\succeq xx^T,\ X\geq 0,\ x\geq 0\end{array}\right\}.$$

The next example is a disjunctive mixed-integer set in $\mathbb{R}^2$. The closed convex hull of mixed-integer sets in the lifted space have been widely studied, e.g. in [10] and [16]. We hope the derivation the basic example can shed some light on future study of the geometry of the lifted closed convex hull for more complicated mixed-integer sets.

Example 4. *Let $\mathcal{G}:=\{\,x\in\mathbb{R}^2\mid x_1-x_2\in\{-1,0,1\}\,\}$ be a disjunctive set composed of the union of three parallel lines, see Figure 4.3. We can rewrite $\mathcal{G}$ as a set defined by non-intersecting quadratic*

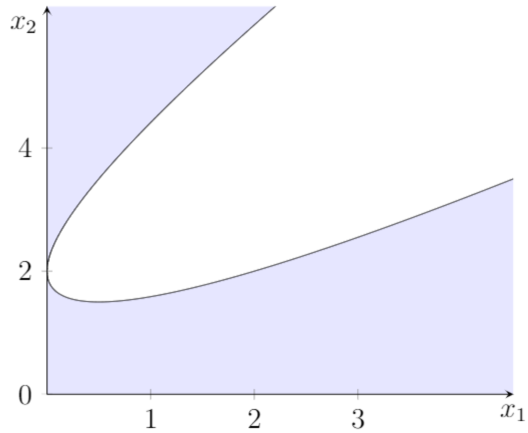


Figure 4.2: The first quadrant with a parabolic hollow.

inequalities. That is,

$$\mathcal{G} = \left\{ x \in \mathbb{R}^2 \;\middle|\; \begin{array}{l} (x_1 - x_2 - 1)(x_1 - x_2 + 1) \le 0, \\ -(x_1 - x_2 - 1)(x_1 - x_2) \le 0, \\ -(x_1 - x_2 + 1)(x_1 - x_2) \le 0 \end{array} \right\}.$$

Since the Hessians of the defining quadratic inequalites are nonzero, Corollary 5 indicates that

$$\overline{\mathcal{C}}(\mathcal{G}) = \mathcal{S}(\mathcal{G}) = \left\{ (x, X) \;\middle|\; \begin{array}{l} X_{11} + X_{22} - 2X_{12} - 1 \le 0 \\ X_{11} + X_{22} - 2X_{12} - x_1 + x_2 \ge 0 \\ X_{11} + X_{22} - 2X_{12} + x_1 - x_2 \ge 0 \\ X \succeq xx^T \end{array} \right\}.$$

The problem we consider in the next example arises from an extension of the Weber Problem [13] with restricted regions [1, 18].

Example 5. *The Weber problem determines the location of a facility that minimizes the sum of the transportation costs from this facility to n sites. In a traditional Weber problem, the transportation costs are proportional to the Euclidean distance. However, in some realistic situations, it would be more appropriate to assume that the transportation costs were proportional to the squared Euclidean distance [13]. On the other hand, restricted regions have been taken into considerations for the*

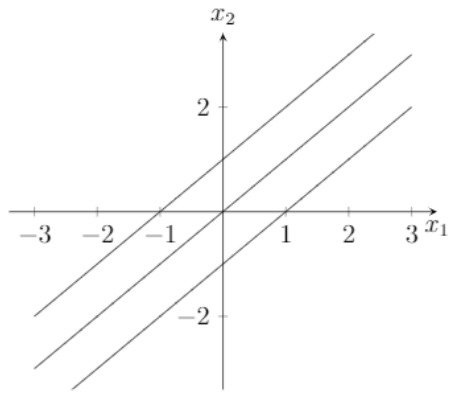


Figure 4.3: Three parallel lines.

Weber problem and facility location problems [1, 18].

These restricted regions, while allowing travel through them, prohibit the placement of a facility. Different shapes of the restricted regions have been considered, e.g. polyhedral restricted regions and circular restricted regions [26].

In this example, we consider an extended Weber problem with squared Euclidean distance and disjoint circular restricted regions. Let $a_1, \ldots, a_n \in \mathbb{R}^2$ *be the location of* n *sites. The transportation costs from a facility* $x \in \mathbb{R}^2$ *to* a_i $(i = 1, \ldots, n)$ *is assumed to be* $w_i\|x - a_i\|_2^2$*, where* $w_i > 0$ *is a weight. Let* b_k *and* r_k *be the center and radius, respectively, of the* k*-th restricted region* $(k = 1, \ldots, K)$*. We seek for an optimal location* x *of a facility, so that the total transportation costs are minimized and* x *is not located in the interior of any of the* K *disjoint regions. The problem can be formulated as a QCQP:*

$$\begin{aligned} \inf \quad & \sum_{i=1}^{n} w_i \|x - a_i\|_2^2 \\ s.t. \quad & \|x - b_k\|_2^2 \geq r_k^2, \quad k = 1, \ldots, K. \end{aligned}$$

Despite of being nonconvex, the feasible region $\mathcal{G} := \{\, x \in \mathbb{R}^2 \mid \|x - b_k\|_2^2 \geq r_k^2 \,\}$ *is defined by non-intersecting quadratic inequalities with nonzero Hessians. Therefore,* $\overline{\mathcal{C}}(\mathcal{G}) = \mathcal{S}(\mathcal{G})$ *by Corollary 5.*

The problem is then equivalent to a semidefinite program:

$$\begin{aligned}
\inf \quad & \sum_{i=1}^{n} w_i(tr(X) - 2a_i^T x + a_i^T a_i) \\
s.t. \quad & tr(X) - 2b_k^T x + b_k^T b_k \geq r_k^2, \quad k = 1, \ldots, K, \\
& X \succeq xx^T.
\end{aligned}$$

In the last example, we regenerate a semidefinite reformulation for a generalized trust-region subproblem with a milder assumption.

Example 6. *The trust-region subproblem (TRS) minimizes a quadratic function over the unit ball. It is well known that the standard semidefinite relaxation of TRS is exact. In this example, we consider a generalized TRS with interval bounds.*

$$\begin{aligned}
\inf \quad & x^T Q x + 2q^T x \\
s.t. \quad & \ell \leq x^T A x + 2a^T x \leq u,
\end{aligned} \tag{GTRS}$$

where $Q, A \in \mathcal{S}^n$, $q, a \in \mathbb{R}^n$, *and* $-\infty < \ell \leq u < \infty$. *Note that* A *is not necessarily positive semidefinite.*

This problem has been widely studied in the literature, e.g. [35, 27, 34]. It is shown in [34] that the following semidefinite relaxation

$$\begin{aligned}
\inf \quad & Q \bullet X + 2q^T x \\
s.t. \quad & \ell \leq A \bullet X + 2a^T x \leq u, \\
& X \succeq xx^T
\end{aligned} \tag{SDP-GTRS}$$

is exact under two assumptions:

1. *(nonzero)* $A \neq 0$*;*

2. *(Slater's condition) There exists* $\hat{x}$ *such that* $\ell < \hat{x}^T A \hat{x} + 2a^T \hat{x} < u$ *in the case when* $\ell < u$*; there exist* $\hat{x}$ *and* $\bar{x}$ *such that* $\hat{x}^T A \hat{x} + 2a^T \hat{x} < 0 < \bar{x}^T A \bar{x} + 2a^T \bar{x}$ *in the case when* $\ell = u$.

With our approach, since $\mathcal{G} := \{\, x \in \mathbb{R}^n \mid \ell \leq x^T A x + 2a^T x \leq u \,\}$ *is defined by two non-*

intersecting quadratic inequalities, we have

$$\overline{\mathcal{C}}(\mathcal{G}) = \mathcal{S}(\mathcal{G}) = \{\, (x, X) \mid \ell \leq A \bullet X + 2a^T x \leq u,\ X \succeq xx^T \,\}$$

if $A \neq 0$*. As a result,* (SDP-GTRS) *is exact as long as* $A \neq 0$*. Note that the second assumption (Slater's condition) in [34] is not required in our approach.*

To illustrate the difference, we denote the Lagrangian dual problem of (GTRS) *by (D-GTRS) and denote the conic dual problem of* (SDP-GTRS) *by (SDD-GTRS). We also use* $v(\cdot)$ *to represent the optimal value of each problem. The approach in [34] proves that* $v(GTRS) = v(D\text{-}GTRS)$ *with both the nonzero assumption and the Slater's condition. It is also shown in [34] that the Slater's condition for* (GTRS) *is equivalent to the one for* (SDP-GTRS)*. Since* $v(\text{(SDP-GTRS)}) = v(SDD\text{-}GTRS)$ *under the latter Slater's condition and (SDD-GTRS) is an equivalent reformulation of (D-GTRS), it is concluded that* $v(\text{(SDP-GTRS)} = v(SDD\text{-}GTRS) = v(D\text{-}GTRS) = v(GTRS)$*. On the other hand, our approach directly connects* (GTRS) *and* (SDP-GTRS) *without considering the dual problem. Therefore, the Slater's condition is not required to prove the exactness of* (SDP-GTRS)*. See Figure 4.4.*

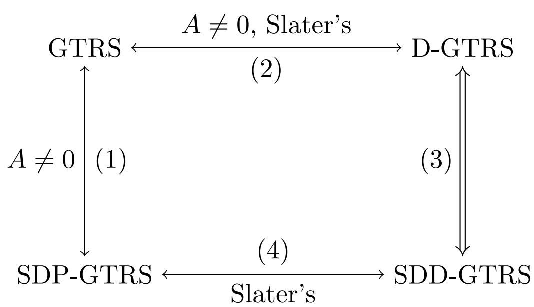


Figure 4.4: A flowchart showing the two approaches mentioned in Example 6. Our direct approach is only concerned with arc (1) whereas the method in [34] traverses arcs (2),(3) and (4).

4.4 Conclusion

For closed sets $\mathcal{F}$ and $\mathcal{H}'$, we consider the relation between $\overline{\mathcal{C}}(\mathcal{F} \cap \mathcal{H}')$ and $\overline{\mathcal{C}}(\mathcal{F})$. We show that $\overline{\mathcal{C}}(\mathcal{F} \cap \mathcal{H}') = \overline{\mathcal{C}}(\mathcal{F}) \cap \mathcal{L}(\mathcal{H}') = \overline{\mathcal{C}}(\mathcal{F}) \cap \overline{\mathcal{C}}(\mathcal{H}')$ when $\mathcal{H}'$ is defined by quadratic constraints with nonzero Hessians and the non-intersecting assumption is satisfied. This result generalizes the bounded case in [37] and other non-intersecting cases captured by Corollary 5. To prove the result, we provide a complete characterization of the asymptotic cones of sets defined by a single quadratic equality as a byproduct.

Chapter 5

Sets with One Non-Intersecting Linear Constraint

Chapter 4 decomposes a set G into two sets: $\mathcal{F}$ which possesses qualities with known results and $\mathcal{H}$, a quadratically defined complicating set defined by non-intersecting constraints ($\text{bd}(\mathcal{H}) \subseteq \mathcal{F}$). The method of proving these results fail when considering a non-intersecting affine constraint. Example 2, in Chapter 4, highlights how the results in the previous chapter require refinement in order to analyze the case with linear constraints. This chapter aims to approach this in two directions. Direction one is the direct approach which considers the conditions for a halfspace to induce a non-intersecting constraint. Under these conditions, the relation of asymptotic directions between $\mathcal{F}$ and $\mathcal{H}$ can be used to extend the results of Chapter 4. The second approach is using homogenizations mentioned in [31]. Defining a new non-intersecting relationship in the lifted, conic space may produce more insight about the relationship between $\mathcal{F}$ and $\mathcal{H}$.

5.1 Lifted Convex Hull Approach

Relaxing $\mathcal{H}$ to an affine linear constraint raises the question of if the boundary, $\text{bd}(\mathcal{H})$, can be contained in $\mathcal{F}$. To simplify this analysis, unless stated otherwise, we assume $\mathcal{F}$ to be at most quadratic and define $\mathcal{F}$, $\mathcal{H}$ as follows:

$$\mathcal{F} = \{\, x \mid x^T Ax + 2a^T x + \alpha \leq 0 \,\},$$
$$\mathcal{H} = \{\, x \mid w^T x + \omega \leq 0 \,\},$$

where, without loss of generality, $\|w\| = 1$. Answering this question is an application of the S-Lemma [36], presented in Lemma 1 in Chapter 2. This theorem of alternatives has been used in many comparisons of quadratic sets and is used to derive results relating objective functions to sets defined by one constraint. Simplifying this to a quadratic constraint and a linear constraint, we have the following lemma.

Lemma 16. *Let* $\mathcal{F} := \{\, x \in \mathbb{R}^n \mid x^T Ax + 2a^T x + \alpha \leq 0 \,\}$ *and* $\mathcal{H} := \{\, x \in \mathbb{R}^n \mid w^T x + \omega \leq 0 \,\}$. *Then* $\mathcal{H}$ *induces a non-intersecting constraint if and only if*

$$\begin{pmatrix} \gamma & (a - \omega Aw)^T B \\ (a - \omega Aw)B^T & B^T AB \end{pmatrix} \preceq 0,$$

where $\gamma = \alpha - 2\omega a^T w + \omega^2 w^T Aw$ *and the columns of* $B \in \mathbb{R}^{n\times(n-1)}$ *form an orthonormal basis of the null space of* w.

Proof. $\mathcal{H}$ induces a non-intersecting constraint in $\mathcal{F}$ if and only if

$$\max \{\, x^T Ax + 2a^T x + \alpha \mid w^T x + \omega = 0 \,\} \leq 0. \tag{5.1}$$

Using a null-space representation, we have that

$$\{\, x \in \mathbb{R}^n \mid w^T x + \omega = 0 \,\} = \{\, By - \omega w \mid y \in \mathbb{R}^{n-1} \,\}.$$

We can rewrite (5.1) as

$$\max \{\, y^T B^T ABy + 2(a - \omega Aw)^T By + \gamma \,\} \leq 0, \tag{5.2}$$

where $\gamma = \alpha - 2\omega a^T w + \omega^2 w^T Aw$.

From [31], we know that (5.2) is satisfied when

$$\begin{pmatrix} \gamma & (a-\omega Aw)^T B \\ (a-\omega Aw)B^T & B^T AB \end{pmatrix} \preceq 0.$$

□

Lemma 16 can be extended inductively to compare a set $\mathcal{F}$ defined by countably finite quadratic inequalities by checking the requirement with each constraint individually. After determining whether $\mathcal{H}$ induces a non-intersecting constraint, the next goal is to define the relationship of the asymptotic directions between $\mathcal{F}$, $\mathcal{H}$, and the intersection : $\mathcal{G} = \mathcal{F} \cap \mathcal{H}$.

Proposition 16. *Let* $\mathcal{H} = \{ x \in \mathbb{R}^n \mid w^T x + \omega \leq 0 \}$, *and* $\text{bd}(\mathcal{H}) \subseteq \mathcal{F}$, *then* $\mathcal{F}_\infty \cap \mathcal{H}_\infty = \mathcal{G}_\infty$.

Proof. The reverse containment is trivial. Consider $d \in \mathcal{F}_\infty \cap \mathcal{H}_\infty$. If $w^T d = 0$, then $d \in \text{bd}(\mathcal{H})_\infty \subseteq \mathcal{G}_\infty$. Otherwise, $w^T d < 0$. Then there exists a sequence $\{ x_i \}_{i \in I} \subseteq \mathcal{F}$ such that $\|x_i\| \to \infty$ and

$$d = \lim_{i\to\infty} \frac{x_i}{\|x_i\|}.$$

We have that

$$\begin{aligned} 0 &> \lim_{i\to\infty} \frac{1}{\|x_i\|} w^T x_i \\ &= \lim_{i\to\infty} \frac{1}{\|x_i\|} (w^T x_i + \omega). \end{aligned}$$

Since $w^T d < 0$ and $\|x_i\| > 0$ for all i, there exists a subsequence $\{x_j\}$ such that $w^T x_j + \omega < 0$. Therefore, this subsequence is contained in $\mathcal{G}$ and $d \in \mathcal{G}_\infty$.

□

Proposition 16 has no assumptions on the definition of the set $\mathcal{F}$ other than the containment of the boundary of $\mathcal{H}$. Recall that in Chapter 4, the asymptotic cone provides insight into the recession cone of sets in the lifted space. That is, $\mathcal{F}_\infty$ provides at least a partial understanding of $\text{Rec}(\overline{\mathcal{C}}(\mathcal{F}))$. In fact, since $\mathcal{F}$ is defined by a single quadratic constraint, the following proposition provides a complete understanding of $\text{Rec}(\overline{\mathcal{C}}(\mathcal{F}))$.

Proposition 17. *Let* $\mathcal{F} \coloneqq \{\, x \in \mathbb{R}^n \mid x^T Ax + 2a^T x + \alpha \le 0 \,\}$*. Then*

$$\operatorname{Rec}(\overline{\mathcal{C}}(\mathcal{F})) = \operatorname{conv}\{\, (0, dd^T) \mid d \in \mathcal{F}_\infty \,\}.$$

Proof. For the forward containment, note that

$$\operatorname{Rec}(\overline{\mathcal{C}}(\mathcal{F})) = \operatorname{Rec}(\mathcal{S}(\mathcal{F})) = \{\, (0, D) \mid A \bullet D \le 0,\ D \succeq 0 \,\},$$

where $\mathcal{S}(\mathcal{F})$ is the Shor relaxation of $\mathcal{F}$. Since $D \succeq 0$, we can decompose D as

$$D = \sum_{i=1}^{r} d_i d_i^T, \quad \text{where } d_i^T A d_i \le 0\ \forall i = 1, \dots, r,\ r = \operatorname{rank}(D).$$

Since $d_i^T A d_i \le 0$, $d \in \mathcal{F}_\infty$ for all $i = 1, \dots, r$ and $(0, D) \in \operatorname{conv}\left\{\, (0, dd^T) \mid d \in \mathcal{F}_\infty \,\right\}$. To see the reverse containment, note that for $(0, D) \in \operatorname{conv}\left\{\, (0, dd^T) \mid d \in \mathcal{F}_\infty \,\right\}$, D is a convex combination of rank-one products of elements in $\mathcal{F}_\infty$. Therefore $D \succeq 0$ where

$$A \bullet D = A \bullet \sum_{i}^{r} \lambda_i d_i d_i^T = \sum_{i}^{r} \lambda_i A \bullet d_i d_i^T \le 0.$$

□

Expanding Proposition 17 for an $\mathcal{F}$ defined by multiple constraints runs into issues for the forward containment of the previous proof. That is, after performing a rank-one decomposition with regard to one constraint, a relation for the remaining constraints has yet to be found.

For the analysis, $\mathcal{F}$ is a quadratically defined set. The non-intersecting assumption provides a rather uninteresting case for when $\mathcal{F}$ is defined as an affine, linear set. Geometrically, the assumption limits the boundaries of $\mathcal{F}$ and $\mathcal{H}$ to being parallel hyperplanes. Algebraically, we have that for $\mathcal{F} = \left\{\, x \mid a^T x + \alpha \le 0 \,\right\}$, then $\mathcal{H} = \left\{\, x \mid \beta a^T x + \omega \le 0 \,\right\}$.

Proposition 18. *Let* $\mathcal{F} = \{\, x \in \mathbb{R}^n \mid a^T x + \alpha \le 0 \,\}$ *and* $\mathcal{H} = \{\, x \in \mathbb{R}^n \mid \beta a^T x + \omega \le 0 \,\}$ *where,* $a \neq 0$*,* $\beta \in \mathbb{R} \setminus \{0\}$*, and* $\mathcal{G} = \mathcal{F} \cap \mathcal{H} \neq \emptyset$*. If* $\operatorname{bd}(\mathcal{H}) \subseteq \mathcal{F}$*, then one of the following is true:*

1. $\beta > 0$ *and* $\mathcal{G} = \mathcal{F} \cap \mathcal{H} = \mathcal{H}$*, or*

2. $\beta < 0$ *and* $\overline{\mathcal{C}}(\mathcal{G}) \neq \overline{\mathcal{C}}(\mathcal{F}) \cap \mathcal{L}(\mathcal{H})$*,*

where

$$\mathcal{L}(\mathcal{H}) := \left\{ (x, X) \;\middle|\; W \bullet X + \beta a^T x + \omega \leq 0 \right\} = \left\{ (x, X) \;\middle|\; \beta a^T x + \omega \leq 0 \right\}.$$

Proof. Item 1 is trivial since $\beta > 0$, so $\mathcal{H} \subseteq \mathcal{F}$. If $\beta < 0$, then, using Proposition 3, $\mathcal{G}_\infty = \left\{ d \in \mathbb{R}^n \;\middle|\; a^T d = 0 \right\} \subsetneq \left\{ d \in \mathbb{R}^n \;\middle|\; a^T d \leq 0 \right\} = \mathcal{F}_\infty$. Therefore, by Proposition 17, $\mathrm{Rec}(\overline{\mathcal{C}}(\mathcal{F})) \neq \mathrm{Rec}(\overline{\mathcal{C}}(\mathcal{G}))$ it is easy to see $\mathrm{Rec}(\overline{\mathcal{C}}(\mathcal{F}) \cap \mathcal{L}(\mathcal{H})) \neq \mathrm{Rec}(\overline{\mathcal{C}}(\mathcal{G}))$. □

Therefore, in order for the results of Chapter 4 to hold when $\mathcal{H}$ is defined as a half-space, $\mathcal{F}$, assuming $\mathcal{F}$ is defined by a single constraint, must either be defined by a strictly quadratic constraint ($A \neq 0$) or $\mathcal{G} = \mathcal{H}$. In particular, Proposition 18 highlights the issue with Example 2 describing how $\mathcal{G}$, and $\overline{\mathcal{C}}(\mathcal{G})$ as consequence, is bounded. However, for the sake of continuing an analysis, $\mathcal{F}$ will be defined with $A \neq 0$. This leads to the question of under what conditions of (a, A) will $\mathcal{H}$ induce a non-intersecting constraint in $\mathcal{F}$. There are only three cases of $\mathcal{F}$ such that a half-space $\mathcal{H}$ may possibly induce a non-intersecting constraint:

1. $A \preceq 0$,
2. A indefinite, and
3. $A \succeq 0$ and $a \in \mathrm{Range}(A)$.

Note that the case of $A \succeq 0$ where $a \notin \mathrm{Range}(A)$ is not on the list of possible cases. This case can be ruled out with the following proposition:

Proposition 19. *Let* $\mathcal{F} = \left\{ x \in \mathbb{R}^n \;\middle|\; x^T A x + 2a^T x + \alpha \leq 0 \right\}$ *be a nonempty set with* $A \succeq 0$ *nonzero and* $\mathcal{H} = \left\{ x \in \mathbb{R}^n \;\middle|\; w^T x + \omega \leq 0 \right\}$. *Then* $\mathrm{bd}(\mathcal{H}) \subseteq \mathcal{F}$ *only if* $a \in \mathrm{Range}(A)$ *and* $\mathrm{rank}(A) \leq 1$.

Proof. Suppose that $\mathrm{bd}(\mathcal{H}) \subseteq \mathcal{F}$. By Lemma 16, $\mathrm{bd}(\mathcal{H}) \subseteq \mathcal{F}$ if and only if

$$T = \begin{pmatrix} \gamma & (a - \omega A w)^T B \\ (a - \omega A w) B^T & B^T A B \end{pmatrix} \preceq 0,$$

where $\gamma = \alpha - 2\omega a^T w + \omega^2 w^T A w$ and the columns of $B \in \mathbb{R}^{n \times (n-1)}$ form an orthonormal basis of the null space of w. This implies that the principal minor $B^T A B \preceq 0$. Combining this with $A \succeq 0$, we have that $B^T A B = 0$ and $AB = 0$. This has two implications. First, since the columns of B

form an orthonormal basis of w, $\text{rank}(B) = n - 1$ and as such, $\text{rank}(A) \leq 1$ where $A = \kappa w w^T$ for some $\kappa \geq 0$. Second, since $T \preceq 0$ and $B^T A B = 0$, then the corresponding rows and columns are also 0, i.e $(a - \omega A w)^T B = 0$. Then

$$0 = (a - \omega A w)^T B = a^T B - \omega w^T A^T B = a^T B - \omega w^T A B = a^T B.$$

As a result of this, $a \in \text{Range}(A)$. Otherwise, $a^T B \neq 0$ which is a contradiction to $T \preceq 0$.

□

From Proposition 19, Case 3 is not as interesting as the previous two cases. From Proposition 19, we know that if $\text{bd}(\mathcal{H}) \subseteq \mathcal{F}$ then $A = \kappa_1 w w^T$ for some $\kappa_1 \geq 0$ and $a = \kappa_2 w$ for $\kappa_2 \in \mathbb{R}$. If $A = 0$, then $\mathcal{F}$ is reduced to a halfspace (assuming $a = 0$) which is discussed in Proposition 18. Otherwise, $\mathcal{F}$ can be rewritten as follows:

$$\begin{aligned}
\mathcal{F} &= \left\{ x \mid x^T A x + 2a^T x + \alpha \leq 0 \right\} \\
&= \left\{ x \mid x^T \kappa_1 w w^T x + 2\kappa_2 w^T x + \alpha \leq 0 \right\} \\
&= \left\{ x \mid \kappa_1 (w^T x)^2 + 2\kappa_2 w^T x + \alpha \leq 0 \right\} \\
&= \left\{ x \mid \ell \leq w^T x \leq u \right\}, \quad \text{for some } \ell \leq u.
\end{aligned}$$

That is, if $\text{bd}(\mathcal{H}) \subseteq \mathcal{F}$, then $\mathcal{F}$ can be described as the region between two parallel hyperplanes. With this in mind, we only consider the case when $A \not\succeq 0$ and have the following propositions relating to the symmetry of the asymptotic directions of $\mathcal{F}$ and its impact on the recession cone $\text{Rec}(\overline{\mathcal{C}}(\mathcal{G}))$.

Proposition 20. *Let $\mathcal{F} = \left\{ x \mid x^T A x + 2a^T x + \alpha \leq 0 \right\}$ such that $A \not\succeq 0$. Then for $\mathcal{H} = \{ x \in \mathbb{R}^n \mid w^T x + \omega \leq 0 \}$ such that $\mathcal{H}$ induces a non-intersecting constraint in $\mathcal{F}$,*

$$\{ (0, dd^T) \mid d \in \mathcal{F}_\infty \} = \{ (0, dd^T) \mid d \in \mathcal{F}_\infty \cap \mathcal{H}_\infty \}.$$

Proof. Since $A \not\succeq 0$, by Proposition 3, if $d \in \mathcal{F}_\infty$ then $-d \in \mathcal{F}_\infty$. Since $\mathcal{H}$ is a halfspace, $d \in \mathcal{H}_\infty$ or $-d \in \mathcal{H}_\infty$. Therefore,

$$(0, dd^T) \in \{ (0, dd^T) \mid d \in \mathcal{F}_\infty \cap \mathcal{H}_\infty \} = \{ (0, dd^T) \mid d \in \mathcal{F}_\infty \}.$$

□

Proposition 20, along with Proposition 12, fills in the picture for when $\mathcal{F}$ is a quadratically defined set where $A \neq 0$ and $\mathcal{H}$ is defined by a halfspace where $\mathrm{bd}(\mathcal{H}) \subseteq \mathcal{F}$, the set $\{ (0, dd^T) \mid d \in \mathcal{F}_\infty \cap \mathcal{H}_\infty \}$ is no more difficult to calculate than $\{ (0, dd^T) \mid d \in \mathcal{F}_\infty \}$. This relationship helps define the recession cone of the lifted convex hull of the intersection.

Proposition 21. *Let* $\mathcal{F} := \{ x \in \mathbb{R}^n \mid x^T Ax + 2a^T x + \alpha \leq 0 \}$ *for* $A \not\succeq 0$, $\mathcal{H} := \{ x \in \mathbb{R}^n \mid w^T x + \omega \leq 0 \}$, *and* $\mathrm{bd}(\mathcal{H}) \subseteq \mathcal{F}$. *Then, for* $\mathcal{G} = \mathcal{F} \cap \mathcal{H}$, $\mathrm{Rec}(\overline{\mathcal{C}}(\mathcal{G})) = \mathrm{Rec}(\overline{\mathcal{C}}(\mathcal{F}))$.

Proof. We know that $\mathrm{Rec}(\overline{\mathcal{C}}(\mathcal{G})) \subseteq \mathrm{Rec}(\overline{\mathcal{C}}(\mathcal{F}))$. For the other direction,

$$\mathrm{Rec}(\overline{\mathcal{C}}(\mathcal{F})) = \mathrm{conv}\,\{ (0, dd^T) \mid d \in \mathcal{F}_\infty \} = \mathrm{conv}\,\{ (0, dd^T) \mid d \in \mathcal{G}_\infty \} \subseteq \mathrm{Rec}(\overline{\mathcal{C}}(\mathcal{G})),$$

where the second equality comes from Propositions 16 and 20. □

With a complete understanding of $\mathrm{Rec}(\overline{\mathcal{C}}(\mathcal{G}))$ with regards to $\overline{\mathcal{C}}(\mathcal{F})$, the relationship between $\overline{\mathcal{C}}(\mathcal{G})$ and $\overline{\mathcal{C}}(\mathcal{F})$ can be completed with the following proposition.

Proposition 22. *Let* $\mathcal{F} := \{ x \in \mathbb{R}^n \mid x^T Ax + 2a^T x + \alpha \leq 0 \}$, $A \not\succeq 0$, *and* $\mathcal{H} := \{ x \in \mathbb{R}^n \mid w^T x + \omega \leq 0 \}$ *where bd*$(\mathcal{H}) \subseteq \mathcal{F}$. *Then* $\overline{\mathcal{C}}(\mathcal{G}) = \overline{\mathcal{C}}(\mathcal{F}) \cap \mathcal{L}(\mathcal{H})$.

Proof. The forward containment is straightforward. For the reverse containment, let $(x, X) \in \overline{\mathcal{C}}(\mathcal{F}) \cap \mathcal{L}(\mathcal{H})$. If $X = xx^T$, then $x \in \mathcal{F} \cap \mathcal{H} = \mathcal{G}$ and $(x, xx^T) \in \overline{\mathcal{C}}(\mathcal{G})$. Otherwise, since $(x, X) \in \overline{\mathcal{C}}(\mathcal{F})$, then

$$(x, X) = \sum_{i \in I} \lambda_i (x^i, x^i (x^i)^T) + \sum_{j \in J} \lambda_j (x^j, x^j (x^j)^T),$$

where $I = \{ i \mid w^T x^i + \omega \leq 0 \}$, $J = \{ j \mid w^T x^j + \omega > 0 \}$, and (x, X) is a convex combination of $(x^i, x^i (x^i)^T) \in \overline{\mathcal{C}}(\mathcal{F})$ for all $i \in I \cup J$. Since $(x, X) \in \mathcal{L}(\mathcal{H})$, $|I| \geq 1$ where $(x^i, x^i (x^i)^T) \in \mathcal{L}(\mathcal{H})$ for $i \in I$. For any $j \in J$, say $(x^j, x^j (x^j)^T) \notin \mathcal{L}(\mathcal{H})$, then there exists $(y_j, Y_j) \in \mathrm{bd}(\mathcal{L}(\mathcal{H}))$ such that

$$(y^j, Y^j) = \mu_j (x, X) + (1 - \mu_j)(x^j, x^j (x^j)^T),$$

where $\mu_j \in [0,1]$. Solving for (x, X) for all $j \in J$, we have

$$(x, X) = \sum_{i \in I} \frac{\lambda_i}{\sigma}(x^i, x^i(x^i)^T) + \sum_{j \in J} \frac{\lambda_j}{\sigma(1-\mu_j)}(y^j, Y^j),$$

where $\sigma := 1 + \sum_{j \in J} \frac{\lambda_j \mu_j}{1-\mu_j}$. Since $\mathrm{bd}(\mathcal{H}) \subseteq \mathcal{F}$, for all $j \in J$, $(y^j, Y^j) = (y^j, y^j(y^j)^T) + \kappa_j(0, D^j)$, where $(y^j, y^j(y^j)^T) \in \overline{\mathcal{C}}(\mathcal{F}) \cap \mathcal{L}(\mathcal{H})$, $\kappa_j \geq 0$, and $(0, D^j) \in \mathrm{Rec}(\overline{\mathcal{C}}(\mathcal{F}))$. Since $(y^j, y^j(y^j)^T) \in \overline{\mathcal{C}}(\mathcal{F}) \cap \mathcal{L}(\mathcal{H})$, $(y^j, y^j(y^j)^T) \in \overline{\mathcal{C}}(\mathcal{G})$. Also, by Proposition 21, $(0, D^j) \in \mathrm{Rec}(\overline{\mathcal{C}}(\mathcal{G}))$. Therefore, since (x, X) can be represented by a convex combination of extreme points and a conic combination of recession directions of $\overline{\mathcal{C}}(\mathcal{G})$, $(x, X) \in \overline{\mathcal{C}}(\mathcal{G})$. □

In conclusion, if the complicating set $\mathcal{H}$ is defined by a non-intersecting affine linear constraint, then the results of Chapter 4 hold. However, there are two drawbacks. The first is that as $\mathcal{F}$ is composed of more constraints, the non-intersecting assumption becomes harder to attain. The second is addressed with Proposition 18, when the addition of an affine linear constraint carries the possibility of removing the unboundedness of $\mathcal{G}$. This is the issue presented in Example 2.

5.2 Homogenizations

The previous section demonstrates the need for extra resources to extend the theory of Chapter 4. Instead of applying concepts and theory directly to the intersection of sets, perhaps an intermediary step can derive a more explicit description. For a given set $\mathcal{F}$, we consider the homogenization $\mathscr{H}(\mathcal{F})$ as follows:

Definition 3. *[Homogenization] Let $\mathcal{F}$ be a closed set, then*

$$\mathscr{H}(\mathcal{F}) := \mathrm{cl}\left\{ \begin{pmatrix} t \\ x \end{pmatrix} \;\middle|\; t > 0, \frac{x}{t} \in \mathcal{F} \right\}. \tag{5.3}$$

This set has another representation with a more intuitive form.

Lemma 17 ([15])**.** *Let $\mathcal{F} \subseteq \mathbb{R}^n$ be a nonempty closed set. Then*

$$\mathscr{H}(\mathcal{F}) = \mathrm{cl}\,\mathrm{cone}(\{1\} \times \mathcal{F}) = \mathrm{cone}(\{1\} \times \mathcal{F}) \cup (\{0\} \times \mathcal{F}_\infty).$$

After a quick inspection, it is easy to see that $\mathscr{H}(\mathcal{F})$ is a cone. In fact, $\mathscr{H}(\mathcal{F})$ has a similar decomposition to the closed positive hull described in [29]: one piece related to the set and the second related to the asymptotic cone. Instances of the homogenization of a set has occurred multiple times in literature. For example, describing the lifted convex hull of a set defined by a single quadratic inequality [31] and in the analysis of the semidefinite representation of nonconvex quadratic programs [15]. Using homogenizations in relation to Chapter 4 with the non-intersecting assumption requires knowledge of the boundary of a homogenized set. Equation (5.3), while being straightforward, does not provide a direct understanding of $\mathscr{H}(\mathcal{F})$ when t approaches 0. Consider the following proposition:

Proposition 23. *Let $\mathcal{F} \subseteq \mathbb{R}^n$ be a closed nonempty set. Then*

$$\operatorname{bd}\left\{ \begin{pmatrix} t \\ x \end{pmatrix} \;\middle|\; \frac{x}{t} \in \mathcal{F}, t > 0 \right\} \subseteq \left\{ \begin{pmatrix} t \\ x \end{pmatrix} \;\middle|\; \frac{x}{t} \in \operatorname{bd}(\mathcal{F}), t > 0 \right\} \cup \left\{ \begin{pmatrix} 0 \\ x \end{pmatrix} \;\middle|\; x \in \mathcal{F}_\infty \right\}.$$

Proof. Denote $A = \left\{ (t,x) \;\middle|\; \frac{x}{t} \in \mathcal{F},\ t > 0 \right\}$ and let $(t,x) \in \operatorname{bd}(A)$. If

1. $t = 0$, then since $\mathscr{H}(\mathcal{F}) = \operatorname{cl}(A)$, $\operatorname{bd}(A) \subseteq \mathscr{H}(\mathcal{F})$ and $(0,x) \in (\{0\} \times \mathcal{F}_\infty)$.

2. $t > 0$, then $y = \frac{x}{t} \notin \operatorname{int}(\mathcal{F})$. Otherwise, there exists $\varepsilon > 0$ such that $\mathcal{B}_\varepsilon(\frac{x}{t}) \subseteq \operatorname{int}(\mathcal{F})$. For M being the minimum distance between $\frac{x}{t}$ and the boundary of $\mathcal{F}$, $0 < \varepsilon < M$. Consider $\delta > 0$ such that for any $t > 0$, $t\varepsilon < (t-\delta)M$ and the set

$$\hat{\mathcal{B}} = [t-\delta, t+\delta] \times t\mathcal{B}_\varepsilon\left(\frac{x}{t}\right).$$

   Then for any given $\hat{t} \in [t-\delta, t+\delta]$, the distance between any point $(\hat{t}, y) \in \hat{\mathcal{B}}$ and the boundary of A is strictly greater than $t\varepsilon$ and $\hat{\mathcal{B}} \subseteq \operatorname{int}(A)$. Hence, (t,x) is not a boundary point of A. Therefore, $\frac{x}{t} \in \operatorname{bd}(\mathcal{F})$ and $(t,x) \in \operatorname{cone}(\{1\} \times \operatorname{bd}(\mathcal{F}))$.

□

Proposition 23 highlights that as t approaches 0, the behavior of $\mathscr{H}(\mathcal{F})$ is determined by the asymptotic cone $\mathcal{F}_\infty$. With this understanding, the boundary, $\operatorname{bd}(\mathscr{H}(\mathcal{F}))$, can be fully defined.

Proposition 24. *Let $\mathcal{F} \subseteq \mathbb{R}^n$ be a closed nonempty set. Then*

$$\operatorname{bd}(\mathscr{H}(\mathcal{F})) = \left\{ \begin{pmatrix} t \\ x \end{pmatrix} \;\middle|\; \frac{x}{t} \in \operatorname{bd}(\mathcal{F}), t > 0 \right\} \cup \left\{ \begin{pmatrix} 0 \\ x \end{pmatrix} \;\middle|\; x \in \mathcal{F}_\infty \right\}.$$

Proof. (⊆). From Proposition 23, we know that

$$\operatorname{bd}\operatorname{cl}\left\{ \begin{pmatrix} t \\ x \end{pmatrix} \;\middle|\; \frac{x}{t} \in \mathcal{F}, t > 0 \right\} = \operatorname{bd}\left\{ \begin{pmatrix} t \\ x \end{pmatrix} \;\middle|\; \frac{x}{t} \in \mathcal{F}, t > 0 \right\}$$
$$\subseteq \left\{ \begin{pmatrix} t \\ x \end{pmatrix} \;\middle|\; \frac{x}{t} \in \operatorname{bd}(\mathcal{F}), t > 0 \right\} \cup \left\{ \begin{pmatrix} 0 \\ x \end{pmatrix} \;\middle|\; x \in \mathcal{F}_\infty \right\}.$$

(⊇). Let $(t, x) \in \left\{ (t, x) \;\middle|\; \frac{x}{t} \in \operatorname{bd}(\mathcal{F}), t > 0 \right\}$. First, we show that $(t, x) \in \mathscr{H}(\mathcal{F})$. For all $\varepsilon > 0$, $\mathcal{B}_\varepsilon\left(\frac{x}{t}\right) \cap \mathcal{F} \neq \emptyset$ and $\mathcal{B}_\varepsilon\left(\frac{x}{t}\right) \cap \mathcal{F}^C \neq \emptyset$, where $\mathcal{F}^C$ is the complement of $\mathcal{F}$. Also, since $\frac{x}{t} \in \operatorname{bd}(\mathcal{F})$, $1 \times \frac{x}{t} \in \operatorname{cone}(\{1\} \times \operatorname{bd}(\mathcal{F})) \subseteq \operatorname{cone}(\{1\} \times \mathcal{F}) \subseteq \mathscr{H}(\mathcal{F})$ and $(t, x) \in \mathscr{H}(\mathcal{F})$.

To show that $(t, x) \in \operatorname{bd}\mathscr{H}(\mathcal{F})$, note that $\mathcal{B}_\varepsilon\left(\frac{x}{t}\right) \cap \mathcal{F} \neq \emptyset$, $\mathcal{B}_\varepsilon\left(\frac{x}{t}\right) \cap \mathcal{F}^C \neq \emptyset$, and there exists $\hat{y} \in \mathcal{B}_\varepsilon\left(\frac{x}{t}\right) \cap \mathcal{F}$ and $\bar{y} \in \mathcal{B}_\varepsilon\left(\frac{x}{t}\right) \cap \mathcal{F}^C$. Consider the set

$$t \times t\mathcal{B}_\varepsilon(x) = t \times \mathcal{B}_{t\varepsilon}(x).$$

Since $\hat{y} \in \mathcal{B}_\varepsilon\left(\frac{x}{t}\right) \cap \mathcal{F} \neq \emptyset$, $(t, t\hat{y}) \in (t \times \mathcal{B}_{t\varepsilon}(x)) \cap \mathscr{H}(\mathcal{F})$. Similarly, $(t, t\bar{y}) \in (t \times \mathcal{B}_{t\varepsilon}(x)) \cap \operatorname{int}\mathscr{H}(\mathcal{F}^C)$. Since $\operatorname{int}\mathscr{H}(\mathcal{F}^C) \subseteq \mathscr{H}(\mathcal{F})^C$, $(t, t\bar{y}) \notin \mathscr{H}(\mathcal{F})$. Therefore, $(t, x) \in \operatorname{bd}(\mathscr{H}(\mathcal{F}))$.

Since $\{ (0, x) \mid x \in \mathcal{F}_\infty \}$ is an exposed face of $\mathscr{H}(\mathcal{F})$, any element $(0, x) \in \{ (0, x) \mid x \in \mathcal{F}_\infty \}$ is in the boundary of $\mathscr{H}(\mathcal{F})$. □

Proposition 24 quickly disproves the idea that $\operatorname{bd}(\mathscr{H}(\mathcal{F})) = \mathscr{H}(\operatorname{bd}(\mathcal{F}))$. This only occurs when $\mathcal{F}_\infty = \operatorname{bd}(\mathcal{F}_\infty)$. In order to build a connection to Chapter

4 with the non-intersecting assumption, we need to understand how non-intersecting in the homogenized space affects the original space.

Proposition 25. *Let $\mathcal{F}$, $\mathcal{H}$ be closed, nonempty sets. If $\operatorname{bd}(\mathscr{H}(\mathcal{H})) \subseteq \mathscr{H}(\mathcal{F})$ then $\operatorname{bd}(\mathcal{H}) \subseteq \mathcal{F}$.*

Proof. Let $x \in \operatorname{bd}(\mathcal{H})$. Then, by Proposition 24, $(t, tx) \in \operatorname{bd}(\mathscr{H}(\mathcal{H})) \subseteq \mathscr{H}(\mathcal{F})$ for all $t > 0$. Setting $t = 1$, we have that $(1, x) \in \mathscr{H}(\mathcal{F})$ and $\frac{x}{1} = x \in \mathcal{F}$. □

Proposition 25 provides only one direction in the relation. This is due to the asymptotic cone. Suppose there are two sets $\mathcal{F}$, $\mathcal{H}$ such that

$$\left\{ \begin{pmatrix} 0 \\ x \end{pmatrix} \;\middle|\; x \in \mathcal{H}_\infty \right\} \not\subset \left\{ \begin{pmatrix} 0 \\ x \end{pmatrix} \;\middle|\; x \in \mathcal{F}_\infty \right\},$$

and thus $\operatorname{bd}(\mathscr{H}(\mathcal{H}))$ is not contained in $\mathscr{H}(\mathcal{F})$. Viewing this directly in the scope of Chapter 4, the direct extension to homogenizations may not provide the same quality of results. However, there is a level of symmetry in quadratic constraints that could be used.

5.2.1 Set Defined by a Single Quadratic Constraint

Focusing this analysis, let $\mathcal{F} \subseteq \mathbb{R}^n$ be a closed, quadratically defined set,

$$\mathcal{F} := \{ x \mid x^T A_i x + 2a_i^T x + \alpha_i \leq 0, \ i \in I \},$$

with the following homogenization:

$$\mathscr{H}(\mathcal{F}) := \operatorname{cl}\left\{ \begin{pmatrix} t \\ x \end{pmatrix} \;\middle|\; t > 0, x^T A_i x + 2ta_i^T x + \alpha_i t^2 \leq 0, \ i \in I \right\}. \tag{5.4}$$

For this discussion, set $|I| = 1$ such that $\mathcal{F}$ is defined by a single quadratic constraint with parameters $A \in \mathcal{S}^n$, $a \in \mathbb{R}^n$, and $\alpha \in \mathbb{R}$. In general, describing the homogenization of $\mathcal{F}$ requires the closure presented in (5.4). In terms of a quadratically defined set, this is due to the asymptotic cone, $\mathcal{F}_\infty$, and its relation to the Hessian A. Consider the inequality

$$x^T Ax + 2ta^T x + \alpha t^2 \leq 0. \tag{5.5}$$

When $t = 0$, $x^T Ax \leq 0$. By Proposition 3, we know that $\mathcal{F}_\infty \subseteq \left\{ x \;\middle|\; x^T Ax \leq 0 \right\}$ with equality if and only if $A \not\succeq 0$. This is due to, when $A \succeq 0$, $\mathcal{F}_\infty$ also depends on the linear inequality $a^T x \leq 0$. This additional information is lost in (5.5) when $t = 0$. Therefore, when $\mathcal{F}$ is defined by a single quadratic inequality, we have the following proposition:

Proposition 26. *Let $\mathcal{F} = \left\{ x \;\middle|\; x^T Ax + 2a^T x + \alpha \leq 0 \right\}$ be a nonempty set. Then*

$$\mathscr{H}(\mathcal{F}) \subseteq \left\{ \begin{pmatrix} t \\ x \end{pmatrix} \;\middle|\; t \geq 0, \ x^T Ax + 2ta^T x + \alpha t^2 \leq 0 \right\},$$

with equality when A is not a positive semidefinite matrix.

Proof. It is straightforward that

$$\operatorname{cone}(\{ 1 \} \times \mathcal{F}) \subseteq \left\{ \begin{pmatrix} t \\ x \end{pmatrix} \;\middle|\; t \geq 0, \ x^T Ax + 2ta^T x + \alpha t^2 \leq 0 \right\}.$$

By Lemma 17, we only need to consider $\{ 0 \} \times \mathcal{F}_\infty$. Let A be a non positive semidefinite matrix. Then $\mathcal{F}_\infty = \left\{ x \mid x^T Ax \leq 0 \right\}$ and

$$\left\{ \begin{pmatrix} t \\ x \end{pmatrix} \,\middle|\, t = 0, x^T Ax \leq 0 \right\} = \left\{ \begin{pmatrix} 0 \\ x \end{pmatrix} \,\middle|\, x \in \mathcal{F}_\infty \right\}.$$

Otherwise, if $A \succeq 0$ (A could be the zero matrix), then $\mathcal{F}_\infty \subseteq \left\{ x \mid x^T Ax \leq 0 \right\}$ and the claim holds. $\square$

When $\mathcal{F}$ is defined by a single, quadratic constraint, $\mathscr{H}(\mathcal{F})$ can be explicitly defined by two constraints. If the Hessian A is not positive-semidefinite, then its formulation is presented in Proposition 26. Since Chapter 4 relied on the complicating set being defined a single constraint, this implies that Theorem 10 may have difficulties extending to the homogenized case. Similar to Chapter 4, the boundary of $\operatorname{conv}\left\{ yy^T \mid y \in \mathcal{G} \right\} \cap \operatorname{conv}\left\{ yy^T \mid y \in \mathscr{H}(\mathcal{F}) \right\}$ is defined in terms of $\operatorname{bd}(\mathscr{H}(\mathcal{F}))$ for some cone $\mathcal{G}$. Then, when considering a point X in the interior of the intersection, a rank one decomposition of X is used where the points are contained in $\operatorname{conv}\left\{ yy^T \mid y \in \mathcal{G} \right\}$. If all of the rank-1 points are contained in $\operatorname{conv}\left\{ yy^T \mid y \in \mathscr{H}(\mathcal{F}) \right\}$, then X is contained in the convex hull of the intersection. Otherwise, the rank-1 point will be written as a convex combination of X and some point in $\operatorname{bd}\operatorname{conv}\left\{ yy^T \mid y \in \mathscr{H}(\mathcal{F}) \right\}$.

Proposition 27. *Let $\mathcal{G} \subseteq \mathbb{R}^{n+1}$ be a cone and $\mathcal{F} = \left\{ x \in \mathbb{R}^n \mid x^T Ax + 2a^T x + \alpha \leq 0 \right\}$ be a closed set such that $\operatorname{bd}(\mathscr{H}(\mathcal{F})) \subseteq \mathcal{G}$. Then*

$$\operatorname{conv}\left\{ yy^T \mid y \in \mathcal{G} \cap \mathscr{H}(\mathcal{F}) \right\} = \operatorname{conv}\left\{ yy^T \mid y \in \mathcal{G} \right\} \cap \operatorname{conv}\left\{ yy^T \mid y \in \mathscr{H}(\mathcal{F}) \right\}.$$

Proof. For ease of notation, for any set S, let

$$\hat{\mathcal{C}}(S) = \left\{ yy^T \mid y \in S \right\}.$$

The forward containment is straightforward. As for the reverse containment, denote

$$\hat{A} = \begin{pmatrix} \alpha & a^T \\ a & A \end{pmatrix}.$$

It suffices to show that $\hat{\mathcal{C}}(\mathcal{G} \cap \mathscr{H}(\mathcal{F})) \supseteq \hat{\mathcal{C}}(\mathcal{G}) \cap \mathcal{S}(\mathscr{H}(\mathcal{F}))$, where $\mathcal{S}(\mathscr{H}(\mathcal{F}))$ is the Shor Relaxation of $\mathscr{H}(\mathcal{F})$ defined below [31]:

$$\mathcal{S}(\mathscr{H}(\mathcal{F})) = \left\{ Y \in \mathcal{S}^{n+1} \;\middle|\; Y \bullet \hat{A} \leq 0,\ Y \succeq 0 \right\}.$$

First, consider elements of $\hat{\mathcal{C}}(\mathcal{G})$ that exist along the boundary of $\mathcal{S}(\mathscr{H}(\mathcal{F}))$ pertaining to $\left\{ Y \;\middle|\; \hat{A} \bullet Y = 0, Y \succeq 0 \right\}$. Let $Y \in S_+^{n+1}$ with rank r such that $\hat{A} \bullet Y = 0$. Since $Y \succeq 0$, a rank-one decomposition yields

$$Y = \sum_{i=1}^{r} y_i y_i^T,\ y_i = \begin{pmatrix} t_i \\ x_i \end{pmatrix}, y_i^T \hat{A} y_i = 0,\ i = 1, \ldots, r.$$

If $t_i = 0$, then $x_i^T A x_i = 0$. If $A \not\succeq 0$, then $x_i \in \mathcal{F}_\infty$ and, by Definition 24, $y_i \in \mathrm{bd}(\mathscr{H}(\mathcal{F})) \subseteq \mathcal{G}$. Therefore, $y_i \in \mathcal{G} \cap \mathscr{H}(\mathcal{F})$. Otherwise, by Proposition 4, we have that $x_i \in \mathcal{F}_\infty$ or $-x_i \in \mathcal{F}_\infty$. Since

$$y_i y_i^T = -y_i(-y_i)^T, \tag{5.6}$$

then

$$y_i y_i = \begin{pmatrix} 0 \\ x_i \end{pmatrix} \begin{pmatrix} 0 \\ x_i \end{pmatrix} = \begin{pmatrix} 0 \\ -x_i \end{pmatrix} \begin{pmatrix} 0 \\ -x_i \end{pmatrix}.$$

Without a loss of generality, choose x_i or $-x_i$ such that $y_i \in \mathcal{G} \cap \mathrm{bd}(\mathscr{H}(\mathcal{F}))$.

If $t_i > 0$, then

$$\begin{pmatrix} t_i \\ x_i \end{pmatrix} = t_i \begin{pmatrix} 1 \\ \frac{x_i}{t_i} \end{pmatrix}, \quad \frac{1}{t_i^2}(x_i^T A x_i + 2 t_i a^T x_i + \alpha t_i^2) = 0,$$

and $y_i \in \mathrm{cone}(\{1\} \times \mathrm{bd}(\mathcal{F})) \subseteq \mathrm{bd}(\mathscr{H}(\mathcal{F})) \subseteq \mathcal{G}$.

If $t_i < 0$, then by using (5.6), choosing $y_i = -y_i$ and following the $t_i > 0$ case, $y_i \in \mathrm{cone}(\{1\} \times \mathrm{bd}(\mathcal{F})) \subseteq \mathrm{bd}(\mathscr{H}(\mathcal{F})) \subseteq \mathcal{G}$. Therefore, since Y is a conic combination of elements in $\mathcal{G} \cap \mathscr{H}(\mathcal{F})$, then $Y \in \hat{\mathcal{C}}(\mathcal{G} \cap \mathcal{F})$.

Now, consider $Y \in \hat{\mathcal{C}}(\mathcal{G})$ such that $\hat{A} \bullet Y < 0$. Since $Y \in \hat{\mathcal{C}}(\mathcal{G})$, then Y can be expressed by the following conic combination

$$Y = \sum_{i \in I} y_i y_i^T + \sum_{j \in J} y_j y_j^T, \tag{5.7}$$

where $\hat{A} \bullet y_i y_i^T \leq 0$ for all $i \in I$ and $\hat{A} \bullet y_j y_j^T > 0$ for all $j \in J$. For each $j \in J$, since the set

$\text{bd}(\mathscr{H}(\mathcal{F}))$ separates $y_j y_j^T$ and Y, then there exists $\hat{Y}_j \in \text{bd}(\mathscr{H}(\mathcal{F})) \subseteq \mathcal{G}$ such that

$$\hat{Y}_j = \gamma_j Y + (1-\gamma_j) y_j y_j^T, \; \gamma \in (0,1).$$

Solving this for $y_j y_j^T = (1-\gamma)^{-1}(\hat{Y}_j - \gamma Y)$ and substituting this into (5.7), we have that

$$Y = \left(1 + \sum_{j \in J} \frac{\gamma_j}{(1-\gamma_j)}\right)^{-1} \left[\sum_{i \in I} \lambda_i y_i y_i^T + \sum_{j \in J} \frac{\hat{Y}_j}{(1-\gamma_j)}\right].$$

Therefore, Y is a conic combination of elements in $\hat{\mathcal{C}}(\mathcal{G} \cap \mathscr{H}(\mathcal{F}))$.

□

In relation to the results presented in Chapter 4, Proposition 27 requires the stronger assumption of $\text{bd}(\mathscr{H}(\mathcal{F})) \subseteq \mathcal{G}$. In Chapter 4, only the boundary of $\mathcal{H}$ needs to be contained in $\mathcal{F}$ and as such, $\text{bd}(\mathcal{H})_\infty \subseteq (\mathcal{F})_\infty$. Under the assumption for Proposition 27, $(\mathcal{H})_\infty \subseteq (\mathcal{F})_\infty$.

5.3 Conclusion

This chapter explored the non-intersecting linear constraint in two ways. First, a direct approach was utilized when the original set, $\mathcal{F}$, is defined by a single quadratic constraint. This method explored the symmetry of the asymptotic cone over a halfspace. However, with more constraints defining $\mathcal{F}$, this analysis falls short without an explicit description of the recession cone $\text{Rec}(\overline{\mathcal{C}}(\mathcal{F}))$. In the second section the non-intersecting assumption is explored in terms of the homogenization $\mathscr{H}(\mathcal{F})$. This assumption led to a sufficient condition for when the closed, lifted convex hull of the intersection of a cone and a non-intersecting homogenization is equal to the intersection of closed, lifted convex hulls similar to Chapter 4. Despite providing a new result with a non-intersecting set defined by two constraints, the downside of the strictness of the non-intersecting assumption in the homogenized space makes finding applications more difficult.

Chapter 6

Existence of Optimal Solutions

While the previous chapters were concerned with reformulating problems in order to find an optimal solution, this chapter focuses on whether that optimal solution exists or not. According to the Weierstrass Theorem, optimizing a continuous function over a compact set will attain it's optimal value. Extending this to unbounded feasible regions, there are three possibilities: the objective function is unbounded over the set, the optimal solution can be attained, or the optimal value is finite but it is not obtainable. Consider the following program

$$\begin{aligned} \min \quad & f(x) \\ s.t. \quad & g_i(x) \leq 0, \quad i = 1, \ldots, m, \end{aligned} \tag{Opt}$$

where the set defined by $\{\, x \mid g_i(x) \leq 0,\ i = 1, \ldots, m \,\}$ is non-empty. If f and g_i are affine linear functions, then we are solving a linear program, which will always obtain its optimal solution if the optimal value is finite. Extending this to a quadratic program with a quadratically defined objective function over affine linear constraints, the Frank-Wolfe Theorem states that if $\min f(x)$ is bounded over the feasible region then the optimal solution is obtained [23]. The goal of this chapter is to explore and expand on the Frank-Wolfe theorem in terms of quadratically constrained quadratic programs (QCQP) :

$$\begin{aligned} \inf \quad & f(x) = x^T A_0 x + 2a_0^T x \\ s.t. \quad & g_i(x) = x^T A_i x + 2a_i^T x + \alpha_i \leq 0, i = 1, \ldots, m; \end{aligned} \tag{QCQP}$$

where $A_i \in \mathcal{S}^n$, $a_i \in \mathbb{R}^n$ for $i = 0, 1, \ldots, m$, and $\alpha_i \in \mathbb{R}$ for $i = 1, \ldots, m$. For ease in describing the asymptotic cones related to (QCQP), we define the lower level sets of $f(x)$ as

$$S_k := \{ x \mid f(x) \leq \gamma_k \},$$

and the feasible region as

$$\mathcal{F} := \{ x \mid g_i(x) \leq 0, i = 1, \ldots, m \}.$$

For example, consider the QP1QC, an instance of (QCQP) where $m = 1$. Hsia, Lin, and Sheu [19] provided results based on the matrix pencil, a tool used in the generalized eigenvalue problem, and the set below.

$$I_{\succeq}(A_0, A_1) = \{ \sigma \in \mathbb{R} \mid A_0 + \sigma A_1 \succeq 0 \}.$$

When investigating the existence and attainability of optimal solutions for QP1QC, their results are listed in Figure 6.1. Extending this to more than one constraint, Luo and Zhang [22] provided

QP1QC / $I_{\succeq}(A_0, A_1)$	unbounded	attainable	unattainable
$\emptyset$	✓	x	x
$\{ \sigma \}$	✓	✓	✓
$[\sigma_{min}, \sigma_{max}]$	✓	✓	x

Figure 6.1: Table of results from Hsia, Lin, and Sheu[19] using the matrix pencil to determine the attainability of an optimal solution for QP1QC.

several positive and negative results in relation to the initial problem, (Opt), that also apply to (QCQP).

1. If $f(x)$ is convex and at least one of the constraint functions $g_i(x)$ is nonlinear and nonconvex, then the optimal solution to (QCQP) is not attainable in general.

2. If $f(x)$ is non-convex and at least two or more functions $g_i(x)$ are nonlinear (but convex), then the optimal solution of (QCQP) is not attainable in general.

3. If $f(x)$ is non-convex and at most one of the constraint functions $g_i(x)$ is nonlinear (but convex), then the problem is unbounded or the optimal solution to (QCQP) is always attained.

4. If $f(x)$ is quasi-convex over the feasible region and all of the constraint functions $g_i(x)$ are convex, then the problem is unbounded or the optimal solution of (QCQP) is always attained.

It requires mention that item 4 does not extend to the feasible region being convex. Consider the problem

$$\begin{aligned} \inf \quad & x_1^2 \\ s.t. \quad & x_1 x_2 \geq 1 \\ & x_2 \geq 0. \end{aligned}$$

This problem is bounded below by 0, but the solution cannot be attained.

These results can be expanded upon even further. Tam and Nghi [32] provided existence results based on a stronger relationship between the objective function and the quadratic constraints. With Proposition 28 below, they extended the results for an arbitrary quadratic objective function $f(x)$ and a finite amount of quadratic constraints with a positive semidefinite Hessian. Consider the following notation and proposition:

$$I_1 = \{\, i \geq 1 \mid A_i \neq 0,\ A_i \succeq 0 \,\}.$$

Proposition 28 ([32]). *Consider problem* (QCQP)*. Assume that $\mathcal{F}$ is nonempty and $f_i(x)$ is convex for all $i = 1, \ldots, m$, $f(x)$ is bounded from below over $\mathcal{F}$, and one of the following conditions is satisfied:*

1. *The set I_1 contains at most one element;*

2. *If $v \in \mathrm{Rec}(\mathcal{F})$ such that $v^T A_0 v = 0$ then $a_i^T v = 0$ for all $i \in I_1$.*

Then (QCQP) *has a solution.*

When $|I_1| = 1$, Proposition 28 is the same result as item 3 from [22]. When $|I_1| > 1$, then there is a relationship between the recession directions of the quadratic constraints $g_i(x)$, $i \in I_1$ and the asymptotic directions of the lower level sets S_k. While the proof of their result uses a modified definition of retractiveness mentioned earlier in Chapter 3, it can be recovered from [5] with the following proposition:

Proposition 29 ([5]). *Let $f\colon \mathbb{R}^n \to (-\infty, \infty]$ be a closed proper function, and let $\mathcal{F}$ be a closed set such that $\mathcal{F} \cap dom(f) \neq \emptyset$. Assume that:*

1. *All the asymptotic directions of $\mathcal{F}$ are retractive, local horizon directions.*

2. *For every decreasing scalar sequence $\{\gamma_k\}$ such that the sets*

$$S_k = \mathcal{F} \cap \{ x \mid f(x) \leq \gamma_k \}, \quad k = 0, 1, \ldots,$$

   are nonempty, for every asymptotic direction d of $\{S_k\}$, and for each $x \in \mathcal{F}$, we either have $lim_{\alpha \to \infty} f(x + \alpha d) = -\infty$, or else $f(x - d) \leq f(x)$.

Then f attains a minimum over $\mathcal{F}$ if and only if the optimal value $\inf_{x \in \mathcal{F}} f(x)$ is finite.

With the analysis of asymptotic directions in Chapter 3, we know that for a quadratically defined set, assumption 1 of Proposition 29 is not true in general. However, this assumption can be relaxed to only requiring the asymptotic directions of $\mathcal{F} \cap S_k$ to be retractive, local horizon directions with respect to $\mathcal{F}$. Also, assumption 2 is always true when f is defined as a quadratic function [5].

In this chapter, we are interested in (QCQP) when one of the quadratic constraints is defined by an indefinite Hessian. In Section 6.1, the assumptions of Proposition 28 are analyzed and expanded upon. In Section 6.2, we explore assumption 1 of Proposition 29. In particular, how this assumption relates to the intersection of halfspaces and one quadratic constraint defined by an indefinite Hessian. A specific instance of (QCQP) is examined under the lens of the "center" of a quadratically defined set. A conjecture of this property will provide a means of checking assumption 1 of Proposition 29.

6.1 Convex Constraints and One Indefinite Constraint

With Proposition 28, Tam and Nghi find a relation between an arbitrary quadratic objective function and a convex set defined by halfspaces and convex quadratic constraints. The interesting part behind this proposition is assumption 2. That is, if $v \in \text{Rec}(\mathcal{F})$ such that $v^T A_0 v = 0$ then $a_i^T v = 0$ for all $i \in I_1$. In terms of asymptotic directions, if $v \in \text{Rec}(\mathcal{F})$ then v is an asymptotic direction of $\mathcal{F}$. Also, if $v^T A_0 v = 0$, then v is possibly an asymptotic direction of the lower level sets of the objective function (requires $a_0^T d \leq 0$ if $A_0 \succeq 0$). Combining these two concepts with the idea of retractive, local directions, the assumption states that if on has an asymptotic direction of the feasible region that is an asymptotic direction of the boundary of the lower level sets of the objective function, then it should be a retractive, local horizon direction of $\mathcal{F}$. This can be seen in the following proof of Proposition 28 using Proposition 29.

Recovering Proposition 28 with Proposition 29. To show that Proposition 28 is a direct consequence of Proposition 29, it suffices to prove that all the asymptotic directions of $\mathcal{F}$ that are also asymptotic directions of some sub-level set of f are retractive local horizon directions of $\mathcal{F}$. Since $\mathcal{F}$ is convex, all asymptotic directions of $\mathcal{F}$ are local horizon directions. Therefore, we only show the retractiveness of such a direction.

For any sub-level set $S_k := \{\, x \mid f(x) \leq k \,\}$ of f,

$$(S_k)_\infty = \{\, d \mid d^T Q d \leq 0 \,\}.$$

Let $v \in \mathcal{F}_\infty \cap (S_k)_\infty = \bigcap_{i \in I} (F_i)_\infty \cap (S_k)_\infty$, and let $\{x_k\} \subseteq \mathcal{F}$ be a sequence such that

$$\|x_k\| \to \infty \qquad \text{and} \qquad \lim_{k \to \infty} \frac{x_k}{\|x_k\|} = \frac{v}{\|v\|}.$$

For each $i \in I \setminus I_1$, by item 1 of Proposition 9, F_i is retractive because it is a closed half space. Therefore, v is a retractive direction of F_i. That is, there exists $k_i \geq 0$ such that

$$g_i(x_k - v) \leq 0 \qquad \forall\, k \geq k_i.$$

Now consider $i \in I_1$. If $v^T Q v < 0$, for any $x \in \mathcal{F}$, $x + \alpha d \in \mathcal{F}$ for all $\alpha \geq 0$ and

$$\lim_{\alpha \to \infty} f(x + \alpha v) = -\infty,$$

which contradicts to the assumption that f is bounded from below over $\mathcal{F}$. If $v^T Q v = 0$, then the assumption in Proposition 28 indicates that $q_i^T v = 0$ for all $i \in I_1$. In this case, for any $x \in \mathcal{F}$ and $i \in I_1$,

$$g_i(x - v) = \frac{1}{2} x^T Q_i x - x^T Q_i v + \frac{1}{2} v^T Q_i v + q_i^T x - q_i^T v + c_i = g_i(x) \leq 0.$$

In particular, $g_i(x_k - v) \leq 0$ for all k. Overall, we have shown that v is a retractive direction of $\mathcal{F}$. $\square$

Geometrically, if v is a local horizon direction of $\mathcal{F}$ and an asymptotic direction in relation to the objective function, then it must be a retractive direction in relation to all nonlinear constraints.

In theory, this proposition can be extended to cover feasible regions defined by two quadratic constraints g_1 and g_2, where g_1 is defined by an indefinite Hessian and g_2 is defined by a positive semidefinite Hessian. The first question is how do we define the asymptotic directions of their intersection? Consider the following set of lemmas and propositions.

Lemma 18 ([5]). *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be closed sets such that $\mathcal{F}_1 \cap \mathcal{F}_2 \neq \emptyset$. Then a vector which is a horizon direction of both $\mathcal{F}_1$ and $\mathcal{F}_2$ with respect to a common set $\mathcal{G}$ is also a horizon direction of $\mathcal{F}_1 \cap \mathcal{F}_2$ with respect to $\mathcal{G}$.*

Proposition 30. *Let $\mathcal{F} = \{ x \mid f_i(x) \leq 0,\ i = 1, 2 \} \neq \emptyset$ where A_1 indefinite and $A_2 \succeq 0$. If $d \in \mathcal{F}_\infty$ such that $d^T A_1 d < 0$ then d is a local horizon direction of $\mathcal{F}$.*

Proof. We have shown in Proposition 7 that for any $d \in (\mathcal{F}_1)_\infty$ such that $d^T A_1 d < 0$, d is a global horizon direction of $\mathcal{F}_1$. Therefore, for any $d \in (\mathcal{F}_1)_\infty$ such that $d^T A_1 d < 0$, d is a horizon direction of $\mathcal{F}_1$ with respect to $\mathcal{F}$. Now, since $\mathcal{F}_2$ is convex, any $d \in (\mathcal{F}_2)_\infty$ is a local horizon direction of $\mathcal{F}_2$, and, thus, a horizon direction of $\mathcal{F}_2$ with respect to $\mathcal{F}$. Combining the observation above with, for any $d \in \mathcal{F}_\infty \subseteq (\mathcal{F}_1)_\infty \cap (\mathcal{F}_2)_\infty$, and Lemma 18 implies that d is a local horizon direction of $\mathcal{F}$. $\square$

Proposition 30 builds the relation for when the regions $\mathcal{F}_1$ and $\mathcal{F}_2$ share local horizon directions with respect to their intersection, then those directions are local horizon directions of said intersection. This concept can be extended to having multiple indefinite constraints as long as $d^T A_j d < 0$ for all indefinite Hessians A_j. Likewise, multiple positive semidefinite constraints can be added with only the precaution that $\mathcal{F}$ can become bounded. However, being a local horizon direction is only one requirement of Propositions 28 and 29 and the assumption of retractiveness must be checked.

Proposition 31. *Let $\mathcal{F} = \bigcap_i \mathcal{F}_i$ and $d \in \mathcal{F}_\infty$ be a retractive asymptotic direction of $\mathcal{F}_i$, $i = 1, \ldots, m$. Then d is retractive direction of $\mathcal{F}$.*

Proof. Let $d \in \mathcal{F} = (\bigcap_i \mathcal{F}_i)_\infty$ such that d is a retractive direction of $\mathcal{F}_i$ for all $i = 1, \ldots, m$. Since d is a retractive direction of $\mathcal{F}_1$, then for every corresponding asymptotic sequence $\{ x_k \} \subseteq \mathcal{F}_1$ with respect to d, $f_1(x_k - d) \leq 0$ for all $k \geq \bar{k}$. Since $d \in \mathcal{F}_\infty$, then every corresponding asymptotic sequence $\{\, \bar{x}_k \,\} \subseteq \mathcal{F} \subseteq \mathcal{F}_1$ is a subsequence of some $\{\, x_k \,\} \subseteq \mathcal{F}_1$. Therefore, for all $k \geq \bar{k}_1$, $f_1(x_k - d) \leq 0$.

Continuing this process for all $\mathcal{F}_i$, $i = 1, \ldots, m$. Then for all $k \geq \max\{\bar{k}_1, \ldots, \bar{k}_m\}$, $f_i(x_k - d) \leq 0$ for all $i = 1, \ldots, m$. Therefore, d is a retractive asymptotic sequence of $\mathcal{F}$.

□

With the previous proposition, a direction of the intersection of sets is a retractive direction with respect to each of the components then the direction is retractive with respect to the intersection. If we are able to find properties that satisfy both Propositions 30 and 31 with respect to the intersection, then we satisfy the assumptions of Proposition 29. From Chapter 3, we know that for a quadratic set defined by an indefinite Hessian, $d \in \mathrm{bd}(\mathcal{F}_\infty)$ is not retractive in general. Likewise, if d is an asymptotic direction of a quadratic set defined by a positive semi-definite Hessian A_i, then $a_i^T d = 0$ for d to be retractive. This leads to the following result.

Theorem 11. *Let $\mathcal{F} = \{\, x \mid f_i(x) \leq 0,\ i = 1, 2 \,\}$ where A_1 indefinite and $A_2 \succeq 0$. Assume that $g(x)$ is bounded from below over $\mathcal{F}$ and the following conditions are satisfied:*

1. *For all $d \in \mathcal{F}_\infty$ such that $d^T Q d = 0$, if $A_2 \neq 0$, then $a_2^T d = 0$. That is, if A_2 is nonzero then $\left\{\, d \;\middle|\; A_2 d = 0, a_2^T d < 0, d^T A_1 d \leq 0, d^T Q d = 0 \,\right\}$;*
2. *$(\mathcal{F}_2)_\infty \subseteq \mathrm{int}(\mathcal{F}_1)_\infty$, or $\left\{\, d \;\middle|\; A_2 d = 0, a_2^T d \leq 0, d^T A_1 d = 0 \,\right\} = \{0\}$.*

Then min $g(x)$ *attains an optimal solution over $\mathcal{F}$.*

Proof. Denote the lower level sets of $g(x)$ as $S_k = \{\, x \mid g(x) \leq k \,\}$ with asymptotic cone $(S_k)_\infty \subseteq \{\, d \mid d^T Q d \leq 0 \,\}$.

By Proposition 29, it suffices to show that all asymptotic directions of $\mathcal{F} \cap S_k$ are retractive, local horizon directions with respect to $\mathcal{F}$. However, this can be further restricted. Let $d \in \mathcal{F}_\infty \neq \{0\}$, then by condition 2, $d^T A_1 d < 0$. By Proposition 30, all directions of $\mathcal{F}$ are local horizon directions with respect to $\mathcal{F}$. Hence, for all $x \in \mathcal{F}$, there exists $\Lambda > 0$ such that

$$x + \lambda d \in \mathcal{F}, \forall \lambda \geq \Lambda.$$

If $d^T Qd < 0$, then

$$g(x + \lambda d) = g(x) + \lambda^2 d^T Qd + 2\lambda q^T d \to -\infty \quad \text{as } \lambda \to \infty,$$

and $g(x)$ is unbounded over $\mathcal{F}$. Therefore, it suffices to consider the asymptotic directions when $d^T Qd = 0$, that is, the directions in $\mathcal{F}_\infty \cap \text{bd}((S_k)_\infty)$.

Since $d^T A_1 d < 0$, d is a retractive direction with respect to $\mathcal{F}_1$. If $A_2 = 0$, then $\mathcal{F}_2$ is a halfspace and is retractive. Then by Proposition 31, d is a retractive direction with respect to $\mathcal{F}$.

Let $0 \neq A_2 \succeq 0$. Then by condition 1, $a_2^T d = 0$, and d is a retractive direction of $\mathcal{F}_2$ because f_2 stays constant along the direction of d. Therefore, by Proposition 31, d is a retractive direction with respect to $\mathcal{F}$.

Therefore, any asymptotic direction of $\mathcal{F} \cap \text{bd}(S_k)$ is a retractive, local horizon direction of $\mathcal{F}$ and by Proposition 29, $g(x)$ attains an optimal solution over $\mathcal{F}$. □

Example 7 explores the conditions of Theorem 11 while Figure 6.2 provides a graphical representation of Example 7. In (a), the region of $\mathcal{F}_2$ (the band) is defined by a positive semidefinite Hessian with $a_2 \in \text{Range}(A_2)$. In this $(\mathcal{F}_2)_\infty \subseteq \text{int}(\mathcal{F}_1)_\infty$. Both (b) and (c) highlight another generalization with issues that need to be further addressed. In both cases, the asymptotic directions of $\mathcal{F}_2$ are in the boundary of $(\mathcal{F}_1)_\infty$ and may not be retractive over $\mathcal{F}$. If $\mathcal{F}_2 \cap \text{bd}(\mathcal{F}_1)$ is bounded as in (b), then the directions of $\mathcal{F}$ are retractive. However, if $\mathcal{F}_2 \cap \text{bd}(\mathcal{F}_1)$ is unbounded as in (c), then the boundary asymptotic directions are not retractive.

Example 7. *Consider the following three formulations of $\mathcal{F} = \mathcal{F}_1 \cap \mathcal{F}_2$ graphically represented in Figure 6.2:*

1. *Figure 6.2(a).*

$$\mathcal{F}_1 = \{ x \mid x_1 x_2 + 1 \leq 0 \}, \quad \mathcal{F}_2 = \left\{ x \mid x_1^2 + x_2^2 - 2x_1 x_2 - 1 \leq 0 \right\}.$$

The asymptotic cone is $\mathcal{F}_\infty = \text{cone}\left\{ (1,1)^T,\ (-1,-1)^T \right\}$. All nonzero directions are retractive over $\mathcal{F}_1$, $\mathcal{F}_2$, and $\mathcal{F}$.

2. *Figure 6.2(b).*

$$\mathcal{F}_1 = \{\, x \mid x_1x_2 + 1 \le 0 \,\}, \quad \mathcal{F}_2 = \left\{\, x \;\middle|\; x_2^2 - 4x_2 - 3 \le 0 \,\right\}.$$

The asymptotic cone is $\mathcal{F}_\infty = \operatorname{cone}\left\{\, (1,0)^T \,\right\}$*. The nonzero direction* $d = (1,0)^T$ *is not retractive over* $\mathcal{F}_1 = \{\, x \mid -x_1x_2 + 1 \le 0 \,\}$ *but is retractive over* $\mathcal{F}$*.*

3. *Figure 6.2(c).*

$$\mathcal{F}_1 = \{\, x \mid x_1x_2 + 1 \le 0 \,\}, \quad \mathcal{F}_2 = \left\{\, x \;\middle|\; x_2^2 - 1 \le 0 \,\right\}.$$

The asymptotic cone is $\mathcal{F}_\infty = \operatorname{cone}\left\{\, (-1,0)^T,\ (1,0)^T \,\right\}$*. The nonzero direction* $d = (1,0)^T$ *has a corresponding asymptotic sequence* $\{\, x_k \,\} \subseteq \operatorname{bd}(\mathcal{F}_1)$ *that is also contained in* $\mathcal{F}$*. Therefore, by Proposition 10,* d *is not retractive.*

These examples can be seen in Figure 6.2.

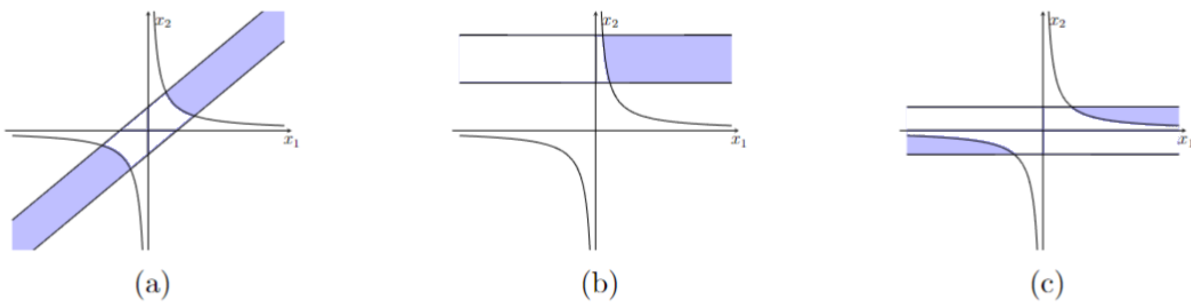


Figure 6.2: 2D graphs of a feasible region (in purple) defined by two quadratic regions $\mathcal{F}_1$ and $\mathcal{F}_2$. $\mathcal{F}_1$ is defined by an indefinite Hessian and $\mathcal{F}_2$ is defined by positive semidefinite Hessian such that $a_2 \in \operatorname{Range}(A_2)$. (a) is an example of Theorem 11 where $(\mathcal{F}_2)_\infty \subseteq \operatorname{int}(\mathcal{F}_1)_\infty$.

For a better understanding of Figure 6.2, consider the following definition:

Definition 4 (Center of Quadratic Set Defined with Indefinite Hessian)**.** *Let* $\mathcal{F} \subseteq \mathbb{R}^n$ *be a closed nonempty set where* $\mathcal{F} = \{\, x \mid (x-c)^T A(x-c) + \alpha \le 0 \,\}$ *with* A *invertible, indefinite. Then* c *is the center of* $\mathcal{F}$*.*

To see why this definition is important, consider the set $\mathcal{F} = \left\{\, x \;\middle|\; x^TAx + \alpha \le 0 \,\right\}$ where A is indefinite. Then the boundary of $\mathcal{F}$ converges to $\mathcal{F}_\infty$ as it tends further away from the origin. Transforming this set to $\mathcal{F}_c = \left\{\, x \;\middle|\; (x-c)^TA(x-c) + \alpha \le 0 \,\right\}$ is nothing more than a translation in the c direction. Note that $\mathcal{F}_c$ has the same asymptotic cone as $\mathcal{F}$, but the boundary of $\mathcal{F}_c$ no longer converges to $\mathcal{F}_\infty$ as it tends away from the origin. Instead, it converges to $\mathcal{F}_\infty$ that is also shifted in the same manner as $\mathcal{F}_c$. Consider 6.2 with figures (b) and (c). In (b), the asymptotic directions, are retractive over $\mathcal{F}$. However, there exists $d \in \mathcal{F}$ that is not retractive over the set $\mathcal{F}_1$ defined by the indefinite Hessian. This is due to the center of $\mathcal{F}_1$ not being contained in $\mathcal{F}_2$ and the nonretractive behavior is lost in the intersection. This is not the case in (c). In (c), the center $c \in \mathcal{F}_2$ and as such, the nonretractive behavior of $\mathcal{F}_1$ is in $\mathcal{F}$.

6.2 Importance of the Center

The goal of this section is to understand the "center" of a quadratically defined set and how it can provide more sufficient conditions for the existence of an optimal solution. Consider a set in $\mathbb{R}^2$ defined as a hyperbola. Then the center of this set is the midpoint between the two foci of that set. This same concept expands to higher dimensions with one consideration; the center of a set may be a hyperplane. For example, consider a quadratic set $\mathcal{F} \in \mathbb{R}^2$ where the Hessian is positive semidefinite ($A \succeq 0$) and $a \in \text{Range}(A)$ (See Figure 3.3(b)). In this set, the center could be interpreted as the hyperplane parallel to the boundary as well as equidistant from both bands of the bounds. In general, the center of a quadratic set does not have an obvious definition. This document focuses on the center of quadratic sets $\mathcal{F}$ with an indefinite, invertible Hessian that has a definite center c of the form

$$\mathcal{F} = \left\{ x \mid (x-c)^T A(x-c) + \alpha \leq 0 \right\}.$$

The rest of this section proceeds as follows: first we provide a low dimensional example showing how the center of a quadratic curve can help identify the asymptotic directions related to the objective function and feasible region. Next, with the concept of the center, a conjecture will be presented that expands Theorem 11 to cover the cases in Figure 6.2.

Given an objective function and its lower level sets S_k, the goal is to use the center to define the asymptotic directions of $S_k \cap \mathcal{F}$. Consider (QCQP) in the case where A_1 is indefinite and $A_i = 0$

for all $i > 1$. For example, consider the following 2D problem:

$$\begin{aligned} \min \quad & f(x) = ax_1^2 + 2bx_1x_2 + cx_2^2 + dx_1 + ex_2 \\ s.t. \quad & g_1(x) = -x_1x_2 + 1 \le 0 \\ & g_2(x) = -x_1 \le 0, \end{aligned} \tag{P}$$

where $a, b, c, d, e \in \mathbb{R}$. In the case of (P), the feasible region $\mathcal{F}$ is a convex feasible region. While there are many cases where (P) will be unbounded, there is a subset of values for a, b, c, d, e such that $\nabla^2 f$ is indefinite and (P) attains an optimal solution over $\mathcal{F}$. In particular,

$$\begin{aligned} \min \quad & f(x) = x_1^2 + x_1x_2 + \frac{79}{9}x_1 + \frac{1}{9}x_2 \\ s.t. \quad & g_1(x) = -x_1x_2 + 1 \le 0 \\ & g_2(x) = -x_1 \le 0, \end{aligned} \tag{Ind}$$

has an attainable optimal solution of $\left(\frac{1}{9}, 9\right)$. Applying Proposition 29 to this, we see that $(S_k)_\infty \cap \mathcal{F}_\infty = \text{cone}\{(0,0)^T,\ (0,1)^T\}$ which is not retractive of $\mathcal{F}$. However, $(S_k \cap \mathcal{F})_\infty = \{(0,0)^T\}$ which is trivially set and vacuously retractive over $\mathcal{F}$. To answer how this is the case, recall Figure 3.1. The feasible region $\mathcal{F}$ is defined by an indefinite Hessian and as such, the boundary of the region approaches the boundary of the asymptotic cone $\mathcal{F}_\infty$. This is because the center of $\mathcal{F}$ is the same as the center of $\mathcal{F}_\infty$. Now consider the case when the center of $\mathcal{F}$ is located at c. Then the boundary of $\mathcal{F}$ converge to the boundary of the shifted asymptotic cone $\mathcal{F}_{\infty,c}$, denoted as

$$\mathcal{F}_{\infty,c} = \left\{ x \;\middle|\; (x-c)^T A(x-c) \le 0 \right\}.$$

In relation to (Ind), the center of $\mathcal{F}_1 = \{ x \mid g_1(x) \le 0 \}$ is located at the origin and the boundary of $\mathcal{F}$ will converge to positive axes of $\mathbb{R}^2$. However, for the lower level sets $S_k = \{ f(x) \le k \}$, the center is located at $c = \frac{1}{9}(-1, -81)$ and the boundary of S_k converges to $(S_k)_{\infty,c}$. With this in mind, one can see that $S_k \cap \mathcal{F}$ is a bounded set with the trivial element $\{0, 0\}$.

The result in Theorem 11 provided a check to make sure that $d \in (S_k \cap \mathcal{F})_\infty$ is retractive by removing the nonretractive directions that are related to the boundary of a quadratic set defined by an indefinite Hessian. That is, if $\in (S_k \cap \mathcal{F})_\infty$ then $d^T A_i d < 0$ for all A_i indefinite. This case can be seen in Figure 6.2(a). Using the center of the indefinite constraint, however, we can begin to allow directions d such that $d^T A_i d = 0$ for A_i indefinite that are retractive over $\mathcal{F}$. Consider Figure 6.2(b). The center of the indefinite constrait, say c, is not located in $\mathcal{F}_2$ defined by the positive semidefinite constraint. In this case, the boundary of $\mathcal{F}$ does not approach the boundary of the shifted asymptotic cone $(\mathcal{F}_1)_{\infty,c}$ and $\mathcal{F}$ is retractive. This is not the case of (c). In this case, $c \in \mathcal{F}_2$ and as a result, there exists a sequence contained in the boundary of $\mathcal{F}_1$ that is also contained in $\mathcal{F}$ and as a result, $\mathcal{F}$ is not retractive.

Consider the following conjecture:

Conjecture 1. *Let* $\mathcal{F} = \{\, x \mid (x-c)^T A_1 (x-c) + \alpha_1 \leq 0,\ x^T A_2 x + 2a_2^T x + \alpha_2 \leq 0 \,\}$ *be nonempty where* A_1 *indefinite and* $A_2 \succeq 0$ *with* $\operatorname{rank}(A_2) = n-1$*. Assume that* $g(x)$ *is bounded from below over* $\mathcal{F}$ *and the following conditions are satisfied:*

1. *For all* $d \in \mathcal{F}_\infty$ *such that* $d^T Q d = 0$*, if* $A_2 \neq 0$*, then* $a_2^T d = 0$*;*
2. *For all* $d \in \mathcal{F}_\infty$ *such that* $d^T Q d = 0$*, either*
   - *(a)* $d^T A_1 d < 0$*, or*
   - *(b) if* $d^T A_1 d = 0$*, then* $c \notin \mathcal{F}$*.*

Then min $g(x)$ *attains an optimal solution over* $\mathcal{F}$*.*

This conjecture captures the usefulness of the center of a set defined by a quadratic constraint. The restriction of $\operatorname{rank}(A_2) = n-1$ is to enforce that there is not a free variable in $\mathcal{F}_2 = \left\{\, x \;\middle|\; x^T A_2 x + 2a_2^T x + \alpha_2 \leq 0 \,\right\}$ that allows $\mathcal{F}$ to have a nonretractive boundary. This requires showing that the assumptions restrict the asymptotic directions of $S_k \cap \mathcal{F}$ to being retractive, local horizon directions over $\mathcal{F}$. For simplicity and without loss of generality, we can consider $\mathcal{F}_2$ to be centered at the origin (i.e no a_2 term). For local horizon directions in assumption 2(b), it must be shown that if $d \in \mathcal{F}_\infty$ such that $d^T A_1 d = 0$, then $-d \notin \mathcal{F}_\infty$. This may only require comparing the asymptotic cone $(\mathcal{F}_2)_\infty$ to the shifted asymptotic cone $(\mathcal{F}_1)_{\infty,c}$. Under this consideration, there could be the argument that $\mathcal{F}_\infty = (\mathcal{F}_1)_{\infty,c} \cap (\mathcal{F}_2)_\infty$. Finally, showing the retractiveness of the asymptotic directions only requires demonstrating that, for any sequence $\{\, x_k \,\} \subset \operatorname{bd}(\mathcal{F}_1) \cap \mathcal{F}_2$, the norm $\|x_k\|$ is bounded as $k \to \infty$. This proves that there are no nonretractive directions in $S_k \cap \mathcal{F}$ for all k.

Chapter 7

Conclusion

Quadratically constrained quadratic programs are inherently difficult due to many factors. Two of these factors that can be addressed are the nonconvexity and the unboundedness of the feasible region. Both of these factors lend a necessity of the asymptotic cone. The asymptotic cone provides a generalization of the recession cone for a nonconvex feasible region as well as describes the behavior as the set tends away from the origin. Chapter 3 provides a description of the asymptotic cone for a set defined by a single quadratic at inequality [15] and equality. This description is the foundation for not only the document as a whole, but also for future research into the asymptotic cone of the intersection of multiple quadratic constraints.

The results of Chapter 4 yield conditions for when the lifted convex hull of the intersection equals the intersection of the lifted convex hull. The proofs behind these conditions show the direct connection between the asymptotic cone of the original set and the recession cone of the lifted convex hull. Not only does this expands and/or recovers results in Chapter 2, it also states that if one can decompose a complicated problem into the non-intersecting intersection of already known results, the lifted convex hull is no more difficult than that of the decomposition.

However, the proofs of Chapter 4 fall short if a complicating constraint is linear. Chapter 5 explored two paths, one is a direct approach using the symmetry of the asymptotic cone while the other used homogenizations discussed in [31]. Under the direct approach, a modified S-Lemma is used to provide necessary conditions for when a halfspace induces a non-intersecting constraint in the feasible region. When the non-intersecting property is satisfied, the symmetry of the asymptotic cone is used to give similar results to Chapter 4. To expand this further, finalizing the relationship convex cone generated by the lifted asymptotic cone and the recession cone of the lifted convex hull would cover the case when the original set is defined by multiple quadratic constraints. With the homogenization approach, the non-intersecting assumption is too strong in the homogenized space and provided weaker results compared to the direct method.

Chapter 6 explored results for the existence of an optimal solution in regards to the asymptotic cone. The extension of the Frank-Wolfe theorem in [5] laid the groundwork for conditions to check for QCQPs. This chapter extended the results in [32] to cover a feasible region with one convex quadratic constraint and at most one quadratic constraint defined by an indefinite Hessian. Also, the chapter guides future research into how the "center" of a constraint may be an important piece to consider. Moving the center of a constraint shifts the end behavior of said constraint. This shift influences how two constraints interact with each other and as a result, influences the asymptotic cone of the intersection.

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