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Algebraic and Integral Closure of a Polynomial Ring in its Power Series Ring

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by
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Abstract

Let R be a domain. We look at the algebraic and integral closure of a polynomial ring, $R[x]$, in its power series ring, $R[[x]]$. A power series $\alpha(x) \in R[[x]]$ is said to be an *algebraic power series* if there exists $F(x, y) \in R[x][y]$ such that $F(x, \alpha(x)) = 0$, where $F(x, y) \neq 0$. If $F(x, y)$ is monic, then $\alpha(x)$ is said to be an *integral power series.* We characterize the units of algebraic and integral power series. We show that the only algebraic power series with infinite radii of convergence are polynomials. We also show which algebraic numbers appear as radii of convergence for algebraic power series.

Additionally, we provide a new characterization of algebraic power series by showing that a convergent power series, $\alpha(x)$, is algebraic over L if and only if $\alpha(a)$ is algebraic over $L(a)$ for every a in the domain of convergence of $\alpha(x)$, where L is a countable subfield of $\mathbb{C}$.

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Chapter 1

Introduction

1.1 Historical Background

Algebraic power series have been studied from different perspectives. The first is through the lens of functional analysis. Here, the singularities, cycles, double points, and analytic continuation are analyzed. For more details, we refer the reader to [1] and [13]. Newton's Diagram is also given as a tool to compute coefficients of algebraic power series. We look into Newton's Diagram more in Chapter 3.

Algebraic power series have also been looked at from a combinatorical viewpoint for generating functions. For example, the enumeration of plane binary trees with n internal nodes involves the sequence: $1, 1, 2, 5, 14 \ldots$, the well known Catalan numbers [17]. The generating function of the Catalan numbers is

$$f(x) = 1 + x + 2x^2 + 5x^3 + 14x^4 + \cdots = \sum_{n=0}^{\infty} \frac{1}{n+1}\binom{2n}{n} x^n.$$

This power series satisfies the relation

$$F(x, y) = xy^2 - y + 1 = 0.$$

Additionally, the well-known Fibonacci generating function satisfies

$$F(x,y) = (1 - x - x^2)y - x.$$

It is known that the radius of convergence of an algebraic power series over $\mathbb{Q}$ is the root of a monic polynomials with rational coefficients [1], [13] but it is still unknown which algebraic numbers appear as a radius of convergence. Banderier and Drmota considered this question [2] and conjectured that all algebraic numbers appear as a radii of convergence when the coefficients are taken over $\mathbb{Q}$. We investigate this further, show which radii of convergence are known so far, highlight some new results and insights on the structure of algebraic power series, and give a new characterization of algebraic power series over a countable field of characteristic 0.

1.2 Mathematical Background

In this section, we review some necessary background. Recall the definition of a polynomial and a power series.

Definition 1.2.1. We say that $f(x)$ is a *polynomial* if it has the form

$$f(x) = \sum_{i=0}^{n} a_i x^i = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0,$$

where each coefficient a_i is in a ring R. We write $f(x) \in R[x]$.

For example, $4x^3 + \frac{7}{2}x^2 + x + 1 \in \mathbb{Q}[x]$, while $\sqrt{x+1} \notin \mathbb{Q}[x]$. Given a polynomial $f(x) \in R[x]$, a *root* of $f(x)$ is a number r such that $f(r) = 0$. The highest power of x with a nonzero coefficient of f is the *degree* of $f(x)$ which is written $\deg(f(x))$. For example,

$$\deg(f(x)) = \deg(4x^3 + \tfrac{7}{2}x^2 + x + 1) = 3.$$

By the fundamental theorem of algebra, if $f(x) \in \mathbb{Q}[x]$ has degree $n \geq 1$ then it has n (not necessarily distinct) roots in $\mathbb{C}$. We will see that the rudimentary concept that the number of roots is equal to the degree of the polynomial continues to hold when we introduce another variable and our roots become fractional power series. In order to discuss power series as roots of polynomials, we need an additional variable.

Definition 1.2.2. A polynomial in two variables has the form

$$f(x,y) = \sum_{\substack{\alpha=0,\ldots,m \\ \beta=0,\ldots,n}} A_{\alpha,\beta} x^{\alpha} y^{\beta}$$

where $m, n \in \mathbb{N}_0$ and $A_{\alpha,\beta} \in R$. A polynomial in two variables may also be referred to as a *bivariate polynomial*, and we write $R[x,y]$ for the ring of all such polynomials.

Example 1.2.3. The following are examples of bivariate polynomials:

1. $f(x,y) = y^3 - 6xy + x^3$

2. $g(x,y) = (x^2 + x + 1)y^5 - (5x - 1)y^4 + 3y^3 + (x^4 - 1)y^2 + (2x^2 - 3x + 1)y + x.$

When working with bivariate polynomials, we will typically want them to have a similar form of $f(x,y)$ and $g(x,y)$ in the previous example. That is, we will view a bivariate polynomial as a polynomial in y with coefficients in $R[x]$. In general they will have the form,

$$F(x,y) = f_n(x)y^n + f_{n-1}y^{n-1} + \cdots + f_1(x)y + f_0(x).$$

When we want to view a bivariate polynomial in this way we denote it by writing $F(x,y) \in R[x][y]$. The rings $R[x][y]$ and $R[x,y]$ contain the same elements, but we use the notation $R[x][y]$ to signal that the variable is y while the coefficients come from $R[x]$. This idea generalizes to higher dimensions. For example, if we wanted a third variable z and wanted the coefficients to be bivariate polynomials, we would notate the ring by $R[x,y][z]$.

When we want to talk about the degree of a bivariate polynomial, we specify the variable by adding a subscript. Given $f(x,y) \in R[x,y]$, the highest power on the variable y is denoted $\deg_y(f(x,y))$ and the highest power on the variable x is denoted $\deg_x(f(x,y))$. For example, consider $f(x,y)$ and

$g(x, y)$ in Example 1.2.3. We have that

$$\begin{aligned}
\deg_y(f(x,y)) &= \deg_y(y^3 - 6xy + x^3) = 3 \\
\deg_x(f(x,y)) &= \deg_y(y^3 - 6xy + x^3) = 3 \\
\deg_y(g(x,y)) &= \deg_y((x^2+x+1)y^5 - (5x-1)y^4 + 3y^3 + (x^4-1)y^2 + (2x^2-3x+1)y + x) = 5 \\
\deg_x(g(x,y)) &= \deg_y((x^2+x+1)y^5 - (5x-1)y^4 + 3y^3 + (x^4-1)y^2 + (2x^2-3x+1)y + x) = 4
\end{aligned}$$

One key characteristic of a polynomial, in any number of variables, is that it only has a finite number of terms. If we want to allow an infinite number of terms, then we have a power series.

Definition 1.2.4. We say that $\alpha(x)$ is a *power series* if it has the form

$$\alpha(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots = \sum_{i=0}^{\infty} a_i x^i,$$

where each a_i is from a ring R. We write $\alpha(x) \in R[[x]]$.

Definition 1.2.5. Let $\alpha(x) = \sum_{n=0}^{\infty} a_n x^n \in R[[x]]$. When working over a subring of $\mathbb{C}$, the *radius of convergence* of $\alpha(x)$ is

$$\text{r.o.c}(\alpha(x)) := \frac{1}{\limsup\limits_{n\to\infty} \sqrt[n]{|a_n|}},$$

where $\text{r.o.c}(\alpha(x)) = \infty$ if $\limsup\limits_{n\to\infty} \sqrt[n]{|a_n|} = 0$.

Example 1.2.6. The following are examples of power series:

1. $\alpha(x) = 1 + x + x^2 + x^3 + \cdots = \sum\limits_{i=0}^{\infty} x^i = \frac{1}{1-x}$ for $|x| < 1$
2. $\beta(x) = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{5x^4}{128} + \cdots = \sqrt{1+x}$ for $|x| < 1$.

We will later see that $\alpha(x)$ is an example of an algebraic power series, while $\beta(x)$ is an an integral power series. We can also discuss power series in more than one variable. If we want to allow two variables, then it takes the form

$$\alpha(x, y) = \sum_{a_1, a_2 = 0}^{\infty} A_{a_1,a_2} x^{a_1} y^{a_2}$$

where $A_{a_1,a_2} \in R$. This concept can easily be generalized uniquely to any finite number of variables, $x_1, \ldots, x_n$.

Remark 1.2.7. Though our discussion of polynomials and power series has only been over a finite number of variables, we can allow an infinite number of variables. In the case of polynomials, the ring $R[x_1, x_2, x_3, \ldots]$ is well defined, because each term of a polynomial is allowed finitely many variables along with the fact that polynomials only have a finite number of terms. In the case of power series, however, the ring $R[[x_1, x_2, x_3, \ldots]]$ has been generalized in at least three distinct ways in the literature, but as it is far afield of our goals, we will not delve into this here.

In order to look at algebraic (or integral) power series, we shift from looking at the roots of polynomials and view the the polynomials or power series as roots themselves. Recall the definition of an algebraic element and properties of the set of algebraic elements.

Definition 1.2.8. Let $R \subset T$ be integral domains. Then $t \in T$ is said to be *algebraic over* R if it is a root of a polynomial in $R[x]$, i.e., there exists a polynomial $f(x) \in R[x]$ such that

$$f(t) = r_n t^n + r_{n-1} t^{n-1} + \cdots + r_1 t + r_0 = 0.$$

If $f(t)$ is monic, then t is said to be *integral* over R.

Definition 1.2.9. Let $R \subset T$ be domains. If T is the quotient field of R then

$$\overline{R}_{\text{Alg}} := \{t \in T \mid t \text{ is algebraic over } R\}$$

is called the *algebraic closure* of R in T and

$$\overline{R}_{\text{Int}} := \{t \in T \mid t \text{ is integral over } R\}$$

is called the *integral closure* of R in T.

Example 1.2.10. Consider $\mathbb{Z} \subset \mathbb{Q}$. Observe that $g(x) = 2x - 1 \in \mathbb{Z}[x]$ and

$$g\left(\frac{1}{2}\right) = 2\left(\frac{1}{2}\right) - 1 = 1 - 1 = 0.$$

Therefore, $\frac{1}{2}$ is algebraic over $\mathbb{Z}$. In fact, $\overline{\mathbb{Z}}_{\text{Alg}} = \mathbb{Q}$. By definition, we have that $\overline{\mathbb{Z}}_{\text{Alg}} \subset \mathbb{Q}$. Let

$\frac{p}{q} \in \mathbb{Q}$ where $p, q \in \mathbb{Z}$ and $q \neq 0$. We simply note that $f(x) = qx - p \in \mathbb{Z}[x]$ and

$$f\left(\frac{p}{q}\right) = q\left(\frac{p}{q}\right) - p = p - p = 0.$$

Hence $\mathbb{Q} \subset \overline{\mathbb{Z}}_{\text{Alg}}$, and thus $\overline{\mathbb{Z}}_{\text{Alg}} = \mathbb{Q}$.

For an example of an integral element, we consider $\mathbb{Z} \subset \mathbb{Z}[\sqrt{2}]$. Note that $f(x) = x^2 - 2 \in \mathbb{Z}[x]$ is monic and

$$f(\sqrt{2}) = (\sqrt{2})^2 - 2 = 2 - 2 = 0.$$

Therefore $\sqrt{2}$ is integral over $\mathbb{Z}$.

In order to show that the collection of all integral (or algebraic) elements form a ring, we establish an equivalent condition of an element being integral.

Theorem 1.2.11. [16] Let $R \subset T$ be domains and let $t \in T$. Then the following are equivalent.

1. t is integral over R.
2. There exists a finitely generated R-submodule $A \subset T$ such that $tA \subset A$.

Proof. (1) $\Rightarrow$ (2) By definition, there exists a monic polynomial

$$f(x) = x^n + r_{n-1}x^{n-1} + \cdots + r_1 x + r_0 \in R[x]$$

such that $f(t) = t^n + r_{n-1}t^{n-1} + \cdots + r_1 t + r_0 = 0$. We take $A \subset T$ generated by $\{1, t, \ldots, t^{n-1}\}$.

(2) $\Rightarrow$ (1) Let $A \subset T$ be generated by $\{a_1, \ldots, a_n\}$ where not all of the a_i's are zero. Then we have the following set of equations

$$ta_j = \sum_{i=1}^{n} r_{j,i} a_i$$

which leads to

$$\begin{pmatrix} (-t + r_{1,1}) & r_{1,2} & \cdots & r_{1,n-1} & r_{1,n} \\ r_{2,1} & (-t + r_{2,2}) & \cdots & r_{2,n-1} & r_{2,n} \\ \vdots & & \ddots & \vdots & \vdots \\ r_{n,1} & \cdots & \cdots & \cdots & (-t + r_{n,n}) \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Since not all of the a_i's are zero, the determinant of the coefficient matrix is zero which leads to a monic equation in t of degree n. □

The statement and proof of Theorem 1.2.7 and the following corollary can be adapted for an algebraic element.

Corollary 1.2.12. Let $R \subset T$ be domains. If $t_1, t_2 \in T$ are integral (or algebraic) over R, then so are $t_1 + t_2$ and $t_1 t_2$.

Proof. Let $R \subset T$ be domains. Suppose $t_1, t_2 \in T$ are integral over R. Then there exists finitely generated R-submodules $A_1, A_2 \subset T$ such that $t_1 A_1 \subset A_1$ and $t_2 A_2 \subset A_2$. Then $A := A_1 A_2$ is a finitely generated R-submodule and

$$(t_1 + t_2)A \subset A$$
$$t_1 t_2 A \subset A$$

So $t_1 + t_2$ and $t_1 t_2$ are integral over R. □

As stated earlier, given a polynomial $f(x) \in \mathbb{Q}[x]$, the roots of f may or may not lie in the field $\mathbb{Q}$. This is because the field $\mathbb{Q}$ is not algebraically closed. In order to guarantee we have all our roots, we must consider the algebraic closure of $\mathbb{Q}$, denoted $\mathbb{A}$.

We endeavor to discover the nature of power series as potential roots of polynomials in two variables. To begin, we need to know in which ring all the roots of bivariate polynomials lie. The ring $\mathbb{Q}[[x]]$ is not large enough as it does not contain the root of $F(x, y) = xy - 1$. If we include this root by allowing division by x and consider the field $\mathbb{Q}[[x]][\frac{1}{x}]$, this is still not enough. The root of $F(x, y) = y^2 - x$ is not in $\mathbb{Q}[[x]][\frac{1}{x}]$. We additionally need to allow fractional exponents on our power series. These are called *Puiseux series*. The algebraic closure of $\mathbb{A}[[x]]$ is the ring of all Puiseux series, denoted $\mathbb{A}\langle\langle x\rangle\rangle$.

Remark 1.2.13. We make a quick note that not every power series can be a root of a polynomial equation. This can be done by a simple cardinality argument. The number of power series over $\mathbb{Q}$ is uncountable while the number roots of polynomial functions over $\mathbb{Q}[x]$ is countable, so there must be uncountably many power series over $\mathbb{Q}$ which are not a root of a polynomial equation.

We summarize the previous sets and introduce any other relevant sets of power series along with their notation.

Definition 1.2.14. If we consider an algebraic equation of the form

$$F(x,y) = f_n(x)y^n + f_{n-1}(x)y^{n-1} + \cdots + f_1(x)y + f_0(x) \tag{1.1}$$

where $n \in \mathbb{N}$ and $f_i(x) \in R[x]$ for $1 \leq i \leq n$, the set of all power series solutions in $R[[x]]$ to an equation of the form (1.1) form a ring denoted $R_{\text{Alg}}[[x]]$.

All elements of $R_{\text{Alg}}[[x]]$ that are solutions to monic algebraic equations of the form (1.1) form the ring of integral power series, denoted $R_{\text{Int}}[[x]]$. In other words, all elements $\alpha(x) \in R_{\text{Int}}[[x]]$ satisfy an equation of the form

$$F(x,y) = y^n + f_{n-1}(x)y^{n-1} + \cdots + f_1(x)y + f_0(x) = 0. \tag{1.2}$$

Definition 1.2.15. Let K be the algebraic closure of the quotient field of R. The algebraic closure of $K[[x]]$ consists of fractional power series called *Puiseux series* and are denoted $K\langle\langle x\rangle\rangle$ [17]. The elements of $K\langle\langle x\rangle\rangle$ have the form

$$\sum_{i=m}^{\infty} r_i x^{i/n}$$

where $r_i \in k, m \in \mathbb{Z}, n \in \mathbb{N}$ are fixed.

Here are the notations that we will use for some central ring constructions:

$$\text{The polynomial ring } R[x] = \left\{ \sum_{i=0}^{n} r_i x^i \mid r_i \in R, n \in \mathbb{N}_0 \right\}$$

$$\text{The formal power series ring } R[[x]] = \left\{ \sum_{i=0}^{\infty} r_i x^i \mid r_i \in R \right\}$$

$$\text{The formal Laurent series ring } R[[x]][\tfrac{1}{x}] = \left\{ \sum_{i=m}^{\infty} r_i x^i \mid r_i \in R, m \in \mathbb{Z} \right\}$$

$$\text{The Puiseux series ring } R\langle\langle x\rangle\rangle = \left\{ \sum_{i=m}^{\infty} r_i x^{i/n} \mid r_i \in R, m \in \mathbb{Z}, n \in \mathbb{N} \text{ fixed} \right\}$$

$$\text{The algebraic series ring } R_{\text{Alg}}[[x]] = \{f(x) \in R[[x]] \mid f(x) \text{ is algebraic}\}$$

$$\text{The integral series ring } R_{\text{Int}}[[x]] = \{f(x) \in R[[x]] \mid f(x) \text{ is integral}\}$$

Example 1.2.16. Notice that

$$f(x) = \frac{1}{x^2} + \frac{1}{x} + 1 + x + x^2 + \cdots = \sum_{i=-2}^{\infty} x^i \in R[[x]][\tfrac{1}{x}],$$

however,

$$g(x) = \cdots + \frac{1}{x^2} + \frac{1}{x} + 1 + x + x^2 + \cdots = \sum_{i=-\infty}^{\infty} x^i \notin R[[x]][\tfrac{1}{x}].$$

That is, the ring $R[[x]][\frac{1}{x}]$ contains series which start at a given integer m.

Example 1.2.17. The denominator of the exponents in the powers of a Puiseux series is bounded by a given natural number. For example,

$$f(x) = 1 + x^{1/2} + x + x^{3/2} + x^2 + \cdots = \sum_{i=0}^{\infty} x^{i/2}$$

$$g(x) = x^{-1} + x^{-2/3} + x^{-1/2} + x^{-1/3} + 1 + x^{1/3} + x^{1/2} + x^{2/3} + x + \ldots$$

are both Puiseux series. It is worth pointing out that not all series with fractional exponents are necessarily Puiseux series. For instance,

$$x + x^{1/2} + x^{1/4} + x^{1/8} + \ldots = \sum_{i=0}^{\infty} x^{1/2^i}$$

$$x^2 + x^{5/2} + x^{10/3} + x^{17/4} + \ldots = \sum_{n=1}^{\infty} x^{\frac{n^2+1}{n}}$$

are not Puiseux series.

Example 1.2.18. We recall our first power series from Example 1.2.6

$$\alpha(x) = 1 + x + x^2 + \cdots = \frac{1}{1-x}.$$

This is an algebraic power series because it is a root of $F(x, y) = (1 - x)y - 1$. We have

$$F(x, \alpha(x)) = (1 - x)(\alpha(x)) - 1 = (1 - x)(\frac{1}{1-x}) - 1 = 1 - 1 = 0.$$

The second power series from Example, 1.2.6

$$\beta(x) = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{5x^4}{128} + \cdots = \sqrt{1+x},$$

is a root of $F(x,y) = y^2 - (1+x)$, because

$$F(x, \beta(x)) = (\beta(x))^2 - (x+1) = (\sqrt{1+x})^2 - (1+x) = 1 + x - (1+x) = 0.$$

Thus, $\beta(x)$ is an integral power series.

Example 1.2.19. Notice that even though $f(x) = \sum_{i=-2}^{\infty} x^i$ from Example 1.2.16 is a root of the polynomial

$$F(x,y) = x^2(1-x)y - 1 \in R[x][y],$$

$f(x) \notin R_{\text{Alg}}[[x]]$ because $f(x)$ is not a power series. In this case, we say that $f(x)$ is an *algebraic Laurent series.*

Similarly, the Puiseux series in Example 1.2.17 are all algebraic. For example, the first Puiseux series given,

$$\sum_{i=0}^{\infty} x^{i/2},$$

satisfies

$$(1-x)y^2 - 2y + 1 = 0.$$

Remark 1.2.20. Notice that $R[x] \subset R_{\text{Int}}[[x]]$ as $f(x) \in R[x]$ is a root of

$$F(x,y) = y - f(x) = 0.$$

Also, $\frac{f(x)}{g(x)} \in R_{\text{Alg}}[[x]]$ given that $g(0) \in U(R)$ since it satisfies

$$F(x,y) = g(x)y - f(x) = 0.$$

The condition $g(0) \in U(R)$ assures that $\frac{f(x)}{g(x)} \in R[[x]]$. Also observe that $R_{\text{Int}}[[x]]$ is a proper subset of $R_{\text{Alg}}[[x]]$ as $\frac{1}{1-x} \in R_{\text{Alg}}[[x]]$ but $\frac{1}{1-x} \notin R_{\text{Int}}[[x]]$.

Observation 1.2.21. We observe that every algebraic power series is an integral power series divided by a polynomial. To see this, let $\alpha(x) \in R_{\text{Alg}}[[x]]$ and let

$$F(x,y) = f_n(x)y^n + f_{n-1}(x)y^{n-1} + \cdots + f_1(x)y + f_0(x) \tag{1.3}$$

be the bivariate polynomial such that $f_n(x) \neq 0$ and $F(x, \alpha(x)) = 0$. Multiplying Equation (1.3) by $f_n(x)^{n-1}$ gives

$$f_n(x)^n y^n + f_n(x)^{n-1} f_{n-1}(x) y^{n-1} + \cdots + f_n(x)^{n-1} f_1(x) y + f_n(x)^{n-1} f_0(x).$$

Letting $v = f_n(x)y$, we get

$$v^n + f_{n-1}(x)v^{n-1} + f_n(x)f_{n-2}(x)v^{n-2} + \cdots + f_n(x)^{n-2} f_1(x)v + f_n(x)^{n-1} f_0(x). \tag{1.4}$$

We notice that Equation (1.4) is monic and that $f_n(x)\alpha(x)$ is a root of it. Therefore, $f_n(x)\alpha(x)$ is an integral power series.

Since we are concerned with power series solutions to equations of the form (1.1), it is good to know when a given bivariate polynomial has a power series solution. In general, given an equation of the form (1.1) it is difficult to tell whether a power series solution exists. There are some ways to alleviate this problem, and one way is to use the implicit function theorem. We first recall the following definitions.

Definition 1.2.22. A function f is *holomorphic* on an open set $U \subset \mathbb{C}$ if it is differentiable at every point within U.

Definition 1.2.23. A function f is *entire* if it is holomorphic on all of $\mathbb{C}$.

This statement of the implicit function theorem is taken from [18] and provided for completeness.

Theorem 1.2.24. (Implicit Function Theorem) Suppose that $F(x, y)$ is holomorphic in the bidisc $D(x_0, R_1) \times D(y_0, R_2) \in \mathbb{C}^2$. If

$$F(x_0, y_0) = 0 \text{ and } \left.\frac{\partial F}{\partial y}\right|_{(x_0,y_0)} \neq 0,$$

then there is a disc $D(x_0, r_0)$ and a unique holomorphic function $f(x)$ defined on $D(x_0, r_0)$ with $f(x_0) = y_0$ for which

$$F(x, f(x)) = 0$$

holds for $x \in D(x_0, r_0)$. Moreover, that function $f(x)$ is represented by

$$\frac{1}{2\pi i} \oint_{C_i} y \frac{F_y(x, y)}{F(x, y)} dy, \tag{1.5}$$

where C is the boundary of the disc $D(y_0, r_1)$ is a suitably chosen circle.

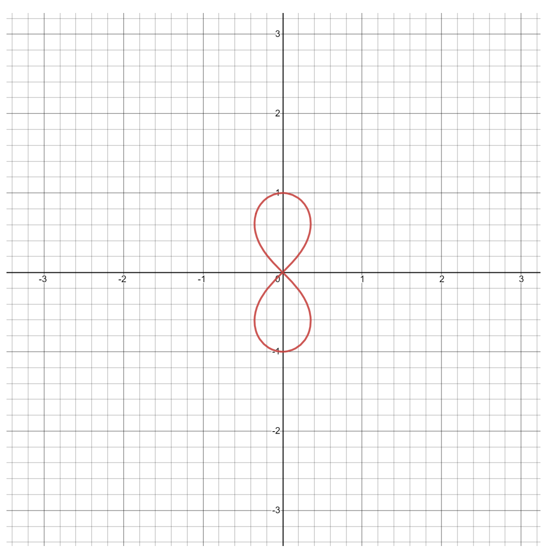
The implicit function theorem also holds when working over the field $\mathbb{R}$ with the additional condition that coefficients of $F(x, y)$ come from $\mathbb{R}$.

Remark 1.2.25. When working over $\mathbb{Q}$ or $\mathbb{R}$, the implicit function theorem has a geometric interpretation. If the graph of $F(x, y)$ is differentiable at any point on the y-axis, then there exists a power series solution to $F(x, y)$.

Example 1.2.26. We consider

$$F(x, y) = \left(y^2 + x^2\right)^2 - (y^2 - x^2) = 0.$$

We note the graph (made in Desmos) of $F(x, y)$ in $\mathbb{R}$ is:



Its y-intercepts are $y = 1, 0, -1$. The relation $F(x, y)$ has four power series solutions centered at $x = 0$. Observe that

$$F_y(x, y) = 4y(x^2 + y^2) - 2y$$

and

$$F_y(0, 1) = -2, \quad F_y(0, -1) = 2, \quad F_y(0, 0) = 0.$$

By the implicit functional theorem, there are at least two power series solutions. When the partial derivative is zero, more work needs to be done to find out if $F(x, y)$ has additional power series solutions. For this example, we simply use the transformation $y = xv$, yielding

$$F(x, xv) = -x^2(1 + x^2 - v^2 + 2x^2v^2 + x^2v^4).$$

Dividing both sides by $-x^2$ gives

$$G(x, v) = 1 + x^2 - v^2 + 2x^2v^2 + x^2v^4$$

which has v-intercepts $v = -1, 1$. Now,

$$G_v(x, v) = -2v + 4x^2v + 4x^2v^3, \quad G_v(0, -1) = 2, \quad G_v(0, 1) = 2.$$

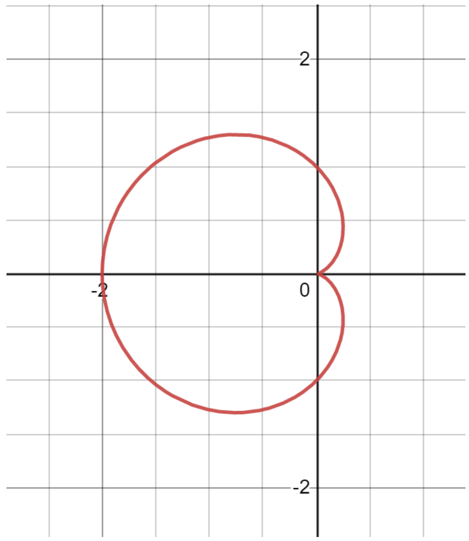
Thus $G(x, y)$ has two power series solutions, $g_1(x)$ and $g_2(x)$ which means that $xg_1(x)$ and $xg_2(x)$ are power series solutions for $F(x, y)$, and F has four power series solutions in total.

Remark 1.2.27. In the previous example, the partial derivative with respect to y at the origin is zero because the graph intersects itself. The substitution $y = xv$ is a type of *blowing up a curve at a point.* The blow up separates the intersection into two curves referred to as the *exceptional curve* (in this case, $x = 0$) and the *blow up* (in this case, $1 + x^2 - v^2 + 2x^2v^2 + x^2v^4 = 0$). For additional reading, we refer the reader to [12].

Example 1.2.28. We consider

$$F(x, y) = 2x\left(x^2 + y^2\right) + \left(x^2 + y^2\right)^2 - y^2$$

and its graph.



We have that

$$F_y(x,y) = 2y - 4xy - 4y(x^2+y^2), \quad F_y(0,1) = -2, \quad F_y = (0,-1) = 2$$

which gives us two power series solutions. Because the graph has a cusp at the origin, the two solutions at $(0,0)$ correspond to Puiseux series solutions.

In the previous examples, we are guaranteed power series solutions by the implicit function theorem but we do not know what they are. If one wanted to compute the series solutions (power series of Puiseux series) one can utilize Newton's Diagram. We explore this concept further in Chapter 3, so given an equation of the form (1.2), we will have a practical technique to determine power series solutions, if any. The harder direction is trying to determine if a given power series is a solution to an equation of the form (1.1). Excluding polynomials, the solutions to algebraic equations must have finite radii of convergence, as a consequence of Theorem 1.2.24. In order to show where the representation (1.5) in comes from, we recall some concepts from complex analysis. The following argument closely follows from [1] and [13]. The theorems and definitions can be found in [19], [25], or [29] some theorems are presented without proof.

Definition 1.2.29. If a function f fails to be analytic at a point z_0, then this point is said to be *singular point*. There are three classifications of singular points: removable singularities, poles of order n, and essential singularities. In order to define the type of singular point, we first write f in its Laurent series expansion centered at $z = z_0$, i.e.,

$$f(z) = \sum_{k=1}^{\infty} a_{-k}(z - z_0)^{-k} + \sum_{k=0}^{\infty} a_k(z - z_0)^k.$$

The point $z = z_0$ is called:

1. a *removable singularity* if $a_{-k} = 0$ for all k.
2. a *pole of order* n if $a_{-n} \neq 0$ and $a_{-k} = 0$ for $k > n$.
3. an *essential singularity* if $\sum_{k=1}^{\infty} a_{-k}(z - z_0)^{-k}$ contains infinitely many nonzero terms.

Example 1.2.30. The function $f(z) = \frac{\cos(z)-1}{z}$ has a removable singularity at $z = 0$. To see this we write

$$f(z) = \frac{\cos(z) - 1}{z} = \frac{z}{2!} - \frac{z^3}{4!} + \frac{z^5}{6!} - \cdots = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{z^{2k-1}}{(2k)!}.$$

As $a_{-k} = 0$ for all k, $f(z)$ has a removable singularity.

The function $f(z) = \frac{1}{1-z}$ has a pole of order 1 at $z = 1$ because it is equal to its Laurent series expansion at $z = 1$. That is,

$$f(z) = \cdots + \frac{0}{(1 - z)^3} + \frac{0}{(1 - z)^2} + \frac{1}{(1 - z)}.$$

The function $\sin(\frac{1}{z})$ has an essential singularity at $z = 0$. This can be seen by its Laurent series expansion

$$f(z) = \cdots + \frac{1}{5!z^5} - \frac{1}{3!z^3} + \frac{1}{z} = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(2k - 1)!z^{2k-1}}.$$

With poles being defined, we can now define a meromorphic function.

Definition 1.2.31. A function f defined on an open set U is said to be *meromorphic* on U if it is analytic everywhere except at a discrete set of points which are poles.

Theorem 1.2.32. (ML Inequality) If f is continuous on a smooth curve C and if $|f(z)| \leq M$ for all z on C, then $|\int_C f(z)dz| \leq ML$, where L is the length of the curve C.

We can now establish the theorems needed to see where (1.5) comes from.

Theorem 1.2.33. (Argument Principle) If f is meromorphic on and within a closed curve C and if $f(z) \neq 0$ on C, then

$$\frac{1}{2\pi i} \oint_C \frac{f'(z)}{f(z)} dz = Z - P$$

where C is traversed counterclockwise, Z is the number of zeros of f within C, and P is the number of poles within C.

Corollary 1.2.34. Let f have zeros z_i and poles p_j within the curve C. Assuming f satisfies the conditions of the argument principle, an immediate consequence is

$$\frac{1}{2\pi i} \oint_C g(z) \frac{f'(z)}{f(z)} dz = \sum_{z_i} g(z_i) n(C; z_i) - \sum_{p_j} g(p_j) n(C; p_j) \tag{1.6}$$

where $n(C; z_i)$ is the multiplicity of z_i and $n(C; p_j)$ is the order of p_j.

We use (1.6) when given a bivariate polynomial $F(x, y) \in R[x][y]$. Suppose $\deg_y(F(x, y)) = n$ and that $F(0, y) = 0$ has n distinct solutions, say $y_1, \ldots, y_n \in \mathbb{C}$. Let

$$r = \frac{1}{2} \left(\min_{1 \leq i < j \leq n} \{|y_i - y_j|\} \right).$$

Then the disks defined by $|y - y_i| \leq r$ are pairwise disjoint. Let C_i be the circle $|y - y_i| = r$. By the argument principle we have that

$$\frac{1}{2\pi i} \oint_{C_i} \frac{F_y(x, y)}{F(x, y)} dy = 1,$$

because there is only one zero within C_i. Denote this zero as $f_i(x)$. Then by (1.6),

$$\frac{1}{2\pi i} \oint_{C_i} y \frac{F_y(x, y)}{F(x, y)} dy = f_i(x), \tag{1.7}$$

which is the representation given in Theorem 1.2.24.

Now we have established that an algebraic power series must have a positive radius of convergence. Additionally, we know that polynomials are algebraic power series and have an infinite radius of convergence, and in fact, polynomials are the only algebraic power series that have an infinite radius of convergence. In order to show that having an infinite radius of convergence implies being a polynomial, we establish the following lemma and proposition.

Lemma 1.2.35. (Cauchy's inequality) Let

$$f(z) = \sum_{i=0}^{\infty} a_i (z-a)^i$$

have radius of convergence $R > 0$. Then

$$\left| \frac{f^{(n)}(a)}{n!} \right| \leq \frac{M(r)}{r^n}$$

for all $0 < r < R$, where

$$M(r) = \sup\{|f(z)| \ : \ |z-a| = R\}.$$

Proof. The theorem follows directly from

$$\frac{f^{(n)}(a)}{n!} = \frac{1}{2\pi i} \int_{\gamma} \frac{f(t)}{(t-a)^{n+1}} dt$$

where $\gamma(\theta) = a + Re^{i\theta}$ for $0 \leq \theta \leq 2\pi$. □

Proposition 1.2.36. If $f(z)$ is entire and $|f(z)| \leq M|z|^m$ for an $M > 0$, $m \in \mathbb{N}_0$ and for all $z \in \mathbb{C}$ such that $|z| \geq R \in \mathbb{R}$, then $f(z)$ is a polynomial of degree less than or equal to m.

Proof. Suppose $f(z)$ is entire and $|f(z)| \leq N|z|^m$ for some $N > 0$, $m \in \mathbb{N}_0$ and for all $z \in \mathbb{C}$ such that $|z| \geq R \in \mathbb{R}$. Since $f(z)$ is entire, it is equal to its Taylor series expansion centered at zero, i.e.,

$$f(z) = \sum_{i=0}^{\infty} \frac{f^{(i)}(0)}{i!} z^i.$$

Applying Lemma 1.2.35, we have that

$$|f^{(n)}(0)| \leq \frac{M(r)n!}{r^n}$$

for $r > 0$. When $r > R$, we have that

$$M(r) \leq N|z|^m \leq Nr^m$$

and thus

$$|f^{(n)}(0)| \leq \frac{M(r)n!}{r^n} \leq \frac{Nr^m n!}{r^n} = \frac{Nn!}{r^{n-m}}.$$

When $n > m$ we have that

$$|f^{(n)}(0)| \leq \frac{Nr!}{r^{n-m}} \to 0 \text{ as } r \to \infty.$$

Therefore,

$$f(z) = \sum_{i=0}^{m} \frac{f^{(i)}(0)}{i!} z^i \in R[z].$$

□

Remark 1.2.37. We point out that Liouville's theorem is a direct corollary of Proposition 1.2.36 when $m = 0$.

We are now able to show that an algebraic power series has an infinite radius of convergence if and only if it is a polynomial.

Theorem 1.2.38. Let R be a subfield of the complex numbers and $\alpha \in R_{\text{Alg}}[[x]]$. Then $\alpha(x) \in R[x]$ if and only if r.o.c$(\alpha) = \infty$.

Proof. The forward direction is trivial. Conversely, suppose that r.o.c$(\alpha(x)) = \infty$. By definition there exists an irreducible polynomial

$$F(x, y) = f_n(x)y^n + f_{n-1}(x)y^{n-1} + \cdots + f_1(x)y + f_0(x) \in R[x][y]$$

such that $F(x, \alpha) = 0$. Let $m = \max_{0 \leq i \leq n}\{\deg(f_i(x))\}$. Then there exists $M_1 > 0$ such that

$$\sum_{i=0}^{n} |f_i(x)| \leq |x|^{m+1} \text{ for } |x| \geq M_1.$$

Let

$$c = \begin{cases} |f_n(0)| & \text{if } f_n(0) \neq 0 \\ 1 & \text{otherwise.} \end{cases}$$

There exists $M_2 > 0$ such that $c \leq |f_n(x)|$ for all $|x| \geq M$. Let $M = \max\{M_1, M_2\}$.

We have that

$$f_n(x)\alpha(x)^n = -\left(\sum_{i=0}^{n-1} f_i(x)\alpha(x)^i\right) \Rightarrow |f_n(x)||\alpha(x)|^n = \left|\sum_{i=0}^{n-1} f_i(x)\alpha(x)^i\right| \leq \sum_{i=0}^{n-1} |f_i(x)||\alpha(x)|^i.$$

If $|\alpha(x)| \leq 1$, then

$$|f_n(x)||\alpha(x)|^n \leq |f_n(x)||\alpha(x)| \leq |f_n(x)| \leq \sum_{i=0}^{n} |f_i(x)|.$$

If $|\alpha(x)| > 1$, then

$$|f_n(x)||\alpha(x)|^n \leq \sum_{i=0}^{n-1} |f_i(x)||\alpha(x)|^i \leq |\alpha(x)|^{n-1} \sum_{i=0}^{n-1} |f_i(x)|.$$

Therefore,

$$|f_n(x)||\alpha(x)| \leq \sum_{i=0}^{n-1} |f_i(x)| \leq \sum_{i=0}^{n} |f_i(x)|$$

holds regardless of the modulus of $\alpha(x)$. Thus, when $|x| \geq M$, it is the case that

$$c|\alpha(x)| \leq |f_n(x)||\alpha(x)| \leq \sum_{i=0}^{n} |f_i(x)|,$$

which implies that

$$|\alpha(x)| \leq \frac{1}{c} \sum_{i=0}^{n} |f_i(x)| \leq \frac{1}{c}|x|^{m+1}.$$

We note that $\alpha(x)$ is entire because its radius of convergence is infinite. Therefore, by Proposition 1.2.36, we have that $\alpha(x)$ is a polynomial, i.e., $\alpha(x) \in R[x]$. □

Corollary 1.2.39. Any power series with an infinite radius of convergence that is not a polynomial is transcendental. For example, e^x, $\sin(x)$, and $\cos(x)$ are transcendental functions.

Chapter 2

Minimal Polynomials and Unitary Elements

2.1 Preliminary Results

We will begin by establishing some preliminary results on minimal polynomials and units.

Definition 2.1.1. An element u in a ring R is a *unit* if there exists $v \in R$ such that $uv = 1$. We denote the set of all units of R by $U(R)$.

For our context, the ring R will be an integral domain, typically $\mathbb{Q}$.

Definition 2.1.2. We say that $p(x, y) \in R[x][y]$ is *irreducible* if whenever $p(x, y) = a(x, y)b(x, y)$, either $a(x, y)$ or $b(x, y)$ is a unit.

Theorem 2.1.3. Given an algebraic power series $\alpha(x) \in R_{\mathrm{Alg}}[[x]]$, then there exists a unique (up to scalar multiples) irreducible polynomial of minimal degree $p(x, y) \in R[x][y]$, such that $p(x, \alpha(x)) = 0$.

Proof. Let K be the quotient field of R. Since $K(x) \subset K(x)[\alpha(x)]$ is an algebraic field extension, there exists a unique minimal polynomial, up to associates, $G(x, y) \in K(x)[y]$ such that $G(x, \alpha(x)) = 0$. We write

$$G(x, y) = \frac{f_n(x)}{g_n(x)} y^n + \cdots + \frac{f_1(x)}{g_1(x)} y + \frac{f_0(x)}{g_0(x)}.$$

Let

$$g(x) = \operatorname*{lcm}_{0 \le i \le n}(g_i(x))$$

and define $F(x,y) = g(x) \cdot G(x,y)$. Clearly $F(x,y) \in R[x][y]$, so this is the minimal polynomial of $\alpha(x)$. Uniqueness of $F(x,y)$ follows from the uniqueness of $G(x,y)$. $\square$

An alternative proof that does not assume the existence of a unique minimal polynomial for an algebraic field extension is as follows:

Proof. Let

$$S = \{\deg_y(p(x,y)) \mid p(x,y) \neq 0 \text{ is irreducible and } p(x,\alpha(x)) = 0\}.$$

Since $\alpha(x)$ is algebraic, there exists an irreducible $F[x][y] \in R[x][y]$ of the form

$$f_m(x)y^m + f_{m-1}(x)y^{m-1} + \cdots + f_1(x)y + f_0$$

where $f_i \in R[x]$ and $m \geq 1$ such that $F(x,\alpha(x)) = 0$. Then $m \in S$, so it is nonempty. We observe that S is in one to one correspondence with a subset of the natural numbers. By the well-ordering principle, there exists a minimal element $n > 0$. Suppose there exists two irreducible polynomials $a(x,y)$ and $b(x,y)$ of degree n in the variable y such that $a(x,\alpha(x)) = b(x,\alpha(x)) = 0$. Writing

$$a(x,y) = f_n(x)y^n + \cdots + f_1(x)y + f_0(x)$$
$$b(x,y) = g_n(x)y^n + \cdots + g_1(x)y + g_0(x),$$

we define $r(x,y) = g_n(x)a(x,y) - f_n(x)b(x,y)$. Clearly, $r(x,\alpha(x)) = g_n(x)a(x,\alpha(x)) - f_n(x)b(x,\alpha(x)) = 0$ and $\deg_y(r(x,y)) < n$. The only way this is possible is if $r(x,y) = 0$. This implies that

$$f_n(x)g_m(x) = g_n(x)f_m(x)$$

for all $m < n$. Therefore,

$$g_m(x) = \frac{g_n(x)f_m(x)}{f_n(x)}.$$

Let $c(x) = \gcd(f_n(x), g_n(x))$. Writing $g_n(x) = c(x)\tilde{g}_n(x)$ and $f_n(x) = c(x)\tilde{f}_n(x)$, we see that

$$g_m(x) = \tilde{g}_n(x)\left(\frac{f_m(x)}{\tilde{f}_n(x)}\right) \in R[x] \quad \Rightarrow \quad \tilde{f}_n(x) \mid f_m(x)$$

for all $m < n$. Since $a(x,y)$ is irreducible, the only way this is possible is if $\tilde{f}_n(x) \in U(R)$. Similarly,

$\tilde{g}_n(x) \in U(R)$, and therefore we have $f_n(x) = cg_n(x)$ for some $c \in U(R)$, which implies that $b(x, y) = ca(x, y)$, and this completes the proof. □

We now characterize the units of integral power series and algebraic power series. We begin with the algebraic power series.

Theorem 2.1.4. Let $a(x) \in R[[x]]$. We have that $a(x) \in U(R_{\text{Alg}}[[x]])$ if and only if $a(x) \in U(R[[x]]) \cap R_{\text{Alg}}[[x]]$.

Proof. (⇒) The forward direction is trivial.
(⇐) Suppose $a(x) \in U(R[[x]]) \cap R_{\text{Alg}}[[x]]$. Since $a(x) \in R_{\text{Alg}}[[x]]$, there exists

$$F(x, y) = f_n(x)y^n + f_{n-1}(x)y^{n-1} + \cdots + f_1(x)y + f_0(x)$$

such that

$$F(x, a(x)) = f_n(x)(a(x))^n + f_{n-1}(x)(a(x))^{n-1} + \cdots + f_1(x)a(x) + f_0(x) = 0. \tag{2.1}$$

Since $a(x) \in U(R[[x]])$, there exists $(a(x))^{-1} \in R[[x]]$ such that $a(x)(a(x))^{-1} = 1$. Multiplying both sides of (2.1) by $((a(x))^{-1})^n$ yields

$$f_n(x) + f_{n-1}(x)((a(x))^{-1}) + \cdots + f_1(x)((a(x))^{-1})^{n-1} + f_0(x)((a(x))^{-1})^n = 0.$$

Therefore $(a(x))^{-1}$ satisfies $G(x, y) := f_n(x) + f_{n-1}(x)y + \cdots + f_1(x)y^{n-1} + f_0(x)y^n \in R[x][y]$. Hence $a(x) \in U(R_{\text{Alg}}[[x]])$. □

In other words, the unitary algebraic power series are precisely the unitary power series, which are also algebraic. The same result does not hold for integral power series, because if $a(x) \in U(R[[x]]) \cap R_{\text{Int}}[[x]]$, it is not necessarily true that $a(x) \in U(R_{\text{Int}}[[x]])$. This is because the power series $(a(x))^{-1}$ is algebraic but often fails to be integral. The easiest example is if $a(x)$ is a nonconstant polynomial. Then $(a(x))^{-1}$ is a root of $a(x)y - 1$, which implies that $(a(x))^{-1}$ is algebraic and not integral. To ensure the inverse is also integral, we have to impose a stronger restriction on the minimal polynomial of $a(x)$.

Theorem 2.1.5. Let $a(x) \in R[[x]]$ have minimal polynomial $F(x, y)$. Then

$$a(x) \in U(R_{\text{Int}}[[x]]) \iff a(x) \in R_{\text{Int}}[[x]] \text{ and } F(x, 0) \in U(R).$$

Proof. Let

$$F(x, y) = y^n + f_{n-1}(x)y^{n-1} + \cdots + f_1(x)y + f_0(x) = 0$$

be the minimal polynomial of $a(x)$.

($\Rightarrow$) Let

$$G(x, y) = y^m + g_{m-1}(x)y^{m-1} + \cdots + g_1(x)y + g_0(x) = 0$$

be the minimal polynomial of $a(x)^{-1}$. We have that

$$F(x, a) = a^n + f_{n-1}(x)a^{n-1} + \cdots + f_1(x)a + f_0(x) = 0,$$

which implies

$$1 + f_{n-1}(x)a(x)^{-1} + \cdots + f_1(x)a(x)^{-(n-1)} + f_0(x)a(x)^{-n} = 0.$$

Since $G(x, y)$ is the minimal polynomial of $a(x)^{-1}$, we have that $m \leq n$. Reversing the roles of $a(x)$ and $(a(x))^{-1}$ gives us $n \leq m$, hence $n = m$. Now we have that

$$a^n + f_{n-1}(x)a^{n-1} + \cdots + f_1(x)a + f_0(x) = 0$$
$$g_0(x)a^n + g_1(x)a^{n-1} + \cdots + g_{n-1}(x)a + 1 = 0$$

By uniqueness of minimal polynomials, the only way this is possible is if the two polynomials

$$y^n + f_{n-1}(x)y^{n-1} + \cdots + f_1(x)y + f_0(x) = 0$$
$$g_0(x)y^n + g_1(x)y^{n-1} + \cdots + g_{n-1}(x)y + 1 = 0$$

are associates. Thus, it must be the case that $g_0(x), f_0(x) \in U(R)$.

($\Leftarrow$) We write $f_0(x)$ as f_0 to emphasize that it is in $U(R)$. We know that $F(x, 0) \in U(R)$, which implies that $a(0) \neq 0$. Therefore $a(x)^{-1} \in R[[x]]$. Additionally, we have that

$$1 + f_{n-1}(x)a(x)^{-1} + \cdots + f_1(x)a(x)^{-(n-1)} + f_0 a(x)^{-n} = 0.$$

Hence $(a(x))^{-1}$ is a root of

$$G(x,y) = y^n + g_{n-1}y^{n-1} + \cdots + g_1(x)y + g_0(x) = 0,$$

where $g_i(x) = f_0^{-1} f_{n-i}(x) \in R[x]$ for $i = 0, 1, \ldots, n-1$. Therefore, $(a(x))^{-1} \in R_{\text{Int}}[[x]]$. $\square$

Remark 2.1.6. We give a more sophisticated example to demonstrate that $a(x) \in U(R[[x]]) \cap R_{\text{Int}}[[x]] \not\Rightarrow a(x) \in U(R_{\text{Int}}[[x]])$. We recall Example 1.2.26,

$$F(x,y) = (y^2 + x^2)^2 - (y^2 - x^2) = 0,$$

which is a lemniscate. We previously showed that it has four power series solutions. We consider the power series which passes through $(0,1)$. The first few terms are

$$a(x) = 1 - \frac{3x^2}{2} - \frac{25x^4}{8} - \frac{203x^6}{16} - \frac{8181x^8}{128} - \frac{92037x^{10}}{256} - \cdots.$$

We notice that $(a(x))^{-1} \in U(R[[x]])$, however its minimal polynomial is

$$G(x,y) = \left(x^2 + x^4\right) y^4 - \left(1 - 2x^2\right) y^2 + 1,$$

which implies that $(a(x))^{-1} \in R_{\text{Alg}}[[x]]$ but $(a(x))^{-1} \notin R_{\text{Int}}[[x]]$.

Chapter 3

Puiseux Series

3.1 Newton-Puiseux Algorithm

Let R be an integral domain and K its algebraic closure. We are interested in roots which satisfy an equation of the form

$$F(x,y) = f_n(x)y^n + f_{n-1}(x)y^{n-1} + \cdots + f_1(x)y + f_0(x) = 0. \tag{3.1}$$

We have seen that if $F_y(x_0, y_0) \neq 0$, then the implicit function theorem guarantees a power series solution. However, the ring of power series $R[[x]]$ is not enough to contain all of our roots. Even the ring allowing division by x, $R[[x]][\frac{1}{x}]$, does not contain all of the roots. We need to allow fractional exponents on our series. The set of fractional Laurent series are called *Puisuex series* and are denoted

$$R\langle\langle x\rangle\rangle = \left\{ \sum_{i=m}^{\infty} r_i x^{i/n} | r_i \in R, m \in \mathbb{Z}, n \in \mathbb{N} \text{ fixed} \right\}.$$

The result that there are n roots which lie in $K\langle\langle x\rangle\rangle$ is due to Victor Puiseux [17].

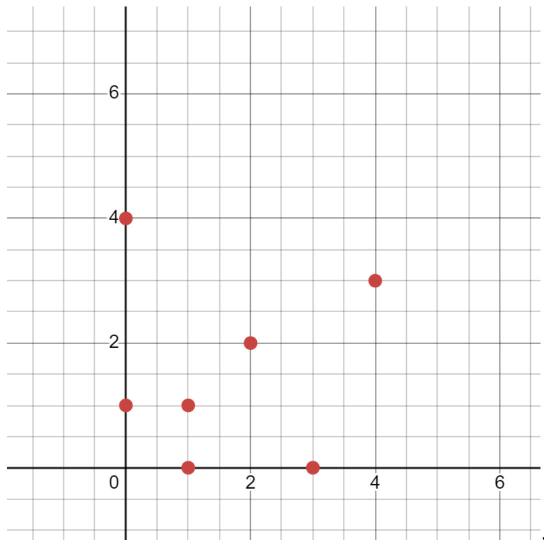
Theorem 3.1.1. (Puiseux's Theorem) The field $K\langle\langle x\rangle\rangle$ is algebraically closed.

Given an equation in the form of (3.1), the solutions can be obtained term by term using the Newton-Puiseux algorithm, first used by Isaac Newton and then later by Victor Puiseux. We provide a loose sketch of the algorithm, and we refer the reader to [4] for a more in-depth view. We will use an example to establish working definitions needed to apply the algorithm.

Example 3.1.2. We consider the set

$$S = \{(1,0),(3,0),(0,4),(0,1),(1,1),(4,3),(2,2)\}$$

and its graph



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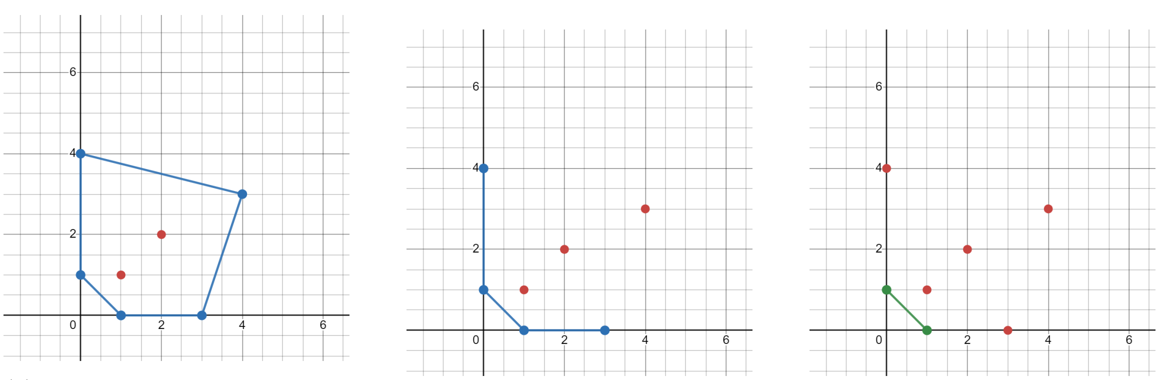
The *boundary of the convex hull* is the boundary of the smallest convex polygon containing all of S, which is given below in figure (a). The *boundary of the lower hull* is the boundary of the lower section of the convex hull from the top left point to the lower right point of the convex hull, which is given below in figure (b). An *edge* is a line of the convex hull that goes through at least two points, an example is given below in figure (c).



(a) The boundary of the convex hull

(b) The lower convex hull

(c) An edge of the lower hull

We will work over $\mathbb{C}$. The Newton-Puisuex Algorithm applied to

$$F(x,y) = \sum_{\substack{\alpha=0,\dots,m \\ \beta=0,\dots,n}} A_{\alpha,\beta} x^{\alpha} y^{\beta},$$

consists of the following steps:

1. Compute and plot $\Delta(F) := \{(\alpha, \beta) \mid A_{\alpha,\beta} \neq 0\}$.

2. Compute the boundary of the convex hull.

3. Pick an edge, E, of the lower hull with a nonzero slope.

4. For every point E passes through, set the sum of the corresponding monomials equal to zero and solve for y.

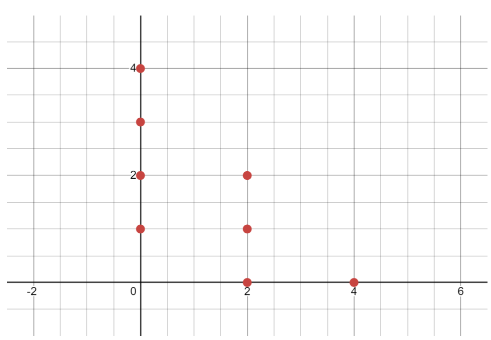
5. Use $y = x^{a_1}$ found in step four and repeat the algorithm for $F(x, y + x^{a_1})$.

Example 3.1.3. We will compute the first two terms of the power series expansion which goes through the point $(0, 1)$ from Example 1.2.16. We know this power series must satisfy $f(0) = 1$, so we start the algorithm at $1 + y$. We have that

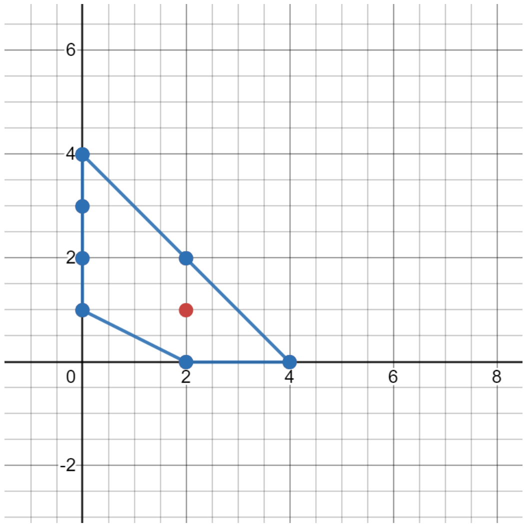
$$F(x, 1+y) = -3x^2 - x^4 - 2y - 4x^2y - 5y^2 - 2x^2y^2 - 4y^3 - y^4 = 0.$$

1. In the first step, we compute and graph

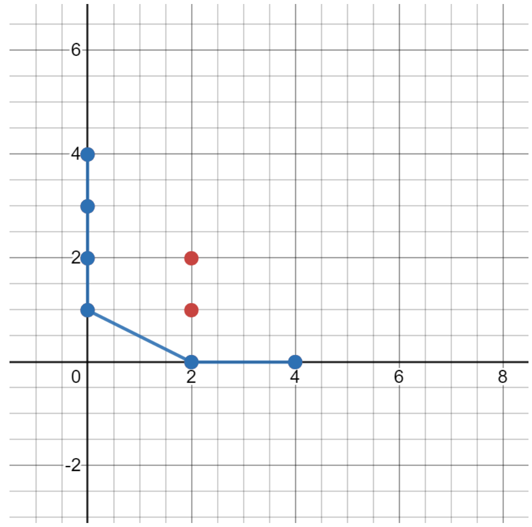
$$\Delta(F) = \{(2,0), (4,0), (0,1), (2,1), (0,2), (2,2), (0,3), (0,4)\}.$$



2. Next we compute the boundary of the convex hull.



(a) The boundary of the convex hull



(b) The boundary of the lower hull

3. Next we pick an edge of the boundary of the lower hull with a nonzero slope. We are forced to choose the edge which passes through $(0, 1)$ and $(2, 0)$.

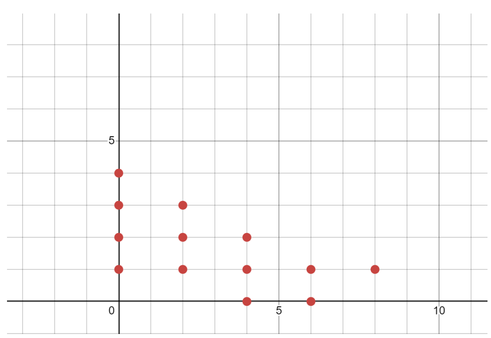
4. The point $(0, 1)$ corresponds to $-2y$ and the point $(2, 0)$ corresponds to $-3x^2$. Setting the sum of the corresponding monomials gives

$$-2y - 3x^2 = 0 \quad \Rightarrow \quad y = -\frac{3}{2}x^2.$$

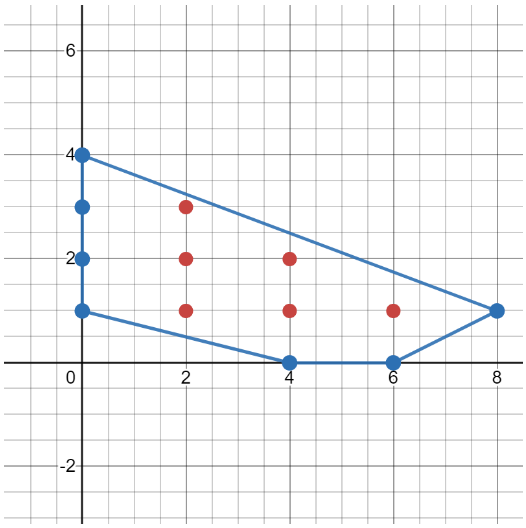
We repeat the process with $F(x, 1 - \frac{3}{2}x^2 + y) = -x^2 + (1 - (3x^2)/2 + y)^2 - (x^2 + (1 - (3x^2)/2 + y)^2)^2$:

1. We compute and graph

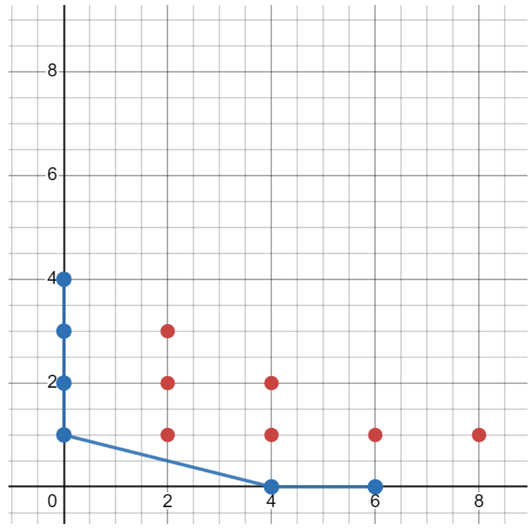
$$\Delta(F) = \{(4, 0), (6, 0), (8, 0), (0, 1), (2, 1), (4, 1), (6, 1), (0, 2), (2, 2), (4, 2), (0, 3), (2, 3), (0, 4)\}.$$



2. Next we compute the boundary of the convex hull.



(a) The boundary of the convex hull



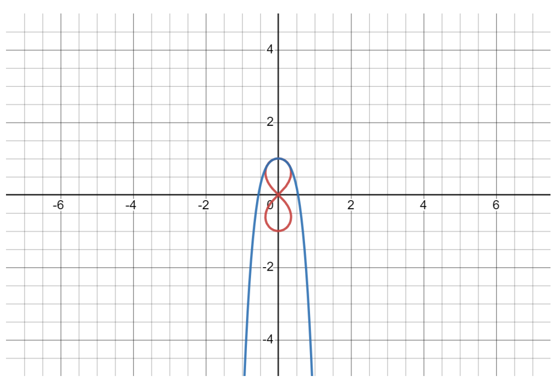
(b) The boundary of the lower hull

3. We pick an edge of the boundary of the lower hull with a nonzero slope. We are forced to take the edge which passes through $(0, 1)$ and $(4, 0)$.

4. The point $(0, 1)$ corresponds to the term $-2y$ and the point $(4, 0)$ corresponds to the term $\frac{25}{4}x^4$. Setting the sum of these monomials to zero yields

$$-2y - \frac{25}{4}x^4 \quad \Rightarrow \quad y = -\frac{25}{8}x^4.$$

Thus far we have determined that $y = 1 - 3/2x^2 - 25/8x^4$ are the first three terms of the power series solution. Graphically we have:

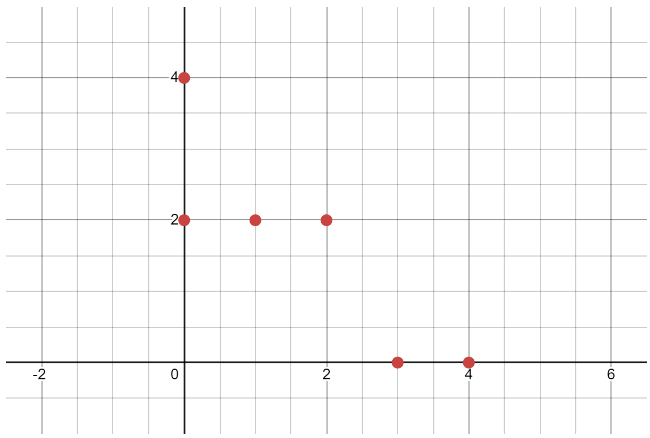


This algorithm also works for fractional power series. We will demonstrate this with our second example.

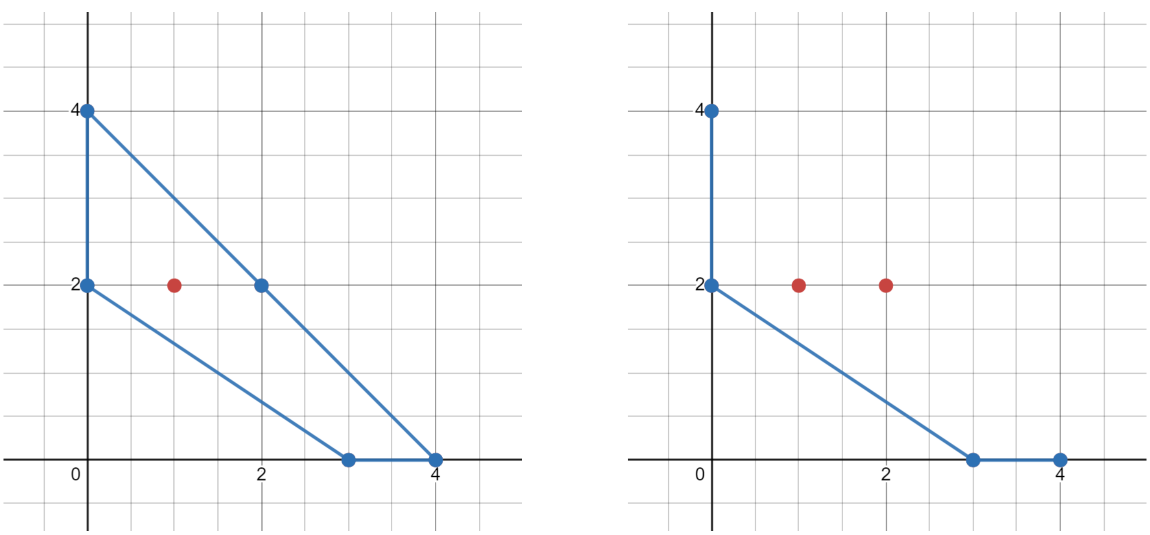
Example 3.1.4. We consider the curve in Example 1.2.17

$$F(x,y) = 2x\left(x^2+y^2\right)+\left(x^2+y^2\right)^2-y^2.$$

1. We compute $\Delta(F) = \{(3,0),(4,0),(0,2),(1,2),(2,2),(0,4)\}$ and provide its graph.



2. Next we compute the boundary of the convex hull.



(a) The boundary of the convex hull

(b) The boundary of the lower hull

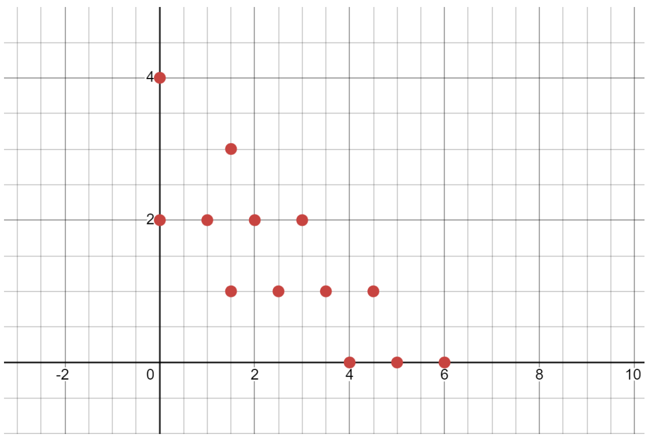
3. We are forced to choose the edge which passes through $(0,2)$ and $(3,0)$.

4. The point $(0,2)$ corresponds to $-y^2$ and the point $(3,0)$ corresponds to $2x^3$. Setting the sum of these monomials to zero yields

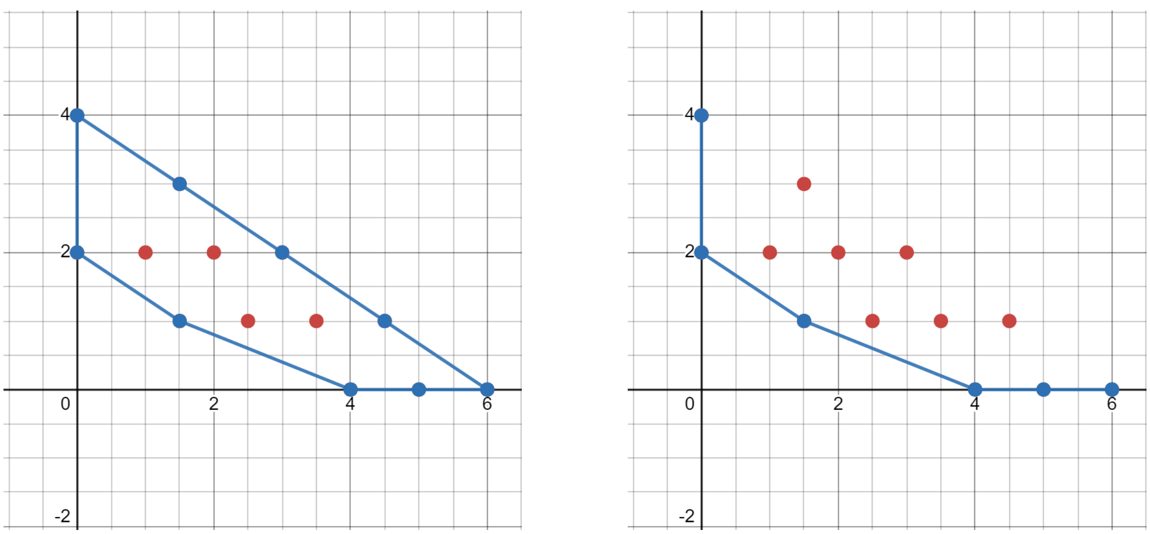
$$-y^2 + 2x^3 = 0 \quad \Rightarrow \quad y = \pm\sqrt{2}x^{3/2}.$$

Now we repeat the algorithm with $F(x, y + \sqrt{2}x^{3/2})$:

1. We compute $\Delta(F) = \{(4,0),(5,0),(6,0),(3/2,1),(5/2,1),(7/2,1),(9/2,1),(0,2),(1,2),(2,2),(3,2),(3/2,3),(0,4)\}$ and provide its graph.



2. Next we compute the boundary of the convex hull.



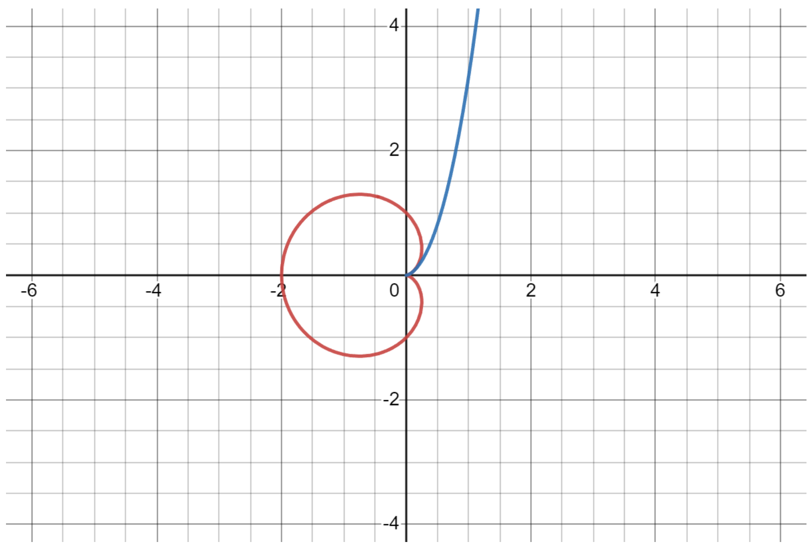
(a) The boundary of the convex hull (b) The boundary of the lower hull

3. Picking the edge which passes through $(0,2), (\frac{3}{2}, 1)$ leads us back to the negative Puiseux solution. Instead, we choose the second edge which passes through $(\frac{3}{2}, 1), (4, 0)$.

4. The point $(\frac{3}{2}, 1)$ corresponds to $2\sqrt{2}x^{3/2}y$ and the point $(4, 0)$ corresponds to $-5x^4$. Setting the sum of these monomials to zero yields

$$-5x^4 + 2\sqrt{2}x^{3/2}y = 0 \quad \Rightarrow \quad y = \frac{5x^{5/2}}{2\sqrt{2}}.$$

So we have that $y = \sqrt{2}x^{3/2} + \frac{5x^{5/2}}{2\sqrt{2}}$ are the first two terms of the power series solution. Graphically we have:



It has been shown in [4] that every solution to $F(x, y) = 0$ can be found by using this algorithm.

Chapter 4

Radius of Convergence

4.1 Radii of Convergence for Algebraic Power Series

In this section, we investigate the radii of convergence of algebraic power series. Recall that $\alpha(x)$ is an algebraic power series if there exists $F(x, y) \in R[x][y]$ such that $F(x, \alpha(x)) = 0$. We write

$$F(x, y) = f_n(x)y^n + f_{n-1}(x)y^{n-1} + \cdots + f_1(x)y + f_0(x),$$

where $f_n(x) \neq 0$, and we denote the partial derivative of F with respect to x and y as $F_x(x, y)$ and $F_y(x, y)$ respectively. We first establish that the radius of convergence for an algebraic power series is indeed an algebraic number. In order to show this, we recall the resultant of two polynomials and the discriminate of polynomial. We refer the reader to [5] and [7] for more details. Let R denote an integral domain.

Definition 4.1.1. The resultant of $f(x) = \sum_{i=0}^{n} a_i x^i$ and $g(x) = \sum_{i=0}^{m} b_i x^i$ in $R[x]$ is the determinant of the Sylvester matrix:

$$\begin{pmatrix} a_n & & & & b_m & & & \\ a_{n-1} & a_n & & & b_{m-1} & b_m & & \\ a_{n-2} & a_{n-1} & \ddots & & b_{m-2} & b_{m-1} & \ddots & \\ \vdots & a_2 & \ddots & a_n & \vdots & b_{m-2} & \ddots & b_m \\ a_0 & \vdots & \ddots & a_{n-1} & b_0 & \vdots & \ddots & b_{m-1} \\ & a_0 & & a_{n-2} & & b_0 & & b_{m-2} \\ & & \ddots & \vdots & & & \ddots & \vdots \\ & & & a_0 & & & & b_0 \end{pmatrix}$$

We denote the result of f and g by $\text{Res}(f, g)$.

The resultant has many useful properties [7]. One property we highlight is that $\text{Res}(f, g) = 0$ if and only if f and g have a common root in $\mathbb{C}$.

Example 4.1.2. The resultant of $f(x) = 1 + 2x + x^2$ and $g(x) = 3 + 3x$ is

$$\text{Res}(f, g) = \begin{vmatrix} 1 & 3 & 0 \\ 2 & 3 & 3 \\ 1 & 0 & 3 \end{vmatrix} = 1 \cdot \begin{vmatrix} 3 & 3 \\ 0 & 3 \end{vmatrix} - 3 \cdot \begin{vmatrix} 2 & 3 \\ 1 & 3 \end{vmatrix} = 9 - 3(6 - 3) = 9 - 9 = 0$$

This shows that f and g have a common root. Upon inspection, one sees that $x = -1$ is a shared root of f and g.

The resultant of $f(x) = 2 + x$ and $g(x) = x - 5$ is

$$\text{Res}(f, g) = \begin{vmatrix} 1 & 1 \\ 2 & -5 \end{vmatrix} = -5 - 2 = -7.$$

When working with polynomials in two variables, we specify which indeterminate is viewed as the variable by adding a subscript and treating the other variable as a coefficient.

Example 4.1.3. Let $F(x,y) = x^2y^2 + 1$ and $G(x,y) = (x+1)y$. Then

$$\text{Res}_y(F,G) = \begin{vmatrix} x^2 & x+1 & 0 \\ 0 & 0 & x+1 \\ 1 & 0 & 0 \end{vmatrix} = x^2 \cdot \begin{vmatrix} 0 & x+1 \\ 0 & 0 \end{vmatrix} - (x+1) \cdot \begin{vmatrix} 0 & x+1 \\ 1 & 0 \end{vmatrix} = 0 + (x+1)^2 = (x+1)^2$$

and

$$\text{Res}_x(F,G) = \begin{vmatrix} y^2 & y & 0 \\ 0 & y & y \\ 1 & 0 & y \end{vmatrix} = y^2 \cdot \begin{vmatrix} y & y \\ 0 & y \end{vmatrix} - y \cdot \begin{vmatrix} 0 & y \\ 1 & y \end{vmatrix} = y^4 + y^2$$

With the resultant defined we can now define the discriminant [14].

Definition 4.1.4. Given a polynomial $f(x) = a_0 + a_1x + \cdots + a_nx^n$, the discriminant of f is

$$\text{Disc}(f) = \frac{(-1)^{n(n-1)/2}}{a_n}\text{Res}(f, f').$$

Because the discriminant is defined via the resultant, it inherits the property that $\text{Disc}(f) = 0$ if and only if f and f' have a common root.

Observation 4.1.5. An equivalent condition that f and f' have a common root is that f has a repeated root. To see this, let $\deg(f) = d$ and denote the roots of f as r_i, $i = 1, \ldots, d$. Then we have that

$$f(x) = \prod_{i=1}^{d}(x - r_i) \quad \text{and} \quad f'(x) = \sum_{i=1}^{d} \prod_{\substack{1 \le j \le d \\ j \ne i}} (x - r_j).$$

Hence,

$$\begin{aligned} f(x) \text{ and } f'(x) \text{ have a repeated root } r_i &\iff f'(r_i) = \prod_{\substack{1 \le j \le d \\ j \ne i}} (r_i - r_j) = 0 \\ &\iff r_j = r_i \text{ for some } j \ne i \\ &\iff r_i \text{ has multiplicity at least 2.} \end{aligned}$$

Example 4.1.6. The derivative of $f(x) = ax^2 + bx + c$ is $f'(x) = 2ax + b$, and then

$$\operatorname{Disc}(f) = \frac{(-1)^3}{a} \begin{vmatrix} a & 2a & 0 \\ b & b & 2a \\ c & 0 & b \end{vmatrix} = \frac{-1}{a} \left(a \cdot \begin{vmatrix} b & 2a \\ 0 & b \end{vmatrix} - 2a \cdot \begin{vmatrix} b & 2a \\ c & b \end{vmatrix} \right) = \frac{-1}{a}(ab^2 - (2ab^2 - 4a^2c)) = b^2 - 4ac$$

Example 4.1.7. Let $F(x, y) = y^2 - x^2 + x^3$. Then $F_x(x, y) = -2x + 3x^2$ and $F_y(x, y) = 2y$. We have

$$\operatorname{Disc}_y(F) = \frac{(-1)^3}{1} \begin{vmatrix} 1 & 2 & 0 \\ 0 & 0 & 2 \\ -x^2 + x^3 & 0 & 0 \end{vmatrix} = -1(-2(0 - 2(-x^2 + x^3))) = -4(-x^2 + x^3)$$

$$\operatorname{Disc}_x(F) = \frac{(-1)^{10}}{1} \begin{vmatrix} 1 & 0 & 3 & 0 & 0 \\ -1 & 1 & -2 & 3 & 0 \\ 0 & -1 & 0 & -2 & 3 \\ y^2 & 0 & 0 & 0 & -2 \\ 0 & y^2 & 0 & 0 & 0 \end{vmatrix} = -y^2(-4 + 27y^2).$$

Remark 4.1.8. Given a bivariate polynomial $F(x, y) \in R[x][y]$ with $\deg_y(F) = n$ and $\deg_x(F) = m$, $\deg(\operatorname{Disc}_y(F)) \leq 2m(n - 1)$. We write

$$F(x, y) = f_n(x)y^n + f_{n-1}(x)y^{n-1} + \cdots + f_1(x)y + f_0(x).$$

The degree of the discriminant being bounded by $2m(n - 1)$ follows from the fact that the resultant comes from taking the determinant of the $2n-1 \times 2n-1$ matrix

$$\begin{pmatrix}
f_n(x) & & & & nf_n(x) & & & \\
f_{n-1}(x) & f_n(x) & & & (n-1)f_{n-1}(x) & nf_n(x) & & \\
f_{n-2}(x) & f_{n-1}(x) & \ddots & & (n-2)f_{n-2}(x) & (n-1)f_{n-1}(x) & \ddots & \\
\vdots & f_2(x) & \ddots & f_n(x) & \vdots & (n-2)f_{n-2}(x) & \ddots & nf_n(x) \\
f_0(x) & \vdots & \ddots & f_{n-1}(x) & f_1(x) & \vdots & \ddots & (n-1)f_{n-1}(x) \\
 & f_0(x) & & f_{n-2}(x) & & f_1(x) & & (n-2)f_{n-2}(x) \\
 & & \ddots & \vdots & & & \ddots & \vdots \\
 & & & f_0(x) & & & & f_1(x)
\end{pmatrix}$$

By writing

$$\operatorname{Res}(F, F_y) = \sum_{\pi \in S_{2n-1}} \operatorname{sign}(\pi) a_{1,\pi(1)}, a_{2,\pi(2)} \dots a_{n,\pi(n)}$$

We see that $f_n(x)$ must divide every term of the sum of $\operatorname{Res}(F, F_y)$. Since the highest degree on any $f_i(x)$ is m, the largest degree from the determinant is $m(2n-2) + \deg(f_n)$. Upon dividing the discriminant by $f_n(x)$, the highest possible degree of $\operatorname{Disc}_y(F)$ is $2m(n-1)$.

With the definition of discriminant established, we return to showing that the radius of convergence for an algebraic power series is an algebraic number. Every algebraic power series is a root of a bivariate polynomial of the form

$$F(x, y) = f_n(x)y^n + f_{n-1}(x)y^{n-1} + \cdots + f_1(x)y + f_0(x).$$

The singularities of $F(x, y)$ are of three types [1], [13]:

1. roots of $f_n(x)$

2. $x_0 \in \mathbb{C}$ such that $F(x_0, y)$ has a root of multiplicity greater than one.

3. points at infinity.

We have already established a necessary and sufficient condition for an algebraic power series to have an infinite radius of convergence, so we will consider algebraic power series which are not a polynomials. Thus, we only focus on the first two scenarios. The singularities from the first two scenarios will mark potential radii of convergence for algebraic power series [13]. The first are all the roots of $f_n(x)$. If $F(x, y)$ has a repeated root, then $\text{Disc}_y(F) = 0$. By Remark 4.1.8, there are only finitely many x-values that satisfy this condition. Because the singularities mark the potential radii of convergence and they are either roots of $f_n(x)$ or $\text{Disc}_y(F)$, the radius of convergence must be an algebraic number.

4.2 The $\mathcal{C}$-set

It is natural to ask if every algebraic number appears as the radius of convergence of an algebraic power series. The question was asked under a different context by Banderier and Drmota in [2]. In that paper, the authors conjecture this to be true when working over the rational numbers. In order to establish which algebraic numbers appear as radii of convergence of algebraic power series, we first establish some definitions and observations. Throughout, let F be a field.

Definition 4.2.1. A field F is said to have characteristic n if n is the smallest positive integer such that $n1_F = 0$. If no such n exists, the field is said to have characteristic 0.

Definition 4.2.2. The *splitting field* of $f(x) \in F[x]$ is the smallest extension field L of F such that $f(x)$ splits into linear factors, i.e.,

$$f(x) = a(x - \alpha_1)(x - \alpha_2) \dots (x - \alpha_n) \text{ where } a \in F,\ \alpha_i \in L.$$

Definition 4.2.3. A nonconstant irreducible polynomial $f(x) \in R[x]$ is *separable* if its factorization in a splitting field has distinct linear factors.

A field extension $F \subseteq L$ is *separable* if for every $\alpha \in L$, the minimal polynomial of α is separable over F.

Definition 4.2.4. Let $F \subset L$ be a field extension. The *separable closure* of F in L is the set

$$\{\alpha \in L \mid \alpha \text{ is separable over } F\}.$$

The separable closure of F is the separable closure of F in its algebraic closure.

Remark 4.2.5. When working over a field of characteristic zero, every algebraic extension of F is separable [14]. Therefore, the algebraic closure and the separable closure are equal.

Definition 4.2.6. The *absolute Galois closure* of a field K is the set of all automorphisms of the separable closure of K that fix K.

Let $F \subset E$ be a field extension. Fraleigh showed in [10] that every isomorphism on E can be lifted to an isomorphism on the algebraic closure of F. In particular, if E is a splitting field, then every automorphism of E leaving F fixed can be extended to an automorphism of $\overline{F}$ which leaves F fixed.

In this paper, we are only concerned with the absolute Galois closure of $\mathbb{Q}$ denoted $G := \mathrm{Gal}(\mathbb{A}/\mathbb{Q})$. Since $\mathbb{Q}$ has characteristic zero, by Remark 4.2.5 its separable closure is the algebraic closure of $\mathbb{Q}$. We note that any automorphism σ in a Galois group can be lifted to an automorphism $\hat{\sigma} \in G$ [10].

Let $a \in \mathbb{A}$. Throughout the rest of this paper, we denote the minimal polynomial of a over $\mathbb{Q}$ by f_a, and we the set of roots by R_a, i.e., $R_a = \{r \in \mathbb{C} | f_a(r) = 0\}$. Every field automorphism that fixes the coefficient field permutes the roots of a polynomial, because if $\sigma \in \mathrm{Gal}(f_a)$,

$$f(\sigma(a)) = \sum_{i=0}^{n} a_i \sigma(a)^i = \sum_{i=0}^{n} a_i \sigma(a^i) = \sum_{i=0}^{n} \sigma(a_i a^i) = \sigma\left(\sum_{i=0}^{n} a_i a^i\right) = \sigma(f(a)) = \sigma(0) = 0.$$

Hence, there is a natural action of $\mathrm{Gal}(K/\mathbb{Q})$ on the set of roots of any $f(x) \in \mathbb{Q}[x]$ with $\mathbb{Q} \subset K \subset \mathbb{A}$. This action is transitive, i.e.,

$$R_a = \mathrm{orb}_G(a) := \{g(a) \mid g \in G\}.$$

We are interested in the set of algebraic numbers that have minimal modulus over all common roots of their minimal polynomial. Formally, this set is

$$\mathcal{C} := \{a \in \mathbb{A} : |a| = \min_{r \in R_a} |r|\},$$

and we call $\mathcal{C}$ the $\mathcal{C}$-set. Another set of interest is the modulus of every element of $\mathcal{C}$, i.e.,

$$\mathrm{mod}(\mathcal{C}) := \{|a| : a \in \mathcal{C}\}.$$

It is not obvious from the definition that $\mathrm{mod}(\mathcal{C})$ is a proper subset of $\mathrm{mod}(\mathbb{A})$. We prove this claim now.

Theorem 4.2.7. If $a \in \mathcal{C}$, then $|a| \in \mathcal{C}$.

Proof. Let $a \in \mathcal{C}$. We first consider the cause of $a \in \mathbb{R}$. If $a > 0$ we are done. If $a < 0$, then $|a| = -a > 0$. Since $-a$ is a root of $f_a(-x)$ and the roots of $f_a(-x)$ and $f_a(x)$ have the same moduli, $|a| \in \mathcal{C}$. Next assume $a \notin \mathbb{R}$, and let $\alpha = |a|$. We first note that α is algebraic. Suppose that $\alpha \notin \mathcal{C}$. Then there exists $\beta \in R_\alpha$ such that $|\beta| < \alpha$. Since f_α is irreducible, there exists $g \in G$ such that $g(\alpha) = \beta$. Writing $a = \alpha e^{i\theta}$ and $\overline{a} = \alpha e^{-i\theta}$, we have

$$g(a) = g(\alpha e^{i\theta}) = g(\alpha)g(e^{i\theta}) = \beta g(e^{i\theta})$$
$$g(\overline{a}) = g(\alpha e^{-i\theta}) = \frac{g(\alpha)}{g(e^{i\theta})} = \frac{\beta}{g(e^{i\theta})}.$$

Taking the modulus yields

$$|g(a)| = |g(\alpha e^{i\theta})| = |g(\alpha)||g(e^{i\theta})| = |\beta||g(e^{i\theta})|$$
$$|g(\overline{a})| = |g(\alpha e^{-i\theta})| = |\frac{g(\alpha)}{g(e^{i\theta})}| = \frac{|\beta|}{|g(e^{i\theta})|}.$$

To reach our conclusion, we simply look at the modulus of $g(e^{i\theta})$. If $|g(e^{i\theta})| \geq 1$, then $|g(\overline{a})| \leq |\beta| < \alpha$. If $|g(e^{i\theta})| < 1$, then $|g(a)| < |\beta| < \alpha$. Observing that $g(a), g(\overline{a}) \in R_a$, we conclude that R_a contains an element of modulus less than α. Therefore $a \notin \mathcal{C}$. $\square$

Remark 4.2.8. An immediate consequence of Theorem 4.2.7 is that

$$\mathrm{mod}(\mathcal{C}) \subsetneq \mathrm{mod}(\mathbb{A}).$$

For example, note that $f(x) = x^2 - 2x - 1$ is irreducible and has roots $1 + \sqrt{2}, 1 - \sqrt{2}$. Since $|1 - \sqrt{2}| < |1 + \sqrt{2}|$, by the previous theorem, there does not exist any polynomial with the smallest root having modulus $|1 + \sqrt{2}|$, i.e., $(1 + \sqrt{2})e^{i\theta} \notin \mathcal{C}$ for $0 \leq \theta < 2\pi$.

Theorem 4.2.9. The $\mathcal{C}$-set is closed under multiplication.

Proof. Let $a, b \in \mathcal{C}$. Then $|a| \leq |g(a)|$ and $|b| \leq |g(b)|$ for all $g \in G$. Therefore, if $c \in R_{ab}$, then $c = g(ab)$ for some $g \in G$. Thus, we have that

$$|c| = |g(ab)| = |g(a)g(b)| = |g(a)||g(b)| \geq |a||b| = |ab|$$

and thus, $ab \in \mathcal{C}$. □

Remark 4.2.10. We can alternatively look at the set with maximum modulus, i.e.,

$$\mathcal{D} := \{a \in \mathbb{A} : \ |a| = \max_{r \in R_a} |r|\}.$$

By a similar argument, the same results hold for this set as well. That is, if $a \in \mathcal{D}$, then $|a| \in \mathcal{D}$ and $\mathcal{D}$ is closed under multiplication.

Remark 4.2.11. Let $a \in \mathcal{C}$. In order to see that there exists an algebraic power series with radius of convergence $|a|$, we simply consider

$$F(x, y) = f_a(x)y - 1 = 0.$$

Notice that $y = \frac{1}{f_a(x)} \in R_{\text{Alg}}[[x]]$ and has radius of convergence $|a|$.

If we require that $F(x, y)$ is monic, then we are working with integral power series. We can also achieve an integral power series with radius of convergence $|a|$ by considering

$$F(x, y) = y^3 - f_a(x) = 0.$$

Then $y = \sqrt[3]{f_a(x)} \in R_{\text{Int}}[[x]]$ and has radius of convergence $|a|$.

4.3 Rational Power Series and Hadamard Product

We have already established that the radius of convergence of an algebraic power series is an algebraic number. In [2], the authors conjecture which algebraic numbers appear as the radii of convergence for algebraic power series, but this remains an open problem. The authors conjecture that if $R = \mathbb{Q}$, then every positive algebraic number appears as a radius of convergence. We look at the known results so far. First, we establish some necessary machinery such as properties of rational functions and the Hadamard product.

Definition 4.3.1. A rational power series $r(x)$ is of the form $\frac{f(x)}{g(x)}$ where $f(x),\ g(x) \in R[[x]]$ and

$g(0) \in U(R)$. It satisfies

$$F(x,y) = g(x)y - f(x).$$

We begin with a well-known property of rational power series, from [27].

Theorem 4.3.2. Let $A_1, \ldots, A_d$ be a fixed sequence of complex numbers, $d \geq 1$, and $A_d \neq 0$. The following conditions on a function $f : \mathbb{N} \to \mathbb{C}$ are equivalent:

1. $$\sum_{i=0}^{\infty} f(i)x^i = \frac{P(x)}{Q(x)}$$

   where $Q(x) = 1 + A_1x + A_2x^2 + \cdots + A_dx^d$ and $P(x)$ is a polynomial of degree less than d.

2. For all $n \geq 0$,

   $$f(n+d) + A_1f(n+d-1) + A_2f(n+d-2) + \cdots + A_df(n) = 0.$$

3. For all $n \geq 0$,

   $$f(n) = \sum_{i=1}^{k} P_i(n)\gamma_i^n,$$

   where $1 + A_1x + \cdots + A_dx^d = \prod_{i=1}^{k}(1-\gamma_ix)^{d_i}$, the γ_i's are distinct and nonzero, and $P_i(n)$ is a polynomial of degree less than d_i.

The previous theorem was proven by Richard Stanley in [27] using vector spaces. We provide an algebraic proof. We only consider the case where $Q(x)$ is irreducible, i.e., for the case when $d_i = 1$ for every $i = 1, \ldots, k$. First, we recall that if $\gamma \in \mathbb{C}$ and $n \in \mathbb{N}$, then

$$\frac{1}{(1-\gamma x)^n} = \sum_{i=0}^{\infty}(-\gamma)^i\binom{-n}{i}x^i = \sum_{i=0}^{\infty}\binom{n+i-1}{n}\gamma^ix^i. \tag{4.1}$$

Remark 4.3.3. The binomial coefficient $\binom{n+i-1}{n}$ is a polynomial in i of degree n. To see this, we write

$$\binom{n+i-1}{n} = \frac{(n+i-1)!}{n!(i-1)!} = \frac{(n+i-1)(n+i-2)\ldots(n+i-(n-1))(n+i-n)(i-1)!}{n!(i-1)!}$$

$$= \frac{(n+i-1)(n+i-2)\ldots(i+1)i}{n!}.$$

Since there are n factors in the numerator, it can be viewed as a polynomial in i of degree n.

For future notational convenience, given a power series $a(x) = \sum_{i=0}^{\infty} a_i x^i$, we denote the coefficient on x^k where $k > 0$ by $[x^k] : a_k$. For example, if $a(x) = 1 + 2x^2$, then $[x^2] : 2$.

Proof. $(2 \Rightarrow 1)$ Assume that the following holds for $n \geq 0$:

$$f(n+d) + A_1 f(n+d-1) + A_2 f(n+d-2) + \cdots + A_d f(n) = 0.$$

Let $r(x) = \sum\limits_{n=0}^{\infty} f(n)x^n$. Recall that

$$Q(x) = 1 + A_1 x + A_2 x^2 + \cdots + A_d x^d,$$

and define the following polynomial

$$P(x) := \sum_{k=0}^{d-1} \left(\sum_{i+j=k} A_i f(j) x^k \right)$$

where $A_0 := 1$. Then $r(x)$ satisfies $F(x, y) = Q(x)y - P(x)$.

To see this, we write the coefficient on x^i for $Q(x)r(x)$. For $k < d$, we have that

$$[x^k] : \sum_{i+j=k} A_i f(j).$$

So the coefficient on $Q(x)r(x) - P(x)$ is

$$[x^k] : \sum_{i+j=k} A_i f(j) - \sum_{i+j=k} A_i f(j) = 0.$$

For $k \geq d$ the coefficient on $Q(x)r(x)$ is

$$[x^k] : \sum_{i=0}^{d} A_i f(k+d-i) = 0.$$

Since $\deg(P(x)) = d - 1$, $P(x)$ contributes no terms to the kth coefficient of $Q(x)r(x) - P(x)$.

Therefore, the coefficient on x^k is 0 for $k > d$. This proves that

$$Q(x)r(x) - P(x) = 0 \quad \Rightarrow \quad r(x) = \frac{P(x)}{Q(x)}.$$

$(1 \Rightarrow 3)$ Let

$$r(x) = \frac{P(x)}{Q(x)} = \sum_{i=0}^{\infty} f(i)x^i$$

be a rational power series. By partial fraction decomposition and using Equation (4.1), we write

$$r(x) = \sum_{k=1}^{\ell} \frac{b_k}{(1-\gamma_k x)^{d_k}} = \sum_{k=1}^{\ell}\sum_{i=0}^{\infty} b_k \binom{d_k+i-1}{d_k-1}\gamma_k^i x^i = \sum_{i=0}^{\infty}\sum_{k=1}^{\ell} b_k \binom{d_k+i-1}{d_k-1}\gamma_k^i x^i.$$

Using Remark 6.1.8, we let $P_k(i) = b_k\binom{d_k+i-1}{d_k-1}$, and so we have

$$r(x) = \sum_{i=0}^{\infty} a_i x^i = \sum_{i=0}^{\infty}\sum_{k=1}^{\ell} P_k(i)\gamma_k^i x^i,$$

which implies $a_i = \sum_{k=1}^{\ell} P_k(i)\gamma_k^i$.

$(3) \Rightarrow (2)$ Since $d_i = 1$ for $i = 1, \ldots, k$, $P_i(n)$ is a constant polynomial, and we write $P_i(n) = p_i$. Now we have that

$$\begin{aligned} f(n+d) + A_1 f(n+d-1) + \cdots + A_d f(n) &= \sum_{i=1}^{\ell} p_i\gamma_i^{n+d} + A_1\sum_{i=1}^{\ell} p_i\gamma_i^{n+d-1} + \cdots + A_d\sum_{i=1}^{\ell} p_i\gamma_i^{n} \\ &= \sum_{i=1}^{\ell}\left(p_i\gamma_i^{n+d} + A_1\gamma_i^{n+d-1} + \cdots + p_i A_d\gamma_i^n\right). \end{aligned} \tag{4.2}$$

Fix $1 \leq j \leq \ell$ and consider the jth index of the sum on the right-hand-side of Equation (4.2). We have that

$$p_j\gamma_j^{n+d} + A_1 p_j\gamma_j^{n+d-1} + \cdots + A_d p_j\gamma_j^n = p_j\gamma^n\left(1 + A_1\gamma_j +_d \gamma_j^d\right) = p_j\gamma^n(0) = 0.$$

As j was arbitrary, this holds for every $1 \leq j \leq \ell$ and

$$\begin{aligned} f(n+d) + A_1 f(n+d-1) + \cdots + A_d f(n) &= \sum_{i=1}^{\ell} p_i \gamma_i^{n+d} + A_1 \sum_{i=1}^{\ell} p_i \gamma_i^{n+d-1} + \cdots + A_d \sum_{i=1}^{\ell} p_i \gamma_i^{n} \\ &= \sum_{i=1}^{\ell} \left(p_i \gamma_i^{n+d} + A_1 \gamma_i^{n+d-1} + \cdots + p_i A_d \gamma_i^{n} \right) = 0. \end{aligned}$$

□

Rational power series play an important role in determining which algebraic numbers appear as a radii of convergence for algebraic power series. We introduce an operation for power series called the Hadamard product [27].

Definition 4.3.4. Let $f(x) = \sum_{i=0}^{\infty} a_i x^i$ and $g(x) = \sum_{i=0}^{\infty} b_i x^i$ be power series. The *Hadamard product* of f and g, denoted $f * g$, is

$$f * g = \sum_{i=0}^{\infty} a_i b_i x^i.$$

The radius of conbergence is multiplicative in the following sense.

Theorem 4.3.5. If $f, g \in R[[x]]$ with nonzero finite radii of convergence R_1 and R_2 respectively, then r.o.c$(f * g) = R_1 R_2$ [2].

Proof. Suppose $f(x) = \sum_{i=0}^{\infty} a_i x^i, g(x) = \sum_{i=0}^{\infty} b_i x^i \in R[[x]]$ with radii of convergence R_1 and R_2 respectively. Then

$$R_1 = \frac{1}{\limsup\limits_{n \to \infty} \sqrt[n]{|a_n|}} \quad \text{and} \quad R_2 = \frac{1}{\limsup\limits_{n \to \infty} \sqrt[n]{|b_n|}}.$$

Hence,

$$\begin{aligned}
\text{r.o.c}(f * g) &= \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|(ab))^n|}} \\
&= \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|a_n b_n|}} \\
&= \frac{1}{\limsup_{n\to\infty} \left(\sqrt[n]{|a_n|} \sqrt[n]{|b_n|} \right)} \\
&= \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|a_n|}} \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|b_n|}} \\
&= R_1 R_2.
\end{aligned}$$

□

Given two rational functions, then their Hadamard product is also a rational function [27]. The same question can be asked of algebraic power series. Unfortunately, the Hadamard product of two algebraic functions is not necessarily algebraic. The Hadamard product of an algebraic power series and a rational function, however, is algebraic [27]. To show this, we first establish some lemmas.

Lemma 4.3.6. Algebraic power series are closed under differentiation. That is, if $\alpha(x) \in R_{\text{Alg}}[[x]]$, then $\frac{d}{dx}(\alpha(x)) \in R_{\text{Alg}}[[x]]$.

Proof. Assume that $\alpha(x) \in R_{\text{Alg}}[[x]]$. Then there exists $F(x, y) \in R[x][y]$ such that $F(x, \alpha(x)) = 0$. It is easy to show that by implicit differentiation,

$$\frac{dF}{dx} = \frac{\partial F}{\partial x} + \frac{dy}{dx}\frac{\partial F}{\partial y} = 0 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{-\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}},$$

hence,

$$\left.\frac{dy}{dx}\right|_{y=\alpha(x)} = \left.\left(\frac{-\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} \right)\right|_{y=\alpha(x)}. \tag{4.3}$$

On the right-hand-side of Equation (4.3) we notice that both $-\frac{\partial F}{\partial x}|_{y=\alpha(x)}$ and $\frac{\partial F}{\partial y}|_{y=\alpha(x)}$ involve terms of the form $x^n \alpha(x)^m$, where $n, m \in \mathbb{N}_0$. Since algebraic power series are closed under multiplication and addition, $-\frac{\partial F}{\partial x}|_{y=\alpha(x)}$ and $\frac{\partial F}{\partial y}|_{y=\alpha(x)}$ are algebraic power series themselves. It is not the case that algebraic power series are closed under division, however, we can consider $\frac{dy}{dx}|_{y=\alpha(x)}$ in its quotient field $R[[x]][\frac{1}{x}]$. Since we are working over a field,

$$\left.\left(\frac{-\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}\right)\right|_{y=\alpha(x)} \in R[[x]][\tfrac{1}{x}]$$

and hence $\frac{dy}{dx}|_{y=\alpha(x)} \in R[[x]][\frac{1}{x}]$. Therefore, $\frac{d}{dx}(\alpha(x))$ is an algebraic Laurent series and hence it is a root of some $G(x,y) \in R(x)[y]$. This is equivalent to $\frac{dy}{dx}|_{y=\alpha(x)}$ satisfying a bivariate polynomial in $R[x][y]$, say $H(x,y)$. To see this, simply multiply G by the least common multiple of all denominators of its coefficients. We also know

$$\frac{d}{dx}(\alpha(x)) = \sum_{n=1}^{\infty} n a_n x^{n-1},$$

which is a power series. Therefore, $\frac{d}{dx}(\alpha(x)) \in R_{\text{Alg}}[[x]]$. □

Remark 4.3.7. We point out that algebraic power series are not necessarily closed under integration. The simplest example is the geometric series

$$f(x) = \frac{1}{1-x}.$$

We have that,

$$\int f(x)\ dx = \int \frac{1}{1-x}dx = -\ln(|1-x|) + C.$$

As the natural logarithm is not algebraic, $\int f(x)dx \notin R_{\text{Alg}}[[x]]$. Notice that $h(x) = -\ln(1-x)$ does satisfy the differential equation

$$(1-x)y' - 1 = 0.$$

Any type of function which satisfies a differential equation in y with coefficients in $R[x]$ is said to be D-finite. D-finite series are closed under integration. For more on the topic of D-finite series, we refer the reader to [20], [21], and [27].

It is worth noting that some algebraic power series are closed under integration. For example, let $g(x) = -\sqrt{1-x}$. For simplicity we will assume $C = 0$. Integrating $g(x)$ gives

$$G^{(1)}(x) := \int g(x)dx = \int -\sqrt{1-x}\ dx = \frac{1}{2\sqrt{1-x}}.$$

$G^{(1)}(x)$ satisfies

$$4(1-x)y^2 - 1 = 0.$$

Let $G^{(n)}(x)$ denote the nth integral of of $g(x)$. Then, assuming $C = 0$ for each integral,

$$G^{(n)}(x) = \frac{\prod_{i=1}^{n}(2i-1)}{2^n(1-x)^{(2n-1)/2}}$$

satisfies

$$\frac{4^n(1-x)^{2n-1}}{(\prod_{i=1}^{n}(2i-1)^2)}y^2 - 1 = 0.$$

Lemma 4.3.8. If $\alpha(x) \in R_{\text{Alg}}[[x]]$, then $\alpha(\gamma x) \in R_{\text{Alg}}[[x]]$ for $\gamma \in R$.

Proof. Suppose $\alpha(x) \in R_{\text{Alg}}[[x]]$. By definition, there exists $F(x,y) \in R[x][y]$ such that $F(x, \alpha(x)) = 0$. One can easily verify that $F(\gamma x, \alpha(\gamma x)) = 0$. Clearly $\alpha(\gamma x) \in R[[x]]$ and thus $\alpha(\gamma x) \in R_{\text{Alg}}[[x]]$. □

Remark 4.3.9. If the radius of convergence for $a(x)$ is $0 < R < \infty$, then the radius of convergence of $a(\gamma x)$ is $\frac{R}{|\gamma|}$.

We are now ready to prove our result that the Hadamard product between an algebraic power series and a rational function is an algebraic power series. This proof follows the one given by Stanley in [27], but we will expand on some concepts.

Theorem 4.3.10. If $\alpha(x) \in R_{\text{Alg}}[[x]]$ and $r(x)$ is a rational power series, then $(a * r)(x) \in R_{\text{Alg}}[[x]]$.

Proof. Write $\alpha(x) = \sum\limits_{i=0}^{\infty} a_i x^i$, and $r(x) = \sum\limits_{i=0}^{\infty} r_i x^i$. By Theorem 4.3.2, there exists polynomials $P_i(x)$ and $\gamma_1, \gamma_2, \ldots, \gamma_\ell \in \mathbb{C}$ such that

$$r_k = \sum_{i=1}^{\ell} P_i(k)\gamma_i^k. \tag{4.4}$$

Let $P_s(n) = p_d n^d + p_{d-1}n^{d-1} + \cdots + p_1 n + p_0$, where $1 \le s \le \ell$. By Lemma 4.3.6, nx^{n-1} is an algebraic power series and therefore so is $x \cdot nx^{n-1} = nx^n$. Consequently, $n^k x^n$ is an algebraic power series for any $k \in \mathbb{N}_0$. Therefore,

$$\sum_{n=0}^{\infty} a_n P_s(n) x^n = \sum_{n=0}^{\infty} a_n \left(p_d n^d x^n + p_{d-1} n^{d-1} x^n + \cdots + p_1 n x^n + p_0 x^n\right)$$

is an algebraic power series. By Lemma 4.3.8,

$$\sum_{n=0}^{\infty} a_n P_s(n)\gamma_s^n x^n$$

is an algebraic power series. Since s is arbitrary, we have that

$$\sum_{i=1}^{\ell} a_n P_i(n)\gamma_i^n x^n$$

is a finite sum of algebraic power series and hence is algebraic. Therefore $(a * r)(x) \in R_{\text{Alg}}[[x]]$. $\square$

This theorem shows that rational power series play a crucial role in the radii of convergence of algebraic power series. It is worth noting that Remark 4.2.11 tells us that there exists rational power series with radii of convergence $|c|$ for any $c \in \mathcal{C}$. In fact, $\text{mod}(\mathcal{C})$ could alternatively be defined this way. We state the following without proof.

Theorem 4.3.11. We have that

$$\left\{ R \mid r(x) = \frac{f(x)}{g(x)} \in \mathbb{Q}(x),\ g(0) \neq 0,\ \text{and r.o.c}(r(x)) = R < \infty \right\} = \text{mod}(\mathcal{C})$$

We now summarize the significance of the previous theorems with regards to their impact on the possible radii of convergence an algebraic power series may have.

Theorem 4.3.12. Given an algebraic power series with radius of convergence R, there also exists algebraic power series of radius of convergence of the following:

1. $R^{1/n}$ for $n \in \mathbb{N}$.

2. $|c|R$ where $c \in \mathcal{C}$.

Proof. (1) Suppose $\alpha(x) \in R_{\text{Alg}}[[x]]$ with radius of convergence $R > 0$. Then there exists $F(x, y) \in R[x][y]$ such that $F(x, \alpha(x)) = 0$. Clearly,

$$F(x^n, a(x^n)) = 0 \quad \Rightarrow \quad a(x^n) \in R_{\text{Alg}}[[x]]$$

and $a(x^n)$ has radius of convergence $R^{1/n}$.

(2) Let $\alpha(x) \in R_{\text{Alg}}[[x]]$ with r.o.c$(a) = R$. By Theorem 4.3.12, $(a * r)(x) \in R_{\text{Alg}}[[x]]$ for any rational power series $r(x)$. By Remark 4.2.11, there exists a rational power series of radius of convergence $|c|$ for any $c \in \mathcal{C}$. Using Theorem 4.3.10, we have that r.o.c$(a * r)(x) = |c|R$. □

We have already seen examples of finding an algebraic (or integral) power series with radius of convergence $|c|$, where $c \in \mathcal{C}$ in Remark 4.2.11. It is difficult to find an algebraic power series with radius of convergence $|c|$ where $c \notin \mathcal{C}$. The authors of [2] attempt to do this for quadratic irrational numbers. They provide a proof that all positive quadratic irrational numbers are radii of an algebraic power series. Unfortunately, the proof contains an error which we point out.

Remark 4.3.13. We point out an error in [2, Theorem 7.4 (iii)]. The authors state that all positive quadratic irrational numbers are radii of an algebraic power series. In the proof they incorrectly factor

$$\alpha + \beta\sqrt{m} = \frac{1}{\alpha^2 - \beta^2 m}(\alpha - \beta\sqrt{m}).$$

Observation 4.3.14. Although an error was found in the proof of [2], we provide numerical evidence to support the claim. For example, if one considers,

$$F(x, y) = x + (y)(y - 2)(y - 3),$$

then the roots of $\text{Disc}_y(F)$ are

$$x = \frac{2(-10 \pm 7\sqrt{7})}{27} \approx -2.113, 0.631.$$

By the implicit function theorem, all three roots of $F(x, y)$ are power series. If we consider the power series expansion at $y = 0$, then using Mathematica to estimate $\frac{1}{|a_n|^{1/n}}$ for $n = 1000$ yields an approximate radius of convergence of 2.13594, suggesting that the radius of convergence is $\frac{2(10+7\sqrt{7})}{27}$.

By changing the y-intercepts in the example above, we can get numerical evidence for other quadratic numbers with larger magnitude appearing as radii of convergence of algebraic power series.

Chapter 5

Characterization of Algebraic Power Series

5.1 Algebraic Power Series

In this chapter we discuss an equivalent condition of a power series being algebraic. Let $\alpha(x) \in \mathbb{Q}[[x]]$ be a convergent power series with domain of convergence $D \subseteq \mathbb{C}$. If $\alpha(x)$ is algebraic, then there exist polynomials $f_i \in \mathbb{Q}[x]$ where $i = 1, \ldots, n$ such that

$$f_n(x)\alpha(x)^n + \cdots + f_1(x)\alpha(x) + f_0(x) = 0. \tag{5.1}$$

We would like to characterize this property in terms of the structure of the image of $\alpha(x)$. At first, we notice that Equation (5.1) implies that $\alpha(a)$ is algebraic for every algebraic number a in D. If this were not so, then $\alpha(a)$ would be transcendental, however, plugging in a into Equation (5.1) yields

$$f_n(a)\alpha(a)^n + \cdots + f_1(a)\alpha(a) + f_0(a) = 0. \tag{5.2}$$

This implies that $\alpha(a)$ is algebraic over the algebraic extension $\mathbb{Q}(a)$, a contradiction. Since the algebraic numbers are dense in $\mathbb{C}$, then it is natural to ask if this condition is enough to imply that $\alpha(x)$ is algebraic. Unfortunately, we find that $\alpha(a)$ being algebraic for every $a \in D \cap \mathbb{A}$ is not enough to imply that $\alpha(x)$ is an algebraic power series. In [22], the authors prove that for every

$\rho \in (0, \infty]$, there exists uncountably many transcendental and analytic functions $\beta \in \mathbb{Q}[[x]]$ with radii of convergence ρ such that $\beta(a) \in \mathbb{A}$ for all algebraic numbers a in the domain of convergence of β. This is due to the fact that although $\mathbb{A}$ is dense in $\mathbb{C}$, they only form a countable set.

Instead, we notice that Equation (5.2) holds for all $a \in D$. In other words, $\alpha(a)$ is algebraic over $\mathbb{Q}(a)$. This condition is sufficient for $\alpha(x)$ to be algebraic.

Theorem 5.1.1. Let $\alpha(x) \in \mathbb{C}[[x]]$ be a power series with radius of convergence $0 < R \leq \infty$ and let L be a countable subfield of $\mathbb{C}$. Let D be the domain of convergence of $\alpha(x)$. The following conditions are equivalent.

1. $\alpha(x)$ is algebraic over $L[x]$.
2. For all $a \in D$, $\alpha(a)$ is algebraic over $L(a)$.
3. There exists $N \in \mathbb{N}$ such that for all $a \in D$, $\alpha(a)$ is algebraic over $L(a)$ of degree at most N.

Before proving this theorem, we establish the definitions of an isolated point and a discrete set as well as two lemmas from complex analysis [19].

Definition 5.1.2. Let $S \subset \mathbb{C}$ be a set of points and $z_0 \in S$. The point z_0 is isolated if there exists $\epsilon > 0$ such that $|z - z_0| < \epsilon$ does not contain any other points of S. We say that S is discrete if every point of S is isolated.

Lemma 5.1.3. [19, Theorem 1.2 pg. 91] Let U be a connected open set. If f is analytic on U and not constant, then the set of zeros of f is discrete.

Lemma 5.1.4. Any discrete set of points of $\mathbb{C}$ is countable.

Proof. Let S be a discrete set of points of $\mathbb{C}$. Then for $z_i \in S$, there exists $\epsilon_i > 0$ such that $|z - z_i| < \epsilon_i$ contains no other point of S. Since $\mathbb{Q}$ is dense in $\mathbb{R}$ we select $(a_i, b_i) \in \mathbb{Q} \times \mathbb{Q}$ such that $a_i \in (\text{Re}(z_i) - \frac{\epsilon_i}{2}, \text{Re}(z_i) + \frac{\epsilon_i}{2})$ and $b_i \in (\text{Im}(z_i) - \frac{\epsilon_i}{2}, \text{Im}(z_i) + \frac{\epsilon_i}{2})$. Thus (a_i, b_i) is in the open disc $|z - z_i| < \epsilon$. We associate the point z_i with (a_i, b_i). Therefore, every point in S is associated with a distinct point $(a_i, b_i) \in \mathbb{Q} \times \mathbb{Q}$. Since $\mathbb{Q} \times \mathbb{Q}$ is countable, then so is S. □

We now provide the proof of Theorem 5.1.1.

Proof. (1) $\Rightarrow$ (3). Assume $f(x)$ is algebraic over $L[x]$. By definition, there exists polynomials $f_n(x), \ldots, f_0(x) \in L[x]$ with $f_n(x) \neq 0$ such that

$$f_n(x)\alpha(x)^n + \cdots + f_1(x)\alpha(x) + f_0(x) = 0.$$

Given $a \in D$,

$$f_n(a)\alpha(a)^n + \cdots + f_1(a)\alpha(a) + f_0(a) = 0$$

which implies that $\alpha(a)$ is algebraic over $L(a)$ of degree at most $n \in \mathbb{N}$.

(3) $\Rightarrow$ (2) This implication is trivial.

(2) $\Rightarrow$ (1) Assume that $\alpha(a)$ is algebraic over $L(a)$ for all $a \in D$. Define the function $\phi : L[x][y] \to D$ by

$$\phi(F(x,y)) = \{a \in D \mid F(a, f(a)) = 0\}.$$

Since L is countable, $L[x][y]$ is also countable. We note that D is uncountable. It must follow that there exists at least one $F(x,y) \in L[x][y]$ such that $\phi(F(x,y))$ is uncountable. If this were not the case, then $\phi(F(x,y))$ would be countable for all $F(x,y) \in L[x][y]$ and

$$D = \bigcup_{F(x,y) \in L[x][y]} \phi(F(x,y)).$$

As the countable union of countable sets is countable, D is countable, giving us the desired contradiction.

Choose any $F(x,y) \in L[x][y]$ such that $\phi(F(x,y))$ is uncountable. Write

$$F(x,y) = f_n(x)y^n + \cdots + f_1(x)y + f_0(x).$$

The function

$$f(x) := f_n(x)\alpha(x)^n + \cdots + f_1(x)\alpha(x) + f_0(x)$$

is analytic in D. For every $a \in \phi(F(x,y))$ we have that

$$f(a) = f_n(a)\alpha(a)^n + \cdots + f_1(a)\alpha(a) + f_0(a) = 0.$$

Since $f(x)$ is analytic and D is a connected open set, Lemma 5.1.3 implies that either $\phi(F(x,y))$ is discrete or $f(x)$ is constant. By Lemma 5.1.4, $\phi(F(x,y))$ is not discrete and it follows that $f(x)$ is constant. The only way this is possible is if $f(x)$ is identically zero. Therefore,

$$f(x) = f_n(x)\alpha(x)^n + \cdots + f_1(x)\alpha(x) + f_0(x) = 0$$

and $\alpha(x) \in R_{\mathrm{Alg}}[[x]]$.

$\square$

Observation 5.1.5. Given a Puiseux series $p(x) \in R\langle\langle x\rangle\rangle$, there exists $m \in \mathbb{N}$ such that $p(x^m) \in R[[x]][\frac{1}{x}]$, because every series in $R\langle\langle x\rangle\rangle$ has the form

$$p(x) = \sum_{i=-\ell}^{\infty} a_i x^{i/k}$$

for $\ell, k \in \mathbb{N}$. Letting $m = k$, we have that

$$p(x^k) = \sum_{i=-\ell}^{\infty} a_i (x^k)^{i/k} = \sum_{i=-\ell}^{\infty} a_i x^i.$$

Theorem 5.1.6. Let $p(x) \in \mathbb{C}\langle\langle x\rangle\rangle$ and L be a countable subfield of $\mathbb{C}$. By Observation 5.1.5, there exists $m \in \mathbb{N}$ such that $p(x^m)$ is a formal Laurent series. Let $h(x) = p(x^m)$. Assume that there exists $r, R > 0$ such that $r < R$ and $h(x)$ is convergent on the annular disc $r < |z| < R$. Let D be the domain of convergence of $h(x)$, and let $E = \{z \in \mathbb{C} \mid p(z) \text{ converges}\}$. The following are equivalent.

1. $p(x)$ is algebraic over $L[x]$.

2. For all $a \in E$, $p(a)$ is algebraic over $L(a)$.

3. There exists $N \in \mathbb{N}$ such that for all $a \in E$, $p(a)$ is algebraic over $L(a)$ of degree at most N.

Proof. We note that (1) $\Rightarrow$ (3) and (3) $\Rightarrow$ (2) are identical to the proof given for Theorem 5.1.1. It remains to show (2) $\Rightarrow$ (1).

(2) $\Rightarrow$ (1) Since $h(x)$ is convergent on $r < |z| < R$, $p(x)$ is convergent on $r^{\frac{1}{m}} < |z| < R^{\frac{1}{m}}$. Using the same argument and notation as in Theorem 5.1.1 for $p(x)$ on E, we choose any

$F(x,y) \in L[x][y]$ such that $\phi(F(x,y))$ is uncountable. The Laurent series

$$g(x) = F(x^k, h(x)) = f_t(x^k)(h(x))^t + \cdots + f_1(x^k)h(x) + f_0(x^k)$$

is analytic on D, and for every $a \in D$ we have that

$$g(a) = f_t(a^k)(h(a))^t + \cdots + f_1(a^k)h(a) + f_0(a^k) = 0.$$

Since D is uncountable, then $g(x)$ is identically zero and we have that

$$f_t(x^k)(h(x))^t + \cdots + f_1(x^k)h(x) + f_0(x^k) = 0.$$

Replacing x with $x^{1/k}$, we get

$$f_t(x)(p(x))^t + \cdots + f_1(x)p(x) + f_0(x) = 0.$$

Therefore $p(x)$ is algebraic. □

Bibliography

[1] Lars V Ahlfors. "Complex analysis: an introduction to the theory of analytic functions of one complex variable". In: *New York, London* 177 (1953).

[2] Cyril Banderier and Michael Drmota. "Formulae and asymptotics for coefficients of algebraic functions". In: *Combinatorics, Probability and Computing* 24.1 (2015), pp. 1–53.

[3] Eduardo Casas-Alvero et al. *Algebraic Curves, the Brill and Noether Way*. Springer, 2019.

[4] Eduardo Casas-Alvero et al. *Singularities of plane curves*. Vol. 276. Cambridge University Press, 2000.

[5] David Cox, John Little, and Donal OShea. *Ideals, varieties, and algorithms: an introduction to computational algebraic geometry and commutative algebra*. Springer Science & Business Media, 2013.

[6] David A Cox. *Galois theory*. Vol. 61. John Wiley & Sons, 2011.

[7] David A Cox, John Little, and Donal O'shea. *Using algebraic geometry*. Vol. 185. Springer Science & Business Media, 2006.

[8] David Steven Dummit and Richard M Foote. *Abstract algebra*. Vol. 3. Wiley Hoboken, 2004.

[9] Philippe Flajolet and Robert Sedgewick. *Analytic combinatorics*. cambridge University press, 2009.

[10] John B Fraleigh. *A first course in abstract algebra*. Pearson Education India, 2003.

[11] L.C. Grove. *Algebra*. Dover Books on Mathematics. Dover Publications, 2012. ISBN: 9780486142135. URL: `https://books.google.com/books?id=C4TByeUh9A4C`.

[12] Robin Hartshorne. *Algebraic geometry*. Vol. 52. Springer Science & Business Media, 2013.

[13] Einar Hille. *Analytic function theory*. Vol. 2. American Mathematical Soc., 2002.

[14] Thomas W Hungerford. *Algebra*. Vol. 73. Springer Science & Business Media, 2012.

[15] Thomas W. Hungerford. *Algebra / Thomas W. Hungerford*. English. Springer-Verlag New York [N.Y.], 1980, xxiii, 502 p. ISBN: 0387905189.

[16] Irving Kaplansky. "Commutative rings". In: *Conference on Commutative Algebra: Lawrence, Kansas 1972*. Springer. 2006, pp. 153–166.

[17] Manuel Kauers. "The concrete tetrahedron". In: *Proceedings of the 36th international symposium on Symbolic and algebraic computation*. 2011, pp. 7–8.

[18] Steven George Krantz and Harold R Parks. *The implicit function theorem: history, theory, and applications*. Springer Science & Business Media, 2002.

[19] Serge Lang. *Complex analysis*. Vol. 103. Springer Science & Business Media, 2013.

[20] Leonard Lipshitz. "D-finite power series". In: *Journal of algebra* 122.2 (1989), pp. 353–373.

[21] Leonard Lipshitz. "The diagonal of a D-finite power series is D-finite". In: *Journal of algebra* 113.2 (1988), pp. 373–378.

[22] Diego Marques and Carlos Gustavo Moreira. "A note on a complete solution of a problem posed by K. Mahler". In: *Bulletin of the Australian Mathematical Society* 98.1 (2018), pp. 60–63.

[23] Arnold Ostebee. "Topics in Number Theory: In Honor of B. Gordon and S Chowla". English. In: *The American Mathematical Monthly* 106.9 (Nov. 1999). Copyright - Copyright Mathematical Association Of America Nov 1999; Last updated - 2017-11-08; CODEN - AMMYAE, p. 876. URL: http://libproxy.clemson.edu/login?url=https://www-proquest-com.libproxy.clemson.edu/scholarly-journals/topics-number-theory-honor-b-gordon-s-chowla/docview/203725837/se-2?accountid=6167.

[24] Igor Rivin. *Galois Groups of Generic Polynomials*. 2015. arXiv: 1511.06446 [math.NT].

[25] Rubı E Rodrıguez, Irwin Kra, and Jane P Gilman. *Complex analysis: in the spirit of Lipman Bers*. Springer, 2013.

[26] Saul Stahl. *Introductory modern algebra: a historical approach*. Wiley, 2013.

[27] Richard P Stanley. "Differentiably finite power series". In: *European journal of combinatorics* 1.2 (1980), pp. 175–188.

[28] Doron Zeilberger. "A holonomic systems approach to special functions identities". In: *Journal of computational and applied mathematics* 32.3 (1990), pp. 321–368.

[29] Dennis Zill and Patrick Shanahan. *A first course in complex analysis with applications*. Jones & Bartlett Learning, 2009.