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Analysis of finite element based numerical methods for acoustic waves, elastic waves, and fluid-solid interactions in the frequency domain

Peter Cummings

To the Graduate Council:

I am submitting herewith a dissertation written by Peter Cummings entitled "Analysis of finite element based numerical methods for acoustic waves, elastic waves, and fluid-solid interactions in the frequency domain." I have examined the final electronic copy of this dissertation for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Mathematics.

Xiaobing Feng, Major Professor

We have read this dissertation and recommend its acceptance:

Ohannes Karakashian, Henry Simpson, Chris Pionke

Accepted for the Council:

Carolyn R. Hodges

Vice Provost and Dean of the Graduate School

(Original signatures are on file with official student records.)

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Analysis of Finite Element Based Numerical Methods for Acoustic Waves, Elastic Waves, and Fluid–Solid Interactions in the Frequency Domain

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ABSTRACT

The following thesis studies the acoustic wave equation, the elastic wave equations, a fluid–solid interaction problem, and their finite element approximations in the frequency domain. The focus is on how the solutions depend on the frequency ω, how the error bounds for the finite element approximations depend on the frequency ω, and how the mesh size h is constrained by the frequency ω in the finite element approximations. Particular emphasis is on results for high frequency waves.

A Rellich identity technique is used to derive an elliptic regularity estimate for the acoustic Helmholtz equation with a first order absorbing boundary condition. The estimate is optimal with respect to the frequency ω. The finite element method for the problem is formulated and analyzed. The finite element analysis leads to two main results. The first is a constraint on the mesh size h in terms of the frequency ω which is necessary to guarantee existence of finite element approximations. The second is an error bound on the finite element approximations which shows explicit ω dependence.

Analogous techniques achieve similar results for the elastic Helmholtz equations. An additional difficulty appears in the elastic case because the Lamé operator is only semi–positive definite. The difficulty is overcome first with a regularity argument, and the result is then improved with a Korn–type inequality on the boundary.

A fluid–solid interaction problem, which is described by a coupled system of acoustic and elastic Helmholtz equations, is considered next. Finite element approximations are proposed and analyzed, and optimal order error estimates are established. Parallelizable iterative algorithms are proposed for solving the corresponding finite element equations. The algorithms are based on domain decomposition methods. Strong convergence in the energy norm of the algorithms is proved.

Finally, the acoustic Helmholtz equation with a second order absorbing boundary condition is studied. Again, the finite element method is formulated and analyzed, and optimal error estimates are derived with explicit dependence on the frequency, ω.

A procedure for recovering the solution in the time domain by numerically approximating the inverse Fourier transform is formulated. The procedure is implemented for both the acoustic Helmholtz problem with the first order absorbing boundary condition, and for the acoustic Helmholtz problem with a second order absorbing boundary condition. A computational comparison of the resulting approximate solutions is given.

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Introduction

Acoustic and elastic wave equations are the governing equations of wave propagation through inviscid fluids and elastic solids. The Helmholtz equation, which is time independent, appears when one applies the Fourier transform in the time variable to the wave equation, or when one seeks time–harmonic solutions of the wave equation. An alternative method for numerically solving wave propagation problems is therefore to first solve the corresponding Helmholtz problem, and then recover the solution in the time domain by calculating the inverse Fourier transform of the solution.

This approach, known as the frequency domain approach or method, is attractive for two reasons. First, it is a parallelizable method; one can exploit parallel algorithms and computers to solve a sequence of Helmholtz problems necessary to approximate the Fourier inversion. Second, it eliminates the need for time stepping. In order to approximate solutions in the time domain, one must solve a discrete problem for all previous time steps before advancing to the next step. Furthermore, the time step cannot be chosen arbitrarily. With a frequency domain approach, however, one only needs to solve the sequence of Helmholtz problems once, and one can then calculate the solution at arbitrary times without any knowledge of the solution at previous time values.

In the following thesis, we are interested in the frequency domain approach to the study of waves, and focus on analysis of Helmholtz problems which arise from the study of wave propagation in fluid and solid media. As governing partial differential equations, we will use the scalar or acoustic Helmholtz equation

$$-\frac{\omega^2}{c^2}p - \Delta p = g_f,$$

and the elastic Helmholtz equation

$$-\omega^2 \rho_s \underset{\sim}{u} - \underset{\sim}{\operatorname{div}} \left(\underset{\approx}{\sigma}(\underset{\sim}{u}) \right) = \underset{\sim}{g_s} \ ,$$

so called because of their respective relationships to the aforementioned acoustic and elastic wave equations. We will use absorbing boundary conditions, which are meant to minimize reflections of outgoing waves, hence simulating the absence of a physical boundary. Our emphasis is on analysis of finite element based numerical solution methods. When possible, we derive elliptic regularity estimates, which bound appropriate norms of the solution of the Helmholtz equation in terms of its source with explicit dependence on the frequency, ω. It is worth noting that the variational form associated with Helmholtz problems is not coercive in general, and we therefore cannot appeal to results such as Céa's Lemma and the Lax-Milgram Theorem to reach our conclusions.

In Chapter 1, we analyze the acoustic Helmholtz equation with a first order absorbing boundary condition, a problem which has received considerable attention in the literature. In the one dimensional case, [10] establishes a regularity estimate using the Green's function representation of the solution. This paper also presents finite element analysis based on the argument of Schatz [34], analysis of the total error of approximate solutions in the time domain, and numerical experiments. [18] derives regularity estimates in the two and three dimensional cases using the fundamental solution of the Helmholtz operator. In [23], the authors derive regularity estimates for the attenuated case, analyze total error, and present results of numerical experiments in both constant and variable wave speed domains. In the two dimensional case, [28] derives a regularity estimate equivalent to that of [10]; the author uses a test function originally developed in [27].

In our analysis of the acoustic Helmholtz problem, we generalize the regularity result of [28] to both two and three dimensions using a technique based on a Rellich identity for the Laplacian. The Rellich identity for the Laplacian originally appeared in [33] for real valued functions; we derive a complex version of this identity for the acoustic Helmholtz problem. We then state the variational or weak formulation of the problem, and prove an existence/uniqueness theorem. Next, we formulate the finite element method, and derive optimal order estimates for the finite element method.

In Chapter 2, we analyze the elastic wave equation with a first order absorbing boundary condition which arises from a model for wave propagation through elastic media. Our analysis parallels that of the acoustic problem in Chapter 1. We first derive a Rellich identity for $\underset{\sim}{\operatorname{div}}\left(\underset{\approx}{\sigma}(\cdot)\right)$, which generalizes the Rellich identities found in [31] and [7] to complex case. We use the Rellich identity to derive two elliptic regularity estimates for solutions to the elastic Helmholtz problem. Our estimates improve results for elastic and nearly elastic waves in [16]. We then prove existence and uniqueness of solutions to the variational formulation, formulate the finite element method and derive optimal order finite element error estimates.

In Chapter 3, we study a coupled system of acoustic and elastic Helmholtz equations which arises from a model for wave propagation through composite fluid–solid media. We use first order absorbing boundary conditions on the outer boundary of the whole domain, and impose additional conditions on the interface to describe the interaction between the fluid and solid domains. The corresponding model in the time domain is the subject of [15] and [17]. The model is developed in [15], where existence, uniqueness, and regularity results also appear. Finite element analysis for the time dependent interaction problem appears in [17]. [27] treats the one dimensional case of the interaction problem in the frequency domain. The authors establish existence and uniqueness of solutions to the variational formulation, derive a regularity estimate for solutions, and provide finite element analysis.

In this work, we generalize the existence and uniqueness result for the coupled

Helmholtz system to the two and three dimensional cases. We also provide finite element analysis for the two and three dimensional cases, and derive optimal order finite element error estimates.

In Chapter 4, we continue our analysis of the coupled system of Helmholtz equations with the presentation and analysis of non–overlapping domain decomposition iterative methods for solving the fluid–solid interaction problem. The key to each method is to replace the physical interface conditions with equivalent relaxation conditions, a technique which is used successfully in [2, 9, 14]. We use an energy method to show convergence of the iterative methods, and provide numerical results to support the analysis. Similar results and techniques for the time dependent interaction problem appear in [17]. Chapter 4 completes an earlier work, [6], with greater detail and exposition.

We revisit the acoustic Helmholtz equation in Chapter 5. It is well known that second order absorbing boundary conditions yield smaller reflections, and we therefore consider the acoustic Helmholtz problem with a second order absorbing boundary condition. A similar analysis of the acoustic problem with a first order absorbing boundary condition appears in [23]. We formulate the finite element method, and derive error estimates. Finally we present and implement a procedure to numerically approximate solutions to the time–dependent acoustic wave equation using both the first order and the second order absorbing boundary conditions, and give a computational comparison of the performance of the first and second order absorbing boundary conditions.

Chapter 1

Acoustic Waves

Consider the unattenuated wave equation,

$$\frac{1}{c^2}P_{tt} - \Delta P = G_f \qquad x \in \Re^N, \quad t > 0 \tag{1.1}$$

where P is a real-valued function of the spatial variable $\underset{\sim}{x} \in \Re^N$ $(N = 2, 3)$ and the time variable t. Equation (1.1) describes the propagation of disturbances in an acoustic medium and solutions are commonly known as "waves." It is not feasible to approximate waves in large or infinite domains, and in practice one often truncates the computational domain by imposing an artificial boundary condition. Ideally, such a condition should "absorb" waves striking the boundary - that is, minimize reflections back into the computational domain - thus simulating the *absence* of a physical boundary. The equation

$$\frac{1}{c}\frac{\partial P}{\partial t} + \frac{\partial P}{\partial \underset{\sim}{n}_f} = 0 \tag{1.2}$$

is known as a first order absorbing boundary condition; when imposed on an artificial boundary, it absorbs outgoing waves arriving normally at the boundary. For a more complete discussion of absorbing boundary conditions, see [11].

Combining equations (1.1) and (1.2) yields the following model for computing wave propagation in an infinite or large acoustic medium

$$(1.3) \qquad \begin{cases} \frac{1}{c^2}P_{tt} - \Delta P & = g_f \quad \underset{\sim}{x} \in \Omega_f,\ t > 0, \\ \frac{1}{c}\frac{\partial P}{\partial t} + \frac{\partial P}{\partial \underset{\sim}{n}_f} & = 0 \quad \underset{\sim}{x} \in \partial\Omega_f,\ t > 0, \\ P = P_t & = 0 \quad \underset{\sim}{x} \in \Omega_f, t \leq 0, \end{cases}$$

where $\Omega_f \subset \Re^N$ is a bounded domain, and $\underset{\sim}{n}_f$ is the unit outward normal to $\partial\Omega_f$. Throughout Chapter One, we will assume that $\Omega_f \subset \Re^N$ is a bounded star-shaped domain with Lipschitz boundary. Recall that if Ω_f is star shaped, there exists a positive constant β_1, and a point $\underset{\sim}{x}_0 \in \Omega_f$ such that

$$(1.4) \qquad \beta_1 \leq (\underset{\sim}{x} - \underset{\sim}{x}_0) \cdot \underset{\sim}{n}_f \qquad \forall\ \underset{\sim}{x} \in \partial\Omega_f.$$

Assume without loss of generality that $\underset{\sim}{x}_0 = \underset{\sim}{0}$

Applying the Fourier transform to (1.3), or seeking time-harmonic solutions to (1.3), yields the following Helmholtz problem

$$(1.5) \qquad \begin{cases} -\frac{\omega^2}{c^2}p - \Delta p & = g_f \quad \text{in } \Omega_f, \\ i\frac{\omega}{c}p + \frac{\partial p}{\partial \underset{\sim}{n}_f} & = 0 \quad \text{on } \partial\Omega_f, \end{cases}$$

where $p = \hat{P} = \int_{-\infty}^{\infty} e^{i\omega t} P(t, \underset{\sim}{x})dt$. Because of its relationship to equation (1.1), the Helmholtz equation above is often referred to as the acoustic wave equation in the frequency domain or the acoustic Helmholtz equation.

Chapter One begins this thesis with an investigation of (1.5). In Section 1.1, we derive several identities which lead to a Rellich-type identity for the Laplacian. In Section 1.2, we use the Rellich identity to derive an elliptic regularity estimate for solutions to (1.5). In Section 1.3, we define the weak or variational formulation of (1.5), and prove an existence/uniqueness theorem for the variational formulation. Section 1.4 contains finite element analysis for (1.5); we present the finite element method, and derive an optimal order error estimate for the finite element solution.

1.1 Preliminary and Rellich-Type Identities

The Rellich identity for the Laplacian relates the L^2-norm of ∇p on the boundary to a particular integral involving Δp. Rellich identities for Poisson's equation appear in [33], and are named for the author. Nečas applies the identities to second and fourth order elliptic problems in [29] and [30]. Generalizations of Rellich's identity appear in [31]. Note that the Rellich identities in [29, 30, 31, 33] only consider real valued functions.

To derive the Rellich identity, we first collect three preliminary identities which are true with minimal hypotheses on the relevant functions.

Lemma 1.1. *Given $a : \mathcal{C}^N \longrightarrow \mathcal{C}$ and $\underset{\sim}{b} : \mathcal{C}^N \longrightarrow \mathcal{C}^N$,*

$$\operatorname{div}(a\underset{\sim}{b}) = \nabla a \cdot \underset{\sim}{b} + a \operatorname{div}(\underset{\sim}{b}).$$

Proof :

$$\begin{aligned}\operatorname{div}(a\underset{\sim}{b}) &= \sum_{i=1}^{N} \frac{\partial}{\partial x_i}(ab_i) = \sum_{i=1}^{N} \frac{\partial a}{\partial x_i}(b_i) + a \sum_{i=1}^{N} \frac{\partial b_i}{\partial x_i} \\ &= \nabla a \cdot \underset{\sim}{b} + a \operatorname{div}(\underset{\sim}{b}).\end{aligned}$$

■

Lemma 1.2. *For any $p, q : \Re^N \longrightarrow \Re$*

$$\nabla p \cdot \overline{\nabla q} = \operatorname{div}((\nabla p)\ \overline{q}) - (\Delta p)\overline{q}.$$

Proof : This result follows from Lemma 1.1 with $a = \overline{q}$ and $\underset{\sim}{b} = \nabla p$. ■

Lemma 1.2 will also be useful in understanding the variational formulation of (1.5). The next lemma is a technical lemma that we will apply directly in the proof of the Rellich identity.

Lemma 1.3. *Given* $p : \Re^N \longrightarrow C$ *and* $\underset{\sim}{\alpha} : \Re^N \longrightarrow \Re^N$,

$$\underset{\sim}{\alpha} \cdot \nabla(\nabla p \cdot \overline{\nabla p}) = 2\,\mathrm{Re}\left\{\mathrm{div}\left((\nabla p)\,\overline{(\underset{\sim}{\alpha} \cdot \nabla p)}\right) - (\Delta p)\,\overline{(\underset{\sim}{\alpha} \cdot \nabla p)}\right\}$$
$$- 2\,\mathrm{Re}\left\{\sum_{i=1}^{N} \frac{\partial p}{\partial x_i} \sum_{j=1}^{N} \frac{\partial \alpha_j}{\partial x_i} \overline{\frac{\partial p}{\partial x_j}}\right\}.$$

Proof : Since

$$\underset{\sim}{\alpha} \cdot \nabla(\nabla p \cdot \overline{\nabla p}) = \sum_{i=1}^{N} \alpha_i \sum_{j=1}^{N} \frac{\partial p}{\partial x_j} \overline{\frac{\partial^2 p}{\partial x_i \partial x_j}} + \sum_{i=1}^{N} \alpha_i \sum_{j=1}^{N} \overline{\frac{\partial p}{\partial x_j}} \frac{\partial^2 p}{\partial x_i \partial x_j}, \tag{1.6}$$

then

$$\begin{aligned}
\nabla p \cdot \nabla\overline{(\underset{\sim}{\alpha} \cdot \nabla p)} &= \sum_{i=1}^{N} \frac{\partial p}{\partial x_i} \frac{\partial}{\partial x_i} \sum_{j=1}^{N} \alpha_j \overline{\frac{\partial p}{\partial x_j}} \\
&= \sum_{i=1}^{N} \frac{\partial p}{\partial x_i} \sum_{j=1}^{N} \frac{\partial \alpha_j}{\partial x_i} \overline{\frac{\partial p}{\partial x_j}} + \sum_{i=1}^{N} \frac{\partial p}{\partial x_i} \sum_{j=1}^{N} \alpha_j \overline{\frac{\partial^2 p}{\partial x_i \partial x_j}} \\
&= \sum_{i=1}^{N} \frac{\partial p}{\partial x_i} \sum_{j=1}^{N} \frac{\partial \alpha_j}{\partial x_i} \overline{\frac{\partial p}{\partial x_j}} + \sum_{j=1}^{N} \sum_{i=1}^{N} \frac{\partial p}{\partial x_i} \alpha_j \overline{\frac{\partial^2 p}{\partial x_i \partial x_j}} \\
&= \sum_{i=1}^{N} \frac{\partial p}{\partial x_i} \sum_{j=1}^{N} \frac{\partial \alpha_j}{\partial x_i} \overline{\frac{\partial p}{\partial x_j}} + \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{\partial p}{\partial x_j} \alpha_i \overline{\frac{\partial^2 p}{\partial x_j \partial x_i}}
\end{aligned}$$

implies

$$\nabla p \cdot \nabla\overline{(\underset{\sim}{\alpha} \cdot \nabla p)} = \sum_{i=1}^{N} \frac{\partial p}{\partial x_i} \sum_{j=1}^{N} \frac{\partial \alpha_j}{\partial x_i} \overline{\frac{\partial p}{\partial x_j}} + \sum_{i=1}^{N} \alpha_i \sum_{j=1}^{N} \frac{\partial p}{\partial x_j} \overline{\frac{\partial^2 p}{\partial x_j \partial x_i}}. \tag{1.7}$$

Equations (1.6) and (1.7) imply

$$2\,\mathrm{Re}\left\{\nabla p \cdot \nabla\overline{(\underset{\sim}{\alpha} \cdot \nabla p)}\right\} = \underset{\sim}{\alpha} \cdot \nabla(\nabla p \cdot \overline{\nabla p}) + 2\,\mathrm{Re}\left\{\sum_{i=1}^{N} \frac{\partial p}{\partial x_i} \sum_{j=1}^{N} \frac{\partial \alpha_j}{\partial x_i} \overline{\frac{\partial p}{\partial x_j}}\right\}.$$

Applying Lemma 1.2 to this equation (with $q = \underset{\sim}{\alpha} \cdot \nabla p$) and rearranging yields:

$$\underset{\sim}{\alpha} \cdot \nabla(\nabla p \cdot \overline{\nabla p}) = 2\,\mathrm{Re}\left\{\mathrm{div}((\nabla p)\,\overline{(\underset{\sim}{\alpha} \cdot \nabla p)}) - (\Delta p)\,\overline{(\underset{\sim}{\alpha} \cdot \nabla p)}\right\}$$
$$- 2\,\mathrm{Re}\left\{\sum_{i=1}^{N} \frac{\partial p}{\partial x_i} \sum_{j=1}^{N} \frac{\partial \alpha_j}{\partial x_i} \overline{\frac{\partial p}{\partial x_j}}\right\}.$$

■

We now use Lemma 1.1 and Lemma 1.3 to derive the Rellich identity for the Laplacian. Note that the Rellich identity and the above Lemmas are true for any H^2 function.

Theorem 1.4. *(Rellich Identity) For any* $p : \Re^N \longrightarrow C$, $\underset{\sim}{\alpha} : \Re^N \longrightarrow \Re^N$,

$$\int_{\partial\Omega_f} \underset{\sim}{\alpha} \cdot \underset{\sim}{n}_f |\nabla p|^2 = \int_{\Omega_f} (\operatorname{div} \underset{\sim}{\alpha}) |\nabla p|^2 + 2 \operatorname{Re} \int_{\partial\Omega_f} \frac{\partial p}{\partial \underset{\sim}{n}_f} (\overline{\underset{\sim}{\alpha} \cdot \nabla p}) - 2 \operatorname{Re} \int_{\Omega_f} \sum_{i=1}^{N} \frac{\partial p}{\partial x_i} \sum_{j=1}^{N} \frac{\partial \alpha_j}{\partial x_i} \overline{\frac{\partial p}{\partial x_j}} - 2 \operatorname{Re} \int_{\Omega_f} (\Delta p)(\overline{\underset{\sim}{\alpha} \cdot \nabla p}).$$

Proof : By Lemma 1.1,

$$\operatorname{div}\{\underset{\sim}{\alpha} |\nabla p|^2\} = \operatorname{div}(\underset{\sim}{\alpha})(\nabla p \cdot \overline{\nabla p}) + \underset{\sim}{\alpha} \cdot \nabla(\nabla p \cdot \overline{\nabla p}).$$

Applying Lemma 1.3 yields

$$\operatorname{div}\{\underset{\sim}{\alpha} |\nabla p|^2\} = \operatorname{div}(\underset{\sim}{\alpha})(\nabla p \cdot \overline{\nabla p}) + 2 \operatorname{Re} \left\{ \operatorname{div} \left((\nabla p) \, \overline{(\underset{\sim}{\alpha} \cdot \nabla p)} \right) - (\Delta p) \, \overline{(\underset{\sim}{\alpha} \cdot \nabla p)} \right\} - 2 \operatorname{Re} \left\{ \sum_{i=1}^{N} \frac{\partial p}{\partial x_i} \sum_{j=1}^{N} \frac{\partial \alpha_j}{\partial x_i} \overline{\frac{\partial p}{\partial x_j}} \right\}.$$

Integrating over Ω_f and applying the divergence theorem implies

$$\int_{\partial\Omega_f} \underset{\sim}{\alpha} \cdot \underset{\sim}{n}_f |\nabla p|^2 = \int_{\Omega_f} (\operatorname{div} \underset{\sim}{\alpha}) |\nabla p|^2 + 2 \operatorname{Re} \int_{\partial\Omega_f} \underset{\sim}{n}_f \cdot \left(\nabla p \overline{(\underset{\sim}{\alpha} \cdot \nabla p)} \right) - 2 \operatorname{Re} \int_{\Omega_f} (\Delta p)(\overline{\underset{\sim}{\alpha} \cdot \nabla p}) - 2 \operatorname{Re} \int_{\Omega_f} \left\{ \sum_{i=1}^{N} \frac{\partial p}{\partial x_i} \sum_{j=1}^{N} \frac{\partial \alpha_j}{\partial x_i} \overline{\frac{\partial p}{\partial x_j}} \right\},$$

which is equivalent to Theorem 1.4. ■

Corollary 1.5. *For any* $p : \Re^N \longrightarrow C$,

$$\int_{\partial\Omega_f} \underset{\sim}{x} \cdot \underset{\sim}{n}_f |\nabla p|^2 = (N-2) \|\nabla p\|^2_{L^2(\Omega_f)} + 2 \operatorname{Re} \int_{\partial\Omega_f} \frac{\partial p}{\partial \underset{\sim}{n}_f} (\overline{\underset{\sim}{x} \cdot \nabla p}) - 2 \operatorname{Re} \int_{\Omega_f} (\Delta p)(\overline{\underset{\sim}{x} \cdot \nabla p}).$$

Proof : Apply Theorem 1.4 with $\underset{\sim}{\alpha} = \underset{\sim}{x}$. ■

Notice that the Rellich identity contains an integral over Ω_f of $\Delta p\overline{(\underset{\sim}{\alpha} \cdot \nabla p)}$. Before we can apply the Rellich identity to the Helmholtz equation, we need the following identity for $p\overline{(\underset{\sim}{\alpha} \cdot \nabla p)}$.

Lemma 1.6. *For any* $p : \Re^N \longrightarrow C$, $\underset{\sim}{\alpha} : \Re^N \longrightarrow \Re^N$,

$$\int_{\Omega_f} (\operatorname{div} \underset{\sim}{\alpha})|p|^2 = \int_{\partial\Omega_f} \underset{\sim}{\alpha} \cdot \underset{\sim}{n}_f |p|^2 - 2\operatorname{Re}\int_{\Omega_f} p\overline{(\underset{\sim}{\alpha} \cdot \nabla p)}.$$

Proof : By Lemma 1.1

$$\begin{aligned}\operatorname{div}\left\{\underset{\sim}{\alpha}|p|^2\right\} &= \operatorname{div}(\underset{\sim}{\alpha})|p|^2 + \underset{\sim}{\alpha} \cdot \nabla|p|^2 \\ &= \operatorname{div}(\underset{\sim}{\alpha})|p|^2 + 2\operatorname{Re}\left\{p\overline{(\underset{\sim}{\alpha} \cdot \nabla p)}\right\}.\end{aligned}$$

Integrating over Ω_f and applying the divergence theorem yields

$$\int_{\partial\Omega_f} \underset{\sim}{\alpha} \cdot \underset{\sim}{n}_f |p|^2 = \int_{\Omega_f} (\operatorname{div} \underset{\sim}{\alpha})|p|^2 + 2\operatorname{Re}\int_{\Omega_f} p\overline{(\underset{\sim}{\alpha} \cdot \nabla p)}.$$

■

Corollary 1.7. *For any* $p : \Re^N \longrightarrow C$, $\underset{\sim}{\alpha} : \Re^N \longrightarrow \Re^N$,

$$N\|p\|^2_{L^2(\Omega_f)} = \int_{\partial\Omega_f} \underset{\sim}{x} \cdot \underset{\sim}{n}_f |p|^2 - 2\operatorname{Re}\int_{\Omega_f} p\overline{(\underset{\sim}{x} \cdot \nabla p)}.$$

Proof : Apply Lemma 1.6 with $\underset{\sim}{\alpha} = \underset{\sim}{x}$. ■

1.2 A Priori Estimates for the Helmholtz Problem

Next, we derive identities and estimates which bound the solution of the Helmholtz problem in terms of its source function g_f, paying particular attention to how the identities and estimates depend on the frequency, ω. The principle result is a regularity estimate for the Helmholtz problem (1.5), which estimates the L^2, H^1 and H^2 norms of the solution in terms of the source function g_f.

The first lemma employs a simple test function technique to establish estimates for the L^2 norm of the gradient of a solution, and for the boundary L^2 norm of a solution. We will use these results to derive the regularity estimate.

Lemma 1.8. *Suppose p is a solution of (1.5); then* $\forall \epsilon_1, \epsilon_2 > 0$,

(1) $||\nabla p||^2_{L^2(\Omega_f)} = \text{Re}\int_{\Omega_f} g_f\overline{p} + \frac{\omega^2}{c^2}||p||^2_{L^2(\Omega_f)} \leq \frac{1}{2\epsilon_1}||g_f||^2_{L^2(\Omega_f)} + \left(\frac{\epsilon_1}{2} + \frac{\omega^2}{c^2}\right)||p||^2_{L^2(\Omega_f)}$.

(2) $\frac{\omega}{c}||p||^2_{L^2(\partial\Omega_f)} = \text{Im}\int_{\Omega_f} g_f\overline{p} \leq \frac{1}{2\epsilon_2}||g_f||^2_{L^2(\Omega_f)} + \frac{\epsilon_2}{2}||p||^2_{L^2(\Omega_f)}$.

Proof : Multiplying the Helmholtz equation in (1.5) by $\overline{p}$ and integrating over Ω_f yields

$$\frac{-\omega^2}{c^2}||p||^2_{L^2(\Omega_f)} + ||\nabla p||^2_{L^2(\Omega_f)} + \frac{i\omega}{c}||p||^2_{L^2(\partial\Omega_f)} = \int_{\Omega_f} g_f\overline{p}.$$

Taking real and imaginary parts of this equation yields

$$\frac{-\omega^2}{c^2}||p||^2_{L^2(\Omega_f)} + ||\nabla p||^2_{L^2(\Omega_f)} = \text{Re}\int_{\Omega_f} g_f\overline{p} \leq \left|\int_{\Omega_f} g_f\overline{p}\right|,$$

and

$$\frac{\omega}{c}||p||^2_{L^2(\partial\Omega_f)} = \text{Im}\int_{\Omega_f} g_f\overline{p} \leq \left|\int_{\Omega_f} g_f\overline{p}\right|.$$

Subsequent applications of Cauchy's inequality and Young's inequality to the right hand side of the above equations yields the result. ■

The regularity estimates for the solution now follow from Lemma 1.8 above, Corollary 1.7, and the Rellich identity in Corollary 1.5.

Theorem 1.9. *Suppose p is a solution of (1.5). Then p satisfies the following regularity estimates:*

(1) $||p||_{L^2(\Omega_f)} \leq C(\frac{1}{\omega} + \frac{1}{\omega^2})||g_f||_{L^2(\Omega_f)}$.

(2) $||p||_{H^1(\Omega_s)} \leq C(1 + \frac{1}{\omega} + \frac{1}{\omega^2})||g_f||_{L^2(\Omega_f)}$.

(3) *If* Ω_s *is a convex polygonal domain, then*

$$\|p\|_{H^2(\Omega_s)} \le C(\omega + 1 + \tfrac{1}{\omega})\|g_f\|_{L^2(\Omega_f)}.$$

Proof : Combining Corollaries 1.5 and 1.7 implies

$$\tfrac{N\omega^2}{2c^2}\|p\|^2_{L^2(\Omega_f)} + \tfrac{1}{2}\int_{\partial\Omega_f}(\underset{\sim}{x}\cdot\underset{\sim}{n}_f)|\nabla p|^2 = \tfrac{\omega^2}{2c^2}\int_{\partial\Omega_f}\underset{\sim}{x}\cdot\underset{\sim}{n}_f|p|^2 + \left(\tfrac{N}{2}-1\right)\|\nabla p\|^2_{L^2(\Omega_f)}$$
$$+\operatorname{Re}\int_{\partial\Omega_f}\frac{\partial p}{\partial \underset{\sim}{n}_f}(\overline{\underset{\sim}{x}\cdot\nabla p}) + \operatorname{Re}\int_{\Omega_f}\left(-\tfrac{\omega^2}{c^2}p - \Delta p\right)(\overline{\underset{\sim}{x}\cdot\nabla p}).$$

Since p solves (1.5), the above becomes

$$\tfrac{N\omega^2}{2c^2}\|p\|^2_{L^2(\Omega_f)} + \tfrac{1}{2}\int_{\partial\Omega_f}(\underset{\sim}{x}\cdot\underset{\sim}{n}_f)|\nabla p|^2 = \tfrac{\omega^2}{2c^2}\int_{\partial\Omega_f}\underset{\sim}{x}\cdot\underset{\sim}{n}_f|p|^2 + \left(\tfrac{N}{2}-1\right)\|\nabla p\|^2_{L^2(\Omega_f)}$$
$$+\operatorname{Re}\int_{\partial\Omega_f}\tfrac{-i\omega}{c}p(\overline{\underset{\sim}{x}\cdot\nabla p}) + \operatorname{Re}\int_{\Omega_f}g_f(\overline{\underset{\sim}{x}\cdot\nabla p}).$$

Applying (1.4), Cauchy's inequality and the boundedness of Ω_f implies

$$\tfrac{N\omega^2}{2c^2}\|p\|^2_{L^2(\Omega_f)} + \tfrac{\beta_1}{2}\|\nabla p\|^2_{L^2(\partial\Omega_f)} \le \tfrac{\omega^2\beta_2}{2c^2}\|p\|^2_{L^2(\partial\Omega_f)} + \left(\tfrac{N}{2}-1\right)\|\nabla p\|^2_{L^2(\Omega_f)}$$
$$+\tfrac{M\omega}{c}\|p\|_{L^2(\partial\Omega_f)}\|\nabla p\|_{L^2(\partial\Omega_f)}$$
$$+M\|g_f\|_{L^2(\Omega_f)}\|\nabla p\|_{L^2(\Omega_f)}$$

for some $M > 0$.

Next, we use Young's inequality and Lemma 1.8 to bound the right side of the above inequality in terms of $\|p\|^2_{L^2(\Omega_f)}$, $\|p\|^2_{L^2(\partial\Omega_f)}$ and $\|g_f\|^2_{L^2(\Omega_f)}$. Carefully manipulating the arbitrary constants will allow of the resulting $\|p\|^2_{L^2(\Omega_f)}$ and $\|p\|^2_{L^2(\partial\Omega_f)}$ terms into the left side while retaining non-negative coefficients.

First, applying Young's inequality above implies

$$\tfrac{N\omega^2}{2c^2}\|p\|^2_{L^2(\Omega_f)} + \tfrac{\beta_1}{2}\|\nabla p\|^2_{L^2(\partial\Omega_f)} \le \tfrac{\omega^2\beta_2}{2c^2}\|p\|^2_{L^2(\partial\Omega_f)} + \left(\tfrac{N}{2}-1\right)\|\nabla p\|^2_{L^2(\Omega_f)}$$
$$+\tfrac{M\omega}{c}\left\{\tfrac{\epsilon_1}{2}\|p\|^2_{L^2(\partial\Omega_f)} + \tfrac{1}{2\epsilon_1}\|\nabla p\|^2_{L^2(\partial\Omega_f)}\right\}$$
$$+M\left\{\tfrac{\epsilon_2}{2}\|\nabla p\|^2_{L^2(\Omega_f)} + \tfrac{1}{2\epsilon_1}\|g_f\|^2_{L^2(\Omega_f)}\right\},$$

or equivalently,

$$\tfrac{N\omega^2}{2c^2}\|p\|^2_{L^2(\Omega_f)} + \left(\tfrac{\beta_1}{2} - \epsilon_1\tfrac{M\omega}{2c}\right)\|\nabla p\|^2_{L^2(\partial\Omega_f)} \le \left(\tfrac{\omega\beta_2}{2c} + \tfrac{M}{2\epsilon_1}\right)\tfrac{\omega}{c}\|p\|^2_{L^2(\partial\Omega_f)} +$$
$$\left(\tfrac{N-2}{2} + \tfrac{M\epsilon_2}{2}\right)\|\nabla p\|^2_{L^2(\Omega_f)} + \tfrac{M}{2\epsilon_2}\|g_f\|_{L^2(\Omega_f)}.$$

Applying Lemma 1.8 implies

$$\frac{N\omega^2}{2c^2}\|p\|^2_{L^2(\Omega_f)} + \left(\frac{\beta_1}{2} - \epsilon_1\frac{M\omega}{2c}\right)\|\nabla p\|^2_{L^2(\partial\Omega_f)}$$
$$\leq \left(\frac{\omega\beta_2}{2c} + \frac{M}{2\epsilon_1}\right)\left\{\frac{1}{2\epsilon_3}\|g_f\|^2_{L^2(\Omega_f)} + \frac{\epsilon_3}{2}\|p\|^2_{L^2(\Omega_f)}\right\}$$
$$+ \left(\frac{N-2}{2} + \frac{M\epsilon_2}{2}\right)\left\{\frac{1}{2\epsilon_4}\|g_f\|^2_{L^2(\Omega_f)} + \left(\frac{\epsilon_4}{2} + \frac{\omega^2}{c^2}\right)\|p\|^2_{L^2(\Omega_f)}\right\}$$
$$+ \frac{M}{2\epsilon_2}\|g_f\|_{L^2(\Omega_f)}$$
$$= \left\{\left(\frac{\omega\beta_2}{2c} + \frac{M}{2\epsilon_1}\right)\frac{\epsilon_3}{2} + \left(\frac{N-2}{2} + \frac{M\epsilon_2}{2}\right)\left(\frac{\epsilon_4}{2} + \frac{\omega^2}{c^2}\right)\right\}\|p\|^2_{L^2(\Omega_f)}$$
$$+ \left\{\left(\frac{\omega\beta_2}{2c} + \frac{M}{2\epsilon_1}\right)\frac{1}{2\epsilon_3} + \left(\frac{N-2}{2} + \frac{M\epsilon_2}{2}\right)\frac{1}{2\epsilon_4} + \frac{M}{2\epsilon_2}\right\}\|g_f\|^2_{L^2(\Omega_f)}.$$

Choose $\epsilon_1 = \frac{c\beta_1}{M\omega}$. Then $\left(\frac{\beta_1}{2} - \epsilon_1\frac{M\omega}{2c}\right) = 0$, and the above becomes

$$\frac{N\omega^2}{2c^2}\|p\|^2_{L^2(\Omega_f)} \leq \left\{\left(\frac{\omega\beta_2}{2c} + \frac{M^2}{2c\beta_1}\right)\frac{\epsilon_3}{2} + \left(\frac{N-2}{2} + \frac{M\epsilon_2}{2}\right)\left(\frac{\epsilon_4}{2} + \frac{\omega^2}{c^2}\right)\right\}\|p\|^2_{L^2(\Omega_f)}$$
$$+ \left\{\left(\frac{\omega\beta_2}{2c} + \frac{M^2}{2c\beta_1}\right)\frac{1}{2\epsilon_3} + \left(\frac{N-2}{2} + \frac{M\epsilon_2}{2}\right)\frac{1}{2\epsilon_4} + \frac{M}{2\epsilon_2}\right\}\|g_f\|^2_{L^2(\Omega_f)}$$

or equivalently

$$\left\{\frac{N\omega^2}{2c^2} - \left[\left(\beta_2 + \frac{M^2}{\beta_1}\right)\frac{\omega}{2c}\frac{\epsilon_3}{2} + \left(\frac{N-2}{2} + \frac{M\epsilon_2}{2}\right)\left(\frac{\epsilon_4}{2} + \frac{\omega^2}{c^2}\right)\right]\right\}\|p\|^2_{L^2(\Omega_f)}$$
$$\leq \left\{\left(\beta_2 + \frac{M^2}{\beta_1}\right)\frac{\omega}{2c}\frac{1}{2\epsilon_3} + \left(\frac{N-2}{2} + \frac{M\epsilon_2}{2}\right)\frac{1}{2\epsilon_4} + \frac{M}{2\epsilon_2}\right\}\|g_f\|^2_{L^2(\Omega_f)}$$

or

$$\left\{\frac{\omega^2}{c^2} - \left[\left(\beta_2 + \frac{M^2}{\beta_1}\right)\frac{\omega}{2c}\frac{\epsilon_3}{2} + \frac{N-2}{2}\frac{\epsilon_4}{2} + \frac{M\epsilon_2\epsilon_4}{4} + \frac{M\epsilon_2}{2}\frac{\omega^2}{c^2}\right]\right\}\|p\|^2_{L^2(\Omega_f)}$$
$$\leq \left\{\left(\beta_2 + \frac{M^2}{\beta_1}\right)\frac{\omega}{2c}\frac{1}{2\epsilon_3} + \left(\frac{N-2}{2} + \frac{M\epsilon_2}{2}\right)\frac{1}{2\epsilon_4} + \frac{M}{2\epsilon_2}\right\}\|g_f\|^2_{L^2(\Omega_f)}.$$

Choose $\epsilon_3 = \frac{\omega}{2c}\left(\beta_2 + \frac{M^2}{\beta_1}\right)^{-1}$ and note that $\frac{N-2}{2} \leq \frac{1}{2}$. The above inequality becomes

$$\left\{\frac{\omega^2}{c^2} - \left[\frac{\omega^2}{8c^2} + \frac{\epsilon_4}{4} + \frac{M\epsilon_2\epsilon_4}{4} + \frac{M\epsilon_2}{2}\frac{\omega^2}{c^2}\right]\right\}\|p\|^2_{L^2(\Omega_f)}$$
$$\leq \left\{\frac{1}{2}\left(\beta_2 + \frac{M}{\beta_1}\right)^2 + \left(\frac{1}{2} + \frac{M\epsilon_2}{2}\right)\frac{1}{2\epsilon_4} + \frac{M}{2\epsilon_2}\right\}\|g_f\|^2_{L^2(\Omega_f)}.$$

Choosing $\epsilon_4 = \frac{\omega^2}{2c^2}$, and $\epsilon_2 = \frac{1}{4M}$ yields

$$\left\{\frac{\omega^2}{c^2} - \frac{13\omega^2}{32c^2}\right\}\|p\|^2_{L^2(\Omega_f)} \leq \left\{\frac{1}{2}\left(\beta_2 + \frac{M}{\beta_1}\right)^2 + \frac{5}{8}\frac{c^2}{\omega^2} + 2M^2\right\}\|g_f\|^2_{L^2(\Omega_f)}.$$

That is,

$$\omega^2\|p\|^2_{L^2(\Omega_f)} \le C\left(1+\tfrac{1}{\omega^2}\right)\|g_f\|^2_{L^2(\Omega_f)}$$

or equivalently,

$$\omega\|p\|_{L^2(\Omega_f)} \le C\left(1+\tfrac{1}{\omega}\right)\|g_f\|_{L^2(\Omega_f)},$$

where C is independent of ω. It follows from the above inequality and Lemma 1.8 that

$$\begin{aligned}\|p\|^2_{H^1(\Omega_f)} &= \|p\|^2_{L^2(\Omega_f)} + \|\nabla p\|^2_{L^2(\Omega_f)}\\ &\le C\left(\tfrac{1}{\omega^2}+\tfrac{1}{\omega^4}\right)\|g_f\|^2_{L^2(\Omega_f)} + \tfrac{1}{2\epsilon}\|g_f\|^2_{L^2(\Omega_f)} + \left(\tfrac{\epsilon}{2}+\tfrac{\omega^2}{c^2}\right)\|p\|^2_{L^2(\Omega_f)}.\end{aligned}$$

Choosing $\epsilon = O(\omega^2)$ implies

$$\|p\|^2_{H^1(\Omega_f)} \le C\left(\tfrac{1}{\omega^2}+\tfrac{1}{\omega^4}\right)\|g_f\|^2_{L^2(\Omega_f)} + C\left(1+\tfrac{1}{\omega^2}\right)\|g_f\|^2_{L^2(\Omega_f)}.$$

Therefore,

$$\|p\|_{H^1(\Omega_f)} \le C\left(1+\tfrac{1}{\omega}+\tfrac{1}{\omega^2}\right)\|g_f\|^2_{L^2(\Omega_f)}.$$

Finally, if Ω_f is a convex polygonal domain, then regularity theory for the Laplace problem (see [19]) implies

$$\begin{aligned}\|p\|_{H^2(\Omega_f)} &\le C\left(\|g_f+\omega^2 p\|_{L^2(\Omega_f)} + |i\omega p|_{H^{1/2}(\partial\Omega_f)}\right)\\ &\le C\left(\|g_f\|_{L^2(\Omega_f)} + \omega^2\|p\|_{L^2(\Omega_f)} + \omega\|p\|_{H^1(\Omega_f)}\right)\\ &\le C\left(\|g_f\|_{L^2(\Omega_f)} + (\omega+1)\|g_f\|_{L^2(\Omega_f)} + C\left(\omega+1+\tfrac{1}{\omega}\right)\|g_f\|_{L^2(\Omega_f)}\right)\end{aligned}$$

or

$$\|p\|_{H^2(\Omega_f)} \le C\left(\omega+1+\tfrac{1}{\omega}\right)\|g_f\|^2_{L^2(\Omega_f)}.$$

■

Regularity estimates for (1.5) appear in several other papers. In the one dimensional case, Douglas, Santos, Sheen and Schreyer [10] use a Green's function technique to establish regularity estimates that are equivalent to those in Theorem 1.9 . In three dimensions, Feng and Sheen [18] use the fundamental solution for the Helmholtz operator to show that the solution satisfies the following weaker estimates,

$$||p||_{L^2(\Omega_f)} \leq C(\omega + \tfrac{1}{\omega})||g_f||_{L^2(\Omega_f)}.$$
$$||p||_{H^1(\Omega_s)} \leq C(\omega^2 + \tfrac{1}{\omega})||g_f||_{L^2(\Omega_f)}.$$
$$||p||_{H^2(\Omega_s)} \leq C(\omega^3 + \tfrac{1}{\omega})||g_f||_{L^2(\Omega_f)}.$$

In [28], Melenk derives estimates equivalent to those in Theorem 1.9 in the two dimensional case.

1.3 Variational Formulation and Well-Posedness

The variational formulation of (1.5) is defined as

$$(1.8) \qquad \begin{cases} \text{Find} \quad p \in H^1(\Omega_f) \quad \text{such that} \\ a(p,q) = (g_f, q) \quad \forall q \in H^1(\Omega_f), \end{cases}$$

where

$$a(p,q) = \frac{-\omega^2}{c^2}\int_{\Omega_f} p\overline{q} + \int_{\Omega_f} \nabla p \cdot \overline{\nabla q} + \frac{i\omega}{c}\int_{\partial\Omega_f} p\overline{q}$$

and

$$(g_f, q) = \int_{\Omega_f} g_f\overline{q} \qquad \forall q \in H^1(\Omega_f).$$

The bilinear form $a(\cdot,\cdot)$ is not coercive so we cannot use the Lax-Milgram Theorem to show existence and uniqueness of solutions. It is easy to show, however, that $a(\cdot,\cdot)$ satisfies a Gårding inequality; specifically,

$$\text{Re}\{a(p,p)\} + \left(\tfrac{\omega^2}{c^2} + 1\right) ||p||^2_{L^2(\Omega_f)} \geq ||p||^2_{H^1(\Omega_f)}.$$

We can therefore use the Fredholm Alternative Theorem ([3],[38]) and the Unique Continuation Principle ([25],[24]) to show that the problem (1.8) is well-posed.

Theorem 1.10. *Suppose $g_f \in L^2(\Omega_f)$, $\omega \neq 0$. Then there exists a unique solution to (1.8).*

Proof : Since the bilinear form $a(\cdot,\cdot)$ satisfies a Gårding inequality, the Fredholm Alternative Theorem implies that a solution to (1.8) exists if the adjoint problem

$$(1.9) \qquad \begin{cases} \text{Find} \quad \varphi \in H^1(\Omega_f) \quad \text{such that} \\ a^*(\varphi, q) = (g_f, q) \quad \forall q \in H^1(\Omega_f), \end{cases}$$

has only the zero solution when $g_f = 0$ (see [1], pg 102). If $g_f = 0$, then choosing $q = \varphi$ in the variational form and taking the imaginary part implies that $\varphi = 0$ on $\partial\Omega_f$. Integrating by parts in (1.9) then implies that $\frac{\partial \varphi}{\partial \underset{\sim}{n}_f} = 0$ on $\partial\Omega_f$. By the Unique Continuation Principle, $\varphi = 0$ in Ω_f, hence solutions exist. The same argument shows $g_f = 0$ in (1.8) implies that $\varphi = 0$, therefore solutions are also unique. ■

1.4 Finite Element Procedures

Let $\mathcal{T}_h$ be a quasi-uniform triangulation of Ω_f with mesh size $h > 0$. Suppose V_h is the P_{m-1} conforming finite element space of $H^1(\Omega_f)$ associated with $\mathcal{T}_h$. It is well known that V_h has the following simultaneous approximation property (see [5]) :

$$(1.10) \qquad \inf_{q \in V_h} \left\{ \|p - q\|_{L^2(\Omega_f)} + h\|p - q\|_{H^1(\Omega_f)} \right\} \leq C_A h^m \|p\|_{H^m(\Omega_f)} \quad \forall\ p \in H^m(\Omega_f).$$

Then the finite element method for (1.8) is defined as

$$(1.11) \qquad \begin{cases} \text{Find} \quad p_h \in V_h \quad \text{such that} \\ a(p_h, q) = (g_f, q) \quad \forall q \in V_h. \end{cases}$$

Let $C_{R,m}$ denote an abstract regularity constant for solutions to (1.5), i.e.

$$\|p\|_{H^m(\Omega_f)} \leq C_{R,m} \|g_f\|_{H^r(\Omega_f)}.$$

where $r = \max\{0, m-2\}$. From Theorem 1.9, we know that $C_{R,m} = O(\omega^{m-1})$ for $m \geq 2$.

The main results in this section are estimates for the finite element error. These estimates bound the L^2 and H^1 norms of the finite element error in terms of the source function g_f, the mesh size h and the frequency ω. Because the bilinear form $a(\cdot,\cdot)$ is not coercive, we cannot appeal to Céa's Lemma to derive error estimates. Instead, we will apply a duality argument to bound the L^2 norm of the error in terms of its H^1 norm, and then use the argument of Schatz [34] to establish the error estimates. A similar approach appears in [10] for the one dimensional case.

Lemma 1.11. *Suppose p solves (1.8) and p_h solves (1.11). Then there are constants C_1 and C_2 independent of ω and h such that $h \leq \frac{1}{\omega\sqrt{C_{R,2}}}C_1 = O(\frac{1}{w^{3/2}})$ implies that*

$$\|p - p_h\|_{L^2(\Omega_f)} \leq C_2 C_{R,2} h \|p - p_h\|_{H^1(\Omega_f)}.$$

Proof : Suppose that p is a solution of (1.8) and p_h is a solution of (1.11). Let φ be a solution to the adjoint problem with source $p - p_h$, i.e. φ solves

$$\begin{cases} \text{Find} \quad \varphi \in H^1(\Omega_f) \quad \text{such that} \\ a^*(\varphi, q) = (p - p_h, q) \quad \forall q \in H^1(\Omega_f), \end{cases}$$

where

$$a^*(\varphi, q) = \frac{-\omega^2}{c^2}\int_{\Omega_f} \varphi\bar{q} + \int_{\Omega_f} \nabla\varphi \cdot \overline{\nabla q} - \frac{i\omega}{c}\int_{\partial\Omega_f} \varphi\bar{q}.$$

Then φ satisfies

$$a(q, \varphi) = (q, p - p_h) \qquad \forall q \in H^1(\Omega_f).$$

Taking $q = p - p_h$ implies

$$\|p - p_h\|^2_{L^2(\Omega_f)} = a(p - p_h, \varphi). \tag{1.12}$$

The fundamental orthogonality identity states that

$$a(p - p_h, \varphi_h) = 0 \qquad \forall \varphi_h \in V_h.$$

This is true because $a(p,\varphi_h) = (g_f,\varphi_h)$ and $a(p_h,\varphi_h) = (g_f,\varphi_h)$ whenever $\varphi_h \in V_h$. Therefore

$$a(p-p_h,\varphi) = a(p-p_h,\varphi-\varphi_h) \qquad \forall\varphi_h \in V_h,$$

and (1.12) becomes

$$\begin{aligned}\|p-p_h\|^2_{L^2(\Omega_f)} &= a(p-p_h,\varphi-\varphi_h)\\ &= -\tfrac{\omega^2}{c^2}(p-p_h,\varphi-\varphi_h) + (\nabla(p-p_h),\nabla(\varphi-\varphi_h))\\ &\quad + \tfrac{i\omega}{c} < p-p_h,\varphi-\varphi_h > .\end{aligned}$$

Schwarz's inequality implies

$$\begin{aligned}\|p-p_h\|^2_{L^2(\Omega_f)} &\le \tfrac{\omega^2}{c^2}\|p-p_h\|_{L^2(\Omega_f)}\|\varphi-\varphi_h\|_{L^2(\Omega_f)}\\ &+ \|\nabla(p-p_h)\|_{L^2(\Omega_f)}\|\nabla(\varphi-\varphi_h)\|_{L^2(\Omega_f)} + \tfrac{\omega}{c}\|p-p_h\|_{L^2(\partial\Omega_f)}\|\varphi-\varphi_h\|_{L^2(\partial\Omega_f)}.\end{aligned}$$

Since Ω_f has a Lipschitz boundary, the trace inequality (see [4])

$$\|p\|_{L^2(\partial\Omega_f)} \le C_f\|p\|^{1/2}_{L^2(\Omega_f)}\|p\|^{1/2}_{H^1(\Omega_f)}. \tag{1.13}$$

holds for all $p \in H^1(\Omega_f)$ (see [4]). Therefore,

$$\begin{aligned}&\tfrac{\omega}{c}\|p-p_h\|_{L^2(\partial\Omega_f)}\|\varphi-\varphi_h\|_{L^2(\partial\Omega_f)}\\ &\qquad \le \tfrac{\omega}{c}C_f^2\|p-p_h\|^{1/2}_{L^2(\Omega_f)}\|\varphi-\varphi_h\|^{1/2}_{L^2(\Omega_f)}\|p-p_h\|^{1/2}_{H^1(\Omega_f)}\|\varphi-\varphi_h\|^{1/2}_{H^1(\Omega_f)},\end{aligned}$$

and by Young's inequality,

$$\begin{aligned}\tfrac{\omega}{c}\|p-p_h\|_{L^2(\partial\Omega_f)}\|\varphi-\varphi_h\|_{L^2(\partial\Omega_f)} &\le \tfrac{\omega^2}{c^2}\|p-p_h\|_{L^2(\Omega_f)}\|\varphi-\varphi_h\|_{L^2(\Omega_f)}\\ &+ \tfrac{C_f^4}{4}\|p-p_h\|_{H^1(\Omega_f)}\|\varphi-\varphi_h\|_{H^1(\Omega_f)}.\end{aligned}$$

Plugging this inequality in above yields

$$\begin{aligned}
||p - p_h||^2_{L^2(\Omega_f)} \leq \tfrac{2\omega^2}{c^2}||p - p_h||_{L^2(\Omega_f)}||\varphi - \varphi_h||_{L^2(\Omega_f)} \\
+ ||\nabla(p - p_h)||_{L^2(\Omega_f)}||\nabla(\varphi - \varphi_h)||_{L^2(\Omega_f)} \\
+ \tfrac{C_f^4}{4}||(p - p_h)||_{H^1(\Omega_f)}||\varphi - \varphi_h||_{H^1(\Omega_f)} \\
\leq \tfrac{2\omega^2}{c^2}||p - p_h||_{L^2(\Omega_f)}||\varphi - \varphi_h||_{L^2(\Omega_f)} \\
+ max\left\{\tfrac{C_f^4}{4}, 1\right\}||p - p_h||_{H^1(\Omega_f)}||\varphi - \varphi_h||_{H^1(\Omega_f)}.
\end{aligned}$$

The approximation property of V_h implies

$$\begin{aligned}
||p - p_h||^2_{L^2(\Omega_f)} \leq \tfrac{2\omega^2}{c^2}||p - p_h||_{L^2(\Omega_f)}C_A h^2||\varphi||_{H^2(\Omega_f)} \\
+ max\left\{\tfrac{C_f^4}{4}, 1\right\}||p - p_h||_{H^1(\Omega_f)}C_A h||\varphi||_{H^2(\Omega_f)}.
\end{aligned}$$

Applying the regularity estimate (recall that φ solves the dual problem with source $p - p_h$) yields

$$\begin{aligned}
||p - p_h||^2_{L^2(\Omega_f)} \leq \tfrac{2\omega^2}{c^2}||p - p_h||_{L^2(\Omega_f)}C_A h^2 C_{R,2}||p - p_h||_{L^2(\Omega_f)} \\
+ max\left\{\tfrac{C_f^4}{4}, 1\right\}||p - p_h||_{H^1(\Omega_f)}C_A h C_{R,2}||p - p_h||_{L^2(\Omega_f)}
\end{aligned}$$

or equivalently,

$$\left\{1 - \tfrac{2\omega^2}{c^2}C_A h^2 C_{R,2}\right\}||p - p_h||_{L^2(\Omega_f)} \leq max\left\{\tfrac{C_f^4}{4}, 1\right\}C_A h C_{R,2}||p - p_h||_{H^1(\Omega_f)}.$$

Choose $h \leq \frac{1}{\omega\sqrt{C_{R,2}}}C_1$ where $C_1 = \frac{c}{2\sqrt{C_A}}$. Then

$$\left\{1 - \tfrac{2\omega^2}{c^2}C_A h^2 C_{R,2}\right\} \geq \tfrac{1}{2}$$

which implies that

$$\tfrac{1}{2}||p - p_h||_{L^2(\Omega_f)} \leq max\left\{\tfrac{C_f^4}{4}, 1\right\}C_A h C_{R,2}||p - p_h||_{H^1(\Omega_f)},$$

or

$$||p - p_h||_{L^2(\Omega_f)} \leq C_2 C_{R,2} h||p - p_h||_{H^1(\Omega_f)},$$

where $C_2 = max\left\{\frac{C_f^4}{4}, 1\right\}C_A$. ■

Theorem 1.12. *Suppose p solves (1.8) and p_h solves (1.11). Then there are constants C_3 and C_4 independent of ω and h such that*

$$h \le C_3 \frac{1}{C_{R,2}\sqrt{\frac{\omega^2}{c^2}+1}} = O(\omega^{-2})$$

implies that

$$\|p - p_h\|_{H^1(\Omega_f)} \le C_4 C_{R,m} \left(\omega h^m + h^{m-1}\right) \|g_f\|_{H^{m-2}(\Omega_f)}$$

and

$$\|p - p_h\|_{L^2(\Omega_f)} \le C C_{R,2} C_{R,m} (\omega h^{m+1} + h^m) \|g_f\|_{H^{m-2}(\Omega_f)}$$

for $m \ge 2$.

Proof : Suppose that p is a solution of (1.8) and p_h is a solution of (1.11). Then

$$\|p - p_h\|^2_{H^1(\Omega_f)} = \operatorname{Re}\{a(p - p_h, p - p_h)\} + \left(\frac{\omega^2}{c^2} + 1\right) \|p - p_h\|^2_{L^2(\Omega_f)}.$$

Note that $a(p - p_h, p - p_h) = a(p - p_h, p - q)$ for every $q \in V_h$ by fundamental orthogonality. Therefore,

$$\begin{aligned}
\|p - p_h\|^2_{H^1(\Omega_f)} &= \operatorname{Re}\{a(p - p_h, p - q)\} + \left(\frac{\omega^2}{c^2} + 1\right) \|p - p_h\|^2_{L^2(\Omega_f)} \\
&\le \frac{\omega^2}{c^2} \|p - p_h\|_{L^2(\Omega_f)} \|p - q\|_{L^2(\Omega_f)} + \|\nabla(p - p_h)\|_{L^2(\Omega_f)} \|\nabla(p - q)\|_{L^2(\Omega_f)} \\
&\quad + \frac{\omega}{c} \|p - p_h\|_{L^2(\partial\Omega_f)} \|p - q\|_{L^2(\partial\Omega_f)} + \left(\frac{\omega^2}{c^2} + 1\right) \|p - p_h\|^2_{L^2(\Omega_f)} \\
&\le \frac{\omega^2}{c^2} \|p - p_h\|_{L^2(\Omega_f)} \|p - q\|_{L^2(\Omega_f)} + \|\nabla(p - p_h)\|_{L^2(\Omega_f)} \|\nabla(p - q)\|_{L^2(\Omega_f)} \\
&\quad + \frac{\omega^2}{c^2} \|p - p_h\|_{L^2(\Omega_f)} \|p - q\|_{L^2(\Omega_f)} + \frac{C_f^4}{4} \|p - p_h\|_{H^1(\Omega_f)} \|p - q\|_{H^1(\Omega_f)} \\
&\quad + \left(\frac{\omega^2}{c^2} + 1\right) \|p - p_h\|^2_{L^2(\Omega_f)}
\end{aligned}$$

or

$$\begin{aligned}
\|p - p_h\|^2_{H^1(\Omega_f)} &\le \frac{\omega^2}{c^2} \|p - p_h\|_{L^2(\Omega_f)} \|p - q\|_{L^2(\Omega_f)} \\
&\quad + \left(\frac{C_f^4}{4} + 1\right) \|p - p_h\|_{H^1(\Omega_f)} \|p - q\|_{H^1(\Omega_f)} + \left(\frac{\omega^2}{c^2} + 1\right) \|p - p_h\|^2_{L^2(\Omega_f)}
\end{aligned}$$

Applying Young's inequality,

$$\begin{aligned}
\|p-p_h\|^2_{H^1(\Omega_f)} \le{}& \tfrac{2\omega^2}{c^2}\left\{\tfrac{1}{2}\|p-p_h\|^2_{L^2(\Omega_f)} + \tfrac{1}{2}\|p-q\|^2_{L^2(\Omega_f)}\right\} \\
&+ \tfrac{1}{2}\|p-p_h\|^2_{H^1(\Omega_f)} + \tfrac{1}{2}\left(\tfrac{C_f^4}{4}+1\right)^2\|p-q\|^2_{H^1(\Omega_f)} \\
&+ \left(\tfrac{\omega^2}{c^2}+1\right)\|p-p_h\|^2_{L^2(\Omega_f)} \\
={}& \left(\tfrac{2\omega^2}{c^2}+1\right)\|p-p_h\|^2_{L^2(\Omega_f)} + \tfrac{\omega^2}{c^2}\|p-q\|^2_{L^2(\Omega_f)} \\
&+ \tfrac{1}{2}\|p-p_h\|^2_{H^1(\Omega_f)} + \tfrac{1}{2}\left(\tfrac{C_f^4}{4}+1\right)^2\|p-q\|^2_{H^1(\Omega_f)}
\end{aligned}$$

or equivalently,

$$\begin{aligned}
\tfrac{1}{2}\|p-p_h\|^2_{H^1(\Omega_f)} \le{}& \left(\tfrac{2\omega^2}{c^2}+1\right)\|p-p_h\|^2_{L^2(\Omega_f)} + \tfrac{\omega^2}{c^2}\|p-q\|^2_{L^2(\Omega_f)} \\
&+ \tfrac{1}{2}\left(\tfrac{C_f^4}{4}+1\right)^2\|p-q\|^2_{H^1(\Omega_f)}.
\end{aligned}$$

The regularity estimate for (1.5) and the approximation property (1.10) implies that

$$\begin{aligned}
\|p-p_h\|_{L^2(\Omega_f)} &\le C_A h^m\|p\|_{H^m(\Omega_f)} \le C_A h^m C_{R,m}\|g_f\|_{H^{m-2}(\Omega_f)} \\
\|p-q\|_{H^1(\Omega_f)} &\le C_A h^{m-1}\|p\|_{H^m(\Omega_f)} \le C_A h^{m-1} C_{R,m}\|g_f\|_{H^{m-2}(\Omega_f)}.
\end{aligned}$$

Therefore

$$\begin{aligned}
\tfrac{1}{2}\|p-p_h\|^2_{H^1(\Omega_f)} \le{}& \left(\tfrac{2\omega^2}{c^2}+1\right)\|p-p_h\|^2_{L^2(\Omega_f)} + \tfrac{\omega^2}{c^2}C_A^2 h^{2m} C_{R,m}^2\|g_f\|^2_{H^{m-2}(\Omega_f)} \\
&+ \tfrac{1}{2}\left(\tfrac{C_f^4}{4}+1\right)^2 C_A^2 h^{2m-2} C_{R,m}^2\|g_f\|^2_{H^{m-2}(\Omega_f)} \\
={}& \left(\tfrac{2\omega^2}{c^2}+1\right)\|p-p_h\|^2_{L^2(\Omega_f)} \\
&+ C_A^2 C_{R,m}^2\left\{\tfrac{\omega^2}{c^2}h^{2m} + \tfrac{1}{2}\left(\tfrac{C_f^4}{4}+1\right)^2 h^{2m-2}\right\}\|g_f\|^2_{H^{m-2}(\Omega_f)}.
\end{aligned}$$

Applying the duality estimate from Lemma 1.11 implies

$$\begin{aligned}
\tfrac{1}{2}\|p-p_h\|^2_{H^1(\Omega_f)} \le{}& \left(\tfrac{2\omega^2}{c^2}+1\right) C_2^2 C_{R,2}^2 h^2\|p-p_h\|^2_{H^1(\Omega_f)} \\
&+ C_A^2 C_{R,m}^2\left\{\tfrac{\omega^2}{c^2}h^{2m} + \tfrac{1}{2}\left(\tfrac{C_f^4}{4}+1\right)^2 h^{2m-2}\right\}\|g_f\|^2_{H^{m-2}(\Omega_f)},
\end{aligned}$$

or equivalently,

$$\left\{\tfrac{1}{2}-\left(\tfrac{2\omega^2}{c^2}+1\right)C_2^2C_{R,2}^2h^2\right\}\|p-p_h\|^2_{H^1(\Omega_f)}$$
$$\le C_A^2C_{R,m}^2\left\{\tfrac{\omega^2}{c^2}h^{2m}+\tfrac{1}{2}\left(\tfrac{C_f^4}{4}+1\right)^2h^{2m-2}\right\}\|g_f\|^2_{H^{m-2}(\Omega_f)},$$

Choose $h\le C_3\frac{1}{C_{R,2}\sqrt{\frac{2\omega^2}{c^2}+1}}$ where $C_3=\frac{1}{2C_2}$. Then

$$\tfrac{1}{2}-\left(\tfrac{2\omega^2}{c^2}+1\right)C_2^2h^2C_{R,2}^2\ge\tfrac{1}{4}$$

and therefore

$$\|p-p_h\|^2_{H^1(\Omega_f)}\le 4C_A^2C_{R,m}^2\left\{\tfrac{\omega^2}{c^2}h^{2m}+\tfrac{1}{2}\left(\tfrac{C_f^4}{4}+1\right)^2h^{2m-2}\right\}\|g_f\|^2_{H^{m-2}(\Omega_f)},$$

or

$$\|p-p_h\|_{H^1(\Omega_f)}\le C_4C_{R,m}\left(\omega h^m+h^{m-1}\right)\|g_f\|_{H^{m-2}(\Omega_f)}.$$

For some constant C_4 independent of h and ω. Combining this estimate with Lemma 1.11 implies that

$$\|p-p_h\|_{L^2(\Omega_f)}\le\|p-p_h\|_{H^1(\Omega_f)}$$
$$\le C_2C_{R,2}hC_4C_{R,m}\left(\omega h^m+h^{m-1}\right)\|g_f\|_{H^{m-2}(\Omega_f)}.$$

■

Chapter 2

Elastic Waves

Consider the elastic wave equations,

$$\rho_s \underset{\sim}{U}_{tt} - \underset{\sim}{\mathrm{div}} \left(\underset{\approx}{\sigma}(\underset{\sim}{U}) \right) = \underset{\sim}{G}_s \quad \underset{\sim}{x} \in \Omega_s, \quad t > 0. \tag{2.1}$$

where $\underset{\sim}{U}$ is a real, vector–valued function of the spatial variable $\underset{\sim}{x} \in \Re^N$ $(N = 2, 3)$ and the time variable t, and

$$\underset{\approx}{\sigma}(\underset{\sim}{U}) = \lambda \,\mathrm{div}\, \underset{\sim}{U} \underset{\approx}{I} + 2\mu \underset{\approx}{\epsilon}(\underset{\sim}{U}), \quad \underset{\approx}{\epsilon}(\underset{\sim}{U}) = \frac{1}{2}\left(\underset{\approx}{\nabla} \underset{\sim}{U} + (\underset{\approx}{\nabla} \underset{\sim}{U})^T \right). \tag{2.2}$$

Equation (2.1) describe the propagation of disturbances in a linearly elastic medium. Equation (2.2) is the constitutive relation for Ω_s where $\lambda \geq 0$ and $\mu > 0$ are the Lamé constants of the elastic medium, and $\underset{\approx}{I}$ denotes the $N \times N$ identity matrix. $\underset{\approx}{A}$ is a $N \times N$ symmetric positive definite matrix whose entries depend only on $\underset{\sim}{n}_s, \rho_s, \lambda$ and μ. When $N = 2$, $\underset{\approx}{A}$ has the form

$$\begin{pmatrix} n_{s,1} & n_{s,2} \\ n_{s,2} & -n_{s,1} \end{pmatrix} \begin{pmatrix} \alpha_s & 0 \\ 0 & \beta_s \end{pmatrix} \begin{pmatrix} n_{s,1} & n_{s,2} \\ n_{s,2} & -n_{s,1} \end{pmatrix}$$

where

$$\alpha_s = \sqrt{\frac{\lambda + 2\mu}{\rho_s}}, \quad \beta_s = \sqrt{\frac{\mu}{\rho_s}}, \quad .$$

The boundary condition

$$
\underset{\approx}{A}\underset{\sim}{U}_t + \underset{\approx}{\sigma}(\underset{\sim}{U})\underset{\sim}{n}_s = 0 \quad \underset{\sim}{x} \in \partial\Omega_s, \quad t \geq 0, \tag{2.3}
$$

is the standard first order absorbing boundary condition (see [13]). When it is imposed on the boundary of Ω_s, waves which arrive normally at the boundary are completely absorbed.

Equations (2.1) and (2.3) lead to the following model for wave propagation in an elastic medium

$$
\begin{cases} \rho_s \underset{\sim}{U}_{tt} - \underset{\sim}{\text{div}}\left(\underset{\approx}{\sigma}(\underset{\sim}{U})\right) &= \underset{\sim}{G}_s \qquad \underset{\sim}{x} \in \Omega_s, \quad t > 0, \\ \underset{\approx}{A}\underset{\sim}{U}_t + \underset{\approx}{\sigma}(\underset{\sim}{U})\underset{\sim}{n}_s &= 0 \qquad \underset{\sim}{x} \in \partial\Omega_s, \quad t \geq 0, \\ \underset{\sim}{U}_t = \underset{\sim}{U} &= 0 \qquad t \leq 0. \end{cases} \tag{2.4}
$$

Throughout Chapter Two, we will assume that $\Omega_s \subset \Re^N$ is a bounded star-shaped domain with a Lipschitz boundary. Since Ω_s is star-shaped, there exists a positive constant γ_1 and a point $\underset{\sim}{x}_0 \in \Omega_s$ such that $\gamma_1 \leq (\underset{\sim}{x} - \underset{\sim}{x}_0) \cdot \underset{\sim}{n}_s$ for all $\underset{\sim}{x} \in \partial\Omega_f$.

Assume without loss of generality that $\underset{\sim}{x}_0 = \underset{\sim}{0}$. Applying the Fourier Transform to (2.4), or seeking time harmonic solutions, yields

$$
\begin{cases} -\omega^2 \rho_s \underset{\sim}{u} - \underset{\sim}{\text{div}}\left(\underset{\approx}{\sigma}(\underset{\sim}{u})\right) &= \underset{\sim}{g}_s \text{ in } \Omega_s, \\ i\omega \underset{\approx}{A}\underset{\sim}{u} + \underset{\approx}{\sigma}(\underset{\sim}{u})\underset{\sim}{n}_s &= 0 \text{ on } \partial\Omega_s, \end{cases} \tag{2.5}
$$

where $\underset{\sim}{u} = \underset{\sim}{\hat{U}} = \int_{-\infty}^{\infty} e^{i\omega t}\underset{\sim}{U}(t, \underset{\sim}{x})$. The above Helmholtz equations are also known as the elastic Helmholtz equations or the elastic wave equations in the frequency domain.

In Chapter Two, we present an analysis of the elastic Helmholtz problem (2.5) which parallels our analysis of the acoustic problem in Chapter One. In Section 2.1, we derive a Rellich identity for the operator $\text{div}(\underset{\approx}{\sigma}(\cdot))$. We then use the Rellich identity in Section 2.2 to derive two elliptic regularity estimates for solutions to (2.5). The second estimate, which we believe is optimal, uses a Korn-type inequality on the boundary of Ω_s. Section 2.3 contains the variational formulation of (2.5), and an existence/uniqueness theorem. Section 2.4 concludes Chapter Two with analysis of the finite element method for the elastic Helmholtz problem.

2.1 Preliminary and Rellich-Type Identities

The Rellich identity for $\operatorname{div}(\underset{\approx}{\sigma}(\cdot))$ relates the L^2 norms of $\operatorname{div}\underset{\sim}{u}$ and $\underset{\approx}{\epsilon}(\underset{\sim}{u})$ on the boundary of Ω_s to a certain integral involving $\operatorname{div}(\underset{\approx}{\sigma}(\underset{\sim}{u}))$. We first gather several identities which hold for general vector-valued functions. These identities – many of which are analogous to identities in section 1.1 – will lead us to the Rellich identity. Rellich-type identities were established for $\operatorname{div}(\underset{\approx}{\sigma}(\cdot))$ by Payne and Weinberger in [31] for real, vector–valued functions (also see [7]).

Define the matrix inner product,

$$\underset{\approx}{A} : \underset{\approx}{M} = \sum_{j=1}^{N}\sum_{k=1}^{N} a_{jk} m_{jk}.$$

Lemma 2.1. *Given* $\underset{\approx}{A} : \Re^N \to \mathcal{C}^{N\times N}$, $\underset{\sim}{b} : \Re^N \to \mathcal{C}^N$, *and* $a : \Re^N \to \mathcal{C}$

(1) $\quad \operatorname{div}(a\underset{\sim}{b}) = (\nabla a)\cdot \underset{\sim}{b} + a \operatorname{div} \underset{\sim}{b}.$

(2) $\quad \operatorname{div}(\underset{\approx}{A}\underset{\sim}{b}) = \sum_{i=1}^{N}\sum_{j=1}^{N} \frac{\partial a_{ij}}{\partial x_i} b_j + \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}\frac{\partial b_j}{\partial x_i}.$

(3) $\quad \operatorname{div}(\underset{\approx}{A}\underset{\sim}{b}) = \left(\underset{\sim}{\operatorname{div}}\left(\underset{\approx}{A}\right)\right)\cdot \underset{\sim}{b} + \underset{\approx}{A} : \underset{\approx}{\nabla}\underset{\sim}{b}$ $\qquad$ *if* $\underset{\approx}{A}$ *is symmetric.*

Proof : The first two equations are easy to verify. The following is a proof of the third:

$$\begin{aligned}
\operatorname{div}(\underset{\approx}{A}\underset{\sim}{b}) &= \sum_{i=1}^{N}\sum_{j=1}^{N} \frac{\partial a_{ij}}{\partial x_i} b_j + \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}\frac{\partial b_j}{\partial x_i} \\
&= \sum_{i=1}^{N}\sum_{j=1}^{N} \frac{\partial a_{ji}}{\partial x_i} b_j + \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ji}\frac{\partial b_j}{\partial x_i} \\
&= \sum_{j=1}^{N}\sum_{i=1}^{N} \frac{\partial a_{ji}}{\partial x_i} b_j + \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ji}\frac{\partial b_j}{\partial x_i} \\
&= (\underset{\sim}{\operatorname{div}}\left(\underset{\approx}{A}\right))\cdot \underset{\sim}{b} + \underset{\approx}{A} : \underset{\approx}{\nabla}\underset{\sim}{b}.
\end{aligned}$$

■

Lemma 2.2. *For any* $\underset{\sim}{u},\ \underset{\sim}{v} : \Re^N \to C^N$

(1) $\underset{\approx}{\sigma}(\underset{\sim}{u}) : \overline{\underset{\approx}{\nabla}\underset{\sim}{v}} = \lambda \operatorname{div}\underset{\sim}{u}\ \overline{\operatorname{div}\underset{\sim}{v}} + 2\mu\underset{\approx}{\epsilon}(\underset{\sim}{u}) : \overline{\underset{\approx}{\epsilon}(\underset{\sim}{v})}$,

(2) $\underset{\approx}{\sigma}(\underset{\sim}{u}) : (\overline{\underset{\approx}{\nabla}\underset{\sim}{v}})^T = \lambda \operatorname{div}\underset{\sim}{u}\ \overline{\operatorname{div}\underset{\sim}{v}} + 2\mu\underset{\approx}{\epsilon}(\underset{\sim}{u}) : \overline{\underset{\approx}{\epsilon}(\underset{\sim}{v})}$,

(3) $\underset{\approx}{\sigma}(\underset{\sim}{u}) : \overline{\underset{\approx}{\epsilon}(\underset{\sim}{v})} = \lambda \operatorname{div}\underset{\sim}{u}\ \overline{\operatorname{div}\underset{\sim}{v}} + 2\mu\underset{\approx}{\epsilon}(\underset{\sim}{u}) : \overline{\underset{\approx}{\epsilon}(\underset{\sim}{v})}$.

Proof :

$$\begin{aligned}
\underset{\approx}{\sigma}(\underset{\sim}{u}) : \overline{\underset{\approx}{\nabla}\underset{\sim}{v}} &= \left(\lambda \operatorname{div}\underset{\sim}{u}\ \underset{\approx}{I} + 2\mu\underset{\approx}{\epsilon}(\underset{\sim}{u})\right) : \overline{\underset{\approx}{\nabla}\underset{\sim}{v}} \\
&= \lambda \operatorname{div}\underset{\sim}{u}\ \underset{\approx}{I} : \overline{\underset{\approx}{\nabla}\underset{\sim}{v}} + 2\mu\ \underset{\approx}{\epsilon}(\underset{\sim}{u}) : \overline{\underset{\approx}{\nabla}\underset{\sim}{v}} \\
&= \lambda \operatorname{div}\underset{\sim}{u}\,\overline{\operatorname{div}\underset{\sim}{v}} + 2\mu\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\overline{\frac{\partial v_i}{\partial x_j}} \\
&= \lambda \operatorname{div}\underset{\sim}{u}\,\overline{\operatorname{div}\underset{\sim}{v}} + 2\mu\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\frac{1}{2}\overline{\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_i}{\partial x_j}\right)} \\
&= \lambda \operatorname{div}\underset{\sim}{u}\ \overline{\operatorname{div}\underset{\sim}{v}} + 2\mu\underset{\approx}{\epsilon}(\underset{\sim}{u}) : \underset{\approx}{\epsilon}(\underset{\sim}{v}).
\end{aligned}$$

A similar proof shows the second identity. Identity (3) is a direct consequence of the first two. ■

Lemma 2.3. *For any* $\underset{\sim}{u},\ \underset{\sim}{v} : \Re^N \to C^N$

$$\lambda \operatorname{div}\underset{\sim}{u}\ \overline{\operatorname{div}\underset{\sim}{v}} + 2\mu\ \underset{\approx}{\epsilon}(\underset{\sim}{u}) : \overline{\underset{\approx}{\epsilon}(\underset{\sim}{v})} = \operatorname{div}\left(\underset{\approx}{\sigma}(\underset{\sim}{u})\,\overline{\underset{\sim}{v}}\right) - \underset{\sim}{\operatorname{div}}\left(\underset{\approx}{\sigma}(\underset{\sim}{u})\right)\cdot\overline{\underset{\sim}{v}}.$$

Proof : By Lemma 2.1,

$$\begin{aligned}
\underset{\sim}{\operatorname{div}}\left(\underset{\approx}{\sigma}(\underset{\sim}{u})\,\overline{\underset{\sim}{v}}\right) &= \underset{\sim}{\operatorname{div}}\left(\underset{\approx}{\sigma}(\underset{\sim}{u})\right)\cdot\overline{\underset{\sim}{v}} + \underset{\approx}{\sigma}(\underset{\sim}{u}) : \overline{\underset{\approx}{\nabla}\underset{\sim}{v}} \\
&= \underset{\sim}{\operatorname{div}}\left(\underset{\approx}{\sigma}(\underset{\sim}{u})\right)\cdot\overline{\underset{\sim}{v}} + \left(\lambda \operatorname{div}\underset{\sim}{u}\ \overline{\operatorname{div}\underset{\sim}{v}} + 2\mu\underset{\approx}{\epsilon}(\underset{\sim}{u}) : \overline{\underset{\approx}{\epsilon}(\underset{\sim}{v})}\right)
\end{aligned}$$

■

Lemma 2.1 is similar to Lemma 1.1 of Chapter One. Lemmas 2.2 and 2.3 are analogous to Lemma 1.2. The following lemma, which corresponds to Lemma 1.3, is a technical lemma which we will use in the proof of the Rellich identity.

Lemma 2.4. *For any* $\underset{\sim}{u} : \Re^N \to \mathcal{C}^N$, $\underset{\sim}{\alpha} : \Re^N \to \Re^N$

$$\begin{aligned}
&\underset{\sim}{\alpha} \cdot \nabla \left(\lambda \,|\operatorname{div} \underset{\sim}{u}|^2 + 2\mu \, \underset{\approx}{\epsilon}(\underset{\sim}{u}) : \overline{\underset{\approx}{\epsilon}(\underset{\sim}{u})} \right) = \\
&\qquad 2\,\mathrm{Re} \left\{ \operatorname{div} \left(\underset{\approx}{\sigma}(\underset{\sim}{u}) \; \left((\overline{\underset{\approx}{\nabla} \underset{\sim}{u}}) \underset{\sim}{\alpha} \right) \right) - \underset{\sim}{\operatorname{div}} \left(\underset{\approx}{\sigma}(\underset{\sim}{u}) \right) \cdot \left((\overline{\underset{\approx}{\nabla} \underset{\sim}{u}}) \underset{\sim}{\alpha} \right) \right\} \\
&\qquad - 2\,\mathrm{Re} \left\{ \lambda (\operatorname{div} \underset{\sim}{u}) \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{\partial \alpha_j}{\partial x_i} \overline{\frac{\partial u_i}{\partial x_j}} + \mu \sum_{i=1}^{N} \sum_{j=1}^{N} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \sum_{k=1}^{N} \overline{\frac{\partial u_i}{\partial x_k}} \frac{\partial \alpha_k}{\partial x_j} \right\}.
\end{aligned}$$

Proof : The identity

$$\begin{aligned}
&\underset{\sim}{\alpha} \cdot \nabla \left(\lambda \,|\operatorname{div} \underset{\sim}{u}|^2 + 2\mu \, \underset{\approx}{\epsilon}(\underset{\sim}{u}) : \overline{\underset{\approx}{\epsilon}(\underset{\sim}{u})} \right) \\
&\quad = \lambda \underset{\sim}{\alpha} \cdot \left\{ \operatorname{div} \underset{\sim}{u} \nabla (\overline{\operatorname{div} \underset{\sim}{u}}) + (\overline{\operatorname{div} \underset{\sim}{u}}) \nabla \operatorname{div} \underset{\sim}{u} \right\} \\
&\quad + 2\mu \underset{\sim}{\alpha} \cdot \begin{bmatrix} \frac{\partial}{\partial x_1} \underset{\approx}{\epsilon}(\underset{\sim}{u}) : \overline{\underset{\approx}{\epsilon}(\underset{\sim}{u})} & + & \frac{\partial}{\partial x_1} \overline{\underset{\approx}{\epsilon}(\underset{\sim}{u})} : \underset{\approx}{\epsilon}(\underset{\sim}{u}) \\ & \vdots & \\ \frac{\partial}{\partial x_N} \underset{\approx}{\epsilon}(\underset{\sim}{u}) : \overline{\underset{\approx}{\epsilon}(\underset{\sim}{u})} & + & \frac{\partial}{\partial x_N} \overline{\underset{\approx}{\epsilon}(\underset{\sim}{u})} : \underset{\approx}{\epsilon}(\underset{\sim}{u}) \end{bmatrix}
\end{aligned}$$

implies that

$$\begin{aligned}
(2.6) \quad &\underset{\sim}{\alpha} \cdot \nabla \left(\lambda \,|\operatorname{div} \underset{\sim}{u}|^2 + 2\mu \, \underset{\approx}{\epsilon}(\underset{\sim}{u}) : \overline{\underset{\approx}{\epsilon}(\underset{\sim}{u})} \right) = \\
&2\,\mathrm{Re} \left\{ \lambda (\operatorname{div} \underset{\sim}{u}) \underset{\sim}{\alpha} \cdot \nabla (\overline{\operatorname{div} \underset{\sim}{u}}) + 2\mu \underset{\sim}{\alpha} \cdot \begin{bmatrix} \frac{\partial}{\partial x_1} \underset{\approx}{\epsilon}(\underset{\sim}{u}) & : & \overline{\underset{\approx}{\epsilon}(\underset{\sim}{u})} \\ & \vdots & \\ \frac{\partial}{\partial x_N} \underset{\approx}{\epsilon}(\underset{\sim}{u}) & : & \overline{\underset{\approx}{\epsilon}(\underset{\sim}{u})} \end{bmatrix} \right\}.
\end{aligned}$$

Also,

$$\begin{aligned}
\lambda \operatorname{div} \underset{\sim}{u} \operatorname{div}\left(\overline{(\underset{\approx}{\nabla}\underset{\sim}{u})\underset{\sim}{\alpha}}\right) &= \lambda \operatorname{div} \underset{\sim}{u}\left\{\sum_{i=1}^{N}\frac{\partial}{\partial x_i}\sum_{j=1}^{N}\overline{\frac{\partial u_i}{\partial x_j}}\alpha_j\right\} \\
&= \lambda \operatorname{div} \underset{\sim}{u}\left\{\sum_{i=1}^{N}\sum_{j=1}^{N}\overline{\frac{\partial^2 u_i}{\partial x_i \partial x_j}}\alpha_j + \sum_{i=1}^{N}\sum_{j=1}^{N}\overline{\frac{\partial u_i}{\partial x_j}}\frac{\partial \alpha_j}{\partial x_i}\right\} \\
&= \lambda \operatorname{div} \underset{\sim}{u}\sum_{j=1}^{N}\alpha_j\sum_{i=1}^{N}\overline{\frac{\partial^2 u_i}{\partial x_i \partial x_j}} + \lambda \operatorname{div} \underset{\sim}{u}\sum_{i=1}^{N}\sum_{j=1}^{N}\overline{\frac{\partial u_i}{\partial x_j}}\frac{\partial \alpha_j}{\partial x_i} \\
&= \lambda(\operatorname{div} \underset{\sim}{u})\, \underset{\sim}{\alpha}\cdot\nabla(\overline{\operatorname{div} \underset{\sim}{u}}) + \lambda \operatorname{div} \underset{\sim}{u}\sum_{i=1}^{N}\sum_{j=1}^{N}\overline{\frac{\partial u_i}{\partial x_j}}\frac{\partial \alpha_j}{\partial x_i},
\end{aligned}$$

that is,

$$\lambda(\operatorname{div} \underset{\sim}{u})\, \underset{\sim}{\alpha}\cdot\nabla(\overline{\operatorname{div} \underset{\sim}{u}}) = \lambda \operatorname{div} \underset{\sim}{u} \operatorname{div}\left(\overline{(\underset{\approx}{\nabla}\underset{\sim}{u})\underset{\sim}{\alpha}}\right) - \lambda \operatorname{div} \underset{\sim}{u}\sum_{i=1}^{N}\sum_{j=1}^{N}\overline{\frac{\partial u_i}{\partial x_j}}\frac{\partial \alpha_j}{\partial x_i}. \tag{2.7}$$

Finally,

$$\begin{aligned}
&2\mu\underset{\approx}{\epsilon}(\underset{\sim}{u}) : \underset{\approx}{\epsilon}\left(\overline{(\underset{\approx}{\nabla}\underset{\sim}{u})\underset{\sim}{\alpha}}\right) = \\
&\qquad 2\mu\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)\frac{1}{2}\left(\frac{\partial}{\partial x_j}\sum_{k=1}^{N}\overline{\frac{\partial u_i}{\partial x_k}}\alpha_k + \frac{\partial}{\partial x_i}\sum_{k=1}^{N}\overline{\frac{\partial u_j}{\partial x_k}}\alpha_k\right) \\
&\quad = 2\mu\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)\frac{1}{2}\sum_{k=1}^{N}\left(\overline{\frac{\partial^2 u_i}{\partial x_j\partial x_k}}+\overline{\frac{\partial^2 u_j}{\partial x_i\partial x_k}}\right)\alpha_k \\
&\qquad\quad + \frac{1}{2}\mu\sum_{i=1}^{N}\sum_{j=1}^{N}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)\sum_{k=1}^{N}\left(\overline{\frac{\partial u_i}{\partial x_k}}\frac{\partial \alpha_k}{\partial x_j}+\overline{\frac{\partial u_j}{\partial x_k}}\frac{\partial \alpha_k}{\partial x_j}\right) \\
&\quad = 2\mu\sum_{k=1}^{N}\alpha_k\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)\frac{1}{2}\frac{\partial}{\partial x_k}\left(\overline{\frac{\partial u_i}{\partial x_j}}+\overline{\frac{\partial u_j}{\partial x_i}}\right) \\
&\qquad\quad + \mu\sum_{i=1}^{N}\sum_{j=1}^{N}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)\sum_{k=1}^{N}\overline{\frac{\partial u_i}{\partial x_k}}\frac{\partial \alpha_k}{\partial x_j} \\
&\quad = 2\mu\underset{\sim}{\alpha}\cdot\begin{bmatrix}\frac{\partial}{\partial x_1}\underset{\approx}{\epsilon}(\underset{\sim}{u}) : \overline{\underset{\approx}{\epsilon}(\underset{\sim}{u})} \\ \vdots \\ \frac{\partial}{\partial x_N}\underset{\approx}{\epsilon}(\underset{\sim}{u}) : \overline{\underset{\approx}{\epsilon}(\underset{\sim}{u})}\end{bmatrix} + \mu\sum_{i=1}^{N}\sum_{j=1}^{N}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)\sum_{k=1}^{N}\overline{\frac{\partial u_i}{\partial x_k}}\frac{\partial \alpha_k}{\partial x_j},
\end{aligned}$$

or

$$(2.8)\quad 2\mu\underset{\sim}{\alpha}\cdot\begin{bmatrix}\frac{\partial}{\partial x_1}\underset{\approx}{\epsilon}(\underset{\sim}{u}) & : & \overline{\underset{\approx}{\epsilon}(\underset{\sim}{u})}\\ & \vdots & \\ \frac{\partial}{\partial x_N}\underset{\approx}{\epsilon}(\underset{\sim}{u}) & : & \overline{\underset{\approx}{\epsilon}(\underset{\sim}{u})}\end{bmatrix} = 2\mu\underset{\approx}{\epsilon}(\underset{\sim}{u}):\underset{\approx}{\epsilon}\left(\overline{(\underset{\approx}{\nabla}\underset{\sim}{u})\underset{\sim}{\alpha}}\right)$$

$$-\mu\sum_{i=1}^{N}\sum_{j=1}^{N}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)\sum_{k=1}^{N}\frac{\overline{\partial u_i}}{\partial x_k}\frac{\partial\alpha_k}{\partial x_j}.$$

Therefore, plugging (2.7) and (2.8) into (2.6) yields

$$\underset{\sim}{\alpha}\cdot\nabla\left(\lambda\,|\operatorname{div}\underset{\sim}{u}|^2+2\mu\,\underset{\approx}{\epsilon}(\underset{\sim}{u}):\overline{\underset{\approx}{\epsilon}(\underset{\sim}{u})}\right)=$$

$$2\operatorname{Re}\left\{\lambda\operatorname{div}\underset{\sim}{u}\ \operatorname{div}\left(\overline{(\underset{\approx}{\nabla}\underset{\sim}{u})\underset{\sim}{\alpha}}\right)+2\mu\underset{\approx}{\epsilon}(\underset{\sim}{u}):\underset{\approx}{\epsilon}\left(\overline{(\underset{\approx}{\nabla}\underset{\sim}{u})\underset{\sim}{\alpha}}\right)\right\}$$

$$-2\operatorname{Re}\left\{\lambda(\operatorname{div}\underset{\sim}{u})\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{\partial\alpha_j}{\partial x_i}\frac{\overline{\partial u_i}}{\partial x_j}+\mu\sum_{i=1}^{N}\sum_{j=1}^{N}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)\sum_{k=1}^{N}\frac{\overline{\partial u_i}}{\partial x_k}\frac{\partial\alpha_k}{\partial x_j}\right\}.$$

Applying Lemma 2.3 with $\underset{\sim}{v}=(\underset{\approx}{\nabla}\underset{\sim}{u})\underset{\sim}{\alpha}$ yields the result. ■

We are now ready to use the above lemmas to prove the Rellich identity.

Theorem 2.5. *(Rellich Identity) For any* $\underset{\sim}{u}:\Re^N\to C^N$, $\underset{\sim}{\alpha}:\Re^N\to\Re^N$

$$\int_{\partial\Omega_s}\underset{\sim}{\alpha}\cdot\underset{\sim}{n}_s\left(\lambda\,|\operatorname{div}\underset{\sim}{u}|^2+2\mu\,\underset{\approx}{\epsilon}(\underset{\sim}{u}):\overline{\underset{\approx}{\epsilon}(\underset{\sim}{u})}\right)$$

$$=\int_{\Omega_s}(\operatorname{div}\underset{\sim}{\alpha})\left(\lambda\,|\operatorname{div}\underset{\sim}{u}|^2+2\mu\,\underset{\approx}{\epsilon}(\underset{\sim}{u}):\overline{\underset{\approx}{\epsilon}(\underset{\sim}{u})}\right)+2\operatorname{Re}\int_{\partial\Omega_s}\left(\underset{\approx}{\sigma}(\underset{\sim}{u})\underset{\sim}{n}_s\right)\cdot\left(\overline{(\underset{\approx}{\nabla}\underset{\sim}{u})\underset{\sim}{\alpha}}\right)$$

$$-2\operatorname{Re}\int_{\Omega_s}\left\{\lambda(\operatorname{div}\underset{\sim}{u})\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{\partial\alpha_j}{\partial x_i}\frac{\overline{\partial u_i}}{\partial x_j}+\mu\sum_{i=1}^{N}\sum_{j=1}^{N}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)\sum_{k=1}^{N}\frac{\overline{\partial u_i}}{\partial x_k}\frac{\partial\alpha_k}{\partial x_j}\right\}$$

$$-2\operatorname{Re}\int_{\Omega_s}\underset{\sim}{\operatorname{div}}\left(\underset{\approx}{\sigma}(\underset{\sim}{u})\right)\cdot\left(\overline{(\underset{\approx}{\nabla}\underset{\sim}{u})\underset{\sim}{\alpha}}\right).$$

Proof : By Lemma 2.1

$$\operatorname{div}\left\{\underset{\sim}{\alpha}\left(\lambda\,|\operatorname{div}\underset{\sim}{u}|^2+2\mu\,\underset{\approx}{\epsilon}(\underset{\sim}{u}):\overline{\underset{\approx}{\epsilon}(\underset{\sim}{u})}\right)\right\}=(\operatorname{div}\underset{\sim}{\alpha})\left(\lambda\,|\operatorname{div}\underset{\sim}{u}|^2+2\mu\,\underset{\approx}{\epsilon}(\underset{\sim}{u}):\overline{\underset{\approx}{\epsilon}(\underset{\sim}{u})}\right)$$

$$+\underset{\sim}{\alpha}\cdot\nabla\left(\lambda\,|\operatorname{div}\underset{\sim}{u}|^2+2\mu\,\underset{\approx}{\epsilon}(\underset{\sim}{u}):\overline{\underset{\approx}{\epsilon}(\underset{\sim}{u})}\right).$$

Applying Lemma 2.4 to the right side yields

$$\operatorname{div}\left\{\underset{\sim}{\alpha}\left(\lambda\,|\operatorname{div}\underset{\sim}{u}|^2+2\mu\,\underset{\approx}{\epsilon}(\underset{\sim}{u}):\overline{\underset{\approx}{\epsilon}(\underset{\sim}{u})}\right)\right\}=(\operatorname{div}\underset{\sim}{\alpha})\left(\lambda\,|\operatorname{div}\underset{\sim}{u}|^2+2\mu\,\underset{\approx}{\epsilon}(\underset{\sim}{u}):\overline{\underset{\approx}{\epsilon}(\underset{\sim}{u})}\right)$$
$$+2\operatorname{Re}\left\{\operatorname{div}\left(\underset{\approx}{\sigma}(\underset{\sim}{u})\left((\overline{\underset{\approx}{\nabla}\underset{\sim}{u}})\underset{\sim}{\alpha}\right)\right)-\underset{\sim}{\operatorname{div}}\left(\underset{\approx}{\sigma}(\underset{\sim}{u})\right)\cdot\left((\overline{\underset{\approx}{\nabla}\underset{\sim}{u}})\underset{\sim}{\alpha}\right)\right\}$$
$$-2\operatorname{Re}\left\{\lambda(\operatorname{div}\underset{\sim}{u})\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{\partial\alpha_j}{\partial x_i}\overline{\frac{\partial u_i}{\partial x_j}}+\mu\sum_{i=1}^{N}\sum_{j=1}^{N}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)\sum_{k=1}^{N}\overline{\frac{\partial u_i}{\partial x_k}}\frac{\partial\alpha_j}{\partial x_j}\right\}.$$

Integrating over Ω_s and applying the divergence theorem implies

$$\int_{\partial\Omega_s}\underset{\sim}{\alpha}\cdot\underset{\sim}{n}_s\left(\lambda|\operatorname{div}\underset{\sim}{u}|^2+2\,\underset{\approx}{\epsilon}(\underset{\sim}{u}):\overline{\underset{\approx}{\epsilon}(\underset{\sim}{u})}\right)=\int_{\Omega_s}\operatorname{div}\underset{\sim}{\alpha}\left(\lambda|\operatorname{div}\underset{\sim}{u}|^2+2\mu\underset{\approx}{\epsilon}(\underset{\sim}{u}):\overline{\underset{\approx}{\epsilon}(\underset{\sim}{u})}\right)$$
$$+2\operatorname{Re}\int_{\partial\Omega_s}\underset{\sim}{n}_s\cdot\left(\underset{\approx}{\sigma}(\underset{\sim}{u})\left((\overline{\underset{\approx}{\nabla}\underset{\sim}{u}})\underset{\sim}{\alpha}\right)\right)-2\operatorname{Re}\int_{\Omega_s}\underset{\sim}{\operatorname{div}}\left(\underset{\approx}{\sigma}(\underset{\sim}{u})\right)\cdot\left((\overline{\underset{\approx}{\nabla}\underset{\sim}{u}})\underset{\sim}{\alpha}\right)$$
$$-2\operatorname{Re}\int_{\Omega_s}\left\{\lambda(\operatorname{div}\underset{\sim}{u})\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{\partial\alpha_j}{\partial x_i}\overline{\frac{\partial u_i}{\partial x_j}}+\mu\sum_{i=1}^{N}\sum_{j=1}^{N}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)\sum_{k=1}^{N}\overline{\frac{\partial u_i}{\partial x_k}}\frac{\partial\alpha_j}{\partial x_j}\right\}.$$

Since $\underset{\approx}{\sigma}(\underset{\sim}{u})$ is symmetric,

$$\underset{\sim}{n}_s\cdot\left(\underset{\approx}{\sigma}(\underset{\sim}{u})\left((\overline{\underset{\approx}{\nabla}\underset{\sim}{u}})\underset{\sim}{\alpha}\right)\right)=\left(\underset{\approx}{\sigma}(\underset{\sim}{u})\underset{\sim}{n}_s\left((\overline{\underset{\approx}{\nabla}\underset{\sim}{u}})\underset{\sim}{\alpha}\right)\right).$$

Applying this equation above yields the result. ■

Corollary 2.6. *For any* $\underset{\sim}{u}:\Re^N\to C^N$,

$$\int_{\partial\Omega_s}\underset{\sim}{x}\cdot\underset{\sim}{n}_s\left(\lambda\,|\operatorname{div}\underset{\sim}{u}|^2+2\mu\underset{\approx}{\epsilon}(\underset{\sim}{u}):\overline{\underset{\approx}{\epsilon}(\underset{\sim}{u})}\right)$$
$$=(N-2)\left(\lambda\|\operatorname{div}\underset{\sim}{u}\|^2_{L^2(\Omega_s)}+2\mu\|\underset{\approx}{\epsilon}(\underset{\sim}{u})\|^2_{L^2(\Omega_s)}\right)$$
$$+2\operatorname{Re}\int_{\partial\Omega_s}\left(\underset{\approx}{\sigma}(\underset{\sim}{u})\underset{\sim}{n}_s\right)\cdot\left((\overline{\underset{\approx}{\nabla}\underset{\sim}{u}})\underset{\sim}{x}\right)-2\operatorname{Re}\int_{\Omega_s}\underset{\sim}{\operatorname{div}}\left(\underset{\approx}{\sigma}(\underset{\sim}{u})\right)\cdot\left((\overline{\underset{\approx}{\nabla}\underset{\sim}{u}})\underset{\sim}{x}\right).$$

Proof : Apply Theorem 2.5 with $\underset{\sim}{\alpha}=\underset{\sim}{x}$. ■

As in Chapter One, we will apply the Rellich identity to the elastic Helmholtz problem, and therefore need the following identity for $\underset{\sim}{u}\cdot((\overline{\underset{\approx}{\nabla}\underset{\sim}{u}})\underset{\sim}{\alpha})$ which corresponds to Lemma 1.6.

Lemma 2.7. *For any* $\underset{\sim}{u} : \Re^N \to \mathcal{C}^N$, $\underset{\sim}{\alpha} : \Re^N \to \Re^N$

$$\int_{\Omega_s} (\operatorname{div} \underset{\sim}{\alpha})|\underset{\sim}{u}|^2 = \int_{\partial\Omega_s} \underset{\sim}{\alpha} \cdot \underset{\sim}{n}_s |\underset{\sim}{u}|^2 - 2\operatorname{Re}\int_{\Omega_s} \underset{\sim}{u} \cdot \left((\overline{\underset{\approx}{\nabla}\underset{\sim}{u}})\underset{\sim}{\alpha} \right).$$

Proof : By Lemma 2.1,

$$\operatorname{div}\left\{ \underset{\sim}{\alpha}(\underset{\sim}{u}\cdot\overline{\underset{\sim}{u}}) \right\} = (\operatorname{div}\ \underset{\sim}{\alpha})\underset{\sim}{u}\cdot\overline{\underset{\sim}{u}} + \underset{\sim}{\alpha}\cdot\nabla(\underset{\sim}{u}\cdot\overline{\underset{\sim}{u}})$$

and

$$\begin{aligned}
\underset{\sim}{\alpha}\cdot\nabla(\underset{\sim}{u}\cdot\overline{\underset{\sim}{u}}) &= \sum_{i=1}^N \alpha_i \frac{\partial}{\partial x_i}\sum_{j=1}^N u_j\overline{u_j} \\
&= \sum_{i=1}^N \alpha_i \sum_{j=1}^N u_j \overline{\frac{\partial u_j}{\partial x_i}} + \sum_{i=1}^N \alpha_i \sum_{j=1}^N \overline{u_j}\frac{\partial u_j}{\partial x_i} \\
&= \sum_{j=1}^N u_j \sum_{i=1}^N \overline{\frac{\partial u_j}{\partial x_i}}\alpha_i + \sum_{j=1}^N \overline{u_j}\sum_{i=1}^N \frac{\partial u_j}{\partial x_i}\alpha_i \\
&= \underset{\sim}{u}\cdot\left((\overline{\underset{\approx}{\nabla}\underset{\sim}{u}})\underset{\sim}{\alpha} \right) + \overline{\underset{\sim}{u}}\cdot\left((\underset{\approx}{\nabla}\underset{\sim}{u})\underset{\sim}{\alpha} \right) \\
&= 2\operatorname{Re}\left\{ \underset{\sim}{u}\cdot\left((\overline{\underset{\approx}{\nabla}\underset{\sim}{u}})\underset{\sim}{\alpha} \right) \right\}.
\end{aligned}$$

Therefore,

$$\operatorname{div}\left\{ \underset{\sim}{\alpha}(\underset{\sim}{u}\cdot\overline{\underset{\sim}{u}}) \right\} = (\operatorname{div}\ \underset{\sim}{\alpha})\underset{\sim}{u}\cdot\overline{\underset{\sim}{u}} + 2\operatorname{Re}\left\{ \underset{\sim}{u}\cdot\left((\overline{\underset{\approx}{\nabla}\underset{\sim}{u}})\underset{\sim}{\alpha} \right) \right\}.$$

Integrating over Ω_s and applying the divergence theorem yields the result. ■

Corollary 2.8. *For any* $\underset{\sim}{u} : \Re^N \to \mathcal{C}^N$,

$$N\|\underset{\sim}{u}\|^2_{L^2(\Omega_s)} = \int_{\partial\Omega_s} \underset{\sim}{x}\cdot\underset{\sim}{n}_s|\underset{\sim}{u}|^2 - 2\operatorname{Re}\int_{\Omega_s}\underset{\sim}{u}\cdot\left((\overline{\underset{\approx}{\nabla}\underset{\sim}{u}})\underset{\sim}{x} \right).$$

Proof : Apply Lemma 2.7 with $\underset{\sim}{\alpha} = \underset{\sim}{x}$. ■

2.2 A Priori Estimates for the Elastic Helmholtz Problem

In this section, we present identities and estimates for solutions to the elastic Helmholtz problem, again paying careful attention to how the estimates depend on the frequency

ω. The principle results are the regularity estimates which bound the L^2, H^1 and H^2 norms of the solution in terms of the source.

We begin by stating the well-known Korn's inequality from elasticity which relates the norm of $\underset{\approx}{\epsilon}(\underset{\sim}{u})$ to the H^1 norm of $\underset{\sim}{u}$.

Lemma 2.9. *(Korn's Inequality) There exists a positive constant K such that*

$$\|\underset{\approx}{\epsilon}(\underset{\sim}{v})\|_{L^2(\Omega_s)} + \|\underset{\sim}{v}\|_{L^2(\Omega_s)} \geq K\|\underset{\sim}{v}\|_{H^1(\Omega_s)} \quad \forall \underset{\sim}{v} \in (H^1(\Omega_s))^N.$$

Proof : See [4], pages 222-223. ∎

The next lemma establishes some fundamental estimates for solutions to (2.5) using a simple test function technique. We will make repeated use of these estimates, particularly when we derive the regularity estimates.

Lemma 2.10. *Suppose $\underset{\sim}{u}$ solves (2.5). Then $\underset{\sim}{u}$ satisfies the following inequalities*

(1) $\omega\|\underset{\approx}{A}\underset{\sim}{u}\|^2_{L^2(\partial\Omega_s)} \leq \operatorname{Im}\int_{\Omega_s} \underset{\sim}{g_s} \cdot \overline{\underset{\sim}{u}}$.

(2) $\lambda\|\operatorname{div}\underset{\sim}{u}\|^2_{L^2(\Omega_s)} + 2\mu\|\underset{\approx}{\epsilon}(\underset{\sim}{u})\|^2_{L^2(\Omega_s)} \leq \omega^2\rho_s\|\underset{\sim}{u}\|^2_{L^2(\Omega_s)} + \operatorname{Re}\int_{\Omega_s} \underset{\sim}{g_s} \cdot \overline{\underset{\sim}{u}}$.

Proof : Multiply the Helmholtz equation in (2.5) by $\overline{\underset{\sim}{u}}$ to get

$$-\omega^2\rho_s\underset{\sim}{u}\cdot\overline{\underset{\sim}{u}} - \underset{\sim}{\operatorname{div}}\left(\underset{\approx}{\sigma}(\underset{\sim}{u})\right)\cdot\overline{\underset{\sim}{u}} = \underset{\sim}{g_s}\cdot\overline{\underset{\sim}{u}}.$$

Since $\underset{\approx}{\sigma}(\underset{\sim}{u})$ is symmetric, Lemma 2.1 implies that

$$-\omega^2\rho_s\underset{\sim}{u}\cdot\overline{\underset{\sim}{u}} - \operatorname{div}(\underset{\approx}{\sigma}(\underset{\sim}{u})\cdot\overline{\underset{\sim}{u}}) + \underset{\approx}{\sigma}(\underset{\sim}{u}) : \overline{\underset{\approx}{\nabla}\underset{\sim}{u}} = \underset{\sim}{g_s}\cdot\overline{\underset{\sim}{u}},$$

so by Lemma 2.2,

$$-\omega^2\rho_s\underset{\sim}{u}\cdot\overline{\underset{\sim}{u}} - \operatorname{div}(\underset{\approx}{\sigma}(\underset{\sim}{u})\cdot\overline{\underset{\sim}{u}}) + \lambda|\operatorname{div}\underset{\sim}{u}| + 2\mu\underset{\approx}{\epsilon}(\underset{\sim}{u}) : \overline{\underset{\approx}{\epsilon}(\underset{\sim}{u})} = \underset{\sim}{g_s}\cdot\overline{\underset{\sim}{u}}$$

Integrating over Ω_s, and applying the divergence theorem yields

$$-\omega^2\rho_s\|\underset{\sim}{u}\|^2_{L^2(\Omega_s)} - \int_{\partial\Omega_s}\underset{\sim}{n_s}\cdot\underset{\approx}{\sigma}(\underset{\sim}{u})\overline{\underset{\sim}{u}} + \lambda\|\operatorname{div}\underset{\sim}{u}\|^2_{L^2(\Omega_s)} + 2\mu\|\underset{\approx}{\epsilon}(\underset{\sim}{u})\|^2_{L^2(\Omega_s)} = \int_{\Omega_s}\underset{\sim}{g_s}\cdot\overline{\underset{\sim}{u}}.$$

Applying the boundary condition from (2.5) yields:

$$-\omega^2 \rho_s \|\underset{\sim}{u}\|^2_{L^2(\Omega_s)} - i\omega \int_{\partial\Omega_s} \underset{\approx}{A} \underset{\sim}{u}\, \overline{\underset{\sim}{u}} + \lambda \| \operatorname{div} \underset{\sim}{u}\|^2_{L^2(\Omega_s)} + 2\mu \| \underset{\approx}{\epsilon}(\underset{\sim}{u})\|^2_{L^2(\Omega_s)} = \int_{\Omega_s} \underset{\sim}{g}_s \cdot \overline{\underset{\sim}{u}}.$$

Treating the real and imaginary parts separately yields the result. ■

In order to derive a regularity estimate, we will need to have a bound for the $\|\nabla \underset{\sim}{u}\|_{L^2(\Omega_s)}$ in terms of $\|\underset{\sim}{u}\|_{L^2(\Omega_s)}$ and $\|\underset{\sim}{g}_s\|_{L^2(\Omega_s)}$. Unlike the acoustic case, the basic test function identities in Lemma 2.10 do not give us such an estimate directly. We can, however, apply Korn's inequality and estimate (2) of Lemma 2.10 to get a bound for $\|\underset{\sim}{u}\|_{H^1(\Omega_s)}$ which will be sufficient.

Lemma 2.11. *Suppose $\underset{\sim}{u}$ solves (2.5). Then for every $\epsilon > 0$*

$$2\mu K \|\underset{\sim}{u}\|^2_{H^1(\Omega_s)} \le (\omega^2 \rho_s + 2\mu + \tfrac{\epsilon}{2}) \|\underset{\sim}{u}\|^2_{L^2(\Omega_s)} + \tfrac{1}{2\epsilon} \|\underset{\sim}{g}_s\|^2_{L^2(\Omega_s)}.$$

Proof : Lemma 2.10 implies for all $\epsilon > 0$

$$\lambda \| \operatorname{div} \underset{\sim}{u}\|^2_{L^2(\Omega_s)} + 2\mu \| \underset{\approx}{\epsilon}(\underset{\sim}{u})\|^2_{L^2(\Omega_s)} \le \omega^2 \rho_s \|\underset{\sim}{u}\|^2_{L^2(\Omega_s)} + \tfrac{1}{2\epsilon} \|\underset{\sim}{g}_s\|^2_{L^2(\Omega_s)} + \tfrac{\epsilon}{2} \|\underset{\sim}{u}\|^2_{L^2(\Omega_s)}.$$

Dropping the divergence term from the left and adding $2\mu \|\underset{\sim}{u}\|^2_{L^2(\Omega_s)}$ to both sides yields:

$$2\mu \left(\| \underset{\approx}{\epsilon}(\underset{\sim}{u})\|^2_{L^2(\Omega_s)} + \|\underset{\sim}{u}\|^2_{L^2(\Omega_s)} \right) \le (\omega^2 \rho_s + \tfrac{\epsilon}{2} + 2\mu) \|\underset{\sim}{u}\|^2_{L^2(\Omega_s)} + \tfrac{1}{2\epsilon} \|\underset{\sim}{g}_s\|^2_{L^2(\Omega_s)}.$$

Applying Korn's inequality (Lemma 2.9) implies

$$2\mu K \|\underset{\sim}{u}\|^2_{H^1(\Omega_s)} \le (\omega^2 \rho_s + \tfrac{\epsilon}{2} + 2\mu) \|\underset{\sim}{u}\|^2_{L^2(\Omega_s)} + \tfrac{1}{2\epsilon} \|\underset{\sim}{g}_s\|^2_{L^2(\Omega_s)}.$$

■

2.2.1 A Regularity Estimate for the Elastic Problem

Our goal is to apply the Rellich identity (Lemma 2.5) to derive a regularity estimate for solutions to (2.5). If we examine the Rellich identity and look back at the argument for the acoustic problem in Theorem 1.9, it is not hard to anticipate that this approach forces us to handle a term of the form $\|\underset{\approx}{\nabla}\underset{\sim}{u}\|_{L^2(\partial\Omega_s)}$. A simple application of Young's inequality was sufficient to handle the corresponding term in the acoustic case. The elastic case is far more delicate, and will require the following lemma.

Lemma 2.12. *Suppose $\underset{\sim}{u}$ solves (2.5) where $\omega \neq 0$ and Ω_s is a convex polygonal domain. Then there is a constant C, independent of ω such that*

(1) $\|\underset{\sim}{u}\|_{H^2(\Omega_s)} \leq C\omega^2\|\underset{\sim}{u}\|_{L^2(\Omega_s)} + C\|\underset{\sim}{g_s}\|_{L^2(\Omega_s)}$.

(2) $\|\underset{\approx}{\nabla}\underset{\sim}{u}\|^2_{L^2(\partial\Omega_s)} \leq C\omega^3\|\underset{\sim}{u}\|_{L^2(\Omega_s)} + C\frac{1}{\omega}\|\underset{\sim}{g_s}\|_{L^2(\Omega_s)}$.

Proof : Regularity theory for elliptic problems (see [19]) implies

$$\begin{aligned}\|\underset{\sim}{u}\|_{H^2(\Omega_s)} &\leq \|\underset{\sim}{g_s} + \omega^2\rho_s\underset{\sim}{u}\|_{L^2(\Omega_s)} + \omega a_2\|\underset{\sim}{u}\|_{H^{1/2}(\partial\Omega_s)}\\ &\leq \|\underset{\sim}{g_s}\|_{L^2(\Omega_s)} + \omega^2\rho_s\|\underset{\sim}{u}\|_{L^2(\Omega_s)} + \omega a_2\|\underset{\sim}{u}\|_{H^1(\Omega_s)}.\end{aligned}$$

Applying Lemma 2.11,

$$\begin{aligned}\|\underset{\sim}{u}\|_{H^2(\Omega_s)} &\leq \|\underset{\sim}{g_s}\|_{L^2(\Omega_s)} + \omega^2\rho_s\|\underset{\sim}{u}\|_{L^2(\Omega_s)}\\ &\quad + \tfrac{\omega a_2}{\sqrt{2\mu K}}\left\{(\omega^2\rho_s + 2\mu + \tfrac{\epsilon}{2})\|\underset{\sim}{u}\|^2_{L^2(\Omega_s)} + \tfrac{1}{2\epsilon}\|\underset{\sim}{g_s}\|^2_{L^2(\Omega_s)}\right\}^{\frac{1}{2}}\\ &\leq \|\underset{\sim}{g_s}\|_{L^2(\Omega_s)} + \omega^2\rho_s\|\underset{\sim}{u}\|_{L^2(\Omega_s)}\\ &\quad + \tfrac{\omega a_2}{\sqrt{2\mu K}}\sqrt{\omega^2\rho_s + 2\mu + \tfrac{\epsilon}{2}}\|\underset{\sim}{u}\|_{L^2(\Omega_s)} + \tfrac{\omega a_2}{\sqrt{2\mu K}}\tfrac{1}{\sqrt{2\epsilon}}\|\underset{\sim}{g_s}\|_{L^2(\Omega_s)}\\ &\leq \left(\omega^2\rho_s + \tfrac{\omega a_2}{\sqrt{2\mu K}}\sqrt{\omega^2\rho_s + 2\mu + \tfrac{\epsilon}{2}}\right)\|\underset{\sim}{u}\|_{L^2(\Omega_s)} + \left(1 + \tfrac{\omega a_2}{\sqrt{2\mu K}}\tfrac{1}{2\epsilon}\right)\|\underset{\sim}{g_s}\|_{L^2(\Omega_s)}.\end{aligned}$$

for all $\epsilon > 0$. Choosing $\epsilon = O(\omega^2)$ yields (1).

Since Ω_s is a Lipschitz domain, the trace inequality,

$$\|\underset{\sim}{u}\|_{L^2(\partial\Omega_s)} \leq C_s\|\underset{\sim}{u}\|^{1/2}_{L^2(\Omega_s)}\|\underset{\sim}{u}\|^{1/2}_{H^1(\Omega_s)} \tag{2.9}$$

holds for all $\underset{\sim}{u} \in H^1(\Omega_s)$. Therefore,

$$\|\underset{\approx}{\nabla}\underset{\sim}{u}\|^2_{L^2(\partial\Omega_s)} \leq C_s\ \|\underset{\approx}{\nabla}\underset{\sim}{u}\|_{L^2(\Omega_s)}\|\underset{\approx}{\nabla}\underset{\sim}{u}\|_{H^1(\Omega_s)}\ \leq\ C_s\ \|\underset{\approx}{\nabla}\underset{\sim}{u}\|_{L^2(\Omega_s)}\|\underset{\sim}{u}\|_{H^2(\Omega_s)}.$$

Applying Lemma 2.11 and inequality (1) implies for all $\epsilon > 0$

$$\|\underset{\approx}{\nabla}\underset{\sim}{u}\|^2_{L^2(\partial\Omega_s)} \le C_s\left\{(\omega^2\rho_s + 2\mu + \tfrac{\epsilon}{2})\|\underset{\sim}{u}\|^2_{L^2(\Omega_s)} + \tfrac{1}{2\epsilon}\|\underset{\sim}{g}_s\|^2_{L^2(\Omega_s)}\right\}^{\frac{1}{2}}$$
$$\cdot\left\{\omega^2\|\underset{\sim}{u}\|_{L^2(\Omega_s)} + \|\underset{\sim}{g}_s\|_{L^2(\Omega_s)}\right\}$$
$$\le C_s\left\{C\omega^2\sqrt{\omega^2\rho_s + 2\mu + \tfrac{\epsilon}{2}}\|\underset{\sim}{u}\|^2_{L^2(\Omega_s)}\right.$$
$$\left. + \left(C\sqrt{\omega^2\rho_s + 2\mu + \tfrac{\epsilon}{2}} + \tfrac{C\omega^2}{\sqrt{2\epsilon}}\right)\|\underset{\sim}{u}\|_{L^2(\Omega_s)}\|\underset{\sim}{g}_s\|_{L^2(\Omega_s)} + \tfrac{C}{\sqrt{2\epsilon}}\|\underset{\sim}{g}_s\|^2_{L^2(\Omega_s)}\right\}.$$

Therefore,

$$\|\underset{\approx}{\nabla}\underset{\sim}{u}\|^2_{L^2(\partial\Omega_s)} \le C_s\left\{C\omega^2\sqrt{\omega^2\rho_s + 2\mu + \tfrac{\epsilon}{2}}\|\underset{\sim}{u}\|^2_{L^2(\Omega_s)}\right.$$
$$+ \left(C\sqrt{\omega^2\rho_s + 2\mu + \tfrac{\epsilon}{2}} + \tfrac{C\omega^2}{\sqrt{2\epsilon}}\right)\left(\tfrac{\epsilon_1}{2}\|\underset{\sim}{u}\|_{L^2(\Omega_s)} + \tfrac{1}{2\epsilon_1}\|\underset{\sim}{g}_s\|_{L^2(\Omega_s)}\right)$$
$$\left. + \tfrac{C}{\sqrt{2\epsilon}}\|\underset{\sim}{g}_s\|^2_{L^2(\Omega_s)}\right\}$$

or equivalently,

$$\|\underset{\approx}{\nabla}\underset{\sim}{u}\|^2_{L^2(\partial\Omega_s)} \le C_s\left\{C\omega^2\sqrt{\omega^2\rho_s + 2\mu + \tfrac{\epsilon}{2}} + \left(C\sqrt{\omega^2\rho_s + 2\mu + \tfrac{\epsilon}{2}} + \tfrac{C\omega^2}{\sqrt{2\epsilon}}\right)\tfrac{\epsilon_1}{2}\right\}\|\underset{\sim}{u}\|^2_{L^2(\Omega_s)}$$
$$+ C_s\left\{\left(C\sqrt{\omega^2\rho_s + 2\mu + \tfrac{\epsilon}{2}} + \tfrac{C\omega^2}{\sqrt{2\epsilon}}\right)\tfrac{1}{2\epsilon_1} + \tfrac{C}{\sqrt{2\epsilon}}\right\}\|\underset{\sim}{g}_s\|^2_{L^2(\Omega_s)}.$$

Choosing $\epsilon = O(\omega)$ and $\epsilon_1 = O(\omega^2)$ yields (2). ■

Theorem 2.13. *Suppose $\underset{\sim}{u}$ solves (2.5) where Ω_s is a convex polygonal domain. Then $\underset{\sim}{u}$ satisfies the following regularity estimates:*

(1) $\|\underset{\sim}{u}\|_{L^2(\Omega_s)} \le C(1 + \frac{1}{\omega})\|\underset{\sim}{g}_s\|_{L^2(\Omega_s)}$.

(2) $\|\underset{\sim}{u}\|_{H^1(\Omega_s)} \le C(\omega + 1)\|\underset{\sim}{g}_s\|_{L^2(\Omega_s)}$.

(3) $\|\underset{\sim}{u}\|_{H^2(\Omega_s)} \le C(\omega^2 + 1)\|\underset{\sim}{g}_s\|_{L^2(\Omega_s)}$.

Proof : Combining Corollaries 2.6 and 2.8 implies

$$\tfrac{N\omega^2\rho_s}{2}\|\underset{\sim}{u}\|^2_{L^2(\Omega_s)} + \frac{1}{2}\int_{\partial\Omega_s}\underset{\sim}{x}\cdot\underset{\sim}{n}_s\left(\lambda\,|\operatorname{div}\underset{\sim}{u}|^2 + 2\mu\underset{\approx}{\epsilon}(\underset{\sim}{u}) : \overline{\underset{\approx}{\epsilon}(\underset{\sim}{u})}\right) = \tfrac{\omega^2\rho_s}{2}\int_{\partial\Omega_s}\underset{\sim}{x}\cdot\underset{\sim}{n}_s|\underset{\sim}{u}|^2$$
$$+ \left(\tfrac{N}{2} - 1\right)\left(\lambda\|\operatorname{div}\underset{\sim}{u}\|^2_{L^2(\Omega_s)} + 2\mu\|\underset{\approx}{\epsilon}(\underset{\sim}{u})\|^2_{L^2(\Omega_s)}\right) + \operatorname{Re}\int_{\partial\Omega_s}\left(\underset{\approx}{\sigma}(\underset{\sim}{u})\underset{\sim}{n}_s\right)\cdot\left((\overline{\underset{\approx}{\nabla}\underset{\sim}{u}})\underset{\sim}{x}\right)$$
$$+ \operatorname{Re}\int_{\Omega_s}\left(-\omega^2\rho_s\underset{\sim}{u} - \underset{\sim}{\operatorname{div}}\left(\underset{\approx}{\sigma}(\underset{\sim}{u})\right)\right)\cdot\left((\overline{\underset{\approx}{\nabla}\underset{\sim}{u}})\underset{\sim}{x}\right).$$

Since $\underset{\sim}{u}$ solves (2.5), the above becomes

$$\tfrac{N\omega^2\rho_s}{2}\|\underset{\sim}{u}\|^2_{L^2(\Omega_s)} + \frac{1}{2}\int_{\partial\Omega_s} \underset{\sim}{x}\cdot\underset{\sim}{n}_s\left(\lambda\,|\operatorname{div}\underset{\sim}{u}|^2 + 2\mu\underset{\approx}{\epsilon}(\underset{\sim}{u}):\overline{\underset{\approx}{\epsilon}(\underset{\sim}{u})}\right) = \tfrac{\omega^2\rho_s}{2}\int_{\partial\Omega_s}\underset{\sim}{x}\cdot\underset{\sim}{n}_s|\underset{\sim}{u}|^2$$
$$+\left(\tfrac{N}{2}-1\right)\left(\lambda\|\operatorname{div}\underset{\sim}{u}\|^2_{L^2(\Omega_s)} + 2\mu\|\underset{\approx}{\epsilon}(\underset{\sim}{u})\|^2_{L^2(\Omega_s)}\right)$$
$$-\operatorname{Re} i\omega\int_{\partial\Omega_s}\underset{\approx}{A}\underset{\sim}{u}\cdot\left((\overline{\underset{\approx}{\nabla}\underset{\sim}{u}})\underset{\sim}{x}\right) + \operatorname{Re}\int_{\Omega_s}\underset{\sim}{g}_s\cdot\left((\overline{\underset{\approx}{\nabla}\underset{\sim}{u}})\underset{\sim}{x}\right).$$

Applying the trace inequality (2.9) implies

$$\begin{aligned}
&\tfrac{\omega^2\rho_s N}{2}\|\underset{\sim}{u}\|^2_{L^2(\Omega_s)} + \tfrac{\gamma_1\lambda}{2}\|\operatorname{div}\underset{\sim}{u}\|^2_{L^2(\partial\Omega_s)} + \tfrac{\gamma_1 2\mu}{2}\|\underset{\approx}{\epsilon}(\underset{\sim}{u})\|^2_{L^2(\partial\Omega_s)}\\
&\quad\le \tfrac{\omega^2\rho_s}{2}\gamma_2\|\underset{\sim}{u}\|^2_{L^2(\partial\Omega_s)} + \left(\tfrac{N}{2}-1\right)\left\{\lambda\,\|\operatorname{div}\underset{\sim}{u}\|^2_{L^2(\Omega_s)} + 2\mu\,\|\underset{\approx}{\epsilon}(\underset{\sim}{u})\|^2_{L^2(\Omega_s)}\right\}\\
&\qquad-\operatorname{Re}\, i\omega\int_{\partial\Omega_s}\underset{\approx}{A}\underset{\sim}{u}\cdot\left((\overline{\underset{\approx}{\nabla}\underset{\sim}{u}})\underset{\sim}{x}\right) + \operatorname{Re}\int_{\Omega_s}\underset{\sim}{g}_s\cdot\overline{(\underset{\approx}{\nabla}\underset{\sim}{u})}\underset{\sim}{x}.
\end{aligned}\tag{2.10}$$

Hence,

$$\begin{aligned}
\tfrac{\omega^2\rho_s N}{2}\|\underset{\sim}{u}\|^2_{L^2(\Omega_s)} &\le \tfrac{\omega^2\rho_s}{2}\gamma_2\|\underset{\sim}{u}\|^2_{L^2(\partial\Omega_s)}\\
&\quad+\left(\tfrac{N}{2}-1\right)\left[\lambda\,\|\operatorname{div}\underset{\sim}{u}\|^2_{L^2(\Omega_s)} + 2\mu\,\|\underset{\approx}{\epsilon}(\underset{\sim}{u})\|^2_{L^2(\Omega_s)}\right]\\
&\quad-\operatorname{Re}\, i\omega\int_{\partial\Omega_s}\underset{\approx}{A}\underset{\sim}{u}\cdot\left[(\overline{\underset{\approx}{\nabla}\underset{\sim}{u}})\underset{\sim}{x}\right] + \operatorname{Re}\int_{\Omega_s}\underset{\sim}{g}_s\cdot\overline{(\underset{\approx}{\nabla}\underset{\sim}{u})}\underset{\sim}{x}.
\end{aligned}\tag{2.11}$$

Applying Schwarz's inequality and Lemma 2.10 yields

$$\begin{aligned}
\tfrac{\omega^2\rho_s N}{2}\|\underset{\sim}{u}\|^2_{L^2(\Omega_s)} &\le \tfrac{\omega^2\rho_s}{2}\gamma_2\|\underset{\sim}{u}\|^2_{L^2(\partial\Omega_s)}\\
&\quad+\left(\tfrac{N}{2}-1\right)\omega^2\rho_s\,\|\underset{\sim}{u}\|^2_{L^2(\Omega_s)} + \left(\tfrac{N}{2}-1\right)\,\|\underset{\sim}{g}_s\|_{L^2(\Omega_s)}\|\underset{\sim}{u}\|_{L^2(\Omega_s)}\\
&\quad+\omega a_2 M\|\underset{\sim}{u}\|_{L^2(\partial\Omega_s)}\|\underset{\approx}{\nabla}\underset{\sim}{u}\|_{L^2(\partial\Omega_s)} + M\|\underset{\sim}{g}_s\|_{L^2(\Omega_s)}\|\underset{\approx}{\nabla}\underset{\sim}{u}\|_{L^2(\Omega_s)}.
\end{aligned}$$

Moving the $\left(\frac{N}{2}-1\right)\omega^2\rho_s\,\|\underset{\sim}{u}\|^2_{L^2(\Omega_s)}$ term to the left side gives

$$\begin{aligned}
\omega^2\rho_s\|\underset{\sim}{u}\|^2_{L^2(\Omega_s)} &\le \tfrac{\omega^2\rho_s}{2}\gamma_2\|\underset{\sim}{u}\|^2_{L^2(\partial\Omega_s)}\\
&\quad+\left(\tfrac{N}{2}-1\right)\,\|\underset{\sim}{g}_s\|_{L^2(\Omega_s)}\|\underset{\sim}{u}\|_{L^2(\Omega_s)}\\
&\quad+M\|\underset{\sim}{g}_s\|_{L^2(\Omega_s)}\|\underset{\approx}{\nabla}\underset{\sim}{u}\|_{L^2(\Omega_s)} + \omega a_2 M\|\underset{\sim}{u}\|_{L^2(\partial\Omega_s)}\|\underset{\approx}{\nabla}\underset{\sim}{u}\|_{L^2(\partial\Omega_s)}.
\end{aligned}$$

Applying Young's inequality implies for any ϵ_1, ϵ_2, $\epsilon_3 > 0$

$$\begin{aligned}\omega^2\rho_s\|\underset{\sim}{u}\|^2_{L^2(\Omega_s)} \leq\ & \tfrac{\omega^2\rho_s}{2}\gamma_2\|\underset{\sim}{u}\|^2_{L^2(\partial\Omega_s)}\\ & + \left(\tfrac{N}{2}-1\right)\left(\tfrac{1}{2\epsilon_3}\|\underset{\sim}{g}_s\|^2_{L^2(\Omega_s)} + \tfrac{\epsilon_3}{2}\|\underset{\sim}{u}\|^2_{L^2(\Omega_s)}\right)\\ & + M\left(\tfrac{1}{\epsilon_2}\|\underset{\sim}{g}_s\|^2_{L^2(\Omega_s)} + \tfrac{\epsilon_2}{2}\|\underset{\approx}{\nabla}\underset{\sim}{u}\|^2_{L^2(\Omega_s)}\right)\\ & + \omega a_2 M\left(\tfrac{1}{\epsilon_1}\|\underset{\sim}{u}\|^2_{L^2(\partial\Omega_s)} + \tfrac{\epsilon_1}{2}\|\underset{\approx}{\nabla}\underset{\sim}{u}\|^2_{L^2(\partial\Omega_s)}\right).\end{aligned}$$

After rearranging, we get

$$\begin{aligned}\omega^2\rho_s\|\underset{\sim}{u}\|^2_{L^2(\Omega_s)} \leq\ & \left(\tfrac{\omega^2\rho_s}{2}\gamma_2 + \tfrac{\omega a_2 M}{2\epsilon_1}\right)\|\underset{\sim}{u}\|^2_{L^2(\partial\Omega_s)} + \left(\tfrac{N}{2}-1\right)\tfrac{\epsilon_3}{2}\|\underset{\sim}{u}\|^2_{L^2(\Omega_s)}\\ & + \left(\left(\tfrac{N}{2}-1\right)\tfrac{1}{2\epsilon_3} + \tfrac{M}{\epsilon_2}\right)\|\underset{\sim}{g}_s\|^2_{L^2(\Omega_s)}\\ & + \tfrac{M\epsilon_2}{2}\|\underset{\approx}{\nabla}\underset{\sim}{u}\|^2_{L^2(\Omega_s)} + \tfrac{\omega a_2 M\epsilon_1}{2}\|\underset{\approx}{\nabla}\underset{\sim}{u}\|^2_{L^2(\partial\Omega_s)}.\end{aligned}$$

Applying Lemma 2.11,

$$\begin{aligned}\omega^2\rho_s\|\underset{\sim}{u}\|^2_{L^2(\Omega_s)} \leq\ & \left(\tfrac{\omega^2\rho_s}{2}\gamma_2 + \tfrac{\omega a_2 M}{2\epsilon_1}\right)\|\underset{\sim}{u}\|^2_{L^2(\partial\Omega_s)} + \left(\tfrac{N}{2}-1\right)\tfrac{\epsilon_3}{2}\|\underset{\sim}{u}\|^2_{L^2(\Omega_s)}\\ & + \left(\left(\tfrac{N}{2}-1\right)\tfrac{1}{2\epsilon_3} + \tfrac{M}{\epsilon_2}\right)\|\underset{\sim}{g}_s\|^2_{L^2(\Omega_s)}\\ & + \tfrac{M\epsilon_2}{2}\tfrac{1}{2\mu K}\left\{(\omega^2\rho_s + 2\mu + \tfrac{\epsilon_4}{2})\|\underset{\sim}{u}\|^2_{L^2(\Omega_s)} + \tfrac{1}{2\epsilon_4}\|\underset{\sim}{g}_s\|^2_{L^2(\Omega_s)}\right\}\\ & + \tfrac{\omega a_2 M\epsilon_1}{2}\|\underset{\approx}{\nabla}\underset{\sim}{u}\|^2_{L^2(\partial\Omega_s)},\end{aligned}$$

or equivalently,

$$\begin{aligned}\omega^2\rho_s\|\underset{\sim}{u}\|^2_{L^2(\Omega_s)} \leq\ & \left(\tfrac{\omega^2\rho_s}{2}\gamma_2 + \tfrac{\omega a_2 M}{2\epsilon_1}\right)\|\underset{\sim}{u}\|^2_{L^2(\partial\Omega_s)}\\ & + \left\{\tfrac{M\epsilon_2}{4\mu K}(\omega^2\rho_s + 2\mu + \tfrac{\epsilon_4}{2}) + \tfrac{(N-2)\epsilon_3}{4}\right\}\|\underset{\sim}{u}\|^2_{L^2(\Omega_s)}\\ & + \left(\tfrac{M\epsilon_2}{4\mu K\epsilon_4} + \tfrac{N-2}{4\epsilon_3} + \tfrac{M}{2\epsilon_2}\right)\|\underset{\sim}{g}_s\|^2_{L^2(\Omega_s)}\\ & + \tfrac{\omega a_2 M\epsilon_1}{2}\|\underset{\approx}{\nabla}\underset{\sim}{u}\|^2_{L^2(\partial\Omega_s)}.\end{aligned}$$

Since $\underset{\approx}{A}$ is constant, symmetric and positive definite, there exists a positive constant a_1 such that

$$a_1\overline{\underset{\sim}{x}}\cdot\underset{\sim}{x} \leq \overline{\underset{\sim}{x}}^T\underset{\approx}{A}\underset{\sim}{x} \quad \text{for all } \underset{\sim}{x}\in\mathcal{C}. \tag{2.12}$$

Hence using Lemma 2.10 to bound $\|\underset{\sim}{u}\|^2_{L^2(\partial\Omega_s)}$,

$$\omega^2\rho_s\|\underset{\sim}{u}\|^2_{L^2(\Omega_s)} \le \left(\tfrac{\omega^2\rho_s}{2}\gamma_2 + \tfrac{\omega a_2 M}{2\epsilon_1}\right)\tfrac{1}{\omega a_1}\left(\tfrac{1}{2\epsilon_5}\|\underset{\sim}{g}_s\|^2_{L^2(\Omega_s)} + \tfrac{\epsilon_5}{2}\|\underset{\sim}{u}\|^2_{L^2(\Omega_s)}\right)$$
$$+\left\{\tfrac{M\epsilon_2}{4\mu K}(\omega^2\rho_s + 2\mu + \tfrac{\epsilon_4}{2}) + \tfrac{(N-2)\epsilon_3}{4}\right\}\|\underset{\sim}{u}\|^2_{L^2(\Omega_s)}$$
$$+\left(\tfrac{M\epsilon_2}{4\mu K\epsilon_4} + \tfrac{N-2}{4\epsilon_3} + \tfrac{M}{2\epsilon_2}\right)\|\underset{\sim}{g}_s\|^2_{L^2(\Omega_s)}$$
$$+\tfrac{\omega a_2 M\epsilon_1}{2}\|\underset{\approx}{\nabla}\underset{\sim}{u}\|^2_{L^2(\partial\Omega_s)}.$$

Rearranging again,

$$\omega^2\rho_s\|\underset{\sim}{u}\|^2_{L^2(\Omega_s)} \le \left\{\left(\tfrac{\omega^2\rho_s}{2}\gamma_2 + \tfrac{\omega a_2 M}{2\epsilon_1}\right)\tfrac{1}{\omega a_1}\tfrac{\epsilon_5}{2} + \tfrac{M\epsilon_2}{4\mu K}(\omega^2\rho_s + 2\mu + \tfrac{\epsilon_4}{2}) + \tfrac{(N-2)\epsilon_3}{4}\right\}\|\underset{\sim}{u}\|^2_{L^2(\Omega_s)}$$
$$+\left\{\left(\tfrac{\omega^2\rho_s}{2}\gamma_2 + \tfrac{\omega a_2 M}{2\epsilon_1}\right)\tfrac{1}{\omega a_1}\tfrac{1}{2\epsilon_5} + \tfrac{M\epsilon_2}{4\mu K\epsilon_4} + \tfrac{N-2}{4\epsilon_3} + \tfrac{M}{2\epsilon_2}\right\}\|\underset{\sim}{g}_s\|^2_{L^2(\Omega_s)}$$
$$+\tfrac{\omega a_2 M\epsilon_1}{2}\|\underset{\approx}{\nabla}\underset{\sim}{u}\|^2_{L^2(\partial\Omega_s)}.$$

Applying Lemma 2.12,

$$\omega^2\rho_s\|\underset{\sim}{u}\|^2_{L^2(\Omega_s)} \le \left\{\left(\tfrac{\omega^2\rho_s}{2}\gamma_2 + \tfrac{\omega a_2 M}{2\epsilon_1}\right)\tfrac{1}{\omega a_1}\tfrac{\epsilon_5}{2} + \tfrac{M\epsilon_2}{4\mu K}(\omega^2\rho_s + 2\mu + \tfrac{\epsilon_4}{2}) + \tfrac{(N-2)\epsilon_3}{4}\right\}\|\underset{\sim}{u}\|^2_{L^2(\Omega_s)}$$
$$+\left\{\left(\tfrac{\omega^2\rho_s}{2}\gamma_2 + \tfrac{\omega a_2 M}{2\epsilon_1}\right)\tfrac{1}{\omega a_1}\tfrac{1}{2\epsilon_5} + \tfrac{M\epsilon_2}{4\mu K\epsilon_4} + \tfrac{N-2}{4\epsilon_3} + \tfrac{M}{2\epsilon_2}\right\}\|\underset{\sim}{g}_s\|^2_{L^2(\Omega_s)}$$
$$+\tfrac{\omega a_2 M\epsilon_1}{2}\left\{C\omega^3\|\underset{\sim}{u}\|^2_{L^2(\Omega_s)} + C\tfrac{1}{w}\|\underset{\sim}{g}_s\|^2_{L^2(\Omega_s)}\right\},$$

which implies

$$\omega^2\rho_s\|\underset{\sim}{u}\|^2_{L^2(\Omega_s)} \le \left\{\left(\tfrac{\omega\rho_s\gamma_2}{2} + \tfrac{a_2 M}{2\epsilon_1 a_1}\right)\tfrac{1}{2\epsilon_5} + \tfrac{M\epsilon_2}{4\mu K\epsilon_4} + \tfrac{N-2}{4\epsilon_3} + \tfrac{M}{2\epsilon_2} + \tfrac{a_2 MC}{2}\epsilon_1\right\}\|\underset{\sim}{g}_s\|^2_{L^2(\Omega_s)}$$
$$+\left\{\left(\tfrac{\omega\rho_s\gamma_2}{2a_1} + \tfrac{a_2 M}{2\epsilon_1 a_1}\right)\tfrac{\epsilon_5}{2} + \tfrac{m\epsilon_2}{4\mu k}(\omega^2\rho_s + 2\mu + \tfrac{\epsilon_4}{2}) + \tfrac{(N-2)\epsilon_3}{4} + \tfrac{\omega^4 a_2 Mc}{2}\epsilon_1\right\}\|\underset{\sim}{u}\|^2_{L^2(\Omega_s)}.$$

Choosing the constants as follows

$$\epsilon_1 = \tfrac{\rho_s}{4\omega^2 a_2 MC}, \qquad \epsilon_2 = \tfrac{\omega^2\rho_s\mu K}{4M(2\omega^2\rho_s + 2\mu)},$$
$$\epsilon_3 = \tfrac{\omega^2\rho_s}{2(N-2)}, \qquad \epsilon_4 = 2\omega^2\rho_s,$$
$$\epsilon_5 = \tfrac{\omega^2\rho_s}{2}\tfrac{a_1\rho_s}{4\omega^2 C{a_2}^2 M^2 + \omega{\rho_s}^2\gamma_2},$$

implies that

$$\omega^2\rho_s\|\underset{\sim}{u}\|^2_{L^2(\Omega_s)} \le \tfrac{\omega^2\rho_s}{2}\|\underset{\sim}{u}\|^2_{L^2(\Omega_s)}$$
$$+\left\{\left(\tfrac{\omega\rho_s\gamma_2}{2} + \tfrac{a_2 M}{2\epsilon_1 a_1}\right)\tfrac{1}{2\epsilon_5} + \tfrac{M\epsilon_2}{4\mu K\epsilon_4} + \tfrac{N-2}{4\epsilon_3} + \tfrac{M}{2\epsilon_2} + \tfrac{a_2 MC}{2}\epsilon_1\right\}\|\underset{\sim}{g}_s\|^2_{L^2(\Omega_s)}$$

or equivalently,

$$\omega^2 \|\underset{\sim}{u}\|^2_{L^2(\Omega_s)} \leq C \left(\omega^2 + 1\right) \|\underset{\sim}{g}_s\|^2_{L^2(\Omega_s)}$$

It follows from the above equation and Lemma 2.11 that (for all $\epsilon > 0$)

$$2\mu K \|\underset{\sim}{u}\|^2_{H^1(\Omega_s)} \leq \left(\omega^2 \rho_s + 2\mu + \tfrac{\epsilon}{2}\right) C \left(1 + \tfrac{1}{\omega^2} + \tfrac{1}{\omega^2}\right) \|\underset{\sim}{g}_s\|^2_{L^2(\Omega_s)} + \tfrac{1}{2\epsilon} \|\underset{\sim}{g}_s\|^2_{L^2(\Omega_s)}.$$

Choosing $\epsilon = O(\omega^2)$ implies

$$\|\underset{\sim}{u}\|_{H^1(\Omega_s)} \leq C(\omega + 1) \|\underset{\sim}{g}_s\|_{L^2(\Omega_s)}$$

for some constant C independent of ω.

Finally, regularity for $\underset{\sim}{\operatorname{div}} \left(\underset{\approx}{\sigma}(\cdot)\right)$ implies

$$\begin{aligned}
\|\underset{\sim}{u}\|_{H^2(\Omega_s)} &\leq B \left(\|\underset{\sim}{g}_s + \omega^2 \underset{\sim}{u}\|_{L^2(\Omega_s)} + \omega \|\underset{\approx}{A}\underset{\sim}{u}\|_{H^{1/2}(\partial\Omega_s)} \right) \\
&\leq B \left(\|\underset{\sim}{g}_s\|_{L^2(\Omega_s)} + \omega^2 \|\underset{\sim}{u}\|_{L^2(\Omega_s)} + \omega a_2 \|\underset{\sim}{u}\|_{H^1(\Omega_s)} \right) \\
&\leq B \left(\|\underset{\sim}{g}_s\|_{L^2(\Omega_s)} + \omega^2 C (1 + \tfrac{1}{\omega}) \|\underset{\sim}{g}_s\|_{L^2(\Omega_s)} + \omega a_1 C (1 + \omega) \|\underset{\sim}{g}_s\|_{L^2(\Omega_s)} \right) \\
&\leq B \left\{ 1 + C(\omega^2 + \omega) + \omega a_1 C(\omega + 1) \right\} \|\underset{\sim}{g}_s\|_{L^2(\Omega_s)}.
\end{aligned}$$

That is,

$$\|\underset{\sim}{u}\|_{H^2(\Omega_s)} \leq C(\omega^2 + 1) \|\underset{\sim}{g}_s\|_{L^2(\Omega_s)}$$

for some C independent of ω. ■

2.2.2 An Improved Regularity Estimate

Although the estimate of Theorem 2.13 improves previously known estimates obtained in [16] for the elastic Helmholtz problem (2.5), we believe that the estimate is not optimal. As we mentioned in the previous section, handling the $\|\underset{\approx}{\nabla}\underset{\sim}{u}\|_{L^2(\partial\Omega_s)}$ term creates a difficulty that is not an issue in the acoustic problem. In fact, this term is the main obstacle to achieving an optimal regularity estimate in the elastic case. In this section, we improve the regularity estimate to what we believe to be an optimal estimate. In the proof of our improved estimate, we will not simply discard the "good" boundary integral from the Rellich identity as we did in the proof of Theorem 2.13 (see inequality 2.11). We will handle the difficult term with a technique similar to that of Theorem 1.9. To do so, we need the following Korn-type inequality on the boundary of Ω_s.

Conjecture 1. *Suppose $\underset{\sim}{u}$ solves (2.5). Then $\underset{\sim}{u}$ satisfies one of these inequalities*

(1) $\|\underset{\approx}{\nabla}\underset{\sim}{u}\|^2_{L^2(\partial\Omega_s)} \leq K_1 \left\{ \|\underset{\approx}{\epsilon}(\underset{\sim}{u})\|^2_{L^2(\partial\Omega_s)} + \|\underset{\approx}{\nabla}\underset{\sim}{u}\|^2_{L^2(\Omega_s)} \right\}.$

(2) $\|\underset{\approx}{\nabla}\underset{\sim}{u}\|^2_{L^2(\partial\Omega_s)} \leq K_1 \left\{ \|\underset{\approx}{\epsilon}(\underset{\sim}{u})\|^2_{L^2(\partial\Omega_s)} + \|\underset{\approx}{\nabla}\underset{\sim}{u}\|^2_{L^2(\Omega_s)} + \|\operatorname{div}\underset{\sim}{u}\|^2_{L^2(\partial\Omega_s)} \right\}.$

As of this writing, we have been unable to establish the above inequalities. However, a similar result appears in [7] for solutions to the traction boundary value problem for the Lamé systems of elastostatics. The Lamé systems are similar to the Helmholtz system (2.5), and we are therefore confident that the techniques in [7] can be applied to show that the above estimates are valid.

Theorem 2.14. *Suppose $\underset{\sim}{u}$ solves (2.5). Then $\underset{\sim}{u}$ satisfies the following regularity estimates*

(1) $\|\underset{\sim}{u}\|_{L^2(\Omega_s)} \leq C(\frac{1}{\omega} + \frac{1}{\omega^2})\|\underset{\sim}{g}_s\|_{L^2(\Omega_s)}.$

(2) $\|\underset{\sim}{u}\|_{H^1(\Omega_s)} \leq C(1 + \frac{1}{\omega})\|\underset{\sim}{g}_s\|^2_{L^2(\Omega_s)}.$

(3) *If Ω_s is a convex polygonal domain, then*

$$\|\underset{\sim}{u}\|_{H^2(\Omega_s)} \leq C(\omega + 1)\|\underset{\sim}{g}_s\|_{L^2(\Omega_s)}.$$

Proof : Recall inequality (2.10) from Theorem 2.13

$$\frac{\omega^2\rho_s N}{2}\|\underset{\sim}{u}\|^2_{L^2(\Omega_s)} + \frac{\gamma_1\lambda}{2}\|\operatorname{div}\underset{\sim}{u}\|^2_{L^2(\partial\Omega_s)} + \frac{\gamma_1 2\mu}{2}\|\underset{\approx}{\epsilon}(\underset{\sim}{u})\|^2_{L^2(\partial\Omega_s)}$$
$$\leq \frac{\omega^2\rho_s}{2}\gamma_2\|\underset{\sim}{u}\|^2_{L^2(\partial\Omega_s)} + \left(\frac{N}{2} - 1\right)\left[\lambda\,\|\operatorname{div}\underset{\sim}{u}\|^2_{L^2(\Omega_s)} + 2\mu\,\|\underset{\approx}{\epsilon}(\underset{\sim}{u})\|^2_{L^2(\Omega_s)}\right]$$
$$- \operatorname{Re}\, i\omega \int_{\partial\Omega_s} \underset{\approx}{A}\underset{\sim}{u} \cdot \left((\underset{\approx}{\nabla}\underset{\sim}{u})\underset{\sim}{x}\right) + \operatorname{Re}\int_{\Omega_s} \underset{\sim}{g}_s \cdot \overline{(\underset{\approx}{\nabla}\underset{\sim}{u})\underset{\sim}{x}}.$$

Now drop the divergence term from the left hand side and add $\xi\|\underset{\approx}{\nabla}\underset{\sim}{u}\|^2_{L^2(\Omega_s)}$ (where $\xi \le \gamma_1\mu$) to both sides to get

$$
\begin{aligned}
&\tfrac{\omega^2\rho_s N}{2}\|\underset{\sim}{u}\|^2_{L^2(\Omega_s)} + \gamma_1\mu\|\underset{\approx}{\epsilon}(\underset{\sim}{u})\|^2_{L^2(\partial\Omega_s)} + \xi\|\underset{\approx}{\nabla}\underset{\sim}{u}\|^2_{L^2(\Omega_s)} \\
&\quad \le \tfrac{\omega^2\rho_s}{2}\gamma_2\|\underset{\sim}{u}\|^2_{L^2(\partial\Omega_s)} + \left(\tfrac{N}{2}-1\right)\left[\lambda\,\|\operatorname{div}\underset{\sim}{u}\|^2_{L^2(\Omega_s)} + 2\mu\,\|\underset{\approx}{\epsilon}(\underset{\sim}{u})\|^2_{L^2(\Omega_s)}\right] \\
&\qquad - \operatorname{Re} i\omega\int_{\partial\Omega_s}\underset{\approx}{A}\underset{\sim}{u}\cdot\left[\overline{(\underset{\approx}{\nabla}\underset{\sim}{u})}\underset{\sim}{x}\right] + \operatorname{Re}\int_{\Omega_s}\underset{\sim}{g}_s\cdot\overline{(\underset{\approx}{\nabla}\underset{\sim}{u})}\underset{\sim}{x} + \xi\|\underset{\approx}{\nabla}\underset{\sim}{u}\|^2_{L^2(\Omega_s)},
\end{aligned}
$$

which implies

$$
\begin{aligned}
&\tfrac{\omega^2\rho_s N}{2}\|\underset{\sim}{u}\|^2_{L^2(\Omega_s)} + \xi\left(\|\underset{\approx}{\epsilon}(\underset{\sim}{u})\|^2_{L^2(\partial\Omega_s)} + \|\underset{\approx}{\nabla}\underset{\sim}{u}\|^2_{L^2(\Omega_s)}\right) \\
&\quad \le \tfrac{\omega^2\rho_s}{2}\gamma_2\|\underset{\sim}{u}\|^2_{L^2(\partial\Omega_s)} + \left(\tfrac{N}{2}-1\right)\left[\lambda\,\|\operatorname{div}\underset{\sim}{u}\|^2_{L^2(\Omega_s)} + 2\mu\,\|\underset{\approx}{\epsilon}(\underset{\sim}{u})\|^2_{L^2(\Omega_s)}\right] \\
&\qquad - \operatorname{Re} i\omega\int_{\partial\Omega_s}\underset{\approx}{A}\underset{\sim}{u}\cdot\left[\overline{(\underset{\approx}{\nabla}\underset{\sim}{u})}\underset{\sim}{x}\right] + \operatorname{Re}\int_{\Omega_s}\underset{\sim}{g}_s\cdot\overline{(\underset{\approx}{\nabla}\underset{\sim}{u})}\underset{\sim}{x} + \xi\|\underset{\approx}{\nabla}\underset{\sim}{u}\|^2_{L^2(\Omega_s)}.
\end{aligned}
$$

Applying Schwarz's inequality and Conjecture 1 yields

$$
\begin{aligned}
&\tfrac{\omega^2\rho_s N}{2}\|\underset{\sim}{u}\|^2_{L^2(\Omega_s)} + \tfrac{\xi}{K_1}\|\underset{\approx}{\nabla}\underset{\sim}{u}\|^2_{L^2(\partial\Omega_s)} \\
&\quad \le \tfrac{\omega^2\rho_s}{2}\gamma_2\|\underset{\sim}{u}\|^2_{L^2(\partial\Omega_s)} + \left(\tfrac{N}{2}-1\right)\left[\lambda\,\|\operatorname{div}\underset{\sim}{u}\|^2_{L^2(\Omega_s)} + 2\mu\,\|\underset{\approx}{\epsilon}(\underset{\sim}{u})\|^2_{L^2(\Omega_s)}\right] \\
&\qquad + \omega M\|\underset{\approx}{A}\underset{\sim}{u}\|_{L^2(\partial\Omega_s)}\|\underset{\approx}{\nabla}\underset{\sim}{u}\|_{L^2(\partial\Omega_s)} + M\|\underset{\sim}{g}_s\|_{L^2(\Omega_s)}\|\underset{\approx}{\nabla}\underset{\sim}{u}\|_{L^2(\Omega_s)} + \xi\|\underset{\approx}{\nabla}\underset{\sim}{u}\|^2_{L^2(\Omega_s)}.
\end{aligned}
$$

It follows from Young's inequality that for any $\epsilon_1,\ \epsilon_2 > 0$

$$
\begin{aligned}
&\tfrac{\omega^2\rho_s N}{2}\|\underset{\sim}{u}\|^2_{L^2(\Omega_s)} + \tfrac{\xi}{K_1}\|\underset{\approx}{\nabla}\underset{\sim}{u}\|^2_{L^2(\partial\Omega_s)} \\
&\quad \le \tfrac{\omega^2\rho_s}{2}\gamma_2\|\underset{\sim}{u}\|^2_{L^2(\partial\Omega_s)} + \left(\tfrac{N}{2}-1\right)\left[\lambda\,\|\operatorname{div}\underset{\sim}{u}\|^2_{L^2(\Omega_s)} + 2\mu\,\|\underset{\approx}{\epsilon}(\underset{\sim}{u})\|^2_{L^2(\Omega_s)}\right] \\
&\qquad + \omega M\left[\tfrac{1}{2\epsilon_1}\|\underset{\approx}{A}\underset{\sim}{u}\|^2_{L^2(\partial\Omega_s)} + \tfrac{\epsilon_1}{2}\|\underset{\approx}{\nabla}\underset{\sim}{u}\|^2_{L^2(\partial\Omega_s)}\right] + M\left[\tfrac{1}{2\epsilon_2}\|\underset{\sim}{g}_s\|^2_{L^2(\Omega_s)} + \tfrac{\epsilon_2}{2}\|\underset{\approx}{\nabla}\underset{\sim}{u}\|^2_{L^2(\Omega_s)}\right] \\
&\qquad + \xi\|\underset{\approx}{\nabla}\underset{\sim}{u}\|^2_{L^2(\Omega_s)}.
\end{aligned}
$$

Applying Lemma 2.10 and Schwarz's inequality implies for any $\epsilon_3 > 0$

$$
\begin{aligned}
&\tfrac{\omega^2\rho_s N}{2}\|\underset{\sim}{u}\|^2_{L^2(\Omega_s)} + \tfrac{\xi}{K_1}\|\underset{\approx}{\nabla}\underset{\sim}{u}\|^2_{L^2(\partial\Omega_s)} \\
&\quad \le \tfrac{\omega^2\rho_s}{2}\gamma_2\|\underset{\sim}{u}\|^2_{L^2(\partial\Omega_s)} + \left(\tfrac{N}{2}-1\right)\left[\omega^2\rho_s\|\underset{\sim}{u}\|^2_{L^2(\Omega_s)} + \tfrac{1}{2\epsilon_3}\|\underset{\sim}{g}_s\|^2_{L^2(\Omega_s)} + \tfrac{\epsilon_3}{2}\|\underset{\sim}{u}\|^2_{L^2(\Omega_s)}\right] \\
&\qquad + \tfrac{\omega M a_2}{2\epsilon_1}\|\underset{\sim}{u}\|^2_{L^2(\partial\Omega_s)} + \tfrac{M\omega\epsilon_1}{2}\|\underset{\approx}{\nabla}\underset{\sim}{u}\|^2_{L^2(\partial\Omega_s)} + \tfrac{M}{2\epsilon_2}\|\underset{\sim}{g}_s\|^2_{L^2(\Omega_s)} + \left(\tfrac{M\epsilon_2}{2}+\xi\right)\|\underset{\approx}{\nabla}\underset{\sim}{u}\|^2_{L^2(\Omega_s)}.
\end{aligned}
$$

Rearranging yields

$$\omega^2\rho_s\left(\frac{N}{2}-\left(\frac{N}{2}-1\right)\right)\|\underset{\sim}{u}\|^2_{L^2(\Omega_s)}+\left(\frac{\xi}{K_1}-\frac{M\omega\epsilon_1}{2}\right)\|\underset{\approx}{\nabla}\underset{\sim}{u}\|^2_{L^2(\partial\Omega_s)}$$
$$\leq\left(\frac{\omega^2\rho_s}{2}\gamma_2+\frac{\omega Ma_2}{2\epsilon_1}\right)\|\underset{\sim}{u}\|^2_{L^2(\partial\Omega_s)}+\left(\frac{N}{2}-1\right)\frac{\epsilon_3}{2}\|\underset{\sim}{u}\|^2_{L^2(\Omega_s)}$$
$$+\left(\left(\frac{N}{2}-1\right)\frac{1}{2\epsilon_3}+\frac{M}{2\epsilon_2}\right)\|\underset{\sim}{g}_s\|^2_{L^2(\Omega_s)}+\left(\frac{M\epsilon_2}{2}+\xi\right)\|\underset{\approx}{\nabla}\underset{\sim}{u}\|^2_{L^2(\Omega_s)}.$$

Applying Lemma 2.11 to handle the $\|\underset{\approx}{\nabla}\underset{\sim}{u}\|^2_{L^2(\Omega_s)}$ term,

$$\omega^2\rho_s\|\underset{\sim}{u}\|^2_{L^2(\Omega_s)}+\left(\frac{\xi}{K_1}-\frac{M\omega\epsilon_1}{2}\right)\|\underset{\approx}{\nabla}\underset{\sim}{u}\|^2_{L^2(\partial\Omega_s)}$$
$$\leq\left(\frac{\omega^2\rho_s}{2}\gamma_2+\frac{\omega Ma_2}{2\epsilon_1}\right)\|\underset{\sim}{u}\|^2_{L^2(\partial\Omega_s)}+\left(\frac{N}{2}-1\right)\frac{\epsilon_3}{2}\|\underset{\sim}{u}\|^2_{L^2(\Omega_s)}+\left(\frac{N-2}{4\epsilon_3}+\frac{M}{2\epsilon_2}\right)\|\underset{\sim}{g}_s\|^2_{L^2(\Omega_s)}$$
$$+\left(\frac{M\epsilon_2}{2}+\xi\right)\frac{1}{2\mu K}\left(\omega^2\rho_s+2\mu+\frac{\epsilon_4}{2}\right)\|\underset{\sim}{u}\|^2_{L^2(\Omega_s)}+\left(\frac{M\epsilon_2}{2}+\xi\right)\frac{1}{2\mu K}\frac{1}{2\epsilon_4}\|\underset{\sim}{g}_s\|^2_{L^2(\Omega_s)},$$

or equivalently

$$\omega^2\rho_s\|\underset{\sim}{u}\|^2_{L^2(\Omega_s)}+\left(\frac{\xi}{K_1}-\frac{M\omega\epsilon_1}{2}\right)\|\underset{\approx}{\nabla}\underset{\sim}{u}\|^2_{L^2(\partial\Omega_s)}$$
$$\leq\left(\frac{\omega^2\rho_s}{2}\gamma_2+\frac{\omega Ma_2}{2\epsilon_1}\right)\|\underset{\sim}{u}\|^2_{L^2(\partial\Omega_s)}+\left(\frac{M\epsilon_2}{2}+\xi\right)\left(\frac{2\omega^2\rho_s+4\mu+\epsilon_4}{4\mu K}\right)\|\underset{\sim}{u}\|^2_{L^2(\Omega_s)}$$
$$+\left(\frac{(N-2)\epsilon_3}{2}\right)\|\underset{\sim}{u}\|^2_{L^2(\Omega_s)}+\left(\frac{N-2}{4\epsilon_3}+\frac{M}{2\epsilon_2}+\left(\frac{M\epsilon_2}{2}+\xi\right)\frac{1}{4\mu K\epsilon_4}\right)\|\underset{\sim}{g}_s\|^2_{L^2(\Omega_s)}.$$

Using Lemma 2.10 to bound $\|\underset{\sim}{u}\|^2_{L^2(\partial\Omega_s)}$

$$\|\underset{\sim}{u}\|^2_{L^2(\partial\Omega_s)}\leq\frac{1}{\omega a_1}\left(\frac{1}{2\epsilon_5}\|\underset{\sim}{g}_s\|^2_{L^2(\Omega_s)}+\frac{\epsilon_5}{2}\|\underset{\sim}{u}\|^2_{L^2(\Omega_s)}\right).$$

Applying this estimate above yields

$$\omega^2\rho_s\|\underset{\sim}{u}\|^2_{L^2(\Omega_s)}+\left(\frac{\xi}{K_1}-\frac{M\omega\epsilon_1}{2}\right)\|\underset{\approx}{\nabla}\underset{\sim}{u}\|^2_{L^2(\partial\Omega_s)}$$
$$\leq\left(\frac{\omega^2\rho_s}{2}\gamma_2+\frac{\omega Ma_2}{2\epsilon_1}\right)\frac{1}{\omega a_1}\left(\frac{1}{2\epsilon_5}\|\underset{\sim}{g}_s\|^2_{L^2(\Omega_s)}+\frac{\epsilon_5}{2}\|\underset{\sim}{u}\|^2_{L^2(\Omega_s)}\right)$$
$$+\left(\left(\frac{M\epsilon_2}{2}+\xi\right)\frac{2\omega^2\rho_s+4\mu+\epsilon_4}{4\mu K}+\frac{(N-2)\epsilon_3}{4}\right)\|\underset{\sim}{u}\|^2_{L^2(\Omega_s)}$$
$$+\left(\frac{N-2}{4\epsilon_3}+\frac{M}{2\epsilon_2}+\left(\frac{M\epsilon_2}{2}+\xi\right)\frac{1}{4\mu K\epsilon_4}\right)\|\underset{\sim}{g}_s\|^2_{L^2(\Omega_s)}.$$

Combining like terms,

$$\begin{aligned}
&\omega^2\rho_s\|\underset{\sim}{u}\|^2_{L^2(\Omega_s)} + \left(\frac{\xi}{K_1} - \frac{M\omega\epsilon_1}{2}\right)\|\underset{\approx}{\nabla}\underset{\sim}{u}\|^2_{L^2(\partial\Omega_s)} \\
&\quad \le \left(\left(\frac{\omega^2\rho_s}{2}\gamma_2 + \frac{\omega M a_2}{2\epsilon_1}\right)\frac{1}{\omega a_1}\frac{\epsilon_5}{2} + \left(\frac{M\epsilon_2}{2} + \xi\right)\frac{2\omega^2\rho_s + 4\mu + \epsilon_4}{4\mu K} + \frac{(N-2)\epsilon_3}{4}\right)\|\underset{\sim}{u}\|^2_{L^2(\Omega_s)} \\
&\qquad \left(\left(\frac{\omega^2\rho_s}{2}\gamma_2 + \frac{\omega M a_2}{2\epsilon_1}\right)\frac{1}{\omega a_1}\frac{1}{2\epsilon_5} + \frac{(N-2)}{4\epsilon_3} + \frac{M}{2\epsilon_2} + \left(\frac{M\epsilon_2}{2} + \xi\right)\frac{1}{4\mu K\epsilon_4}\right)\|\underset{\sim}{g}_s\|^2_{L^2(\Omega_s)}.
\end{aligned}$$

Simplifying,

$$\begin{aligned}
&\omega^2\rho_s\|\underset{\sim}{u}\|^2_{L^2(\Omega_s)} + \left(\frac{\xi}{K_1} - \frac{M\omega\epsilon_1}{2}\right)\|\underset{\approx}{\nabla}\underset{\sim}{u}\|^2_{L^2(\partial\Omega_s)} \\
&\quad \le \left\{\left(\frac{\omega\rho_s}{2a_1}\gamma_2 + \frac{M a_2}{2\epsilon_1 a_1}\right)\frac{\epsilon_5}{2} + \left(\frac{M\epsilon_2}{2} + \xi\right)\frac{2\omega^2\rho_s + 4\mu + \epsilon_4}{4\mu K} + \frac{(N-2)\epsilon_3}{4}\right\}\|\underset{\sim}{u}\|^2_{L^2(\Omega_s)} \\
&\qquad \left\{\left(\frac{\omega\rho_s}{2a_1}\gamma_2 + \frac{M a_2}{2\epsilon_1 a_1}\right)\frac{1}{2\epsilon_5} + \left(\frac{M\epsilon_2}{2} + \xi\right)\frac{1}{4\mu K\epsilon_4} + \frac{M}{2\epsilon_2} + \frac{(N-2)}{4\epsilon_3}\right\}\|\underset{\sim}{g}_s\|^2_{L^2(\Omega_s)},
\end{aligned}$$

or

$$(\omega^2\rho_s - C_1)\|\underset{\sim}{u}\|^2_{L^2(\Omega_s)} + \left(\frac{\xi}{K_1} - \frac{M\omega\epsilon_1}{2}\right)\|\underset{\approx}{\nabla}\underset{\sim}{u}\|^2_{L^2(\partial\Omega_s)} \le C_2\|\underset{\sim}{g}_s\|^2_{L^2(\Omega_s)},$$

where

$$C_1 = \left\{\left(\frac{\omega\rho_s}{2a_1}\gamma_2 + \frac{M a_2}{2\epsilon_1 a_1}\right)\frac{\epsilon_5}{2} + \left(\frac{M\epsilon_2}{2} + \xi\right)\frac{2\omega^2\rho_s + 4\mu + \epsilon_4}{4\mu K} + \frac{(N-2)\epsilon_3}{4}\right\},$$

$$C_2 = \left\{\left(\frac{\omega\rho_s}{2a_1}\gamma_2 + \frac{M a_2}{2\epsilon_1 a_1}\right)\frac{1}{2\epsilon_5} + \left(\frac{M\epsilon_2}{2} + \xi\right)\frac{1}{4\mu K\epsilon_4} + \frac{M}{2\epsilon_2} + \frac{(N-2)}{4\epsilon_3}\right\}.$$

Choosing the constants as follows

$$\begin{aligned}
\epsilon_1 &= \frac{\mu K\omega\rho_s}{4MK_1(2\omega^2\rho_s + 4\mu + \epsilon_4)}, \quad \epsilon_2 = \frac{2}{M}\frac{4\mu K}{2\omega^2\rho_s + 4\mu + \epsilon_4}\frac{\omega^2\rho_s}{16}, \\
\epsilon_3 &= \frac{\omega^2\rho_s}{4}, \qquad \epsilon_5 = \frac{\omega^2\rho_s}{4}\left\{\frac{\omega\rho_s\gamma_2}{a_1} + \frac{4M^2 a_2 K_1(2\omega^2\rho_s + 4\mu + \epsilon_4)}{\mu K\omega\rho_s}\right\}^{-1}, \\
\xi &= \frac{\omega^2\rho_s}{16}\frac{4\mu K}{2\omega^2\rho_s + 4\mu + \epsilon_4},
\end{aligned}$$

implies that $C_1 \le \frac{\omega^2\rho_s}{2}$, $C_2 = O(1 + \frac{1}{\omega^2})$, and $\left(\frac{\xi}{K_1} - \frac{M\omega\epsilon_1}{2}\right) \ge 0$. Therefore

(2.13) $$\omega^2\rho_s\|\underset{\sim}{u}\|^2_{L^2(\Omega_s)} \le \hat{C}(1 + \frac{1}{\omega^2})\|\underset{\sim}{g}_s\|^2_{L^2(\Omega_s)}$$

for some $\hat{C}$ independent of ω. Applying the above inequality to Lemma 2.11 implies

$$2\mu K\|\underset{\sim}{u}\|^2_{H^1(\Omega_s)} \le \left(\omega^2\rho_s + \tfrac{\epsilon}{2} + 2\mu\right)\hat{C}\left(\tfrac{1}{\omega^2} + \tfrac{1}{\omega^4}\right)\|\underset{\sim}{g_s}\|^2_{L^2(\Omega_s)} + \tfrac{1}{2\epsilon}\|\underset{\sim}{g_s}\|^2_{L^2(\Omega_s)} \quad \forall \epsilon > 0$$

or

$$2\mu K\|\underset{\sim}{u}\|^2_{H^1(\Omega_s)} \le \left\{\left(\omega^2\rho_s + \tfrac{\epsilon}{2} + 2\mu\right)\hat{C}\left(\tfrac{1}{\omega^2} + \tfrac{1}{\omega^4}\right) + \tfrac{1}{2\epsilon}\right\}\|\underset{\sim}{g_s}\|^2_{L^2(\Omega_s)} \quad \forall \epsilon > 0.$$

Choosing $\epsilon = O(\omega^2)$ implies that

$$\|\underset{\sim}{u}\|^2_{H^1(\Omega_s)} \le C\left(1 + \tfrac{1}{\omega^2}\right)\|\underset{\sim}{g_s}\|^2_{L^2(\Omega_s)} \tag{2.14}$$

for some constant C independent of ω. If Ω_s is a convex polygonal domain, then regularity for $\underset{\sim}{\text{div}}\left(\underset{\approx}{\sigma}(\cdot)\right)$ implies that

$$\begin{aligned}
\|\underset{\sim}{u}\|_{H^2(\Omega_s)} &\le B\left(\|\underset{\sim}{g_s} + \omega^2\underset{\sim}{u}\|_{L^2(\Omega_s)} + \omega\|\underset{\approx}{A}\underset{\sim}{u}\|_{H^{1/2}(\partial\Omega_s)}\right)\\
&\le B\left(\|\underset{\sim}{g_s}\|_{L^2(\Omega_s)} + \omega^2\|\underset{\sim}{u}\|_{L^2(\Omega_s)} + \omega a_1\|\underset{\sim}{u}\|_{H^1(\Omega_s)}\right)\\
&\le B\left(\|\underset{\sim}{g_s}\|_{L^2(\Omega_s)} + \omega^2 C(\tfrac{1}{\omega} + \tfrac{1}{\omega^2})\|\underset{\sim}{g_s}\|_{L^2(\Omega_s)} + \omega a_1 C(1 + \tfrac{1}{\omega})\|\underset{\sim}{g_s}\|_{L^2(\Omega_s)}\right).
\end{aligned}$$

That is,

$$\|\underset{\sim}{u}\|_{H^2(\Omega_s)} \le C(\omega + 1)\|\underset{\sim}{g_s}\|_{L^2(\Omega_s)}.$$

■

Regularity estimates for the equations of motion for elastic and nearly elastic solids were derived in [16]. They are as follows

(1) $\|\underset{\sim}{u}\|_{L^2(\Omega_s)} \le C(\omega + \frac{1}{\omega})\|\underset{\sim}{g_s}\|_{L^2(\Omega_s)}$.

(2) $\|\underset{\sim}{u}\|_{H^1(\Omega_s)} \le C(\omega^2 + \frac{1}{\omega})\|\underset{\sim}{g_s}\|^2_{L^2(\Omega_s)}$.

(3) $\|\underset{\sim}{u}\|_{H^2(\Omega_s)} \le C(\omega^3 + \frac{1}{\omega})\|\underset{\sim}{g_s}\|_{L^2(\Omega_s)}$.

Clearly, the estimates proved in Theorem 2.14 and 2.13 improve the above estimates.

2.3 Variational Formulation and Well-Posedness

The variational formulation of (2.5) is defined as

$$(2.15) \qquad \begin{cases} \text{Find } \quad \underset{\sim}{u} \in (H^1(\Omega_s))^N \quad \text{such that} \\ b(\underset{\sim}{u}, \underset{\sim}{v}) = (\underset{\sim}{g_s}, \underset{\sim}{v}) \quad \forall \underset{\sim}{v} \in (H^1(\Omega_s))^N. \end{cases}$$

where

$$b(\underset{\sim}{u}, \underset{\sim}{v}) = -\omega^2 \rho_s(\underset{\sim}{u}, \underset{\sim}{v}) + 2\mu(\underset{\approx}{\epsilon}(\underset{\sim}{u}), \underset{\approx}{\epsilon}(\underset{\sim}{v})) + \lambda(\text{div}(\underset{\sim}{u}), \text{div}(\underset{\sim}{v})) + i\omega < \underset{\approx}{A}\underset{\sim}{u}, \underset{\sim}{v} > .$$

The bilinear form $b(\cdot, \cdot)$ is not coercive, so we cannot use the Lax-Milgram Theorem to show existence and uniqueness of solutions to (1.8). We can use Korn's inequality (Lemma 2.9), however, to show that $b(\cdot, \cdot)$ satisfies a Gårding inequality. Specifically,

$$\text{Re}\, b(\underset{\sim}{u}, \underset{\sim}{u}) + (\omega^2 \rho_s + \delta)\|\underset{\sim}{u}\|^2_{L^2(\Omega_s)} \geq \delta K \|\underset{\sim}{u}\|^2_{H^1(\Omega_s)}$$

for any $0 < \delta \leq 2\mu$. Now we can apply the Unique Continuation Principle and the Fredholm Alternative Theorem to show that (2.5) is well-posed.

Lemma 2.15. *Suppose* $\underset{\sim}{\psi} = \underset{\approx}{\sigma}(\underset{\sim}{\psi})\underset{\sim}{n_s} = 0$ *on* $\partial\Omega_s$. *Then* $\underset{\approx}{\nabla}\underset{\sim}{\psi} = 0$ *on* $\partial\Omega_s$.

Proof : Fix $\underset{\sim}{x} \in \partial\Omega_s$. To show that $\underset{\approx}{\nabla}\underset{\sim}{\psi} = 0$ on $\partial\Omega_s$, we will show that $\underset{\sim}{b}^T(\underset{\approx}{\nabla}\underset{\sim}{\psi})\underset{\sim}{a} = 0$ for all $\underset{\sim}{a}, \underset{\sim}{b} \in \Re^N$ Let $\underset{\sim}{n_s}$ be the unit normal to $\partial\Omega_s$ at $\underset{\sim}{x}$, and let $\underset{\sim}{\tau_1}, ..., \underset{\sim}{\tau_{N-1}}$ be unit orthogonal tangent vectors to $\partial\Omega_s$ at $\underset{\sim}{x}$. Then $\underset{\sim}{\psi} = 0$ on $\partial\Omega_s$ implies that $\nabla\psi_k \cdot \underset{\sim}{\tau_j} = 0$ for $k = 1, ..., N$ and for $j = 1, ..., N-1$. Therefore,

$$(2.16) \qquad (\underset{\approx}{\nabla}\underset{\sim}{\psi})\underset{\sim}{\tau_j} = 0 \quad \text{at } \underset{\sim}{x} \quad \text{for } j = 1, ...N-1.$$

It follows from (2.16) that

$$(2.17) \qquad \underset{\sim}{b}^T(\underset{\approx}{\nabla}\underset{\sim}{\psi})\underset{\sim}{\tau_j} = 0 \quad \text{at } \underset{\sim}{x} \quad \text{for all } \underset{\sim}{b} \in \Re^N.$$

Since $\underset{\approx}{\sigma}(\underset{\sim}{\psi})\underset{\sim}{n_s} = 0$ at $\underset{\sim}{x}$,

$$(2.18) \qquad \lambda\, \text{div}(\underset{\sim}{\psi})\underset{\sim}{n_s} + \mu(\underset{\approx}{\nabla}\underset{\sim}{\psi})\underset{\sim}{n_s} + \mu(\underset{\approx}{\nabla}\underset{\sim}{\psi})^T\underset{\sim}{n_s} = 0 \quad \text{at } \underset{\sim}{x}.$$

Dot (2.18) with $\underset{\sim}{\tau}_j$ to get

$$\mu \underset{\sim}{\tau}_j^T (\underset{\approx}{\nabla} \underset{\sim}{\psi}) \underset{\sim}{n}_s + \mu \underset{\sim}{\tau}_j^T (\underset{\approx}{\nabla} \underset{\sim}{\psi})^T \underset{\sim}{n}_s = 0 \quad \text{at } \underset{\sim}{x}. \tag{2.19}$$

Since $\mu \neq 0$, (2.16) and (2.19) imply that

$$\underset{\sim}{\tau}_j^T (\underset{\approx}{\nabla} \underset{\sim}{\psi}) \underset{\sim}{n}_s = 0 \quad \text{at } \underset{\sim}{x} \text{ for } j = 1, ..., N-1. \tag{2.20}$$

Equations (2.16) and (2.20) imply that

$$\underset{\sim}{\tau}_j^T (\underset{\approx}{\nabla} \underset{\sim}{\psi}) \underset{\sim}{a} = 0 \quad \text{at } \underset{\sim}{x} \quad \text{for all } \underset{\sim}{a} \in \Re^N, \tag{2.21}$$

therefore,

$$\underset{\sim}{\tau}_j^T (\underset{\approx}{\nabla} \underset{\sim}{\psi}) = 0 \quad \text{at } \underset{\sim}{x}. \tag{2.22}$$

Since div is rotation invariant,

$$\text{div}(\underset{\sim}{\psi}) = \underset{\sim}{n}_s^T (\underset{\approx}{\nabla} \underset{\sim}{\psi}) \underset{\sim}{n}_s + \sum_{j=1}^{N-1} \underset{\sim}{\tau}_j^T (\underset{\approx}{\nabla} \underset{\sim}{\psi}) \underset{\sim}{\tau}_j.$$

Therefore by (2.16),

$$\text{div}(\underset{\sim}{\psi}) = \underset{\sim}{n}_s^T (\underset{\approx}{\nabla} \underset{\sim}{\psi}) \underset{\sim}{n}_s.$$

Dot (2.18) with $\underset{\sim}{n}_s$ and apply the above expression for $\text{div}(\underset{\sim}{\psi})$to get

$$(\lambda + 2\mu) \underset{\sim}{n}_s^T (\underset{\approx}{\nabla} \underset{\sim}{\psi}) \underset{\sim}{n}_s = 0.$$

Therefore,

$$\underset{\sim}{n}_s^T (\underset{\approx}{\nabla} \underset{\sim}{\psi}) \underset{\sim}{n}_s = 0 \tag{2.23}$$

since $(\lambda + 2\mu) > 0$. Equations (2.20) and (2.23) imply that

$$\underset{\sim}{b}^T (\underset{\approx}{\nabla} \underset{\sim}{\psi}) \underset{\sim}{n}_s = 0 \ \forall \ \underset{\sim}{b} \in \Re^N. \tag{2.24}$$

Combining equations (2.17) and (2.24),

$$\underset{\sim}{b}^T (\underset{\approx}{\nabla} \underset{\sim}{\psi}) \underset{\sim}{a} = 0 \qquad \forall \ \underset{\sim}{a}, \underset{\sim}{b} \in \Re^N.$$

■

Theorem 2.16. *Suppose $\underset{\sim}{g}_s \in L^2(\Omega_s)$ and $\omega \neq 0$. Then there exists a unique solution to (2.15).*

Proof : Since the bilinear form $b(\cdot,\cdot)$ satisfies a Gårding inequality, the Fredholm Alternative Theorem implies that a solution to (2.15) exists if the adjoint problem

$$\text{(2.25)} \qquad \begin{cases} \text{Find} \quad \underset{\sim}{\psi} \in H^1(\Omega_s) \quad \text{such that} \\ b^*(\underset{\sim}{\psi}, \underset{\sim}{v}) = (\underset{\sim}{g}_s, \underset{\sim}{v}) \quad \forall \underset{\sim}{v} \in H^1(\Omega_s). \end{cases}$$

has only the zero solution when $\underset{\sim}{g}_s = 0$ (see [1], pg 102). If $\underset{\sim}{g}_s = 0$ in (2.25), then choosing $\underset{\sim}{v} = \underset{\sim}{\psi}$ in the variational formulation and taking imaginary part implies that $\underset{\sim}{\psi} = 0$ on $\partial\Omega_s$. Integrating by parts in (2.5) then implies that $\underset{\approx}{\sigma}(\underset{\sim}{\psi})\underset{\sim}{n}_s = 0$ on $\partial\Omega_s$, so by Lemma 5.21, $\underset{\approx}{\nabla}(\underset{\sim}{\psi}) = 0$ on $\partial\Omega_s$. By the Unique Continuation Principle, $\underset{\sim}{\psi} = 0$ in Ω_s, and the Fredholm Alternative Theorem therefore implies that solutions exist. The same argument shows that $\underset{\sim}{g}_s = 0$ in (2.15) implies that $\underset{\sim}{u} = 0$. Solutions are therefore unique. ■

2.4 Finite Element Procedures

Let $\mathcal{T}_h$ be a quasi-uniform triangulation of Ω_s with mesh size $h > 0$. Suppose V_h is the P_{m-1} conforming finite element space of $H^1(\Omega_s)$ associated with $\mathcal{T}_h$. It is well known that V_h has the following simultaneous approximation property [5]:

(2.26)

$$\inf_{\underset{\sim}{v} \in V_h} \left\{ ||\underset{\sim}{u} - \underset{\sim}{v}||_{L^2(\Omega_f)} + h||\underset{\sim}{u} - \underset{\sim}{v}||_{H^1(\Omega_f)} \right\} \leq C_A h^m ||\underset{\sim}{u}||_{H^m(\Omega_f)} \quad \forall \ \underset{\sim}{u} \in (H^m(\Omega_f))^N.$$

The finite element method is then defined as

$$\text{(2.27)} \qquad \begin{cases} \text{Find} \quad \underset{\sim}{v}_h \in (V_h)^N \quad \text{such that} \\ b(\underset{\sim}{u}_h, \underset{\sim}{v}) = (\underset{\sim}{g}_s, \underset{\sim}{v}) \quad \forall \underset{\sim}{v}_h \in (V_h)^N. \end{cases}$$

Let $C_{R,m}$ denote an abstract regularity constant for solutions to (2.5), i.e.

$$||\underset{\sim}{u}||_{H^m(\Omega_s)} \leq C_{R,m} ||\underset{\sim}{g}_s||_{H^r(\Omega_s)}$$

where $r = \max\{0, m-2\}$. Theorem 2.14 implies that $C_{R,m} = O(\omega^{m-1})$ for $m \geq 2$.

Lemma 2.17. *For any* $\underset{\sim}{v} : \Re^N \to C^N$

(1) $\quad \|\underset{\approx}{\epsilon}(\underset{\sim}{v})\|_{L^2(\Omega_s)} \leq \|\underset{\approx}{\nabla} \underset{\sim}{v}\|_{L^2(\Omega_s)}$.

(2) $\quad \|\operatorname{div}(\underset{\sim}{v})\|_{L^2(\Omega_s)} \leq \sqrt{N}\|\underset{\approx}{\nabla} \underset{\sim}{v}\|_{L^2(\Omega_s)}$.

Lemma 2.18. *Suppose* $\underset{\sim}{u}$ *solves (2.15) and* $\underset{\sim}{u_h}$ *solves (2.27). Then there are constants* C_1 *and* C_2*, independent of* ω *and* h*, such that* $\; h \leq C_1 \frac{1}{\omega\sqrt{C_{R,2}}} = O(\frac{1}{\omega^{3/2}})$, *implies that*

$$\|\underset{\sim}{u} - \underset{\sim}{u_h}\|_{L^2(\Omega_s)} \leq C_2 C_{R,2} h \|\underset{\sim}{u} - \underset{\sim}{u_h}\|_{H^1(\Omega_s)}.$$

Proof : Suppose $\underset{\sim}{u}$ solves (2.15) and $\underset{\sim}{u_h}$ solves (2.27). Let $\underset{\sim}{\psi}$ be a solution to the adjoint problem with source $\underset{\sim}{u} - \underset{\sim}{u_h}$, i.e. $\underset{\sim}{\psi}$ solves

$$\begin{cases} \text{Find} \quad \underset{\sim}{\psi} \in (V_h)^N \quad \text{such that} \\ b^*(\underset{\sim}{\psi}, \underset{\sim}{v}) = (\underset{\sim}{u} - \underset{\sim}{u_h}, \underset{\sim}{v}) \quad \forall \underset{\sim}{v} \in (H^1(\Omega_s))^N, \end{cases}$$

where

$$b^*(\underset{\sim}{u}, \underset{\sim}{v}) = -\omega^2 \rho_s(\underset{\sim}{u}, \underset{\sim}{v}) + 2\mu(\underset{\approx}{\epsilon}(\underset{\sim}{u}), \underset{\approx}{\epsilon}(\underset{\sim}{v})) + \lambda(\operatorname{div}(\underset{\sim}{u}), \operatorname{div}(\underset{\sim}{v})) - i\omega < \underset{\approx}{A}\underset{\sim}{u}, \underset{\sim}{v} > .$$

Then $\underset{\sim}{\psi}$ satisfies $b(\underset{\sim}{v}, \underset{\sim}{\psi}) = (\underset{\sim}{v}, \underset{\sim}{u} - \underset{\sim}{u_h})$. Taking $\underset{\sim}{v} = \underset{\sim}{u} - \underset{\sim}{u_h}$ implies

$$\|\underset{\sim}{u} - \underset{\sim}{u_h}\|^2_{L^2(\Omega_s)} = b(\underset{\sim}{u} - \underset{\sim}{u_h}, \underset{\sim}{\psi}). \tag{2.28}$$

The fundamental orthogonality identity states that

$$b(\underset{\sim}{u} - \underset{\sim}{u_h}, \underset{\sim}{\psi_h}) = 0 \qquad \text{for any } \underset{\sim}{\psi_h} \in V_h.$$

which is true since $b(\underset{\sim}{u}, \underset{\sim}{\psi_h}) = (\underset{\sim}{g_s}, \underset{\sim}{\psi_h})$ and $b(\underset{\sim}{u_h}, \underset{\sim}{\psi_h}) = (\underset{\sim}{g_s}, \underset{\sim}{\psi_h})$ whenever $\underset{\sim}{\psi_h} \in V_h$ Therefore,

$$b(\underset{\sim}{u} - \underset{\sim}{u_h}, \underset{\sim}{\psi}) = b(\underset{\sim}{u} - \underset{\sim}{u_h}, \underset{\sim}{\psi} - \underset{\sim}{\psi_h}) \qquad \text{for any } \underset{\sim}{\psi_h} \in V_h$$

and (2.28) becomes

$$\begin{aligned}\|\underset{\sim}{u} - \underset{\sim}{u_h}\|^2_{L^2(\Omega_s)} &= b(\underset{\sim}{u} - \underset{\sim}{u_h}, \underset{\sim}{\psi} - \underset{\sim}{\psi_h}) \\ &= 2\mu(\underset{\approx}{\epsilon}(\underset{\sim}{u} - \underset{\sim}{u_h}), \underset{\approx}{\epsilon}(\underset{\sim}{\psi} - \underset{\sim}{\psi_h})) + \lambda(\text{div}(\underset{\sim}{u} - \underset{\sim}{u_h}), \text{div}(\underset{\sim}{\psi} - \underset{\sim}{\psi_h})) \\ &\quad - \omega^2 \rho_s (\underset{\sim}{u} - \underset{\sim}{u_h}, \underset{\sim}{\psi} - \underset{\sim}{\psi_h}) + i\omega < \underset{\approx}{A}(\underset{\sim}{u} - \underset{\sim}{u_h}), \underset{\sim}{\psi} - \underset{\sim}{\psi_h} >,\end{aligned}$$

which implies

$$\begin{aligned}\|\underset{\sim}{u} - \underset{\sim}{u_h}\|^2_{L^2(\Omega_s)} \leq\ & 2\mu \|\underset{\approx}{\epsilon}(\underset{\sim}{u} - \underset{\sim}{u_h})\|_{L^2(\Omega_s)} \|\underset{\approx}{\epsilon}(\underset{\sim}{\psi} - \underset{\sim}{\psi_h})\|_{L^2(\Omega_s)} \\ &+ \lambda \|\text{div}(\underset{\sim}{u} - \underset{\sim}{u_h})\|_{L^2(\Omega_s)} \|\text{div}(\underset{\sim}{\psi} - \underset{\sim}{\psi_h})\|_{L^2(\Omega_s)} \\ &+ \omega^2 \rho_s \|\underset{\sim}{u} - \underset{\sim}{u_h}\|_{L^2(\Omega_s)} \|\underset{\sim}{\psi} - \underset{\sim}{\psi_h}\|_{L^2(\Omega_s)} \\ &+ \omega a \|\underset{\sim}{u} - \underset{\sim}{u_h}\|_{L^2(\partial\Omega_s)} \|\underset{\sim}{\psi} - \underset{\sim}{\psi_h}\|_{L^2(\partial\Omega_s)}.\end{aligned}$$

The trace theorem on Ω_s implies

$$\begin{aligned}&\omega \|\underset{\sim}{u} - \underset{\sim}{u_h}\|_{L^2(\partial\Omega_s)} \|\underset{\sim}{\psi} - \underset{\sim}{\psi_h}\|_{L^2(\partial\Omega_s)} \\ &\quad \leq \omega C_s^2 \|\underset{\sim}{u} - \underset{\sim}{u_h}\|^{1/2}_{L^2(\Omega_s)} \|\underset{\sim}{u} - \underset{\sim}{u_h}\|^{1/2}_{H^1(\Omega_s)} \|\underset{\sim}{\psi} - \underset{\sim}{\psi_h}\|^{1/2}_{L^2(\Omega_s)} \|\underset{\sim}{\psi} - \underset{\sim}{\psi_h}\|^{1/2}_{H^1(\Omega_s)} \\ &\quad \leq \tfrac{\omega^2 C_s^4}{4} \|\underset{\sim}{u} - \underset{\sim}{u_h}\|_{L^2(\Omega_s)} \|\underset{\sim}{\psi} - \underset{\sim}{\psi_h}\|_{L^2(\Omega_s)} + \|\underset{\sim}{u} - \underset{\sim}{u_h}\|_{H^1(\Omega_s)} \|\underset{\sim}{\psi} - \underset{\sim}{\psi_h}\|_{H^1(\Omega_s)}.\end{aligned}$$

Inserting this inequality into the above implies

$$\begin{aligned}\|\underset{\sim}{u} - \underset{\sim}{u_h}\|^2_{L^2(\Omega_s)} \leq\ & 2\mu \|\underset{\approx}{\epsilon}(\underset{\sim}{u} - \underset{\sim}{u_h})\|_{L^2(\Omega_s)} \|\underset{\approx}{\epsilon}(\underset{\sim}{\psi} - \underset{\sim}{\psi_h})\|_{L^2(\Omega_s)} \\ &+ \lambda \|\text{div}(\underset{\sim}{u} - \underset{\sim}{u_h})\|_{L^2(\Omega_s)} \|\text{div}(\underset{\sim}{\psi} - \underset{\sim}{\psi_h})\|_{L^2(\Omega_s)} \\ &+ \omega^2 \rho_s \|\underset{\sim}{u} - \underset{\sim}{u_h}\|_{L^2(\Omega_s)} \|\underset{\sim}{\psi} - \underset{\sim}{\psi_h}\|_{L^2(\Omega_s)} \\ &+ a \Big\{ \tfrac{\omega^2 C_s^4}{4} \|\underset{\sim}{u} - \underset{\sim}{u_h}\|_{L^2(\Omega_s)} \|\underset{\sim}{\psi} - \underset{\sim}{\psi_h}\|_{L^2(\Omega_s)} \\ &\qquad + \|\underset{\sim}{u} - \underset{\sim}{u_h}\|_{H^1(\Omega_s)} \|\underset{\sim}{\psi} - \underset{\sim}{\psi_h}\|_{H^1(\Omega_s)} \Big\}.\end{aligned}$$

Applying Lemma 2.17 implies

$$\|\underset{\sim}{u} - \underset{\sim}{u_h}\|^2_{L^2(\Omega_s)} \le (2\mu + \lambda N + a)\|\underset{\sim}{u} - \underset{\sim}{u_h}\|_{H^1(\Omega_s)}\|\underset{\sim}{\psi} - \underset{\sim}{\psi_h}\|_{H^1(\Omega_s)}$$
$$+ \omega^2 \left(\rho_s + a\frac{C_s^4}{4}\right) \|\underset{\sim}{u} - \underset{\sim}{u_h}\|_{L^2(\Omega_s)}\|\underset{\sim}{\psi} - \underset{\sim}{\psi_h}\|_{L^2(\Omega_s)}.$$

Applying the approximation property of V_h,

$$\|\underset{\sim}{u} - \underset{\sim}{u_h}\|^2_{L^2(\Omega_s)} \le (2\mu + \lambda N + a)\|\underset{\sim}{u} - \underset{\sim}{u_h}\|_{H^1(\Omega_s)} C_A h \|\underset{\sim}{\psi}\|_{H^2(\Omega_s)}$$
$$+ \omega^2 \left(\rho_s + a\frac{C_s^4}{4}\right) \|\underset{\sim}{u} - \underset{\sim}{u_h}\|_{L^2(\Omega_s)} C_A h^2 \|\underset{\sim}{\psi}\|_{H^2(\Omega_s)}.$$

Applying the regularity estimate to $\|\underset{\sim}{\psi}\|_{H^2(\Omega_s)}$, (recall that $\underset{\sim}{\psi}$ solves the dual problem with source $\underset{\sim}{u} - \underset{\sim}{u_h}$),

$$\|\underset{\sim}{u} - \underset{\sim}{u_h}\|^2_{L^2(\Omega_s)} \le (2\mu + \lambda N + a) C_A h C_{R,2} \|\underset{\sim}{u} - \underset{\sim}{u_h}\|_{H^1(\Omega_s)} \|\underset{\sim}{u} - \underset{\sim}{u_h}\|_{L^2(\Omega_s)}$$
$$+ \omega^2 \left(\rho_s + a\frac{C_s^4}{4}\right) C_A h^2 C_{R,2} \|\underset{\sim}{u} - \underset{\sim}{u_h}\|^2_{L^2(\Omega_s)},$$

which implies

$$\left(1 - \omega^2 \left(\rho_s + a\frac{C_s^4}{4}\right) C_A C_{R,2} h^2\right) \|\underset{\sim}{u} - \underset{\sim}{u_h}\|_{L^2(\Omega_s)} \le$$
$$(2\mu + \lambda N + a) C_A C_{R,2} h \|\underset{\sim}{u} - \underset{\sim}{u_h}\|_{H^1(\Omega_s)}.$$

Choose $h \le C_1 \frac{1}{\omega\sqrt{C_{R,2}}}$, where $C_1 = \frac{1}{\sqrt{\left(\rho_s + a_2\frac{C_s^4}{4}\right) C_A}}$. Then

$$\left(1 - \omega^2 \left(\rho_s + a\frac{C_s^4}{4}\right) C_A C_{R,2} h^2\right) \ge \tfrac{1}{2}.$$

Therefore,

$$\tfrac{1}{2}\|\underset{\sim}{u} - \underset{\sim}{u_h}\|_{L^2(\Omega_s)} \le (2\mu + \lambda N + a_2) C_A C_{R,2} h \|\underset{\sim}{u} - \underset{\sim}{u_h}\|_{H^1(\Omega_s)}$$

or

$$\|\underset{\sim}{u} - \underset{\sim}{u_h}\|_{L^2(\Omega_s)} \le C_2 C_{R,2} h \|\underset{\sim}{u} - \underset{\sim}{u_h}\|_{H^1(\Omega_s)},$$

where

$$C_2 = (2\mu + \lambda N + a_2) C_A C_{R,2}.$$

■

Theorem 2.19. *Suppose $\underset{\sim}{u}$ solves 2.15 and $\underset{\sim}{u_h}$ solves 2.27. Then there are constants C_3 and C_4, independent of ω and h, such that $h \leq C_3 \frac{1}{\omega C_{R,2}} = O(\frac{1}{\omega^2})$ implies that*

$$\|\underset{\sim}{u} - \underset{\sim}{u_h}\|_{H^1(\Omega_s)} \leq C_4 C_{R,m} (h^m \omega + h^{m-1}) \|\underset{\sim}{g_s}\|_{H^{m-2}(\Omega_s)}$$

and

$$\|\underset{\sim}{u} - \underset{\sim}{u_h}\|_{L^2(\Omega_s)} \leq C C_{R,2} C_{R,m} (h^{m+1} \omega + h^m) \|\underset{\sim}{g_s}\|_{H^{m-2}(\Omega_s)}$$

for $m \geq 2$.

Proof :

$$\begin{aligned} b(\underset{\sim}{u} - \underset{\sim}{u_h}, \underset{\sim}{u} - \underset{\sim}{u_h}) &= -\omega^2 \rho_s \|\underset{\sim}{u} - \underset{\sim}{u_h}\|^2_{L^2(\Omega_s)} + 2\mu \|\underset{\approx}{\epsilon}(\underset{\sim}{u} - \underset{\sim}{u_h})\|^2_{L^2(\Omega_s)} \\ &+ \lambda \|\operatorname{div}(\underset{\sim}{u} - \underset{\sim}{u_h})\|^2_{L^2(\Omega_s)} + i\omega < \underset{\approx}{A}(\underset{\sim}{u} - \underset{\sim}{u_h}), \underset{\sim}{u} - \underset{\sim}{u_h} > . \end{aligned}$$

Taking the real part on both sides and using the fact that $< \underset{\approx}{A}(\underset{\sim}{u} - \underset{\sim}{u_h}), \underset{\sim}{u} - \underset{\sim}{u_h} >$ is real since $\underset{\approx}{A}$ is symmetric, we get

$$\begin{aligned} &\operatorname{Re} b(\underset{\sim}{u} - \underset{\sim}{u_h}, \underset{\sim}{u} - \underset{\sim}{u_h}) + \omega^2 \rho_s \|\underset{\sim}{u} - \underset{\sim}{u_h}\|^2_{L^2(\Omega_s)} \\ &\quad = 2\mu \|\underset{\approx}{\epsilon}(\underset{\sim}{u} - \underset{\sim}{u_h})\|^2_{L^2(\Omega_s)} + \lambda \|\operatorname{div}(\underset{\sim}{u} - \underset{\sim}{u_h})\|^2_{L^2(\Omega_s)} \geq 2\mu \|\underset{\approx}{\epsilon}(\underset{\sim}{u} - \underset{\sim}{u_h})\|^2_{L^2(\Omega_s)}. \end{aligned}$$

Applying Korn's inequality,

$$\begin{aligned} &\operatorname{Re} b(\underset{\sim}{u} - \underset{\sim}{u_h}, \underset{\sim}{u} - \underset{\sim}{u_h}) + (\omega^2 \rho_s + 2\mu) \|\underset{\sim}{u} - \underset{\sim}{u_h}\|^2_{L^2(\Omega_s)} \\ &\quad \geq 2\mu \left(\|\underset{\approx}{\epsilon}(\underset{\sim}{u} - \underset{\sim}{u_h})\|^2_{L^2(\Omega_s)} + \|\underset{\sim}{u} - \underset{\sim}{u_h}\|^2_{L^2(\Omega_s)} \right) \geq 2\mu K \|\underset{\sim}{u} - \underset{\sim}{u_h}\|^2_{H^1(\Omega_s)}. \end{aligned}$$

So for ω sufficiently large,

$$\text{Re } b(\underset{\sim}{u} - \underset{\sim}{u_h}, \underset{\sim}{u} - \underset{\sim}{u_h}) + 2\omega^2 \rho_s \|\underset{\sim}{u} - \underset{\sim}{u_h}\|^2_{L^2(\Omega_s)} \geq 2\mu K \|\underset{\sim}{u} - \underset{\sim}{u_h}\|^2_{H^1(\Omega_s)}.$$

Note that $\text{Re } b(\underset{\sim}{u} - \underset{\sim}{u_h}, \underset{\sim}{u} - \underset{\sim}{u_h}) = \text{Re } b(\underset{\sim}{u} - \underset{\sim}{u_h}, \underset{\sim}{u} - \underset{\sim}{v})$ for all $\underset{\sim}{v} \in V_h$ by fundamental orthogonality; therefore, for all $\underset{\sim}{v} \in V_h$,

$$\begin{aligned} 2\mu K \|\underset{\sim}{u} - \underset{\sim}{u_h}\|^2_{H^1(\Omega_s)} &\leq \text{Re}\, b(\underset{\sim}{u} - \underset{\sim}{u_h}, \underset{\sim}{u} - \underset{\sim}{v}) + (2\omega^2 \rho_s)\|\underset{\sim}{u} - \underset{\sim}{u_h}\|^2_{L^2(\Omega_s)} \\ &\leq \omega^2 \rho_s \|\underset{\sim}{u} - \underset{\sim}{u_h}\|_{L^2(\Omega_s)} \|\underset{\sim}{u} - \underset{\sim}{v}\|_{L^2(\Omega_s)} \\ &+ 2\mu \|\underset{\approx}{\epsilon}(\underset{\sim}{u} - \underset{\sim}{u_h})\|_{L^2(\Omega_s)} \|\underset{\approx}{\epsilon}(\underset{\sim}{u} - \underset{\sim}{v})\|_{L^2(\Omega_s)} \\ &+ \lambda \|\operatorname{div}(\underset{\sim}{u} - \underset{\sim}{u_h})\|_{L^2(\Omega_s)} \|\operatorname{div}(\underset{\sim}{u} - \underset{\sim}{v})\|_{L^2(\Omega_s)} \\ &+ \omega \|\underset{\approx}{A}(\underset{\sim}{u} - \underset{\sim}{u_h})\|_{L^2(\partial\Omega_s)} \|(\underset{\sim}{u} - \underset{\sim}{v})\|_{L^2(\partial\Omega_s)} + (2\omega^2 \rho_s)\|\underset{\sim}{u} - \underset{\sim}{u_h}\|^2_{L^2(\Omega_s)}. \end{aligned}$$

Applying Lemma 2.17 implies

$$\begin{aligned} 2\mu K \|\underset{\sim}{u} - \underset{\sim}{u_h}\|^2_{H^1(\Omega_s)} &\leq \omega^2 \rho_s \|\underset{\sim}{u} - \underset{\sim}{u_h}\|_{L^2(\Omega_s)} \|\underset{\sim}{u} - \underset{\sim}{v}\|_{L^2(\Omega_s)} \\ &+ (2\mu + \lambda N)\|\underset{\sim}{u} - \underset{\sim}{u_h}\|_{H^1(\Omega_s)} \|\underset{\sim}{u} - \underset{\sim}{v}\|_{H^1(\Omega_s)} \\ &+ \omega a_2 \|(\underset{\sim}{u} - \underset{\sim}{u_h})\|_{L^2(\partial\Omega_s)} \|(\underset{\sim}{u} - \underset{\sim}{v})\|_{L^2(\partial\Omega_s)} + (2\omega^2 \rho_s)\|\underset{\sim}{u} - \underset{\sim}{u_h}\|^2_{L^2(\Omega_s)} \end{aligned}$$

or

$$\begin{aligned} 2\mu K \|\underset{\sim}{u} - \underset{\sim}{u_h}\|^2_{H^1(\Omega_s)} &\leq \tfrac{\omega^2 \rho_s}{2} \|\underset{\sim}{u} - \underset{\sim}{u_h}\|^2_{L^2(\Omega_s)} + \tfrac{\omega^2 \rho_s}{2} \|\underset{\sim}{u} - \underset{\sim}{v}\|^2_{L^2(\Omega_s)} \\ &+ (2\mu + \lambda N)\left(\tfrac{1}{2\epsilon}\|\underset{\sim}{u} - \underset{\sim}{u_h}\|^2_{H^1(\Omega_s)} + \tfrac{\epsilon}{2}\|\underset{\sim}{u} - \underset{\sim}{v}\|^2_{H^1(\Omega_s)}\right) \\ &+ a_2 \left(\tfrac{\omega^2 C_s^4}{4} \|(\underset{\sim}{u} - \underset{\sim}{u_h})\|_{L^2(\Omega_s)} \|(\underset{\sim}{u} - \underset{\sim}{v})\|_{L^2(\Omega_s)} \right. \\ &\left. + \|(\underset{\sim}{u} - \underset{\sim}{u_h})\|_{H^1(\Omega_s)}\ \|(\underset{\sim}{u} - \underset{\sim}{v})\|_{H^1(\Omega_s)}\right) + (2\omega^2 \rho_s)\|\underset{\sim}{u} - \underset{\sim}{u_h}\|^2_{L^2(\Omega_s)}. \end{aligned}$$

Choose $\epsilon = \frac{2\mu+\lambda N}{\mu K}$. Then

$$\begin{aligned}
2\mu K\|\underset{\sim}{u} - \underset{\sim}{u_h}\|^2_{H^1(\Omega_s)} &\leq \left(\tfrac{5\omega^2\rho_s}{2}\right)\|\underset{\sim}{u} - \underset{\sim}{u_h}\|^2_{L^2(\Omega_s)} + \tfrac{\omega^2\rho_s}{2}\|\underset{\sim}{u} - \underset{\sim}{v}\|^2_{L^2(\Omega_s)} \\
&\quad + \tfrac{\mu K}{2}\|\underset{\sim}{u} - \underset{\sim}{u_h}\|^2_{H^1(\Omega_s)} + \tfrac{(2\mu+\lambda N)^2}{2\mu k}\|\underset{\sim}{u} - \underset{\sim}{v}\|^2_{H^1(\Omega_s)} \\
&\quad + a_2\tfrac{\omega^2 C_s^8}{8}\|(\underset{\sim}{u} - \underset{\sim}{u_h})\|_{L^2(\Omega_s)} + a_2\tfrac{\omega^2}{8}\|(\underset{\sim}{u} - \underset{\sim}{v})\|_{L^2(\Omega_s)} \\
&\quad + \tfrac{\mu k}{2}\|(\underset{\sim}{u} - \underset{\sim}{u_h})\|_{H^1(\Omega_s)} + \tfrac{1}{2\mu k}\|(\underset{\sim}{u} - \underset{\sim}{v})\|_{H^1(\Omega_s)} \\
&= \left(\tfrac{5\omega^2\rho_s}{2} + a_2\tfrac{\omega^2 C_s^8}{8}\right)\|\underset{\sim}{u} - \underset{\sim}{u_h}\|^2_{L^2(\Omega_s)} \\
&\quad + \left(\tfrac{\omega^2\rho_s}{2} + a_2\tfrac{\omega^2}{8}\right)\|\underset{\sim}{u} - \underset{\sim}{v}\|^2_{L^2(\Omega_s)} + \mu K\|\underset{\sim}{u} - \underset{\sim}{u_h}\|^2_{H^1(\Omega_s)} \\
&\quad + \tfrac{(2\mu+\lambda N)^2+1}{2\mu k}\|\underset{\sim}{u} - \underset{\sim}{v}\|^2_{H^1(\Omega_s)},
\end{aligned}$$

which implies

$$\begin{aligned}
\mu K\|\underset{\sim}{u} - \underset{\sim}{u_h}\|^2_{H^1(\Omega_s)} &\leq \left(\tfrac{20\rho_s + a_2 C_s^8}{8}\omega^2\right)\|\underset{\sim}{u} - \underset{\sim}{u_h}\|^2_{L^2(\Omega_s)} + \tfrac{4\rho_s + a_2}{8}\omega^2\|\underset{\sim}{u} - \underset{\sim}{v}\|^2_{L^2(\Omega_s)} \\
&\quad + \tfrac{(2\mu+\lambda N)^2+1}{2\mu k}\|\underset{\sim}{u} - \underset{\sim}{v}\|^2_{H^1(\Omega_s)}.
\end{aligned}$$

Regularity of (2.5) and the approximation property of V_h implies (for $\underset{\sim}{v}$ appropriately chosen)

$$\|\underset{\sim}{u} - \underset{\sim}{v}\|_{L^2(\Omega_s)} \leq C_A h^m \|\underset{\sim}{u}\|_{H^m(\Omega_s)} \leq C_A h^m C_{R,m}\|\underset{\sim}{g_s}\|_{H^{m-2}(\Omega_s)}.$$

$$\|\underset{\sim}{u} - \underset{\sim}{v}\|_{H^1(\Omega_s)} \leq C_A h^{m-1} \|\underset{\sim}{u}\|_{H^m(\Omega_s)} \leq C_A h^{m-1} C_{R,m}\|\underset{\sim}{g_s}\|_{H^{m-2}(\Omega_s)}.$$

Using these inequalities in the previous inequality gives

$$\begin{aligned}
\mu K\|\underset{\sim}{u} - \underset{\sim}{u_h}\|^2_{H^1(\Omega_s)} &\leq \left(\tfrac{20\rho_s + a_2 C_s^8}{8}\omega^2\right)\|\underset{\sim}{u} - \underset{\sim}{u_h}\|^2_{L^2(\Omega_s)} \\
&\quad + \tfrac{4\rho_s + a_2}{8}\omega^2 (C_A h^m C_{R,m})^2 \|\underset{\sim}{g_s}\|^2_{H^{m-2}(\Omega_s)} \\
&\quad + \tfrac{(2\mu+\lambda N)^2+1}{2\mu k}(C_A h^{m-1} C_{R,m})^2 \|\underset{\sim}{g_s}\|^2_{H^{m-2}(\Omega_s)},
\end{aligned}$$

which implies

$$\begin{aligned}
\mu K\|\underset{\sim}{u} - \underset{\sim}{u_h}\|^2_{H^1(\Omega_s)} &\leq \left(\tfrac{20\rho_s + a_2 C_s^8}{8}\omega^2\right)\|\underset{\sim}{u} - \underset{\sim}{u_h}\|^2_{L^2(\Omega_s)} \\
&\quad + \left\{\tfrac{4\rho_s + a_2}{8}\omega^2 h^2 + \tfrac{(2\mu+\lambda N)^2+1}{2\mu k}\right\} {C_A}^2 (C_{R,m} h^{m-1})^2 \|\underset{\sim}{g_s}\|^2_{H^{m-2}(\Omega_s)}.
\end{aligned}$$

Applying the duality estimate to $\|\underset{\sim}{u} - \underset{\sim}{u}_h\|^2_{L^2(\Omega_s)}$ (cf. Lemma 2.18),

$$\mu K \|\underset{\sim}{u} - \underset{\sim}{u}_h\|^2_{H^1(\Omega_s)} \leq \left(\frac{20\rho_s + a_2 C_s^8}{8}\omega^2\right) {C_2}^2 {C_{R,2}}^2 h^2 \|\underset{\sim}{u} - \underset{\sim}{u}_h\|^2_{H^1(\Omega_s)}$$
$$+ \left\{\frac{4\rho_s + a_2}{8}\omega^2 h^2 + \frac{(2\mu + \lambda N)^2 + 1}{2\mu k}\right\} {C_A}^2 (C_{R,m} h^{m-1})^2 \|\underset{\sim}{g}_s\|^2_{H^{m-2}(\Omega_s)}$$

or

$$\left\{\mu K - \left(\frac{20\rho_s + a_2 C_s^8}{8}\omega^2\right) C_2^2 C_{R,2}^2 h^2\right\} \|\underset{\sim}{u} - \underset{\sim}{u}_h\|^2_{H^1(\Omega_s)}$$
$$\leq \left(\frac{4\rho_s + a_2}{8}\omega^2 h^2 + \frac{(2\mu + \lambda N)^2 + 1}{2\mu k}\right) {C_A}^2 (C_{R,m} h^{m-1})^2 \|\underset{\sim}{g}_s\|^2_{H^{m-2}(\Omega_s)}.$$

Choose $h \leq C_3 \frac{1}{\omega C_{R,2}}$ where $C_3 = \frac{1}{C_2}\sqrt{\frac{4\mu K}{20\rho_s + a_2 C_s^8}}$. Then

$$\left\{\mu K - \left(\frac{20\rho_s + a_2 C_s^8}{8}\omega^2\right) {C_2}^2 {C_{R,2}}^2 h^2\right\} \geq \frac{\mu K}{2}.$$

Therefore,

$$\frac{\mu K}{2} \|\underset{\sim}{u} - \underset{\sim}{u}_h\|^2_{H^1(\Omega_s)} \leq \left(\frac{4\rho_s + a_2}{8}\omega^2 h^2 + \frac{(2\mu + \lambda N)^2 + 1}{2\mu k}\right) {C_A}^2 (C_{R,m} h^{m-1})^2 \|\underset{\sim}{g}_s\|^2_{H^{m-2}(\Omega_s)}$$

or

$$\|\underset{\sim}{u} - \underset{\sim}{u}_h\|_{H^1(\Omega_s)} \leq C_4 C_{R,m} (h^m \omega + h^{m-1}) \|\underset{\sim}{g}_s\|_{H^{m-2}(\Omega_s)},$$

where C_4 is independent of h and ω. Combining this estimate with Lemma 2.18 implies that

$$\|\underset{\sim}{u} - \underset{\sim}{u}_h\|_{L^2(\Omega_s)} \leq C_2 C_{R,2} h \|\underset{\sim}{u} - \underset{\sim}{u}_h\|_{H^1(\Omega_s)}$$
$$\leq C_2 C_{R,2} h \left\{C_4 C_{R,m} (h^m \omega + h^{m-1}) \|\underset{\sim}{g}_s\|_{H^{m-2}(\Omega_s)}\right\}.$$

■

Chapter 3

Fluid Solid Interactions

In Chapter One, interest in wave propagation through fluid media motivated our analysis of the acoustic Helmholtz equation. In Chapter Two, we analyzed the elastic Helmholtz equation, motivated primarily by our interest in wave propagation through solid media. In the following two chapters, we study a system of coupled acoustic and elastic Helmholtz equations which emerges from a model for wave propagation through a composite fluid-solid medium.

Suppose Ω is a subset of $\Re^N$ $(N = 2, 3)$ where $\Omega = \Omega_f \cup \Omega_s$ and let Γ denote $\Omega_f \cap \Omega_s$. For our purposes, Ω represents a composite fluid-solid medium; Ω_f represents the fluid part of the composite medium, and Ω_s represents the solid part. In order to model wave propagation through such a medium, one begins with the following system of equations

$$(3.1) \qquad \begin{cases} \frac{1}{c^2} P_{tt} - \Delta P &= G_f \qquad \underset{\sim}{x} \in \Omega_f, \quad t > 0, \\ \rho_s \underset{\sim}{U}_{tt} - \underset{\sim}{\text{div}} \left(\underset{\approx}{\sigma}(\underset{\sim}{U}) \right) &= \underset{\sim}{G}_s \qquad \underset{\sim}{x} \in \Omega_s, \quad t > 0. \end{cases}$$

The acoustic wave equation thus governs the propagation of waves in the fluid medium, and the elastic wave equation governs the propagation of waves in the solid medium.

One must then impose boundary conditions on $\partial\Omega$. Following the example of Chapters One and Two, we will use the appropriate version of the first order absorbing boundary condition, that is,

$$(3.2)\qquad \begin{cases} \frac{1}{c}\frac{\partial P}{\partial t} + \frac{\partial P}{\partial \underset{\sim}{n}_f} &= 0 \qquad \underset{\sim}{x} \in \Gamma_f, \quad t > 0, \\ \underset{\approx}{A}\underset{\sim}{U}_t + \underset{\approx}{\sigma}(\underset{\sim}{U})\underset{\sim}{n}_s &= 0 \qquad \underset{\sim}{x} \in \Gamma_s, \quad t > 0, \end{cases}$$

where $\Gamma_f = \partial\Omega_f \cap \partial\Omega$, $\Gamma_s = \partial\Omega_s \cap \partial\Omega$

Finally, one must address the question, "what happens to waves or disturbances which pass from one sub-domain to the other, or strike the interface $\Gamma = \partial\Omega_f \cap \partial\Omega_s$ between the fluid and solid subdomains?" In mathematical terms, one must impose conditions which describe the interaction between the fluid component and the solid component of the composite medium. The following interface conditions

$$(3.3)\qquad \begin{cases} \frac{1}{c}\frac{\partial P}{\partial \underset{\sim}{n}_f} - \rho_f \underset{\sim}{U}_{tt} \cdot \underset{\sim}{n}_s &= 0 \qquad \underset{\sim}{x} \in \Gamma, \quad t > 0, \\ \underset{\approx}{\sigma}(\underset{\sim}{U})\underset{\sim}{n}_s - p\underset{\sim}{n}_f &= 0 \qquad \underset{\sim}{x} \in \Gamma, \quad t > 0, \end{cases}$$

which were derived in [17] and in [8], are frequently used in the literature. The physical meaning amounts to continuity of the normal stress and the normal acceleration on the interface.

Together, equations (3.1),(3.2) and (3.3) yield the following model for wave propagation through a composite fluid-solid medium

$$(3.4)\qquad \begin{cases} \frac{1}{c^2}P_{tt} - \Delta P &= G_f \qquad \underset{\sim}{x} \in \Omega_f, \quad t > 0, \\ \frac{1}{c}\frac{\partial P}{\partial t} + \frac{\partial P}{\partial \underset{\sim}{n}_f} &= 0 \qquad \underset{\sim}{x} \in \partial\Omega_f, \quad t > 0, \\ \frac{1}{c}\frac{\partial P}{\partial \underset{\sim}{n}_f} - \rho_f \underset{\sim}{U}_{tt} \cdot \underset{\sim}{n}_s &= 0 \qquad \underset{\sim}{x} \in \Gamma, \quad t > 0, \\ P = P_t &= 0 \qquad \underset{\sim}{x} \in \Omega_f, \quad t \le 0; \end{cases}$$

$$(3.5)\qquad \begin{cases} \rho_s \underset{\sim}{U}_{tt} - \underset{\sim}{\mathrm{div}}\left(\underset{\approx}{\sigma}(\underset{\sim}{U})\right) &= \underset{\sim}{G}_s \qquad \underset{\sim}{x} \in \Omega_s, \quad t > 0, \\ \underset{\approx}{A}\underset{\sim}{U}_t + \underset{\approx}{\sigma}(\underset{\sim}{U})\underset{\sim}{n}_s &= 0 \qquad \underset{\sim}{x} \in \partial\Omega_s, \quad t > 0, \\ \underset{\approx}{\sigma}(\underset{\sim}{U})\underset{\sim}{n}_s - p\underset{\sim}{n}_f &= 0 \qquad \underset{\sim}{x} \in \Gamma, \quad t > 0, \\ \underset{\sim}{U}_t = \underset{\sim}{U} &= 0 \qquad \underset{\sim}{x} \in \Omega_s, \quad t \le 0. \end{cases}$$

In the system, P is the pressure in Ω_f, $\underset{\sim}{U}$ is the displacement vector in Ω_s, and c is the wave speed in the fluid medium. ρ_i $(i = f, s)$ denotes the density of Ω_i, n_i $(i = f, s)$ denotes the unit outward normal to $\partial\Omega_i$. Throughout Chapter Three, we will assume that the domains Ω_f and Ω_s are subsets of $\Re^N$ with Lipschitz boundaries. Furthermore, we assume that they are star-shaped domains, that is, there exists $\underset{\sim}{\hat{x}} \in \Omega_f$, $\underset{\sim}{\hat{\hat{x}}} \in \Omega_s$ such that

$$
(3.6) \qquad \begin{aligned} (\underset{\sim}{x} - \underset{\sim}{\hat{x}}) \cdot \underset{\sim}{n}_f \geq c_0 > 0 \text{ for all } \underset{\sim}{x} \in \Omega_f, \\ (\underset{\sim}{x} - \underset{\sim}{\hat{\hat{x}}}) \cdot \underset{\sim}{n}_s \geq c_0 > 0 \text{ for all } \underset{\sim}{x} \in \Omega_s. \end{aligned}
$$

The derivation of the above model can be found in [15], where a detailed mathematical analysis concerning the existence, uniqueness and regularity of the solutions also appears. [17] studies the finite element approximations of the model, proposes both semi–discrete and fully–discrete finite element methods, and establishes optimal order error estimates for both discretizations of the fluid-solid interaction model.

Applying the Fourier transform to (3.4) and (3.5), or seeking time harmonic solutions $P(\underset{\sim}{x}, t) = p(\underset{\sim}{x})e^{i\omega t}$ and $\underset{\sim}{U}(\underset{\sim}{x}, t) = \underset{\sim}{u}(\underset{\sim}{x})e^{i\omega t}$, yields the following frequency domain representation of the problem

$$
(3.7) \qquad \begin{cases} -\frac{\omega^2}{c^2}p - \Delta p & = g_f \text{ in } \Omega_f, \\ i\frac{\omega}{c}p + \frac{\partial p}{\partial \underset{\sim}{n}_f} & = 0 \text{ on } \Gamma_f, \\ \frac{\partial p}{\partial \underset{\sim}{n}_f} + \omega^2 \rho_f \underset{\sim}{u} \cdot \underset{\sim}{n}_s & = 0 \text{ on } \Gamma; \end{cases}
$$

$$
(3.8) \qquad \begin{cases} -\omega^2 \rho_s \underset{\sim}{u} - \underset{\sim}{\operatorname{div}} \left(\underset{\approx}{\sigma}(\underset{\sim}{u}) \right) & = \underset{\sim}{g}_s \text{ in } \Omega_s, \\ i\omega \underset{\approx}{A} \underset{\sim}{u} + \underset{\approx}{\sigma}(\underset{\sim}{u}) \underset{\sim}{n}_s & = 0 \text{ on } \Gamma_s, \\ \underset{\approx}{\sigma}(\underset{\sim}{u}) \underset{\sim}{n}_s - p \underset{\sim}{n}_f & = 0 \text{ on } \Gamma, \end{cases}
$$

which is the coupled system of acoustic and elastic Helmholtz equations that we studied in Chapters One and Two.

We continue our study of Helmholtz problems in Chapter 3 with an analysis of (3.7)-(3.8). In Section 3.1, we derive identities for solutions to the interaction problem.

Several are direct consequences of the Rellich identities from Chapters One and Two. We present the variational formulation for (3.7)-(3.8) in Section 3.2, and prove the existence and uniqueness of solutions. In Section 3.3, we formulate the finite element method, and derive finite element error estimates.

3.1 Some Basic and Rellich Type Identities

We first collect some preliminary identities which describe the solution to the interaction problem in terms of its source function. We prove the first lemmas using the solution as a test function. The second lemma follows when we recognize that some cancellation occurs on the interface between subdomains.

Lemma 3.1. *Suppose p and $\underset{\sim}{u}$ solve the interaction problem (3.7)-(3.8). Then*

(1) $$-\tfrac{1}{c^2\rho_f}\|p\|^2_{L^2(\Omega_f)} + \tfrac{1}{\omega^2\rho_f}\|\nabla p\|^2_{L^2(\Omega_f)} + \tfrac{i}{c\omega\rho_f}\|p\|^2_{L^2(\Gamma_f)} + \int_\Gamma (\underset{\sim}{u}\cdot\underset{\sim}{n}_s)\overline{p} = \tfrac{1}{\omega^2\rho_f}\int_{\Omega_f} g_f\overline{p}.$$

(2) $$-\omega^2\rho_s\|\underset{\sim}{u}\|^2_{L^2(\Omega_s)} + \lambda\|\operatorname{div}\underset{\sim}{u}\|^2_{L^2(\Omega_s)} + 2\mu\|\underset{\approx}{\epsilon}(\underset{\sim}{u})\|^2_{L^2(\Omega_s)} + i\omega\int_{\Gamma_s}(\underset{\approx}{A}\underset{\sim}{u})\cdot\overline{\underset{\sim}{u}} - \int_\Gamma (p\underset{\sim}{n}_f)\cdot\overline{\underset{\sim}{u}} = \int_{\Omega_s}\underset{\sim}{g}_s\cdot\overline{\underset{\sim}{u}}.$$

Proof : To prove (1), multiply the Helmholtz equation in (3.7) by $\overline{p}$ and integrate over Ω_f to get

$$-\tfrac{\omega^2}{c^2}\int_{\Omega_f}|p|^2 - \int_{\Omega_f}(\Delta p)\overline{p} = \int_{\Omega_f} g_f\overline{p}.$$

By Lemma 1.2 and the divergence theorem,

$$-\tfrac{\omega^2}{c^2}\|p\|^2_{L^2(\Omega_f)} + \|\nabla p\|^2_{L^2(\Omega_f)} - \int_{\partial\Omega_f}\frac{\partial p}{\partial \underset{\sim}{n}_f}\overline{p} = \int_{\Omega_f} g_f\overline{p}.$$

Applying the boundary and interface conditions in (3.7) yields (1).

To prove (2), multiply the Helmholtz equation in (3.8) by $\overline{\underset{\sim}{u}}$ and integrate over Ω_s to get

$$-\omega^2\rho_s\int_{\Omega_s}|\underset{\sim}{u}|^2 - \int_{\Omega_s}\underset{\sim}{\operatorname{div}}\left(\underset{\approx}{\sigma}(\underset{\sim}{u})\right)\cdot\overline{\underset{\sim}{u}} = \int_{\Omega_s}\underset{\sim}{g}_s\cdot\overline{\underset{\sim}{u}}.$$

By Lemma 2.3 and the divergence theorem,

$$-\omega^2\rho_s\|\underset{\sim}{u}\|^2_{L^2(\Omega_s)} + \lambda\|\operatorname{div}\underset{\sim}{u}\|^2_{L^2(\Omega_s)} + 2\mu\|\underset{\approx}{\epsilon}(\underset{\sim}{u})\|^2_{L^2(\Omega_s)} - \int_{\partial\Omega_s}(\underset{\approx}{\sigma}(\underset{\sim}{u})\underset{\sim}{n}_s)\cdot\overline{\underset{\sim}{u}} = \int_{\Omega_s}\underset{\sim}{g}_s\overline{\underset{\sim}{u}}.$$

Applying the boundary and interface conditions in (3.8) yields (2). ■

Lemma 3.2. *Suppose p and* $\underset{\sim}{u}$ *solve the interaction problem (3.7)-(3.8). Then*

$$(1)\qquad \tfrac{1}{c\omega\rho_f}\|p\|^2_{L^2(\Gamma_f)} + \int_{\Gamma_s}(\underset{\approx}{A}\underset{\sim}{u})\cdot\underset{\sim}{u} = \operatorname{Im}\left(\tfrac{1}{\omega^2\rho_f}\int_{\Omega_f} g_f\overline{p} + \int_{\Gamma_s}\underset{\sim}{g}_s\cdot\overline{\underset{\sim}{u}}\right).$$

$$(2)\qquad \begin{aligned} &-\tfrac{1}{c^2\rho_f}\|p\|^2_{L^2(\Omega_f)} + \omega^2\rho_s\|\underset{\sim}{u}\|^2_{L^2(\Omega_s)} + \tfrac{1}{\omega^2\rho_f}\|\nabla p\|^2_{L^2(\Omega_f)} \\ &\quad - (\lambda\|\operatorname{div}\underset{\sim}{u}\|^2_{L^2(\Omega_s)} + 2\mu\|\underset{\approx}{\epsilon}(\underset{\sim}{u})\|^2_{L^2(\Omega_s)}) = \operatorname{Re}\left(\tfrac{1}{\omega^2\rho_f}\int_{\Omega_f} g_f\overline{p} - \int_{\Omega_s}\underset{\sim}{g}_s\cdot\overline{\underset{\sim}{u}}\right).\end{aligned}$$

Proof : On Γ, $\underset{\sim}{n}_f = -\underset{\sim}{n}_s$. Therefore,

$$\begin{aligned}\int_\Gamma(\underset{\sim}{u}\cdot\underset{\sim}{n}_s)\overline{p} - \int_\Gamma(p\underset{\sim}{n}_f)\cdot\overline{\underset{\sim}{u}} &= \int_\Gamma(\underset{\sim}{u}\cdot\underset{\sim}{n}_s)\overline{p} + \int_\Gamma(p\underset{\sim}{n}_s)\cdot\overline{\underset{\sim}{u}} \\ &= \int_\Gamma(\underset{\sim}{u}\cdot\underset{\sim}{n}_s)\overline{p} + \overline{\int_\Gamma(\underset{\sim}{u}\cdot\underset{\sim}{n}_s)\overline{p}} \\ &= 2\operatorname{Re}\left\{\int_\Gamma(\underset{\sim}{u}\cdot\underset{\sim}{n}_s)\overline{p}\right\}.\end{aligned}$$

Adding the equations from Lemma 3.1 and taking the imaginary part yields (1). Similarly,

$$\int_\Gamma(\underset{\sim}{u}\cdot\underset{\sim}{n}_s)\overline{p} + \int_\Gamma(p\underset{\sim}{n}_f)\cdot\overline{\underset{\sim}{u}} = 2i\operatorname{Im}\left\{\int_\Gamma(\underset{\sim}{u}\cdot\underset{\sim}{n}_s)\overline{p}\right\}.$$

Subtracting the equations in lemma 3.1 and taking the real part yields (2). ■

The remaining identities in this section are direct application of the Rellich identities in Chapters One and Two to the interaction problem (3.7)-(3.8).

Lemma 3.3. *Suppose p and $\underset{\sim}{u}$ solve the interaction problem (3.7)-(3.8). Then for any $\underset{\sim}{\alpha}(x) \in C^1(\Re^N)$ and $\underset{\sim}{\beta}(x) \in C^1(\Re^N)$*

$$
\begin{aligned}
(1)\quad & \tfrac{\omega^2}{2c^2}\int_{\Omega_f} (\operatorname{div}\underset{\sim}{\alpha})|p|^2 + \tfrac{1}{2}\int_{\partial\Omega_f} (\underset{\sim}{\alpha}\cdot\underset{\sim}{n}_f)|\nabla p|^2 \\
&= \tfrac{\omega^2}{2c^2}\int_{\partial\Omega_f} (\underset{\sim}{\alpha}\cdot\underset{\sim}{n}_f)|p|^2 + \tfrac{1}{2}\int_{\Omega_f}(\operatorname{div}\underset{\sim}{\alpha})|\nabla p|^2 \\
&\quad - \operatorname{Re}\left\{\tfrac{i\omega}{c}\int_{\Gamma_f} p(\overline{\underset{\sim}{\alpha}\cdot\nabla p})\right\} - \operatorname{Re}\left\{\omega^2\rho_f\int_{\Gamma}(\underset{\sim}{u}\cdot\underset{\sim}{n}_s)(\overline{\underset{\sim}{\alpha}\cdot\nabla p})\right\} \\
&\quad - \operatorname{Re}\left\{\int_{\Omega_f}\sum_{i=1}^N \frac{\partial p}{\partial x_i}\sum_{j=1}^N \frac{\partial\alpha_j}{\partial x_i}\overline{\frac{\partial p}{\partial x_j}}\right\} + \operatorname{Re}\int_{\Omega_f} g_f(\overline{\underset{\sim}{\alpha}\cdot\nabla p}).
\end{aligned}
$$

$$
\begin{aligned}
(2)\quad & \tfrac{\omega^2\rho_s}{2}\int_{\Omega_s}\operatorname{div}\underset{\sim}{\beta}\,|\underset{\sim}{u}|^2 + \tfrac{1}{2}\int_{\partial\Omega_s}\underset{\sim}{\beta}\cdot\underset{\sim}{n}_s\left(\lambda\,|\operatorname{div}\underset{\sim}{u}|^2 + 2\mu\,\underset{\approx}{\epsilon}(\underset{\sim}{u}) : \overline{\underset{\approx}{\epsilon}(\underset{\sim}{u})}\right) \\
&= \tfrac{\omega^2\rho_s}{2}\int_{\partial\Omega_s}\underset{\sim}{\beta}\cdot\underset{\sim}{n}_s|\underset{\sim}{u}|^2 + \tfrac{1}{2}\int_{\Omega_s}(\operatorname{div}\underset{\sim}{\beta})\left(\lambda\,|\operatorname{div}\underset{\sim}{u}|^2 + 2\mu\,\underset{\approx}{\epsilon}(\underset{\sim}{u}) : \overline{\underset{\approx}{\epsilon}(\underset{\sim}{u})}\right) \\
&\quad - \operatorname{Re}\int_{\Omega_s}\left\{\lambda\sum_{i=1}^N\sum_{j=1}^N \frac{\partial\alpha_j}{\partial x_i}\overline{\frac{\partial u_i}{\partial x_j}} + \mu\sum_{i=1}^N\sum_{j=1}^N\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\sum_{k=1}^N \overline{\frac{\partial u_i}{\partial x_k}}\frac{\partial\alpha_k}{\partial x_j}\right\} \\
&\quad - \operatorname{Re}\left\{i\omega\int_{\Gamma_s}(\underset{\approx}{A}\underset{\sim}{u})\cdot(\overline{(\nabla\underset{\sim}{u})\underset{\sim}{\beta}})\right\} + \operatorname{Re}\left\{\int_{\Gamma} p\underset{\sim}{n}_f\cdot(\overline{(\underset{\approx}{\nabla}\underset{\sim}{u})\underset{\sim}{\beta}})\right\} \\
&\quad + \operatorname{Re}\int_{\Omega_s}\underset{\sim}{g}_s\cdot(\overline{(\underset{\approx}{\nabla}\underset{\sim}{u})\underset{\sim}{\beta}}).
\end{aligned}
$$

Proof : By Theorems 1.4 and 1.6,

$$
\begin{aligned}
\tfrac{\omega^2}{2c^2}\int_{\partial\Omega_f}\underset{\sim}{\alpha}\cdot\underset{\sim}{n}_f|p|^2 + \tfrac{1}{2}\int_{\partial\Omega_f}(\underset{\sim}{\alpha}\cdot\underset{\sim}{n}_f)|\nabla p|^2 &= \tfrac{\omega^2}{2c^2}\int_{\partial\Omega_f}\underset{\sim}{\alpha}\cdot\underset{\sim}{n}_f|p|^2 \\
&\quad + \int_{\Omega_f}(\operatorname{div}\underset{\sim}{\alpha})|\nabla p|^2 + \operatorname{Re}\int_{\partial\Omega_f}\frac{\partial p}{\partial \underset{\sim}{n}_f}(\overline{\underset{\sim}{\alpha}\cdot\nabla p}) \\
&\quad - \operatorname{Re}\int_{\Omega_f}\sum_{i=1}^N\frac{\partial p}{\partial x_i}\sum_{j=1}^N\frac{\partial\alpha_j}{\partial x_i}\overline{\frac{\partial p}{\partial x_j}} + \operatorname{Re}\int_{\Omega_f}\left(-\tfrac{\omega^2}{c^2}p - \Delta p\right)(\overline{\underset{\sim}{\alpha}\cdot\nabla p}).
\end{aligned}
$$

Since p satisfies (3.7), the above implies (1). By Theorems 2.5 and 2.7,

$$\begin{aligned}
&\tfrac{\omega^2\rho_s}{2}\int_{\Omega_s}(\operatorname{div}\underset{\sim}{\beta})|\underset{\sim}{u}|^2+\tfrac{1}{2}\int_{\partial\Omega_s}\underset{\sim}{\beta}\cdot\underset{\sim}{n}_s\left(\lambda\,|\operatorname{div}\underset{\sim}{u}|^2+2\mu\underset{\approx}{\epsilon}(\underset{\sim}{u}):\overline{\underset{\approx}{\epsilon}(\underset{\sim}{u})}\right)\\
&=\tfrac{\omega^2\rho_s}{2}\int_{\partial\Omega_s}\underset{\sim}{\beta}\cdot\underset{\sim}{n}_s|\underset{\sim}{u}|^2+\int_{\Omega_s}(\operatorname{div}\underset{\sim}{\beta})\left(\lambda\,|\operatorname{div}\underset{\sim}{u}|^2+2\mu\,\underset{\approx}{\epsilon}(\underset{\sim}{u}):\overline{\underset{\approx}{\epsilon}(\underset{\sim}{u})}\right)\\
&-\operatorname{Re}\int_{\Omega_s}\left\{\lambda(\operatorname{div}\underset{\sim}{u})\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{\partial\beta_j}{\partial x_i}\overline{\frac{\partial u_i}{\partial x_j}}+\mu\sum_{i=1}^{N}\sum_{j=1}^{N}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)\sum_{k=1}^{N}\overline{\frac{\partial u_i}{\partial x_k}}\frac{\partial\beta_k}{\partial x_j}\right\}\\
&+\operatorname{Re}\int_{\partial\Omega_s}\left(\underset{\approx}{\sigma}(\underset{\sim}{u})\underset{\sim}{n}_s\right)\cdot\left((\overline{\underset{\approx}{\nabla}\underset{\sim}{u}})\underset{\sim}{\beta}\right)+\operatorname{Re}\int_{\Omega_s}\left(-\omega^2\rho_s\underset{\sim}{u}-\underset{\sim}{\operatorname{div}}\left(\underset{\approx}{\sigma}(\underset{\sim}{u})\right)\right)\cdot\left((\overline{\underset{\approx}{\nabla}\underset{\sim}{u}})\underset{\sim}{\beta}\right).
\end{aligned}$$

Since $\underset{\sim}{u}$ satisfies (3.8), the above implies (2). ■

Lemma 3.4. *Suppose p and $\underset{\sim}{u}$ solve the interaction problem (3.7)-(3.8). Then*

(1)
$$\begin{aligned}
&\frac{\omega^2 N}{2c^2}\|p\|^2_{L^2(\Omega_f)}+\frac{1}{2}\int_{\partial\Omega_f}(\underset{\sim}{x}-\underset{\sim}{\hat{x}})\cdot\underset{\sim}{n}_f|\nabla p|^2=\frac{\omega^2}{2c^2}\int_{\partial\Omega_f}(\underset{\sim}{x}-\underset{\sim}{\hat{x}})\cdot\underset{\sim}{n}_f|p|^2\\
&\quad+\left(\frac{N}{2}-1\right)\|\nabla p\|^2_{L^2(\Omega_f)}+\operatorname{Re}\int_{\Omega_f}g_f\overline{((\underset{\sim}{x}-\underset{\sim}{\hat{x}})\cdot\nabla p)}\\
&\quad-\operatorname{Re}\left\{\frac{i\omega}{c}\int_{\Gamma_f}p\overline{((\underset{\sim}{x}-\underset{\sim}{\hat{x}})\cdot\nabla p)}\right\}-\operatorname{Re}\left\{\omega^2\rho_f\int_{\Gamma}(\underset{\sim}{u}\cdot\underset{\sim}{n}_s)\overline{((\underset{\sim}{x}-\underset{\sim}{\hat{x}})\cdot\nabla p)}\right\}.
\end{aligned}$$

(2)
$$\begin{aligned}
&\frac{\omega^2\rho_s N}{2}\|\underset{\sim}{u}\|^2_{L^2(\Omega_s)}+\frac{1}{2}\int_{\partial\Omega_s}(\underset{\sim}{x}-\underset{\sim}{\hat{\hat{x}}})\cdot\underset{\sim}{n}_s\left(\lambda\,|\operatorname{div}\underset{\sim}{u}|^2+2\mu\,\underset{\approx}{\epsilon}(\underset{\sim}{u}):\overline{\underset{\approx}{\epsilon}(\underset{\sim}{u})}\right)\\
&=\frac{\omega^2\rho_s}{2}\int_{\partial\Omega_s}(\underset{\sim}{x}-\underset{\sim}{\hat{\hat{x}}})\cdot\underset{\sim}{n}_s|\underset{\sim}{u}|^2+\left(\frac{N}{2}-1\right)\int_{\Omega_s}\left(\lambda\,|\operatorname{div}\underset{\sim}{u}|^2+2\mu\,\underset{\approx}{\epsilon}(\underset{\sim}{u}):\overline{\underset{\approx}{\epsilon}(\underset{\sim}{u})}\right)\\
&+\operatorname{Re}\int_{\Omega_s}\underset{\sim}{g}_s\cdot\overline{((\underset{\approx}{\nabla}\underset{\sim}{u})(\underset{\sim}{x}-\underset{\sim}{\hat{\hat{x}}})}-\operatorname{Re}\left\{i\omega\int_{\Gamma_s}(\underset{\approx}{A}\underset{\sim}{u})\cdot\overline{((\underset{\approx}{\nabla}\underset{\sim}{u})(\underset{\sim}{x}-\underset{\sim}{\hat{\hat{x}}}))}\right\}\\
&+\operatorname{Re}\left\{\int_{\Gamma}(p\underset{\sim}{n}_f)\cdot\overline{((\underset{\approx}{\nabla}\underset{\sim}{u})(\underset{\sim}{x}-\underset{\sim}{\hat{\hat{x}}}))}\right\}.
\end{aligned}$$

Proof : Follows from the Lemma 3.3 using $\underset{\sim}{\alpha}=\underset{\sim}{x}-\underset{\sim}{\hat{x}}$, $\underset{\sim}{\beta}=\underset{\sim}{x}-\underset{\sim}{\hat{\hat{x}}}$. ■

Lemma 3.5. *Suppose p and $\underset{\sim}{u}$ solve the interaction problem (3.7)-(3.8). Then*

$$\begin{aligned}
&\frac{\omega^2 N}{2c^2}\|p\|^2_{L^2(\Omega_f)} + \frac{\omega^2\rho_s N}{2}\|\underset{\sim}{u}\|^2_{L^2(\Omega_s)} \\
&\quad + \frac{1}{2}\int_{\partial\Omega_f} (\underset{\sim}{x}-\underset{\sim}{\hat{x}})\cdot \underset{\sim}{n}_f |\nabla p|^2 + \frac{1}{2}\int_{\partial\Omega_s} (\underset{\sim}{x}-\underset{\sim}{\hat{\hat{x}}})\cdot \underset{\sim}{n}_s \left(\lambda\,|\operatorname{div}\underset{\sim}{u}|^2 + 2\mu\,\underset{\approx}{\epsilon}(\underset{\sim}{u}) : \overline{\underset{\approx}{\epsilon}(\underset{\sim}{u})}\right) \\
&= \frac{\omega^2}{2c^2}\int_{\partial\Omega_f} (\underset{\sim}{x}-\underset{\sim}{\hat{x}})\cdot \underset{\sim}{n}_f |p|^2 + \frac{\omega^2\rho_s}{2}\int_{\partial\Omega_s} (\underset{\sim}{x}-\underset{\sim}{\hat{\hat{x}}})\cdot \underset{\sim}{n}_s |\underset{\sim}{u}|^2 \\
&\quad + \left(\frac{N}{2}-1\right)\|\nabla p\|^2_{L^2(\Omega_f)} + \left(\frac{N}{2}-1\right)\int_{\Omega_s}\left(\lambda\,|\operatorname{div}\underset{\sim}{u}|^2 + 2\mu\,\underset{\approx}{\epsilon}(\underset{\sim}{u}) : \overline{\underset{\approx}{\epsilon}(\underset{\sim}{u})}\right) \\
&\quad - \operatorname{Re}\left\{\frac{i\omega}{c}\int_{\Gamma_f} p\overline{((\underset{\sim}{x}-\underset{\sim}{\hat{x}})\cdot\nabla p)}\right\} - \operatorname{Re}\left\{i\omega\int_{\Gamma_s}(\underset{\approx}{A}\underset{\sim}{u})\cdot\overline{((\underset{\approx}{\nabla}\underset{\sim}{u})(\underset{\sim}{x}-\underset{\sim}{\hat{\hat{x}}}))}\right\} \\
&\quad - \operatorname{Re}\left\{\omega^2\rho_f\int_{\Gamma}(\underset{\sim}{u}\cdot\underset{\sim}{n}_s)\overline{((\underset{\sim}{x}-\underset{\sim}{\hat{x}})\cdot\nabla p)}\right\} + \operatorname{Re}\left\{\int_{\Gamma}(p\underset{\sim}{n}_f)\cdot\overline{((\underset{\approx}{\nabla}\underset{\sim}{u})(\underset{\sim}{x}-\underset{\sim}{\hat{\hat{x}}}))}\right\} \\
&\quad + \operatorname{Re}\int_{\Omega_f} g_f\overline{((\underset{\sim}{x}-\underset{\sim}{\hat{x}})\cdot\nabla p)} + \operatorname{Re}\int_{\Omega_s}\underset{\sim}{g}_s\cdot\overline{((\underset{\approx}{\nabla}\underset{\sim}{u})(\underset{\sim}{x}-\underset{\sim}{\hat{\hat{x}}}))}.
\end{aligned}$$

Proof : Add the equations in Lemma 3.4. ■

3.2 Variational Formulation and Well-Posedness

The variational formulation of (3.7)-(3.8) is defined as:

$$(3.9)\qquad \begin{cases} \text{Find}\quad [p,\underset{\sim}{u}] \in H^1(\Omega_f)\times(H^1(\Omega_s))^N \quad \text{such that} \\ L([p,\underset{\sim}{u}],[q,\underset{\sim}{v}]) = G([q,\underset{\sim}{v}]) \quad \forall [q,\underset{\sim}{v}] \in H^1(\Omega_f)\times(H^1(\Omega_s))^N, \end{cases}$$

where

$$L([p,\underset{\sim}{u}],[q,\underset{\sim}{v}]) = L_f(p,q,\underset{\sim}{u}) + L_s(\underset{\sim}{u},\underset{\sim}{v},p),$$

$$\begin{aligned}
L_f(p,q,\underset{\sim}{u}) &= \frac{-1}{c^2\rho_f}\int_{\Omega_f} p\overline{q} + \frac{1}{\omega^2\rho_f}\int_{\Omega_f}\nabla p\cdot\overline{\nabla q} + \frac{i}{c\omega\rho_f}\int_{\Gamma_f} p\overline{q} + \int_{\Gamma}\underset{\sim}{u}\cdot\underset{\sim}{n}_s\overline{q}, \\
L_s(\underset{\sim}{u},\underset{\sim}{v},p) &= -\omega^2\rho_s\int_{\Omega_s}\underset{\sim}{u}\cdot\underset{\sim}{v} + 2\mu\int_{\Omega_s}\underset{\approx}{\epsilon}(\underset{\sim}{u}):\overline{\underset{\approx}{\epsilon}(\underset{\sim}{v})} + \lambda\int_{\Omega_s}\operatorname{div}(\underset{\sim}{u})\overline{\operatorname{div}(\underset{\sim}{v})} \\
&\quad + i\omega\int_{\Gamma_s}\underset{\approx}{A}\underset{\sim}{u}\cdot\overline{\underset{\sim}{v}} - \int_{\Gamma} p\underset{\sim}{n}_f\cdot\overline{\underset{\sim}{v}}, \\
G([q,\underset{\sim}{v}]) &= \frac{1}{\omega^2\rho_f}\int_{\Omega_f} g_f\overline{q} + \int_{\Omega_s}\underset{\sim}{g}_s\cdot\overline{\underset{\sim}{v}}.
\end{aligned}$$

To show existence and uniqueness for (3.9), we will first show that the bilinear form $L(\cdot,\cdot)$ satisfies a Gårding inequality, and then appeal to the Unique Continuation Principle [25].

Lemma 3.6. *The bilinear form $L(\cdot,\cdot)$ satisfies the Gårding inequality.*

$$\begin{aligned}\operatorname{Re}\{L_f(p,q,\underset{\sim}{u}) + L_s(\underset{\sim}{u},\underset{\sim}{v},p)\} + C\left(\tfrac{1}{\omega^2}+1\right)\|p\|^2_{L^2(\Omega_f)} + C(\omega^2+1)\|\underset{\sim}{u}\|^2_{L^2(\Omega_s)} \\ \geq \tfrac{1}{2\omega^2\rho_f}\|p\|^2_{H^1(\Omega_f)} + \mu K\|\underset{\sim}{u}\|^2_{H^1(\Omega_s)}.\end{aligned}$$

Proof :

$$\begin{aligned}\operatorname{Re}\{L_f(p,q,\underset{\sim}{u}) + L_s(\underset{\sim}{u},\underset{\sim}{v},p)\} = -\tfrac{1}{c^2\rho_f}\|p\|^2_{L^2(\Omega_f)} + \tfrac{1}{\omega^2\rho_f}\|\nabla p\|^2_{L^2(\Omega_f)} - \omega^2\rho_s\|\underset{\sim}{u}\|^2_{L^2(\Omega_s)} \\ + \lambda\|\operatorname{div}\underset{\sim}{u}\|^2_{L^2(\Omega_s)} + 2\mu\|\underset{\approx}{\epsilon}(\underset{\sim}{u})\|^2_{L^2(\Omega_s)} + 2\operatorname{Re}\int_\Gamma(\underset{\sim}{u}\cdot\underset{\sim}{n}_s)\overline{p},\end{aligned}$$

which implies that

$$\begin{aligned}\operatorname{Re}\{L_f(p,q,\underset{\sim}{u}) + L_s(\underset{\sim}{u},\underset{\sim}{v},p)\} + \left(\tfrac{1}{\omega^2\rho_f}+\tfrac{1}{c^2\rho_f}\right)\|p\|^2_{L^2(\Omega_f)} + (\omega^2\rho_s+2\mu)\|\underset{\sim}{u}\|^2_{L^2(\Omega_s)} \\ = \tfrac{1}{\omega^2\rho_f}\|p\|^2_{H^1(\Omega_f)} + \lambda\|\operatorname{div}\underset{\sim}{u}\|^2_{L^2(\Omega_s)} + 2\mu\|\underset{\approx}{\epsilon}(\underset{\sim}{u})\|^2_{L^2(\Omega_s)} \\ + 2\mu\|\underset{\sim}{u}\|^2_{L^2(\Omega_s)} + 2\operatorname{Re}\int_\Gamma(\underset{\sim}{u}\cdot\underset{\sim}{n}_s)\overline{p}.\end{aligned}$$

Therefore by Korn's Inequality (Lemma 2.9),

$$\begin{aligned}\operatorname{Re}\{L_f(p,q,\underset{\sim}{u}) + L_s(\underset{\sim}{u},\underset{\sim}{v},p)\} + \left(\tfrac{1}{\omega^2\rho_f}+\tfrac{1}{c^2\rho_f}\right)\|p\|^2_{L^2(\Omega_f)} + (\omega^2\rho_s+2\mu)\|\underset{\sim}{u}\|^2_{L^2(\Omega_s)} \\ = \tfrac{1}{\omega^2\rho_f}\|p\|^2_{H^1(\Omega_f)} + 2\mu K\|\underset{\sim}{u}\|^2_{H^1(\Omega_s)} + 2\operatorname{Re}\int_\Gamma(\underset{\sim}{u}\cdot\underset{\sim}{n}_s)\overline{p}.\end{aligned}$$

Note that

$$\begin{aligned}\tfrac{1}{\omega^2\rho_f}\|p\|^2_{H^1(\Omega_f)} + 2\mu K\|\underset{\sim}{u}\|^2_{H^1(\Omega_s)} + 2\operatorname{Re}\int_\Gamma(\underset{\sim}{u}\cdot\underset{\sim}{n}_s)\overline{p} \\ \geq \tfrac{1}{\omega^2\rho_f}\|p\|^2_{H^1(\Omega_f)} + 2\mu K\|\underset{\sim}{u}\|^2_{H^1(\Omega_s)} - 2\left|\int_\Gamma(\underset{\sim}{u}\cdot\underset{\sim}{n}_s)\overline{p}\right|,\end{aligned}$$

which implies

(3.10)

$$\begin{aligned}\operatorname{Re}\{L_f(p,q,\underset{\sim}{u}) + L_s(\underset{\sim}{u},\underset{\sim}{v},p)\} + \left(\tfrac{1}{\omega^2\rho_f}+\tfrac{1}{c^2\rho_f}\right)\|p\|^2_{L^2(\Omega_f)} + (\omega^2\rho_s+2\mu)\|\underset{\sim}{u}\|^2_{L^2(\Omega_s)} \\ \geq \tfrac{1}{\omega^2\rho_f}\|p\|^2_{H^1(\Omega_f)} + 2\mu K\|\underset{\sim}{u}\|^2_{H^1(\Omega_s)} - 2\left|\int_\Gamma(\underset{\sim}{u}\cdot\underset{\sim}{n}_s)\overline{p}\right|.\end{aligned}$$

Applying Schwarz's inequality, the trace inequalities on Ω_f and Ω_s, and Young's inequality yields

$$2\left|\int_\Gamma (\underset{\sim}{u} \cdot \underset{\sim}{n}_s)\overline{p}\right| \leq 2C_f C_s \left(\tfrac{\epsilon_1\epsilon_2}{4}\|p\|^2_{H^1(\Omega_f)} + \tfrac{\epsilon_1}{4\epsilon_2}\|p\|^2_{L^2(\Omega_f)} + \tfrac{\epsilon_3}{4\epsilon_1}\|\underset{\sim}{u}\|^2_{L^2(\Omega_s)} + \tfrac{1}{4\epsilon_1\epsilon_3}\|\underset{\sim}{u}\|^2_{H^1(\Omega_s)}\right)$$

for all $\epsilon_1, \epsilon_2, \epsilon_3 > 0$. Choosing $\epsilon_1 = \frac{1}{C_s w}$, $\epsilon_2 = \frac{1}{C_f \rho_f w}$, and $\epsilon_3 = \frac{C_f C_s^2 w}{2\mu K}$ implies that

$$2\left|\int_\Gamma (\underset{\sim}{u} \cdot \underset{\sim}{n}_s)\overline{p}\right| \leq \tfrac{1}{2\omega^2\rho_f}\|p\|^2_{H^1(\Omega_f)} + \tfrac{C_f^2\rho_f}{2}\|p\|^2_{L^2(\Omega_f)} + \tfrac{C_f^2 C_s^4\omega^2}{4\mu K}\|\underset{\sim}{u}\|^2_{L^2(\Omega_s)} + \mu K\|\underset{\sim}{u}\|^2_{H^1(\Omega_s)}.$$

Applying this expression to (3.10) above yields

$$\mathrm{Re}\{L_f(p, q, \underset{\sim}{u}) + L_s(\underset{\sim}{u}, \underset{\sim}{v}, p)\} + C\left(\tfrac{1}{\omega^2} + 1\right)\|p\|^2_{L^2(\Omega_f)} + C(\omega^2 + 1)\|\underset{\sim}{u}\|^2_{L^2(\Omega_s)} \geq \tfrac{1}{2\omega^2\rho_f}\|p\|^2_{H^1(\Omega_f)} + \mu K\|\underset{\sim}{u}\|^2_{H^1(\Omega_s)},$$

where C is independent of ω. ■

Theorem 3.7. *Suppose $g_f \in L^2(\Omega_f)$, $\underset{\sim}{g}_s \in L^2(\Omega_s)$ and $\omega \neq 0$. Then there exists a unique solution to (3.7)-(3.8).*

Proof : Since L satisfies a Gårding's inequality, again the Fredholm Alternative Theorem implies that a solution to (3.9) exists if the corresponding adjoint problem

$$\text{(3.11)}\qquad \begin{cases} \text{Find} \quad [p, \underset{\sim}{u}] \in H^1(\Omega_f) \times (H^1(\Omega_s))^N \quad \text{such that} \\ L^*([\varphi, \underset{\sim}{\psi}], [q, \underset{\sim}{v}]) = G([q, \underset{\sim}{v}]) \quad \forall [q, \underset{\sim}{v}] \in H^1(\Omega_f) \times (H^1(\Omega_s))^N, \end{cases}$$

has only the zero solution when the source is zero. If $g_f = 0$ and $\underset{\sim}{g}_s = 0$, then choosing $[q, \underset{\sim}{v}] = [\varphi, \underset{\sim}{\psi}]$ in the variational form and taking the imaginary part implies that $\varphi = 0$ and $\underset{\sim}{\psi} = \underset{\sim}{0}$ on Γ_f and Γ_s respectively. Integrating by parts in (3.11) implies that $\frac{\partial\varphi}{\partial \underset{\sim}{n}_f} = 0$ on Γ_f and $\underset{\approx}{\sigma}(\underset{\sim}{\psi})\underset{\sim}{n}_s = 0$ on Γ_s. By the Unique Continuation Principle that $\varphi = 0$ and in Ω_f and $\underset{\sim}{\psi} = \underset{\sim}{0}$ in Ω_s, hence solutions exist. The same argument shows that $g_f = 0$ and $\underset{\sim}{g}_s = 0$ in (3.9) implies that $p = 0$ in Ω_f and $\underset{\sim}{u} = 0$ in Ω_s, so solutions are therefore unique. ■

3.3 Finite Element Procedures

Let $\mathcal{T}^f_{h_1}$, $\mathcal{T}^s_{h_2}$ be quasi-uniform triangulations of Ω_f and Ω_s respectively with mesh sizes $h_1 > 0$ and $h_2 > 0$. Notice that we do not require the triangulations $\mathcal{T}^f_{h_1}$ and $\mathcal{T}^s_{h_2}$ to be aligned along the interface Γ. Suppose V_{h_1} is the P_{m-1} conforming finite element space of $H^1(\Omega_f)$ associated with $\mathcal{T}^f_{h_1}$ and W_{h_2} is the P_{k-1} conforming finite element space of $H^1(\Omega_s)$ associated with $\mathcal{T}^s_{h_2}$. It is well known that V_{h_1} and W_{h_2} have the following approximation properties [5]:

$$(3.12) \qquad \inf_{q\in V_{h_1}} \left\{\|p-q\|_{L^2(\Omega_f)} + h_1\|p-q\|_{H^1(\Omega_f)}\right\} \le C_A h_1^m \|p\|_{H^m(\Omega_f)}$$

and

$$(3.13) \qquad \inf_{\underset{\sim}{v}\in W_{h_2}} \left\{\|\underset{\sim}{u}-\underset{\sim}{v}\|_{L^2(\Omega_s)} + h_2\|\underset{\sim}{u}-\underset{\sim}{v}\|_{H^1(\Omega_s)}\right\} \le C_A h_2^k \|\underset{\sim}{u}\|_{H^k(\Omega_s)}.$$

The finite element method for (3.9) is then defined as

$$(3.14) \qquad \begin{cases} \text{Find} \quad [p_h, \underset{\sim}{u_h}] \in V_{h_1} \times (W_{h_2})^N \quad \text{such that} \\ L([p_h, \underset{\sim}{u_h}], [q, \underset{\sim}{v}]) = G([q, \underset{\sim}{v}]) \quad \forall [q, \underset{\sim}{v}] \in V_{h_1} \times (W_{h_2})^N. \end{cases}$$

Suppose solutions to (3.9) satisfy the following abstract regularity condition:

$$(3.15) \qquad \|p\|^2_{H^m(\Omega_f)} + \omega^2\|\underset{\sim}{u}\|^2_{H^k(\Omega_s)} \le R_{f,m}\|g_f\|^2_{H^{m-2}(\Omega_f)} + R_{s,k}\|\underset{\sim}{g_s}\|^2_{H^{k-2}(\Omega_s)},$$

where $m, k \ge 2$ and $R_{f,m}, R_{s,k}$ depend on ω.

We now apply the duality argument to first bound the L^2 norm of the solution to (3.7)-(3.8) in terms of its H^1 norm. We will then apply the argument of Schatz to derive an estimate for the finite element error.

Lemma 3.8. *Suppose p and $\underset{\sim}{u}$ are solutions to the interaction problem (3.7)-(3.8), p_h and $\underset{\sim}{u_h}$ are their finite element solutions. Then there exists a constant C, independent of ω and h such that*

$$h_1 + h_2 \le C \cdot min\left\{\frac{1}{\omega\sqrt{R_{f,2}}}, \frac{1}{R_{f,2}}, \frac{1}{\sqrt{R_{s,2}}}, \frac{\omega^2}{R_{s,2}}\right\}$$

implies that

$$\|p - p_h\|^2_{L^2(\Omega_f)} + \omega^2 \|\underset{\sim}{u} - \underset{\sim}{u_h}\|^2_{L^2(\Omega_s)} \leq C(h_1 + h_2) \left\{ \|p - p_h\|^2_{H^1(\Omega_f)} + \omega^2 \|\underset{\sim}{u} - \underset{\sim}{u_h}\|^2_{H^1(\Omega_s)} \right\}.$$

Proof : Let $[p, \underset{\sim}{u}]$ be the solution of the interaction problem, and let $[p_h, \underset{\sim}{u_h}]$ be the finite element approximations. Let $[\varphi, \underset{\sim}{\psi}]$ be a solution to the adjoint problem with source $[p - p_h, \underset{\sim}{u} - \underset{\sim}{u_h}]$, i.e. $[\varphi, \underset{\sim}{\psi}]$ solves

$$\begin{cases} \text{Find} \quad [\varphi, \underset{\sim}{\psi}] \in H^1(\Omega_f) \times (H^1(\Omega_s))^N \quad \text{such that} \\ L^*([\varphi, \underset{\sim}{\psi}], [q, \underset{\sim}{v}]) = ([p - p_h, \underset{\sim}{u} - \underset{\sim}{u_h}], [q, \underset{\sim}{v}]) \quad \forall [q, \underset{\sim}{v}] \in H^1(\Omega_f) \times (H^1(\Omega_s))^N, \end{cases}$$

where

$$L^*([\varphi, \underset{\sim}{\psi}], [q, \underset{\sim}{v}]) = L([q, \underset{\sim}{v}], [\varphi, \underset{\sim}{\psi}]).$$

Then $[\varphi, \underset{\sim}{\psi}]$ satisfies

$$L([q, \underset{\sim}{v}], [\varphi, \underset{\sim}{\psi}]) = ([q, \underset{\sim}{v}], [p - p_h, \underset{\sim}{u} - \underset{\sim}{u_h}]) \quad \forall [q, \underset{\sim}{v}] \in H^1(\Omega_f) \times (H^1(\Omega_s))^N.$$

That is,

$$L_f(q, \varphi, \underset{\sim}{v}) + L_s(\underset{\sim}{v}, \psi, q) = \frac{1}{\omega^2 \rho_f} \int_{\Omega_f} q \overline{(p - p_h)} + \int_{\Omega_s} \underset{\sim}{v} \cdot \overline{(\underset{\sim}{u} - \underset{\sim}{u_h})}$$

for all $q \in H^1(\Omega_f)$ and all $\underset{\sim}{v} \in (H^1(\Omega_s))^N$. Taking $q = p - p_h$ and $\underset{\sim}{v} = \underset{\sim}{u} - \underset{\sim}{u_h}$ and applying the fundamental orthogonality property,

$$\frac{1}{\omega^2 \rho_f} \|p - p_h\|^2_{L^2(\Omega_f)} = L_f(p - p_h, \varphi, \underset{\sim}{u} - \underset{\sim}{u_h}) = L_f(p - p_h, \varphi - \varphi_h, \underset{\sim}{u} - \underset{\sim}{u_h}),$$

$$\|\underset{\sim}{u} - \underset{\sim}{u_h}\|^2_{L^2(\Omega_s)} = L_s(\underset{\sim}{u} - \underset{\sim}{u_h}, \underset{\sim}{\psi}, p - p_h) = L_s(\underset{\sim}{u} - \underset{\sim}{u_h}, \underset{\sim}{\psi} - \underset{\sim}{\psi_h}, p - p_h),$$

therefore,

$$
\begin{aligned}
\|p - p_h\|^2_{L^2(\Omega_f)} \le \tfrac{\omega^2}{c^2} \|p - p_h\|_{L^2(\Omega_f)} \|\varphi - \varphi_h\|_{L^2(\Omega_f)} \\
+ \|\nabla(p - p_h)\|_{L^2(\Omega_f)} \|\nabla(\varphi - \varphi_h)\|_{L^2(\Omega_f)} \\
+ \tfrac{\omega}{c} \|p - p_h\|_{L^2(\Gamma_f)} \|\varphi - \varphi_h\|_{L^2(\Gamma_f)} \\
+ \omega^2 \rho_f \|(\underset{\sim}{u} - \underset{\sim}{u_h}) \cdot \underset{\sim}{n_s}\|_{L^2(\Gamma)} \|\varphi - \varphi_h\|_{L^2(\Gamma)}
\end{aligned}
$$

and

$$
\begin{aligned}
\|\underset{\sim}{u} - \underset{\sim}{u_h}\|^2_{L^2(\Omega_s)} \le \omega^2 \rho_s \|\underset{\sim}{u} - \underset{\sim}{u_h}\|_{L^2(\Omega_s)} \|\underset{\sim}{\psi} - \underset{\sim}{\psi_h}\|_{L^2(\Omega_s)} \\
+ 2\mu \|\underset{\approx}{\epsilon}(\underset{\sim}{u} - \underset{\sim}{u_h})\|_{L^2(\Omega_s)} \|\underset{\approx}{\epsilon}(\underset{\sim}{\psi} - \underset{\sim}{\psi_h})\|_{L^2(\Omega_s)} \\
+ \lambda \| \operatorname{div}(\underset{\sim}{u} - \underset{\sim}{u_h})\|_{L^2(\Omega_s)} \| \operatorname{div}(\underset{\sim}{\psi} - \underset{\sim}{\psi_h})\|_{L^2(\Omega_s)} \\
+ \omega a \|\underset{\sim}{u} - \underset{\sim}{u_h}\|_{L^2(\Gamma_s)} \|\underset{\sim}{\psi} - \underset{\sim}{\psi_h}\|_{L^2(\Gamma_s)} + \|p - p_h\|_{L^2(\Gamma)} \|\underset{\sim}{\psi} - \underset{\sim}{\psi_h}\|_{L^2(\Gamma)}.
\end{aligned}
$$

Note that

$$
\tfrac{\omega}{c} \|p - p_h\|_{L^2(\Gamma_f)} \|\varphi - \varphi_h\|_{L^2(\Gamma_f)} \le \tfrac{\omega}{c} \|p - p_h\|_{L^2(\partial\Omega_f)} \|\varphi - \varphi_h\|_{L^2(\partial\Omega_f)}
$$

so by the trace inequality on $\partial\Omega_f$ (1.13),

$$
\begin{aligned}
&\tfrac{\omega}{c} \|p - p_h\|_{L^2(\Gamma_f)} \|\varphi - \varphi_h\|_{L^2(\Gamma_f)} \\
&\quad \le \tfrac{\omega}{c} C_f^2 \|p - p_h\|^{1/2}_{L^2(\Omega_f)} \|\varphi - \varphi_h\|^{1/2}_{L^2(\Omega_f)} \|p - p_h\|^{1/2}_{H^1(\Omega_f)} \|\varphi - \varphi_h\|^{1/2}_{H^1(\Omega_f)} \\
&\quad \le \tfrac{\omega^2}{2c^2} C_f^2 \|p - p_h\|_{L^2(\Omega_f)} \|\varphi - \varphi_h\|_{L^2(\Omega_f)} \\
&\qquad + \tfrac{C_f^2}{2} \|p - p_h\|_{H^1(\Omega_f)} \|\varphi - \varphi_h\|_{H^1(\Omega_f)}.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\omega^2\rho_f \|(\underset{\sim}{u} - \underset{\sim}{u}_h)\cdot \underset{\sim}{n}_s\|_{L^2(\Gamma)}\|\varphi-\varphi_h\|_{L^2(\Gamma)} \\
&\le \omega^2\rho_f \|(\underset{\sim}{u} - \underset{\sim}{u}_h)\cdot \underset{\sim}{n}_s\|_{L^2(\partial\Omega_s)}\|\varphi-\varphi_h\|_{L^2(\partial\Omega_f)} \\
&\le \omega^2\rho_f C_f C_s \|\underset{\sim}{u} - \underset{\sim}{u}_h\|^{1/2}_{L^2(\Omega_s)}\|\varphi-\varphi_h\|^{1/2}_{L^2(\Omega_f)}\|\underset{\sim}{u} - \underset{\sim}{u}_h\|^{1/2}_{H^1(\Omega_s)}\|\varphi-\varphi_h\|^{1/2}_{H^1(\Omega_f)} \\
&\le \tfrac{\omega^3\rho_f}{2} C_f C_s \|\underset{\sim}{u} - \underset{\sim}{u}_h\|_{L^2(\Omega_s)}\|\varphi-\varphi_h\|_{L^2(\Omega_f)} \\
&\quad + \tfrac{\omega\rho_f}{2} C_f C_s \|\underset{\sim}{u} - \underset{\sim}{u}_h\|_{H^1(\Omega_s)}\|\varphi-\varphi_h\|_{H^1(\Omega_f)}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\|p-p_h\|^2_{L^2(\Omega_f)} &\le \tfrac{\omega^2}{c^2}\|p-p_h\|_{L^2(\Omega_f)}\|\varphi-\varphi_h\|_{L^2(\Omega_f)} \\
&\quad + \|\nabla(p-p_h)\|_{L^2(\Omega_f)}\|\nabla(\varphi-\varphi_h)\|_{L^2(\Omega_f)} \\
&\quad + \tfrac{\omega^2}{2c^2} C_f^2 \|p-p_h\|_{L^2(\Omega_f)}\|\varphi-\varphi_h\|_{L^2(\Omega_f)} \\
&\quad + \tfrac{C_f^2}{2}\|p-p_h\|_{H^1(\Omega_f)}\|\varphi-\varphi_h\|_{H^1(\Omega_f)} \\
&\quad + \tfrac{\omega^3\rho_f}{2} C_f C_s \|\underset{\sim}{u} - \underset{\sim}{u}_h\|_{L^2(\Omega_s)}\|\varphi-\varphi_h\|_{L^2(\Omega_f)} \\
&\quad + \tfrac{\omega\rho_f}{2} C_f C_s \|\underset{\sim}{u} - \underset{\sim}{u}_h\|_{H^1(\Omega_s)}\|\varphi-\varphi_h\|_{H^1(\Omega_f)},
\end{aligned}
$$

which implies

$$
\begin{aligned}
\|p-p_h\|^2_{L^2(\Omega_f)} &\le \tfrac{\omega^2}{c^2}\left(1+\tfrac{C_f^2}{2}\right)\|p-p_h\|_{L^2(\Omega_f)}\|\varphi-\varphi_h\|_{L^2(\Omega_f)} \\
&\quad + \left(1+\tfrac{C_f^2}{2}\right)\|p-p_h\|_{H^1(\Omega_f)}\|\varphi-\varphi_h\|_{H^1(\Omega_f)} \\
&\quad + \tfrac{\omega^3\rho_f}{2} C_f C_s \|\underset{\sim}{u} - \underset{\sim}{u}_h\|_{L^2(\Omega_s)}\|\varphi-\varphi_h\|_{L^2(\Omega_s)} \\
&\quad + \tfrac{\omega\rho_f}{2} C_f C_s \|\underset{\sim}{u} - \underset{\sim}{u}_h\|_{H^1(\Omega_s)}\|\varphi-\varphi_h\|_{H^1(\Omega_s)}.
\end{aligned}
$$

Applying the approximation property of V_{h_1} (3.12) implies

$$
\begin{aligned}
\|p-p_h\|^2_{L^2(\Omega_f)} \le \; & \tfrac{\omega^2}{c^2}\left(1+\tfrac{C_f^2}{2}\right)\|p-p_h\|_{L^2(\Omega_f)} C_A h_1^2 \|\varphi\|_{H^2(\Omega_f)} \\
& + \left(1+\tfrac{C_f^2}{2}\right)\|p-p_h\|_{H^1(\Omega_f)} C_A h_1 \|\varphi\|_{H^2(\Omega_f)} \\
& + \tfrac{\omega^3\rho_f}{2} C_f C_s \|\underset{\sim}{u}-\underset{\sim}{u}_h\|_{L^2(\Omega_s)} C_A h_1^2 \|\varphi\|_{H^2(\Omega_s)} \\
& + \tfrac{\omega\rho_f}{2} C_f C_s \|\underset{\sim}{u}-\underset{\sim}{u}_h\|_{H^1(\Omega_s)} C_A h_1 \|\varphi\|_{H^2(\Omega_s)}.
\end{aligned}
\tag{3.16}
$$

It is also easy to show that

$$
\begin{aligned}
\omega a_2 \|\underset{\sim}{u}-\underset{\sim}{u}_h\|_{L^2(\Gamma_s)} \|\underset{\sim}{\psi}-\underset{\sim}{\psi}_h\|_{L^2(\Gamma_s)} \le \; & \omega^2 a_2 C_s^2 \|\underset{\sim}{u}-\underset{\sim}{u}_h\|_{L^2(\Omega_s)} \|\underset{\sim}{\psi}-\underset{\sim}{\psi}_h\|_{L^2(\Omega_s)} \\
& + \tfrac{a_2}{4} C_s^2 \|\underset{\sim}{u}-\underset{\sim}{u}_h\|_{H^1(\Omega_s)} \|\underset{\sim}{\psi}-\underset{\sim}{\psi}_h\|_{H^1(\Omega_s)},
\end{aligned}
$$

and that

$$
\begin{aligned}
\|(p-p_h)\underset{\sim}{n}_f\|_{L^2(\Gamma)} \|\underset{\sim}{\psi}-\underset{\sim}{\psi}_h\|_{L^2(\Gamma)} \le \; & \tfrac{\omega}{2} C_f C_s \|p-p_h\|_{L^2(\Omega_f)} \|\underset{\sim}{\psi}-\underset{\sim}{\psi}_h\|_{L^2(\Omega_s)} \\
& + \tfrac{1}{2\omega} C_f C_s \|p-p_h\|_{H^1(\Omega_f)} \|\underset{\sim}{\psi}-\underset{\sim}{\psi}_h\|_{H^1(\Omega_s)}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\|\underset{\sim}{u}-\underset{\sim}{u}_h\|^2_{L^2(\Omega_s)} \le \; & \omega^2 \rho_s \|\underset{\sim}{u}-\underset{\sim}{u}_h\|_{L^2(\Omega_s)} \|\underset{\sim}{\psi}-\underset{\sim}{\psi}_h\|_{L^2(\Omega_s)} \\
& + 2\mu \|\underset{\approx}{\epsilon}(\underset{\sim}{u}-\underset{\sim}{u}_h)\|_{L^2(\Omega_s)} \|\underset{\approx}{\epsilon}(\underset{\sim}{\psi}-\underset{\sim}{\psi}_h)\|_{L^2(\Omega_s)} \\
& + \lambda \|\operatorname{div}(\underset{\sim}{u}-\underset{\sim}{u}_h)\|_{L^2(\Omega_s)} \|\operatorname{div}(\underset{\sim}{\psi}-\underset{\sim}{\psi}_h)\|_{L^2(\Omega_s)} \\
& + \omega^2 a_2 C_s^2 \|\underset{\sim}{u}-\underset{\sim}{u}_h\|_{L^2(\Omega_s)} \|\underset{\sim}{\psi}-\underset{\sim}{\psi}_h\|_{L^2(\Omega_s)} \\
& + \tfrac{a_2}{4} C_s^2 \|\underset{\sim}{u}-\underset{\sim}{u}_h\|_{H^1(\Omega_s)} \|\underset{\sim}{\psi}-\underset{\sim}{\psi}_h\|_{H^1(\Omega_s)} \\
& + \tfrac{\omega}{2} C_f C_s \|p-p_h\|_{L^2(\Omega_f)} \|\underset{\sim}{\psi}-\underset{\sim}{\psi}_h\|_{L^2(\Omega_s)} \\
& + \tfrac{1}{2\omega} C_f C_s \|p-p_h\|_{H^1(\Omega_f)} \|\underset{\sim}{\psi}-\underset{\sim}{\psi}_h\|_{H^1(\Omega_s)}.
\end{aligned}
$$

Applying Lemma 2.17 and simplifying,

$$\begin{aligned}||\underset{\sim}{u} - \underset{\sim}{u_h}||^2_{L^2(\Omega_s)} \le{}& \omega^2(\rho_s + a_2 C_s^2)||\underset{\sim}{u} - \underset{\sim}{u_h}||_{L^2(\Omega_s)}||\underset{\sim}{\psi} - \underset{\sim}{\psi_h}||_{L^2(\Omega_s)} \\ &+ (2\mu + \lambda N + \tfrac{a_2}{4}C_s^2)||\underset{\sim}{u} - \underset{\sim}{u_h}||_{H^1(\Omega_s)}||\underset{\sim}{\psi} - \underset{\sim}{\psi_h}||_{H^1(\Omega_s)} \\ &+ \tfrac{\omega}{2}C_f C_s ||p - p_h||_{L^2(\Omega_f)}||\underset{\sim}{\psi} - \underset{\sim}{\psi_h}||_{L^2(\Omega_s)} \\ &+ \tfrac{1}{2\omega}C_f C_s ||p - p_h||_{H^1(\Omega_f)}||\underset{\sim}{\psi} - \underset{\sim}{\psi_h}||_{H^1(\Omega_s)}.\end{aligned}$$

Applying the approximation property of W_{h_2} (3.13) implies

$$\begin{aligned}||\underset{\sim}{u} - \underset{\sim}{u_h}||^2_{L^2(\Omega_s)} \le{}& \omega^2(\rho_s + a_2 C_s^2)||\underset{\sim}{u} - \underset{\sim}{u_h}||_{L^2(\Omega_s)} C_A h_2^2 ||\underset{\sim}{\psi}||_{H^2(\Omega_s)} \\ &+ (2\mu + \lambda N + \tfrac{a_2}{4}C_s^2)||\underset{\sim}{u} - \underset{\sim}{u_h}||_{H^1(\Omega_s)} C_A h_2 ||\underset{\sim}{\psi}||_{H^2(\Omega_s)} \\ &+ \tfrac{\omega}{2}C_f C_s ||p - p_h||_{L^2(\Omega_f)} C_A h_2^2 ||\underset{\sim}{\psi}||_{H^2(\Omega_s)} \\ &+ \tfrac{1}{2\omega}C_f C_s ||p - p_h||_{H^1(\Omega_f)} C_A h_2 ||\underset{\sim}{\psi}||_{H^2(\Omega_s)}.\end{aligned} \tag{3.17}$$

Inequalities (3.16) and (3.17) imply there exists C independent of ω, h_1 and h_2 such that

$$\begin{aligned}&||p - p_h||^2_{L^2(\Omega_f)} + \omega^2||\underset{\sim}{u} - \underset{\sim}{u_h}||^2_{L^2(\Omega_s)} \\ &\quad\le C\left\{h_1^2\omega^2||p - p_h||_{L^2(\Omega_f)}||\varphi||_{H^2(\Omega_f)} + h_2^2\omega^3||p - p_h||_{L^2(\Omega_f)}||\underset{\sim}{\psi}||_{H^2(\Omega_s)}\right\} \\ &\quad+ C\left\{h_1^2\omega^3||\underset{\sim}{u} - \underset{\sim}{u_h}||_{L^2(\Omega_s)}||\varphi||_{H^2(\Omega_s)} + h_2^2\omega^4||\underset{\sim}{u} - \underset{\sim}{u_h}||_{L^2(\Omega_s)}||\underset{\sim}{\psi}||_{H^2(\Omega_s)}\right\} \\ &\quad+ C\left\{h_1||p - p_h||_{H^1(\Omega_f)}||\varphi||_{H^2(\Omega_f)} + h_2\omega||p - p_h||_{H^1(\Omega_f)}||\underset{\sim}{\psi}||_{H^2(\Omega_s)}\right\} \\ &\quad+ C\left\{h_1\omega||\underset{\sim}{u} - \underset{\sim}{u_h}||_{H^1(\Omega_s)}||\varphi||_{H^2(\Omega_s)} + h_2\omega^2||\underset{\sim}{u} - \underset{\sim}{u_h}||_{H^1(\Omega_s)}||\underset{\sim}{\psi}||_{H^2(\Omega_s)}\right\}.\end{aligned}$$

Young's inequality implies

$$\|p-p_h\|^2_{L^2(\Omega_f)}+\omega^2\|\underset{\sim}{u}-\underset{\sim}{u_h}\|^2_{L^2(\Omega_s)}$$
$$\leq C\left\{h_1^2\omega^2\left(\tfrac{1}{2}\|p-p_h\|^2_{L^2(\Omega_f)}+\tfrac{1}{2}\|\varphi\|^2_{H^2(\Omega_f)}\right)+h_2^2\omega^3\left(\tfrac{1}{2\omega}\|p-p_h\|^2_{L^2(\Omega_f)}+\tfrac{\omega}{2}\|\underset{\sim}{\psi}\|^2_{H^2(\Omega_s)}\right)\right.$$
$$+h_1^2\omega^3\left(\tfrac{\omega}{2}\|\underset{\sim}{u}-\underset{\sim}{u_h}\|^2_{L^2(\Omega_s)}+\tfrac{1}{2\omega}\|\varphi\|^2_{H^2(\Omega_s)}\right)+h_2^2\omega^4\left(\tfrac{1}{2}\|\underset{\sim}{u}-\underset{\sim}{u_h}\|^2_{L^2(\Omega_s)}+\tfrac{1}{2}\|\underset{\sim}{\psi}\|^2_{H^2(\Omega_s)}\right)$$
$$+h_1\left(\tfrac{1}{2}\|p-p_h\|^2_{H^1(\Omega_f)}+\tfrac{1}{2}\|\varphi\|^2_{H^2(\Omega_f)}\right)+h_2\omega\left(\tfrac{1}{2\omega}\|p-p_h\|^2_{H^1(\Omega_f)}+\tfrac{\omega}{2}\|\underset{\sim}{\psi}\|^2_{H^2(\Omega_s)}\right)$$
$$\left.+h_1\omega\left(\tfrac{\omega}{2}\|\underset{\sim}{u}-\underset{\sim}{u_h}\|^2_{H^1(\Omega_s)}+\tfrac{1}{2\omega}\|\varphi\|^2_{H^2(\Omega_s)}\right)+h_2\omega^2\left(\tfrac{1}{2}\|\underset{\sim}{u}-\underset{\sim}{u_h}\|^2_{H^1(\Omega_s)}+\tfrac{1}{2}\|\underset{\sim}{\psi}\|^2_{H^2(\Omega_s)}\right)\right\}.$$

Equivalently, there exists C independent of ω, h_1 and h_2 such that

$$\|p-p_h\|^2_{L^2(\Omega_f)}+\omega^2\|\underset{\sim}{u}-\underset{\sim}{u_h}\|^2_{L^2(\Omega_s)}$$
$$\leq C\left\{(h_1^2+h_2^2)\omega^2\|p-p_h\|^2_{L^2(\Omega_f)}+(h_1^2+h_2^2)^2\omega^4\|\underset{\sim}{u}-\underset{\sim}{u_h}\|^2_{L^2(\Omega_s)}\right\}$$
$$+C\left\{(h_1+h_2)\|p-p_h\|^2_{H^1(\Omega_f)}+(h_1+h_2)\omega^2\|\underset{\sim}{u}-\underset{\sim}{u_h}\|^2_{H^1(\Omega_s)}\right\}$$
$$+C\omega\left\{(h_1^2\omega^2+h_1)\|\varphi\|^2_{H^2(\Omega_s)}+(h_2^2\omega^4+h_2\omega^2)\|\underset{\sim}{\psi}\|^2_{H^2(\Omega_s)}\right\}$$

or

$$(1-C(h_1^2+h_2^2)\omega^2)\left\{\|p-p_h\|^2_{L^2(\Omega_f)}+\omega^2\|\underset{\sim}{u}-\underset{\sim}{u_h}\|^2_{L^2(\Omega_s)}\right\}$$
$$\leq C(h_1+h_2)\left\{\|p-p_h\|^2_{H^1(\Omega_f)}+\omega^2\|\underset{\sim}{u}-\underset{\sim}{u_h}\|^2_{H^1(\Omega_s)}\right\}$$
$$+C\left((h_1^2+h_2^2)\omega^2+(h_1+h_2)\right)\left\{\|\varphi\|^2_{H^2(\Omega_s)}+\omega^2\|\underset{\sim}{\psi}\|^2_{H^2(\Omega_s)}\right\}.$$

By the regularity estimate assumption (3.15),

$$(1-C(h_1^2+h_2^2)\omega^2)\left\{\|p-p_h\|^2_{L^2(\Omega_f)}+\omega^2\|\underset{\sim}{u}-\underset{\sim}{u_h}\|^2_{L^2(\Omega_s)}\right\}$$
$$\leq C(h_1+h_2)\left\{\|p-p_h\|^2_{H^1(\Omega_f)}+\omega^2\|\underset{\sim}{u}-\underset{\sim}{u_h}\|^2_{H^1(\Omega_s)}\right\}$$
$$+C((h_1^2+h_2^2)\omega^2+(h_1+h_2))\left\{R_{f,2}\|p-p_h\|^2_{L^2(\Omega_f)}+R_{s,2}\|\underset{\sim}{u}-\underset{\sim}{u_h}\|^2_{L^2(\Omega_s)}\right\}.$$

Therefore,

$$\{1 - C((h_1^2 + h_2^2)\omega^2 + R_{f,2}((h_1^2 + h_2^2)\omega^2 + (h_1 + h_2)))\} \|p - p_h\|^2_{L^2(\Omega_f)}$$
$$+ \left\{1 - C\left((h_1^2 + h_2^2)\omega^2 + \tfrac{R_{s,2}}{\omega^2}((h_1^2 + h_2^2)\omega^2 + (h_1 + h_2))\right)\right\} \omega^2 \|\underset{\sim}{u} - \underset{\sim}{u}_h\|^2_{L^2(\Omega_s)}$$
$$\leq C(h_1 + h_2)\left\{\|p - p_h\|^2_{H^1(\Omega_f)} + \omega^2\|\underset{\sim}{u} - \underset{\sim}{u}_h\|^2_{H^1(\Omega_s)}\right\}.$$

Choose h such that

$$C\left((h_1^2 + h_2^2)\omega^2 + R_{f,2}((h_1^2 + h_2^2)\omega^2 + (h_1 + h_2))\right) \leq \tfrac{1}{2},$$
$$C\left((h_1^2 + h_2^2)\omega^2 + \tfrac{R_{s,2}}{\omega^2}((h_1^2 + h_2^2)\omega^2 + (h_1 + h_2))\right) \leq \tfrac{1}{2},$$

that is,

$$h_1 + h_s \leq C \ min\left\{\tfrac{1}{\omega}, \tfrac{1}{\omega\sqrt{R_{f,2}}}, \tfrac{1}{R_{f,2}}, \tfrac{1}{\sqrt{R_{s,2}}}, \tfrac{\omega^2}{R_{s,2}}\right\}.$$

For h_1 and h_2 chosen thusly,

$$\|p - p_h\|^2_{L^2(\Omega_f)} + \omega^2\|\underset{\sim}{u} - \underset{\sim}{u}_h\|^2_{L^2(\Omega_s)}$$
$$\leq C(h_1 + h_2)\left\{\|p - p_h\|^2_{H^1(\Omega_f)} + \omega^2\|\underset{\sim}{u} - \underset{\sim}{u}_h\|^2_{H^1(\Omega_s)}\right\}.$$

■

Theorem 3.9. *Suppose p and $\underset{\sim}{u}$ are solutions to the interaction problem (3.7)-(3.8), p_h and $\underset{\sim}{u}_h$ the finite element approximations. Then there exist a constant C, independent of ω, h_1 and h_2 such that $h_1 + h_2 \leq C\omega^{-2}$ implies*

$$\|p - p_h\|^2_{H^1(\Omega_f)} + \omega^2\|\underset{\sim}{u} - \underset{\sim}{u}_h\|^2_{H^1(\Omega_s)}$$
$$\leq C\omega^2 \ max\left\{(h_1^{m-1})^2(h_1^2 + 1), (h_2^{k-1})^2(h_2^2 + 1)\right\}$$
$$\cdot \left\{R_{f,m}\|g_f\|^2_{H^{m-2}(\Omega_f)} + R_{s,k}\|\underset{\sim}{g}_s\|^2_{H^{k-2}(\Omega_s)}\right\}$$

and

$$\|p - p_h\|^2_{L^2(\Omega_f)} + \omega^2\|\underset{\sim}{u} - \underset{\sim}{u}_h\|^2_{L^2(\Omega_s)}$$
$$\leq C(h_1 + h_2)\omega^2 \ max\left\{(h_1^{m-1})^2(h_1^2 + 1), (h_2^{k-1})^2(h_2^2 + 1)\right\}$$
$$\cdot \left\{R_{f,m}\|g_f\|^2_{H^{m-2}(\Omega_f)} + R_{s,k}\|\underset{\sim}{g}_s\|^2_{H^{k-2}(\Omega_s)}\right\}$$

for $k, m \geq 2$.

Proof : Suppose $[p, \underset{\sim}{u}]$ solves the interaction problem, and $[p_h, \underset{\sim}{u}_h]$ solves the finite element interaction problem. Then

$$L_f(p - p_h, p - p_h, \underset{\sim}{u} - \underset{\sim}{u}_h) = \frac{-1}{c^2\rho_f}||p - p_h||^2_{L^2(\Omega_f)} + \frac{1}{\omega^2\rho_f}||\nabla(p - p_h)||^2_{L^2(\Omega_f)}$$
$$+ \frac{i}{c\omega\rho_f}||p - p_h||^2_{L^2(\Gamma_f)} + \left\langle (\underset{\sim}{u} - \underset{\sim}{u}_h) \cdot \underset{\sim}{n}_s, p - p_h \right\rangle_\Gamma,$$

therefore,

$$\mathrm{Re}\left\{ L_f(p - p_h, p - p_h, \underset{\sim}{u} - \underset{\sim}{u}_h) \right\} + \frac{1}{c^2\rho_f}||p - p_h||^2_{L^2(\Omega_f)}$$
$$= \frac{1}{\omega^2\rho_f}||\nabla(p - p_h)||^2_{L^2(\Omega_f)} + \mathrm{Re}\left\{ \left\langle (\underset{\sim}{u} - \underset{\sim}{u}_h) \cdot \underset{\sim}{n}_s, p - p_h \right\rangle_\Gamma \right\},$$

which implies that

$$\mathrm{Re}\left\{ L_f(p - p_h, p - p_h, \underset{\sim}{u} - \underset{\sim}{u}_h) \right\} + \left(\frac{1}{\omega^2\rho_f} + \frac{1}{c^2\rho_f} \right) ||p - p_h||^2_{L^2(\Omega_f)}$$
$$= \frac{1}{\omega^2\rho_f}||p - p_h||^2_{H^1(\Omega_f)} + \mathrm{Re}\left\{ \left\langle (\underset{\sim}{u} - \underset{\sim}{u}_h) \cdot \underset{\sim}{n}_s, p - p_h \right\rangle_\Gamma \right\}$$

and

$$\mathrm{Re}\left\{ L_s(\underset{\sim}{u} - \underset{\sim}{u}_h, \underset{\sim}{u} - \underset{\sim}{u}_h, p - p_h) \right\} + (\omega^2\rho_s + 2\mu)||\underset{\sim}{u} - \underset{\sim}{u}_h||^2_{L^2(\Omega_s)}$$
$$\geq 2\mu||\underset{\sim}{u} - \underset{\sim}{u}_h||^2_{H^1(\Omega_s)} - \mathrm{Re}\left\{ \left\langle (p - p_h)\underset{\sim}{n}_f, \underset{\sim}{u} - \underset{\sim}{u}_h \right\rangle_\Gamma \right\}.$$

By the fundamental orthogonality identity,

$$L_f(p - p_h, q, \underset{\sim}{u} - \underset{\sim}{u}_h) = 0 \qquad \forall q \in V_{h_1},$$

therefore,

$$\frac{1}{\omega^2\rho_f}||p - p_h||^2_{H^1(\Omega_f)} = -\mathrm{Re}\left\{ \left\langle (\underset{\sim}{u} - \underset{\sim}{u}_h) \cdot \underset{\sim}{n}_s, p - p_h \right\rangle_\Gamma \right\}$$
$$+ \left(\frac{1}{\omega^2\rho_f} + \frac{1}{c^2\rho_f} \right) ||p - p_h||^2_{L^2(\Omega_f)}. \tag{3.18}$$

By the trace inequality,

$$\text{Re}\left\langle (\underset{\sim}{u} - \underset{\sim}{u_h}) \cdot \underset{\sim}{n_s}, p - p_h \right\rangle_\Gamma \leq \|\underset{\sim}{u} - \underset{\sim}{u_h}\|_{L^2(\Gamma)} \|p - p_h\|_{L^2(\Gamma)}$$
$$\leq C \|\underset{\sim}{u} - \underset{\sim}{u_h}\|^{1/2}_{L^2(\Omega_s)} \|p - p_h\|^{1/2}_{L^2(\Omega_f)} \|\underset{\sim}{u} - \underset{\sim}{u_h}\|^{1/2}_{H^1(\Omega_s)} \|p - p_h\|^{1/2}_{H^1(\Omega_f)}.$$

So for any $\epsilon_1, \epsilon_2, \epsilon_3 > 0$,

$$(3.19) \quad \begin{aligned} \text{Re}\left\langle (\underset{\sim}{u} - \underset{\sim}{u_h}) \cdot \underset{\sim}{n_s}, p - p_h \right\rangle_\Gamma &\leq C \tfrac{\epsilon_1 \epsilon_2}{4} \|p - p_h\|^2_{H^1(\Omega_f)} + C \tfrac{\epsilon_1}{4\epsilon_2} \|\underset{\sim}{u} - \underset{\sim}{u_h}\|^2_{H^1(\Omega_s)} \\ &\quad + C \tfrac{\epsilon_3}{4\epsilon_1} \|p - p_h\|^2_{L^2(\Omega_f)} + C \tfrac{1}{4\epsilon_1 \epsilon_3} \|\underset{\sim}{u} - \underset{\sim}{u_h}\|^2_{L^2(\Omega_s)} \end{aligned}$$

and

$$\begin{aligned} \text{Re}\left\{ L_f(p - p_h, p - q, \underset{\sim}{u} - \underset{\sim}{u_h}) \right\} &\leq \tfrac{1}{c^2 \rho_f} \|p - p_h\|_{L^2(\Omega_f)} \|p - q\|_{L^2(\Omega_f)} \\ &\quad + \tfrac{1}{\omega^2 \rho_f} \|\nabla(p - p_h)\|_{L^2(\Omega_f)} \|\nabla(p - q)\|_{L^2(\Omega_f)} \\ &\quad + \tfrac{1}{c\omega \rho_f} \|p - p_h\|_{L^2(\Gamma_f)} \|p - q\|_{L^2(\Gamma_f)} \\ &\quad + \|\underset{\sim}{u} - \underset{\sim}{u_h}\|_{L^2(\Gamma)} \|p - q\|_{L^2(\Gamma)}. \end{aligned}$$

Applying the trace theorem and Young's inequality,

$$\begin{aligned} \text{Re}\left\{ L_f(p - p_h, p - q, \underset{\sim}{u} - \underset{\sim}{u_h}) \right\} &\leq \tfrac{1}{c^2 \rho_f} \|p - p_h\|_{L^2(\Omega_f)} \|p - q\|_{L^2(\Omega_f)} \\ &+ \tfrac{1}{\omega^2 \rho_f} \|\nabla(p - p_h)\|_{L^2(\Omega_f)} \|\nabla(p - q)\|_{L^2(\Omega_f)} \\ &+ \tfrac{1}{c^2 \rho_f} \|p - p_h\|_{L^2(\Omega_f)} \|p - q\|_{L^2(\Omega_f)} \\ &+ C_f^4 \tfrac{1}{\omega^2 \rho_f} \|p - p_h\|_{H^1(\Omega_f)} \|p - q\|_{H^1(\Omega_f)} \\ &+ C_f C_s \|\underset{\sim}{u} - \underset{\sim}{u_h}\|^{1/2}_{L^2(\Omega_s)} \|p - q\|^{1/2}_{L^2(\Omega_f)} \|\underset{\sim}{u} - \underset{\sim}{u_h}\|^{1/2}_{H^1(\Omega_s)} \|p - q\|^{1/2}_{H^1(\Omega_f)}, \end{aligned}$$

which implies

$$(3.20) \quad \begin{aligned} \text{Re}\left\{ L_f(p - p_h, p - q, \underset{\sim}{u} - \underset{\sim}{u_h}) \right\} &\leq \tfrac{2}{c^2 \rho_f} \|p - p_h\|_{L^2(\Omega_f)} \|p - q\|_{L^2(\Omega_f)} \\ &+ \left(\tfrac{C_f^4 + 1}{\omega^2 \rho_f} \right) \|p - p_h\|_{H^1(\Omega_f)} \|p - q\|_{H^1(\Omega_f)} \\ &+ C_f C_s \|\underset{\sim}{u} - \underset{\sim}{u_h}\|^{1/2}_{L^2(\Omega_s)} \|p - q\|^{1/2}_{L^2(\Omega_f)} \|\underset{\sim}{u} - \underset{\sim}{u_h}\|^{1/2}_{H^1(\Omega_s)} \|p - q\|^{1/2}_{H^1(\Omega_f)}. \end{aligned}$$

Applying (3.19) and (3.20) to (3.18) implies

(3.21)

$$\frac{1}{\omega^2\rho_f}||p-p_h||^2_{H^1(\Omega_f)} \le C||p-p_h||_{L^2(\Omega_f)}||p-q||_{L^2(\Omega_f)}$$
$$+ C\frac{1}{\omega^2}||p-p_h||_{H^1(\Omega_f)}||p-q||_{H^1(\Omega_f)}$$
$$+ C||\underset{\sim}{u}-\underset{\sim}{u_h}||^{1/2}_{L^2(\Omega_s)}||p-q||^{1/2}_{L^2(\Omega_f)}||\underset{\sim}{u}-\underset{\sim}{u_h}||^{1/2}_{H^1(\Omega_s)}||p-q||^{1/2}_{H^1(\Omega_f)}$$
$$+ C\frac{\epsilon_1\epsilon_2}{4}||p-p_h||^2_{H^1(\Omega_f)} + C\frac{\epsilon_1}{4\epsilon_2}||\underset{\sim}{u}-\underset{\sim}{u_h}||^2_{H^1(\Omega_s)}$$
$$+ C\frac{\epsilon_3}{4\epsilon_1}||p-p_h||^2_{L^2(\Omega_f)} + C\frac{1}{4\epsilon_1\epsilon_3}||\underset{\sim}{u}-\underset{\sim}{u_h}||^2_{L^2(\Omega_s)}$$
$$+ C\left(1+\frac{1}{\omega^2}\right)||p-p_h||^2_{L^2(\Omega_f)}$$

for some C independent of ω.

Again, since $[p, \underset{\sim}{u}]$ solves the interaction problem and $[p_h, \underset{\sim}{u_h}]$ solves the finite element interaction problem,

$$L_s(\underset{\sim}{u}-\underset{\sim}{u_h}, \underset{\sim}{u}-\underset{\sim}{u_h}, p-p_h) = -\omega^2\rho_s||\underset{\sim}{u}-\underset{\sim}{u_h}||^2_{L^2(\Omega_s)}$$
$$+ \lambda||\operatorname{div}(\underset{\sim}{u}-\underset{\sim}{u_h})||^2_{L^2(\Omega_s)} + 2\mu||\underset{\approx}{\epsilon}(\underset{\sim}{u}-\underset{\sim}{u_h})||^2_{L^2(\Omega_s)}$$
$$+ i\omega\left\langle \underset{\sim}{u}-\underset{\sim}{u_h}, \underset{\sim}{u}-\underset{\sim}{u_h}\right\rangle_{\Gamma_s} - \left\langle (p-p_h)\underset{\sim}{n_f}, \underset{\sim}{u}-\underset{\sim}{u_h}\right\rangle_{\Gamma},$$

$$\operatorname{Re}\left\{L_s(\underset{\sim}{u}-\underset{\sim}{u_h}, \underset{\sim}{u}-\underset{\sim}{u_h}, p-p_h)\right\} + \omega^2\rho_s||\underset{\sim}{u}-\underset{\sim}{u_h}||^2_{L^2(\Omega_s)} = 2\mu||\underset{\approx}{\epsilon}(\underset{\sim}{u}-\underset{\sim}{u_h})||^2_{L^2(\Omega_s)}$$
$$+ \lambda||\operatorname{div}(\underset{\sim}{u}-\underset{\sim}{u_h})||^2_{L^2(\Omega_s)} - \operatorname{Re}\left\{\left\langle (p-p_h)\underset{\sim}{n_f}, \underset{\sim}{u}-\underset{\sim}{u_h}\right\rangle_{\Gamma}\right\}.$$

By the fundamental orthogonality identity,

$$L_s(\underset{\sim}{u}-\underset{\sim}{u_h}, \underset{\sim}{v}, p-p_h) = 0 \qquad \forall \underset{\sim}{v} \in (W_{h_2})^N,$$

therefore,

(3.22)

$$2\mu||\underset{\sim}{u}-\underset{\sim}{u_h}||^2_{H^1(\Omega_s)} \le \operatorname{Re}\left\{L_s(\underset{\sim}{u}-\underset{\sim}{u_h}, \underset{\sim}{u}-\underset{\sim}{v}, p-p_h)\right\} + \operatorname{Re}\left\{\left\langle (p-p_h)\underset{\sim}{n_f}, \underset{\sim}{u}-\underset{\sim}{u_h}\right\rangle_{\Gamma}\right\}$$
$$+ (\omega^2\rho_s + 2\mu)||\underset{\sim}{u}-\underset{\sim}{u_h}||^2_{L^2(\Omega_s)}.$$

By the trace inequality,

$$\operatorname{Re}\left\langle (p-p_h)\underset{\sim}{n}_f, \underset{\sim}{u}-\underset{\sim}{u}_h \right\rangle_\Gamma \le \|p-p_h\|_{L^2(\Gamma)}\|\underset{\sim}{u}-\underset{\sim}{u}_h\|_{L^2(\Gamma)}$$
$$\le C\|p-p_h\|^{1/2}_{L^2(\Omega_f)}\|\underset{\sim}{u}-\underset{\sim}{u}_h\|^{1/2}_{L^2(\Omega_s)}\|p-p_h\|^{1/2}_{H^1(\Omega_f)}\|\underset{\sim}{u}-\underset{\sim}{u}_h\|^{1/2}_{H^1(\Omega_s)}.$$

So for any $\epsilon_4, \epsilon_5, \epsilon_6 > 0$,

(3.23)
$$\operatorname{Re}\left\{\left\langle (p-p_h)\underset{\sim}{n}_f, \underset{\sim}{u}-\underset{\sim}{u}_h \right\rangle_\Gamma\right\} \le C\tfrac{\epsilon_4\epsilon_5}{4}\|p-p_h\|^2_{H^1(\Omega_f)} + C\tfrac{\epsilon_4}{4\epsilon_5}\|\underset{\sim}{u}-\underset{\sim}{u}_h\|^2_{H^1(\Omega_s)}$$
$$+ C\tfrac{\epsilon_6}{4\epsilon_4}\|p-p_h\|^2_{L^2(\Omega_f)} + C\tfrac{1}{4\epsilon_4\epsilon_6}\|\underset{\sim}{u}-\underset{\sim}{u}_h\|^2_{L^2(\Omega_s)}.$$

Applying the trace theorem and Young's inequality,

$$\operatorname{Re}\left\{L_s(\underset{\sim}{u}-\underset{\sim}{u}_h, \underset{\sim}{u}-\underset{\sim}{v}, p-p_h)\right\} \le \omega^2\rho_s\|\underset{\sim}{u}-\underset{\sim}{u}_h\|_{L^2(\Omega_s)}\|\underset{\sim}{u}-\underset{\sim}{v}\|_{L^2(\Omega_s)}$$
$$+ \lambda\|\operatorname{div}(\underset{\sim}{u}-\underset{\sim}{u}_h)\|_{L^2(\Omega_s)}\|\operatorname{div}(\underset{\sim}{u}-\underset{\sim}{v})\|_{L^2(\Omega_s)}$$
$$+ 2\mu\|\underset{\approx}{\epsilon}(\underset{\sim}{u}-\underset{\sim}{u}_h)\|_{L^2(\Omega_s)}\|\underset{\approx}{\epsilon}(\underset{\sim}{u}-\underset{\sim}{v})\|_{L^2(\Omega_s)}$$
$$+ \omega a_2\|\underset{\sim}{u}-\underset{\sim}{u}_h\|_{L^2(\Gamma_s)}\|\underset{\sim}{u}-\underset{\sim}{v}\|_{L^2(\Gamma_s)} + \|p-p_h\|_{L^2(\Gamma)}\|\underset{\sim}{u}-\underset{\sim}{v}\|_{L^2(\Gamma)}$$
$$\le \omega^2\rho_s\|\underset{\sim}{u}-\underset{\sim}{u}_h\|_{L^2(\Omega_s)}\|\underset{\sim}{u}-\underset{\sim}{v}\|_{L^2(\Omega_s)} + (2\mu+\lambda N)\|\underset{\sim}{u}-\underset{\sim}{u}_h\|_{H^1(\Omega_s)}\|\underset{\sim}{u}-\underset{\sim}{v}\|_{H^1(\Omega_s)}$$
$$+ \omega^2\rho_s\|\underset{\sim}{u}-\underset{\sim}{u}_h\|_{L^2(\Omega_s)}\|\underset{\sim}{u}-\underset{\sim}{v}\|_{L^2(\Omega_s)} + \tfrac{a_2^2C_s^4}{4\rho_s}\|\underset{\sim}{u}-\underset{\sim}{u}_h\|_{H^1(\Omega_s)}\|\underset{\sim}{u}-\underset{\sim}{v}\|_{H^1(\Omega_s)}$$
$$+ C_fC_s\|p-p_h\|^{1/2}_{L^2(\Omega_f)}\|p-p_h\|^{1/2}_{H^1(\Omega_f)}\|\underset{\sim}{u}-\underset{\sim}{v}\|^{1/2}_{L^2(\Omega_s)}\|\underset{\sim}{u}-\underset{\sim}{v}\|^{1/2}_{H^1(\Omega_s)},$$

which implies

(3.24)
$$\operatorname{Re} L_s(\underset{\sim}{u}-\underset{\sim}{u}_h, \underset{\sim}{u}-\underset{\sim}{u}_h, p-p_h) \le 2\omega^2\rho_s\|\underset{\sim}{u}-\underset{\sim}{u}_h\|_{L^2(\Omega_s)}\|\underset{\sim}{u}-\underset{\sim}{v}\|_{L^2(\Omega_s)}$$
$$+ (2\mu+\lambda N+\tfrac{a_2^2C_s^4}{4\rho_s})\|\underset{\sim}{u}-\underset{\sim}{u}_h\|_{H^1(\Omega_s)}\|\underset{\sim}{u}-\underset{\sim}{v}\|_{H^1(\Omega_s)}$$
$$+ C_fC_s\|p-p_h\|^{1/2}_{L^2(\Omega_f)}\|p-p_h\|^{1/2}_{H^1(\Omega_f)}\|\underset{\sim}{u}-\underset{\sim}{v}\|^{1/2}_{L^2(\Omega_s)}\|\underset{\sim}{u}-\underset{\sim}{v}\|^{1/2}_{H^1(\Omega_s)}.$$

Applying (3.23) and (3.24) to (3.22) implies

(3.25)

$$
\begin{aligned}
2\mu\|\underset{\sim}{u}-\underset{\sim}{u_h}\|^2_{H^1(\Omega_s)} \le\ & 2\omega^2\rho_s\|\underset{\sim}{u}-\underset{\sim}{u_h}\|_{L^2(\Omega_s)}\|\underset{\sim}{u}-\underset{\sim}{v}\|_{L^2(\Omega_s)} \\
&+ \left(2\mu+\lambda N+\tfrac{a_2^2C_s^4}{4\rho_s}\right)\|\underset{\sim}{u}-\underset{\sim}{u_h}\|_{H^1(\Omega_s)}\|\underset{\sim}{u}-\underset{\sim}{v}\|_{H^1(\Omega_s)} \\
&+ C_fC_s\|p-p_h\|^{1/2}_{L^2(\Omega_f)}\|p-p_h\|^{1/2}_{H^1(\Omega_f)}\|\underset{\sim}{u}-\underset{\sim}{v}\|^{1/2}_{L^2(\Omega_s)}\|\underset{\sim}{u}-\underset{\sim}{v}\|^{1/2}_{H^1(\Omega_s)} \\
&+ C\tfrac{\epsilon_4\epsilon_5}{4}\|p-p_h\|^2_{H^1(\Omega_f)} + C\tfrac{\epsilon_4}{4\epsilon_5}\|\underset{\sim}{u}-\underset{\sim}{u_h}\|^2_{H^1(\Omega_s)} \\
&+ C\tfrac{\epsilon_6}{4\epsilon_4}\|p-p_h\|^2_{L^2(\Omega_f)} + C\tfrac{1}{4\epsilon_4\epsilon_6}\|\underset{\sim}{u}-\underset{\sim}{u_h}\|^2_{L^2(\Omega_s)} \\
&+ (\omega^2\rho_s+2\mu)\|\underset{\sim}{u}-\underset{\sim}{u_h}\|^2_{L^2(\Omega_s)}.
\end{aligned}
$$

Choose $\epsilon_1=\epsilon_4\frac{\mu}{\omega C}\sqrt{\frac{1}{2\mu\rho_f}}$, $\epsilon_2=\epsilon_5\frac{1}{\omega}\sqrt{\frac{1}{2\mu\rho_f}}$, $\epsilon_3=\epsilon_6=\frac{1}{\omega}$. Then $C\frac{\epsilon_1\epsilon_2}{4}=C\frac{\epsilon_4\epsilon_5}{4}=\frac{1}{8\omega^2\rho_f}$, and $C\frac{\epsilon_1}{4\epsilon_2}=C\frac{\epsilon_4}{4\epsilon_5}=\frac{\mu}{4}$, therefore (3.21) becomes

$$
\begin{aligned}
\tfrac{1}{\omega^2\rho_f}\|p-p_h\|^2_{H^1(\Omega_f)} \le\ & C\|p-p_h\|_{L^2(\Omega_f)}\|p-q\|_{L^2(\Omega_f)} \\
&+ C\tfrac{1}{\omega^2}\|p-p_h\|_{H^1(\Omega_f)}\|p-q\|_{H^1(\Omega_f)} \\
&+ C\|\underset{\sim}{u}-\underset{\sim}{u_h}\|^{1/2}_{L^2(\Omega_s)}\|\underset{\sim}{u}-\underset{\sim}{u_h}\|^{1/2}_{H^1(\Omega_s)}\|p-q\|^{1/2}_{L^2(\Omega_f)}\|p-q\|^{1/2}_{H^1(\Omega_f)} \\
&+ \tfrac{1}{8\omega^2\rho_f}\|p-p_h\|^2_{H^1(\Omega_f)} + \tfrac{\mu}{4}\|\underset{\sim}{u}-\underset{\sim}{u_h}\|^2_{H^1(\Omega_s)} + C\omega^2\|\underset{\sim}{u}-\underset{\sim}{u_h}\|^2_{L^2(\Omega_s)} \\
&+ C\left(1+\tfrac{1}{\omega^2}\right)\|p-p_h\|^2_{L^2(\Omega_f)},
\end{aligned}
$$

which implies

$$
\begin{aligned}
\tfrac{7}{8\omega^2\rho_f}\|p-p_h\|^2_{H^1(\Omega_f)} \le\ & C\|p-p_h\|_{L^2(\Omega_f)}\|p-q\|_{L^2(\Omega_f)} \\
&+ C\tfrac{1}{\omega^2}\|p-p_h\|_{H^1(\Omega_f)}\|p-q\|_{H^1(\Omega_f)} \\
&+ C\|\underset{\sim}{u}-\underset{\sim}{u_h}\|^{1/2}_{L^2(\Omega_s)}\|\underset{\sim}{u}-\underset{\sim}{u_h}\|^{1/2}_{H^1(\Omega_s)}\|p-q\|^{1/2}_{L^2(\Omega_f)}\|p-q\|^{1/2}_{H^1(\Omega_f)} \\
&+ \tfrac{\mu}{4}\|\underset{\sim}{u}-\underset{\sim}{u_h}\|^2_{H^1(\Omega_s)} + C\omega^2\|\underset{\sim}{u}-\underset{\sim}{u_h}\|^2_{L^2(\Omega_s)} \\
&+ C\left(1+\tfrac{1}{\omega^2}\right)\|p-p_h\|^2_{L^2(\Omega_f)}.
\end{aligned}
$$

Applying Young's inequality,

$$\begin{aligned}
\tfrac{7}{8\omega^2\rho_f}\|p-p_h\|^2_{H^1(\Omega_f)} \le{}& C\tfrac{1}{2}\|p-p_h\|^2_{L^2(\Omega_f)} + C\tfrac{1}{2}\|p-q\|^2_{L^2(\Omega_f)} \\
&+ \tfrac{1}{8\omega^2}\|p-p_h\|^2_{H^1(\Omega_f)} + \tfrac{2C^2}{\omega^2}\|p-q\|^2_{H^1(\Omega_f)} \\
&+ \tfrac{C}{4}\omega^2\|\underset{\sim}{u}-\underset{\sim}{u_h}\|^2_{L^2(\Omega_s)} + \tfrac{C}{4\omega^2}\|p-q\|^2_{L^2(\Omega_f)} \\
&+ \tfrac{\mu}{4}\|\underset{\sim}{u}-\underset{\sim}{u_h}\|^2_{H^1(\Omega_s)} + \tfrac{C^2}{4\mu}\|p-q\|^2_{H^1(\Omega_f)} \\
&+ \tfrac{\mu}{4}\|\underset{\sim}{u}-\underset{\sim}{u_h}\|^2_{H^1(\Omega_s)} + C\omega^2\|\underset{\sim}{u}-\underset{\sim}{u_h}\|^2_{L^2(\Omega_s)} \\
&+ C\left(1+\tfrac{1}{\omega^2}\right)\|p-p_h\|^2_{L^2(\Omega_f)},
\end{aligned}$$

which implies

$$\begin{aligned}
\tfrac{3}{4\omega^2\rho_f}\|p-p_h\|^2_{H^1(\Omega_f)} \le{}& C(1+\tfrac{1}{\omega^2})\|p-q\|^2_{L^2(\Omega_f)} + C(1+\tfrac{1}{\omega^2})\|p-q\|^2_{H^1(\Omega_f)} \\
&+ C\omega^2\|\underset{\sim}{u}-\underset{\sim}{u_h}\|^2_{L^2(\Omega_s)} + \tfrac{\mu}{2}\|\underset{\sim}{u}-\underset{\sim}{u_h}\|^2_{H^1(\Omega_s)} \\
&+ C\left(1+\tfrac{1}{\omega^2}\right)\|p-p_h\|^2_{L^2(\Omega_f)}
\end{aligned} \tag{3.26}$$

for some C independent of ω. Similarly, (3.25) becomes

$$\begin{aligned}
2\mu\|\underset{\sim}{u}-\underset{\sim}{u_h}\|^2_{H^1(\Omega_s)} \le{}& C\omega^2\|\underset{\sim}{u}-\underset{\sim}{u_h}\|_{L^2(\Omega_s)}\|\underset{\sim}{u}-\underset{\sim}{v}\|_{L^2(\Omega_s)} + C\|\underset{\sim}{u}-\underset{\sim}{u_h}\|_{H^1(\Omega_s)}\|\underset{\sim}{u}-\underset{\sim}{v}\|_{H^1(\Omega_s)} \\
&+ C\|p-p_h\|^{1/2}_{L^2(\Omega_f)}\|\underset{\sim}{u}-\underset{\sim}{v}\|^{1/2}_{L^2(\Omega_s)}\|p-p_h\|^{1/2}_{H^1(\Omega_f)}\|\underset{\sim}{u}-\underset{\sim}{v}\|^{1/2}_{H^1(\Omega_s)} \\
&+ \tfrac{1}{8\omega^2\rho_f}\|p-p_h\|^2_{H^1(\Omega_f)} + \tfrac{\mu}{4}\|\underset{\sim}{u}-\underset{\sim}{u_h}\|^2_{H^1(\Omega_s)} + C\|p-p_h\|^2_{L^2(\Omega_f)} \\
&+ C(\omega^2+1)\|\underset{\sim}{u}-\underset{\sim}{u_h}\|^2_{L^2(\Omega_s)},
\end{aligned}$$

which implies

$$\begin{aligned}
\tfrac{7\mu}{4}\|\underset{\sim}{u}-\underset{\sim}{u_h}\|^2_{H^1(\Omega_s)} \le{}& C\omega^2\|\underset{\sim}{u}-\underset{\sim}{u_h}\|_{L^2(\Omega_s)}\|\underset{\sim}{u}-\underset{\sim}{v}\|_{L^2(\Omega_s)} + C\|\underset{\sim}{u}-\underset{\sim}{u_h}\|_{H^1(\Omega_s)}\|\underset{\sim}{u}-\underset{\sim}{v}\|_{H^1(\Omega_s)} \\
&+ C\|p-p_h\|^{1/2}_{L^2(\Omega_f)}\|\underset{\sim}{u}-\underset{\sim}{v}\|^{1/2}_{L^2(\Omega_s)}\|p-p_h\|^{1/2}_{H^1(\Omega_f)}\|\underset{\sim}{u}-\underset{\sim}{v}\|^{1/2}_{H^1(\Omega_s)} \\
&+ \tfrac{1}{8\omega^2\rho_f}\|p-p_h\|^2_{H^1(\Omega_f)} + C\|p-p_h\|^2_{L^2(\Omega_f)} \\
&+ C(\omega^2+1)\|\underset{\sim}{u}-\underset{\sim}{u_h}\|^2_{L^2(\Omega_s)}.
\end{aligned}$$

Applying Young's inequality,

$$\begin{aligned}\tfrac{7\mu}{4}\|\underset{\sim}{u} - \underset{\sim}{u_h}\|^2_{H^1(\Omega_s)} &\le C\tfrac{\omega^2}{2}\|\underset{\sim}{u} - \underset{\sim}{u_h}\|^2_{L^2(\Omega_s)} + C\tfrac{\omega^2}{2}\|\underset{\sim}{u} - \underset{\sim}{v}\|^2_{L^2(\Omega_s)} \\ &+ \tfrac{\mu}{4}\|\underset{\sim}{u} - \underset{\sim}{u_h}\|^2_{H^1(\Omega_s)} + \tfrac{C^2}{\mu}\|\underset{\sim}{u} - \underset{\sim}{v}\|^2_{H^1(\Omega_s)} \\ &+ \tfrac{C}{4}\|p - p_h\|^2_{L^2(\Omega_f)} + \tfrac{C}{4}\|\underset{\sim}{u} - \underset{\sim}{v}\|^2_{L^2(\Omega_s)} \\ &+ \tfrac{1}{8\omega^2\rho_f}\|p - p_h\|^2_{H^1(\Omega_f)} + \tfrac{c^2\omega^2\rho_f}{2}\|\underset{\sim}{u} - \underset{\sim}{v}\|^2_{H^1(\Omega_s)} \\ &+ \tfrac{1}{8\omega^2\rho_f}\|p - p_h\|^2_{H^1(\Omega_f)} + C\|p - p_h\|^2_{L^2(\Omega_f)} \\ &+ C(\omega^2 + 1)\|\underset{\sim}{u} - \underset{\sim}{u_h}\|^2_{L^2(\Omega_s)},\end{aligned}$$

which implies

$$\begin{aligned}\tfrac{3\mu}{2}\|\underset{\sim}{u} - \underset{\sim}{u_h}\|^2_{H^1(\Omega_s)} &\le C(\omega^2 + 1)\|\underset{\sim}{u} - \underset{\sim}{v}\|^2_{L^2(\Omega_s)} + C(\omega^2 + 1)\|\underset{\sim}{u} - \underset{\sim}{v}\|^2_{H^1(\Omega_s)} \\ &+ C\|p - p_h\|^2_{L^2(\Omega_f)} + \tfrac{1}{4\omega^2\rho_f}\|p - p_h\|^2_{H^1(\Omega_f)} \\ &+ C(\omega^2 + 1)\|\underset{\sim}{u} - \underset{\sim}{u_h}\|^2_{L^2(\Omega_s)}\end{aligned} \tag{3.27}$$

for some C independent of ω.

Inequalities (3.26) and (3.27) together imply

$$\begin{aligned}\tfrac{3}{4\omega^2\rho_f}\|p - p_h\|^2_{H^1(\Omega_f)} + \tfrac{3\mu}{2}\|\underset{\sim}{u} - \underset{\sim}{u_h}\|^2_{H^1(\Omega_s)} &\le C(\omega^2 + 1)\|\underset{\sim}{u} - \underset{\sim}{v}\|^2_{L^2(\Omega_s)} \\ &+ C(\omega^2 + 1)\|\underset{\sim}{u} - \underset{\sim}{v}\|^2_{H^1(\Omega_s)} + C\|p - p_h\|^2_{L^2(\Omega_f)} \\ &+ \tfrac{1}{4\omega^2\rho_f}\|p - p_h\|^2_{H^1(\Omega_f)} + C(\omega^2 + 1)\|\underset{\sim}{u} - \underset{\sim}{u_h}\|^2_{L^2(\Omega_s)} \\ &+ C(1 + \tfrac{1}{\omega^2})\|p - q\|^2_{L^2(\Omega_f)} + C(1 + \tfrac{1}{\omega^2})\|p - q\|^2_{H^1(\Omega_f)} \\ &+ C\omega^2\|\underset{\sim}{u} - \underset{\sim}{u_h}\|^2_{L^2(\Omega_s)} + \tfrac{\mu}{2}\|\underset{\sim}{u} - \underset{\sim}{u_h}\|^2_{H^1(\Omega_s)} \\ &+ C\left(1 + \tfrac{1}{\omega^2}\right)\|p - p_h\|^2_{L^2(\Omega_f)}\end{aligned}$$

or

$$\frac{1}{2\omega^2\rho_f}||p-p_h||^2_{H^1(\Omega_f)} + \mu||\underset{\sim}{u} - \underset{\sim}{u_h}||^2_{H^1(\Omega_s)} \leq C(1+\tfrac{1}{\omega^2})||p-q||^2_{L^2(\Omega_f)}$$

$$+ C(\omega^2+1)||\underset{\sim}{u} - \underset{\sim}{v}||^2_{L^2(\Omega_s)} + C(1+\tfrac{1}{\omega^2})||p-q||^2_{H^1(\Omega_f)}$$

$$+ C(\omega^2+1)||\underset{\sim}{u} - \underset{\sim}{v}||^2_{H^1(\Omega_s)} + C\left(1+\tfrac{1}{\omega^2}\right)||p-p_h||^2_{L^2(\Omega_f)}$$

$$+ C(\omega^2+1)||\underset{\sim}{u} - \underset{\sim}{u_h}||^2_{L^2(\Omega_s)}.$$

Applying the approximation properties (3.12) and (3.13),

$$\frac{1}{2\omega^2\rho_f}||p-p_h||^2_{H^1(\Omega_f)} + \mu||\underset{\sim}{u} - \underset{\sim}{u_h}||^2_{H^1(\Omega_s)} \leq C(1+\tfrac{1}{\omega^2})(C_A h_1^m)^2||p||^2_{H^m(\Omega_f)}$$

$$+ C(\omega^2+1)(C_A h_2^k)^2||\underset{\sim}{u}||^2_{H^k(\Omega_s)} + C(1+\tfrac{1}{\omega^2})(C_A h_1^{m-1})^2||p||^2_{H^m(\Omega_f)}$$

$$+ C(\omega^2+1)(C_A h_2^{k-1})^2||\underset{\sim}{u}||^2_{H^k(\Omega_s)} + C\left(1+\tfrac{1}{\omega^2}\right)||p-p_h||^2_{L^2(\Omega_f)}$$

$$+ C(\omega^2+1)||\underset{\sim}{u} - \underset{\sim}{u_h}||^2_{L^2(\Omega_s)},$$

which implies

$$\frac{1}{2\omega^2\rho_f}||p-p_h||^2_{H^1(\Omega_f)} + \mu||\underset{\sim}{u} - \underset{\sim}{u}||^2_{H^1(\Omega_s)} \leq C(1+\tfrac{1}{\omega^2})(C_A h_1^{m-1})^2(h_1^2+1)||p||^2_{H^m(\Omega_f)}$$

$$+ C(\omega^2+1)(C_A h_2^{k-1})^2(h_2^2+1)||\underset{\sim}{u}||^2_{H^k(\Omega_s)}$$

$$+ C\left(1+\tfrac{1}{\omega^2}\right)||p-p_h||^2_{L^2(\Omega_f)} + C(\omega^2+1)||\underset{\sim}{u} - \underset{\sim}{u_h}||^2_{L^2(\Omega_s)}.$$

Applying Lemma 3.8, for sufficiently small h_1 and h_2,

$$\frac{1}{2\omega^2\rho_f}||p-p_h||^2_{H^1(\Omega_f)} + \mu||\underset{\sim}{u} - \underset{\sim}{u_h}||^2_{H^1(\Omega_s)} \leq C(1+\tfrac{1}{\omega^2})(C_A h_1^{m-1})^2(h_1^2+1)||p||^2_{H^m(\Omega_f)}$$

$$+ C(\omega^2+1)(C_A h_2^{k-1})^2(h_2^2+1)||\underset{\sim}{u}||^2_{H^k(\Omega_s)}$$

$$+ C(h_1+h_2)\left\{||p-p_h||^2_{H^1(\Omega_f)} + \omega^2||\underset{\sim}{u} - \underset{\sim}{u_h}||^2_{H^1(\Omega_s)}\right\}$$

or equivalently,

$$||p-p_h||^2_{H^1(\Omega_f)} + \omega^2||\underset{\sim}{u} - \underset{\sim}{u_h}||^2_{H^1(\Omega_s)}$$

$$\leq C\omega^2\left\{(h_1^{m-1})^2(h_1^2+1)||p||^2_{H^m(\Omega_f)} + (h_2^{k-1})^2(h_2^2+1)\omega^2||\underset{\sim}{u}||^2_{H^k(\Omega_s)}\right\}$$

$$+ C(h_1+h_2)\omega^2\left\{||p-p_h||^2_{H^1(\Omega_f)} + \omega^2||\underset{\sim}{u} - \underset{\sim}{u_h}||^2_{H^1(\Omega_s)}\right\}$$

for some C independent of ω, h_1 and h_2. Applying the regularity estimate and simplifying,

$$\left(1 - C(h_1 + h_2)\omega^2\right) \left\{ ||p - p_h||^2_{H^1(\Omega_f)} + \omega^2 ||\underset{\sim}{u} - \underset{\sim}{u}_h||^2_{H^1(\Omega_s)} \right\}$$
$$\leq C\omega^2 \max\left\{ (h_1^{m-1})^2 (h_1^2 + 1), (h_2^{k-1})^2 (h_2^2 + 1) \right\}$$
$$\cdot \left\{ R_{f,m} ||g_f||^2_{H^{m-2}(\Omega_f)} + R_{s,m} ||\underset{\sim}{g}_s||^2_{H^{k-2}(\Omega_s)} \right\}.$$

Choosing $h_1 + h_2 < C\frac{1}{\omega^2}$ implies

$$||p - p_h||^2_{H^1(\Omega_f)} + \omega^2 ||\underset{\sim}{u} - \underset{\sim}{u}_h||^2_{H^1(\Omega_s)}$$
$$\leq C\omega^2 \max\left\{ (h_1^{m-1})^2 (h_1^2 + 1), (h_2^{k-1})^2 (h_2^2 + 1) \right\}$$
$$\cdot \left\{ R_{f,m} ||g_f||^2_{H^{m-2}(\Omega_f)} + R_{s,m} ||\underset{\sim}{g}_s||^2_{H^{k-2}(\Omega_s)} \right\}.$$

Combining this inequality with Lemma 3.8,

$$||p - p_h||^2_{L^2(\Omega_f)} + \omega^2 ||\underset{\sim}{u} - \underset{\sim}{u}_h||^2_{L^2(\Omega_s)}$$
$$\leq C(h_1 + h_2)\omega^2 \max\left\{ (h_1^{m-1})^2 (h_1^2 + 1), (h_2^{k-1})^2 (h_2^2 + 1) \right\}$$
$$\cdot \left\{ R_{f,m} ||g_f||^2_{H^{m-2}(\Omega_f)} + R_{s,m} ||\underset{\sim}{g}_s||^2_{H^{k-2}(\Omega_s)} \right\}.$$

■

Remark : The global error depends on the maximum of the quantities $(h_1^{m-1})^2(h_1^2+1)$ and $(h_2^{k-1})^2(h_2^2+1)$. Clearly, one should therefore choose $h_1 \approx h_2$ and $k = m$ in order to make the most efficient use of computational resources.

Chapter 4

Domain Decomposition Methods for the Fluid–Solid Interaction Problem

In the following chapter, we develop some parallelizable non–overlapping domain decomposition iterative methods for solving the system of coupled acoustic and elastic Helmholtz equations which we introduced in Chapter Three. We propose two classes of iterative methods for decoupling the whole domain problem into individual fluid and solid subdomain problems. The key of each method is to replace the physical interface condition with equivalent relaxation conditions which act as the transmission conditions. We establish the utility of these methods by showing their strong convergence in the energy norm of the underlying fluid–solid interaction problem. Numerical experiments validate the analysis and show the effectiveness of the methods.

Because of existence of the physical interface, it is natural to use non–overlapping domain decompositions method to solve the fluid–solid interaction problem. In fact, non–overlapping domain decomposition methods have been successfully used to solve several coupled boundary value problems from scientific applications. See [32] and the references therein. The non–overlapping domain decomposition methods developed in this chapter are based on the idea of using convex combinations of the original physical interface conditions to transmit information between subdomains. See [2, 9, 14, 26, 36] for expositions and discussions on this approach for homogeneous problems. It is more delicate to apply the idea to the heterogeneous fluid–solid interaction problem because using straightforward combinations of the original interface conditions as transmission conditions may lead to divergent iterative procedures.

We also address an implementation issue (cf. [17]). We show that the difficulty of explicitly computing fluxes on the interface can be avoided through a simple modification.

The results of this chapter were reported in [6] with few details. Here, we present the results in greater detail; in particular, we include all proofs.

4.1 Non–overlapping methods

As in the comparable methods of [2, 9, 14], the main idea is to replace the original physical interface conditions with equivalent Robin type interface conditions. It is easy to check that the interface conditions in 3.3 are equivalent to

$$\frac{\partial p}{\partial \underset{\sim}{n}_f} + \alpha p = -\omega^2 \rho_f \underset{\sim}{u} \cdot \underset{\sim}{n}_s - \alpha \underset{\approx}{\sigma}(\underset{\sim}{u}) \underset{\sim}{n}_s \cdot \underset{\sim}{n}_s, \quad \text{on } \Gamma, \tag{4.1}$$

$$\beta \underset{\approx}{\sigma}(\underset{\sim}{u}) \underset{\sim}{n}_s \cdot \underset{\sim}{n}_s + \omega^2 \rho_f \underset{\sim}{u} \cdot \underset{\sim}{n}_s = -\beta p - \frac{\partial p}{\partial \underset{\sim}{n}_f}, \quad \text{on } \Gamma, \tag{4.2}$$

$$\underset{\approx}{\sigma}(\underset{\sim}{u}) \underset{\sim}{n}_s \cdot \tau_s = 0, \quad \text{on } \Gamma. \tag{4.3}$$

for any $\alpha, \beta \in C$ such that $\alpha \neq \beta$.

Based on the new form of the interface conditions, we propose the following iterative algorithms. Algorithm 1 resembles a block Jacobi type algorithm, and Algorithm 2 resembles a block Gauss-Seidel type algorithm.

Algorithm 1

Step 1 $\forall\, p^0 \in H^1(\Omega_f)$, $\underset{\sim}{u}^0 \in (H^1(\Omega_s))^N$.

Step 2 For $k \geq 0$, define $(p^{k+1}, \underset{\sim}{u}^{k+1})$ such that

$$
\begin{aligned}
-\tfrac{\omega^2}{c^2} p^{k+1} - \Delta p^{k+1} &= g_f, && \text{in } \Omega_f,\\
\tfrac{i\omega}{c} p^{k+1} + \tfrac{\partial p^{k+1}}{\partial n_f} &= 0, && \text{on } \Gamma_f,\\
\tfrac{\partial p^{k+1}}{\partial n_f} + \alpha p^{k+1} &= -\omega^2 \rho_f \underset{\sim}{u}^k \cdot n_s - \alpha \underset{\approx}{\sigma}(\underset{\sim}{u}^k) n_s \cdot n_s, && \text{on } \Gamma;\\
-\omega^2 \rho_s \underset{\sim}{u}^{k+1} - \operatorname{div}(\underset{\approx}{\sigma}(\underset{\sim}{u}^{k+1})) &= \underset{\sim}{g}_s, && \text{in } \Omega_s,\\
i\omega \underset{\approx}{A} \underset{\sim}{u}^{k+1} + \underset{\approx}{\sigma}(\underset{\sim}{u}^{k+1}) n_s &= 0, && \text{on } \Gamma_s,\\
\beta \underset{\approx}{\sigma}(\underset{\sim}{u}^{k+1}) n_s \cdot n_s + \omega^2 \rho_f \underset{\sim}{u}^{k+1} \cdot n_s &= -\beta p^k - \tfrac{\partial p^k}{\partial n_f}, && \text{on } \Gamma,\\
\underset{\approx}{\sigma}(\underset{\sim}{u}^{k+1}) n_s \cdot \tau_s &= 0, && \text{on } \Gamma.
\end{aligned}
$$

Algorithm 2

Step 1 $\forall\, \underset{\sim}{u}^0 \in (H^1(\Omega_s))^N$.

Step 2 For $k \geq 0$, define $(p^k, \underset{\sim}{u}^{k+1})$ such that

$$
\begin{aligned}
-\tfrac{\omega^2}{c^2} p^{k} - \Delta p^{k} &= g_f, && \text{in } \Omega_f,\\
\tfrac{i\omega}{c} p^{k} + \tfrac{\partial p^{k}}{\partial n_f} &= 0, && \text{on } \Gamma_f,\\
\tfrac{\partial p^{k}}{\partial n_f} + \alpha p^{k} &= -\omega^2 \rho_f \underset{\sim}{u}^k \cdot \underset{\sim}{n}_s - \alpha \underset{\approx}{\sigma}(\underset{\sim}{u}^k) \underset{\sim}{n}_s \cdot \underset{\sim}{n}_s, && \text{on } \Gamma;\\
-\omega^2 \rho_s \underset{\sim}{u}^{k+1} - \operatorname{div}(\underset{\approx}{\sigma}(\underset{\sim}{u}^{k+1})) &= \underset{\sim}{g}_s, && \text{in } \Omega_s,\\
i\omega \underset{\approx}{A} \underset{\sim}{u}^{k+1} + \underset{\approx}{\sigma}(\underset{\sim}{u}^{k+1}) \underset{\sim}{n}_s &= 0, && \text{on } \Gamma_s,\\
\beta \underset{\approx}{\sigma}(\underset{\sim}{u}^{k+1}) \underset{\sim}{n}_s \cdot \underset{\sim}{n}_s + \omega^2 \rho_f \underset{\sim}{u}^{k+1} \cdot \underset{\sim}{n}_s &= -\beta p^k - \tfrac{\partial p^k}{\partial n_f}, && \text{on } \Gamma,\\
\underset{\approx}{\sigma}(\underset{\sim}{u}^{k+1}) \underset{\sim}{n}_s \cdot \tau_s &= 0, && \text{on } \Gamma.
\end{aligned}
$$

Notice that in both algorithms, no information is transmitted tangentially along the interface. Also, well-posedness of the algorithms in Step 2 comes directly from the well-posedness results in Chapters One and Two.

4.2 Convergence Analysis

In this section, we will use an "energy norm" technique to show convergence of the algorithms. Since the proofs are similar, we only present the proof for Algorithm 1.

We will need to use the following two technical lemmas.

Lemma 4.1. *Let $Q \subset \Re^N$ be open and bounded and $\partial Q = \Gamma_1 \cup \Gamma_2$ where Γ_1 is not empty. Let $\underset{\approx}{M} \in \Re^n$ be a symmetric, positive definite matrix, and $a, b_1, b_2 \in \Re$, where $b_1 \neq 0$. Define*

$$\underset{\sim}{S} := \left\{ \underset{\sim}{v} \in \underset{\sim}{H}^1(Q) | -a^2 \underset{\sim}{v} - \underset{\sim}{\operatorname{div}} \left(\underset{\approx}{\sigma}(\underset{\sim}{v}) \right) = 0 \text{ in } Q,\ \underset{\approx}{\sigma}(\underset{\sim}{v}) \underset{\sim}{n} \cdot \underset{\sim}{\tau}_j = 0 \text{ on } \Gamma_2 \right\}$$

where $\underset{\sim}{n}$ is the outward unit normal to ∂Q and $\underset{\sim}{\tau}_j$ $(j = 1, ..., N-1)$ are the unit tangent vectors to ∂Q. Then there exists a positive constant C which depends only on $\underset{\approx}{M}$, Q, a, b and c such that

$$\|\underset{\sim}{v}\|_{H^1(Q)} \leq C \left\{ \|\underset{\approx}{\sigma}(\underset{\sim}{v})\underset{\sim}{n} + ib_1 \underset{\approx}{M} \underset{\sim}{v}\|_{H^{-1/2}(\Gamma_1)} + \|\underset{\approx}{\sigma}(\underset{\sim}{v})\underset{\sim}{n} \cdot \underset{\sim}{n} + ib_2 \underset{\sim}{v} \cdot \underset{\sim}{n}\|_{H^{-1/2}(\Gamma_2)} \right\}$$

for all $\underset{\sim}{v} \in \underset{\sim}{S}$.

Proof : Given $(\underset{\sim}{w}_1, w_2) \in (H^{1/2}(\Gamma_1))^N \times H^{1/2}(\Gamma_2)$, define the function $\underset{\sim}{w}$ as follows:

$$\underset{\sim}{w} = \begin{cases} \underset{\sim}{w}_1 & \text{on } \Gamma_1, \\ w_2 \underset{\sim}{n} & \text{on } \Gamma_2, \end{cases}$$

and define the set

$$V := \left\{ (\underset{\sim}{w}_1, w_2) \in (H^{1/2}(\Gamma_1))^N \times H^{1/2}(\Gamma_2) | \quad \underset{\sim}{w} \in (H^{1/2}(\partial Q))^N \right\}.$$

Note that $\underset{\sim}{S}$ is a closed subset of $H^1(Q)$, and define the operator $T : \underset{\sim}{S} \longrightarrow V^*$ (where V^* is the dual space of V) by

$$(T\underset{\sim}{v})(\underset{\sim}{w}_1, w_2) = \int_Q (\underset{\approx}{\sigma}(\underset{\sim}{v}) : \overline{(\underset{\approx}{\nabla} \underset{\sim}{w})} - a^2 \underset{\sim}{v} \cdot \overline{\underset{\sim}{w}}) + ib_1 \underset{\approx}{M} \int_{\Gamma_1} \underset{\sim}{v} \cdot \overline{\underset{\sim}{w_1}} + ib_2 \int_{\Gamma_2} (\underset{\sim}{v} \cdot \underset{\sim}{n}) \overline{w_2}$$

for each $\underset{\sim}{w} \in V$. Here, we abuse the notation by using $\underset{\sim}{w}$ to also denote its extension in $(H^1(Q))^N$. T is linear and continuous, and by Lemmas 2.2 and 2.3,

$$
\begin{aligned}
(T\underset{\sim}{v})(\underset{\sim}{w}) = \int_Q \operatorname{div}(\underset{\approx}{\sigma}(\underset{\sim}{v})\overline{\underset{\sim}{w}}) + \int_Q \left(-a^2\underset{\sim}{v} - \underset{\sim}{\operatorname{div}}\left(\underset{\approx}{\sigma}(\underset{\sim}{v})\right)\right) \cdot \overline{\underset{\sim}{w}} \\
+ ib_1\underset{\approx}{M}\int_{\Gamma_1} \underset{\sim}{v} \cdot \overline{\underset{\sim}{w_1}} + ib_2\int_{\Gamma_2} (\underset{\sim}{v} \cdot \underset{\sim}{n})\overline{w_2}.
\end{aligned}
$$

Since $\underset{\sim}{v} \in \underset{\sim}{S}$,

$$
\begin{aligned}
(T\underset{\sim}{v})(\underset{\sim}{w_1}, w_2) &= \int_Q \operatorname{div}(\underset{\approx}{\sigma}(\underset{\sim}{v})\overline{\underset{\sim}{w}}) + ib_1\underset{\approx}{M}\int_{\Gamma_1} \underset{\sim}{v} \cdot \overline{\underset{\sim}{w_1}} + ib_2\int_{\Gamma_2} (\underset{\sim}{v} \cdot \underset{\sim}{n})\overline{w_2} \\
&= \int_{\Gamma_1} (\underset{\approx}{\sigma}(\underset{\sim}{v})\underset{\sim}{n}) \cdot \overline{\underset{\sim}{w_1}} + ib_1\underset{\approx}{M}\int_{\Gamma_1} \underset{\sim}{v} \cdot \overline{\underset{\sim}{w_1}} + \int_{\Gamma_2} (\underset{\approx}{\sigma}(\underset{\sim}{v})\underset{\sim}{n}) \cdot \overline{(w_2\underset{\sim}{n})} \\
&\quad + ib_2\int_{\Gamma_2} (\underset{\sim}{v} \cdot \underset{\sim}{n})\overline{w_2} \\
&= \int_{\Gamma_1} (\underset{\approx}{\sigma}(\underset{\sim}{v})\underset{\sim}{n} + ib_1\underset{\approx}{M}\underset{\sim}{v}) \cdot \overline{\underset{\sim}{w_1}} + \int_{\Gamma_2} (\underset{\approx}{\sigma}(\underset{\sim}{v})\underset{\sim}{n}) \cdot \overline{(w_2\underset{\sim}{n})} + ib_2\int_{\Gamma_2} (\underset{\sim}{v} \cdot \underset{\sim}{n})\overline{w_2}.
\end{aligned}
$$

Splitting $\underset{\approx}{\sigma}(\underset{\sim}{v})\underset{\sim}{n}$ into its normal and tangential components,

$$
\begin{aligned}
(T\underset{\sim}{v})(\underset{\sim}{w_1}, w_2) &= \int_{\Gamma_1} (\underset{\approx}{\sigma}(\underset{\sim}{v})\underset{\sim}{n} + ib_1\underset{\approx}{M}\underset{\sim}{v}) \cdot \overline{\underset{\sim}{w_1}} \\
&\quad + \int_{\Gamma_2} \left\{ (\underset{\approx}{\sigma}(\underset{\sim}{v})\underset{\sim}{n} \cdot \underset{\sim}{n})\underset{\sim}{n} + \sum_{j=1}^{N-1} (\underset{\approx}{\sigma}(\underset{\sim}{v})\underset{\sim}{n} \cdot \underset{\sim}{\tau_j})\underset{\sim}{\tau_j} \right\} \cdot \overline{(w_2\underset{\sim}{n})} + ib_2\int_{\Gamma_2} (\underset{\sim}{v} \cdot \underset{\sim}{n})\overline{w_2}.
\end{aligned}
$$

Since $\underset{\sim}{v} \in \underset{\sim}{S}$, $(\underset{\approx}{\sigma}(\underset{\sim}{v})\underset{\sim}{n} \cdot \underset{\sim}{\tau_j}) = 0$ on Γ_2 for $j = 1, \ldots, N-1$, and the above becomes

$$
\begin{aligned}
(T\underset{\sim}{v})(\underset{\sim}{w_1}, w_2) &= \int_{\Gamma_1} (\underset{\approx}{\sigma}(\underset{\sim}{v})\underset{\sim}{n} + ib_1\underset{\approx}{M}\underset{\sim}{v}) \cdot \overline{\underset{\sim}{w_1}} + \int_{\Gamma_2} \left\{ (\underset{\approx}{\sigma}(\underset{\sim}{v})\underset{\sim}{n} \cdot \underset{\sim}{n})\underset{\sim}{n} \right\} \cdot \overline{(w_2\underset{\sim}{n})} \\
&\quad + ib_2\int_{\Gamma_2} (\underset{\sim}{v} \cdot \underset{\sim}{n})\overline{w_2} \\
&= \int_{\Gamma_1} (\underset{\approx}{\sigma}(\underset{\sim}{v})\underset{\sim}{n} + ib_1\underset{\approx}{M}\underset{\sim}{v}) \cdot \overline{\underset{\sim}{w_1}} + \int_{\Gamma_2} \left\{ \underset{\approx}{\sigma}(\underset{\sim}{v})\underset{\sim}{n} \cdot \underset{\sim}{n} + ib_2(\underset{\sim}{v} \cdot \underset{\sim}{n}) \right\} \overline{w_2},
\end{aligned}
$$

that is,

$$
T\underset{\sim}{v} = \begin{cases} \underset{\approx}{\sigma}(\underset{\sim}{v})\underset{\sim}{n} + ib_1\underset{\approx}{M}\underset{\sim}{v} & \text{on } \Gamma_1, \\ \underset{\approx}{\sigma}(\underset{\sim}{v})\underset{\sim}{n} \cdot \underset{\sim}{n} + ib_2(\underset{\sim}{v} \cdot \underset{\sim}{n}) & \text{on } \Gamma_2. \end{cases}
$$

Consider the problem

$$(4.4)\qquad \begin{cases} -a^2\rho_s \underset{\sim}{v} - \underset{\sim}{\operatorname{div}}\left(\underset{\approx}{\sigma}(\underset{\sim}{v})\right) & = 0 \quad \text{in } Q, \\ \underset{\approx}{\sigma}(\underset{\sim}{v})\underset{\sim}{n} + ib_1 \underset{\approx}{M}\underset{\sim}{v} & = \underset{\sim}{g} \quad \text{on } \Gamma_1, \\ \underset{\approx}{\sigma}(\underset{\sim}{v})\underset{\sim}{n} + ib_2(\underset{\sim}{v}\cdot\underset{\sim}{n})\underset{\sim}{n} & = f\underset{\sim}{n} \quad \text{on } \Gamma_2. \end{cases}$$

It is easy to show that the corresponding variational formulation satisfies a Gårding inequality, and the homogeneous adjoint problem has a unique trivial solution. The Fredholm Alternative therefore implies that (4.4) admits exactly one solution $\underset{\sim}{v} \in (H^1(\Omega_s))^N$ for each $(\underset{\sim}{g}, f) \in V^*$ (see chapter 2). Hence, T is one–to–one and onto V^*. By the Open Mapping Theorem (see [22]), T^{-1} exists and is bounded. Hence for $\underset{\sim}{v} \in \underset{\sim}{S}$,

$$\begin{aligned} \|\underset{\sim}{v}\|_{H^1(Q)} &= \|T^{-1}\left(\underset{\approx}{\sigma}(\underset{\sim}{v})\underset{\sim}{n} + ib_1\underset{\approx}{M}\underset{\sim}{v},\ \underset{\approx}{\sigma}(\underset{\sim}{v})\underset{\sim}{n}\cdot\underset{\sim}{n} + ib_2(\underset{\sim}{v}\cdot\underset{\sim}{n})\right)\|_{H^1(Q)} \\ &\leq C\|\left(\underset{\approx}{\sigma}(\underset{\sim}{v})\underset{\sim}{n} + ib_1\underset{\approx}{M}\underset{\sim}{v},\ \underset{\approx}{\sigma}(\underset{\sim}{v})\underset{\sim}{n}\cdot\underset{\sim}{n} + ib_2(\underset{\sim}{v}\cdot\underset{\sim}{n})\right)\|_{V^*} \\ &= C\left\{\|\underset{\approx}{\sigma}(\underset{\sim}{v})\underset{\sim}{n} + ib_1\underset{\approx}{M}\underset{\sim}{v}\|_{H^{-1/2}(\Gamma_1)} + \|\underset{\approx}{\sigma}(\underset{\sim}{v})\underset{\sim}{n}\cdot\underset{\sim}{n} + ib_2(\underset{\sim}{v}\cdot\underset{\sim}{n})\|_{H^{-1/2}(\Gamma_2)}\right\} \end{aligned}$$

for all $\underset{\sim}{v} \in \underset{\sim}{S}$. The proof is complete. ∎

Lemma 4.2. *Let $Q \subset \Re^N$ be open and bounded, $a, b \in \Re$, $b \neq 0$ and define*

$$S := \left\{q \in H^1(Q) | -a^2 q - \Delta q = 0 \ in\ Q\right\}.$$

Then there exists a positive constant C which depends only on Q, a and b such that

$$\|q\|_{H^1(Q)} \leq C\|\tfrac{\partial q}{\partial n} + ibq\|_{H^{-1/2}(\partial Q)} \qquad \forall\ q \in S.$$

Proof : The proof is similar to the proof of Lemma 4.1, and can be found in [9]. ∎

Define the error functions at the kth iteration,

$$r^k = p - p^k, \qquad \underset{\sim}{e}^k = \underset{\sim}{u} - \underset{\sim}{u}^k$$

It is easy to check that r^k and $\underset{\sim}{e}^k$ satisfy the equations

$$-\frac{\omega^2}{c^2}r^{k+1} - \Delta r^{k+1} = 0, \quad \text{in } \Omega_f, \tag{4.5}$$

$$\frac{i\omega}{c}r^{k+1} + \frac{\partial r^{k+1}}{\partial n_f} = 0, \qquad \text{on } \Gamma_f, \tag{4.6}$$

$$\frac{\partial r^{k+1}}{\partial n_f} + \alpha r^{k+1} = -\omega^2 \rho_f \underset{\sim}{e}^k \cdot \underset{\sim}{n}_s - \alpha \underset{\approx}{\sigma}(\underset{\sim}{e}^k)\underset{\sim}{n}_s \cdot \underset{\sim}{n}_s, \qquad \text{on } \Gamma; \tag{4.7}$$

$$-\omega^2 \rho_s \underset{\sim}{e}^{k+1} - \operatorname{div}(\underset{\approx}{\sigma}(\underset{\sim}{e}^{k+1})) = \underset{\sim}{g}_s, \qquad \text{in } \Omega_s, \tag{4.8}$$

$$i\omega \underset{\approx}{A}\underset{\sim}{e}^{k+1} + \underset{\approx}{\sigma}(\underset{\sim}{e}^{k+1})\underset{\sim}{n}_s = 0, \qquad \text{on } \Gamma_s, \tag{4.9}$$

$$\beta \underset{\approx}{\sigma}(\underset{\sim}{e}^{k+1})\underset{\sim}{n}_s \cdot \underset{\sim}{n}_s + \omega^2 \rho_f \underset{\sim}{e}^{k+1} \cdot \underset{\sim}{n}_s = -\beta r^k - \frac{\partial r^k}{\partial n_f}, \qquad \text{on } \Gamma, \tag{4.10}$$

$$\underset{\approx}{\sigma}(\underset{\sim}{e}^{k+1})\underset{\sim}{n}_s \cdot \tau_s = 0, \qquad \text{on } \Gamma. \tag{4.11}$$

Define the "pseudo–energy"

$$E_k = \|\frac{\partial r^k}{\partial n_f} + \alpha r^k\|^2_{L^2(\Gamma)} + \|\beta \underset{\approx}{\sigma}(\underset{\sim}{e}^k)\underset{\sim}{n}_s \cdot \underset{\sim}{n}_s + \omega^2 \rho_f \underset{\sim}{e}^k \cdot \underset{\sim}{n}_s\|^2_{L^2(\Gamma)}. \tag{4.12}$$

Then we have

Lemma 4.3. *The pseudo–energy satisfies*

$$E_{k+1} = E_k - R_k,$$

where

$$R_k = (|\alpha|^2 - |\beta|^2)\left\{\|r^k\|^2_{L^2(\Gamma)} - \|\underset{\approx}{\sigma}(\underset{\sim}{e}^k)\underset{\sim}{n}_s \cdot \underset{\sim}{n}_s\|^2_{L^2(\Gamma)}\right\}$$
$$- \left(2\operatorname{Re}\int_\Gamma \overline{(\alpha-\beta)\underset{\approx}{\sigma}(\underset{\sim}{e}^k)\underset{\sim}{n}_s \cdot \underset{\sim}{n}_s}\omega^2 \rho_f \underset{\sim}{e}^k \cdot \underset{\sim}{n}_s + 2\operatorname{Re}\int_\Gamma \overline{(\alpha-\beta)r^k}\frac{\partial r^k}{\partial n_f}\right).$$

Proof : From the transmission conditions in Algorithm 1,

$$E_{k+1} = \|\frac{\partial r^{k+1}}{\partial n_f} + \alpha r^{k+1}\|^2_{L^2(\Gamma)} + \|\beta \underset{\approx}{\sigma}(\underset{\sim}{e}^{k+1})\underset{\sim}{n}_s \cdot \underset{\sim}{n}_s + \omega^2 \rho_f \underset{\sim}{e}^{k+1} \cdot \underset{\sim}{n}_s\|^2_{L^2(\Gamma)}$$
$$= \| - \omega^2 \rho_f \underset{\sim}{e}^k \cdot \underset{\sim}{n}_s - \alpha \underset{\approx}{\sigma}(\underset{\sim}{e}^k)\underset{\sim}{n}_s \cdot \underset{\sim}{n}_s\|^2_{L^2(\Gamma)} + \| - \beta r^k - \frac{\partial r^k}{\partial n_f}\|^2_{L^2(\Gamma)},$$

or equivalently,

$$E_{k+1} = \|\beta r^k + \frac{\partial r^k}{\partial n_f}\|^2_{L^2(\Gamma)} + \|\omega^2 \rho_f \underset{\sim}{e}^k \cdot \underset{\sim}{n}_s + \alpha \underset{\approx}{\sigma}(\underset{\sim}{e}^k)\underset{\sim}{n}_s \cdot \underset{\sim}{n}_s\|^2_{L^2(\Gamma)}. \tag{4.13}$$

Calculating the two terms in (4.13) separately,

$$\begin{aligned}\|\beta r^k + \tfrac{\partial r^k}{\partial n_f}\|^2_{L^2(\Gamma)} &= \|\tfrac{\partial r^k}{\partial n_f} + \alpha r^k + (\beta-\alpha) r^k\|^2_{L^2(\Gamma)} \\ &= \|\tfrac{\partial r^k}{\partial n_f} + \alpha r^k\|^2_{L^2(\Gamma)} + \|(\beta-\alpha) r^k\|^2_{L^2(\Gamma)} \\ &\quad + 2\,\mathrm{Re}\int_\Gamma (\tfrac{\partial r^k}{\partial n_f} + \alpha r^k)\overline{(\beta-\alpha) r^k}.\end{aligned}$$

Note that

$$\begin{aligned}2\,\mathrm{Re}\int_\Gamma (\tfrac{\partial r^k}{\partial n_f} + \alpha r^k)\overline{(\beta-\alpha) r^k} &= 2\,\mathrm{Re}\int_\Gamma \overline{(\beta-\alpha)}\tfrac{\partial r^k}{\partial n_f}\overline{r^k} \\ &\quad + 2\,\mathrm{Re}(\alpha\overline{\beta})\|r^k\|^2_{L^2(\Gamma)} - 2\|\alpha r^k\|^2_{L^2(\Gamma)},\end{aligned}$$

therefore,

$$\begin{aligned}\|\beta r^k + \tfrac{\partial r^k}{\partial n_f}\|^2_{L^2(\Gamma)} &= \|\tfrac{\partial r^k}{\partial n_f} + \alpha r^k\|^2_{L^2(\Gamma)} + \{|\beta-\alpha|^2 - 2|\alpha|^2 + 2\,\mathrm{Re}(\alpha\overline{\beta}\}\|r^k\|^2_{L^2(\Gamma)} \\ &\quad + 2\,\mathrm{Re}\int_\Gamma \overline{(\beta-\alpha)}\tfrac{\partial r^k}{\partial n_f}\overline{r^k}\end{aligned}$$

or equivalently,

$$\begin{aligned}\|\beta r^k + \tfrac{\partial r^k}{\partial n_f}\|^2_{L^2(\Gamma)} &= \|\tfrac{\partial r^k}{\partial n_f} + \alpha r^k\|^2_{L^2(\Gamma)} + \left(|\beta|^2 - |\alpha|^2\right)\|r^k\|^2_{L^2(\Gamma)} \\ &\quad + 2\,\mathrm{Re}\int_\Gamma \overline{(\beta-\alpha)}\tfrac{\partial r^k}{\partial n_f}\overline{r^k}.\end{aligned} \tag{4.14}$$

Next, simplifying the second term in (4.13),

$$\begin{aligned}&\|\omega^2\rho_f \underset{\sim}{e}^k\cdot \underset{\sim}{n}_s + \alpha\underset{\approx}{\sigma}(\underset{\sim}{e}^k)\underset{\sim}{n}_s\cdot \underset{\sim}{n}_s\|^2_{L^2(\Gamma)} \\ &\quad= \|\beta\underset{\approx}{\sigma}(\underset{\sim}{e}^k)\underset{\sim}{n}_s\cdot \underset{\sim}{n}_s + \omega^2\rho_f \underset{\sim}{e}^k\cdot \underset{\sim}{n}_s + (\alpha-\beta)\underset{\approx}{\sigma}(\underset{\sim}{e}^k)\underset{\sim}{n}_s\cdot \underset{\sim}{n}_s\|^2_{L^2(\Gamma)} \\ &\quad= \|\beta\underset{\approx}{\sigma}(\underset{\sim}{e}^k)\underset{\sim}{n}_s\cdot \underset{\sim}{n}_s + \omega^2\rho_f \underset{\sim}{e}^k\cdot \underset{\sim}{n}_s\|^2_{L^2(\Gamma)} + \|(\alpha-\beta)\underset{\approx}{\sigma}(\underset{\sim}{e}^k)\underset{\sim}{n}_s\cdot \underset{\sim}{n}_s\|^2_{L^2(\Gamma)} \\ &\qquad + 2\,\mathrm{Re}\int_\Gamma (\beta\underset{\approx}{\sigma}(\underset{\sim}{e}^k)\underset{\sim}{n}_s\cdot \underset{\sim}{n}_s + \omega^2\rho_f \underset{\sim}{e}^k\cdot \underset{\sim}{n}_s)\overline{(\alpha-\beta)\underset{\approx}{\sigma}(\underset{\sim}{e}^k)\underset{\sim}{n}_s\cdot \underset{\sim}{n}_s}.\end{aligned}$$

Note that

$$\begin{aligned}&2\,\mathrm{Re}\int_\Gamma (\beta\underset{\approx}{\sigma}(\underset{\sim}{e}^k)\underset{\sim}{n}_s\cdot \underset{\sim}{n}_s + \omega^2\rho_f \underset{\sim}{e}^k\cdot \underset{\sim}{n}_s)\overline{(\alpha-\beta)\underset{\approx}{\sigma}(\underset{\sim}{e}^k)\underset{\sim}{n}_s\cdot \underset{\sim}{n}_s} \\ &\quad= 2\,\mathrm{Re}\int_\Gamma \overline{(\alpha-\beta)\underset{\approx}{\sigma}(\underset{\sim}{e}^k)\underset{\sim}{n}_s\cdot \underset{\sim}{n}_s}\omega^2\rho_f \underset{\sim}{e}^k\cdot \underset{\sim}{n}_s + 2\,\mathrm{Re}(\beta\overline{\alpha})\|\underset{\approx}{\sigma}(\underset{\sim}{e}^k)\underset{\sim}{n}_s\cdot \underset{\sim}{n}_s\|^2_{L^2(\Gamma)} \\ &\qquad + 2|\beta|^2\|\underset{\approx}{\sigma}(\underset{\sim}{e}^k)\underset{\sim}{n}_s\cdot \underset{\sim}{n}_s\|^2_{L^2(\Gamma)},\end{aligned}$$

therefore,

$$
\begin{aligned}
&\|\omega^2\rho_f \underset{\sim}{e}^k\cdot \underset{\sim}{n}_s + \alpha\underset{\approx}{\sigma}(\underset{\sim}{e}^k)\underset{\sim}{n}_s\cdot \underset{\sim}{n}_s\|^2_{L^2(\Gamma)}\\
&= \|\beta\underset{\approx}{\sigma}(\underset{\sim}{e}^k)\underset{\sim}{n}_s\cdot\underset{\sim}{n}_s + \omega^2\rho_f\underset{\sim}{e}^k\cdot\underset{\sim}{n}_s\|^2_{L^2(\Gamma)} + (|\alpha|^2-|\beta|^2)\|\underset{\approx}{\sigma}(\underset{\sim}{e}^k)\underset{\sim}{n}_s\cdot\underset{\sim}{n}_s\|^2_{L^2(\Gamma)}\\
&\quad + 2\,\mathrm{Re}\int_\Gamma \overline{(\alpha-\beta)\underset{\approx}{\sigma}(\underset{\sim}{e}^k)\underset{\sim}{n}_s\cdot\underset{\sim}{n}_s}(\omega^2\rho_f\underset{\sim}{e}^k\cdot\underset{\sim}{n}_s).
\end{aligned}
\tag{4.15}
$$

Plugging (4.14) and (4.15) into (4.13) yields

$$
\begin{aligned}
E_{k+1} &= \|\tfrac{\partial r^k}{\partial n_f} + \alpha r^k\tfrac{\partial r^k}{\partial n_f}\|^2_{L^2(\Gamma)} + \|\beta\underset{\approx}{\sigma}(\underset{\sim}{e}^k)\underset{\sim}{n}_s\cdot\underset{\sim}{n}_s + \omega^2\rho_f\underset{\sim}{e}^k\cdot\underset{\sim}{n}_s\|^2_{L^2(\Gamma)}\\
&\quad + (|\beta|^2-|\alpha|^2)\left\{\|r^k\|^2_{L^2(\Gamma)} - \|\underset{\approx}{\sigma}(\underset{\sim}{e}^k)\underset{\sim}{n}_s\cdot\underset{\sim}{n}_s\|^2_{L^2(\Gamma)}\right\}\\
&\quad + 2\,\mathrm{Re}\int_\Gamma \overline{(\alpha-\beta)\underset{\approx}{\sigma}(\underset{\sim}{e}^k)\underset{\sim}{n}_s\cdot\underset{\sim}{n}_s}(\omega^2\rho_f\underset{\sim}{e}^k\cdot\underset{\sim}{n}_s) + 2\,\mathrm{Re}\int_\Gamma \overline{(\beta-\alpha)\tfrac{\partial r^k}{\partial n_f}}\,\overline{r^k}\\
&= E_k - R_k.
\end{aligned}
$$

■

Lemma 4.4. *R_k has the following expression*

$$
\begin{aligned}
R_k &= (|\alpha|^2-\beta|^2)\left\{\|r^k\|^2_{L^2(\Gamma)} - \|\underset{\approx}{\sigma}(\underset{\sim}{e}^k)\underset{\sim}{n}_s\cdot\underset{\sim}{n}_s\|^2_{L^2(\Gamma)}\right\}\\
&\quad + 2\,\mathrm{Re}\Big\{\overline{(\alpha-\beta)}\Big(\omega^4\rho_f\rho_s\|\underset{\sim}{e}^k\|^2_{L^2(\Omega_s)} - \lambda\omega^2\rho_f\|\,\mathrm{div}\,\underset{\sim}{e}^k\|^2_{L^2(\Omega_s)}\\
&\qquad\qquad - 2\mu\omega^2\rho_f\|\underset{\approx}{\epsilon}(\underset{\sim}{e}^k)\|^2_{L^2(\Omega_s)} - \tfrac{\omega^2}{c^2}\|r^k\|^2_{L^2(\Omega_s)} + \|\nabla r^k\|^2_{L^2(\Omega_s)}\Big)\\
&\qquad\qquad + i\left(\omega^3\int_{\Gamma_s}\underset{\approx}{A}\overline{\underset{\sim}{e}^k}\cdot\underset{\sim}{e}^k + \tfrac{\omega}{c}\|r^k\|^2_{L^2(\Omega_s)}\right)\Big\}.
\end{aligned}
$$

Proof : From Lemma 4.3,

$$
\begin{aligned}
R_k &= (|\alpha|^2-|\beta|^2)\left\{\|r^k\|^2_{L^2(\Gamma)} - \|\underset{\approx}{\sigma}(\underset{\sim}{e}^k)\underset{\sim}{n}_s\cdot\underset{\sim}{n}_s\|^2_{L^2(\Gamma)}\right\}\\
&\quad - \left(2\,\mathrm{Re}\int_\Gamma \overline{(\alpha-\beta)\underset{\approx}{\sigma}(\underset{\sim}{e}^k)\underset{\sim}{n}_s\cdot\underset{\sim}{n}_s}\,\omega^2\rho_f\underset{\sim}{e}^k\cdot\underset{\sim}{n}_s + 2\,\mathrm{Re}\int_\Gamma\overline{(\alpha-\beta)r^k\tfrac{\partial r^k}{\partial n_f}}\right).
\end{aligned}
$$

Testing equation (4.5) against $\overline{r^k}$ implies (by Lemma 1.2) that

$$
-\tfrac{\omega^2}{c^2}\|r^k\|^2_{L^2(\Omega_f)} + \|\nabla r^k\|^2_{L^2(\Omega_f)} = \int_{\Omega_f}\mathrm{div}\left((\nabla r^k)\overline{r^k}\right).
$$

Applying the divergence theorem and equations (4.6) and (4.7) implies

$$\int_{\Gamma} \frac{\partial r^k}{\partial n_f}\overline{r^k} = -\frac{\omega^2}{c^2}\|r^k\|^2_{L^2(\Omega_f)} + \|\nabla r^k\|^2_{L^2(\Omega_f)} + \frac{i\omega}{c}\|r^k\|^2_{L^2(\partial\Omega_f)}. \tag{4.16}$$

Testing the conjugate of equation (4.8) against $\underset{\sim}{e}^k$ implies (by Lemma 2.3) that

$$-\omega^2\rho_s\|\underset{\sim}{e}^k\|^2_{L^2(\Omega_s)} + \lambda\|\operatorname{div}\underset{\sim}{e}^k\|^2_{L^2(\Omega_s)} + 2\mu\|\underset{\approx}{\epsilon}(\underset{\sim}{e}^k)\|^2_{L^2(\Omega_s)} = \int_{\Omega_s} \operatorname{div}\left(\overline{\underset{\approx}{\sigma}(\underset{\sim}{e}^k)}\underset{\sim}{e}^k\right).$$

Applying the divergence theorem, and equations (4.9) and (4.10),

$$\begin{aligned}\int_{\Gamma}\left(\overline{\underset{\approx}{\sigma}(\underset{\sim}{e}^k)\underset{\sim}{n}_s}\right)\cdot\underset{\sim}{e}^k &= -\omega^2\rho_s\|\underset{\sim}{e}^k\|^2_{L^2(\Omega_s)} + \lambda\|\operatorname{div}\underset{\sim}{e}^k\|^2_{L^2(\Omega_s)} \\ &\quad + 2\mu\|\underset{\approx}{\epsilon}(\underset{\sim}{e}^k)\|^2_{L^2(\Omega_s)} + \int_{\Gamma_s}\operatorname{div}\left(\overline{\underset{\approx}{A}\underset{\sim}{e}^k}\right)\underset{\sim}{e}^k.\end{aligned} \tag{4.17}$$

Applying equations (4.16) and (4.17) to the expression for R_k and gathering terms yields the result. ■

Theorem 4.5. *Suppose $\alpha = i\xi$, $\beta = -i\xi$ where $\xi > 0$. Then the sequence $\{(p^k, \underset{\sim}{u}^k)\}$ generated by Algorithm 1 satisfies*

(i) $p^k \longrightarrow p$ strongly in $H^1(\Omega_f)$,

(ii) $\underset{\sim}{u}^k \longrightarrow \underset{\sim}{u}$ strongly in $(H^1(\Omega_s))^N$.

Proof : First, we will show that $p^k \longrightarrow p$ strongly in $H^1(\Omega_f)$. Since $\alpha = i\xi = -\beta$, Lemma 4.4 implies that

$$R_k = 4\xi\left(\omega^3\rho_f\int_{\Gamma_s}(\underset{\approx}{A}\overline{\underset{\sim}{e}^k})\cdot\underset{\sim}{e}^k + \frac{\omega}{c}\|r^k\|^2_{L^2(\Gamma_f)}\right).$$

Therefore $R_k \geq 0$ for all k since $\underset{\approx}{A}$ is positive definite. Lemma 4.3 implies that

$$E_{k+1} + \sum_{j=0}^{k} R_j = E_0 \tag{4.18}$$

for all k, hence $R_k \longrightarrow 0$ as $k \longrightarrow \infty$ and the sequence $\{E_K\}_{k=1}^{\infty}$ is bounded in $L^2(\Gamma)$. $R_k \longrightarrow 0$ implies that $r^k \longrightarrow 0$ strongly in $L^2(\Gamma_f)$. Equation (4.6) then implies that $\frac{\partial r^k}{\partial n_f} \longrightarrow 0$ in $L^2(\Gamma_f)$. E_k bounded implies that $\{\frac{\partial r^k}{\partial n_f} + i\xi r^k\}_{k=1}^{\infty}$ is bounded in $L^2(\Gamma)$. This together with the fact that $r^k \longrightarrow 0$ and $\frac{\partial r^k}{\partial n_f} \longrightarrow 0$ in $L^2(\Gamma_f)$ implies that $\{\frac{\partial r^k}{\partial n_f}+i\xi r^k\}_{k=1}^{\infty}$ is bounded in $L^2(\partial\Omega_f)$. By Lemma 4.2, $\{r^k\}_{k=1}^{\infty}$ is bounded in $H^1(\Omega_f)$. Hence, $\{r^k\}_{k=1}^{\infty}$ has a subsequence which converges weakly in $H^1(\Omega_f)$. Let $\{r^{a_k}\}_{k=1}^{\infty}$ denote the subsequence, and let r denote its weak limit. Since $H^1(\Omega_f) \overset{c}{\hookrightarrow} L^2(\Omega_f)$ (see [1], pg.30), $r^k \longrightarrow r$ strongly in $L^2(\Omega_f)$. By the trace inequality on $\partial\Omega_f$,

$$\|r^k - r\|^2_{L^2(\partial\Omega_f)} \leq C_f\|r^k - r\|_{L^2(\Omega_f)}\|r^k - r\|_{H^1(\Omega_f)}$$

which implies that $r^{a_k} \longrightarrow r$ strongly in $L^2(\partial\Omega_f)$ and hence $r = 0$ on Γ_f. The trace inequality also implies that $\{r^k\}_{k=1}^{\infty}$ is bounded in $L^2(\Gamma)$. Therefore, since $\{\frac{\partial r^k}{\partial n_f} + i\xi r^k\}_{k=1}^{\infty}$ is bounded in $L^2(\Gamma)$, $\{\frac{\partial r^{a_k}}{\partial n_f}\}_{k=1}^{\infty}$ is bounded in $L^2(\partial\Omega_f)$ and must have a subsequence which converges weakly in $L^2(\partial\Omega_f)$. For notational brevity, let $\{\frac{\partial r^{a_k}}{\partial n_f}\}_{k=1}^{\infty}$ denote the subsequence, and let $\chi \in L^2(\partial\Omega_f)$ denote its weak limit.

By equation (4.5), each r^{a_k} satisfies

$$\frac{-\omega^2}{c^2}\int_{\Omega_f} r^{a_k}\overline{q} + \int_{\Omega_f}\nabla r^{a_k}\cdot\overline{\nabla q} = \int_{\partial\Omega_f}\frac{\partial r^{a_k}}{\partial n_f}\overline{q} \qquad \forall q \in H^1(\Omega_f).$$

Taking the limit as $k \longrightarrow \infty$ implies

$$\frac{-\omega^2}{c^2}\int_{\Omega_f} r\overline{q} + \int_{\Omega_f}\nabla r\cdot\overline{\nabla q} = \int_{\partial\Omega_f}\chi\overline{q} \qquad \forall q \in H^1(\Omega_f).$$

Therefore, r satisfies

$$\frac{-\omega^2}{c^2}r - \Delta r = 0 \quad \text{in } \Omega_f,$$

$$\frac{-i\omega}{c}r + \frac{\partial r}{\partial n_f} = \chi \quad \text{on } \partial\Omega_f.$$

We already know that $r = 0$ on Γ_f, and $\frac{\partial r^k}{\partial n_f} + i\xi r^k = 0$ on Γ_f for all k, so $\frac{\partial r}{\partial n_f} = 0$ on Γ_f. By the unique continuation principle, $r = 0$ in Ω_f. This implies that every weakly convergent sequence of $\{r^k\}_{k=1}^{\infty}$ must converge weakly to zero. Therefore, the whole sequence must converge to zero weakly. Finally, test (4.5) against $q = r^k$ to get

$$\|\nabla r^k\|^2_{L^2(\Omega_f)} = \frac{\omega^2}{c^2}\|r^k\|^2_{L^2(\Omega_f)} + \int_{\partial\Omega_f}\frac{\partial r^k}{\partial n_f}\overline{r^k}.$$

Since $r^k \longrightarrow 0$ strongly in $L^2(\Omega_f)$ and in $L^2(\partial\Omega_f)$, this implies that $r^k \longrightarrow 0$ strongly in $H^1(\Omega_f)$.

The proof that $\underset{\sim}{u}^k \longrightarrow \underset{\sim}{u}$ strongly in $(H^1(\Omega_s))^N$ is similar. $R_k \longrightarrow 0$ and $\{E_k\}_{k=1}^{\infty}$ bounded in $L^2(\Gamma)$ implies that

$$\{\beta\underset{\approx}{\sigma}(\underset{\sim}{e}^k)\underset{\sim}{n}_s \cdot \underset{\sim}{n}_s + \omega^2\rho_f\underset{\sim}{e}^k \cdot \underset{\sim}{n}_s\}_{k=0}^{\infty}$$

is a bounded sequence in $L^2(\Gamma)$. Since $i\omega\underset{\approx}{A}\underset{\sim}{e}^{k+1} + \underset{\approx}{\sigma}(\underset{\sim}{e}^{k+1})\underset{\sim}{n}_s = 0$ on Γ_s for all k, Lemma 4.1 implies that $\{\underset{\sim}{e}^k\}_{k=0}^{\infty}$ is bounded in $(H^1(\Omega_s))^N$ and must therefore have a weakly convergent subsequence. Arguing as above, one can show that this weak limit must be zero, and that the whole sequence $\{\underset{\sim}{e}^k\}_{k=0}^{\infty}$ must converge to zero. The proof is complete. ■

Algorithms 1 and 2 have a drawback with respect to implementation. Solving for $(p^k, \underset{\sim}{u}^{k+1})$ in Step 2 of Algorithm 2, for example, requires the normal derivatives $\underset{\approx}{\sigma}(\underset{\sim}{u})\underset{\sim}{n}_s \cdot \underset{\sim}{n}_s$ and $\frac{\partial p^k}{\partial n_f}$ on the interface. Consequently, one is forced to use non–standard or hybrid finite element methods in order to implement Algorithm 1 and 2. This drawback can be easily avoided through a simple modification. The modified algorithms are given as follows.

Algorithm 3

Step 1 $\forall h_f^0,\ h_s^0 \in L^2(\Gamma)$.

Step 2 For $k \geq 0$, define $(p^k, \underset{\sim}{u}^k)$ such that

$$\begin{aligned}
-\tfrac{\omega^2}{c^2}p^k - \Delta p^k &= g_f, && \text{in } \Omega_f,\\
\tfrac{i\omega}{c}p^k + \tfrac{\partial p^k}{\partial n_f} &= 0, && \text{on } \Gamma_f,\\
\tfrac{\partial p^k}{\partial n_f} + \alpha p^k &= h_f^k, && \text{on } \Gamma;\\
-\omega^2\rho_s\underset{\sim}{u}^k - \operatorname{div}(\sigma(\underset{\sim}{u}^k)) &= \underset{\sim}{g}_s, && \text{in } \Omega_s,\\
i\omega A_s\underset{\sim}{u}^k + \sigma(\underset{\sim}{u}^k)n_s &= 0, && \text{on } \Gamma_s,\\
\sigma(\underset{\sim}{u}^k)n_s \cdot n_s + \beta\underset{\sim}{u}^k \cdot n_s &= h_s^k, && \text{on } \Gamma,
\end{aligned}$$

$$\sigma(\underset{\sim}{u}^k)n_s \cdot \tau_s = 0, \qquad \text{on } \Gamma;$$

$$h_s^{k+1} = (\alpha - \beta)p^k - h_f^k, \qquad \text{on } \Gamma,$$

$$h_f^{k+1} = \omega^2 \rho_f (\tfrac{\alpha}{\beta} - 1)\underset{\sim}{u}^k \cdot n_s - \tfrac{\alpha}{\beta} h_s^k, \qquad \text{on } \Gamma.$$

Algorithm 4

<u>Step 1</u> $\forall h_f^0 \in L^2(\Gamma)$.

<u>Step 2</u> For $k \geq 0$, define $(p^k, \underset{\sim}{u}^{k+1})$ such that

$$-\tfrac{\omega^2}{c^2}p^k - \Delta p^k = g_f, \qquad \text{in } \Omega_f,$$

$$\tfrac{i\omega}{c}p^k + \tfrac{\partial p^k}{\partial n_f} = 0, \qquad \text{on } \Gamma_f,$$

$$\tfrac{\partial p^k}{\partial n_f} + \alpha p^k = h_f^k, \qquad \text{on } \Gamma,$$

$$h_s^k = (\alpha - \beta)p^k - h_f^k; \qquad \text{on } \Gamma;$$

$$-\omega^2 \rho_s \underset{\sim}{u}^{k+1} - \text{div}(\sigma(\underset{\sim}{u}^{k+1})) = \underset{\sim}{g}_s, \qquad \text{in } \Omega_s,$$

$$i\omega \mathcal{A}_s \underset{\sim}{u}^{k+1} + \sigma(\underset{\sim}{u}^{k+1})n_s = 0, \qquad \text{on } \Gamma_s,$$

$$\sigma(\underset{\sim}{u}^{k+1})n_s \cdot n_s + \beta \underset{\sim}{u}^{k+1} \cdot n_s = h_s^k, \qquad \text{on } \Gamma;$$

$$\sigma(\underset{\sim}{u}^{k+1})n_s \cdot \tau_s = 0, \qquad \text{on } \Gamma,$$

$$h_f^{k+1} = \omega^2 \rho_f (\tfrac{\alpha}{\beta} - 1)\underset{\sim}{u}^{k+1} \cdot n_s - \tfrac{\alpha}{\beta} h_s^k, \qquad \text{on } \Gamma.$$

The equivalence of Algorithm 2 and 4 can be seen formally from the following

$$\begin{aligned}
\beta\sigma(\underset{\sim}{u}^{k+1})n_s \cdot n_s + \omega^2 \rho_f \underset{\sim}{u}^{k+1} \cdot n_s = h_s^k &= (\alpha - \beta)p^k - h_f^k \\
&= (\alpha - \beta)p^k - \left(\frac{\partial p^k}{\partial n_f} + \alpha p\right) \\
&= -\beta p^k - \frac{\partial p^k}{\partial n_f}.
\end{aligned}$$

Algorithms 1 and 3 are equivalent for a similar reason. Following the proof of Theorem 4.5, it can be shown that the statement of Theorem 4.5 also holds for Algorithm 3 and 4.

4.3 Numerical Experiments

In this section we will present the numerical results of two test problems in order to validate the theoretical analysis established in the previous section and to demonstrate the effectiveness of the proposed domain decomposition algorithms. To test the domain decomposition algorithm, we performed two sets of numerical experiments. In the first set, the true solution and the whole domain finite element solution are known; in the second, the true solution and the finite element solution are not explicitly known. In all experiments, $\Omega_f = [0,1] \times [0,1]$, $\Omega_s = [1,2] \times [0,1]$ and the mesh size is approximately 0.1. We used Algorithm 4 to generate all domain decomposition solutions.

For the first set of experiments, we chose the following source functions

$$\underset{\sim}{g_s} \equiv 0.$$
$$g_f = \sin^2(\pi x)\sin^2(\pi y)(4\pi^2 - \omega^2) - 2\pi^2[\sin^2(\pi x)\cos^2(\pi y) + \cos^2(\pi x)\sin^2(\pi y)].$$

Given the above sources, it is easy to show that the true solution is $\underset{\sim}{u}(x,y) = 0$, $p(x,y) = \sin^2(\pi x)\sin^2(\pi y)$

Figure 4.1 shows the finite element solution and the domain decomposition solution. Table 4.1 shows the L^∞-norm and the L^2-norm of the error. In table 4.1, p_h denotes the (global) finite element solution of the fluid half, and $\underset{\sim}{u_h}$ denotes the

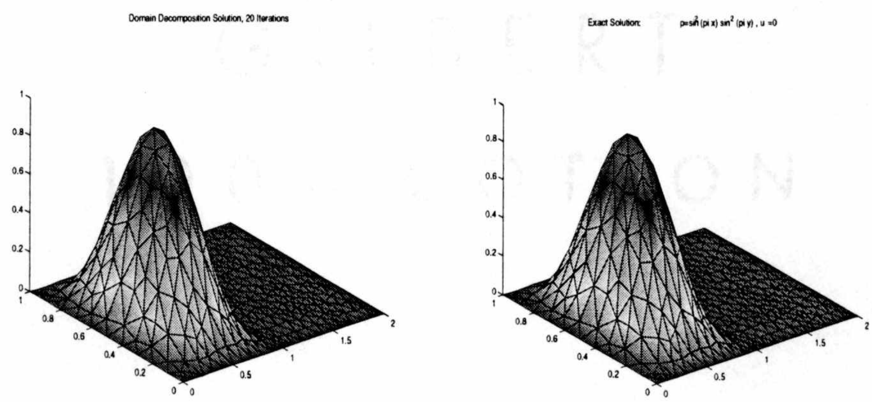

Figure 4.1: Finite element solution vs. domain decomposition solution

Table 4.1: Errors of domain decomposition solution

k	$\|p^k - p\|_{L^2}$	$\|p^k - p\|_{L^\infty}$	$\|\underset{\sim}{u}^k - \underset{\sim}{u}\|_{L^2}$	$\|\underset{\sim}{u}^k - \underset{\sim}{u}\|_{L^\infty}$
5	.00774713	.00150090	.00737340	.00184845
10	.00750596	.00151935	.00731775	.00172242
20	.00750852	.00152213	.00731816	.00172427

(global) finite element solution of the solid half.

In the second experiment, we choose the source functions

$$g_f(x, y) = x^2 e^y, \qquad \underset{\sim}{g}_s(x, y) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

In this test, because the true solution of the problem is not explicitly known and the whole domain finite element solution is not easily obtained, we tested the accuracy of our domain decomposition algorithms by calculating the relative error of successive iterates. Figure 4.2 shows the domain decomposition solution after 30 iterations. Note that the graph shows the real part of the first coordinate of the solution $\underset{\sim}{u}$. Table 4.2 shows the L^2-norm and the L^∞-norm of the relative errors of the domain decomposition solutions after 5, 10, 20 and 30 iterations, respectively.

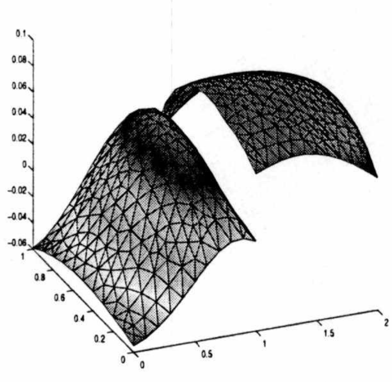

Figure 4.2: Domain decomposition solution after 30 interactions

Table 4.2: Relative errors of domain decomposition solution

k	$\frac{\left\|p^k-p^{k+1}\right\|_{L^2}}{\left\|p^{k+1}\right\|_{L^2}}$	$\frac{\left\|p^k-p^{k+1}\right\|_{L^\infty}}{\left\|p^{k+1}\right\|_{L^\infty}}$	$\frac{\left\|\underset{\sim}{u}^k-\underset{\sim}{u}^{k+1}\right\|_{L^2}}{\left\|\underset{\sim}{u}^{k+1}\right\|_{L^2}}$	$\frac{\left\|\underset{\sim}{u}^k-\underset{\sim}{u}^{k+1}\right\|_{L^\infty}}{\left\|\underset{\sim}{u}^{k+1}\right\|_{L^\infty}}$
5	.00762755	.01312341	.00133775	.00313325
10	6.825e-05	2.292e-05	1.697e-05	7.146e-05
20	6.103e-08	2.175e-07	1.736e-08	7.198e-08
30	8.856e-11	3.640e-10	2.442e-11	9.257e-11

Chapter 5

Approximation of Scalar Waves in the Space-Frequency Domain Using Second Order Absorbing Boundary Conditions

In Chapters One and Two, we studied Helmholtz problems with a first order absorbing boundary conditions. Recall that an absorbing boundary condition is meant to minimize reflections from waves passing through the boundary of the computational domain. Such a property is desirable because it simulates the absence of a physical boundary, yet is computationally practical. We now return to the acoustic wave equation (1.1) which we studied in Chapter One. In this chapter, however, we consider the following second order absorbing boundary condition

$$\frac{1}{c^2}P_{tt} + \frac{\partial}{\partial t}\frac{\partial P}{\partial \underset{\sim}{n}_f} - \frac{1}{2}\Delta_\Gamma P = 0, \tag{5.1}$$

where Δ_Γ denotes the Beltrami-Laplacian on $\Gamma = \partial\Omega_f$. Equation (5.1) is a form of the second order absorbing boundary condition which was proposed by Engquist and Majda in [11], [12] and [13], and studied by Higdon in [20] and [21], and by Sheen in

[35]. Imposing (5.1) on (1.1) then leads to the following model for wave propagation in an acoustic medium:

$$
(5.2) \qquad \begin{cases} \frac{1}{c^2}P_{tt} - \Delta P &= G_f \qquad \underset{\sim}{x} \in \Omega_f,\ t > 0, \\ \frac{1}{c^2}P_{tt} + \frac{\partial}{\partial t}\frac{\partial P}{\partial \underset{\sim}{n}_f} - \frac{1}{2}\Delta_\Gamma P &= 0 \qquad \underset{\sim}{x} \in \Gamma,\ t > 0, \\ P = P_t &= 0 \qquad \underset{\sim}{x} \in \Omega_f;\ t \le 0. \end{cases}
$$

Applying the Fourier transform to (5.2), or seeking time–harmonic solutions to (5.2) yields the following Helmholtz problem

$$
(5.3) \qquad \begin{cases} -\frac{\omega^2}{c^2}p - \Delta p &= g_f \quad \text{in } \Omega_f, \\ -\frac{\omega^2}{c^2}p + i\frac{\omega}{c}\frac{\partial p}{\partial \underset{\sim}{n}_f} - \frac{1}{2}\Delta_\Gamma p &= 0 \quad \text{on } \Gamma. \end{cases}
$$

where $p = \hat{P} = \int_{-\infty}^{\infty} e^{i\omega t} P(t, \underset{\sim}{x})dt$.

5.1 Finite Element Procedure for (5.3)

Introduce the following function space and its associated norm:

$$
V := \{p \in H^1(\Omega_f) | \quad \nabla_\Gamma p \in (L^2(\partial\Omega_f))^{N-1}\},
$$

where ∇_Γ denotes the tangential gradient operator on $\Gamma = \partial\Omega_f$.

$$
||p||_V := \sqrt{||p||^2_{H^1(\Omega_f)} + ||\nabla_\Gamma p||^2_{L^2(\partial\Omega_f)}}.
$$

The variational formation of (5.3) is defined as

$$
(5.4) \qquad \begin{cases} \text{Find} \quad p \in V \quad \text{such that} \\ \Lambda(p, q) = (g_f, q) \quad \forall q \in V, \end{cases}
$$

where

$$
\Lambda(p, q) = \frac{-\omega^2}{c^2}\int_{\Omega_f} p\overline{q} + \int_{\Omega_f} \nabla p \cdot \overline{\nabla q} + \frac{i\omega}{c}\int_{\partial\Omega_f} p\overline{q} - \frac{i}{2\omega}\int_{\partial\Omega_f} \nabla_\Gamma p \overline{\nabla_\Gamma q}
$$

and

$$
(g_f, q) = \int_{\Omega_f} g_f \overline{q} \qquad \forall q \in H^1(\Omega_f).
$$

Let $\mathcal{T}_h$ be a quasi-uniform triangulation of Ω_f with mesh size h. Suppose V_h is a finite element subspace of $H^1(\Omega_f)$ associated with $\mathcal{T}_h$. It is well known that V_h has the following simultaneous approximation property (see [5]): for some $1 < s < 2$,

(5.5)

$$\inf_{q \in V_h} \left\{ \|p - q\|_{L^2(\Omega_f)} + h\|p - q\|_{H^1(\Omega_f)} + h^s \|\nabla_\Gamma (p - q)\|_{L^2(\partial\Omega_f)} \right\} \le C_A h^m \|p\|_{H^m(\Omega_f)}.$$

Let $C_{R,m}$ denote an abstract regularity constant for solutions to (5.3), i.e.

$$\|p\|_{H^m(\Omega_f)} \le C_{R,m} \|g_f\|_{H^{m-2}(\Omega_f)}.$$

where $m \ge 2$. The finite element method for (5.4) is defined as

$$\text{(5.6)} \qquad \begin{cases} \text{Find} \quad p_h \in V_h \quad \text{such that} \\ \Lambda(p_h, q) = (g_f, q) \quad \forall q \in V_h. \end{cases}$$

We will modify the argument of Schatz [34] to derive error estimates for the finite element solutions. First, we prove the following modified Gårding inequality for the bilinear form $\Lambda(\cdot, \cdot)$.

Lemma 5.1. *There is a positive constant K which is independent of ω such that*

$$\left| \Lambda(p, p) + K\omega^2 \|p\|^2_{L^2(\Omega_f)} \right| \ge \frac{1}{4\omega} \|p\|^2_V$$

for all $p \in V$.

Proof : Proving the lemma is equivalent to showing that

$$\text{(5.7)} \qquad \left| \Lambda(p, p) + K\omega^2 \|p\|^2_{L^2(\Omega_f)} \right|^2 \ge \frac{1}{16\omega^2} \|p\|^4_V$$

for some $K > 0$ independent of ω. Expanding the left-hand side of (5.7),

$$\begin{aligned}
&|\Lambda(p,p) + K\omega^2 \|p\|^2_{L^2(\Omega_f)}|^2 \\
&= \left| (K\omega^2 - \tfrac{\omega^2}{c^2}) \|p\|^2_{L^2(\Omega_f)} + \|\nabla p\|^2_{L^2(\Omega_f)} + \tfrac{i\omega}{c} \|p\|^2_{L^2(\partial\Omega_f)} - \tfrac{i}{2\omega} \|\nabla_\Gamma p\|^2_{L^2(\partial\Omega_f)} \right|^2 \\
&= \left\{ (K\omega^2 - \tfrac{\omega^2}{c^2}) \|p\|^2_{L^2(\Omega_f)} + \|\nabla p\|^2_{L^2(\Omega_f)} \right\}^2 + \left\{ \tfrac{\omega}{c} \|p\|^2_{L^2(\partial\Omega_f)} - \tfrac{1}{2\omega} \|\nabla_\Gamma p\|^2_{L^2(\partial\Omega_f)} \right\}^2 \\
&= (K\omega^2 - \tfrac{\omega^2}{c^2})^2 \|p\|^4_{L^2(\Omega_f)} + \|\nabla p\|^4_{L^2(\Omega_f)} + \tfrac{1}{4\omega^2} \|\nabla_\Gamma p\|^4_{L^2(\partial\Omega_f)} \\
&\quad + 2(K\omega^2 - \tfrac{\omega^2}{c^2}) \|p\|^2_{L^2(\Omega_f)} \|\nabla p\|^2_{L^2(\Omega_f)} + \tfrac{\omega^2}{c^2} \|p\|^4_{L^2(\partial\Omega_f)} - \tfrac{1}{c} \|p\|^2_{L^2(\partial\Omega_f)} \|\nabla_\Gamma p\|^2_{L^2(\partial\Omega_f)}.
\end{aligned}$$

By Young's inequality,

$$\tfrac{1}{c}||p||^2_{L^2(\partial\Omega_f)}||\nabla_\Gamma p||^2_{L^2(\partial\Omega_f)} \le \tfrac{2\omega^2}{c^2}||p||^4_{L^2(\partial\Omega_f)} + \tfrac{1}{8\omega^2}||\nabla_\Gamma p||^4_{L^2(\partial\Omega_f)},$$

hence,

$$\begin{aligned}|\Lambda(p,p) + K\omega^2||p||^2_{L^2(\Omega_f)}|^2 \ge{}& (K\omega^2 - \tfrac{\omega^2}{c^2})^2||p||^4_{L^2(\Omega_f)} + ||\nabla p||^4_{L^2(\Omega_f)} + \tfrac{1}{8\omega^2}||\nabla_\Gamma p||^4_{L^2(\partial\Omega_f)}\\ &+ 2(K\omega^2 - \tfrac{\omega^2}{c^2})||p||^2_{L^2(\Omega_f)}||\nabla p||^2_{L^2(\Omega_f)} - \tfrac{\omega^2}{c^2}||p||^4_{L^2(\partial\Omega_f)}\end{aligned}$$

and (5.7) is therefore true if

$$\begin{aligned}\{(K\omega^2 - \tfrac{\omega^2}{c^2})^2 - \tfrac{1}{16\omega^2}\}||p||^4_{L^2(\Omega_f)} &+ (1 - \tfrac{1}{16\omega^2})||\nabla p||^4_{L^2(\Omega_f)} + (\tfrac{1}{8\omega^2} - \tfrac{1}{16\omega^2})||\nabla_\Gamma p||^4_{L^2(\partial\Omega_f)}\\ &+ 2(K\omega^2 - \tfrac{\omega^2}{c^2} - \tfrac{1}{16\omega^2})||p||^2_{L^2(\Omega_f)}||\nabla p||^2_{L^2(\Omega_f)} - \tfrac{\omega^2}{c^2}||p||^4_{L^2(\partial\Omega_f)}\\ &- \tfrac{1}{8\omega^2}||p||^2_{L^2(\Omega_f)}||\nabla_\Gamma p||^2_{L^2(\partial\Omega_f)} - \tfrac{1}{8\omega^2}||\nabla p||^2_{L^2(\Omega_f)}||\nabla_\Gamma p||^2_{L^2(\partial\Omega_f)} \ge 0.\end{aligned}$$

By the trace inequality on $\partial\Omega_f$ and Young's inequality,

$$\begin{aligned}\tfrac{\omega^2}{c^2}||p||^4_{L^2(\partial\Omega_f)} &\le \tfrac{\omega^2}{c^2}C_f^4||p||^2_{L^2(\Omega_f)}||\nabla p||^2_{L^2(\Omega_f)}\\ &\le \tfrac{\omega^2}{c^2}C_f^4\left\{\tfrac{\omega^2 C_f^4}{2c^2}||p||^4_{L^2(\Omega_f)} + \tfrac{c^2}{2\omega^2 C_f^4}||\nabla p||^2_{L^2(\Omega_f)}\right\}\\ &= \tfrac{1}{2}\left(\tfrac{\omega^2 C_f^4}{c^2}\right)^2||p||^4_{L^2(\Omega_f)} + \tfrac{1}{2}||\nabla p||^4_{L^2(\Omega_f)}.\end{aligned}$$

Also by Young's inequality,

$$\tfrac{1}{8\omega^2}||p||^2_{L^2(\Omega_f)}||\nabla_\Gamma p||^2_{L^2(\partial\Omega_f)} \le \tfrac{1}{4\omega^2}||p||^4_{L^2(\Omega_f)} + \tfrac{1}{64\omega^2}||\nabla_\Gamma p||^4_{L^2(\partial\Omega_f)}$$

and

$$\tfrac{1}{8\omega^2}||\nabla p||^2_{L^2(\Omega_f)}||\nabla_\Gamma p||^2_{L^2(\partial\Omega_f)} \le \tfrac{1}{4\omega^2}||\nabla p||^4_{L^2(\Omega_f)} + \tfrac{1}{64\omega^2}||\nabla_\Gamma p||^4_{L^2(\partial\Omega_f)}.$$

Therefore, (5.7) is true since

$$\begin{aligned}\left\{(K\omega^2 - \tfrac{\omega^2}{c^2})^2 - \tfrac{1}{16\omega^2} - \tfrac{1}{2}\left(\tfrac{\omega^2 C_f^4}{c^2}\right)^2 - \tfrac{1}{4\omega^2}\right\}||p||^4_{L^2(\Omega_f)} + \left(\tfrac{1}{2} - \tfrac{1}{16\omega^2} - \tfrac{1}{4\omega^2}\right)||\nabla p||^4_{L^2(\Omega_f)}\\ + (\tfrac{1}{16\omega^2} - \tfrac{1}{32\omega^2})||\nabla_\Gamma p||^4_{L^2(\partial\Omega_f)} + 2(K\omega^2 - \tfrac{\omega^2}{c^2} - \tfrac{1}{16\omega^2})||p||^2_{L^2(\Omega_f)}||\nabla p||^2_{L^2(\Omega_f)} \ge 0.\end{aligned}$$

■

Lemma 5.2. *Suppose p solves (5.4) and p_h solves (5.6). Then there are constants C_1 and C_2 independent of ω and h such that $h \leq \frac{1}{\omega\sqrt{C_{R,2}}} C_1$ implies that*

$$\|p - p_h\|_{L^2(\Omega_f)} \leq C_2 C_{R,2} \left\{ h\|p - p_h\|_{H^1(\Omega_f)} + h^{2-s}\tfrac{1}{\omega}\|\nabla_\Gamma(p - p_h)\|_{L^2(\partial\Omega_f)} \right\}$$

Proof : Suppose that p is a solution of (5.4) and p_h is a solution of (5.6). Let φ be a solution to the adjoint problem with source $p - p_h$, i.e. φ solves

$$\begin{cases} \text{Find} \quad \varphi \in H^1(\Omega_f) \quad \text{such that} \\ \Lambda^*(\varphi, q) = (p - p_h, q) \quad \forall q \in H^1(\Omega_f), \end{cases}$$

where

$$\Lambda^*(\varphi, q) = \frac{-\omega^2}{c^2}\int_{\Omega_f} \varphi\overline{q} + \int_{\Omega_f} \nabla\varphi \cdot \overline{\nabla q} - \frac{i\omega}{c}\int_{\partial\Omega_f} \varphi\overline{q} + \frac{i}{2\omega}\int_{\partial\Omega_f} \nabla_\Gamma p \overline{\nabla_\Gamma q}.$$

Then φ satisfies

$$\Lambda(q, \varphi) = (q, p - p_h) \qquad \forall q \in H^1(\Omega_f).$$

Taking $q = p - p_h$ implies

$$\|p - p_h\|^2_{L^2(\Omega_f)} = \Lambda(p - p_h, \varphi).$$

The fundamental orthogonality identity implies that

$$\Lambda(p - p_h, \varphi) = \Lambda(p - p_h, \varphi - \varphi_h) \qquad \forall \varphi_h \in V_h.$$

Therefore,

$$\begin{aligned} \|p - p_h\|^2_{L^2(\Omega_f)} &= \Lambda(p - p_h, \varphi - \varphi_h) \\ &= -\tfrac{\omega^2}{c^2}(p - p_h, \varphi - \varphi_h) + (\nabla(p - p_h), \nabla(\varphi - \varphi_h)) \\ &\quad + \tfrac{i\omega}{c} < p - p_h, \varphi - \varphi_h > - \tfrac{i}{2\omega} < \nabla_\Gamma(p - p_h), \nabla_\Gamma(\varphi - \varphi_h) > . \end{aligned}$$

Schwarz's inequality implies

$$\begin{aligned} \|p - p_h\|^2_{L^2(\Omega_f)} &\leq \tfrac{\omega^2}{c^2}\|p - p_h\|_{L^2(\Omega_f)}\|\varphi - \varphi_h\|_{L^2(\Omega_f)} \\ &+ \|\nabla(p - p_h)\|_{L^2(\Omega_f)}\|\nabla(\varphi - \varphi_h)\|_{L^2(\Omega_f)} + \tfrac{\omega}{c}\|p - p_h\|_{L^2(\partial\Omega_f)}\|\varphi - \varphi_h\|_{L^2(\partial\Omega_f)} \\ &+ \tfrac{1}{2\omega}\|\nabla_\Gamma(p - p_h)\|_{L^2(\partial\Omega_f)}\|\nabla_\Gamma(\varphi - \varphi_h)\|_{L^2(\partial\Omega_f)}. \end{aligned}$$

The trace theorem on $\partial\Omega_f$ implies

$$\begin{aligned}
&\tfrac{\omega}{c}\|p - p_h\|_{L^2(\partial\Omega_f)}\|\varphi - \varphi_h\|_{L^2(\partial\Omega_f)} \\
&\qquad \le \tfrac{\omega}{c}{C_f}^2\|p - p_h\|^{1/2}_{L^2(\Omega_f)}\|\varphi - \varphi_h\|^{1/2}_{L^2(\Omega_f)}\|p - p_h\|^{1/2}_{H^1(\Omega_f)}\|\varphi - \varphi_h\|^{1/2}_{H^1(\Omega_f)}.
\end{aligned}$$

Therefore, by Young's inequality,

$$\begin{aligned}
\tfrac{\omega}{c}\|p - p_h\|_{L^2(\partial\Omega_f)}\|\varphi - \varphi_h\|_{L^2(\partial\Omega_f)} &\le \tfrac{\omega^2}{c^2}\|p - p_h\|_{L^2(\Omega_f)}\|\varphi - \varphi_h\|_{L^2(\Omega_f)} \\
&\quad + \tfrac{{C_f}^4}{4}\|p - p_h\|_{H^1(\Omega_f)}\|\varphi - \varphi_h\|_{H^1(\Omega_f)}.
\end{aligned}$$

Plugging this inequality in above yields

$$\begin{aligned}
\|p - p_h\|^2_{L^2(\Omega_f)} &\le \tfrac{2\omega^2}{c^2}\|p - p_h\|_{L^2(\Omega_f)}\|\varphi - \varphi_h\|_{L^2(\Omega_f)} \\
&\quad + \left(\tfrac{C_f^4}{4} + 1\right)\|p - p_h\|_{H^1(\Omega_f)}\|\varphi - \varphi_h\|_{H^1(\Omega_f)} \\
&\quad + \tfrac{1}{2\omega}\|\nabla_\Gamma(p - p_h)\|_{L^2(\partial\Omega_f)}\|\nabla_\Gamma(\varphi - \varphi_h)\|_{L^2(\partial\Omega_f)}.
\end{aligned}$$

The approximation property of V_h implies

$$\begin{aligned}
\|p - p_h\|^2_{L^2(\Omega_f)} &\le \tfrac{2\omega^2}{c^2}\|p - p_h\|_{L^2(\Omega_f)}C_Ah^2\|\varphi\|_{H^2(\Omega_f)} \\
&\quad + \left(\tfrac{C_f^4}{4} + 1\right)\|p - p_h\|_{H^1(\Omega_f)}C_Ah\|\varphi\|_{H^2(\Omega_f)} \\
&\quad + \tfrac{1}{2\omega}\|\nabla_\Gamma(p - p_h)\|_{L^2(\partial\Omega_f)}C_Ah^{2-s}\|\varphi\|_{H^2(\Omega_f)}.
\end{aligned}$$

Applying the regularity estimate (recall that φ solves the dual problem with source $p - p_h$) yields

$$\begin{aligned}
\|p - p_h\|^2_{L^2(\Omega_f)} &\le \tfrac{2\omega^2}{c^2}\|p - p_h\|_{L^2(\Omega_f)}C_Ah^2C_{R,2}\|p - p_h\|_{L^2(\Omega_f)} \\
&\quad + \left(\tfrac{C_f^4}{4} + 1\right)\|p - p_h\|_{H^1(\Omega_f)}C_AhC_{R,2}\|p - p_h\|_{L^2(\Omega_f)} \\
&\quad + \tfrac{1}{2\omega}\|\nabla_\Gamma(p - p_h)\|_{L^2(\partial\Omega_f)}C_Ah^{2-s}C_{R,2}\|p - p_h\|_{L^2(\Omega_f)},
\end{aligned}$$

or equivalently,

$$\begin{aligned}
\left\{1 - \tfrac{2\omega^2}{c^2}C_Ah^2C_{R,2}\right\}\|p - p_h\|_{L^2(\Omega_f)} &\le \left(\tfrac{{C_f}^4}{4} + 1\right)C_AhC_{R,2}\|p - p_h\|_{H^1(\Omega_f)} \\
&\quad + \tfrac{1}{2\omega}C_Ah^{2-s}C_{R,2}\|\nabla_\Gamma(p - p_h)\|_{L^2(\partial\Omega_f)}.
\end{aligned}$$

Choose $h \leq \frac{1}{\omega\sqrt{C_{R,2}}} C_1$ where $C_1 = \frac{c}{2\sqrt{C_A}}$. Then

$$\left\{1 - \frac{2\omega^2}{c^2} C_A h^2 C_{R,2}\right\} \geq \frac{1}{2}$$

which implies that

$$\begin{aligned}\|p - p_h\|_{L^2(\Omega_f)} \leq & \left(\frac{C_f{}^4}{4} + 1\right) 2C_A h C_{R,2} \|p - p_h\|_{H^1(\Omega_f)} \\ & + \frac{1}{\omega} C_A h^{2-s} C_{R,2} \|\nabla_\Gamma (p - p_h)\|_{L^2(\partial\Omega_f)},\end{aligned}$$

or

$$\|p - p_h\|_{L^2(\Omega_f)} \leq C_2 C_{R,2} \left\{ h\|p - p_h\|_{H^1(\Omega_f)} + h^{2-s}\frac{1}{\omega}\|\nabla_\Gamma(p - p_h)\|_{L^2(\partial\Omega_f)} \right\},$$

where $C_2 = \left(\frac{C_f^4}{4} + 1\right) C_A$. ■

Theorem 5.3. *Suppose p solves (5.4) and p_h solves (5.6). Then there are constants C_3 and C_4 independent of ω and h such that*

$$h \leq C_3 min \left\{ \frac{1}{\omega^{3/2} C_{R,2}}, \frac{1}{\omega C_{R,2}^2} \right\}$$

implies that

$$\|p - p_h\|_V \leq C_4 C_{R,m} \left(\omega^3 h^m + \omega h^{m-1} + h^{m-s}\right) \|g_f\|_{H^{m-2}(\Omega_f)},$$

and

$$\|p - p_h\|_{L^2(\Omega_f)} \leq C C_{R,2} C_{R,m} \left(h + h^{2-s}\frac{1}{\omega}\right) \left(\omega^3 h^m + \omega h^{m-1} + h^{m-s}\right) \|g_f\|_{H^{m-2}(\Omega_f)}.$$

Proof : Let p be a solution of (5.4) and let p_h be a solution of (5.6). By Lemma 5.1,

$$\frac{1}{\omega}\|p - p_h\|_V^2 \leq \left|\Lambda(p - p_h, p - p_h) + K\omega^2 \|p - p_h\|^2_{L^2(\Omega_f)}\right|.$$

Note that $\Lambda(p - p_h, p - p_h) = \Lambda(p - p_h, p - q)$ for every $q \in V_h$ by fundamental identity. Therefore,

$$\begin{aligned}\|p - p_h\|_V^2 \leq & \ \omega\,|\Lambda(p - p_h, p - q)| + K\omega^3 \|p - p_h\|^2_{L^2(\Omega_f)} \\ \leq & \ \frac{2\omega^3}{c^2}\|p - p_h\|_{L^2(\Omega_f)}\|p - q\|_{L^2(\Omega_f)} + \omega\left(\frac{C_f^4}{4} + 1\right)\|p - p_h\|_{H^1(\Omega_f)}\|p - q\|_{L^2(\Omega_f)} \\ & + \frac{1}{2}\|\nabla_\Gamma(p - p_h)\|_{L^2(\partial\Omega_f)}\|\nabla_\Gamma(p - q)\|_{L^2(\partial\Omega_f)} + K\omega^3\|p - p_h\|^2_{L^2(\Omega_f)}.\end{aligned}$$

Applying the duality estimate of Lemma 5.2 and simplifying,

$$\begin{aligned}\|p-p_h\|_V^2 \le \tfrac{2\omega^3}{c^2}\|p-p_h\|_{L^2(\Omega_f)}\|p-q\|_{L^2(\Omega_f)} + \omega\left(\tfrac{C_f^4}{4}+1\right)\|p-p_h\|_{H^1(\Omega_f)}\|p-q\|_{L^2(\Omega_f)} \\ + \tfrac{1}{2}\|\nabla_\Gamma(p-p_h)\|_{L^2(\partial\Omega_f)}\|\nabla_\Gamma(p-q)\|_{L^2(\partial\Omega_f)} \\ + K\omega^3C_2^2\left\{hC_{R,2}\|p-p_h\|_{H^1(\Omega_f)} + h^{2-s}\tfrac{1}{\omega}C_{R,2}\|\nabla_\Gamma(p-p_h)\|_{L^2(\partial\Omega_f)}\right\}^2,\end{aligned}$$

which implies (since $a^2+b^2 \le 2a^2+2b^2$)

$$\begin{aligned}\|p-p_h\|_V^2 \le \tfrac{2\omega^3}{c^2}\|p-p_h\|_{L^2(\Omega_f)}\|p-q\|_{L^2(\Omega_f)} + \omega\left(\tfrac{C_f^4}{4}+1\right)\|p-p_h\|_{H^1(\Omega_f)}\|p-q\|_{L^2(\Omega_f)} \\ + \tfrac{1}{2}\|\nabla_\Gamma(p-p_h)\|_{L^2(\partial\Omega_f)}\|\nabla_\Gamma(p-q)\|_{L^2(\partial\Omega_f)} \\ + C\left\{\omega^3h^2C_{R,2}^2\|p-p_h\|_{H^1(\Omega_f)} + h^{4-2s}\omega C_{R,2}^2\|\nabla_\Gamma(p-p_h)\|_{L^2(\partial\Omega_f)}\right\},\end{aligned}$$

or equivalently

$$\begin{aligned}\|p-p_h\|_V^2 - C\left(\omega^3h^2C_{R,2}^2\|p-p_h\|_{H^1(\Omega_f)} + h^{4-2s}\omega C_{R,2}^2\|\nabla_\Gamma(p-p_h)\|_{L^2(\partial\Omega_f)}\right) \\ \le \tfrac{2\omega^3}{c^2}\|p-p_h\|_{L^2(\Omega_f)}\|p-q\|_{L^2(\Omega_f)} \\ + \omega\left(\tfrac{C_f^4}{4}+1\right)\|p-p_h\|_{H^1(\Omega_f)}\|p-q\|_{H^1(\Omega_f)} \\ + \tfrac{1}{2}\|\nabla_\Gamma(p-p_h)\|_{L^2(\partial\Omega_f)}\|\nabla_\Gamma(p-q)\|_{L^2(\partial\Omega_f)}.\end{aligned}$$

Applying the approximation property implies

$$\begin{aligned}\|p-p_h\|_V^2 - C\left(\omega^3h^2C_{R,2}^2\|p-p_h\|_{H^1(\Omega_f)} + h^{4-2s}\omega C_{R,2}^2\|\nabla_\Gamma(p-p_h)\|_{L^2(\partial\Omega_f)}\right) \\ \le \tfrac{2\omega^3}{c^2}\|p-p_h\|_{L^2(\Omega_f)}C_Ah^m\|p\|_{H^m(\Omega_f)} \\ + \omega\left(\tfrac{C_f^4}{4}+1\right)\|p-p_h\|_{H^1(\Omega_f)}C_Ah^{m-1}\|p\|_{H^m(\Omega_f)} \\ + \tfrac{1}{2}\|\nabla_\Gamma(p-p_h)\|_{L^2(\partial\Omega_f)}C_Ah^{m-s}\|p\|_{H^m(\Omega_f)}.\end{aligned}$$

Applying the regularity estimate,

$$\begin{aligned}\|p-p_h\|_V^2 - C\left(\omega^3h^2C_{R,2}^2\|p-p_h\|_{H^1(\Omega_f)} + h^{4-2s}\omega C_{R,2}^2\|\nabla_\Gamma(p-p_h)\|_{L^2(\partial\Omega_f)}\right) \\ \le \tfrac{2\omega^3}{c^2}\|p-p_h\|_{L^2(\Omega_f)}C_Ah^mC_{R,m}\|g_f\|_{H^{m-2}(\Omega_f)} \\ + \omega\left(\tfrac{C_f^4}{4}+1\right)\|p-p_h\|_{H^1(\Omega_f)}C_Ah^{m-1}C_{R,m}\|g_f\|_{H^{m-2}(\Omega_f)} \\ + \tfrac{1}{2}\|\nabla_\Gamma(p-p_h)\|_{L^2(\partial\Omega_f)}C_Ah^{m-s}C_{R,m}\|g_f\|_{H^{m-2}(\Omega_f)}.\end{aligned}$$

Therefore,

$$\left\{1 - C\left(\omega^3 h^2 C_{R,2}^2 + h^{4-2s}\omega C_{R,2}^2\right)\right\} \|p - p_h\|_V$$
$$\leq \left\{\tfrac{2\omega^3}{c^2} C_A h^m C_{R,m} + \omega\left(\tfrac{C_f^4}{4} + 1\right) C_A h^{m-1} C_{R,m} + \tfrac{1}{2} C_A h^{m-s} C_{R,m}\right\} \|g_f\|_{H^{m-2}(\Omega_f)}.$$

Choose $h \leq C_3 min\left\{\frac{1}{\omega^{3/2}C_{R,2}}, \frac{1}{\omega^{1/(4-2s)}C_{R,2}^{2/(4-2s)}}\right\}$ where $C_3 = min\left\{\frac{1}{\sqrt{2C_2}}, \frac{1}{(2C_2)^{1/(4-2s)}}\right\}$. Then

$$1 - C\left(\omega^3 h^2 C_{R,2}^2 + h^{4-2s}\omega C_{R,2}^2\right) \geq \frac{1}{2}$$

and therefore,

$$\|p - p_h\|_V \leq 2\left\{\tfrac{2\omega^3}{c^2} C_A h^m C_{R,m} + \omega\left(\tfrac{C_f^4}{4} + 1\right) C_A h^{m-1} C_{R,m} + \tfrac{1}{2} C_A h^{m-s} C_{R,m}\right\} \|g_f\|_{H^{m-2}(\Omega_f)}.$$

Hence,

$$\|p - p_h\|_V \leq C_4 C_{R,m}\left(\omega^3 h^m + \omega h^{m-1} + h^{m-s}\right) \|g_f\|_{H^{m-2}(\Omega_f)},$$

where C_4 is independent of ω and h. Applying the duality estimate from Lemma 5.2 implies

$$\|p - p_h\|_{L^2(\Omega_f)} \leq C C_{R,2} C_{R,m}\left(h + h^{2-s}\tfrac{1}{\omega}\right)\left(\omega^3 h^m + \omega h^{m-1} + h^{m-s}\right) \|g_f\|_{H^{m-2}(\Omega_f)}.$$

■

5.2 Implementation Issues and Total Errors

We are ready to describe a procedure to numerically approximate solutions to the wave equation in the time domain using both the first order and the second order absorbing boundary condition. Our approach is similar to the that of [23], which treats the acoustic Helmholtz problem with the first order absorbing boundary condition. Recall that one can recover the solution in the time domain through the Fourier inversion formula

$$P(\underset{\sim}{x}, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} p(\underset{\sim}{x}, \omega) e^{i\omega t} d\omega \tag{5.8}$$

where $p(\underset{\sim}{x}, \omega)$ is the solution to the Helmholtz problem (1.5) or (5.3). Since $p(\underset{\sim}{x}, \omega)$ satisfies the conjugation relation $p(\underset{\sim}{x}, -\omega) = \overline{p(\underset{\sim}{x}, \omega)}$, equation (5.8) reduces to

$$P(\underset{\sim}{x}, t) = \frac{1}{\pi} \operatorname{Re} \int_{0}^{\infty} p(\underset{\sim}{x}, \omega) e^{i\omega t} d\omega \tag{5.9}$$

Of course, in order to apply equation (5.8) or equation (5.9) directly, one must determine $p(\underset{\sim}{x}, \omega)$ analytically as a function of ω. We will only calculate finite element approximations for discrete values of ω, and it is therefore necessary to approximate the Fourier inversion. Since the integral is improper, we chose to first truncate the integral, and approximate the remaining finite integral with numerical integration, that is

$$\begin{aligned} P(\underset{\sim}{x}, t) &:= \frac{1}{\pi} \operatorname{Re} \int_{0}^{\infty} p(\underset{\sim}{x}, \omega) e^{i\omega t} d\omega \approx \frac{1}{\pi} \operatorname{Re} \int_{0}^{\omega^*} p(\underset{\sim}{x}, \omega) e^{i\omega t} d\omega \\ &\approx \frac{1}{\pi} \operatorname{Re} \sum_{j=1}^{M} \xi_j p(\underset{\sim}{x}, \omega_j) e^{i\omega_j t} \end{aligned} \tag{5.10}$$

for some large frequency ω^*, some set of weights $\{\xi_j\}$ and some set of frequencies $\{\omega_j\} \subset [0, \omega^*]$. We will approximate each $p(\underset{\sim}{x}, \omega)$ in the set $\{\omega_j\}$ by its finite element approximation defined by (1.11) or (5.6) as appropriate. We will therefore approximate the time domain solution $P(\underset{\sim}{x}, t)$ by

$$p^h_{\omega^*, M}(\underset{\sim}{x}, t) = \frac{1}{\pi} \operatorname{Re} \sum_{j=1}^{M} \xi_j p_h(\underset{\sim}{x}, \omega_j) e^{i\omega_j t}. \tag{5.11}$$

The obvious question is then, "how good is such an approximation?" Before proceeding to the numerical results, we present a brief discussion of the error which results from this procedure. Define

$$P_{\omega^*}(\underset{\sim}{x}, t) := \frac{1}{\pi} \operatorname{Re} \int_{0}^{\omega^*} p(\underset{\sim}{x}, \omega) e^{i\omega t} d\omega,$$

and

$$p_{\omega^*,M}(\underset{\sim}{x},t) := \frac{1}{\pi}\,\mathrm{Re}\sum_{j=1}^{M}\xi_j p(\underset{\sim}{x},\omega_j)e^{i\omega_j t}.$$

For a fixed time t, we can decompose the error into three components as follows

$$\begin{aligned}P(\underset{\sim}{x},t) - p^h_{\omega^*,M}(\underset{\sim}{x},t) &= \left\{P(\underset{\sim}{x},t) - P_{\omega^*}(\underset{\sim}{x},t)\right\} + \left\{P_{\omega^*}(\underset{\sim}{x},t) - p_{\omega^*,M}(\underset{\sim}{x},t)\right\}\\ &\quad + \left\{p_{\omega^*,M}(\underset{\sim}{x},t) - p^h_{\omega^*,M}(\underset{\sim}{x},t)\right\}\\ &= E_1(\underset{\sim}{x},t) + E_2(\underset{\sim}{x},t) + E_3(\underset{\sim}{x},t).\end{aligned} \tag{5.12}$$

The first error term is a consequence of the truncation point ω^* of the Fourier integral,

$$\begin{aligned}\|E_1(\cdot,t)\|_{L^2(\Omega_f)} &= \left(\int_{\Omega_f}\left|\tfrac{1}{\pi}\,\mathrm{Re}\int_{\omega^*}^{\infty} p(\underset{\sim}{x},\omega)e^{i\omega t}d\omega\right|^2 d\Omega_f\right)^{1/2}\\ &\le C\left(\int_{\omega^*}^{\infty}\|p(\cdot,\omega)\|^2_{L^2(\Omega_f)}d\omega\right)^{1/2}.\end{aligned}$$

The second term depends on the numerical integration method. If, for example, we use the composite midpoint rule, then

$$E_2(\underset{\sim}{x},t) = \tfrac{1}{\pi}\,\mathrm{Re}\int_{\omega^*}^{\infty} p(\underset{\sim}{x},\omega)e^{i\omega t}d\omega - \tfrac{1}{\pi}\,\mathrm{Re}\sum_{j=0}^{M/2}\left(\tfrac{2\omega^*}{M+2}\right)p(\underset{\sim}{x},\omega_{2j})e^{i\omega_{2j}t},$$

and

$$\begin{aligned}\|E_2(\cdot,t)\|^2_{L^2(\Omega_f)} &\le \tfrac{1}{\pi}\int_{\Omega_f}\left\|\tfrac{\omega^*}{6}\left(\tfrac{2\omega^*}{M+2}\right)^2\tfrac{\partial^2}{\partial\omega^2}(p(\underset{\sim}{x},\omega)e^{i\omega t})\right\|^2_{L^2(0,\omega^*)}d\Omega_f\\ &\le C(\omega^*)^2\left(\tfrac{2\omega^*}{M+2}\right)^4\int_{\Omega_f}\left\|\tfrac{\partial^2}{\partial\omega^2}(p(\underset{\sim}{x},\omega)e^{i\omega t})\right\|^2_{L^2(0,\omega^*)}d\Omega_f\\ &= C(\omega^*)^2\left(\tfrac{2\omega^*}{M+2}\right)^4\int_{\Omega_f}\left\|\{\tfrac{\partial^2}{\partial\omega^2}p(\underset{\sim}{x},\omega) + 2it\tfrac{\partial}{\partial\omega}p(\underset{\sim}{x},\omega) - t^2p(\underset{\sim}{x},\omega)\}e^{i\omega t}\right\|^2_{L^2(0,\omega^*)}d\Omega_f.\end{aligned}$$

Since the Fourier transform satisfies the property

$$(-i)^m\tfrac{\partial^m}{\partial\omega^m}\hat{\varphi}(\omega) = \widehat{t^m\varphi(t)},$$

$$\begin{aligned}\|E_2(\cdot,t)\|^2_{L^2(\Omega_f)} \le C(\omega^*)^2\left(\tfrac{2\omega^*}{M+2}\right)^4\int_{\Omega_f}&\left\{\left\|\widehat{t^2P(\underset{\sim}{x},t)}\right\|_{L^2(0,\omega^*)} + t^2\left\|\widehat{tP(\underset{\sim}{x},t)}\right\|_{L^2(0,\omega^*)}\right.\\ &\left.+t^4\left\|\widehat{P(\underset{\sim}{x},t)}\right\|_{L^2(0,\omega^*)}\right\}d\Omega_f.\end{aligned}$$

Therefore by Parseval's identity (see [37]),

$$\|E_2(\cdot,t)\|^2_{L^2(\Omega_f)} \le C(\omega^*)^2 \left(\tfrac{2\omega^*}{M+2}\right)^4 \left\{ \left\| t^2 P(\underset{\sim}{x},t) \right\|_{L^2(0,\omega^*)} + t^2 \left\| tP(\underset{\sim}{x},t) \right\|_{L^2(0,\omega^*)} + t^4 \left\| P(\underset{\sim}{x},t) \right\|_{L^2(0,\omega^*)} \right\}.$$

The third error term depends on the finite element method.

$$\begin{aligned}
\|E_3(\cdot,t)\|^2_{L^2(\Omega_f)} &\le C \int_{\Omega_f} \left| \sum_{j=1}^{M} p(\underset{\sim}{x},\omega_j) - p_h(\underset{\sim}{x},\omega_j) \right|^2 d\Omega_f \\
&\le C \sum_{j=1}^{M} \|p(\cdot,\omega_j) - p_h(\cdot,\omega_j)\|^2_{L^2(\Omega_f)} \\
&\le C \sum_{j=1}^{M} \left\{ C_{R,2} C_{R,m} \left(\omega_j^2 h^m + \tfrac{1}{\omega_j} h^{m-s}\right) \left(h + \tfrac{h^{2-s}}{\omega_j}\right) \|g_f\|_{H^{m-2}(\Omega_f)} \right\}^2.
\end{aligned}$$

Combining the above, we have the following theorem.

Theorem 5.4. *Suppose $P(\underset{\sim}{x},t)$ is the solution of (1.3) and $Q(\underset{\sim}{x},t)$ is the solution of (5.2). If $p^h_{\omega^*,M}(\underset{\sim}{x},t)$ and $q^h_{\omega^*,M}(\underset{\sim}{x},t)$ are the approximations of $P(\underset{\sim}{x},t)$ and $Q(\underset{\sim}{x},t)$ defined by (5.11), then*

$$\begin{aligned}
\|P(\cdot,t) - p^h_{\omega^*,M}(\cdot,t)\|^2_{L^2(\Omega_f)} &\le C \int_{\omega^*}^{\infty} \|p(\cdot,\omega)\|^2_{L^2(\Omega_f)} d\omega \\
&+ C(\omega^*)^2 \left(\tfrac{2\omega^*}{M+2}\right)^4 \left\{ \left\| t^2 P(\underset{\sim}{x},t) \right\|_{L^2(0,\omega^*)} + t^2 \left\| tP(\underset{\sim}{x},t) \right\|_{L^2(0,\omega^*)} + t^4 \left\| P(\underset{\sim}{x},t) \right\|_{L^2(0,\omega^*)} \right\} \\
&+ C \sum_{j=1}^{M} C_{R,2} C_{R,m} (C_{R,2}\omega_j^2 h^{m+2} + h^m) \|g_f\|_{H^{m-2}(\Omega_f)},
\end{aligned}$$

and

$$\begin{aligned}
\|Q(\cdot,t) - q^h_{\omega^*,M}(\cdot,t)\|^2_{L^2(\Omega_f)} &\le C \int_{\omega^*}^{\infty} \|q(\cdot,\omega)\|^2_{L^2(\Omega_f)} d\omega \\
&+ C(\omega^*)^2 \left(\tfrac{2\omega^*}{M+2}\right)^4 \left\{ \left\| t^2 Q(\underset{\sim}{x},t) \right\|_{L^2(0,\omega^*)} + t^2 \left\| tQ(\underset{\sim}{x},t) \right\|_{L^2(0,\omega^*)} + t^4 \left\| Q(\underset{\sim}{x},t) \right\|_{L^2(0,\omega^*)} \right\} \\
&+ C \sum_{j=1}^{M} C_{R,2} C_{R,m} \left(h + h^{2-s}\tfrac{1}{\omega_j}\right) \left(\omega_j^3 h^m + \omega_j h^{m-1} + h^{m-s}\right) \|g_f\|_{H^{m-2}(\Omega_f)}.
\end{aligned}$$

5.2.1 Notes on Implementation of the Finite Element Method

Recall that the weak formulation of the Helmholtz problem with a first order absorbing boundary condition is

$$(\nabla p, \nabla q) - \tfrac{\omega^2}{c^2}(p,q) + \tfrac{i\omega}{c} < p, q >= (g_f, q),$$

and with a second order absorbing boundary condition, the variational formulation is

$$(\nabla p, \nabla q) - \tfrac{\omega^2}{c^2}(p,q) + \tfrac{i\omega}{c} < p, q > - \tfrac{i}{2\omega} < \nabla_\Gamma p, \nabla_\Gamma q >= (g_f, q).$$

If V_N is an N dimensional finite element space with basis $\{\phi_j\}_{j=1}^N$, then solving the Helmholtz problem at the frequency ω_k with the finite element method is equivalent to solving the linear system

$$(K - \omega_k^2 M + i\omega_k Q)p_h = G_{\omega_k}, \tag{5.13}$$

or for the second order absorbing boundary condition,

$$(K - \omega_k^2 M + i\omega_k Q - \frac{i}{2\omega_k} B)p_h = G_{\omega_k}, \tag{5.14}$$

where the matrices K, M, Q, B and G_{ω_k} are defined as follows:

$$K_{ij} = \int_{\Omega_f} \nabla\phi_j \cdot \nabla\phi_i dx, \qquad M_{ij} = \int_{\Omega_f} \phi_j\phi_i dx, \qquad Q_{ij} = \int_\Gamma \phi_j\phi_i ds; \tag{5.15}$$

$$B_{ij} = \int_\Gamma \nabla_\Gamma\phi_j\nabla_\Gamma\phi_i ds, \qquad [G_{\omega_k}]_j = \int_{\Omega_f} \hat{G}(\underset{\sim}{x}, \omega)\phi_j dx. \tag{5.16}$$

The matrices K, M, Q, and B only depend on the basis $\{\phi_j\}_{j=1}^N$, and not on the frequency ω_k. Moreover, if we assume that the source function $G(\underset{\sim}{x}, t)$ is of the form

$$G(\underset{\sim}{x}, t) = g(t)f(\underset{\sim}{x}), \tag{5.17}$$

which is the case for most applications, then $\hat{G}(\underset{\sim}{x}, \omega) = \hat{g}(\omega)f(\underset{\sim}{x})$ and

$$G_{\omega_k} = \hat{g}(\omega_k)G,$$

where

$$G_j = \int_{\Omega_f} f(\underset{\sim}{x})\phi_j dx.$$

The linear systems (5.13) and (5.14) can be rewritten as:

$$(K - \omega_k^2 M + i\omega_k Q)p_h = \hat{g}(\omega_k)G, \tag{5.18}$$

and

$$(K - \omega_k^2 M + i\omega_k Q - \frac{i}{2\omega_k}B)p_h = \hat{g}(\omega_k)G \tag{5.19}$$

The matrices K, M, Q, B, and F are then independent of ω. After choosing Ω_f and determining the basis $\{\phi_j\}_{j=1}^N$, the matrices only need to be assembled once, and can then be used to solve the Helmholtz problem for each frequency in the set $\{\omega_j\}$.

5.3 Numerical Experiments

In this section, we will implement the procedure outlined in Section 5.2 in order to approximate solutions to (1.3) and (5.2). Our eventual goal is to provide numerical evidence for which boundary condition better approximates a non-reflecting absorbing boundary condition, that is, which more accurately simulates the absence of a physical boundary. To that end, we will approximate solutions to both (1.3) and (5.2) using the same source function, and compare the reflections generated by the resulting wavefronts.

Because we truncate the Fourier inversion, it is important to choose the source function $G(\underset{\sim}{x}, t)$ so that $\hat{G}(\underset{\sim}{x}, \omega) \longrightarrow 0$ rapidly as $\omega \rightarrow \infty$. Following the example in [23], we chose

$$G(x,t) = g(t)f(\underset{\sim}{x}),$$

where

$$g(t) = \begin{cases} 2\zeta(t-t_0)exp(-\zeta(t-t_0)^2) & t \geq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{5.20}$$

Figures 5.1 and 5.2 show graphs of the time component of the source function and the absolute value of its Fourier transform, respectively. Note that $|\hat{g}(\omega)| \to 0$ rapidly as $\omega \to \infty$. Based on the profile of $|\hat{g}(\omega)|$, we chose evenly spaced frequencies in the interval $[0, 25]$, with $\Delta\omega = .1$. We then recovered the solution in the time domain by truncating the Fourier inversion integral at $\omega^* = 25$ and applying the composite Simpson's rule,

$$
\begin{aligned}
P(x,t) &= \frac{1}{\pi}\mathrm{Re}\int_0^\infty p(x,\omega)e^{i\omega t}d\omega \approx \frac{1}{\pi}\mathrm{Re}\int_0^{\omega^*} p(x,\omega)e^{i\omega t}d\omega \\
&\approx \frac{1}{\pi}\frac{\Delta\omega}{3}\mathrm{Re}\left\{ \hat{u}(x,\omega_0)e^{i\omega_0 t} + 2\sum_{k=1}^{N/2}\hat{u}(x,\omega_{2k})e^{i\omega_{2k}t} \right. \\
&\qquad \left. +4\sum_{k=1}^{N/2} p(x,\omega_{2k-1})e^{i\omega_{2k-1}t} + p(x,\omega_{2N})e^{i\omega_{2N}t} \right\}.
\end{aligned}
\tag{5.21}
$$

To approximate the solution of the appropriate Helmholtz problem at each frequency, we used the finite element space of piecewise linear C^0 functions, and used Gaussian elimination to solve the systems (5.18) and (5.19) for p_h.

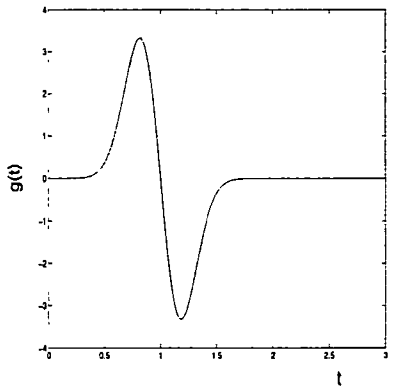

Figure 5.1: $g(t)$

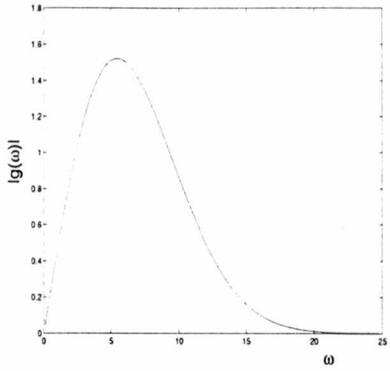

Figure 5.2: $|\hat{g}(\omega)|$

In the first experiment, we chose $\Omega_f = [0,4] \times [0,4]$ as the spatial domain and partitioned Ω_f into a regular triangular mesh (see Figure 5.3) with mesh size $h = .0625$. We used $f(\underset{\sim}{x}) = \delta_{\underset{\sim}{x}-\underset{\sim}{x}_0} - \delta_{\underset{\sim}{x}-\underset{\sim}{x}_1}$ as the source, where $\underset{\sim}{x}_0 = (2, .06)$, $\underset{\sim}{x}_1 = (2, .04)$. Figures 5.4 through 5.13 show snapshots of the approximate solution generated with both the first and the second order absorbing boundary condition at $t = 3, 4, 5, 6$ and 7.

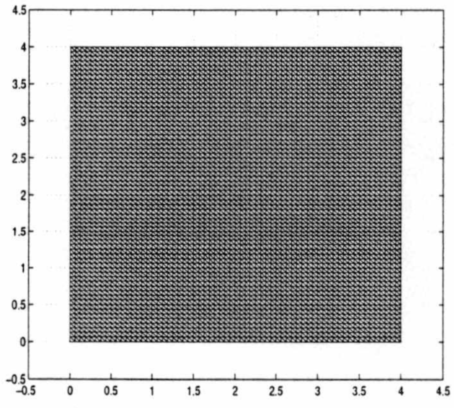

Figure 5.3: Regular Triangular Mesh of Ω_f, $h = .0625$

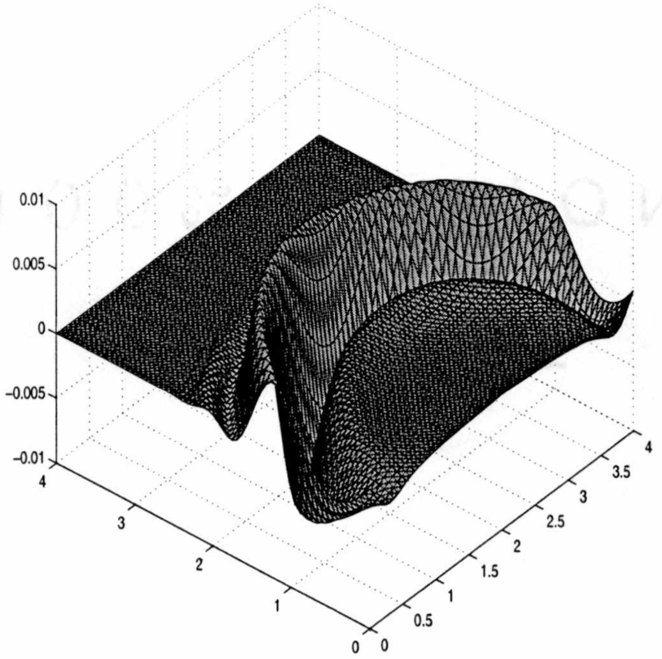

Figure 5.4: Wave profile, first order absorbing boundary condition $t = 3$

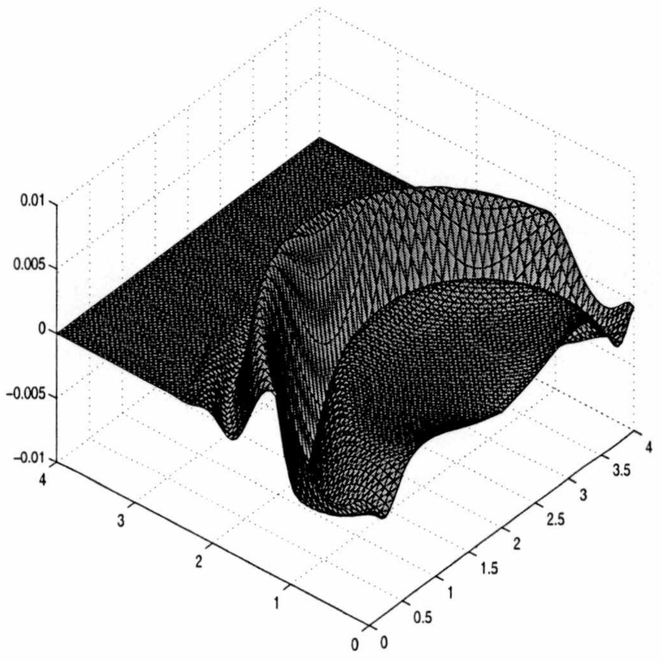

Figure 5.5: Wave profile, second order absorbing boundary condition $t = 3$

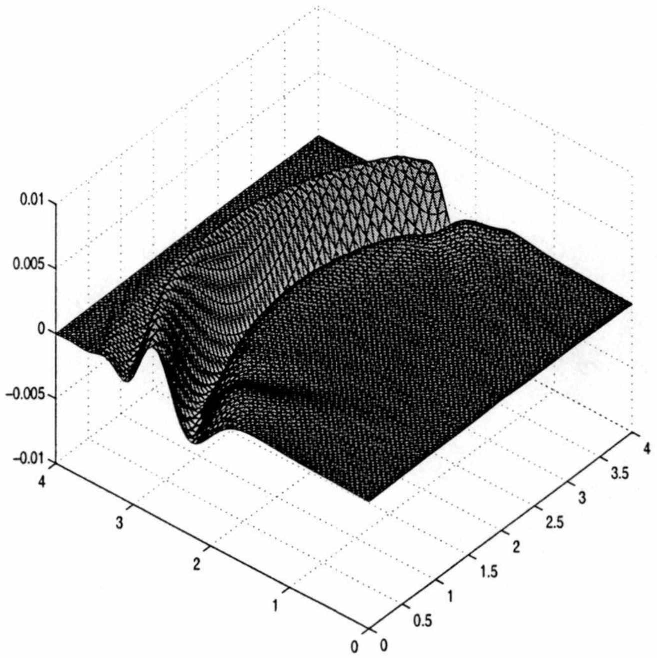

Figure 5.6: Wave profile, first order absorbing boundary condition $t = 4$

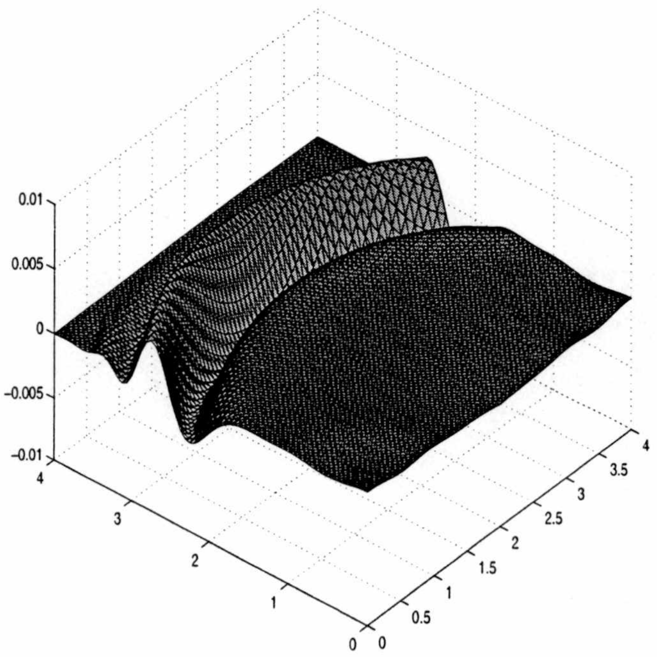

Figure 5.7: Wave profile, second order absorbing boundary condition $t = 4$

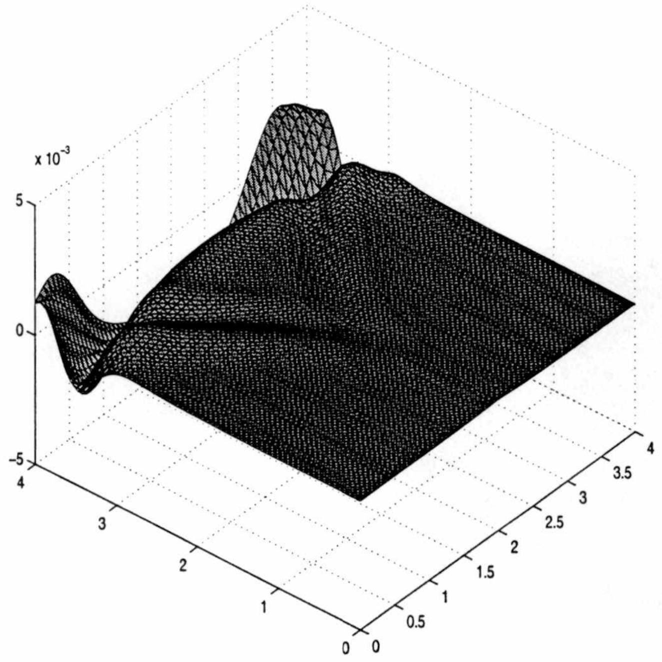

Figure 5.8: Wave profile, first order absorbing boundary condition $t = 5$

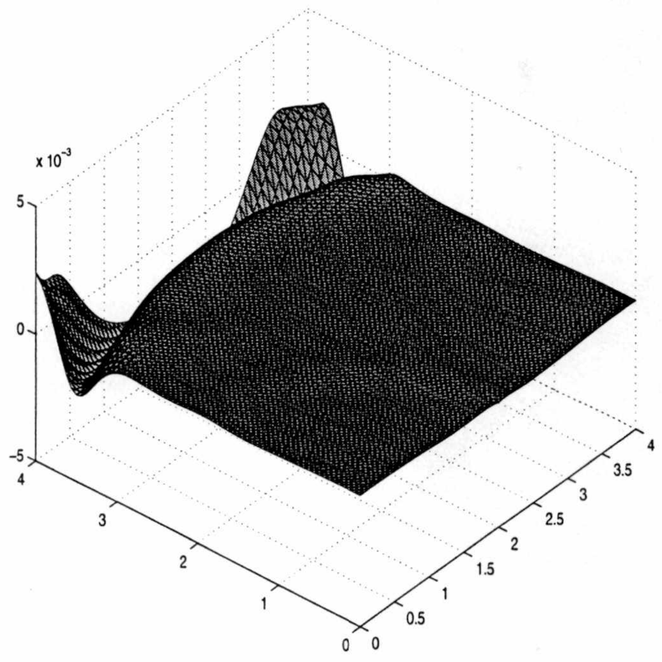

Figure 5.9: Wave profile, second order absorbing boundary condition $t = 5$

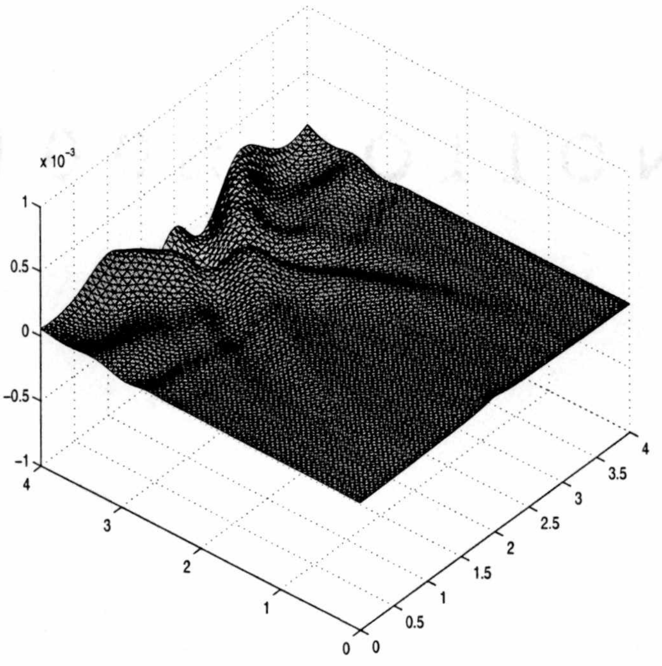

Figure 5.10: Wave profile, first order absorbing boundary condition $t = 6$

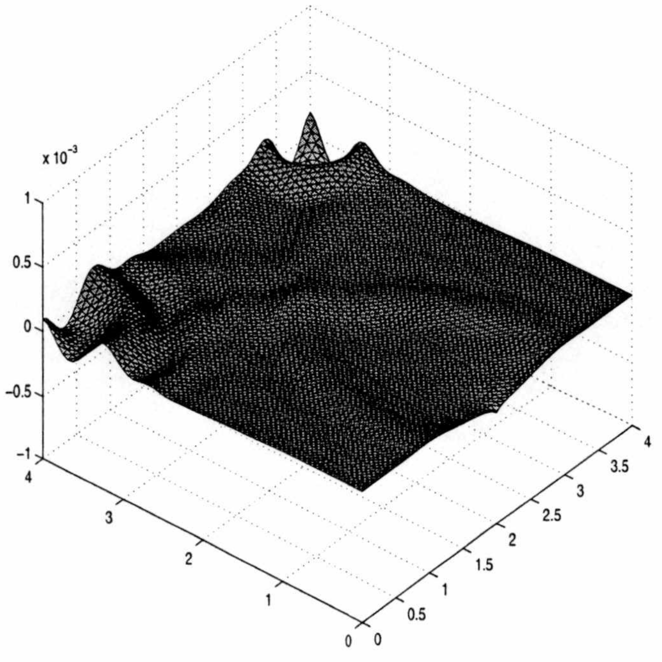

Figure 5.11: Wave profile, second order absorbing boundary condition $t = 6$

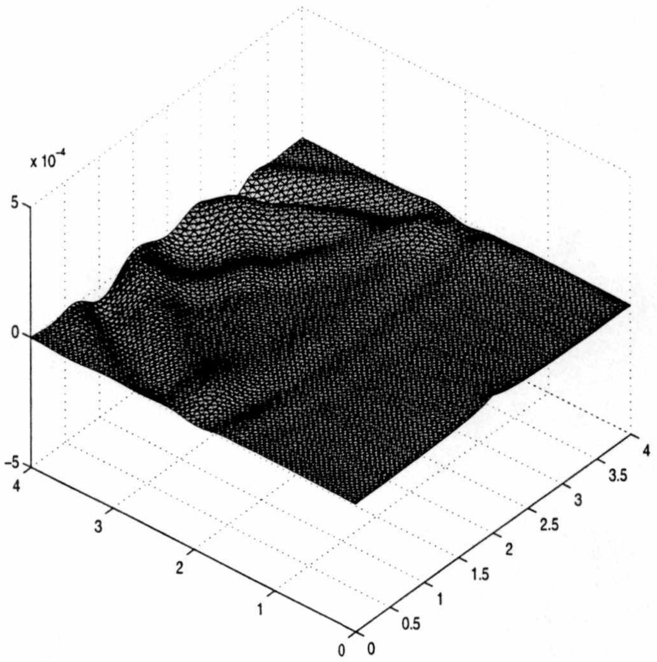

Figure 5.12: Wave profile, first order absorbing boundary condition $t = 7$

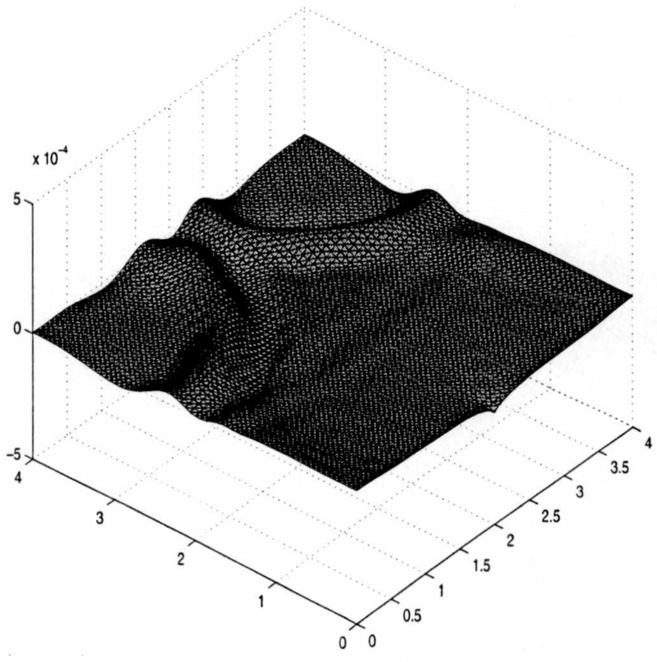

Figure 5.13: Wave profile, second order absorbing boundary condition $t = 7$

In the second experiment, we took a more rigorous approach to comparing reflections. To determine the reflection quantitatively, we applied the following general procedure

(1) Choose $\check{\Omega}_f$ and a source function $G(x,t)$ with property (5.17).

(2) Find $\check{p}$, the solution to the wave equation on $\check{\Omega}_f$

(3) Enlarge $\check{\Omega}_f$ to a domain that contains $\partial\check{\Omega}_f$ in its interior; call this domain Ω_f.

(4) Find p, the solution to the wave equation on Ω_f

(5) For a given value of t, calculate $p(x,t)|_{\check{\Omega}_f} - \check{p}(x,t)$. This is the reflected part of $\check{p}$ at time t.

This procedure should yield a reasonable approximation of the reflected part of $\check{p}$ provided the wave $\check{p}$ has not struck $\partial\Omega_f$ and reflected back into $\check{\Omega}_f$.

For this experiment, we chose $\check{\Omega}_f = [1,3]\times[1,3]$ and $\Omega_f = [0,4]\times[0,4]$ and placed the source at the center, $(2,2)$. We partitioned the domains into regular triangular meshes with mesh sizes $h = .125$ and $h = .0625$. We were careful to partition the domains so that each mesh point of $\check{\Omega}_f$ was also a mesh point of Ω_f. Tables 5.1 and 5.2 compare the L^2 and L^∞ norms of the reflected waves generated in step 5 above. Figures 5.14 through 5.17 show snapshots of the solution on Ω_f at times $t = 1.6, 1.8, 2.0$ and 2.2. Figures 5.18, 5.20 and 5.22 show the reflected part of the wave generated with the first order absorbing boundary condition at times $t = 1.6, 1.8$ and2; figures 5.19, 5.21 and 5.23 show the reflected part of the wave generated with the second order absorbing boundary condition at times $t = 1.6, 1.8$ and 2. In all figures, the mesh size is $h = .025$.

Comparing figures 5.18, 5.20 and 5.22 to figures 5.19, 5.21 and 5.23, it is clear that reflections generated by the second order absorbing boundary condition are smaller than those generated by the first order absorbing boundary condition. The second order boundary condition therefore better approximates a non–reflecting boundary condition, that is, the absence of a physical boundary. It is also worth noting that using the second order boundary condition does not require much additional computational overhead. The only additional calculations that are required are assembling the matrix B from (5.16), and an additional matrix addition when assembling the system (5.19). As tables 5.1 and 5.2 indicate, however, the reflections for both boundary conditions begin to grow as t gets large. Therefore, when one needs a simulation which is long relative to the wave speed, one must use a larger computational domain.

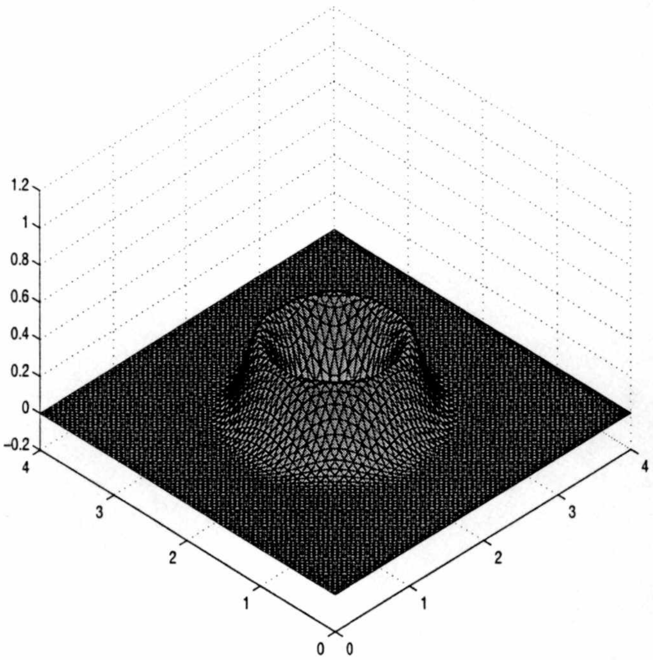

Figure 5.14: Wave profile, first order absorbing boundary condition $t = 1.6$

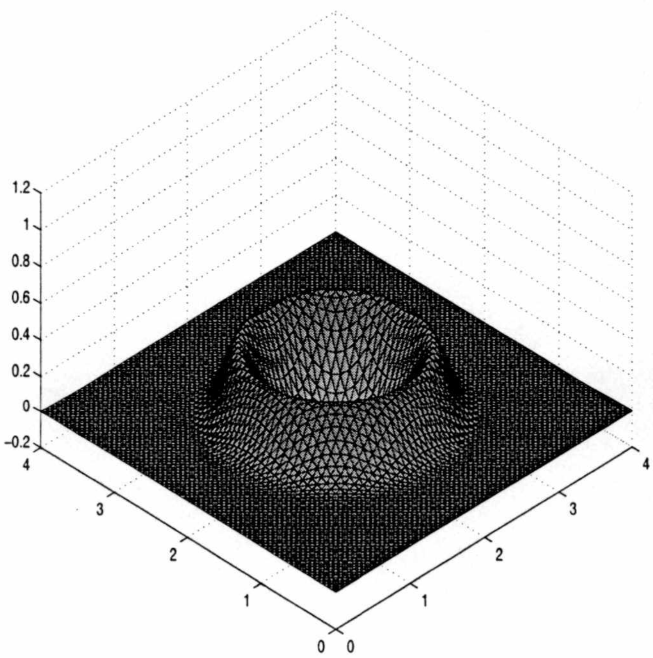

Figure 5.15: Wave profile, first order absorbing boundary condition $t = 1.8$

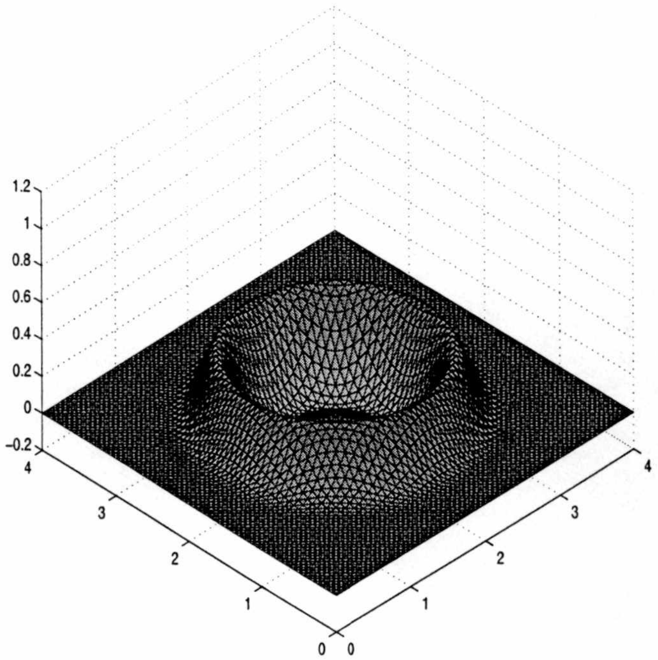

Figure 5.16: Wave profile, first order absorbing boundary condition $t = 2.0$

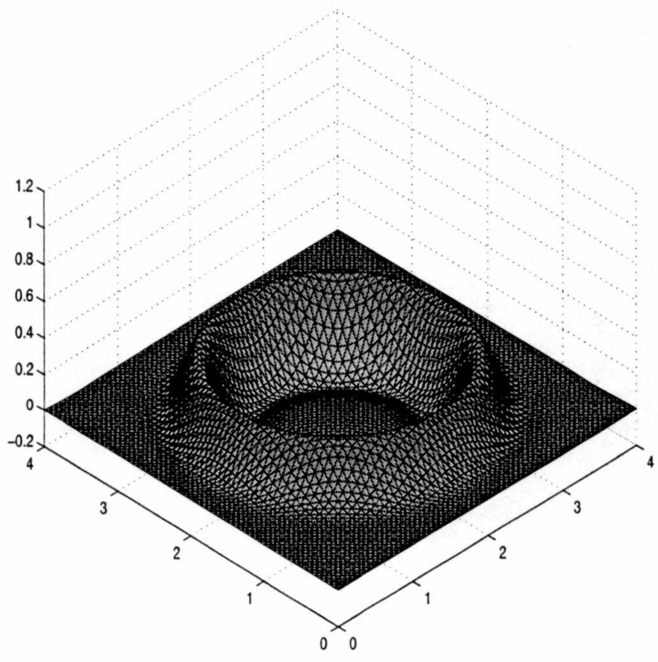

Figure 5.17: Wave profile, first order absorbing boundary condition $t = 2.2$

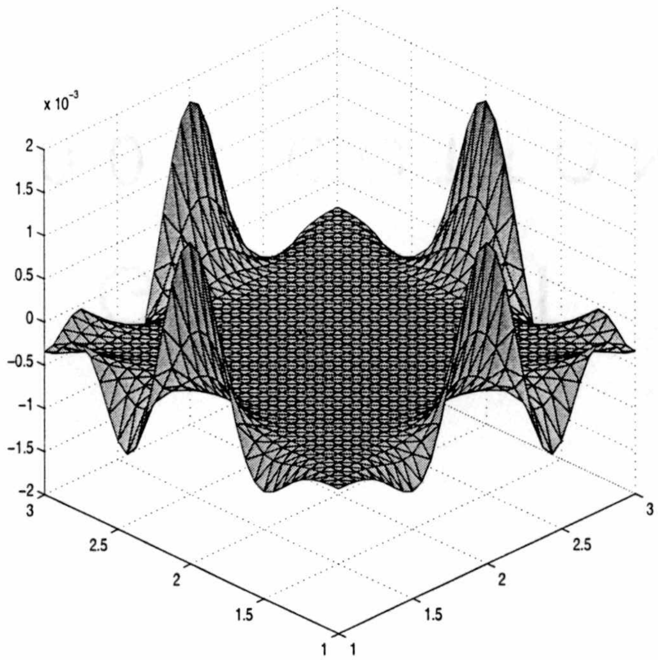

Figure 5.18: Reflected waves, first order absorbing boundary condition $t = 1.6$

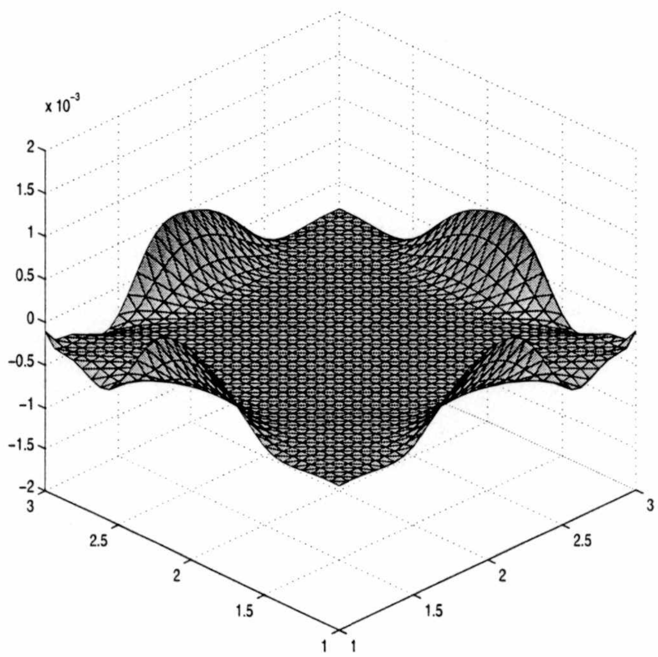

Figure 5.19: Reflected waves, second order absorbing boundary condition $t = 1.6$

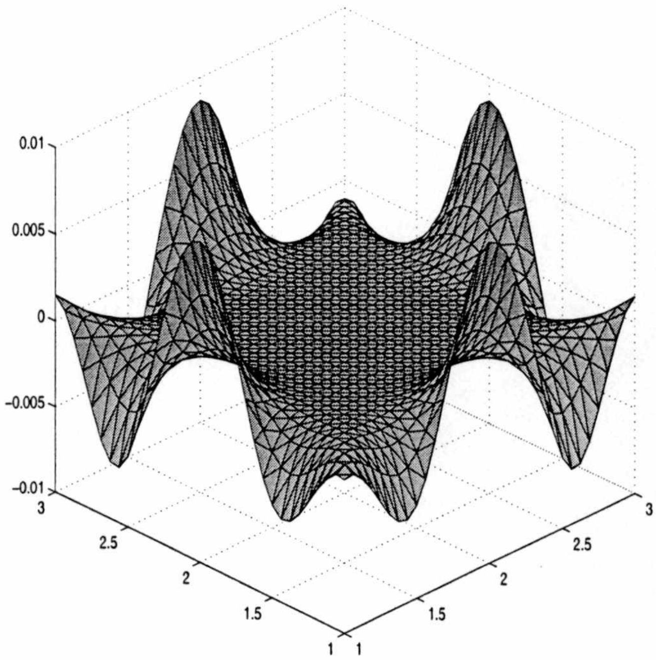

Figure 5.20: Reflected waves, first order absorbing boundary condition $t = 1.8$

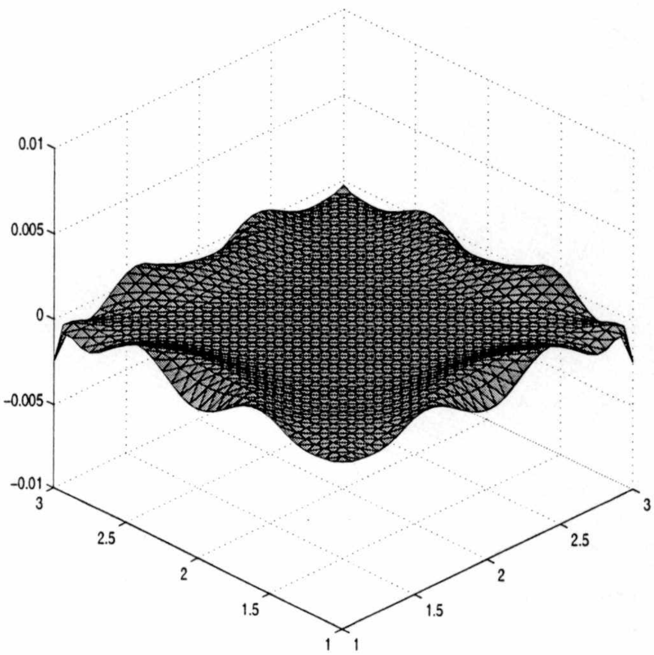

Figure 5.21: Reflected waves, second order absorbing boundary condition $t = 1.8$

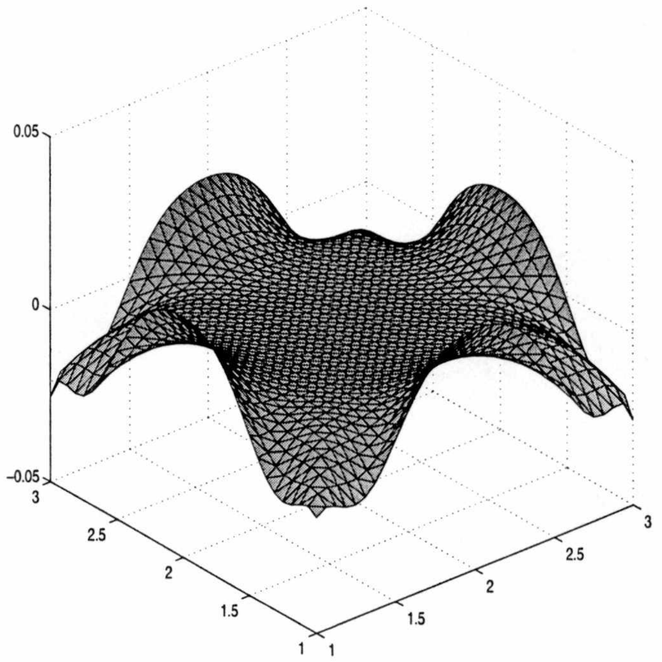

Figure 5.22: Reflected waves, first order absorbing boundary condition $t = 2$

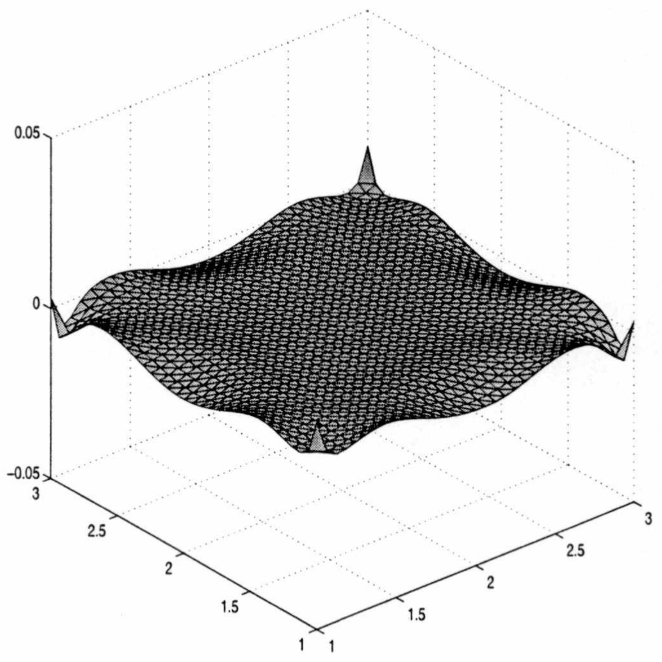

Figure 5.23: Reflected waves, second order absorbing boundary condition $t = 2$

Table 5.1: Norms of reflected waves, $h = .125$

	L^2-norm		L^∞-norm	
t	ABC1	ABC2	ABC1	ABC2
1.3000	0.0060	0.0062	0.0010	0.0013
1.3500	0.0069	0.0070	0.0018	0.0017
1.4000	0.0078	0.0073	0.0025	0.0016
1.4500	0.0085	0.0088	0.0022	0.0016
1.5000	0.0109	0.0143	0.0026	0.0054
1.5500	0.0176	0.0223	0.0048	0.0097
1.6000	0.0272	0.0278	0.0108	0.0111
1.6500	0.0365	0.0274	0.0157	0.0059
1.7000	0.0425	0.0289	0.0158	0.0077
1.7500	0.0458	0.0478	0.0087	0.0278
1.8000	0.0546	0.0750	0.0118	0.0476
1.8500	0.0776	0.0953	0.0280	0.0580
1.9000	0.1129	0.0999	0.0494	0.0511
1.9500	0.1537	0.0925	0.0658	0.0304
2.0000	0.1939	0.0970	0.0731	0.0302
2.0500	0.2287	0.1357	0.0702	0.0730
2.1000	0.2540	0.1970	0.0581	0.1220
2.1500	0.2659	0.2650	0.0613	0.1578
2.2000	0.2615	0.3311	0.0591	0.1747
2.2500	0.2417	0.3901	0.0480	0.1717
2.3000	0.2140	0.4380	0.0296	0.1603
2.3500	0.1957	0.4723	0.0343	0.1609
2.4000	0.2068	0.4939	0.0450	0.1426

Table 5.2: Norms of reflected waves, $h = .0625$

	L^2-norm		L^∞-norm	
t	ABC1	ABC2	ABC1	ABC2
1.3000	0.0100	0.0101	0.0003	0.0003
1.3500	0.0100	0.0101	0.0003	0.0003
1.4000	0.0100	0.0100	0.0004	0.0003
1.4500	0.0099	0.0099	0.0004	0.0004
1.5000	0.0099	0.0098	0.0005	0.0004
1.5500	0.0106	0.0097	0.0009	0.0004
1.6000	0.0131	0.0098	0.0017	0.0005
1.6500	0.0187	0.0101	0.0029	0.0006
1.7000	0.0286	0.0104	0.0044	0.0007
1.7500	0.0437	0.0108	0.0063	0.0008
1.8000	0.0652	0.0127	0.0087	0.0024
1.8500	0.0938	0.0179	0.0114	0.0052
1.9000	0.1304	0.0260	0.0144	0.0074
1.9500	0.1765	0.0345	0.0173	0.0067
2.0000	0.2337	0.0440	0.0254	0.0106
2.0500	0.3016	0.0639	0.0398	0.0220
2.1000	0.3753	0.1079	0.0527	0.0504
2.1500	0.4433	0.1801	0.0607	0.0843
2.2000	0.4903	0.2765	0.0606	0.1163
2.2500	0.5018	0.3893	0.0582	0.1387
2.3000	0.4720	0.5084	0.0505	0.1458
2.3500	0.4130	0.6229	0.0338	0.1375
2.4000	0.3665	0.7236	0.0233	0.1344

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