Solving Ordinary Differential Equations Using Differential Forms and Lie Groups

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A Senior Thesis submitted in partial fulfillment
of the requirements for graduation
in the Honors Program
Liberty University
Spring 2015

Acceptance of Senior Honors Thesis

This Senior Honors Thesis is accepted in partial fulfillment of the requirements for graduation from the Honors Program of Liberty University.

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Abstract

Differential equations have bearing on practically every scientific field. Though they are prevalent in nature, they can be challenging to solve. Most of the work done in differential equations is dependent on the use of many methods to solve particular types of equations. Sophus Lie proposed a modern method of solving ordinary differential equations in the 19th century along with a coordinate free variation of finding the infinitesimal generator by combining the influential work of Élie Cartan among others in the field of differential geometry. The driving idea behind using symmetries to solve differential equations is that there exists a coordinate system for any given differential equation such that the equation can be directly integrated within the newfound system.

Solving Ordinary Differential Equations Using Differential Forms and Lie Groups

At the first encounter to Ordinary Differential Equations (ODEs), the undergraduate student is flooded with numerous methods to solve an ODE. Some of the most familiar, and simplistic, types of ODEs are of the separable, first order linear varieties, and Bernoulli, among many others. The number of methods to solve these ODEs is proportional to the number of different types of equations. These specific types of equations are easy to solve because, as will be shown, they admit a symmetry group. Up until the work of Sophus Lie, a Norwegian mathematician of the 19th century, the correlation between these methods was not well known. The ability of one to solve any given ODE was dependent on his or her ability to remember the necessary method. He discovered a general method to solve any given ODE based on the equations' inherent group of symmetries, which was later called a Lie Group (Ibragimov, 1999).

Toward the end of Lie's life, a French mathematician by the name of Élie Cartan did momentous work on Lie groups. He popularized the idea of a differential p-form, a concept foundational to modern Differential Geometry, and this notion simplified the idea of Lie groups. He went on to complete many well-known works on combinations of these two ideas, both of which are essential to finding symmetries ODEs in this more modern, coordinate-free way (Cartan, 1970).

By combining the work of these two men, one can solve any given ODE that admits symmetry with an algorithmic approach. First, one must write the given ODE as a system of differential forms by making the necessary variable substitutions. Then, through the methods developed by these two brilliant mathematicians, one can find the Lie group of symmetries of the system of differential forms by the means of the infinitesimal generator. Once this group of symmetries is found, a simpler symmetrical solution can be developed that correlates directly with the original solution. By means of translation, the symmetrical solution can then be used to find the original solution. This method makes no concern out of the particular type of ODE, and through using differential forms, the symmetry can be found without the use of coordinates.

Topological Foundation of Smooth Manifolds

The rigorous formulation of differential forms and vectors is essential to a proper understanding of the methodology used by Sophus Lie to determine the symmetry groups of differential equations. If the reader desires an application-based approach, proceed to the section on Symmetries.

Differential forms and vectors exist on a mathematical space, known as a manifold, which has a particular structure that allows for calculus. Included in this section is a brief discussion on the properties of manifolds that is heavy with definitions as well as what the requirements of a manifold are such that it is considered smooth. This discussion simply lays the foundation for the nature of differential forms, Lie groups, and vectors that are necessary to solve ODEs using Lie groups and differential forms (Conlon, 1993).

Topology

A *topology* on a set X is a collection $\mathcal{T}$ of subsets of X such that

1. $\emptyset$ and X are in $\mathcal{T}$.
2. The union of the elements of any subcollection of $\mathcal{T}$ is in $\mathcal{T}$.
3. The intersection of the elements of any finite subcollection of $\mathcal{T}$ is in $\mathcal{T}$.

For an example, given a set $X = \{a, b, c\}$, four possibilities for a topology could be $\mathcal{T} = \{\emptyset, X\}$, which is the trivial topology, $\mathcal{T} = \{\emptyset, \{c\}, X\}$, $\mathcal{T} = \{\emptyset, \{b, c\}, \{b\}, \{c\}, X\}$, or $\mathcal{T} = \{\emptyset, \{a, b\}, \{b\}, \{b, c\}, X\}$. These examples are only a few of many possibilities. On the other hand, $\mathcal{T} = \{\emptyset, \{a, b\}, \{b, c\}, X\}$ would not be an acceptable topology because $\{a, b\} \cap \{b, c\} = \{b\} \notin \mathcal{T}$. Neither would $\mathcal{T} = \{\emptyset, \{a\}, \{b\}, \{c\}, X\}$ be an acceptable topology since $\{a\} \cup \{b\} = \{a, b\} \notin \mathcal{T}$. See Figure 1.

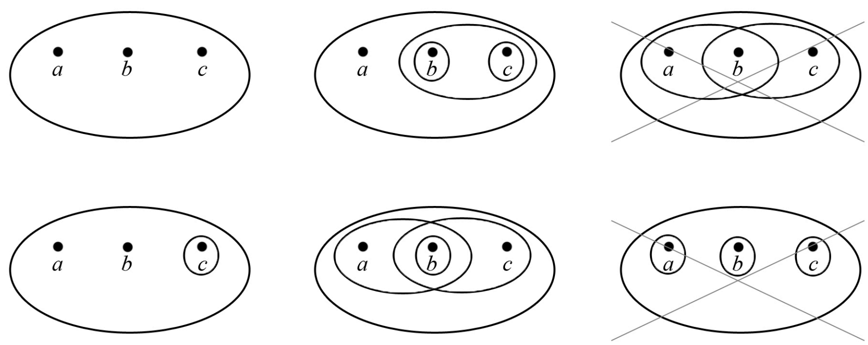

Figure 1. Examples and nonexamples of topologies of the set X.

Much of what is done with differential equations will be done in $\mathbb{R}^n$. The *standard topology* of $\mathbb{R}^n$ is the set of all open n-balls,

$$B_\epsilon(x) = \{v_n \in \mathbb{R}^n || v_n - x| < \epsilon, \epsilon \in \mathbb{R}^+ \cup \{0\}\}$$

centered at each point $x \in \mathbb{R}^n$ (Munkres, 2000).

A topological space X is *locally Euclidean* if $\forall x \in X \; \exists n \geq 0$ where $n \in \mathbb{Z}$, an open neighborhood $U \subseteq X$ of x, denoted $U_\epsilon(x)$, an open subset $\widetilde{U} \subseteq \mathbb{R}^n$, and a homeomorphism $f: \widetilde{U} \to W$. Figure 2 shows how such a homeomorphism takes a 2 dimensional manifold to the Euclidean plane. These homeomorphisms, as will later be discussed, are a key difference that separates a general topological manifold from a smooth manifold. Euclidean n-space is the set $\mathbb{R}^n$ coupled with the standard metric (Auslander & Mackenzie, 1977).

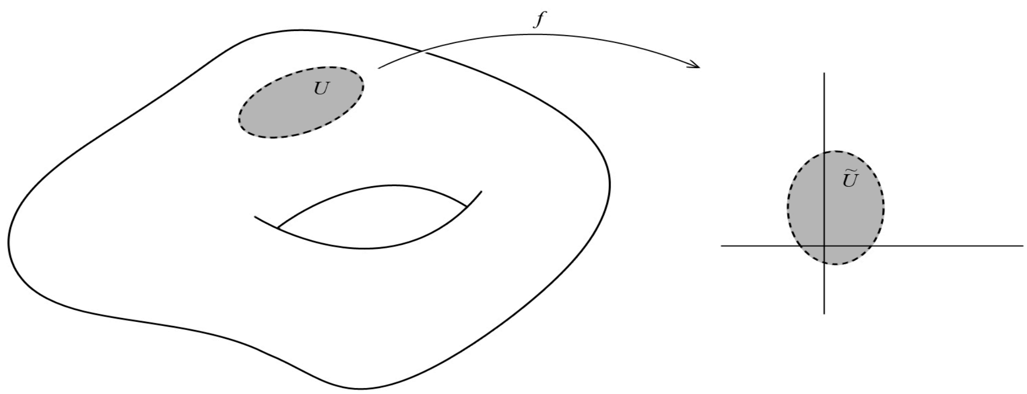

Figure 2. Locally Euclidean.

A topological space X is *Hausdorff* if for each pair a, b of distinct points in X, there exists $U_\epsilon(a)$ and $U_\delta(b)$ such that $U_\epsilon(a) \cup U_\delta(b) = \emptyset$. This property gives each point separation within the topological space. Since the idea of a space being Hausdorff is most likely an unfamiliar concept, next is a lemma that relates a familiar idea of a metric space to a Hausdorff space.

Lemma 1: If X is a metric space, then it is Hausdorff

> Let the metric space X consist of a set A with a metric d. Let $a, b \in A$ such that $a \neq b$. Hence, $d(a, b) > 0$. Now choose $\epsilon = \frac{d(a,b)}{2}$ and let $U_\epsilon(a)$ and $U_\epsilon(b)$ be open balls centered about a and b respectively with radius ϵ. Suppose $U_\epsilon(a)$ and $U_\epsilon(b)$ are not disjoint $\Longrightarrow \exists c \in A$ such that $c \in U_\epsilon(a)$ and $c \in U_\epsilon(b)$. Then $d(a, c) < \epsilon$ and $d(b, c) < \epsilon$. Hence, $d(a, c) + d(b, c) < 2\epsilon = d(a, b)$. This contradicts the definition of a metric, so it must be the case that $U_\epsilon(a)$ and $U_\epsilon(b)$ are disjoint. Therefore, X is Hausdorff. ■

Knowing that any metric space is Hausdorff is a powerful idea. If a given topological space has a notion of distance, it is inherently Hausdorff (Warner, 1971).

A space X is said to have a *countable basis* at $a \in X$ if there is a countable collection C of neighborhoods of a such that each neighborhood of a contains at least one of the elements of C. A space X is second-countable if it has a countable basis for every point in its topology (Munkres, 2000).

Manifolds

A topological space M is a manifold of dimension n if

1. M is locally Euclidean and $\dim M = n$
2. M is Hausdorff
3. M is second-countable.

On a manifold, the homeomorphisms along with the region that they may are referred to as a *chart* and a collection of charts is an *atlas*. Though a manifold must locally resemble Euclidean space, it need not resemble it globally. Take for example the unit circle. The Hausdorff and second-countable properties are inherent from $\mathbb{R}^2$, so it is the locally Euclidean aspect that is of interest. Obviously, a circle is not globally Euclidean, but it is possible to find a proper atlas that shows it is locally Euclidean. One can map the positive y portion of the circle to $\mathbb{R}$, as well as the negative y portion, positive x portion, and negative x portion. These four charts *cover* the entire unit circle and form an atlas. This, of course is a simple example of a manifold. The *dimension* of the Euclidean space that the atlas maps to defines a manifold's dimension. As the dimension of the manifold increases, so does the complexity of its atlas. In this example, the unit circle is a 1-dimensional manifold.

A manifold M is considered smooth if given any two charts in its atlas $(f, U_1), (g, U_2)$ the composition $f \circ g^{-1}$ and $g \circ f^{-1}$ are smooth functions where $g \circ f^{-1}: f(U_1 \cap U_2) \to g(U_1 \cap U_2)$ and $f \circ g^{-1}: g(U_1 \cap U_2) \to f(U_1 \cap U_2)$. To visualize how these composition mappings work, see Figure 3 (Kosinski, 1993).

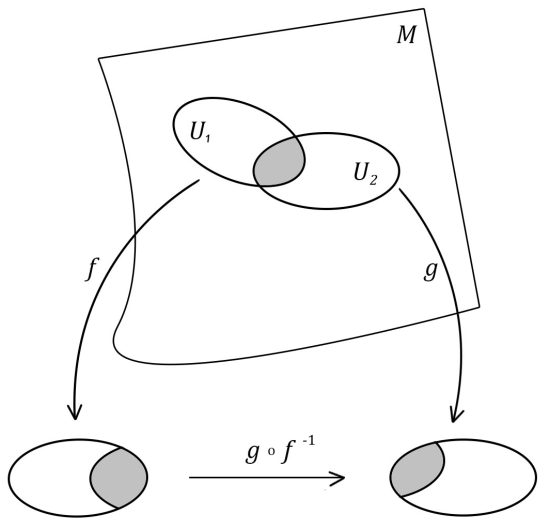

Figure 3. Composition of mappings.

A function is of class C^k if the derivatives $f', f'', \ldots f^{(k)}$ exist and are continuous on the function's domain. A function is of class C^∞, or smooth, if it has derivatives of all orders that are continuous. For the composition functions of a manifold to be smooth, they must be infinitely differentiable. Smooth manifolds are a natural place for the existence of differential forms and vectors. Not only do they provide the necessary differential structure required by these two objects but it does so independent of any specific coordinates. It may be the case that a particular atlas does not permit a smooth structure. To tell if a particular manifold has such an atlas, one can adjoin all possible charts into an atlas, called the *maximal atlas*. This particular atlas is unique to a manifold. If a manifold permits a smooth structure, this maximal atlas will do so. Finding such an atlas will more than likely not be practical, it is rarely used directly in calculations.

There exist many common examples of smooth manifolds. Some well-known examples, with charts omitted for brevity, include

- $\mathbb{R}^n$, the standard Euclidean plane.
- $\mathcal{S}^n = \{(x_1, x_2, \dots, x_{n+1}) \in \mathbb{R}^{n+1} | x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = 1\}$, the unit n-sphere.
- $M \times N$, the Cartesian product of two smooth manifolds.
- U an open subset of a smooth manifold.

These familiar spaces can be paired with an atlas of Diffeomorphisms (Arvanitoyeorgos, 1999).

Lie Groups

A Lie Group is the marriage between a smooth manifold and a group. It inherits the properties of both, the ability to define calculus from the manifold aspect, and symmetry from the group aspect. As Belinfante and Kolman (1972) said, "Basically, a Lie group is the structure which naturally results when analytic machinery is coupled with abstract group theory" (p. 14).

Groups

Before addressing the specific properties of Lie Groups, it is important to understand the general notion of groups and subgroups. A group is a collection of elements, X, paired with a binary operation, $*$, that has the following properties:

1. Binary operation: If $a, b \in X$ then $a * b \in X$;

2. Associativity: If $a, b, c \in X$ then $a * (b * c) = (a * b) * c$;
3. Existence of identity: There exists an identity element $e \in X$ such that $e * a = a * e = a$ for all $a \in X$;
4. Existence of inverses: Given any $a \in X$, there exists $a^{-1} \in X$ such that $a * a^{-1} = a^{-1} * a = e$.

The idea of groups is first applied in elementary school, when students are taught to add. $\mathbb{Z}$ forms a group under addition, where the identity element is 0 and any element a has an inverse $a^{-1} = -a$ (Fraleigh, 2003).

Lie Groups and Lie Subgroups

A *Lie group* is a smooth manifold as well as a group where the operation $(a, b) \mapsto a * b$ is smooth and the inverse function $a \mapsto a^{-1}$ is also smooth (Varadarajan, 1984). To prove a set is a Lie group takes a significant amount of work. There are some important lemmas that make proving a set paired with a specific operation is a Lie group much easier.

Lemma 2: Any open subset of a n-dimensional smooth manifold or n-dimensional vector space is a n-dimensional smooth manifold.

Let M be a smooth n-dimensional manifold with atlas $A = \{(U_a, \varphi_a)\}$ where $\varphi_a: U_a \to \mathbb{R}^n$ and is smooth. Now let $X \subseteq M$. $A_X = \{(U_a \cap X, \phi_a)\}$ is at atlas for X where $\phi_a: U_a \cap X \to \mathbb{R}^n$ which also must be smooth since its domain is a subset of φ_a.
Let V be a n-dimensional vector space with basis $\{e_i\}_{i=1,2,\dots,n}$. Define $\varphi: V \to \mathbb{R}^n$ by $\varphi(\boldsymbol{x}) = \varphi(x_1e_1 + x_2e_2 + \cdots x_ne_n) = (x_1, x_2, \dots, x_n)$. This function is obviously a homeomorphism and smooth, and therefore $A = \{V, \varphi\}$ is an atlas for V. The proof of any open subset of V is a smooth manifold follows in a similar manner as the first part of this proof. ■

As an example of this idea, consider $GL(n, \mathbb{R}) \subseteq \mathbb{R}^{n\times n}$. Since $\mathbb{R}^{n\times n} \cong \mathbb{R}^{n^2}$ it is a n^2-dimensional vector space. Is $GL(n, \mathbb{R})$ open in $\mathbb{R}^{n\times n}$? First, let the topology of $\mathbb{R}^{n\times n}$ be the set of all open balls of $n\times n$ matrices,

$$B_\epsilon(M) = \{X \in \mathbb{R}^{n\times n} | \|A\| < \epsilon\}$$

where

$$\|A\| = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}|^2}.$$

This is called the Frobenius norm. It can also be written in terms of trace:

$$\|A\| = \sqrt{Tr(AA^T)}.$$

From this point, it should be noted that $det: GL(n, \mathbb{R}) \to (0, \infty)$ is a polynomial of entries of the matrix A and is therefore a continuous map. The inverse mapping of the determinate maps the open interval $(0, \infty) \mapsto GL(n, \mathbb{R})$ which shows that $GL(n, \mathbb{R})$ is the inverse image of an open set and consequently must be open since the determinate is a continuous mapping. Consequently $GL(n, \mathbb{R})$ is a smooth manifold (Bump, 2004).

Lemma 3: If set X is a group under $*$, then $V \subseteq X$ where $V \neq \emptyset$ is a subgroup under $*$ if and only if $a * b^{-1} \in V, \forall a, b \in V$.

> Let $V \leq X$. Thus each element has an inverse and since it is closed under $*$, $a * b^{-1} \in V, \forall a, b \in V$.
> Conversely, since $V \neq \emptyset$, let $v \in V$. Then, $v * v^{-1} \in V$ and consequently, $e \in V$. Now, $e * v^{-1} \in V$, so $v^{-1} \in V$. If $v, w \in V$, then $w^{-1} \in V$ which implies $v * (w^{-1})^{-1} = v * w \in V$. Because X is a group and therefore associative, and since $V \subseteq X$, V must also be associative. Therefore, $V \leq G$. ■

Now, to put *Lemma 3* into practice, show $S(\mathbb{C}) = \{\in \mathbb{C}: |x + iy| = 1\} \subseteq \mathbb{C}$, the "complex unit circle," is a group under complex multiplication. Because $1 + i0 \in \mathbb{C}, \mathbb{C} \neq \emptyset$. Let $a + ib, c + id \in S(\mathbb{C})$. Since $e = 1 + 0i$,

$$(c + id)^{-1} = -\frac{c}{-c^2 - d^2} + \frac{id}{-c^2 - d^2}.$$

To show it is in $S(\mathbb{C})$,

$$\left|-\frac{c}{-c^2 - d^2} + \frac{id}{-c^2 - d^2}\right| = \left(-\frac{c}{-c^2 - d^2}\right)^2 + \left(\frac{id}{-c^2 - d^2}\right)^2 = \frac{1}{c^2 + d^2} = \frac{1}{|c + id|}$$

$$= 1.$$

Now consider

$$(a + ib)(c + id)^{-1} = \frac{ca + bd}{c^2 + d^2} + i\left(\frac{-ad + bc}{c^2 + d^2}\right),$$

which can be shown to be an element of $S(\mathbb{C})$ in a manner similar to showing $(c + id)^{-1} \in S(\mathbb{C})$. Therefore, by Lemma 3 it is a subgroup under complex multiplication (Grillet, 2007).

Symmetries

The idea of symmetries, though often abstract, provides a unique tool for solving differential equations. An object is considered symmetrical if one can impose a certain operation under which the object remains virtually the same. In the case of differential equations, the symmetry is dependent upon the solution–two equations are symmetrical if they are solution invariant (Cantwell, 2002). The idea of solution invariance is important to this discussion but will be expounded upon later. Lie groups apply directly to the idea of symmetry, for a group of symmetries is inherently a Lie group (Ovsiannikov, 1982).

A simple example of this applied to differential equations is the symmetry of the equation

$$\frac{dy}{dx} = 0.$$

Integrating this equation directly gives solutions in the form $y = c$ where c is a constant. As seen from Figure 4, this is simply the set of all horizontal lines. A particular type of symmetry, which will be most important in this discussion, called Lie symmetry, maps a point on a particular solution to a point on a different solution and is dependent on a parameter, ϵ. In this case, the Lie symmetry would be $(x, y) \mapsto (x, y + \epsilon)$.

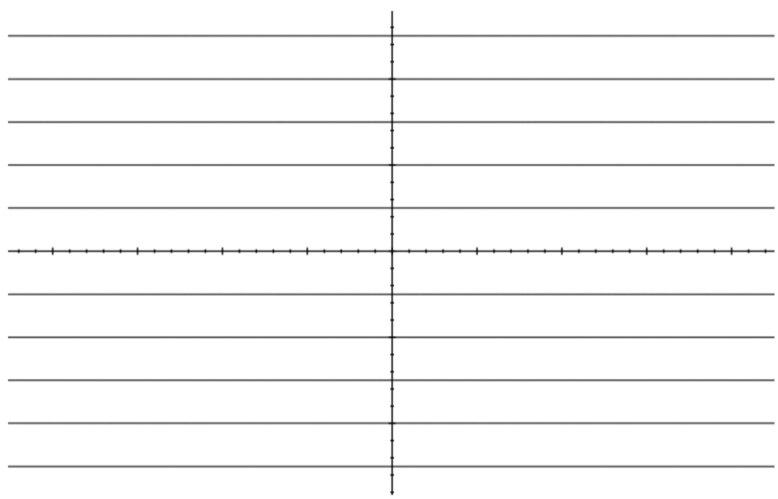

Figure 4. Solutions of $\boldsymbol{y' = 0}$.

In general, the set of transformations

$$\bar{x} = f(x, y, \epsilon) \qquad \bar{y} = g(x, y, \epsilon)$$

depending on $\epsilon \in \mathbb{R}$ is called a *one-parameter group of point transformations* if V contains the identity, $\epsilon = 0$, as well as the inverse of its elements. It also must be the case that if the set of transformations is a Lie group

(LG) and $F \in LG, F: (x, y) \to (f(x, y, \epsilon), g(x, y, \epsilon))$ is injective. This group of transformations forms a *symmetry group*, of a differential equation if the equation is form invariant. It is possible to extend the idea of point transformations to jet transformations, which are make it easier to solve nth-order differential equations by treating the derivatives as variables and transforming them as well. For an nth-order ODE to be solution invariant over a jet transformation, $F\left(\bar{x}, \bar{y}, \bar{y}', \dots, \bar{y}^{(n)}\right) = 0$ when $F\left(x, y, y', \dots y^{(n)}\right) = 0$ - this is called the *symmetry condition* (Bluman & Kumei, 1989). As will soon be shown, one can extend $F \in LG$ from points to jets.

Considering a first order ODE,

$$\frac{dy}{dx} = \Gamma(x, y),$$

a symmetry must satisfy the symmetry condition mentioned. Namely, that

$$\frac{d\bar{y}}{d\bar{x}} = \Gamma(\bar{x}, \bar{y})$$

and since both $\bar{x}$ and $\bar{y}$ are functions of x and y,

$$\frac{d\bar{y}}{d\bar{x}} = \frac{df(x, y, \epsilon)}{dg(x, y, \epsilon)} = \frac{y'\left(\frac{\partial f}{\partial y}\right) + \left(\frac{\partial f}{\partial x}\right)}{y'\left(\frac{\partial g}{\partial y}\right) + \left(\frac{\partial g}{\partial x}\right)} = \frac{\Gamma \partial_y f + \partial_x f}{\Gamma \partial_y g + \partial_x g}.$$

This equation is an extension of the symmetry condition for first-order ODEs and how one can get from points to the first jet.

As an example, consider the group of rotations in the $\mathbb{R}^2$ plane:

$$\bar{x} = x \cos \epsilon - y \sin \epsilon \,; \qquad \bar{y} = x \sin \epsilon + y \cos \epsilon,$$

and show that they are symmetries of the differential equation

$$\frac{dy}{dx} = \frac{-x}{y}.$$

Using the symmetry condition, it must be the case (if indeed it is a symmetry) that

$$\frac{\Gamma \partial_y f + \partial_x f}{\Gamma \partial_y g + \partial_x g} = \Gamma(\bar{x}, \bar{y}).$$

$$\frac{\left(-\frac{x}{y}\right)(\cos \epsilon) + \sin \epsilon}{\left(-\frac{x}{y}\right)(-\sin \epsilon) + \cos \epsilon} = -\frac{\bar{x}}{\bar{y}}$$

$$\frac{-x \cos \epsilon + y \sin \epsilon}{x \sin \epsilon + y \cos \epsilon} = -\frac{x \cos \epsilon - y \sin \epsilon}{x \sin \epsilon + y \cos \epsilon}$$

and since the equality is true, it is the case that this is a symmetry of the given differential equation.

Canonical Coordinates

If an ODE has a symmetry $(x, y) \mapsto (x, y + \epsilon)$, the equation can be solved using integration directly. If an ODE has this symmetry it is necessarily of the form

$$\frac{dy}{dx} = \gamma(x)$$

and can thus be integrated.

Let the ODE $\frac{dy}{dx} = \Gamma(x, y)$ have symmetry $\bar{x} = x$ and $\bar{y} = y + \epsilon$. From the symmetry condition, $F(x, y, y') = y' - \Gamma(x, y)$ and consequently $F(\bar{x}, \bar{y}, \bar{y}') = \bar{y}' - \Gamma(\bar{x}, \bar{y})$. This implies that $\bar{y}' = y' = \Gamma(x, y + \epsilon)$. Differentiating y with respect to ϵ gives

$$\frac{d}{d\epsilon}(y') = \frac{d}{d\epsilon}\big(\Gamma(x, y + \epsilon)\big) = \frac{\partial \Gamma}{\partial x}\frac{dx}{d\epsilon} + \frac{\partial \Gamma}{\partial (y + \epsilon)}\frac{d(y + \epsilon)}{d\epsilon}$$

$$\Rightarrow 0 = \frac{\partial \Gamma}{\partial (y + \epsilon)}$$

so Γ does not depend on the $(y + \epsilon)$ coordinate. Therefore, $\frac{dy}{dx} = \gamma(x)$. ■

It is not common for differential equations to have this type of symmetry in their original coordinates. It is possible to change to a different coordinate system that gives the differential equation a symmetry of this form. The coordinate system that simplifies an ODE is called *canonical coordinates*. Often the new coordinate system is not easy to see, and requires the help of the infinitesimal generator, as will be shown in a later section (Cantwell, 2002).

Consider the differential equation

$$\frac{dy}{dx} = \frac{y^3 + x^2y - x - y}{x^3 + xy^2 - x + y}.$$

Integrating this directly in Cartesian coordinates would require some complicated maneuvering. Given that the canonical coordinates are the standard polar coordinates, this ODE becomes quite simple to integrate using partial fractions

$$\frac{dr}{d\theta} = r(1 - r^2).$$

It can be shown now that this differential equation has symmetry $(\boldsymbol{r}, \boldsymbol{\theta}) \mapsto (\boldsymbol{r}, \boldsymbol{\theta} + \boldsymbol{\epsilon})$. This is a rotational symmetry, similar to the symmetry of the last example. The set of orbits of this differential equation make an interesting design as shown in Figure 5. As is apparent from this figure, one can get from one solution to the next by shifting $\boldsymbol{\theta}$.

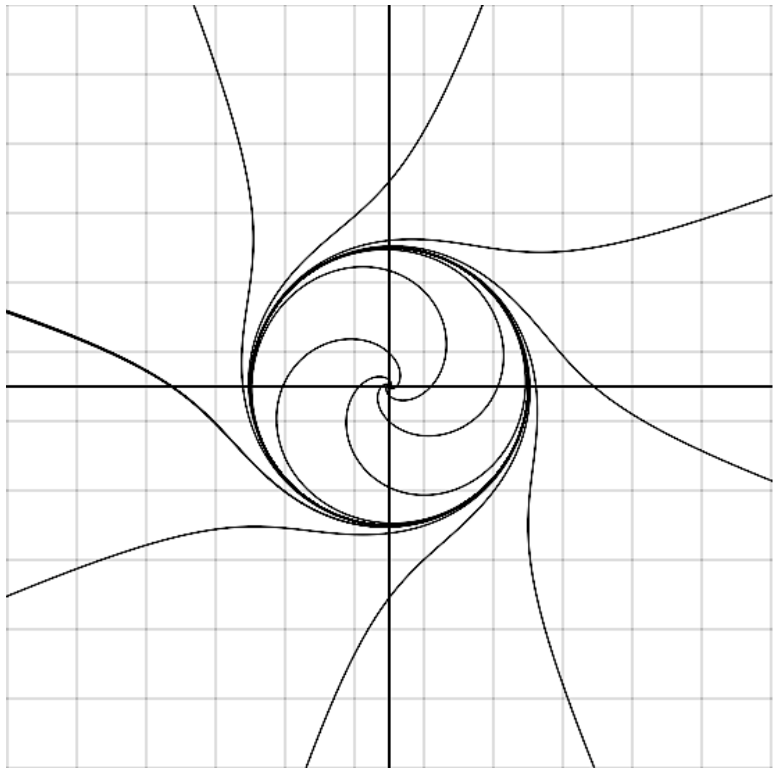

Figure 5. Solutions of $\frac{dy}{dx} = \frac{y^3+x^2y-x-y}{x^3+xy^2-x+y}$.

Orbits

If (x_0, y_0) is a point on a solution curve of a differential equation, the *orbit* of (x_0, y_0) is the set of all points that (x_0, y_0) can be mapped to by ϵ. Under the rotational symmetry and the differential equation

$$\frac{dy}{dx} = -\frac{x}{y}$$

the point (0,1) can be mapped to $\left\{(x,y) \in \mathbb{R}^2 \middle| \sqrt{x^2+y^2} = 1\right\}$ by ϵ. Since this lies entirely on one solution curve to the differential equation, it is called the *trivial symmetry*. On the other hand, the same symmetry applied to

$$\frac{dy}{dx} = 1$$

is not trivial. A symmetry that is trivial for one differential equation may not be trivial for another. That being said, the orbit of a given point under the same symmetry will be the same. The point $(0,1)$ under the same rotational symmetry will still have an orbit $\left\{(x, y) \in \mathbb{R}^2 \middle| \sqrt{x^2 + y^2} = 1\right\}$ (Hydon, 2000).

Symbols

Since the point on a solution curve travels along its orbit as ϵ varies, choosing a coordinate system dependent on the tangents to the orbit at a point would make the coordinate change canonical. However, in general, the tangents, or *symbols*, in the x and y directions respectively are found by, and defined as

$$\frac{\partial \bar{x}}{\partial \epsilon} = \xi(\bar{x}, \bar{y}); \qquad \frac{\partial \bar{y}}{\partial \epsilon} = \eta(\bar{x}, \bar{y}),$$

which can be evaluated at $\epsilon = 0$ (since this deals with an orbit) giving

$$\left.\frac{\partial \bar{x}}{\partial \epsilon}\right|_{\epsilon=0} = \xi(x, y); \qquad \left.\frac{\partial \bar{y}}{\partial \epsilon}\right|_{\epsilon=0} = \eta(x, y).$$

It is the case that sometimes a symmetry is trivial for a certain differential equation. Using the symbols, there is a test that can tell if a symmetry will be trivial. The *characteristic*, G is

$$G = \eta - y'\xi$$

and if the characteristic is zero, then the symmetry is trivial.

To show that the symmetry chosen previously is trivial, first find

$$\eta = x; \quad \xi = -y,$$

which gives the characteristic

$$G = x - \left(-\frac{x}{y}\right)(-y) = 0.$$

If it so happens that a given symmetry has characteristic of zero, if one finds the orbit of a point one has found a particular solution.

Theorem: Let $\bar{x} = f(x, y, \epsilon)$ and $\bar{y} = g(x, y, \epsilon)$ be symmetries of the differential equation $\frac{dy}{dx} = \Gamma(x, y)$ where ξ and η are defined as previously mentioned and $G = \eta - y'\xi$. If $G = 0$ then the curve $\phi(\epsilon) = \big(f(x, y, \epsilon), g(x, y, \epsilon)\big)$ is a solution to the differential equation.

$G = \eta - y'\xi$, so when $G = 0, \eta - y'\xi = 0$. We know from the definition of η and ξ that $\eta = \frac{\partial \bar{y}}{\partial \epsilon} = \frac{d\bar{y}}{d\epsilon}$ and $\xi = \frac{\partial \bar{x}}{\partial \epsilon} = \frac{d\bar{x}}{d\epsilon}$ when x, y are fixed. Thus, $\frac{d\bar{y}}{d\epsilon} - \Gamma(\bar{x}, \bar{y})\frac{d\bar{x}}{d\epsilon} = 0$ which shows that $\bar{x} = \bar{x}(\epsilon)$ and $\bar{y} = \bar{y}(\epsilon)$ are parametric solutions of $d\bar{y} - \Gamma(\bar{x}, \bar{y})d\bar{x} = 0$. This equation of differential forms is equivalent to $\frac{d\bar{y}}{d\bar{x}} = \Gamma(\bar{x}, \bar{y})$. Thus, $\phi(\epsilon)$ is a solution.
■

Using the symbols provides a calculation way to find the canonical coordinates. First, if $(r(x, y), s(x, y))$ are the appropriate canonical coordinates, then the symbols at (r, s) are

$$\left.\frac{\partial \bar{r}}{\partial \epsilon}\right|_{\epsilon=0} = 0; \quad \left.\frac{\partial \bar{s}}{\partial \epsilon}\right|_{\epsilon=0} = 1,$$

since a symmetry of canonical coordinates must be a mapping $(r, s) \mapsto (r, s + \epsilon)$. Since both r and s are dependent on x and y the chain rule applies

$$\left.\frac{\partial \bar{r}}{\partial \epsilon}\right|_{\epsilon=0} = \frac{\partial r}{\partial x}\eta(x, y) + \frac{\partial r}{\partial y}\xi(x, y) = 0$$

$$\left.\frac{\partial \bar{s}}{\partial \epsilon}\right|_{\epsilon=0} = \frac{\partial s}{\partial x}\eta(x, y) + \frac{\partial s}{\partial y}\xi(x, y) = 1.$$

This is a system of first-order partial differential equations (PDEs) and can thus be solved using the *method of characteristics*. A detailed explanation of this method was given by Fritz John (1971) and E.C Zachmanoglou (1976).

Continuing with a previous example, one can derive the canonical coordinates for

$$\frac{dy}{dx} = \frac{y^3 - x^2y - x - y}{x^3 - xy^2 - x + y}$$

given the symmetry. Since $\eta = x$ and $\xi = -y$, the PDEs become

$$\partial_x r\, x - \partial_y r\, y = 0; \quad \partial_x s\, x - \partial_y s\, y = 1$$

and from the method of characteristics, it gives

$$(r, s) = \left(x^2 + y^2, \tan^{-1}\frac{y}{x}\right),$$

which is polar coordinates.

Linearized Symmetry Condition

The examples so far assume the symmetry is given. Though sometimes it may be most practical and efficient to guess and check the symmetry, there is a general method to find a symmetry of a differential equation, called the *linearized symmetry condition* (LSC). The point transformations are symmetry transformations for which

$$\bar{x} = x + \epsilon\left(\left.\frac{\partial\bar{x}}{\partial\epsilon}\right|_{\epsilon=0}\right) + \cdots = x + \epsilon\xi(x, y) + \cdots$$

$$\bar{y} = y + \epsilon\left(\left.\frac{\partial\bar{y}}{\partial\epsilon}\right|_{\epsilon=0}\right) + \cdots = y + \epsilon\eta(x, y) + \cdots.$$

Finding the symmetry of a given ODE can be accomplished by considering the group of transformations that is solution invariant, specifically the symmetry condition. This condition can be solved if one calculates the necessary derivatives of $\bar{y}$. Consider the first derivative, y' and the derivation of its transformation, $\bar{y}'$ given by the symmetry condition

$$\bar{y}' = \frac{d\bar{y}}{d\bar{x}} = \frac{d\bar{y}(x,y,\epsilon)}{d\bar{x}(x,y,\epsilon)} = \frac{y'\left(\frac{\partial \bar{y}}{\partial y}\right) + \left(\frac{\partial \bar{y}}{\partial x}\right)}{y'\left(\frac{\partial \bar{x}}{\partial y}\right) + \left(\frac{\partial \bar{x}}{\partial x}\right)}.$$

Since $y' = f(x,y)$,

$$\bar{y}' = \frac{h(x,y)\left(\frac{\partial \bar{y}}{\partial y}\right) + \left(\frac{\partial \bar{y}}{\partial x}\right)}{h(x,y)\left(\frac{\partial \bar{x}}{\partial y}\right) + \left(\frac{\partial \bar{x}}{\partial x}\right)}.$$

Now, substituting the linear portion of the Taylor expansion into this equation gives,

$$f + \epsilon\left(\partial_x h\xi + \partial_y h\eta\right) = \frac{f + \epsilon(\partial_x \eta + f\partial_y \eta)}{1 + \epsilon(\partial_x \xi + f\partial_y \xi)}$$

and thus

$$\partial_x \eta - \partial_y \xi f^2 + f\left(\partial_y \eta + \partial_x \xi\right) - \xi \partial_x f - \eta \partial_y f = 0.$$

This is the linearized symmetry condition (LSC) for a first-order ODE. In a similar manner, the LSC can be derived for higher-ordered ODEs. Past the first-order ODEs, this becomes messy since the chain rule must be taken into consideration as well. A well-designed computer program could easily derive the LSC for higher-ordered ODEs.

Still, this often is not sufficient. The LSC will regularly give a PDE that is more difficult to solve than the original ODE. It is possible to make an educated guess about the variable dependence of η and ξ. There is a chart on page 158 of Cantwell's (2002) *Introduction to Symmetry Analysis* that takes the guesswork out of many ODEs of specific forms. It is possible that one's guess does not give a PDE with solutions. If this is the case, make another guess and try again.

Infinitesimal Generator

So far, the methods developed only have application to first-order ODEs. The *infinitesimal generator* can be used in general to find canonical coordinates and extended to higher-order ODEs. The infinitesimal generator, $\boldsymbol{X}$, is given by

$$\boldsymbol{X} = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y}.$$

This infinitesimal generator can be seen as the tangent vector field to the symmetry transformations at $\epsilon = 0$ since both η and ξ are involved (Hydon, 2000).

It is also possible to write the infinitesimal generator in terms of canonical coordinates. From the generator, it is apparent that

$$Xr = \xi(x,y)\frac{\partial r}{\partial x} + \eta(x,y)\frac{\partial r}{\partial y} = 0$$

$$Xs = \xi(x,y)\frac{\partial s}{\partial x} + \eta(x,y)\frac{\partial s}{\partial y} = 1.$$

It can easily be show that for a change of coordinates like such, the infinitesimal generator becomes

$$X = Xr\frac{\partial}{\partial r} + Xs\frac{\partial}{\partial s}$$

which implies

$$X = \frac{\partial}{\partial s}.$$

An extension of the infinitesimal generator would be beneficial, since it is often the case that an ODE is not first-order. Recall that

$$\bar{x} = x + \epsilon\xi(x,y) + \cdots = x + \epsilon\boldsymbol{X}x + \cdots$$

$$\bar{y} = y + \epsilon\eta(x,y) + \cdots = y + \epsilon\boldsymbol{X}y + \cdots$$

and this extends to

$$\begin{aligned}\bar{y}' &= y' + \epsilon\eta'(x, y, y') + \cdots = y' + \epsilon\boldsymbol{X}y' + \cdots \\ &\vdots \\ \bar{y}^{(n)} &= y^{(n)} + \epsilon\eta^{(n)}\left(x, y, \ldots, y^{(n)}\right) + \cdots = y' + \epsilon\boldsymbol{X}y^{(n)} + \cdots\end{aligned}$$

where $\eta^{(n)}$ is defined in a manner similar to η. Using the symmetry condition that was shown,

$$\bar{y}' = \frac{d\bar{y}}{d\bar{x}} = \frac{y'\left(\frac{\partial\bar{y}}{\partial y}\right) + \left(\frac{\partial\bar{y}}{\partial x}\right)}{y'\left(\frac{\partial\bar{x}}{\partial y}\right) + \left(\frac{\partial\bar{x}}{\partial x}\right)}$$

one can substitute

$$\bar{y}' = \frac{y' + \epsilon\left(\frac{d\eta}{dx}\right) + \cdots}{1 + \epsilon\left(\frac{d\xi}{dx}\right) + \cdots} = y' + \epsilon\left(\frac{d\eta}{dx} - y'\frac{d\xi}{dx}\right) + \cdots.$$

Equating this to the general extension gives

$$y' + \epsilon\left(\frac{d\eta}{dx} - y'\frac{d\xi}{dx}\right) + \cdots = y' + \epsilon\eta'(x, y, y') + \cdots$$

$$\eta'(x, y, y') = \frac{d\eta}{dx} - y'\frac{d\xi}{dx}.$$

For a general $\eta^{(n)}$ it can be prolonged in a similar manner, resulting in

$$\eta^{(n)} = \frac{d\eta^{(n-1)}}{dx} - y^{(n)}\frac{d\xi}{dx}.$$

From this conclusion, the infinitesimal generator can be extended to a nth-order ODE by

$$\boldsymbol{X} = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \cdots + \eta^{(n)}\frac{\partial}{\partial y^{(n)}}$$

where $\eta^{(n)}$ is as we derived it above (Olver, 1986). Since a given ODE must remain invariant under the symmetrical coordinate change,

$$F\left(x, y, y', \dots, y^{(n)}\right) = F\left(\bar{x}, \bar{y}, \bar{y}', \dots, \bar{y}^{(n)}\right) = 0$$

and if F admits symmetries, then $\boldsymbol{X}F = 0$ independent of coordinates.

In addition, the LSC mentioned before can be applied algorithmically using the recursive definition of $\eta^{(n)}$ but the number of terms will still increase exponentially, making explicit calculations of $\eta^{(n)}$ a computer's job.

If an ODE has order greater than one, the canonical coordinates method mentioned above will reduce the order of the ODE by one. Though this can be repeated multiple times, it is possible that a reduced ODE does not admit a group of symmetry. If the original ODE has two independent symmetry generators, a new generator can be defined by an operation over the set of all generators (Quispel and Sahadevan, 1993).

In general, given two vectors, $\boldsymbol{X}, \boldsymbol{V},$ the *commutator* of $\boldsymbol{X}$ and $\boldsymbol{V}$ is simply

$$[\boldsymbol{X}.\boldsymbol{V}] = \boldsymbol{XV} - \boldsymbol{VX} = -[\boldsymbol{V}, \boldsymbol{X}].$$

An important property of the commutator is the *Jacobi Identity*. Given three vectors, $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}$

$$\left[\boldsymbol{X}, [\boldsymbol{Y}, \boldsymbol{Z}]\right] + \left[\boldsymbol{Y}, [\boldsymbol{Z}, \boldsymbol{X}]\right] + \left[\boldsymbol{Z}, [\boldsymbol{X}, \boldsymbol{Y}]\right] = 0.$$

The commutator of two infinitesimal generators is itself an infinitesimal generator. It is this generator that will make inheriting a Lie symmetry over the reduction of order most likely. This idea extends abstractly to a set of vectors. Under the commutator and in addition to the structure of a vector space, this set forms a *Lie algebra* (Nomizu, 1956).

$\boldsymbol{n}$th-order ODE as a first-order PDE

Treating $y', y'', \dots, y^{(n)}$ as independent variables will become important when working with differential forms. To equate the ideas learned in this section with what will be uncovered through differential forms, an equivalent way of representing nth-order ODEs while regarding $y', y'', \dots, y^{(n)}$ as independent must be established. Given any nth-order ODE, there exists an associated first-order PDE with $n+1$ variables. In Einstein notation, this PDE can be expressed as

$$\boldsymbol{A}\varphi = \sum_{i=1}^{n} a^i \frac{\partial}{\partial x^i}\varphi = a^i \frac{\partial}{\partial x^i}\varphi = 0.$$

Since A is a generator, it too can be transformed to

$$A = \frac{\partial}{\partial s}.$$

The solution to the equation $\boldsymbol{A}\varphi = 0$ is determined by the coordinate change up to a factor, often denoted λ.

To create the equivalent PDE, given the nth-order ODE,

$$y^{(n)} = g(x, y, y', \dots, y^{(n-1)})$$

create

$$\boldsymbol{A}f = \left(\frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + y''\frac{\partial}{\partial y'} + \cdots + g\frac{\partial}{\partial y^{(n-1)}}\right)f = 0.$$

By the method of characteristics, the first integral shows that these two differential equations have the same solutions. Since the two equations are equivalent, the generator, $\boldsymbol{X}$, and the differential operator, $\boldsymbol{A}$, accomplish the same purpose.

There is a condition on $\boldsymbol{X}$ that can ensure it is also a symmetry of $\boldsymbol{A}f$. If φ^a is a set of solutions to $\boldsymbol{A}f = 0$ and consequently the original ODE, then since $\boldsymbol{X}$ maps solutions into solutions, as discussed previously, $\boldsymbol{X}\varphi^a = H(\varphi^b)$ and $\boldsymbol{A}\varphi^a = \boldsymbol{A}H(\varphi^b) = 0$. Now, consider the commutator of $\boldsymbol{A}$ and $\boldsymbol{X}$

$$[\boldsymbol{X}, \boldsymbol{A}] = \boldsymbol{XA} - \boldsymbol{AX}$$

and consequently

$$[\boldsymbol{X}, \boldsymbol{A}]\varphi^a = \boldsymbol{X}(\boldsymbol{A}\varphi^a) - \boldsymbol{A}(\boldsymbol{X}\varphi^a) = 0.$$

This conclusion implies that $[\boldsymbol{X}, \boldsymbol{A}]f = 0$ must have the same solution set as $\boldsymbol{A}f = 0$ up to a factor $\lambda\left(x, y, \dots, y^{(n-1)}\right)$. If it is the case that

$$[\boldsymbol{X}, \boldsymbol{A}] = \lambda \boldsymbol{A}$$

then $\boldsymbol{X}$ is a symmetry generator for $\boldsymbol{A}f = 0$ (Stephani, 1989).

Vectors and Differential Forms

A vector field $\boldsymbol{V}$ is a linear differential operator that maps a smooth function into a real number corresponding to the particular coordinate. In standard notation, $\boldsymbol{V} = v^a(x_i)\frac{\partial}{\partial x^a}$ where $v^a(x_i)$ are functions that are the components of the $\boldsymbol{V}$. In the context of manifold theory, these vector fields are elements of the tangent space, denoted $\mathcal{T}_p(M)$. Within this class is the symmetry generator, which is often denoted $\boldsymbol{X}$, is an essential part of solving a differential equation (Abraham, 1988).

The set of real valued functions on a particular manifold is denoted $\Omega^0(M)$, and these functions are called 0-forms.

A Pfaffian Form, or a 1-form, $\boldsymbol{\omega} \in \Omega^1(M)$, the set of all 1-forms on M. In the conventional notation, a 1-from is given by $\boldsymbol{\omega} = \omega_a\left(x^i\right)dx^a$. The interior product in this context is the *contraction* of a differential form with a vector field (Bowen and Wang, 1976). Given a p-form, $\boldsymbol{\omega}$, and a vector field, $\boldsymbol{V}$, the contraction of $\boldsymbol{\omega}$, a Pfaffian form, with $\boldsymbol{V}$, denoted $\langle \boldsymbol{\omega}, \boldsymbol{V} \rangle$, is a mapping $\langle \cdot \rangle : \Omega^1(M) \to \Omega^0(M)$ where

$$\langle \boldsymbol{\omega}, \boldsymbol{V} \rangle = v^a \omega_a.$$

The existence of differential forms of higher dimensions is directly correlated with the dimension of the manifold. If a manifold is n-dimensional, there exist exactly n-linearly independent Pfaffian forms. In the previous notation, these linearly independent Pfaffian forms are symbolized by dx^a. Using an operation called the *wedge product*, linearly independent Pfaffian forms can be used to construct forms in $\Omega^2(M)$, $\Omega^3(M)$ all the way up to the dimension of the manifold itself, $\Omega^n(M)$ (O'Neill, 2006). The wedge product is a mapping $\wedge: \Omega^q(M) \times \Omega^p(M) \rightarrow \Omega^{p+q}(M)$ by

$$\boldsymbol{\omega} \wedge \boldsymbol{\sigma} = \omega_{i_1 i_2 \dots i_p}(x^i)\sigma_{j_1 j_2 \dots j_q}(x^j) dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_p} \wedge dx^{j_1} \wedge dx^{j_2} \wedge \dots \wedge dx^{j_q}$$

or in a more concise notation, let $I = [i_1 i_2 \dots i_p]$ and $J = [j_1 j_2 \dots j_q]$ the wedge product can be defined by

$$\boldsymbol{\omega} \wedge \boldsymbol{\sigma} = \omega_I \sigma_J dx^I \wedge dx^J.$$

Considering addition and the wedge product, differential forms have some convenient properties under these two operations. The following are said properties:

$$(\boldsymbol{\omega} + \boldsymbol{\sigma}) \wedge \boldsymbol{\zeta} = \boldsymbol{\omega} \wedge \boldsymbol{\zeta} + \boldsymbol{\sigma} \wedge \boldsymbol{\zeta} \text{ (right distributive property)}$$

$$\boldsymbol{\omega} \wedge (\boldsymbol{\sigma} + \boldsymbol{\zeta}) = \boldsymbol{\omega} \wedge \boldsymbol{\sigma} + \boldsymbol{\omega} \wedge \boldsymbol{\zeta} \text{ (left distributive property)}$$

$$(c\boldsymbol{\omega}) \wedge \boldsymbol{\sigma} = \boldsymbol{\omega} \wedge (c\boldsymbol{\sigma}) = c(\boldsymbol{\omega} \wedge \boldsymbol{\sigma}) \text{ (scalars factor out)}$$

$$\boldsymbol{\omega} \wedge (\boldsymbol{\sigma} \wedge \boldsymbol{\zeta}) = (\boldsymbol{\omega} \wedge \boldsymbol{\sigma}) \wedge \boldsymbol{\zeta} \text{ (associativity of the wedge product)}$$

The set of differential forms is closed under addition. The commutation rule for the wedge product of a p-form $\boldsymbol{\omega}$ and q-form $\boldsymbol{\sigma}$ is

$$\boldsymbol{\omega} \wedge \boldsymbol{\sigma} = (-1)^{pq} \boldsymbol{\sigma} \wedge \boldsymbol{\omega}.$$

The anitcommutativity of the wedge product is an essential feature of the tensor product and bilinearity of $\boldsymbol{\omega}$ and $\boldsymbol{\sigma}$. An important property of the wedge product of differential forms is that the wedge product between two linearly dependent Pfaffian forms is always 0 (Carmo, 1994).

> Let $\boldsymbol{\omega}, \boldsymbol{\sigma} \in \Omega^1(M)$ where $\boldsymbol{\omega} = c\boldsymbol{\sigma}$ for some $c \in \mathbb{R}$. By the anticommutativity of $\boldsymbol{\sigma}$, $\boldsymbol{\sigma} \wedge \boldsymbol{\sigma} = -\boldsymbol{\sigma} \wedge \boldsymbol{\sigma} \Rightarrow 2\boldsymbol{\sigma} \wedge \boldsymbol{\sigma} = 0 \Rightarrow \boldsymbol{\sigma} \wedge \boldsymbol{\sigma} = 0$. Now consider $\boldsymbol{\omega} \wedge \boldsymbol{\sigma} = c\boldsymbol{\sigma} \wedge \boldsymbol{\sigma} = c(0) = 0$. ■

The idea of the contraction between a vector and a Pfaffian form can be extended to the contraction between a vector and a p-form. If $\boldsymbol{\omega}$ is a p-form and $\boldsymbol{V}$ is a vector the contraction between $\boldsymbol{\omega}$ and $\boldsymbol{V}$ denoted $\langle \boldsymbol{\omega}, \boldsymbol{V} \rangle$ is a mapping $\langle \cdot \rangle : \Omega^p(M) \to \Omega^{p-1}(M)$ where

$$\langle \boldsymbol{\omega}, \boldsymbol{V} \rangle = \langle \boldsymbol{\omega}_I dx^I, \left(v^b \frac{\partial}{\partial x^b} \right) \rangle = p!\, v^b \omega_{[b i_2 \dots i_p]} dx^{i_2} \wedge \dots \wedge dx^{i_p}$$

in which this contraction has distributive properties under differential form addition and wedge product as follows :

$$\langle \boldsymbol{\omega} + \boldsymbol{\sigma}, \boldsymbol{V} \rangle = \langle \boldsymbol{\omega}, \boldsymbol{V} \rangle + \langle \boldsymbol{\sigma}, \boldsymbol{V} \rangle$$

$$\langle \boldsymbol{\omega} \wedge \boldsymbol{\sigma}, \boldsymbol{V} \rangle = \langle \boldsymbol{\omega}, \boldsymbol{V} \rangle \wedge \boldsymbol{\sigma} + (-1)^p \boldsymbol{\omega} \wedge \langle \boldsymbol{\sigma}, \boldsymbol{V} \rangle.$$

Exterior Derivative

If $\boldsymbol{\omega} \in \Omega^p(M)$, then the *exterior derivative* of $\boldsymbol{\omega}$ is, in the standard notation,

$$d\boldsymbol{\omega} = \omega_{[i_1 i_2 \dots i_p b]} dx^b \wedge dx^{i_1} \wedge dx^i \wedge \dots \wedge dx^{i_p}$$

and is a $(p+1)$-form. This idea coincides with the Calculus I definition of the derivative of a function f because

$$df = \frac{\partial f}{\partial x^a} dx^a = \boldsymbol{\sigma}.$$

It should now be obvious that the exterior derivative $d: \Omega^p(M) \to \Omega^{p+1}(M)$. Just as was the case with the wedge product, there are some properties of the exterior derivative that will often prove to be quite beneficial. First, let $\boldsymbol{\omega}$ denote an arbitrary p-form, $\boldsymbol{\sigma}$ denote an arbitrary q-form, and f be a 0-form:

$$d(d\boldsymbol{\omega}) = 0$$

$$d(\boldsymbol{\omega} \wedge \boldsymbol{\sigma}) = (d\boldsymbol{\omega}) \wedge \boldsymbol{\sigma} + (-1)^{pq}\boldsymbol{\omega} \wedge (d\boldsymbol{\sigma})$$

$$d(f\boldsymbol{\omega}) = df \wedge \boldsymbol{\omega} + fd\boldsymbol{\omega}.$$

Applying the exterior derivative repeatedly will result in a value of zero. The exterior derivative of two forms wedged will distribute with consideration of the anitcommutativity of the wedge product. These properties make performing otherwise difficult calculations rather simple (Schreiber, 1977).

Lie Derivative

The *Lie Derivative* is taken with respect to a given vector field and it quantifies the change in an object along the given vector field. Unlike the exterior derivative it preserves the type of object. If $\boldsymbol{V}$ is the vector the Lie derivative is taken with respect to, the Lie derivative of $\boldsymbol{A}$ denoted $\mathcal{L}_{\boldsymbol{V}}\boldsymbol{A} = [\boldsymbol{V}, \boldsymbol{A}] = -\mathcal{L}_{\boldsymbol{A}}\boldsymbol{V}$ is the commutator of $\boldsymbol{V}$ and $\boldsymbol{A}$. It is apparent that the Lie derivative of a vector field is a vector field.

With forms, however, the Lie derivative of a p-form is still a p-form, unlike the exterior derivative. If $\boldsymbol{V}$ is a vector and $\boldsymbol{\omega}$ a p-form,

$$\mathcal{L}_{\boldsymbol{V}}\boldsymbol{\omega} = \langle d\boldsymbol{\omega}, \boldsymbol{V}\rangle + d\langle \boldsymbol{\omega}, \boldsymbol{V}\rangle.$$

Lie differentiation commutes with the exterior derivative, and one can calculate the Lie derivative of the wedge product of two forms and the contraction between a vector and a form in the natural way

$$d\mathcal{L}_{\boldsymbol{V}}\boldsymbol{\omega} = \mathcal{L}_{\boldsymbol{V}}\boldsymbol{d\omega}$$

$$\mathcal{L}_{\boldsymbol{V}}(\boldsymbol{\omega} \wedge \boldsymbol{\sigma}) = (\mathcal{L}_{\boldsymbol{V}}\boldsymbol{\omega}) \wedge \boldsymbol{\sigma} + \boldsymbol{\omega} \wedge (\mathcal{L}_{\boldsymbol{V}}\boldsymbol{\sigma})$$

$$\mathcal{L}_{\boldsymbol{V}}\langle \boldsymbol{\omega}, \boldsymbol{A} \rangle = \langle \boldsymbol{\omega}, [\boldsymbol{V}, \boldsymbol{A}] \rangle + \langle \mathcal{L}_{\boldsymbol{V}}\boldsymbol{\omega}, \boldsymbol{A} \rangle.$$

The Lie derivative will soon become an essential player in the discovery of symmetry for an ODE (Gołab, 1974).

Writing ODEs as Differential Forms

Differential Forms provide a unique way of finding the symmetries of a given differential equation. Not only do they make use of the operations discussed so far, but solving differential equations in this way allows one to solve in conventional methods since it is possible to map back to Euclidean space via the various charts, but it provides the freedom from coordinates by the nature of a manifold. The method transcends a particular choice of coordinate. This section is based largely on the work of Hans Stephani (1989).

Quasi-linear First Order Differential Equations and Rewriting as Differentials

To write an ODE as an equation of differential forms, the first step is to write the equation as a system of quasi-linear first order differential equation by substituting the necessary higher-order derivatives as variables. This gives an expanded set of variables of which are not all independent of each other. A consequence of this is that their differentials relate linearly. To see how this is done, consider the second-order ODE,

$$\frac{d^2y}{dx^2} = f(y', y, x).$$

To change this ODE appropriately, let $p = y'$ and make the substitution throughout the equation, giving

$$\frac{dp}{dx} = f(p, y, x).$$

This substitution gives you a system of two quasi-linear first order ODEs with an expanded variable set of (p, y, x). Using these two equations, the system of differentials is

$$pdx = dy; \qquad dp = fdx.$$

Finding an Equivalent System of Differential Forms

Once the equivalent system of differentials is found, one must rearrange the equations to find an equivalent system of differential forms such that $\boldsymbol{\omega}_i = 0$ for each differential form. Since this system must be equivalent to the system of differentials, it preserves all of the relationships found by making the variable substitutions. Since the expanded variable set is not linearly independent, the system of differential forms should be closed, that is if $\boldsymbol{\omega_i} = 0$ then $d\boldsymbol{\omega}_i = 0$. Not only should this system be closed, but since the integrability of the original differential equation is preserved by the differentials, so too should the integrability be preserved in the system of differential forms. Using the same general second-order ODE, one can rearrange the differentials

$$\boldsymbol{\omega}_1 = dy - pdx = 0; \qquad \boldsymbol{\omega}_2 = dp - fdx = 0.$$

The equivalence of this system to the original ODE is apparent since it was derived from the equivalent system of quasi-linear first order ODEs. To show that this is closed,

$$d\boldsymbol{\omega}_1 = -dp \wedge dx = dx \wedge \boldsymbol{\omega}_2 = dx \wedge 0 = 0$$

$$d\boldsymbol{\omega}_2 = -df \wedge dx = -\left(\frac{\partial f}{\partial p}dp + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial x}dx\right) \wedge dx = -f_p dp \wedge dx - f_y dy \wedge dx =$$

$$f_p dx \wedge d\boldsymbol{\omega}_2 + f_y dx \wedge \boldsymbol{\omega}_1 = 0.$$

Symmetries of ODEs as Differential Forms

When discussing the representation of an nth order ODE as a PDE, a generator must satisfy

$$[\boldsymbol{X}, \boldsymbol{A}] = \lambda \boldsymbol{A}.$$

This is equivalent to saying

$$\mathcal{L}_{\boldsymbol{X}} \boldsymbol{A} = \lambda \boldsymbol{A}.$$

The Lie derivative can be used to express symmetries. The application of the infinitesimal generator to its respective differential equation should always result in a value of 0. That is if

$$H\left(x, y, y', \dots y^{(n)}\right) = 0$$

is a nth-order ODE with generator

$$\boldsymbol{X} = \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} + \cdots + \eta^{(n)} \frac{\partial}{\partial y^{(n)}}$$

then $\boldsymbol{X}H = 0$. Translated into forms with the Lie derivative, the Lie derivative of all forms $\boldsymbol{\omega}_i$ that represents an ODE should go to zero if $\boldsymbol{\omega}_i = 0$ is true (which it is by construction). A different way of thinking about this is that the Lie derivative of ω_i will be a linear combination of each ω_i.

To show how this method works, consider again the second order ODE

$$\frac{d^2 y}{dx^2} = f(x, y, y').$$

Through the variable change, it was shown that this equation is equivalent to the system of differential forms,

$$\boldsymbol{\omega}_1 = dy - p\,dx; \qquad \boldsymbol{\omega}_2 = dp - f\,dx.$$

Taking the Lie derivative of each of these differential forms with respect to their infinitesimal generator

$$\boldsymbol{X} = \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} + \eta' \frac{\partial}{\partial y'}$$

gives

$$\mathcal{L}_{\boldsymbol{X}} \boldsymbol{\omega}_1 = \lambda_1 \boldsymbol{\omega}_1 + \lambda_2 \boldsymbol{\omega}_2$$

$$\mathcal{L}_{\boldsymbol{X}} \boldsymbol{\omega}_2 = \lambda_3 \boldsymbol{\omega}_1 + \lambda_4 \boldsymbol{\omega}_2.$$

Using the definition of the Lie derivative,

$$\langle \boldsymbol{\omega}_1, \boldsymbol{X} \rangle = \langle dy - pdx, \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} + \eta' \frac{\partial}{\partial y'} \rangle = -p\xi + \eta$$

$$\langle \boldsymbol{d\omega}_1, \boldsymbol{X} \rangle = \langle -dp \wedge dx, \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} + \eta' \frac{\partial}{\partial y'} \rangle = \xi dp - \eta' dx$$

$$\mathcal{L}_{\boldsymbol{X}} \boldsymbol{\omega}_1 = \langle \boldsymbol{d\omega}_1, \boldsymbol{X} \rangle + d\langle \boldsymbol{\omega}_1, \boldsymbol{X} \rangle = -\eta' dx + d\eta - pd\xi = \lambda_1(dy - pdx) + \lambda_2(dp - fdx).$$

Rearranging and comparing coefficients gives

$$\eta' = \partial_x \eta + p\partial_y \eta + f\partial_p \eta - p(\partial_x \xi + p\partial_y \xi + f\partial_p \xi)$$

and since $p = y'$

$$\eta' = \partial_x \eta + y'\partial_y \eta + f\partial_{y'} \eta - y'(\partial_x \xi + py'\xi + f\partial_{y'} \xi)$$

which can be shown to be equivalent to the LSC discussed earlier. From this, canonical coordinates can be found simply.

To conclude this discussion on finding a symmetry of an ODE using differential forms, below is an example of finding the LSC of a differential equation using differential forms. Consider the differential equation

$$y' = (y + 2)(y - 2).$$

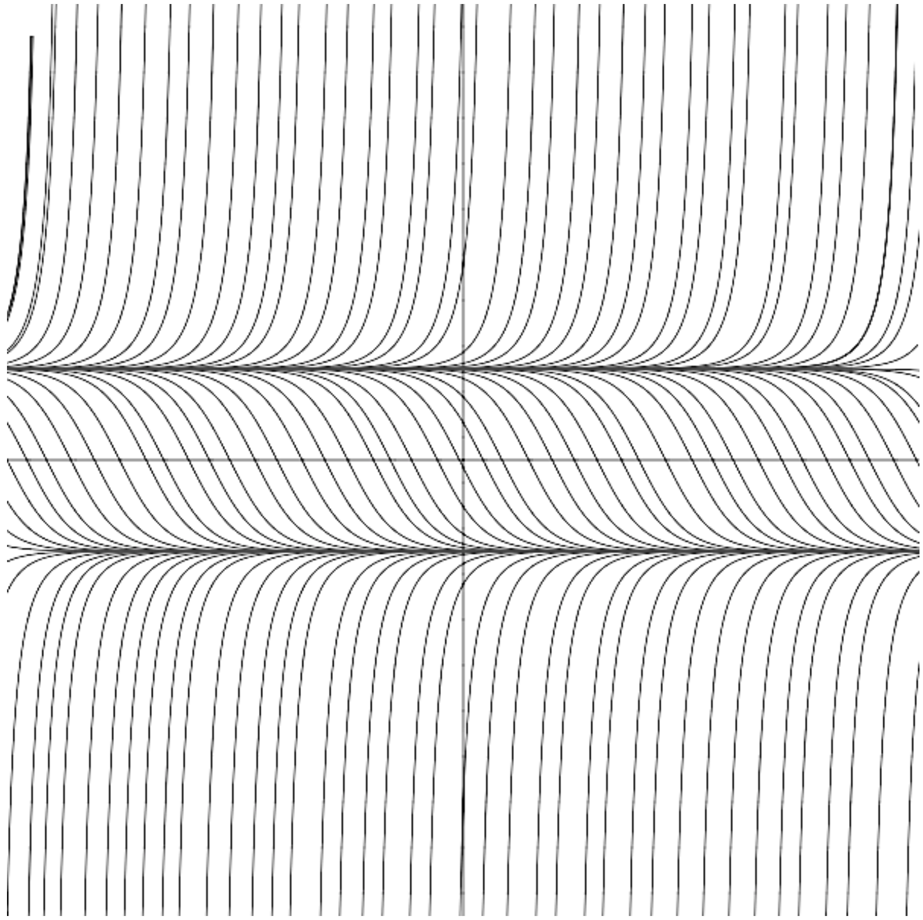

Figure 6. Solutions of $\boldsymbol{y}' = (\boldsymbol{y}+\mathbf{2})(\boldsymbol{y}-\mathbf{2})$.

This equation results in the following equivalent differential form

$$dy = (y+2)(y-2)dx$$

$$\boldsymbol{\omega} = (y+2)(y-2)dx - dy.$$

First, consider,

$$\langle \omega, X \rangle = (y^2-4)\xi - \eta$$

$$\langle d\omega, X \rangle = \langle 2y\, dy \wedge dx, X \rangle = 2y(dy\xi - dx\eta).$$

Taking the Lie Derivative,

$$\mathcal{L}_X\omega = \langle d\omega, X\rangle + d\langle \omega, X\rangle = -2ydx\eta - d\eta + (y^2-4)d\xi$$

$$= -2y\eta dx - \left(\partial_y\eta dy + \partial_x\eta dx\right) + (y^2-4\,)(\partial_y\xi dy + \partial_x\xi dx)$$

and through rearranging and comparing coefficients,

$$\lambda = (y^2-4)\partial_y\xi - \partial_y\eta$$

which results in the condition

$$2y\eta - \partial_x\eta + (y^2-4)\partial_x\xi = (y^2-4)^2\partial_y\xi - (y^2-4)\partial_y\eta.$$

This is the LSC for the given differential equation. Through observation of Figure 6, a potential symmetry is a shift in the horizontal direction. This symmetry would be represented by the coordinate change

$$(\bar{x},\bar{y}) = (x+\epsilon, y).$$

To verify that this is indeed a symmetry, check to see if it satisfies the LSC discovered from the Lie derivative.

$$\eta = 0; \xi = 1$$

so the LSC with these values is

$$0 - 0 + 0 = 0 - 0 \Rightarrow 0 = 0.$$

Therefore, the symmetry condition is satisfied and this transformation is indeed a symmetry of the differential equation.

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