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Fascinating Characteristics and Applications of the Fibonacci Sequence

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Abstract

This thesis offers a brief background on the life of Fibonacci as well as his discovery of the famous Fibonacci sequence. Next, the limit of the ratio of consecutive Fibonacci terms is established and discussed. The Fibonacci sequence is then defined as a recursive function, a linear homogeneous recurrence relation with constant coefficients, and a generating function. Proofs for those particular properties are introduced and proven. Several theorems and identities from the field of number theory concerning the properties of the Fibonacci numbers are also introduced and proven. Finally, the famous Fibonacci puzzle is introduced and critiqued. These fascinating characteristics and applications demonstrate not only the universal nature of the Fibonacci sequence but also the aesthetic nature of God.

Fascinating Characteristics and Applications of the Fibonacci Sequence

Introduction

In the realm of mathematics, many concepts have applications in multiple mathematical fields. Without these important concepts, every field of mathematics would be seemingly disjointed and unrelated to topics from other mathematical fields. One of these concepts was discovered by a man named Leonardo Pisano during the early 13th century. This particular concept is known today as the Fibonacci sequence. Since its official introduction to the world by Leonardo Pisano, the Fibonacci sequence has become one of the most fascinating concepts in the entire realm of mathematics through its remarkable characteristics; its useful applications to various mathematical fields such as number theory, discrete mathematics, and geometry; and its clear demonstration of the aesthetic nature of God.

According to O'Connor and Robertson (1998), Leonardo Pisano was born into a merchant family in Pisa, Italy, in 1175 A.D. More commonly know by his nickname Fibonacci, Leonardo was educated in North Africa rather than Italy; because his father, Guglielmo Bonacci, held a diplomatic post in that area. Fibonacci's father was the representative of the Pisan merchants to the Mediterranean port city of Bugia, which was located in northeastern Algeria. While in Bugia, Fibonacci was educated in the field of mathematics. During his early adulthood, Fibonacci traveled with his father throughout the Mediterranean region, broadening his knowledge of and appreciation for various cultures. Fibonacci's journeys across the Mediterranean introduced him to innovative mathematical ideas and concepts from numerous countries, fueling his love for mathematics (O'Connor & Robertson, 1998).

In 1200 A.D., Fibonacci returned to his birthplace and began to work on his most well-known mathematical masterpiece: *Liber Abaci*, which means "The Book of Calculations" (Knott & Quinney, 1997). Fibonacci's *Liber Abaci* contains many of the mathematical ideas that he encountered during his travels throughout the Mediterranean. Through his work, Fibonacci introduced the Latin-speaking world to the Hindu-Arabic numerals, the decimal system, numerous topics in the field of number theory, and a peculiar sequence of numbers that is now known as the Fibonacci sequence (O'Connor & Robertson, 1998). Fibonacci's initial words in *Liber Abaci* were, "These are the nine figures of the Indians: 9 8 7 6 5 4 3 2 1. With these nine figures, and with this sign 0 which in Arabic is called zephirum, any number can be written, as will be demonstrated" (as cited in Knott & Quinney, 1997, ¶ 6). The introduction of these Hindu-Arabic numerals forever changed the face of mathematics in the western world.

In *Liber Abaci*, Fibonacci also introduced a unique sequence of numbers with interesting characteristics. This sequence eventually became one of the most famous sequences in the realm of mathematics. Although Fibonacci was credited with discovering the sequence, it was not officially named the Fibonacci sequence until after his death in 1250 A.D. (Burton, 2002). In fact, this sequence was not labeled the Fibonacci sequence until the 19th century when a number theorist named Edouard Lucas examined a problem in Fibonacci's *Liber Abaci* and linked Fibonacci's name to the sequence that the problem involves. In his book, Fibonacci (1202) introduced the sequence as the following hypothetical situation:

> A man put one pair of rabbits in a certain place entirely surrounded by a wall. How many pairs of rabbits can be produced from that pair in a year, if the nature of these rabbits is such that every month each pair bears a new pair which from the second month on becomes productive? (as cited in Burton, 2002, p. 271)

As Fibonacci began to examine this particular problem, he discovered a sequence involving the numbers of pairs of rabbits.

The problem begins with a pair of baby rabbits. Once the first month has concluded, that initial pair of baby rabbits has reached adulthood and is now capable of reproducing (Silverman, 2006). Assuming that the average gestation period for a rabbit is one month, the initial pair of rabbits will give birth to a second pair of rabbits at the beginning of the third month. At this point in time in the problem, there currently exist a pair of adult rabbits and a pair of baby rabbits. Fibonacci assumes in his problem that once a pair of rabbits has reached adulthood, they reproduce another pair of rabbits each month afterward (Silverman, 2006). The current pair of baby rabbits in the problem is able to reproduce by the beginning of the fourth month, and they give birth to a pair of baby rabbits each month thereafter. In order to maintain uniformity in his problem, Fibonacci also assumes that none of the rabbits die (Burton, 2002).

After each month of the problem, Fibonacci counted the number of pairs of rabbits; and his conclusions led him to a sequence of numbers with the number of pairs of rabbits as the terms of the sequence and the corresponding month numbers as the subscripts for those terms. Fibonacci's rabbit problem is illustrated in Figure 1 with each rabbit image representing a pair of rabbits (Silverman, 2006). The smaller rabbit images represent rabbits that have been newly birthed while the larger rabbit images represent adult rabbits that are at least one month old.

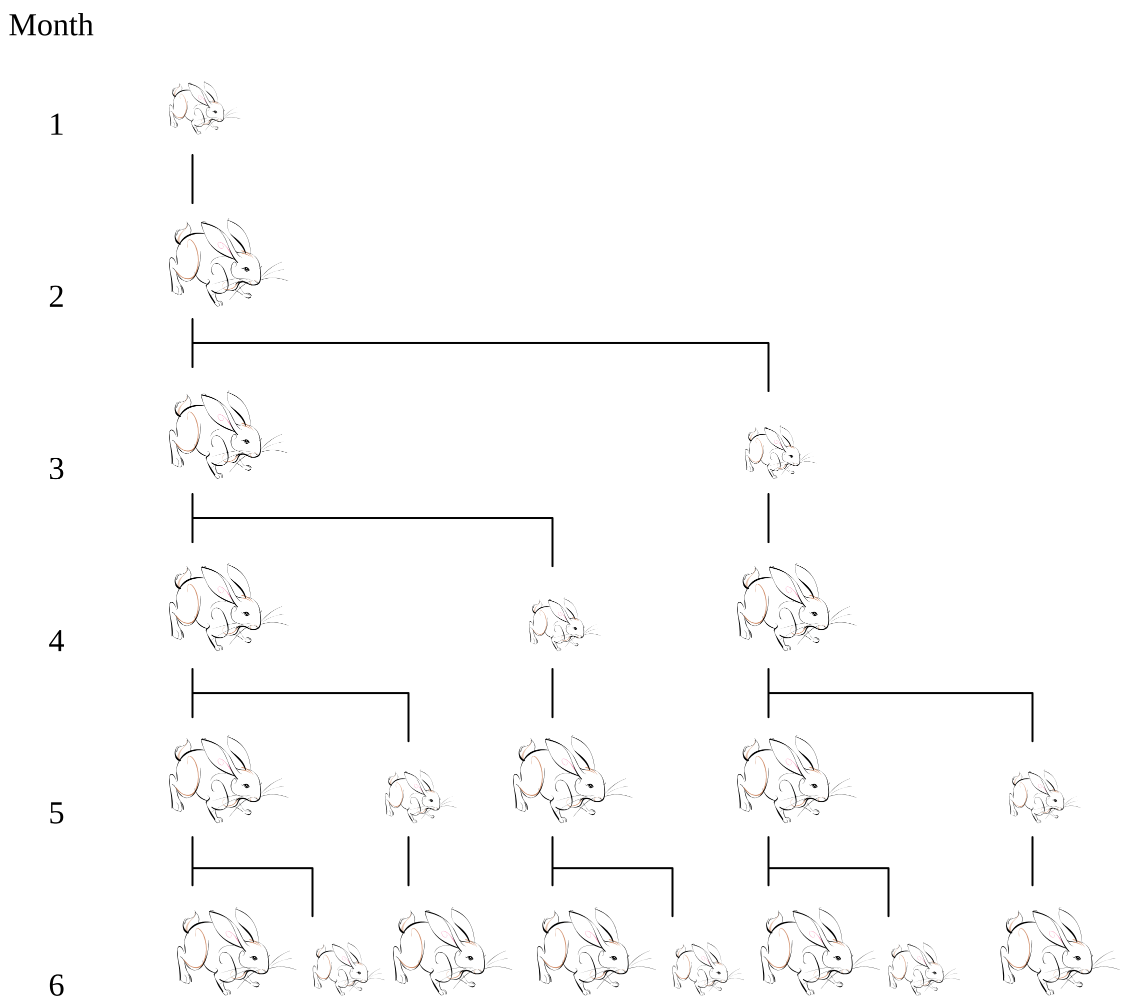

Figure 1. Fibonacci's hypothetical rabbit problem.

The terms of the sequence from Fibonacci's problem were 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144. Based upon Fibonacci's findings, it is clear that each term is a sum of the previous two terms of the sequence (Silverman, 2006). In fact, the Fibonacci sequence can be defined recursively as $f_n = f_{n-1} + f_{n-2}$, $f_0 = 0$, $f_1 = 1$. With this unique recursive definition, terms in the Fibonacci sequence hold interesting characteristics in relation to each other.

Limit of the Ratio of Consecutive Fibonacci Terms

One of the interesting characteristics of the Fibonacci sequence is evidenced in the convergence of the sequence. The limit of the ratio of consecutive Fibonacci terms as the subscripts become infinite converges to the number phi (φ). The number phi has also been known for many centuries as the golden ratio, the golden section, and the golden mean (Nickel, 2001). In fact, Johannes Kepler referred to phi as the "Divine Proportion" (Nickel, 2001, p. 245). In order to determine the limit of the ratio of consecutive Fibonacci terms, one must first let each ratio of consecutive terms equal a unique value a_n with $n \in \mathrm{N}$ so that $a_n = \left(f_{n+1} / f_n\right)$. Therefore,

$$a_1 = \frac{1}{1} = 1$$

$$a_2 = \frac{2}{1} = 2$$

$$a_3 = \frac{3}{2} = 1.5$$

$$a_4 = \frac{5}{3} = 1.66$$

$$\vdots$$

In order to prove the convergence of the sequence of the ratio of consecutive Fibonacci terms, the following conjectures must be proven: 1 is a lower bound for a_n and 2 is an upper bound for a_n $\left(i.e.\, 1 \le a_n \le 2, \forall n \in \mathrm{N}\right)$, $\left(a_{2n+1}\right)$ is an increasing subsequence, $\left(a_{2n}\right)$ is a decreasing subsequence, and $\left(a_n\right)$ is a convergent sequence (Craw, 2002). Algebraic manipulation of the equation $a_n = \left(f_{n+1} / f_n\right)$ yields

$$a_n = \frac{f_{n+1}}{f_n} = \frac{f_n + f_{n-1}}{f_n} = \frac{f_n}{f_n} + \frac{f_{n-1}}{f_n} = 1 + \frac{f_{n-1}}{f_n} = 1 + \frac{1}{\frac{f_n}{f_{n-1}}} = 1 + \frac{1}{a_{n-1}} \tag{1}$$

Since $f_{n+1} \ge f_n \ \forall n \in \mathrm{N}$, $a_n = \dfrac{f_{n+1}}{f_n} \ge 1 \ \forall n \in \mathrm{N}$. Now, the conjecture $a_n \le 2 \ \forall n \in \mathrm{N}$ must be proven inductively (Craw, 2002). Therefore,

$$a_1 = \frac{f_{n+1}}{f_n} = \frac{1}{1} = 1 < 2\,. \tag{2}$$

Now, assume $a_k \le 2$ is true for some $k \in \mathrm{N}$ and prove that $a_{k+1} \le 2$ is true. In order to simplify the proof, $a_{k+1} \le 2$ should be written as $2 - a_{k+1} \ge 0$. Therefore,

$$2 - a_{n+1} = 2 - \left(1 - \frac{1}{a_n}\right) = 1 - \frac{1}{a_n} = 1 - \frac{f_n}{f_{n+1}} \ge 0\,. \tag{3}$$

Thus, 1 is a lower bound for a_n and 2 is an upper bound for a_n. Since it has already been established from Equation 1 that

$$a_n = 1 + \frac{1}{a_{n-1}},$$

the following equation holds:

$$a_{n+2} = 1 + \frac{1}{a_{n+1}} = 1 + \frac{1}{1 + \frac{1}{a_n}} = 1 + \frac{a_n}{1 + a_n}. \tag{4}$$

Using Equation 4, the difference between successive terms in a subsequence can be computed as

$$\begin{aligned} a_{n+2} - a_n &= \frac{a_n}{1 + a_n} - \frac{a_{n-2}}{1 + a_{n-2}} = \frac{a_n(1 + a_{n-2}) - a_{n-2}(1 + a_{n-2})}{(1 + a_{n-2})(1 + a_{n-2})} \\ &= \frac{a_n - a_{n-2}}{(1 + a_{n-2})(1 + a_{n-2})}. \end{aligned} \tag{5}$$

The denominator of Equation 5 is positive, in fact, $4 \le (1 + a_{n-2})(1 + a_{n-2}) \le 9$. Therefore, the numerator $a_n - a_{n-2}$ of Equation 5 must have the same sign as $a_{n+2} - a_n$ (Craw, 2002). This information must be used in order to prove that (a_{2n+1}) is an increasing subsequence and that (a_{2n}) is a decreasing subsequence. These two conjectures must ultimately be proven inductively. First,

$$a_4 < a_2 = 2 \tag{6}$$

Next assume that $a_{2k+2} < a_{2k}$ is true for some $k \in \mathrm{N}$. One must now verify that $a_{2k+4} < a_{2k+2}$.

$$a_{2k+4} = 1 + \frac{1}{a_{2k+3}} = 1 + \frac{1}{1 + \frac{1}{a_{2k+2}}} < 1 + \frac{1}{1 + \frac{1}{a_{2k}}} = a_{2k+2} \tag{7}$$

Therefore, $a_{2n+2} < a_{2n}$ $\forall n \in \mathrm{N}$. Thus, (a_{2n}) is an decreasing subsequence of (a_n). Since (a_{2n}) is decreasing and bounded below by 1, it must converge to some limit α. Also, since adjacent terms in the subsequence (a_{2n}) converge to the same limit, the terms a_{2n+2} and a_{2n} from Equation 4 may be replaced with the limit α: Thus,

$$\alpha = 1 + \frac{\alpha}{1+\alpha} \quad \Rightarrow \quad \alpha = \frac{1+2\alpha}{1+\alpha} \quad \Rightarrow \quad \alpha^2 + \alpha = 1 + 2\alpha$$

$$\Rightarrow \quad \alpha^2 - \alpha - 1 = 0. \tag{8}$$

Now, solving for α in Equation 8 using the quadratic equation yields

$$\alpha = \frac{1 \pm \sqrt{5}}{2}. \tag{9}$$

Since all of the a_n terms for $n \in \mathrm{N}$ are positive, only the positive root is meaningful.

Thus,

$$\alpha = \frac{1 + \sqrt{5}}{2}. \tag{10}$$

Similarly, the subsequence (a_{2n+1}) can be proven to be an increasing subsequence by induction on n. First,

$$a_3 > a_1 = 1. \tag{11}$$

Next, assume that $a_{2k+3} > a_{2k+1}$ is true for some $k \in \mathrm{N}$ and show that $a_{2k+5} > a_{2k+3}$ is true.

$$a_{2k+5} = 1 + \frac{1}{a_{2k+4}} = 1 + \frac{1}{1 + \frac{1}{a_{2k+3}}} > 1 + \frac{1}{1 + \frac{1}{a_{2k+1}}} = a_{2k+3} \tag{12}$$

Therefore, $a_{2n+3} > a_{2n+1}\ \forall n \in \mathrm{N}$. Thus, (a_{2n+1}) is an increasing subsequence of (a_n). Since (a_{2n+1}) is increasing and bounded above by 2, it must converge to some limit β.

Adjacent terms in the subsequence (a_{2n+1}) converge to the same limit; thus, the terms a_{2n+2} and a_{2n} from Equation 4 may be replaced with the limit β. Therefore, the following equations hold:

$$\beta = 1 + \frac{\beta}{1+\beta} \;\Rightarrow\; \beta = \frac{1+2\beta}{1+\beta} \;\Rightarrow\; \beta^2 + \beta = 1 + 2\beta \;\Rightarrow\; \beta^2 - \beta - 1 = 0\,. \quad (13)$$

Solving for β using the quadratic equation yields

$$\beta = \frac{1 \pm \sqrt{5}}{2}\,. \quad (14)$$

Once again, only the positive root $(1+\sqrt{5})/2$ is meaningful in this instance. Therefore, both sequences converge to the same limit $(1+\sqrt{5})/2$. Since the subsequences (a_{2n}) and (a_{2n+1}) of (a_n) converge to $(1+\sqrt{5})/2$ and any subsequence of the two subsequences converges to the same limit,

$$\lim_{n\to\infty} \frac{f_{n+1}}{f_n} = \frac{1+\sqrt{5}}{2}\,. \quad (15)$$

Amazingly, the limit of the ratio of consecutive Fibonacci terms yields number φ. This particular correlation between the Fibonacci sequence and the golden ratio is simply a glimpse into the relationship between these two powerful mathematical concepts.

Fibonacci Numbers and the Golden Ratio

The first mention of the golden section appeared in Euclid's *Elements* in 300 B.C. In his *Elements*, Euclid proposed a problem involving the division of a line into a golden section (Knott, 2007c). The problem begins with a unit segment that is divided into two lengths with one length equal to x and the other length equal to $1-x$. Euclid found that the ratio of the smaller part of the segment to the larger part of the segment is equal to the ratio of the larger part of the segment to the length of the entire segment (Knott, 2007c). Solving the equation for the value of x reveals that x is equal to the number φ.

Although Euclid was the first to study the idea of the golden section, it has been evident for thousands of years. The Greeks were fascinated by this ratio, because it frequently appeared in the field of geometry. One geometric application of the golden ratio can be found in the proportion of the lines of a pentagram (Nickel, 2001). The pentagram can be broken into different lengths where the lines cross each other to form the figure. The ratio of certain lengths in the pentagram will yield the golden ratio (Nickel, 2001). Throughout history, many artists have also used the golden ratio in their works, because it was believed to be aesthetically pleasing (Knott, 2007b). Also, many great architectural structures throughout the world have evidences of the golden ratio in their proportions (Knott, 2007b).

One of the most perplexing features of both the Fibonacci sequence and the Golden Ratio is their frequent appearances in nature. The Fibonacci sequence is often evidenced in the petal arrangement of flowers in that many flowers contain a number of petals that matches a number in the Fibonacci sequence (Knott, 2007a). The Fibonacci sequence can also be found in the spiral arrangement of pine cones, and pineapples, flower petals, and flower seed heads (Nickel, 2001). When looking either at a pine cone, a pineapple, or a sunflower, numerous spirals stemming from the center of the object are evident. According to Knott (2007a), the number of spirals on these objects is typically a Fibonacci number. The Fibonacci sequence is also apparent in the spiraling pattern of the leaves on the stems of plants. Starting from a given leaf at a specific position on a plant, the number of turns required to find another leaf in the same position is typically a Fibonacci number. Moreover, the number of leaves found within those turns is typically a Fibonacci number as well (Knott, 2007a).

According to Nickel (2001), the Fibonacci sequence is also easily found in the realm of music; for example, the keys on a piano are divided into Fibonacci numbers. Including the two notes that are an octave apart, an octave on a piano contains a total of thirteen keys, eight of which are white and five of which are black. Along the keyboard, the black keys are also separated into groups of two and three keys (Nickel, 2001). Also, numerous classical musical compositions implement the golden ratio (Beer, 2005). According to Beer (2005), one such example can be found in the “Hallelujah Chorus” from Handel’s *Messiah*. The entire musical composition consists of ninety-four measures, and one of the most important events in the song occurs during measures fifty seven and fifty eight. This particular event is located approximately eight thirteenths of the way through the composition. This ratio of two Fibonacci numbers yields the golden proportion (Beer, 2005).

God has clearly demonstrated his aesthetic nature to mankind through the golden ratio. Throughout history, many have described this ratio as both “pleasing to the eye” and “ideal” (Knott, 2007b). In His Creation, God used the golden ratio to evidence His purpose and beauty so that the world would be aesthetically pleasing and representative of the nature of God. Once humans discovered the aesthetic qualities of the golden ratio, they began to use it to perform mathematical calculations and create artistic expressions through architecture and music that imitate elements of God's Creation.

Linear Homogeneous Recurrence Relation with Constant Coefficients

One of the interesting properties of the Fibonacci sequence concerning the realm of discrete mathematics is that it is a solution to a linear homogeneous recurrence relation with constant coefficients. A linear homogeneous recurrence relation of degree k with constant coefficients is a recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + ... + c_k a_{n-k} . \qquad (16)$$

Equation 16 can be described as linear, because the left-hand side of the equation is equal to the sum of the previous terms in the sequence (Rosen, 1999). It can be also be described as homogeneous, because there are no terms in the recurrence relation that are not multiples of the $a_j 's$ (Rosen, 1999). Finally, the coefficients of the terms of the sequence in Equation 16 are all constants.

In order to solve a linear homogeneous recurrence relation, one must find a solution of the form $a_n = r^n$, where r is a constant (Rosen, 1999). Implementing this equality in Equation 16 yields:

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + ... + c_k a_{n-k}$$

$$\Rightarrow \quad r^n = c_1 r^{n-1} + c_2 r^{n-2} + \ldots + c_k r^{n-k} \tag{17}$$

Dividing both sides of Equation 17 by r^{n-k} and subtracting the entire left side of the equation from both sides of the equation yields

$$r^k - c_1 r^{k-1} - c_2 r^{k-2} - \ldots - c_{k-1} r - c_k = 0\,, \tag{18}$$

which is known as the characteristic equation of the recurrence relation (Rosen, 1999).

In order to understand more about the nature of the Fibonacci sequence as a solution linear homogeneous recurrence relation with constant coefficients, a theorem must be proven in order to establish the characteristics of recurrence relations. This theorem will be useful in determining the explicit formula for the Fibonacci sequence.

Theorem 1

Let c_1 and c_2 be real numbers. Suppose $r^2 - c_1 r - c_2 = 0$, the characteristic equation of a recurrence relation of degree two, has two distinct roots r_1 and r_2. Then the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if $a_n = \alpha_1 r_1{}^n + \alpha_2 r_2{}^n$ for $n = 0, 1, 2, \ldots$ where α_1 and α_2 are constants (Rosen, 1999).

Proof

In order to prove the theorem, one must first show that if r_1 and r_2 are roots of the characteristic equation $r^2 - c_1 r - c_2 = 0$ of a recurrence relation of degree two, and α_1 and α_2 are constants, then the sequence $\{a_n\}$ with $a_n = \alpha_1 r_1{}^n + \alpha_2 r_2{}^n$ is a solution of the recurrence relation (Rosen, 1999). Since, r_1 and r_2 are roots of $r^2 - c_1 r - c_2 = 0$,

$$r_1{}^2 = c_1 r_1 + c_2 \tag{19}$$

and

$$r_2{}^2 = c_1 r_2 + c_2 \quad . \tag{20}$$

Therefore,

$$\begin{aligned}\therefore a_n &= c_1 a_{n-1} + c_2 a_{n-2} = c_1\left(\alpha_1 r_1{}^{n-1} + \alpha_2 r_2{}^{n-1}\right) + c_2\left(\alpha_1 r_1{}^{n-2} + \alpha_2 r_2{}^{n-2}\right) \\ &= c_1\alpha_1 r_1{}^{n-1} + c_1\alpha_2 r_2{}^{n-1} + c_2\alpha_1 r_1{}^{n-2} + c_2\alpha_2 r_2{}^{n-2} \\ &= c_1\alpha_1 r_1{}^{n-1} + c_2\alpha_1 r_1{}^{n-2} + c_1\alpha_2 r_2{}^{n-1} + c_2\alpha_2 r_2{}^{n-2} \\ &= \alpha_1 r_1{}^{n-2}\left(c_1 r_1 + c_2\right) + a_2 r_2{}^{n-2}\left(c_1 r_2 + c_2\right) \quad .\end{aligned} \tag{21}$$

Now, from Equations 19, 20, and 21, the following equation can be deduced:

$$a_n = \alpha_1 r_1{}^{n-2}\left(r_1{}^2\right) + \alpha_2 r_2{}^{n-2}\left(r_2{}^2\right) = \alpha_1 r_1{}^n + \alpha_2 r_2{}^n \quad . \tag{22}$$

Therefore, the sequence $\{a_n\}$ with $a_n = \alpha_1 r_1{}^n + \alpha_2 r_2{}^n$ is solution of the recurrence relation.

Next, one must show that if the sequence $\{a_n\}$ is a solution of the recurrence relation, then $a_n = \alpha_1 r_1{}^n + \alpha_2 r_2{}^n$ for some constants α_1 and α_2 (Rosen, 1999). First, suppose $\{a_n\}$ is a solution of the recurrence relation and that the following initial conditions hold: $a_0 = C_0$ and $a_1 = C_1$. Next, one must show that there exist constants α_1 and α_2 such that the sequence $\{a_n\}$ with $a_n = \alpha_1 r_1^{\ n} + \alpha_2 r_2^{\ n}$ satisfies these same initial conditions. Thus,

$$a_0 = C_0 = \alpha_1 r_1^{\ 0} + \alpha_2 r_2^{\ 0} = \alpha_1 + \alpha_2 \quad . \tag{23}$$

Therefore,

$$\alpha_2 = C_0 - \alpha_1 , \tag{24}$$

and

$$a_1 = C_1 = \alpha_1 r_1^{\ 1} + \alpha_2 r_2^{\ 1} = \alpha_1 r_1 + \alpha_2 r_2 . \tag{25}$$

From Equations 24 and 25,

$$C_1 = \alpha_1 r_1 + (C_0 - \alpha_1) r_2 = \alpha_1 r_1 + C_0 r_2 - \alpha_1 r_2 = \alpha_1 (r_1 - r_2) + C_0 r_2 . \tag{26}$$

Therefore,

$$C_1 = \alpha_1 (r_1 - r_2) + C_0 r_2 \quad \Rightarrow \quad C_1 - C_0 r_2 = \alpha_1 (r_1 - r_2)$$

$$\Rightarrow \quad \frac{C_1 - C_0 r_2}{r_1 - r_2} = \alpha_1 . \qquad (27)$$

Also, since $\alpha_2 = C_0 - \alpha_1$ from Equation 24,

$$\alpha_2 = C_0 - \frac{C_1 - C_0 r_2}{r_1 - r_2}$$

$$= \frac{C_0(r_1 - r_2)}{r_1 - r_2} - \frac{C_1 - C_0 r_2}{r_1 - r_2} = \frac{C_0 r_1 - C_0 r_2}{r_1 - r_2} - \frac{C_1 - C_0 r_2}{r_1 - r_2}$$

$$= \frac{C_0 r_1 - C_0 r_2 - C_1 + C_0 r_2}{r_1 - r_2} = \frac{C_0 r_1 - C_1}{r_1 - r_2} \text{ when } r_1 \neq r_2 . \qquad (28)$$

Therefore, the sequence $\{a_n\}$ satisfies the initial conditions. Since the sequence is uniquely determined by these initial conditions and this recurrence relation, Equation 22 holds.

Since the Fibonacci sequence is a linear combination of the terms of the sequence, the specific recurrence relation for the Fibonacci sequence can be found in its recursive definition:

$$f_n = 0 \text{ } if \text{ } n = 0; \text{ } f_n = 1 \text{ } if \text{ } n = 1; \text{ } and \text{ } f_n = f_{n-2} + f_{n-1} \text{ } if \text{ } n > 1 . \qquad (29)$$

From Equation 29, the initial conditions of the Fibonacci sequence are $f_0 = 0$ and $f_1 = 1$. Also, the characteristic equation of the Fibonacci sequence is

$$r^2 - c_1 r - c_2 = 0 \quad \Rightarrow \quad r^2 - r - 1 = 0\,. \tag{30}$$

One must now solve the Fibonacci characteristic equation for r in order to find the roots of the equation. Using the quadratic formula yields:

$$r = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2} \tag{31}$$

From Equation 31, let

$$r_1 = \frac{1+\sqrt{5}}{2} \tag{32}$$

and let

$$r_2 = \frac{1-\sqrt{5}}{2}\,. \tag{33}$$

From Theorem 1,

$$f_n = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right)^n \tag{34}$$

for some constants α_1 and α_2. Now, use the initial conditions $f_0 = 0,\ f_1 = 1$ to find α_1 and α_2.

$$f_0 = \alpha_1\left(\frac{1+\sqrt{5}}{2}\right)^0 + \alpha_2\left(\frac{1-\sqrt{5}}{2}\right)^0 = \alpha_1 + \alpha_2 = 0 \quad \Rightarrow \quad \alpha_1 = -\alpha_2 \tag{35}$$

$$f_1 = \alpha_1\left(\frac{1+\sqrt{5}}{2}\right)^1 + \alpha_2\left(\frac{1-\sqrt{5}}{2}\right)^1 = 1 \quad \Rightarrow \quad \alpha_1\left(1+\sqrt{5}\right) + \alpha_2\left(1-\sqrt{5}\right) = 2 \tag{36}$$

Now, substitute $-\alpha_2$ for α_1.

$$-\alpha_2\left(1+\sqrt{5}\right) + \alpha_2\left(1-\sqrt{5}\right) = 2 \quad \Rightarrow \quad -\alpha_2 - \alpha_2\sqrt{5} + \alpha_2 - \alpha_2\sqrt{5} = 2 \tag{37}$$

Therefore,

$$\alpha_2 = -\frac{1}{\sqrt{5}} \text{ and } \alpha_1 = \frac{1}{\sqrt{5}} \tag{38}$$

Substituting these values for α_1 and α_2 into Equation 34 yields the following equation:

$$f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n . \tag{39}$$

Equation 39 is the explicit formula for the Fibonacci numbers. This particular formula can yield any term in the Fibonacci sequence. The value n in Equation 39 denotes the subscript of the particular term in the sequence.

Fibonacci Sequence as a Generating Function

Generally, any sequence of the form $a_0, a_1, a_3, a_4, a_5, ...$ can be grouped into a power series of the form $A(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + ...$ (Silverman, 2006). This particular function is called the generating function for the sequence. The generating function $A(x)$ is a function of the variable x so that if a value is substituted into the equation for that x variable, then a value for $A(x)$ can be determined. The generating function for the Fibonacci sequence is

$$f(x) = f_1x + f_2x^2 + f_3x^3 + f_4x^4 + f_5x^5 + f_6x^6 + ... \tag{40}$$

In order to determine the interval of convergence of the generating function for the Fibonacci sequence, the following limit ratio test must be used:

$$\lim_{n\to\infty}\left|\frac{f_{n+1}}{f_n}\right||x|. \tag{41}$$

Because $|x|$ is considered to be a constant inside the limit,

$$\lim_{n\to\infty}\left|\frac{f_{n+1}}{f_n}\right||x| = |x|\lim_{n\to\infty}\left|\frac{f_{n+1}}{f_n}\right|. \tag{42}$$

Also, from Equation 15,

$$\lim_{n\to\infty}\frac{f_{n+1}}{f_n}=\frac{1+\sqrt{5}}{2}. \tag{43}$$

Therefore,

$$|x|\lim_{n\to\infty}\left|\frac{f_{n+1}}{f_n}\right|=|x|\left(\frac{1+\sqrt{5}}{2}\right) \tag{44}$$

The interval of convergence for a generating function is found by setting the value determined from the limit to be less than 1. Thus,

$$|x|\left(\frac{1+\sqrt{5}}{2}\right)<1 \quad\Rightarrow\quad |x|<\frac{2}{1+\sqrt{5}}. \tag{45}$$

Now, rationalize the denominator.

$$|x|<\frac{2}{1+\sqrt{5}}\cdot\frac{(1-\sqrt{5})}{(1-\sqrt{5})}=\frac{2(1-\sqrt{5})}{1-5}=\frac{2(1-\sqrt{5})}{-4}=\frac{\sqrt{5}-1}{2} \tag{46}$$

Therefore, the interval of convergence for the Fibonacci sequence is

$$|x|<\frac{\sqrt{5}-1}{2}. \tag{47}$$

In order to determine the generating function formula of the Fibonacci sequence, one must substitute the corresponding Fibonacci numbers into the original generating function (Silverman, 2006). Therefore,

$$f(x) = x + x^2 + 2x^3 + 3x^4 + 5x^5 + 8x^6 + 13x^7 + \ldots \qquad (48)$$

Now, one must find a simple expression for $f(x)$. Using the recursive definition of the Fibonacci sequence, $f_n = f_{n-1} + f_{n-2}$, replace all the Fibonacci numbers accordingly (Silverman, 2006). Therefore,

$$f(x) = f_1 x + f_2 x^2 + f_3 x^3 + f_4 x^4 + f_5 x^5 + f_6 x^6 + \ldots$$

$$= f_1 x + f_2 x^2 + (f_2 + f_1)x^3 + (f_3 + f_2)x^4 + (f_4 + f_3)x^5 + (f_5 + f_4)x^6 + \ldots \qquad (49)$$

Now, ignore the first two terms of Equation 49 momentarily and regroup the remaining terms (Silverman, 2006).

$$f(x) = f_1 x + f_2 x^2 + x^2 \underbrace{\{f_1 x + f_2 x^2 + f_3 x^3 + \ldots\}}_{\Downarrow \; f(x)} + x \underbrace{\{f_2 x^2 + f_3 x^3 + f_4 x^4 + \ldots\}}_{\Downarrow \; f(x) - f_1 x} \qquad (50)$$

$$\therefore f(x) = f_1 x + f_2 x^2 + x^2 f(x) + x(f(x) - f_1 x) \qquad (51)$$

Now, use the values $f_1 = 1$ and $f_2 = 1$ to determine the following formula:

$$f(x) = x + x^2 + x^2 \cdot f(x) + x \cdot f(x) - x^2 = x + x \cdot f(x) + x^2 \cdot f(x)$$

Therefore,

$$f(x) = x + x \cdot f(x) + x^2 \cdot f(x). \tag{52}$$

Now, simply solve for $f(x)$ to obtain the Fibonacci generating function formula.

$$\begin{gathered} f(x) = x + x \cdot f(x) + x^2 \cdot f(x) \quad \Rightarrow \quad f(x) - x \cdot f(x) - x^2 \cdot f(x) = x \\ \therefore\ f(x)(1 - x - x^2) = x \quad \Rightarrow \quad f(x) = \frac{x}{1 - x - x^2} \end{gathered} \tag{53}$$

Amazingly, the explicit formula for the Fibonacci sequence can also be derived from the sequence's generating function formula. The two roots of the polynomial expression $1 - x - x^2$ are $(-1 \pm \sqrt{5})/2$, and the reciprocals of these two roots are $(1 \pm \sqrt{5})/2$. In order to separate the two roots, let

$$\alpha = \frac{1 + \sqrt{5}}{2} \tag{54}$$

and let

$$\beta = \frac{1 - \sqrt{5}}{2}. \tag{55}$$

One must first factor the polynomial expression $1 - x - x^2$ using α and β.

$$1 - x - x^2 = (1 - \alpha x)(1 - \beta x) \tag{56}$$

Now, use this factorization to split the generating function using partial fractions.

$$\frac{x}{1 - x - x^2} = \frac{x}{(1 - \alpha x)(1 - \beta x)} = \frac{A}{(1 - \alpha x)} + \frac{B}{(1 - \beta x)} \tag{57}$$

$$x = A \cdot (1 - \beta x) + B \cdot (1 - \alpha x) \quad \Rightarrow \quad x = A - A\beta x + B - B\alpha x \tag{58}$$

$$\therefore x + 0 = x \cdot (-A\beta - B\alpha) + (A + B) \tag{59}$$

Therefore,

$$1 = -A\beta - B\alpha \tag{60}$$

from the coefficients of the x terms on each side of Equation 59. Also,

$$0 = A + B \quad \Rightarrow \quad -A = B. \tag{61}$$

$$\therefore 1 = -A\beta + A\alpha \quad \Rightarrow \quad 1 = A(-\beta + \alpha). \tag{62}$$

$$\therefore A = \frac{1}{\alpha - \beta}. \tag{63}$$

Now, substitute the value for A back into Equation 61. Therefore,

$$B = -\left(\frac{1}{\alpha - \beta}\right) = \frac{1}{\beta - \alpha}. \tag{64}$$

Now, recall from Equations 54 and 55 that

$$\alpha = \frac{1+\sqrt{5}}{2} \text{ and } \beta = \frac{1-\sqrt{5}}{2}.$$

Therefore,

$$A = \frac{1}{\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}} = \frac{1}{\frac{1}{2} + \frac{\sqrt{5}}{2} - \frac{1}{2} + \frac{\sqrt{5}}{2}} = \frac{1}{\frac{2\sqrt{5}}{2}} = \frac{1}{\sqrt{5}}. \tag{65}$$

Similarly,

$$B = \frac{1}{\frac{1-\sqrt{5}}{2} - \frac{1+\sqrt{5}}{2}} = \frac{1}{\frac{1}{2} - \frac{\sqrt{5}}{2} - \frac{1}{2} - \frac{\sqrt{5}}{2}} = \frac{1}{\frac{-2\sqrt{5}}{2}} = -\frac{1}{\sqrt{5}} \tag{66}$$

Now, substitute the values for A and B from Equations 63 and 64 into Equation 57.

$$\therefore \frac{x}{1-x-x^2} = \frac{1}{\sqrt{5}}\left(\frac{1}{1-\alpha x}\right) - \frac{1}{\sqrt{5}}\left(\frac{1}{1-\beta x}\right) \tag{67}$$

However, according to Silverman (2006), $1/(1-\alpha x)$ can be expressed as the geometric series $1+\alpha x+(\alpha x)^2+(\alpha x)^3+\dots$ Similarly, $1/(1-\beta x)$ can be expressed as the geometric series $1+\beta x+(\beta x)^2+(\beta x)^3+\dots$ Now, one must express the Fibonacci generating function formula as a power series. Therefore,

$$\frac{x}{1-x-x^2}=\frac{1}{\sqrt{5}}\left(\frac{1}{1-\alpha x}\right)-\frac{1}{\sqrt{5}}\left(\frac{1}{1-\beta x}\right)$$

$$=\frac{1}{\sqrt{5}}\left(1+\alpha x+(\alpha x)^2+(\alpha x)^3+\dots\right)-\frac{1}{\sqrt{5}}\left(1+\beta x+(\beta x)^2+(\beta x)^3+\dots\right) \tag{68}$$

Now, combine terms appropriately. Therefore,

$$\frac{1}{\sqrt{5}}\left(1+\alpha x+(\alpha x)^2+(\alpha x)^3+\dots\right)-\frac{1}{\sqrt{5}}\left(1+\beta x+(\beta x)^2+(\beta x)^3+\dots\right)$$

$$=\frac{\alpha-\beta}{\sqrt{5}}x+\frac{\alpha^2-\beta^2}{\sqrt{5}}x^2+\frac{\alpha^3-\beta^3}{\sqrt{5}}x^3+\dots \tag{69}$$

Referring back to the original Fibonacci generating function formula

$$f(x)=\frac{x}{1-x-x^2}=f_1x+f_2x^2+f_3x^3+f_4x^4+\dots$$

$$=\frac{\alpha-\beta}{\sqrt{5}}x+\frac{\alpha^2-\beta^2}{\sqrt{5}}x^2+\frac{\alpha^3-\beta^3}{\sqrt{5}}x^3+\frac{\alpha^4-\beta^4}{\sqrt{5}}x^4+\dots \tag{70}$$

Equating the corresponding coefficients from Equation 70 yields

$$f_1 = \frac{\alpha - \beta}{\sqrt{5}},\ f_2 = \frac{\alpha^2 - \beta^2}{\sqrt{5}},\ f_3 = \frac{\alpha^3 - \beta^3}{\sqrt{5}},\ f_4 = \frac{\alpha^4 - \beta^4}{\sqrt{5}}, \ldots$$

Therefore,

$$f_n = \frac{\alpha^n - \beta^n}{\sqrt{5}} = \frac{1}{\sqrt{5}}\left(\alpha^n - \beta^n\right). \tag{71}$$

Now, substitute the values for α and β from Equations 54 and 55 into Equation 71 in order to, once again, arrive at the explicit formula for the Fibonacci numbers:

$$f_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]. \tag{72}$$

The Fibonacci sequence's characteristics as both a linear homogeneous recurrence relation with constant coefficients and a generating function demonstrate once again the relationship between the Fibonacci sequence and the golden ratio. God has used the Fibonacci sequence to exhibit the interrelatedness of specific mathematical fields. Although various mathematical fields involve different topics and different methods of solution, both the Fibonacci sequence and the golden ratio represent mathematical concepts that bridge the gap between particular fields of mathematics. These two concepts reveal the perfect designing nature of God, the Creator of both the universe and knowledge.

Lame's Theorem

Gabriel Lame contributed numerous ideas to the realm of number theory during the nineteenth century (Rosen, 1999). One of Lame's most famous accomplishments was determining an upper bound for the number of divisions used by the Euclidean algorithm (Rosen, 1999). Interestingly, Lame's discovery involved the use of Fibonacci numbers. The following lemma and proof convey Lame's findings.

Lemma

Let f_n denote the nth Fibonacci number. Also let $\alpha = \left(1+\sqrt{5}\right)/2$. Prove that $\alpha^{n-2} < f_n$ whenever $n \geq 3$ (Rosen, 1999).

Proof

The proof of this lemma can be completed by the induction method (Rosen, 1999). Let $P(n)$ be $f_n > \alpha^{n-2}$. One must show that $P(n)$ is true when $n \geq 3$ and $n \in \mathrm{Z}^{+}$. First, verify that $P(3)$ and $P(4)$ are true (Rosen, 1999).

$$P(3):\ \alpha^{3-2} = \alpha = \frac{1+\sqrt{5}}{2} < 2 = f_3 \tag{73}$$

$$P(4):\ \alpha^{4-2} = \alpha^2 = \left(\frac{1+\sqrt{5}}{2}\right)^2 = \frac{3+\sqrt{5}}{2} < 3 = f_4 \tag{74}$$

Therefore, $P(3)$ and $P(4)$ are both true. Now, continue with the inductive step and assume that $P(k)$ is true $\left(i.e.\ \ f_k > \alpha^{k-2}\right)$ for all $k \ni\ 3 \leq k \leq n$, where $n \geq 4$.

Furthermore, one must show that $P(n+1)$ is true $\left(i.e.\ f_{n+1} > \alpha^{n-1}\right)$. Since α is a solution to the quadratic equation $x^2 - x - 1 = 0$,

$$\alpha^2 - \alpha - 1 = 0 \quad \Rightarrow \quad \alpha^2 = \alpha + 1 \tag{75}$$

Now, using the information from Equation 75 and basic algebra,

$$\alpha^{n-1} = \alpha^2 \cdot \alpha^{n-3} = (\alpha + 1)\alpha^{n-3} = \alpha \cdot \alpha^{n-3} + \alpha^{n-3} = \alpha^{n-2} + \alpha^{n-3}. \tag{76}$$

Furthermore, it has already been assumed that both

$$f_{n-1} > \alpha^{n-3} \tag{77}$$

and

$$f_n > \alpha^{n-2} \tag{78}$$

are true. Now, the Fibonacci recursive definition, $f_{n+1} = f_n + f_{n-1}$, can be used to prove the remainder of the lemma. From Equations 76, 77, and 78, and the recursive definition of the Fibonacci sequence,

$$f_{n+1} = f_n + f_{n-1} > \alpha^{n-2} + \alpha^{n-3} = \alpha^{n-1}. \tag{79}$$

Therefore, $P(n+1)$ is also true. Thus,

$$\alpha^{n-2} < f_n \text{ whenever } n \geq 3.$$

Lame's Theorem

Let a and b be positive integers with $a \geq b$. Then the number of divisions used by the Euclidean Algorithm to find $\gcd(a,b)$ is less than or equal to five times the number of decimal digits in b (Rosen, 1999).

Proof

For notational purposes, let f_n denote the nth Fibonacci number. Also, let $a = r_0$ and $b = r_1$ in the Euclidean algorithm (Rosen, 1999). Now, apply the Euclidean algorithm using both a and b.

$$\begin{aligned}
r_0 &= r_1 q_1 + r_2 \qquad 0 \leq r_2 < r_1 \\
r_1 &= r_2 q_2 + r_3 \qquad 0 \leq r_3 < r_2 \\
r_2 &= r_3 q_3 + r_4 \qquad 0 \leq r_4 < r_3 \\
&\vdots
\end{aligned}$$

$$r_{n-2} = r_{n-1} q_{n-1} + r_n \quad 0 \leq r_n < r_{n-1} \tag{80}$$

$$r_{n-1} = r_n q_n \tag{81}$$

From Equations 80 and 81, one can determine that n divisions are needed to evaluate $\gcd(a,b) = r_n$. Also, from the Euclidean algorithm, we know that $r_i < r_{i-1}$. Therefore, each $q_i \geq 1$ for $1 \leq i \leq n$, because from the algorithm $r_{i-1} = r_i q_i + r_{i+1}$ and $r_i < r_{i-1}$. Each

r_i must be multiplied by an integer that is great than or equal to one $(i.e.\ q_i)$. In particular, $r_n < r_{n-1}$. Therefore, $q_n \geq 2$, because the remainder for $r_{n-1} = r_n q_n$ is zero and $q_n \in \mathrm{Z}$. Since $r_n \in \mathrm{Z}^+$, $r_n \geq 1$. Also, $r_n \geq 1$ and $q_n \geq 2$; therefore, $r_{n-1} \geq 2$. Since the third term of the Fibonacci sequence (i.e. f_3) is equal to two, $r_{n-1} \geq f_3$. Using the information from Equation 80, $r_{n-2} = r_{n-1}q_{n-1} + r_n \geq r_{n-1} + r_n$; because $q_{n-1} \geq 1$. Therefore, $r_{n-1} + r_n \geq f_3 + 1$; because $r_{n-1} \geq f_3$ and $r_n \geq 1$. It is clear from the Fibonacci sequence that, $f_2 = 1$; therefore, $f_3 + 1 = f_3 + f_2$. Also, recall from the Fibonacci recursive definition that $f_4 = f_3 + f_2$. Continuing along the same line of reasoning,

$$r_{n-3} = r_{n-2}q_{n-2} + r_{n-1} \geq r_{n-2} + r_{n-1} \geq f_4 + f_3 = f_5. \tag{82}$$

Eventually, this process will lead to

$$r_1 = r_2 q_2 + r_3 \geq r_2 + r_3 \geq f_n + f_{n-1} = f_{n+1}. \tag{83}$$

Since, $b = r_1$,

$$b \geq f_{n+1}. \tag{84}$$

It has already been previously established from the lemma to Lame's Theorem that,

$$f_{n+1} > \alpha^{n-1}, \text{ for } n > 2 \text{ and } \alpha = \frac{1+\sqrt{5}}{2}. \tag{85}$$

Thus, from Equations 84 and 85, it is clear that

$$b > f_{n+1} > \alpha^{n-1} \quad \Rightarrow \quad b > \alpha^{n-1}. \tag{86}$$

Now, one must take the logarithm of both sides of the inequality (Rosen, 1999). The inequality is preserved, because both b and α^{n-1} are positive from Equations 85 and 86 and the logarithm function is an increasing function. Therefore,

$$\log_{10} b > \log_{10}\left(\alpha^{n-1}\right) = (n-1)\log_{10} \alpha. \tag{87}$$

Using the value of α from Equation 85, $\log_{10} \alpha \approx 0.209 > 0.20 = 1/5$ (Rosen, 1999). Now, substitute $1/5$ for the value of $\log_{10} \alpha$. Therefore,

$$\log_{10} b > (n-1)\left(\frac{1}{5}\right) = \frac{n-1}{5} \quad \Rightarrow \quad \log_{10} b > \frac{n-1}{5}. \tag{88}$$

Now, let k be the number of decimal digits in b (Rosen, 1999). Therefore,

$$b < 10^{k}. \tag{89}$$

Now, take the logarithm of both sides of the inequality (Rosen, 1999). The inequality is preserved, because both b and 10^k, $k \in Z^+$ are positive. Therefore,

$$\log_{10} b < k \cdot \log_{10} 10 = k(1) = k \,. \tag{90}$$

Thus, it is clear that

$$\log_{10} b < k \tag{91}$$

Now, combine Equations 88 and 91.

$$k > \log_{10} b > \frac{n-1}{5} \quad \Rightarrow \quad k > \frac{n-1}{5} \quad \Rightarrow \quad 5k > n-1 \quad \Rightarrow \quad 5k+1 > n \tag{92}$$

Since $n \in Z^+$, $n \leq 5k$. Recall that n is the number of divisions used by the Euclidean algorithm to find $\gcd(a,b) = r_n$, and k is the number of decimal digits in b. Therefore, based upon Equation 92, Lame's Theorem has been proven.

Further investigation of Lame's Theorem

From Equation 88, it has been previously established that

$$\log_{10} b > \frac{n-1}{5} \,. \tag{93}$$

Therefore,

$$5 \cdot \log_{10} b > n - 1 \quad \Rightarrow \quad n < 1 + 5 \cdot \log_{10} b \ . \tag{94}$$

Since $b \geq f_{n+1} = f_n + f_{n-1}$ and f_{n-1} must be at least the first term in the Fibonacci sequence,

$$b \geq f_{n+1} \geq 2 \ . \tag{95}$$

Thus, $b \geq 2$. Therefore,

$$5 \cdot \log_{10} b \geq 5 \cdot \log_{10} 2 \approx 1.5 > 1 \quad \Rightarrow \quad 5 \cdot \log_{10} b > 1 \quad . \tag{96}$$

From Equations 94 and 96,

$$n < 1 + 5 \cdot \log_{10} b < 5 \cdot \log_{10} b + 5 \cdot \log_{10} b = 10 \cdot \log_{10} b \ . \tag{97}$$

Therefore, according to Rosen (1999), $n < 10 \cdot \log_{10} b$ and $10 \cdot \log_{10} b = \mathrm{O}(\log_{10} b)$. Thus,

$$n < \mathrm{O}(\log_{10} b). \tag{98}$$

Therefore, $\mathrm{O}(\log_{10} b)$ divisions are needed to compute $\gcd(a,b)$ by the Euclidean algorithm, when $a > b$.

More Applications to Number Theory

The uniqueness of the relationship among the terms in the Fibonacci sequence led Fibonacci and other mathematicians to develop theorems in the field of number theory that evidence the fascinating properties and characteristics of the terms in the Fibonacci sequence. Using the Fibonacci numbers, mathematicians have also developed mathematical identities that are helpful in proving those theorems. This section will address a few of those theorems and identities.

Theorem 2

For the Fibonacci sequence, $\gcd\left(f_n, f_{n+1}\right) = 1$ for every $n \geq 1$ (Burton, 2002).

Proof

Assume there exists an integer $d > 1$ that divides both f_n and f_{n+1} (Burton, 2002). Therefore, their difference $\left(i.e.\ f_{n+1} - f_n\right)$ would be divisible by d. Since $f_{n+1} = f_n + f_{n-1}$, $f_{n+1} - f_n = f_{n-1}$. Also, $f_n - f_{n-1} = f_{n-2}$ must be divisible by d since both f_n and f_{n-1} are divisible by d. Therefore,

$$d \mid f_{n-2} \tag{99}$$

Continuing to work backwards, this method can be used to show that

$$d \mid f_{n-3}\ ,\ d \mid f_{n-4}\ ,\ \ldots \tag{100}$$

Finally, it can be shown that $d \mid f_1$; however, $f_1 = 1$, and d can't divide 1 when $d > 1$. Thus, a contradiction has been encountered in the proof. Therefore, Theorem 2 has been proven.

Identity 1

$$f_{m+n} = f_{m-1} f_n + f_m f_{n+1} \tag{101}$$

Proof

The proof of this identity can also be completed using the induction method on n (Burton, 2002). First, let $n = 1$. Thus,

$$f_{m+1} = f_{m-1} f_1 + f_m f_2 = f_{m-1} + f_m \tag{102}$$

since both f_1 and f_2 are equal to one. Equation 102 is identical to the Fibonacci recursive definition. Now, assume $f_{m+n} = f_{m-1} f_n + f_m f_{n+1}$ for all $n = 1, 2, 3, 4, \ldots, k$. One must now verify that the identity is true when $n = k + 1$. It has already been assumed that

$$f_{m+k} = f_{m-1} f_k + f_m f_{k+1} \tag{103}$$

and

$$f_{m+(k-1)} = f_{m-1} f_{k-1} + f_m f_k \tag{104}$$

Now, add Equations 103 and 104 together.

$$f_{m+k} + f_{m+(k+1)} = f_{m-1} f_k + f_m f_{k+1} + f_{m-1} f_{k-1} + f_m f_k$$
$$= f_{m-1}\left(f_k + f_{k-1}\right) + f_m\left(f_{k+1} + f_k\right) \quad (105)$$

Incorporating the Fibonacci recursive definition once again yields the following equation:

$$f_{m+(k+1)} = f_{m-1} f_{k+1} + f_m f_{k+2} \quad (106)$$

Equation 106 is identical to Equation 101 with $n = k + 1$. Therefore,

$f_{m+n} = f_{m-1} f_n + f_m f_{n+1}$ holds $\forall\, m \geq 2$ and $n \geq 1$.

Theorem 3

For $m \geq 1$, $n \geq 1$, f_{mn} is divisible by f_m (Burton, 2002).

Proof

The proof of this theorem can be completed using the induction method on n (Burton, 2002). First , let $n = 1$ and m be any integer. When $n = 1$, $f_{mn} = f_{m(1)} = f_m$, which, of course, is divisible by f_m. Now, assume f_{mn} is divisible by f_m for the cases $n = 1, 2, 3, 4, ..., k$. One must now show that $f_{m(k+1)}$ is divisible by f_m. Use Identity 1 from Equation 101 $\left(\textit{i.e.}\ f_{m+n} = f_{m-1} f_n + f_m f_{n+1}\right)$ for the following equation:

$$f_{m(k+1)} = f_{mk+m} . \quad (107)$$

Therefore,

$$f_{m(k+1)} = f_{mk+m} = f_{mk-1} f_m + f_{mk} f_{m+1} \quad . \tag{108}$$

Earlier, it was assumed that f_{mk} is divisible by f_m; therefore, an f_m may be factored from each term on the right-hand side of Equation 108. Thus, the entire right-hand side of the Equation 108 above is divisible by f_m. Now, it is clear that the left-hand side of Equation 108 is also divisible by f_m $\left(i.e.\ \ f_m \mid f_{m(k+1)}\right)$. Therefore, for $m \geq 1,\ n \geq 1,\ f_{mn}$ is divisible by f_m.

Identity 2

$$f_{n+3} = 3\, f_{n+1} - f_{n-1}\,,\ n \geq 2 \tag{109}$$

Proof

Apply Identity 1 $\left(f_{m+n} = f_{m-1} f_n + f_m f_{n+1}\right)$ to the left-hand side Equation 109. Therefore,

$$f_{n+3} = f_{n-1} f_3 + f_n f_4 = 2\, f_{n-1} + 3\, f_n \quad . \tag{110}$$

Now, use the Fibonacci recursive definition as follows:

$$f_{n+3} = 2\, f_{n-1} + 3\, f_n = 2\left(f_{n+1} - f_n\right) + 3\, f_n = 2\, f_{n+1} - 2\, f_n + 3\, f_n = 2\, f_{n+1} + f_n \quad . \tag{111}$$

Thus,

$$f_{n+3} = 2f_{n+1} + f_n \ . \tag{112}$$

Now, apply the Fibonacci recursive definition once again.

$$f_{n+3} = 2f_{n+1} + f_n = 2f_{n+1} + \left(f_{n+1} - f_{n-1}\right) = 3f_{n+1} - f_{n-1} \tag{113}$$

Therefore, $f_{n+3} = 3f_{n+1} - f_{n-1}$, $n \geq 2$.

Identity 3

$$f_1 + f_2 + f_3 + f_4 + f_5 + ... + f_n = f_{n+2} - 1 \tag{114}$$

Proof

By the Fibonacci recursive definition, the following relations hold:

$$\begin{aligned} f_1 &= f_3 - f_2 \\ f_2 &= f_4 - f_3 \\ f_3 &= f_5 - f_4 \\ f_4 &= f_6 - f_5 \\ f_5 &= f_7 - f_6 \\ &\vdots \\ f_{n-1} &= f_{n+1} - f_n \\ f_n &= f_{n+2} - f_{n+1} \end{aligned}$$

Add all of the above equations together. The terms on the left sides of the equations yield the sum of the first n Fibonacci numbers, and the terms on the right sides of the equations cancel except for the terms f_2 and f_{n+2}. Therefore,

$$f_1 + f_2 + f_3 + f_4 + f_5 + \ldots + f_n = f_{n+2} - f_2 \quad . \tag{115}$$

Since $f_2 = 1$, $f_1 + f_2 + f_3 + f_4 + f_5 + \ldots + f_n = f_{n+2} - 1$.

Identity 4

$$f_n^{\ 2} = f_{n+1} f_{n-1} + (-1)^{n-1} \tag{116}$$

First subtract $f_{n+1} f_{n-1}$ from both sides of Equation 116 in order to prove the following variation of Identity 4:

$$f_n^{\ 2} - f_{n+1} f_{n-1} = (-1)^{n-1} \tag{117}$$

Start with the right side of Equation 117 and implement the use of the Fibonacci recursive definition (Burton, 2002).

$$f_n^{\ 2} - f_{n+1} f_{n-1} = f_n (f_{n-1} + f_{n-2}) - f_{n+1} f_{n-1} = f_n f_{n-1} + f_n f_{n-2} - f_{n+1} f_{n-1}$$

$$= (f_n - f_{n+1}) f_{n-1} + f_n f_{n-2} \tag{118}$$

Now, since $f_{n+1} = f_n + f_{n-1}$, the expression $f_n - f_{n+1}$ from the right-hand side of the Equation 118 can be expressed as

$$f_n - f_{n+1} = f_n - \left(f_n + f_{n-1}\right) = f_n - f_n - f_{n-1} = -f_{n-1}\ . \tag{119}$$

Now substitute the value from Equation 119 into Equation 118.

$$\begin{aligned} f_n^{\ 2} - f_{n+1} f_{n-1} &= \left(-f_{n-1}\right) f_{n-1} + f_n f_{n-2} \\ &= -\left(f_{n-1}\right)^2 + f_n f_{n-2} = (-1)\left(f_{n-1}^{\ 2} - f_n f_{n-2}\right) \end{aligned} \tag{120}$$

Therefore,

$$f_n^{\ 2} - f_{n+1} f_{n-1} = (-1)\left(f_{n-1}^{\ 2} - f_n f_{n-2}\right). \tag{121}$$

Now, the right and left-hand sides of Equation 121 are identical, except for the initial sign and the fact that all of the subscripts have been decreased by one on the right-hand side. By repeating the same argument,

$$f_n^{\ 2} - f_{n+1} f_{n-1} = (-1)\left(f_{n-1}^{\ 2} - f_n f_{n-2}\right) = (-1)(-1)\left(f_{n-2}^{\ 2} - f_{n-1} f_{n-3}\right)$$

$$= (-1)^2\left(f_{n-2}^{\ 2} - f_{n-1} f_{n-3}\right) \tag{122}$$

Now, continue the same pattern. After $n-2$ steps have been completed,

$$f_n{}^2 - f_{n+1} f_{n-1} = (-1)^{n-2}\left(f_2{}^2 - f_3 f_1\right). \tag{123}$$

From the Fibonacci sequence, $f_1 = 1$, $f_2 = 1$, and $f_3 = 2$. Therefore,

$$f_n{}^2 - f_{n+1} f_{n-1} = (-1)^{n-2}(1-2) = (-1)^{n-2}(-1) = (-1)^{n-1} \tag{124}$$

Therefore, $f_n{}^2 = f_{n+1} f_{n-1} + (-1)^{n-1}$.

Identity 5

$$f_{2k}{}^2 = f_{2k+1} f_{2k-1} - 1 \tag{125}$$

Proof

Identity 5 simply explores the case when $n = 2k$ for Identity 4. First use Identity 4 from Equation 116, but substitute $n = 2k$. Therefore,

$$f_{2k}{}^2 = f_{2k+1} f_{2k-1} + (-1)^{2k-1}. \tag{126}$$

Since, $2k$ is an even number, we know that $2k-1$ is an odd number. An odd power of (-1) simply produces (-1). Therefore, $f_{2k}{}^2 = f_{2k+1} f_{2k-1} - 1$.

Lemma to Theorem 4

If $m = qn + r$, then $\gcd(f_m, f_n) = \gcd(f_r, f_n)$ (Burton, 2002).

Proof

Since $m = qn + r$,

$$\gcd(f_m, f_n) = \gcd(f_{qn+r}, f_n). \tag{127}$$

Now, implement Identity 1 $(f_{m+n} = f_{m-1} f_n + f_m f_{n+1})$ in Equation 127.

$$\therefore \gcd(f_m, f_n) = \gcd(f_{qn+r}, f_n) = \gcd(f_{qn-1} f_r + f_{qn} f_{r+1}, f_n) \tag{128}$$

From Theorem 3, f_{qn} is divisible by f_n when $n \geq 1$, $q \geq 1$. Therefore, $f_{qn} f_{r+1}$ is divisible by f_n. Now, let $f_{qn} f_{r+1}$ be represented by c. Also, let $f_{qn-1} f_r$ be represented by a and let f_n be represented by b (Burton, 2002). Substituting these values into Equation 128 yields

$$\gcd(f_{qn-1} f_r + f_{qn} f_{r+1}, f_n) = \gcd(a + c, b) . \tag{129}$$

Since $f_n \mid f_{qn} f_{r+1}$, we know that $b \mid c$. Because $b \mid c$, $\gcd(a + c, b) = \gcd(a, b)$, because c is simply a multiple of b (Burton, 2002). Therefore,

$$\gcd(f_{qn-1} f_r + f_{qn} f_{r+1}, f_n) = \gcd(f_{qn-1} f_r, f_n) \tag{130}$$

At this time, the claim must be made that $\gcd(f_{qn-1}, f_n) = 1$. The proof of this claim is essential to the completion of the entire proof of the lemma. At this time, let

$$d = \gcd(f_{qn-1}, f_n) \tag{131}$$

From Equation 131, it is clear that $d \mid f_n$. It has also been previously established from Theorem 3 that $f_n \mid f_{qn}$. Therefore, $d \mid f_{qn}$. From Equation 131, it is also clear that $d \mid f_{qn-1}$. Therefore, d divides consecutive Fibonacci numbers f_{qn} and f_{qn-1}. However, Theorem 2 states that consecutive Fibonacci numbers are relatively prime $\left(\textit{i.e.}\ \gcd(f_i, f_{i+1}) = 1\right)$. Therefore, $d = 1$. Thus, $\gcd(f_{qn-1}, f_n) = 1$.

According to Burton (2002), the following number theory property must now be implemented: if $\gcd(x, z) = 1$, then $\gcd(x, yz) = \gcd(x, y)$. Let $x = f_n$, $y = f_r$, and $z = f_{qn-1}$ in this property (Burton, 2002). Since $\gcd(f_{qn-1}, f_n) = 1$, the above property implies that $\gcd(f_n, f_{qn-1}) = \gcd(f_n, f_r)$. Therefore,

$$\begin{aligned} \gcd(f_m, f_n) &= \gcd(f_{qn+r}, f_n) = \gcd(f_{qn-1} f_r + f_{qn} f_{r+1}, f_n) \\ &= \gcd(f_{qn-1} f_r, f_n) = \gcd(f_r, f_n) \end{aligned} \tag{132}$$

Therefore, $\gcd(f_m, f_n) = \gcd(f_r, f_n)$.

Theorem 4

The greatest common divisor of two Fibonacci numbers is again a Fibonacci number; specifically, $\gcd(f_m, f_n) = f_d$ where $d = \gcd(m, n)$ (Burton, 2002).

Proof

First, assume that $m \geq n$ and apply the Euclidean algorithm to m and n.

$$\begin{array}{ll} m = q_1 n + r_1 & 0 < r_1 < n \\ n = q_2 r_1 + r_2 & 0 < r_2 < r_1 \\ r_1 = q_3 r_2 + r_3 & 0 < r_3 < r_2 \\ \quad\vdots & \\ r_{n-2} = q_n r_{n-1} + r_n & 0 < r_n < r_{n-1} \\ r_{n-1} = q_{n+1} r_n + 0 & \end{array} \tag{133}$$

Using the lemma to Theorem 4,

$$\gcd(f_m, f_n) = \gcd(f_r, f_n) = \gcd(f_{r_1}, f_{r_2}) = \ldots = \gcd(f_{r_{n-1}}, f_{r_n}). \tag{134}$$

Because $r_n \mid r_{n-1}$ from Equation 133, Theorem 3 verifies that $f_{r_n} \mid f_{r_{n-1}}$. Therefore,

$$\gcd(f_{r_{n-1}}, f_{r_n}) = f_{r_n} \ . \tag{135}$$

However, r_n is the last non-zero remainder from the Euclidean Algorithm for m and n. Thus, $\gcd(m, n) = r_n$. Therefore,

$$\gcd\left(f_{r_{n-1}}, f_{r_n}\right) = f_{r_n} = f_{\gcd(m,n)} = f_d , \tag{136}$$

where $r_n = \gcd(m,n) = d$.

These theorems and identities in the field of number theory display not only the unique relationships among Fibonacci terms but also the rigid structure of the realm of mathematics. Although many different languages are spoken throughout the world, there is the only one universal language: mathematics. God has given humans the ability to communicate through the succinct, ordered language of mathematics in order to discover both His beauty and His sovereignty.

Fibonacci Puzzles

Identity 4 can be used to explain a famous geometric deception related to the Fibonacci sequence (Burton, 2002). This deception focuses upon the division of a square into four shapes: two triangles and two trapezoids. The specific division of these shapes is evidenced in Figure 2 (Burton, 2002). Once the square has been divided properly, the pieces are then rearranged to form a rectangle, which is exhibited in Figure 3 (Burton, 2002).

Based upon the dimensions of Figure 2, the square has an area of sixty four units. However, when the four shapes are rearranged to create the rectangle in Figure 3, the area has changed. The dimensions of Figure 3 reveal that the rectangle has an area of sixty five units. How can this reconstruction of the square be valid? The same four shapes seemingly produce two different areas when rearranged differently.

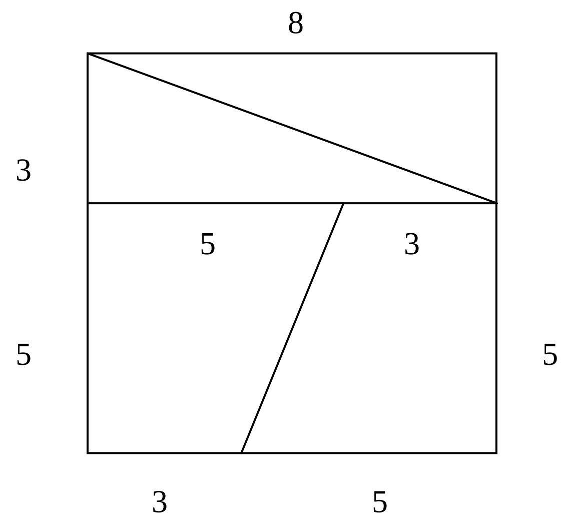

Figure 2. Square broken into 4 shapes with Fibonacci dimensions.

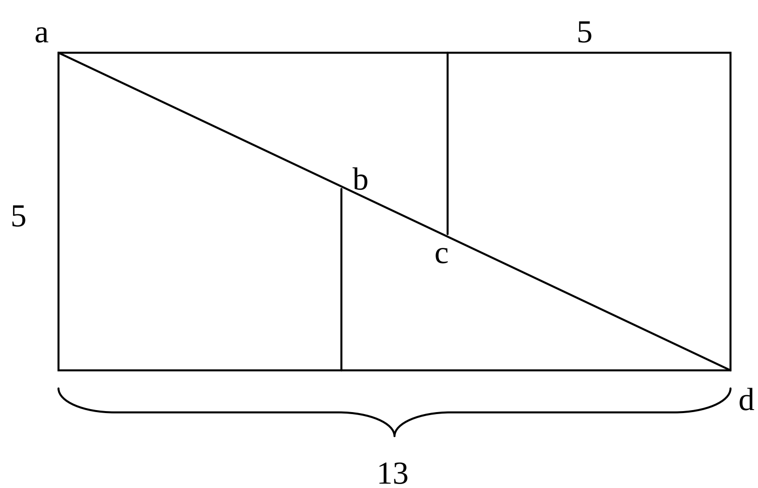

Figure 3. Seemingly unified rectangle composed of 4 shapes.

This anomaly can be explained using Identity 5 $\left(i.e.\ f_{2k}^{\ 2} = f_{2k+1} f_{2k-1} - 1\right)$. The square must be broken into the four shapes using three consecutive Fibonacci numbers. An entire side of the square must have the dimension of an even term in the Fibonacci sequence (i.e. the number has an even subscript). The other two dimensions are the two Fibonacci numbers that precede that even term in the Fibonacci sequence. The appropriate dimensions of the square are illustrated in Figure 4 (Burton, 2002). Figure 5 demonstrates, however, that when the shapes are properly rearranged into a rectangle, the points $a, b, c,$ and d do not lie directly on the diagonal of the rectangle (Burton, 2002). In reality, the points $a, b, c,$ and d form a small parallelogram with an area of exactly one square unit.

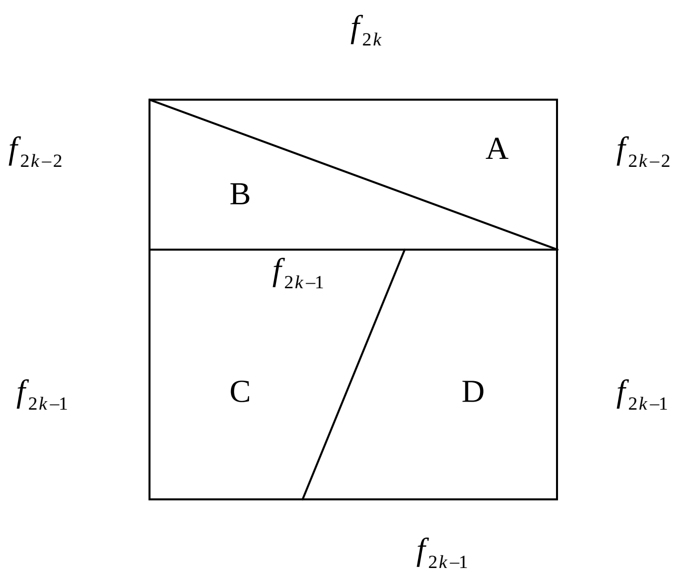

Figure 4. Square with dimensions of specific Fibonacci terms.

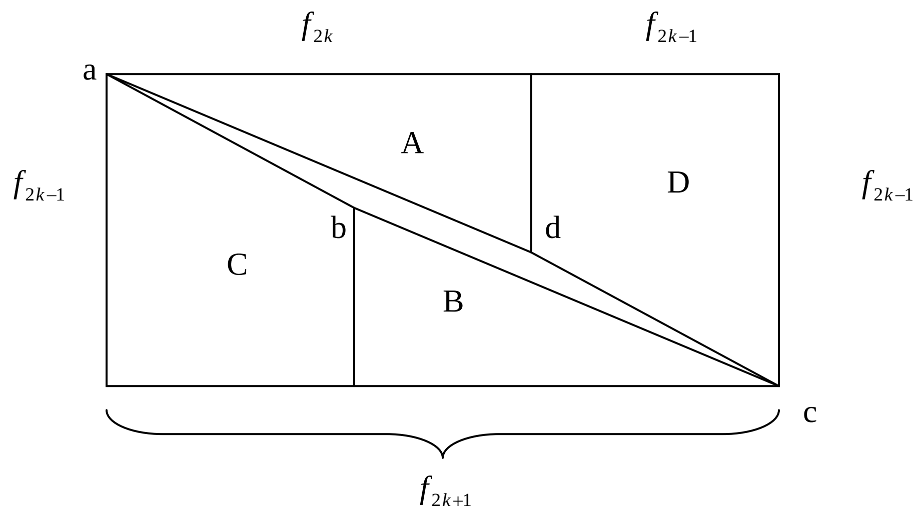

Figure 5. Rectangle Fibonacci puzzle with unit parallelogram.

Conclusion

Ultimately, the Fibonacci sequence appears not only in the various branches of mathematics but also many different aspects of this world. From number theory to nature, the Fibonacci sequence can be found in numerous arenas both inside and outside of the realm of mathematics. The prevalence of this particular sequence is not coincidental; it has been divinely established and maintained. The beauty of the Fibonacci sequence illustrates the aesthetic nature of God, our Creator and Sustainer of the universe. The Fibonacci sequence is simply one of numerous evidences of God's sovereignty over the affairs of mankind.

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