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Sums of Powers and Harmonic Numbers: A new approach

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ABSTRACT

There have been derivations for the Sums of Powers published since the sixteenth century. All techniques have used recursive processes, producing the following formula in the series. I present a new method that calculates the Sums of Powers and Harmonic Numbers. Starting with a novel relationship between Pascal's Numbers and Stirling's Numbers of the First Kind, the Sums of Powers is developed. This formula, published previously using a different methodology, is in terms of Pascal Numbers multiplied by constant coefficients. However, a further step is introduced. A recursive relationship is discovered among the coefficients of these formulae. A double sigma master formula is developed, allowing one to calculate all formulae for Sums of Powers without needing Bernoulli Numbers. Finally, from the Sums of Powers master formula, I derive a formula to calculate the Bernoulli Numbers. I further develop a summation formula for the Harmonic Numbers using the same relationships.

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Bernoulli Numbers; Pascal Triangle; Stirling Numbers; Sums of Powers

1. Introduction

It is difficult to say luck's role in the discovery process, although I suspect it happens more often than is admitted. My journey started with a casual statement made in a popular mathematics book. The statement was that the division of Stirling Numbers of the First Kind, the penultimate by the ultimate number, results in the Harmonic Numbers. I could not believe this was just a coincidence; a personal inquiry resulted in this writing. From the start, I assumed that Pascal Numbers had to be involved. This inquiry led me to a mathematical relationship where I found a new methodology to formulate all Sums of Powers. A long, arduous process with many dead ends, but the thrill of defining a problem and solving it was worth the endeavor.

The Sums of Powers has fascinated mathematicians for centuries, and mathematicians have explored these infinite series dating back to the tenth century (Beery, 2010; Coen, 1996). The first written formula dates to the 16th century with Harriot (Beery, 2010; Coen, 1996), who wrote them in his notebooks but never published them. Faulhaber (Edwards, 1982) was the first to publish formulae for the Sums of Powers and claimed to have found formulae up to the 17th power. Pascal derived his solution for the Sums Powers using the binomial expansion (Edwards, 1982, 1987). The use of Pascal Numbers to express the Sums of Powers was more recently derived using mathematical induction (Thoddi, 1993). Bernoulli was the first to calculate the Sums of Powers using a single formula. However, each series member needed a separate calculation to find the corresponding Bernoulli Number for the subsequent series (Edwards, 1982). It was Euler, almost half a century later, who proved the Bernoulli forms using the calculus of finite differences (Edwards, 1982) and coined the name, Bernoulli Numbers. Other modern decomposition methods and fractional calculus have recently been implemented to solve this age-old problem (Bazso et al., 2012; Nishimoto & Srivastava, 1991; Srivastava et al., 1991). The methodology presented here is derived using a new approach and is applied to solve not only Sums of Powers but also Harmonic Numbers.

2. Derivation of Sums of Powers formula

I present a method by which one can generate all Sums of Powers without needing Bernoulli Numbers. The general Sums of the Powers Formula are presented in Equation 1.

$$\sum_{L=1}^{\infty} L^n = \sum_{t=0}^{\infty} \left(C_1^{t+1}\right)^n \qquad (1)$$

Where L is the index of summation, n is a constant, and L, n, t $\subset \mathbb{Z}^+$. C_1^{t+1} are numbered along the second

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$$
\begin{array}{c}
1_0^0\\
1_0^1\ 1_1^1\\
1_0^2\ 2_1^2\ 1_2^2\\
1_0^3\ 3_1^3\ 3_2^3\ 1_3^3\\
1_0^4\ 4_1^4\ 6_2^4\ 4_3^4\ 1_4^4\\
1_0^5\ 5_1^5\ 10_2^5\ 10_3^5\ 5_4^5\ 1_5^5\\
1_0^6\ 6_1^6\ 15_2^6\ 20_3^6\ 15_4^6\ 10_5^6\ 1_6^6
\end{array}
$$

Figure 1. Pascal's Triangle.

diagonal of the Pascal Triangle; in this case, the counting numbers,e.g., $1_1^1, 2_1^2, 3_1^3, 4_1^4, \ldots C_1^{t+1}$.

The general method by which the Sums of Powers are derived using Pascal Numbers begins with the following relationship.

$$\sum_{L=1}^{\infty} L^2 = \sum_{t=0}^{\infty} [C_1^{t+1} C_1^{t+1}] = \sum_{t=0}^{\infty} [2C_2^{t+2} - 1C_1^{t+1}] \quad (2)$$

Proposition 1. The sum of the product C_1^{t+1} and C_1^{t+1} is equal to the sum $[2C_2^{t+2} - C_1^{t+1}]$, which can be written as:

$$\sum_{t=0}^{\infty}[C_1^{t+1}C_1^{t+1}] = \sum_{t=0}^{\infty}[2C_2^{t+2} - 1C_1^{t+1}] \quad (3)$$

For the general Proof of Equation 3, one has the following:

Proof 1.

$$\sum_{t=0}^{\infty}(q+t)\frac{(q+t)!}{q!t!} = \sum_{t=0}^{\infty}(q+t)\frac{(q+t)!}{q!t!}$$

$$\sum_{t=0}^{\infty}(q+t)\frac{(q+t)!}{q!t!} = \sum_{t=0}^{\infty}(q+t)\frac{(q+t)!}{q!t!} + \frac{(q+t)!}{q!t!} - \frac{(q+t)!}{q!t!}$$

$$\sum_{t=0}^{\infty}(q+t)\frac{(q+t)!}{q!t!} = \sum_{t=0}^{\infty}(q+t+1)\frac{(q+t)!}{q!t!} - \frac{(q+t)!}{q!t!}$$

$$\sum_{t=0}^{\infty}(q+t)\frac{(q+t)!}{q!t!} = \sum_{t=0}^{\infty}\frac{(q+t+1)!}{q!t!} - \frac{(q+t)!}{q!t!}$$

$$\sum_{t=0}^{\infty}\frac{(q+t)!}{1!(q+t-1)!}\frac{(q+t)!}{q!t!} = \sum_{t=0}^{\infty}(q+1)\frac{(q+t+1)!}{(q+1)!t!} - \frac{(q+t)!}{q!t!}$$

$$\sum_{t=0}^{\infty}C_1^{t+q}C_q^{t+q} = \sum_{t=0}^{\infty}(q+1)\,C_{q+1}^{q+t+1} - C_q^{q+t}$$

The right-most equation of Equation 3 is further converted by using the following general relationship:

$$\sum_{t=0}^{\infty}C_1^{t+1} = C_2^{t+2} \quad (4)$$

Where $t \subset \mathbb{Z}^+$, making the final version of Equation 3 into the following equation:

$$\sum_{t=0}^{\infty}[2C_2^{t+2} - 1C_1^{t+1}] = 2C_3^{t+3} - 1C_2^{t+2} \quad (5)$$

Proposition 2. The sum of the series $C_1^{t+1}$is equal to C_2^{t+2}. But I will demonstrate the Proof for the general form.

$$\sum_{t=0}^{\infty}C_q^{t+q} = C_{q+1}^{t+q+1} \quad (6)$$

Proof 2. Using mathematical induction: Step 1, where $t = 0$.

$$\frac{q!}{q!0!} = \frac{(q+1)!}{(q+1)!0!}$$

$$1 = 1$$

Step 2, Where one adds the next t = 1 term.

$$\sum_{t=0}^{1}C_q^{t+q} = C_{q+1}^{q+2}$$

$$\frac{q!}{q!0!} + \frac{(q+1)!}{q!1!} = \frac{(q+2)!}{(q+1)!1!}$$

$$1 + (q+1) = (q+2)$$

The following Sums of Powers is gotten by multiplying Equation 3 by C_1^{t+1}.

$$\sum_{L=1}^{\infty}L^3 = \sum_{t=0}^{\infty}[C_1^{t+1}C_1^{t+1}\,C_1^{t+1}] = \sum_{t=0}^{\infty}[(2C_2^{t+2} - C_1^{t+1})C_1^{t+1}] \quad (7)$$

$$\sum_{t=0}^{\infty}[C_1^{t+1}C_1^{t+1}C_1^{t+1}] = \sum_{t=0}^{\infty}2[C_1^{t+1}C_2^{t+2}] - [C_1^{t+1}C_1^{t+1}] \quad (8)$$

The relation in Equation 9 converts $[C_1^{t+1}C_2^{t+2}]$ to $[C_1^{t+2}C_2^{t+2} - 1C_2^{t+2}]$.

$$\sum_{t=0}^{\infty}[C_1^{n+t}C_m^{m+t}] = \sum_{t=0}^{\infty}[C_1^{m+t}\,C_m^{m+t} - (m-n)\,C_m^{m+t}] \quad (9)$$

Where $m>n$.

Proposition 3. The sum of the series $C_1^{n+t}C_m^{m+t}$ is equal to the sum of the series $C_1^{m+t}C_m^{m+t} - (m-n)\,C_m^{m+t}$. Where m = n+q.

Proof 3.

$$\sum_{t=0}^{\infty}(n+t)\frac{(n+q+t)!}{(n+q)!\,(t)!} = \sum_{t=0}^{\infty}(n+t)\frac{(n+q+t)!}{(n+q)!\,(t)!}$$

$$\sum_{t=0}^{\infty}(n+t)\frac{(n+q+t)!}{(n+q)!\,(t)!} = \sum_{t=0}^{\infty}(n+q+t)\frac{(n+q+t)!}{(n+q)!\,(t)!} - q\frac{(n+q+t)!}{(n+q)!\,(t)!}$$

$$\sum_{t=0}^{\infty}\frac{(n+t)!}{1!\,(n+t-1)!}\frac{(n+q+t)!}{(n+q)!\,(t)!} = \sum_{t=0}^{\infty}\frac{(n+q+t)!}{1!\,(n+q+t-1)!}\frac{(n+q+t)!}{(n+q)!\,(t)!} - q\frac{(n+q+t)!}{(n+q)!\,(t)!}$$

$$\sum_{t=0}^{\infty}C_1^{n+t}C_{n+q}^{n+q+t} = \sum_{t=0}^{\infty}C_1^{n+q+t}C_{n+q}^{n+q+t} - [(n+q)-n]\,C_{n+q}^{n+q+t}$$

$$\sum_{t=0}^{\infty}C_1^{n+t}C_m^{m+t} = \sum_{t=0}^{\infty}[C_1^{m+t}C_m^{m+t} - (m-n)\,C_m^{m+t}]$$

Continuing from Equation 8 and applying the conversion in Equation 9, one gets the following:

$$\sum_{t=0}^{\infty}[C_1^{t+1}C_1^{t+1}C_1^{t+1}] = \sum_{t=0}^{\infty}2[C_1^{t+2}C_2^{t+2} - 1C_2^{t+2}] - [C_1^{t+1}C_1^{t+1}] \quad (10)$$

Using the conversion from Equation 2:

$$\sum_{t=0}^{\infty}[C_1^{t+2}C_2^{t+2}] = \sum_{t=0}^{\infty}[3C_3^{t+3} - 1C_2^{t+2}] \quad (11)$$

Inserting the results of Equation 11 into Equation 10.

$$\sum_{t=0}^{\infty}[C_1^{t+1}C_1^{t+1}C_1^{t+1}] = \sum_{t=0}^{\infty}2\,[3C_3^{t+3} - 2C_2^{t+2}] - [C_1^{t+1}C_1^{t+1}] \quad (12)$$

Using the results of Equation 2 to convert $\sum_{t=0}^{\infty}[C_1^{t+1}C_1^{t+1}]$ in Equation 12 one gets:

$$\sum_{t=0}^{\infty}[C_1^{t+1}C_1^{t+1}C_1^{t+1}] = \sum_{t=0}^{\infty}[6C_3^{t+3} - 4C_2^{t+2}] - [2C_2^{t+2} - 1C_1^{t+1}] \quad (13)$$

$$\sum_{t=0}^{\infty}[C_1^{t+1}C_1^{t+1}C_1^{t+1}] = \sum_{t=0}^{\infty}[6C_3^{t+3} - 6C_2^{t+2} + 1C_1^{t+1}] \quad (14)$$

Using Equation 4 to convert Equation 14, one gets:

$$\sum_{L=1}^{\infty}L^3 = 6C_4^{t+4} - 6C_3^{t+3} + 1C_2^{t+2} \quad (15)$$

If one continues multiplying by C_1^{t+1} and using Equation 2, Equation 4, and Equation 9 to get subsequent Sums of Powers, one obtains the following series of equations seen in Figure 2.

Sums of Powers represented by Pascal Numbers multiplied by a coefficient have been published in the past using a different methodology (Thoddi, 1993). However, here I introduce a new relationship that takes it a step further.

From left to right, the coefficients running along the second diagonal, e.g. 2, 6, 14, 30,... demonstrate a recursive relationship taking the following form:

$$x(t+1) = \mathrm{K}x(t) + Kx_0 \quad (16)$$

In the first recursive relationship, $K = 2$, $x(1) = 2$, the first number of the second diagonal, and $x_0 = 1$ is the coefficient from the first diagonal in Figure 2.

$$x(t+1) = 2x(1) + 2(1) \quad (17)$$

This first and the subsequent recursive relations were solved using MapleSoft Computer Algebra Software vs.11 (MapleSoft) using an algorithm adapted for these recursive sequences (Enns, 2006). The first recursive relation results with the following formula.

$1 \cdot 2^{t+2} - 2 \cdot 1^{t+2} = 2, 6, 14, 30, \ldots$

Using Equation 16, I develop the following recursive relationships.

$$1\,C_2^{t+2}$$
$$2\,C_3^{t+3}\;-1\,C_2^{t+2}$$
$$6\,C_4^{t+4}\;-6\,C_3^{t+3}\;+1\,C_2^{t+2}$$
$$24\,C_5^{t+5}\;-36\,C_4^{t+4}\;+14\,C_3^{t+3}\;-1\,C_2^{t+2}$$
$$120\,C_6^{t+6}\;-240\,C_5^{t+5}\;+150\,C_4^{t+4}\;-30\,C_3^{t+3}\;+1\,C_2^{t+2}$$
$$720\,C_7^{t+7}\;-1800\,C_6^{t+6}\;+1560\,C_5^{t+5}\;-540\,C_4^{t+4}\;+62\,C_3^{t+3}\;-1\,C_2^{t+2}$$
$$5040\,C_8^{t+8}\;-15120\,C_7^{t+7}\;+16800\,C_6^{t+6}\;-8400\,C_5^{t+5}\;+1806\,C_4^{t+4}\;-126\,C_3^{t+3}\;+1\,C_2^{t+2}$$

Figure 2. The Sums of Powers derived from Pascal's Triangle.

$$1\cdot 1^{t+1}$$
$$1\cdot 2^{t+2}-2\cdot 1^{t+2}$$
$$1\cdot 3^{t+3}-3\cdot 2^{t+3}+3\cdot 1^{t+3}$$
$$1\cdot 4^{t+4}-4\cdot 3^{t+4}+6\cdot 2^{t+4}-4\cdot 1^{t+4}$$
$$1\cdot 5^{t+5}-5\cdot 4^{t+5}+10\cdot 3^{t+5}-10\cdot 2^{t+5}+5\cdot 1^{t+5}$$
$$1\cdot 6^{t+6}-6\cdot 5^{t+6}+15\cdot 4^{t+6}-20\cdot 3^{t+6}+15\cdot 2^{t+6}-6\cdot 1^{t+6}$$

Figure 3. The solution of the Recursive Formulae from Figure 2 is derived using the recursive in Equation 16.

$$\sum_{L=1}^{\infty} L^0 = L$$
$$\sum_{L=1}^{\infty} L^1 = \frac{1}{2}L^2+\frac{1}{2}L$$
$$\sum_{L=1}^{\infty} L^2 = \frac{1}{3}L^3+\frac{1}{2}L^2+\frac{1}{6}L$$
$$\sum_{L=1}^{\infty} L^3 = \frac{1}{4}L^4+\frac{1}{2}L^3+\frac{1}{4}L^2$$
$$\sum_{L=1}^{\infty} L^4 = \frac{1}{5}L^5+\frac{1}{2}L^4+\frac{1}{3}L^3-\frac{1}{30}L$$
$$\sum_{L=1}^{\infty} L^5 = \frac{1}{6}L^6+\frac{1}{2}L^5+\frac{5}{12}L^4-\frac{1}{12}L^2$$
$$\sum_{L=1}^{\infty} L^6 = \frac{1}{7}L^7+\frac{1}{2}L^6+\frac{1}{2}L^5-\frac{1}{6}L^3+\frac{1}{42}L$$
$$\sum_{L=1}^{\infty} L^7 = \frac{1}{8}L^8+\frac{1}{2}L^7+\frac{7}{12}L^6-\frac{7}{24}L^4+\frac{1}{12}L^2$$
$$\sum_{L=1}^{\infty} L^8 = \frac{1}{9}L^9+\frac{1}{2}L^8+\frac{2}{3}L^7-\frac{7}{15}L^5+\frac{2}{9}L^3-\frac{1}{30}L$$
$$\sum_{L=1}^{\infty} L^9 = \frac{1}{10}L^{10}+\frac{1}{2}L^9+\frac{3}{4}L^8-\frac{7}{10}L^6+\frac{1}{2}L^4-\frac{1}{12}L^2$$
$$\sum_{L=1}^{\infty} L^{10} = \frac{1}{11}L^{11}+\frac{1}{2}L^{10}+\frac{5}{6}L^9-L^7+L^5-\frac{1}{2}L^3+\frac{5}{66}L$$

Figure 4. Sums of Powers Formulae.

$x(t+1) = 2x(1) + 2(1); x(1)= 2$
$x(t+1)= 3x(1)+3(1{\cdot}2^{t+2} - 1{\cdot}1^{t+2}); x(1)= 6$
$x(t+1)= 4x(1)+4(1{\cdot}3^{t+3} - 3{\cdot}2^{t+3} + 1{\cdot}1^{t+3}); x(1)= 24$
$x(t+1)= 5x(1)+5(1{\cdot}4^{t+4} - 4{\cdot}3^{t+4} + 6{\cdot}2^{t+4} - 4{\cdot}1^{t+4}); x(1)= 120$
$x(t+1)= 6x(1)+6(1{\cdot}5^{t+5} - 5{\cdot}4^{t+5} + 10{\cdot}3^{t+5} - 10{\cdot}2^{t+5} + 5{\cdot}1^{t+5});$
$x(1)= 720$

If one continues with this pattern of recursive relationships, it will result in the following series of formulae in Figure 3.

From these recursive relationships, I derive an equation for all Sums of Powers.

$$\sum_{L=1}^{\infty} L^n = \sum_{k=1}^{t}\sum_{t=1}^{n}\left[(-1)^{n+k}k^n\binom{t}{k}\binom{L+t}{t+1}\right] \quad (18)$$

The following familiar formulae in Figure 4 were generated using Equation 18 in MapleSoft vs.11 (MapleSoft).

3. Derivation of the Bernoulli Numbers formula

Multiplying out the term $\binom{L+t}{t+1}$ from Equation 18, the Sums of Powers Formula can be rewritten in the following way.

$$\sum_{L=1}^{\infty} L^n = \sum_{s=0}^{t}\sum_{k=1}^{t}\sum_{t=1}^{n}\left[(-1)^{n+k}\binom{t}{k}\frac{k^n}{(t+1)!}S_{s,t}L^{t+1-s}\right] \quad (19)$$

Where $S_{s,t}$ are the Stirling Numbers of the First Kind. In Figure 5, Equation 18 is calculated and organized by powers of L.

$$\sum_{L=1}^{\infty} L^1 = \left[\frac{1}{2}S_{0,1}\right]L^2 + \left[\frac{1}{2}S_{1,1}\right]L$$

$$\sum_{L=1}^{\infty} L^2 = \left[\frac{1}{3}S_{0,2}\right]L^3 + \left[\frac{1}{3}S_{1,2} - \frac{1}{2}S_{0,1}\right]L^2 + \left[\frac{1}{3}S_{2,2} - \frac{1}{2}S_{1,1}\right]L$$

$$\sum_{L=1}^{\infty} L^3 = \left[\frac{1}{4}S_{0,3}\right]L^4 + \left[\frac{1}{4}S_{1,3} - \frac{2}{2}S_{0,2}\right]L^3 + \left[\frac{1}{4}S_{2,3} - \frac{2}{2}S_{1,2} + \frac{1}{2}S_{0,1}\right]L^2 + \left[\frac{1}{4}S_{3,3} - \frac{2}{2}S_{2,2} + \frac{1}{2}S_{1,1}\right]L$$

$$\sum_{L=1}^{\infty} L^4 = \left[\frac{1}{5}S_{0,4}\right]L^5 + \left[\frac{1}{5}S_{1,4} - \frac{3}{2}S_{0,3}\right]L^4 + \left[\frac{1}{5}S_{2,4} - \frac{3}{2}S_{1,3} + \frac{7}{3}S_{0,2}\right]L^3$$
$$+ \left[\frac{1}{5}S_{3,4} - \frac{3}{2}S_{2,3} + \frac{7}{3}S_{1,2} - \frac{1}{2}S_{0,1}\right]L^2 + \left[\frac{1}{5}S_{4,4} - \frac{3}{2}S_{3,3} + \frac{7}{3}S_{2,2} - \frac{1}{2}S_{1,1}\right]L$$

$$\sum_{L=1}^{\infty} L^n = \left[\frac{1}{(n+1)}S_{0,n}\right]L^n + \left[\frac{1}{(n+1)}S_{1,n} - \frac{(n-1)}{2}S_{0,n}\right]L^{n-1} + \cdots$$

Figure 5. The coefficients of the Sums of Powers are calculated from Equation 19 where $S_{0,n} = 1$, and $S_{1,n} = \frac{n(n+1)}{2}$. The origin of the first coefficient demonstrates it is equal to $\frac{1}{(n+1)}$, as was known to Bernoulli. The origin of the second coefficient demonstrates that it is always equal to $\frac{1}{2}$.

In Figure 5, the last coefficient corresponds to the Bernoulli Numbers. Note that Stirling Numbers $S_{n,n} = t!$. Thus, one can derive Equation 20 by multiplying the corresponding factorial by the coefficients in Equation 19 and calculating all the Bernoulli Numbers.

$$B_n = \sum_{k=1}^{t}\sum_{t=1}^{n}\left[(-1)^k \binom{t}{k} \frac{k^n}{(t+1)!}t!\right] \tag{20}$$

which simplifies to:

$$B_n = \sum_{k=1}^{t}\sum_{t=1}^{n}\left[(-1)^k \binom{t}{k} \frac{k^n}{(t+1)}\right] \tag{21}$$

$$B_n = 1, -\frac{1}{2}, \frac{1}{6}, 0, -\frac{1}{30}, 0, \frac{1}{42}, 0, -\frac{1}{30}, 0, \frac{5}{66}, 0, -\frac{691}{2730}, 0, \frac{7}{6}, 0, -\frac{3617}{510}, 0, \frac{43867}{798}, 0, \ldots.$$

Numerous publications of explicit formulae calculating Bernoulli Numbers (Apostol, 2008; Gould, 1972) date back over 100 years. Equation 21 was previously published by (Higgins, 1970) and dated back to (Worpitzky, 1883), and here is derived from the first principle.

4. Deriving Stirling Numbers of the first kind from Pascal Numbers

A new method was previously introduced by which, from first principles, a double sigma formula for the Sums of Powers was derived using Pascal Numbers. This same method is applied here to derive the Stirling Numbers of the First Kind, and from this, derive the Harmonic Numbers $\sum_{L=1}^{\infty}\frac{I}{L}$. Both Pascal Numbers and Stirling Numbers of the First Kind originate similarly. The Pascal Numbers originate from the coefficients of the following formula:

$$\prod_{n=1}^{\infty}(x+1)^n = (x+1)(x+1)(x+1)\ldots \tag{22}$$

When the coefficients of Equation 22 are arranged in a vertical stack, it forms what is known as Pascal's Triangle, Figure 1. In Equation 23, the Stirling Numbers of the First Kind are derived, and when the coefficients are stacked, it forms a similar triangle called Stirling's Triangle, Figure 6.

$$\prod_{n=1}^{\infty}(x+n) = (x+1)(x+2)(x+3)\ldots(x+n) \tag{23}$$

The underlying premise is that because both Pascal Numbers and Stirling Numbers of the First Kind originate similarly, one should be able to calculate Stirling Numbers of the First Kind from Pascal Numbers. The general formula used to calculate the Stirling Numbers of the First Kind from Pascal Numbers is the following:

$$\sum_{t=0}^{\infty} C_1^{t+n} \ldots. \sum_{t=0}^{\infty} C_1^{t+4} \sum_{t=0}^{\infty} C_1^{t+3} \sum_{t=0}^{\infty} C_1^{t+2} \sum_{t=0}^{\infty} C_1^{t+1} \tag{24}$$

1

1 1

1 3 2

1 6 11 6

1 10 35 50 24

1 15 85 225 274 120

1 21 175 735 1624 1764 720

Figure 6. Stirling Numbers of the First Kind from the coefficients from Equation 23.

The following three equations, (2), (4), and (9), are used to achieve these conversions. The First Stirling Diagonal in terms of Pascal Numbers is trivial and is derived using Equation 4 on Equation 24.

$$\sum_{t=0}^{\infty} C_1^{t+1} = C_2^{t+2} = 1, 3, 6, 10, 15, 21\ldots \tag{25}$$

For the Second Stirling Diagonal, we apply Equation 2, Equation 4 to $\sum C_1^{t+2}C_2^{t+2}$ from the generalized Stirling formula to obtain:

$$\sum_{t=0}^{\infty}[C_1^{t+2}C_2^{t+2}] = \sum_{t=0}^{\infty}[3C_3^{t+3} - 1C_2^{t+2}] \tag{26}$$

And using Equation 4 converts the Second Stirling Diagonal in terms of Pascal Numbers to:

$$\sum_{t=0}^{\infty}[3C_3^{t+3} - 1C_2^{t+2}] = [3C_4^{t+4} - 1C_3^{t+3}] = 2,\ 11,\ 35,\ 85,\ 175,\ \ldots. \tag{27}$$

The Third, Stirling to Pascal Number conversion in the series, is calculated by multiplying $\sum_{t=0}^{\infty} C_1^{t+3}$ with Equation 27, then using Equation 2, Equation 4, and Equation 9, converting it to:

$$15C_6^{t+6} - 10C_5^{t+5} + 1C_4^{t+4} = 6,\ 50,\ 225,\ 735,\ 1960,\ \ldots. \tag{28}$$

Proceeding with subsequent multiplications and conversions, one gets the following formulae in Figure 7.

A recursive relationship is found among all the diagonals in Figure 7. There is a recursive relationship among the coefficients between the First Diagonal 3,10,25,56,119,246,… and the Zero Diagonal 1,1,1,1,1,1,…. All recursive relationships were solved using MapleSoft Program (MapleSoft) with the following algorithm (Enns & McGuire, 2006):

$Unassign('t', 'x')$:
$eq := x(t+1) = C_1\, x(t) + C_2$
$x := rsolve(eq, x(1) = StartingNumber, x);$
$x := expand(x);$
$n := 1;\ number := x$

The following recursive relationships were used to derive a final formula which calculates the coefficients in the First Diagonal in Figure 7.

$$x(t+1) = 2x(t) + 4(1);\ x(1) = 3;\ \frac{\mathbf{7}\cdot 2^t}{2} - \mathbf{4} = \mathbf{3}, 10 \tag{29}$$

$$x(t+1) = 2x(t) + 5(1);\ x(1) = 10;\ \frac{\mathbf{15}\cdot 2^t}{2} - \mathbf{5} = \mathbf{10}, 25 \tag{30}$$

$$x(t+1) = 2x(t) + 6(1);\ x(1) = 25;\ \frac{\mathbf{31}\cdot 2^t}{2} - \mathbf{6} = \mathbf{25}, 56 \tag{31}$$

$$x(t+1) = 2x(t) + 7(1);\ x(1) = 56;\ \frac{\mathbf{63}\cdot 2^t}{2} - \mathbf{7} = \mathbf{56}, 119 \tag{32}$$

Equations (29-32) are precise only over two numbers and represent stepwise progression down the First Diagonal in Figure 7. The bold number solutions in Equations (29-32) equate to the bold numbers along the vertical **3,10, 25, 56** in Figure 7. The bold numbers **7, 15, 31, 63** also demonstrate a recursive relationship. In Equation (33), we derive the formula for these numbers.

$$x(t+1) = 2x(t) + (1);\ x(1) = 7; = 4\cdot 2^t - 1 = 7, 15, 31, 63\cdots \tag{33}$$

The formula in Equation (33) is in a raw form, and to convert it to its final form, I compared the basic formulae from the Harmonic to the Quartic power and from the First to the Fifth diagonal to get the following consistent format for the First Diagonal in Equation 34.

$$\sum_{t=0}^{\infty}\left[\left(\frac{-\binom{0}{0}1^2}{0!}\right)\binom{3}{0}\right]\binom{t+3}{3} + \left[\left(\frac{-\binom{0}{0}1^2}{0!}\right)\binom{4}{1} + \left(\frac{-\binom{1}{0}1^3}{1!} + \frac{\binom{1}{1}2^3}{1!}\right)\binom{4}{0}\right]\binom{t+4}{4} \tag{34}$$

$$1\,C_2^{t+2}$$

$$3\,C_4^{t+4} - 1\,C_3^{t+3}$$

$$15\,C_6^{t+6} - 10\,C_5^{t+5} + 1\,C_4^{t+4}$$

$$105\,C_8^{t+8} - 105\,C_7^{t+7} + 25\,C_6^{t+6} - 1\,C_5^{t+5}$$

$$945\,C_{10}^{t+10} - 1260\,C_9^{t+9} + 490\,C_8^{t+8} - 56\,C_7^{t+7} + 1\,C_6^{t+6}$$

$$10395\,C_{12}^{t+12} - 17325\,C_{11}^{t+11} + 9450\,C_{10}^{t+10} - 1918\,C_9^{t+9} + 119\,C_8^{t+8} - 1\,C_7^{t+7}$$

$$135135\,C_{14}^{t+14} - 270270\,C_{13}^{t+13} + 190575\,C_{12}^{t+12} - 56980\,C_{11}^{t+11} + 6825\,C_{10}^{t+10} - 246\,C_9^{t+9} + 1\,C_8^{t+8}$$

Figure 7. Stirling Numbers of the First Kind in terms of Pascal Numbers.

$$\frac{1}{2!}L^2+\frac{1}{2!}L$$

$$\frac{3}{4!}L^4+\frac{14}{4!}L^3+\frac{21}{4!}L^2+\frac{10}{4!}L$$

$$\frac{15}{6!}L^6+\frac{165}{6!}L^5+\frac{705}{6!}L^4+\frac{1455}{6!}L^3+\frac{1440}{6!}L^2+\frac{540}{6!}L$$

$$\frac{105}{8!}L^8+\frac{2100}{8!}L^7+\frac{17570}{8!}L^6+\frac{79464}{8!}L^5+\frac{208985}{8!}L^4+\frac{317940}{8!}L^3+\frac{257180}{8!}L^2+\frac{84336}{8!}L$$

$$\frac{945}{10!}L^{10}+\frac{29925}{10!}L^9+\frac{412650}{10!}L^8+\frac{3245130}{10!}L^7+\frac{15997905}{10!}L^6+\frac{51110325}{10!}L^5+\frac{105405300}{10!}L^4+\frac{134614620}{10!}L^3+\frac{95911200}{10!}L^2+\frac{28728000}{10!}L$$

Figure 8. Power Formulae for Five Series of Stirling Numbers of the First Kind.

The Second Diagonal of coefficients, 15,105,490,1918, 6825..., is derived similarly. However, the recursive relationship is now between the Second Diagonal and a multiple of the First Diagonal.

$$x(t+1)=3x(t)+6\left(\frac{15\cdot 2^t}{2}-5\right);x(1)=15;\quad \frac{\mathbf{90}\cdot 3^t}{3}-\frac{\mathbf{6}\cdot\mathbf{15}\cdot 2^t}{2}+\mathbf{15}=\mathbf{15},\ 105 \tag{35}$$

$$x(t+1)=3x(t)+7\left(\frac{31\cdot 2^t}{2}-6\right);x(1)=105;\quad \frac{\mathbf{301}\cdot 3^t}{3}-\frac{\mathbf{7}\cdot\mathbf{31}\cdot 2^t}{2}+\mathbf{21}=\mathbf{105},\ 490 \tag{36}$$

$$x(t+1)=3x(t)+8\left(\frac{63\cdot 2^t}{2}-7\right);x(1)=490;\quad \frac{\mathbf{966}\cdot 3^t}{3}-\frac{\mathbf{8}\cdot\mathbf{63}\cdot 2^t}{2}+\mathbf{28}=\mathbf{490},\ 1918 \tag{37}$$

$$x(t+1)=3x(t)+9\left(\frac{127\cdot 2^t}{2}-8\right);x(1)=1918;\quad \frac{\mathbf{3025}\cdot 3^t}{3}-\frac{\mathbf{9}\cdot\mathbf{127}\cdot 2^t}{2}+\mathbf{36}=\mathbf{1918},\ 6825 \tag{38}$$

The bold numbers again equate to the bold diagonal numbers along the vertical, **15,105**, **490,1918**. For instance, in Equation 35-Equation 38, **90–6·15+15 = 15**, **301–7·31+21 = 105**, **966 – 8 · 63 + 28 = 490**, **3025 – 9 · 127 + 36 = 1918**. In Equation 35-Equation 38, one again finds a recursive relationship among the numbers **90, 301, 966, 3025** (see Equation 39). The recursive relationship among the numbers **15, 31, 63,127** was previously derived in Equation (33).

$$\begin{aligned} x(t+1) &= 3x(t)+(4\cdot 2^{t+2}-1);\ x(1)=90;\\ &=\frac{81\cdot 3^t}{2}-16\cdot 2^t+\frac{1}{2}\\ &=90,\ 301,\ 966,\ 3025,\ \ldots. \end{aligned} \tag{39}$$

Equation 39 solution is in a raw form and is converted into the following equivalent form:

$$\sum_{t=0}^{\infty}\left[\frac{\binom{0}{0}1^3}{0!}\binom{4}{0}\right]\binom{t+4}{4}+\left[\left(\frac{\binom{0}{0}1^3}{0!}\right)\binom{5}{1}+\left(\frac{\binom{1}{0}1^4}{1!}-\frac{\binom{1}{1}2^4}{1!}\right)\binom{5}{0}\right]\binom{t+5}{5}$$
$$+\left[\frac{\binom{0}{0}1^3}{0!}\binom{6}{2}+\left(\frac{\binom{1}{0}1^4}{1!}-\frac{\binom{1}{1}2^4}{1!}\right)\binom{6}{1}+\left(\frac{\binom{2}{0}1^5}{2!}-\frac{\binom{2}{1}2^5}{2!}+\frac{\binom{2}{2}3^5}{2!}\right)\binom{6}{0}\right]\binom{t+6}{6} \tag{40}$$

$$\left[\frac{\tbinom{0}{0}\tbinom{2}{0}1^1}{0!}\right]\binom{n+2}{2}$$

Stirling Numbers, First Diagonal=1,3,6,10,15,...

$$\left[-\frac{\tbinom{0}{0}\tbinom{3}{0}1^2}{0!}\right]\binom{n+3}{3}+\left[\begin{matrix} -\frac{\tbinom{0}{0}\tbinom{4}{1}1^2}{0!} \\ -\frac{\tbinom{1}{0}\tbinom{4}{0}1^3}{1!}+\frac{\tbinom{1}{1}\tbinom{4}{0}2^3}{1!}\end{matrix}\right]\binom{n+4}{4}$$

Stirling Numbers, Second Diagonal=2,11,35,85,175,...

$$\left[\frac{\tbinom{0}{0}\tbinom{4}{0}1^3}{0!}\right]\binom{n+4}{4}+\left[\begin{matrix} \frac{\tbinom{0}{0}\tbinom{5}{1}1^3}{0!} \\ \frac{\tbinom{1}{0}\tbinom{5}{0}1^4}{1!}-\frac{\tbinom{1}{1}\tbinom{5}{0}2^4}{1!}\end{matrix}\right]\binom{n+5}{5}+\left[\begin{matrix} \frac{\tbinom{0}{0}\tbinom{6}{2}1^3}{0!} \\ \frac{\tbinom{1}{0}\tbinom{6}{1}1^4}{1!}-\frac{\tbinom{1}{1}\tbinom{6}{1}2^4}{1!} \\ \frac{\tbinom{2}{0}\tbinom{6}{0}1^5}{2!}-\frac{\tbinom{2}{1}\tbinom{6}{0}2^5}{2!}+\frac{\tbinom{2}{2}\tbinom{6}{0}3^5}{2!}\end{matrix}\right]\binom{n+6}{6}$$

Stirling Numbers, Third Diagonal=6,50,225,735,1960,...

$$\left[-\frac{\tbinom{0}{0}\tbinom{5}{0}1^4}{0!}\right]\binom{n+5}{5}+\left[\begin{matrix} -\frac{\tbinom{0}{0}\tbinom{6}{1}1^4}{0!} \\ -\frac{\tbinom{1}{0}\tbinom{6}{0}1^5}{1!}+\frac{\tbinom{1}{1}\tbinom{6}{0}2^5}{1!}\end{matrix}\right]\binom{n+6}{6}+\left[\begin{matrix} -\frac{\tbinom{0}{0}\tbinom{7}{2}1^4}{0!} \\ -\frac{\tbinom{1}{0}\tbinom{7}{1}1^5}{1!}+\frac{\tbinom{1}{1}\tbinom{7}{1}2^5}{1!} \\ -\frac{\tbinom{2}{0}\tbinom{7}{0}1^6}{2!}+\frac{\tbinom{2}{1}\tbinom{7}{0}2^6}{2!}-\frac{\tbinom{2}{2}\tbinom{6}{0}3^6}{2!}\end{matrix}\right]\binom{n+7}{7}+\left[\begin{matrix} -\frac{\tbinom{0}{0}\tbinom{8}{3}1^4}{0!} \\ -\frac{\tbinom{1}{0}\tbinom{8}{2}1^5}{1!}+\frac{\tbinom{1}{1}\tbinom{8}{2}2^5}{1!} \\ -\frac{\tbinom{2}{0}\tbinom{8}{1}1^6}{2!}+\frac{\tbinom{2}{1}\tbinom{8}{1}2^6}{2!}-\frac{\tbinom{2}{2}\tbinom{8}{1}3^6}{2!} \\ -\frac{\tbinom{3}{0}\tbinom{8}{0}1^7}{3!}+\frac{\tbinom{3}{1}\tbinom{8}{0}2^7}{3!}-\frac{\tbinom{3}{2}\tbinom{8}{0}3^7}{3!}+\frac{\tbinom{3}{3}\tbinom{8}{0}4^7}{3!}\end{matrix}\right]\binom{n+8}{8}$$

Stirling Numbers, Fourth Diagonal=24, 274, 1624, 6769, 22449,...

Figure 9. Demonstrating the symmetry of the coefficients of the Stirling Numbers.

The formulae in Figure 7 are the conversions of the diagonals of the Stirling Numbers of the First Kind in terms of Pascal Numbers. Because recursive relationships were found among the diagonals in Figure 7, a master formula was found in Equation 41.

$$S_{n,1} = \sum_{h=0}^{v} \sum_{v=0}^{t-N} \sum_{t=N}^{2N} \sum_{N=0}^{\infty} \left[\frac{(-1)^{N+h}(h+1)^{N+v+1}\binom{v}{h}\binom{t+2}{t-N-v}}{v!} \binom{n+t+2}{t+2} \right] \tag{41}$$

Here, $n \subset \mathbb{Z}^{+}$ represents the following Stirling Numbers along the diagonal. In Figure (6), $N = 0$ would represent the first diagonal, and n would produce the series 1,3,6,10, 15…, $N = 1$ would represent the second diagonal, and n would produce the series 2,11,35,85, 175…. If n is converted to $L - 1$, we can calculate the Power Series formulae in Figure 8, Figure 9.

5. Deriving the Harmonic Numbers

The Harmonic Numbers (H_n) can be calculated from the Stirling Numbers of the First Kind.

$$H_n = \sum_{n=1}^{\infty} \frac{S_{(n,2)}}{n!} = \frac{S_{(n,2)}}{S_{(n,1)}} = \frac{1}{1}, \frac{3}{2}, \frac{11}{6}, \frac{50}{24}, \frac{274}{120}, \frac{1764}{720}, \frac{13068}{5040}, \ldots, \tag{42}$$

The Harmonic Numbers can now be calculated from Equation 41 and divided by the factorial (see Equation 43).

$$H_n = \sum_{h=0}^{v} \sum_{v=0}^{t-N} \sum_{t=N}^{2N} \sum_{N=0}^{\infty} \left[\frac{(-1)^{N+h}(h+1)^{N+v+1}\binom{v}{h}\binom{t+2}{t-N-v}}{v!(N+2)!} \binom{t+3}{t+2} \right] \tag{43}$$

6. Conclusion

A large body of literature is written about the Sums of Powers and its connections to the Bernoulli Numbers (Apostol, 2008). Using a simple relationship found among the Pascal Numbers, one could build another triangle that derives from each sequential Sum of Powers in terms of Pascal Numbers. I then derived a recursive relationship among the coefficients of these numbers. I calculated a series of these recursive relationships and, from there, developed a double summation formula. This summation formula calculates all Sums of Powers without needing Bernoulli Numbers. I then expanded a term within the Sums of Powers double summation formula to derive a double summation formula to calculate all the Bernoulli Numbers. Because I derived this from a simple relationship between Pascal Numbers, I have demonstrated how Bernoulli Numbers are generated from first principles.

The same technique from Equation 2, Equation 4, and Equation 9 can also derive formulae for Stirling Numbers of the First Kind, in terms of Pascal Numbers, and from this, calculate the Harmonic Numbers.

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No potential conflict of interest was reported by the author(s).

Notes on contributor

Carlos Morton Barrera graduated from Loyola of the South, New Orleans, LA, U.S.A. majoring in Physics with a minor in mathematics. I graduated from Tulane Medical School, doing postgraduate work on the Blood Brain Barrier under Abba Kastin. I obtained a subspecialty in Endocrinology. I eventually left for Boston University and finished a subspecialty in Pulmonary. I finally started a private practice in Miami, FL, U.S.A. This article derives from casual reading in mathematics, which provoked interest in this problem. I hope to continue the method presented on all integer powers of the Zeta function.

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