Algebraic and geometric aspects of two-dimensional Artin groups

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Abstract

In this thesis we study the algebra and the geometry of two-dimensional Artin groups under various aspects. First, we solve the problem of acylindrical hyperbolicity, by proving that all the two-dimensional Artin groups that are not trivially non-acylindrically-hyperbolic are acylindrically hyperbolic. In particular, we prove that every non-spherical Artin group of dimension 2 has trivial centre. Then, we study the structure of parabolic subgroups of large-type Artin groups, and prove various results about their combinatorial structure. We notably show that any intersection of parabolic subgroups is again a parabolic subgroup. Finally, we study the isomorphisms between Artin groups of large-type, and we prove that the family of large-type free-of-infinity Artin groups is rigid. We also fully describe the automorphism groups of these Artin groups.

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Chapter 1

Introduction

Geometric group theory is a branch of mathematics whose principle is to understand finitely generated groups by making them act on suitable spaces. When the spaces at play are well-behaved, one would like to infer from the geometric and topological properties of the space algebraic properties for the group.

Geometric group theory takes its origin from combinatorial group theory in the 1880's, with the early work of Dyck, Klein and Poincaré, who started studying by their presentations finitely presented groups, such as fundamental groups of closed manifolds. In the early 20th century, the celebrated word, conjugacy and isomorphism problems formulated by Dehn and Tietze drew even more attention to the branch.

A fruitful approach has been to study group actions, and thus groups, through the prism of curvature. A prime example of that is the notion of hyperbolic groups introduced by Gromov in the 1980's ([45]). Geometric group theory has since then become a more and more popular theme of interest. Today the branch is highly interdisciplinary, mixing group theory, low-dimensional topology, Riemannian and hyperbolic geometry, formal languages, and even probabilistic approaches.

While it is interesting to study group actions on metric spaces, it is also interesting to see groups as metric spaces themselves by means of their Cayley graphs. In that setting, the notion of hyperbolicity encapsulates the idea of a group having large-scale negative curvature. Hyperbolicity is a powerful property, that has strong geometric and algebraic implications for a group. Although in a sense most finitely presented groups are hyperbolic, a lot of groups are not hyperbolic (for instance, any group containing a $\mathbf{Z}^2$ subgroup). Amongst them however, many behave in a fashion that shares a lot of features with hyperbolic groups. In recent years, various authors have actively worked on notions that capture a weaker form of non-positive curvature, like acylindrical hyperbolicity, CAT(0)-ness, or systolicity.

The goal of the present thesis is to study a large family of groups called Artin groups through the eyes of geometric group theory. Artin groups, or Artin-Tits groups, were introduced by Tits ([90]) as "extensions" of Coxeter groups, who are themselves a generalisation of the symmetry groups of the regular polyhedras. Consider a simplicial graph Γ and suppose that every edge between two vertices a and b has integer coefficient $m_{ab} \geqslant 2$. Then Γ defines an **Artin group** A_Γ whose generators are the vertices of Γ, and for which there is a relation of the form $aba \cdots = bab \cdots$ (with m_{ab} terms on each side) every time there is an edge connecting a and b. The **rank** of A_Γ is the cardinality of $V(\Gamma)$, and is assumed to be finite. The **Coxeter group** associated with Γ is the group W_Γ obtained from A_Γ by additionally requiring every generator to have order 2. While Coxeter groups are generated by "reflections" of order 2, in Artin groups the "reflections" have infinite order.

The class of Artin groups encompasses a large spectrum of groups that can be seen as interpolations between free groups (a discrete graph) and free abelian groups (a complete graph whose labels are all 2). It contains classes such as the right-angled Artin groups (those whose only permitted coefficients are 2), the braid groups (the braid relation having coefficient 3), and many others.

Coxeter groups are well understood. For instance, they are known to have solvable word and conjugacy problem ([91], [64]), to have finite centres (isomorphic to $(\mathbf{Z}/2\mathbf{Z})^n$) ([59]), to be CAT(0) groups ([74])) and to be virtually torsion-free (Tits proved they were linear in characteristic 0, and Selberg's Lemma states that such groups are virtually torsion-free). On the other hand, Artin groups remain quite mysterious. We recall that an Artin group A_Γ is said to be **reducible** if Γ is a **2-join**, that is, a join of two non-trivial subgraphs such that every edge of the join has coefficient 2 (when this happens, A_Γ can be decomposed as a direct product). Then, even the simplest questions cannot yet be solved in full generality:

Conjecture 1.1. Consider an Artin group A_Γ. Then:

(1) A_Γ is torsion-free.

(2) If A_Γ is irreducible and W_Γ is infinite, then the centre of A_Γ is trivial.

(3) A_Γ has solvable word and conjugacy problem.

(4) A_Γ satisfies the $K(\pi, 1)$ conjecture.

These conjectures are explained in more details in Section 2.4.2 (we also refer the reader to [26] for a survey on open questions about Artin groups). Despite not much being known in general, a lot is known about certain classes of Artin groups. We give here three classes of Artin groups for which substantial progress has been made:

• **Spherical Artin groups**. An Artin group A_Γ is **spherical** if its associated Coxeter group W_Γ is finite.

• **Artin groups of dimension 2**. The **dimension** of an Artin group A_Γ is the maximal integer n such that any choice of n vertices of Γ spans a subgraph $\Gamma' \subseteq \Gamma$ such that $A_{\Gamma'}$ is a spherical Artin group. The class of 2-dimensional Artin groups includes the class of large Artin groups (those with coefficients at least 3).

• **Artin groups of type FC**. An Artin group A_Γ is said to be of **type FC** if every complete subgraph $\Gamma' \subseteq \Gamma$ generates an Artin group $A_{\Gamma'}$ of spherical type. Note that spherical Artin groups and right-angled Artin groups are of type FC.

Spherical Artin groups are well understood, notably thanks to the existence of a normal form ([38],[37],[35]). Artin groups of dimension 2 and of type FC are also well understood, and satisfy all of the above conjectures ([28], [42], [41], [58], [22]).

There has recently been an increasing interest in understanding the geometry of Artin groups. In particular, the action of an Artin group on a space with properties that encode some kind of non-positive curvature. While free groups are the only Artin groups to be hyperbolic (every other Artin group contains a $\mathbf{Z}^2$ subgroup), even the "least" negatively curved Artin group, i.e. the free abelian groups, are non-positively curved. There is a reasonable belief that all Artin groups should be non-positively curved in a way or another. Similar to their algebraic and algorithmic behaviour, the geometry of Artin groups remains very mysterious as well. For instance, it is not known whether the following conjectures hold in general:

Conjecture 1.2. Consider an Artin group A_Γ. Then:

(1) A_Γ is CAT(0), i.e. acts properly and cocompactly on a CAT(0) space.

(2) If A_Γ is irreducible, the central quotient $A_\Gamma/Z(A_\Gamma)$ is acylindrically hyperbolic.

There are partial results to the above conjectures. For instance, Conjecture 1.2.(1) has been proved to hold for right-angled Artin groups ([27]), some classes of 2-dimensional Artin groups ([7], [14], [48]) or spherical Artin groups of rank 3 ([13]). As regards to Conjecture 1.2.(2), it is known to hold for Artin groups of spherical type ([33]). It is then enough to look at what happens when the group is non-spherical. In that case, since the centre $Z(A_\Gamma)$ is conjectured to be trivial, the question essentially comes down to asking whether A_Γ is acylindrically hyperbolic. Many results are of this type. For instance, Charney and Morris-Wright proved that Artin groups A_Γ whose defining graph Γ is not a join are acylindrically hyperbolic [28]. In [75], Martin and Przytycki also showed that 2-dimensional Artin groups of hyperbolic type (whose associated Coxeter groups are hyperbolic) are acylindrically hyperbolic.

Before exposing the results obtained in this thesis we want to bring light on the general methods we use. Many natural spaces associated to Artin groups have been introduced and studied over the years. The structure of most of these spaces comes from the combinatorics of important subgroups of Artin groups called **parabolic subgroups**, that are "smaller" Artin groups embedded in the main group and arising from subgraphs of the main defining graph. A prime example is the **modified Deligne complex** due to Deligne ([34]), and extended by Charney and Davis ([27]). For an Artin group A_Γ, this space noted D_Γ is a combinatorial complex that arises from the combinatorial structure of the spherical parabolic subgroups of A_Γ. This complex has become a central tool to understand the structure and geometry of Artin groups, and is at the heart of this thesis.

We now expose the various themes and results of this thesis in more details. All the results obtained concern Artin groups of dimension 2.

Acylindrical hyperbolicity

The notion of acylindrical hyperbolicity was recently introduced by Osin ([80]). A group G is said to be **acylindrically hyperbolic** if it is not virtually cyclic and has an acylindrical action on a hyperbolic space with unbounded orbits. Roughly speaking, acylindrically hyperbolic groups may not be hyperbolic, but still have "hyperbolic directions". The condition of acylindrical hyperbolicity merges many previously known results, bringing together classes such as mapping class groups, $Out(F_n)$ for $n \geqslant 2$, many CAT(0) groups and most of 3-manifold groups. Nevertheless, acylindrical hyperbolicity is still strong enough to ensure interesting properties for the group. For instance, acylindrically hyperbolic groups have finite centres and contain non-abelian free subgroups ([80]).

In this thesis we answer the question of acylindrical hyperbolicity for all Artin groups of dimension 2:

Theorem 1.3. *Every irreducible* 2*-dimensional Artin group of rank at least* 3 *is acylindrically hyperbolic.*

Acylindrically hyperbolic groups can never be decomposed as direct products of infinite groups. Therefore, a first consequence of Theorem 1.3 is that for 2-dimensional Artin groups, decomposability as a direct product is equivalent to irreducibility, which can directly be "read" from their defining graph:

Corollary 1.4. *A* 2*-dimensional Artin group A_Γ can be decomposed as a non-trivial direct product if and only if it is irreducible (equivalently, Γ is a* 2*-join).*

Note that acylindrically hyperbolic groups also have finite centres. Along with the fact that 2-dimensional Artin groups are torsion-free, this proves that the Artin groups from Theorem 1.3 actually have trivial centre. This gives a new proof of Conjecture 1.1.(2), which could already be deduced from [42] although it is not explicitly stated:

Corollary 1.5. *Artin groups of dimension* 2 *and rank at least* 3 *have trivial centre. Moreover, all irreducible Artin groups A_Γ of dimension* 2 *have acylindrically hyperbolic central quotient $A_\Gamma/Z(A_\Gamma)$.*

Parabolic subgroups

It is hard to imagine working with Artin groups without having to mention parabolic subgroups. These subgroups are not only the most "natural" kind of subgroups for an Artin group relatively to a given defining graph, but they are also incredibly useful in studying Artin groups.

Coxeter groups also admit parabolic subgroups whose definition is analogous to that of Artin groups. However, the combinatorics of parabolic subgroups of Coxeter groups are well-understood in general. For instance, it is known that the intersection of any subset of parabolic subgroups is itself a parabolic subgroup ([84]). By contrast, the analogous property for Artin groups is open in general:

Conjecture 1.6. Let A_Γ be any Artin group. Then the set of parabolic subgroups of A_Γ is stable under arbitrary intersections.

This conjecture has been proved true for braid groups using relations between parabolic subgroups of braid groups and isotopy classes of non-degenerated simple closed multicurves in mapping class groups of punctured disks. This result was recently generalised to all Artin groups of spherical type using Garside theory ([23]). For Artin groups of type FC, it was shown that the intersection of two parabolic subgroups of spherical type is again a parabolic subgroup of spherical type ([73]). However, the case of general parabolic subgroups remains open.

Besides being interesting on their own, such results about parabolic subgroups can be valuable tools in studying the structure of Artin groups. For instance, the positive answer to Conjecture 1.6 for spherical Artin groups was a key ingredient in the proof that Artin groups of type FC satisfy the Tits alternative ([77]).

In a joint work with Cumplido and Martin, we studied the combinatorics of the parabolic subgroups of large-type Artin groups, and proved the following:

Theorem 1.7. *Let A_Γ be a large-type Artin group. Then the intersection of an arbitrary subset of parabolic subgroups of A_Γ is itself a parabolic subgroup. Moreover, the set of parabolic subgroups of A_Γ is a lattice for the inclusion.*

A direct consequence of this theorem is that every subset of A_Γ is contained in a unique minimal parabolic subgroup. This generalises to large-type Artin groups the notion of **parabolic closure** known for Coxeter groups ([84]) and Artin groups of spherical type ([23]).

The approach we use to prove Theorem 1.7 is geometric in nature. We associate to each Artin group A_Γ a simplicial complex X_Γ called its **Artin complex**. This complex allows for a geometric study the parabolic subgroups of A_Γ, as they correspond to stabilisers of simplices of the complex. In particular, studying intersections of parabolic subgroups can be done if we have a sufficiently strong control over the (combinatorial) geodesics of X_Γ between two simplices. This is possible for large-type Artin groups, as we show that these complexes are non-positively curved in an appropriate sense. The key geometric result is the following:

Theorem 1.8. *Let A_Γ be a large-type Artin group of rank at least 3. Then the Artin complex X_Γ is systolic.*

As an application, we solve the conjugacy stability problem for parabolic subgroups of large-type Artin groups. A subgroup H of a group G is called **conjugacy stable** if two elements of H conjugated in G are always conjugated in H. This problem had already been solved for parabolic subgroups of spherical Artin groups ([21]), generalising pre-existing results for braid groups ([44]).

Theorem 1.9. *Let $A_{\Gamma'}$ be a standard parabolic subgroup of a large-type Artin group A_Γ. Then $A_{\Gamma'}$ is not conjugacy stable in A_Γ if and only if there exist vertices of Γ' that are connected by an odd-labelled path in Γ and that are not connected by an odd-labelled path in Γ'.*

As another application, we show that parabolic subgroups of large-type Artin groups are stable under taking roots, a result whose analogue for Artin groups of spherical type was proved in [23].

Theorem 1.10. *Let A_Γ be a large-type Artin group, let P be a parabolic subgroup of A_Γ, and let $g \in A_\Gamma$. If $g^n \in P$ for some integer $n \neq 0$, then $g \in P$.*

Beside the intersection properties of parabolic subgroups, the previous results rely on understanding the fixed-point sets and normalisers of parabolic subgroups. Their structure has been studied by various authors, but the results are a bit hidden in the literature. In the case of large-type Artin groups, our approach provides a unifying perspective that allows us to recover all these results within a single framework, giving an explicit description of every normaliser of parabolic subgroups of a large-type Artin group (see Theorem 4.5).

Isomorphism problem

A very natural goal for Artin and Coxeter groups is to want to answer the **isomorphism problem**, which is that of determining which defining graphs give rise to isomorphic Artin or Coxeter groups. A strong notion to consider is that of rigidity. An Artin or a Coxeter group is said to be **rigid** if it cannot be obtained from two non-isomorphic graphs. In [15], the authors proved that Artin and Coxeter groups are not rigid in general: two non-isomorphic graphs that are obtainable from each others by a series of "diagram twists" give rise to isomorphic Artin groups and Coxeter groups. For Coxeter groups, it was even showed that diagram twists are not the only way such a phenomenon can occur ([85]), although the question remains open for Artin groups. That said, studying the rigidity of Artin and Coxeter groups is essential for classes of groups in which there are no such twists. Coxeter groups have been well studied in that regard, and partial answers to the isomorphism problem have been obtained (see [78], [20]). However, not much is known for Artin groups, outside of right-angled Artin groups ([36]), and some large-type Artin groups ([32]).

The usually more accessible problem is to ask whether there are some classes of Artin groups in which we can solve the isomorphism problem and eventually show rigidity. A class of Artin groups is called **rigid** if two non-isomorphic graphs of the class always generate non-isomorphic Artin groups. Note that the rigidity of a class of Artin groups does not imply that Artin groups of the class are rigid.

Question 1.11. What classes of Artin groups are rigid?

For instance, the class of right-angled Artin groups has been proved to be rigid ([36]). In fact, every right-angled Artin group is itself rigid. The question of rigidity is inherently related to the study of isomorphisms between Artin groups. A natural next step in the theory is to try to understand these isomorphisms completely, which essentially comes down to understanding the automorphism groups of the Artin groups. Although Artin groups have been more and more studied over the past three decades, the study of the automorphisms of Artin groups has turned out to be quite difficult. The most famous results that are not only about free groups or free abelian groups are that of right-angled Artin groups ([36], [88], [65]). The situation becomes even more complicated when introducing non-commuting relations. Prior to our work, the only results on Artin groups that are not right-angled concerned the class of "connected large-type triangle-free" Artin groups introduced by Crisp in [32].

In this thesis, we give a partial answer to Question 1.11 by studying large-type Artin groups. The class of large-type Artin groups is known to not be rigid, hence why part of our study focuses on large-type Artin groups that are also **free-of-infinity** (i.e. $m_{ab} < \infty$ for all $a, b \in V(\Gamma)$). In [43], Godelle and Paris made explicit the interests of looking at free-of-infinity Artin groups. They proved that if one can solve any of the first three points of Conjecture 1.1 for all free-of-infinity Artin groups, then one can solve the corresponding conjecture for all Artin groups. It is thus natural to want to first study the Artin groups that are free-of-infinity. In our case, we obtained the following result of rigidity:

Theorem 1.12. *The class of large-type free-of-infinity Artin groups is rigid. In other words, if A_Γ and $A_{\Gamma'}$ are two large-type free-of-infinity Artin groups, then A_Γ and $A_{\Gamma'}$ are isomorphic if and only if Γ and Γ' are isomorphic.*

As a consequence of studying the isomorphisms between Artin groups, we are also able to recover a precise description of their automorphism groups:

Theorem 1.13. *Let A_Γ be a large-type free-of-infinity Artin group of rank at least 3. Then $Aut(A_\Gamma)$ is generated by the conjugations, the graph-induced automorphisms, and the global involution. In particular, $Out(A_\Gamma)$ is finite.*

While it is not possible to extend the two previous theorems to all large-type Artin groups (see [32]), we also prove a strong result of rigidity that holds for all large-type Artin groups. To our knowledge, this is the only result about isomorphisms that concerns all large-type Artin groups.

Theorem 1.14. *Let A_Γ and $A_{\Gamma'}$ be two large-type Artin groups of rank at least 3. Then any isomorphism $\varphi : A_\Gamma \to A_{\Gamma'}$ induces a bijection between the set of spherical parabolic subgroups of A_Γ and the set of spherical parabolic subgroups of $A_{\Gamma'}$.*

Theorem 1.14 has many consequences outside of being a precious tool for proving Theorem 1.12 and Theorem 1.13. For instance, it implies that any isomorphism between large-type Artin groups sends standard generators onto conjugates of standard generators. This gives a form of rigidity of the automorphism group.

While proving Theorem 1.12 and Theorem 1.13, we use the previous result of rigidity of the spherical parabolic subgroups and find a way to "reconstruct" the associated Deligne complex in a purely algebraic manner, i.e. in a way does not depend on the choice of defining graph for A_Γ, but only on the abstract structure of the group. In particular, we show that isomorphic large-type free-of-infinity Artin groups have isomorphic Deligne complexes (see Theorem 5.4). We also give an explicit classification of all the subgroups of large-type Artin groups that are isomorphic to dihedral Artin groups (see Theorem 5.5).

Chapter 2 serves as a preliminary chapter. We will start by recalling some basic notions of geometric group theory, before introducing in more details the various notions of curvature that we will use throughout this thesis. We then define the notion of complex of groups, that will be used to construct several key simplicial complexes. Finally, we recall the basic notions and conjectures related to Artin groups in more details.

Chapter 3 is dedicated to the study of acylindrical hyperbolicity for Artin groups of dimension 2, and follows the results of [93]. Along the way, we will also prove results of independent interests concerning the links of vertices in the Deligne complexes of these Artin groups. These results were a key ingredient that Hagen, Martin and Sisto used to prove that extra-large type Artin groups are hierarchically hyperbolic ([53]).

Chapter 4 is a joint work with Cumplido and Martin, and contains the results of [29]. There we will show the various results concerning the combinatorial structure of parabolic subgroups of large-type Artin groups that we exposed earlier in the introduction.

Finally in Chapter 5 we will focus on the question of rigidity for large-type Artin groups, following the results of [94]. This starts with a in-depth study of all the dihedral Artin subgroups of these Artin groups. We then obtain stronger results about large-type free-of-infinity Artin groups, classifying their automorphisms and proving the rigidity of the class.

Chapter 2

Preliminaries

In this chapter we introduce most of the standard definitions and results that will be used in this thesis. In Section 2.1 we introduce the most basic notions about metric spaces, simplicial complexes and some associated group actions. Section 2.2 is dedicated to various notions of non-positive curvature for spaces and groups. In Section 2.3 we introduce the notion of complexes of groups. Finally, in Section 2.4 we will be talking about Artin groups in more details.

2.1 Basic notions

In this section we introduce the most basic notions about groups as metric spaces, geodesics, abstract simplicial complexes and piecewise-euclidean simplicial complexes. We partially follow [17, Chapter I, Chapter II]. Throughout this section we suppose that G is a group generated by a finite set S.

2.1.1 Groups as metric spaces

Definition 2.1.1. The **free group** on S is the group F_S whose elements are the reduced words in the alphabet $S \sqcup \{s^{-1} \mid s \in S\}$. We denote by φ the epimorphism $\varphi : F_S \twoheadrightarrow G$ that sends every word w to the corresponding element $\varphi(w)$ of the group. A subset $R \subseteq F_S$ of words is called a set of **relations** for G if the smallest normal subgroup $\langle\langle R \rangle\rangle$ of F_S containing every element of R is precisely the kernel of φ. In that case, G is said to have **presentation**

$$G = \langle S \mid R \rangle.$$

A reduced word $w := s_1 \cdots s_n$, where $s_1, \cdots, s_n \in S$, is said of have **length** $\ell(w) = n \geqslant 0$. This yields a notion of **length** for elements of G, saying that

$$\ell(g) := \min\{\ell(w) \mid w \in F_S : \varphi(w) = g\}.$$

In other words, the length of g is the length of the shortest word representing the element g. This defines a metric on G known as the **word metric**:

$$d_S(g, h) = \min\{\ell(w) \mid w \in F_S : \varphi(w) = g^{-1}h\}.$$

The use of the word metric makes any finitely generated group into a metric space. Although this metric space is discrete (the distance between two elements is always an integer), one can extend this space to a metric graph by means of the corresponding Cayley graph:

Definition 2.1.2. The **Cayley graph** $Cay_S(G)$ of G with respect to S is the graph whose vertices are the elements of G, and for which there is an edge of length 1 between g and h if and only if $g^{-1}h \in S \sqcup S^{-1}$.

Note that when restricting to the vertex set of $Cay_S(G)$, the word metric and the metric induced from the Cayley graph coincide. A prime feature of Cayley graphs is that despite being seemingly highly dependent on the choice of (finite) generating set, all the Cayley graphs are "equivalent" in a way. This is made more precise in the following definition and proposition:

Definition 2.1.3. Let X and Y be two metric spaces. A map $f : X \to Y$ is called a **quasi-isometric embedding** if there are constants $A \geqslant 1, B \geqslant 0$ such that for every $p, q \in X$, we have

$$\frac{1}{A} \cdot d_X(p, q) - B \leqslant d_Y(f(p), f(q)) \leqslant A \cdot d_X(p, q) + B.$$

If additionally there exists a constant $C \geqslant 0$ such that the C-neighbourhood of $Im(f)$ is the whole of Y, then f is called a **quasi-isometry**.

Proposition 2.1.4. *Let G be a group with finite generating sets S and S'. Then the spaces $Cay_S(G)$ and $Cay_{S'}(G)$ are quasi-isometric.*

We now introduce some basic notions about metric spaces. Let (X, d) be such a space.

Definition 2.1.5. Let J be a connected subset of $\mathbf{R}$. A curve $\gamma : J \to X$ is called a **geodesic** if for every $t, t' \in J$ we have $d(\gamma(t), \gamma(t')) = |t' - t|$. The space X is called **geodesic** if for every pair $(x, y) \in X^2$ there exists a geodesic $\gamma : [0, D] \to X$ for which $\gamma(0) = x$ and $\gamma(D) = y$, and it is called **uniquely geodesic** if this geodesic is always unique.

Remark 2.1.6. The Cayley graph of any group G is always a geodesic space. However it is not uniquely geodesic, except when G is free.

One would like to define from a metric d on a space X the lengths of curves on X.

Definition 2.1.7. Let $\gamma : [a, b] \to X$ be a curve (or path) in X. The length of γ is defined by

$$\ell(\gamma) := \sup \left\{ \sum_{i=0}^{n-1} d(\gamma(t_i), \gamma(t_{i+1})) \mid \exists n \geqslant 0 : \exists t_0 := a \leqslant t_1 \leqslant \cdots \leqslant t_n := b \right\}.$$

2.1.2 Piecewise-Euclidean simplicial complexes

In this section we introduce simplicial complexes, and more precisely piecewise-Euclidean simplicial complexes. We start with a more general definition:

Definition 2.1.8. An **abstract simplicial complex** X consists of the following data.

(1) A non-empty set V called the set of **vertices** of X.

(2) A collection S of finite non-empty subsets $\Delta \subseteq V$ called the set of **simplices** of X, and satisfying:

(2.1) for every $v \in V$, we have $\{v\} \in S$;

(2.2) for any simplex $\Delta \in S$ and any non-empty set $\Delta' \subseteq \Delta$ we have $\Delta' \in S$.

One often refers Δ as an n-simplex if $|\Delta| = n + 1$ (alternatively, we say Δ has **dimension** n). An abstract simplicial complex Y is a **subcomplex** of X if the vertex set of Y is a subset of V and if every simplex of Y is a simplex of X. A subcomplex of a simplex Δ that is itself a simplex is called a **face** of Δ.

One often thinks of a simplicial complex in a geometric way, and not purely as an abstract set of vertices and simplices. Therefore we would like to be able to realise a given abstract simplicial complex geometrically. This is the goal of the next definition.

Definition 2.1.9. Consider an abstract simplicial complex X with vertex set V and simplex set S. Let $\mathbf{R}^V$ be the real vector space with basis V. The **geometric realisation** $|\Delta|$ of a simplex $\Delta \in S$ is the set of points in $\mathbf{R}^V$ of the form

$$\sum_{v \in \Delta} \lambda_v v \ , \qquad \text{with } \lambda_v \in [0,1] \text{ and } \sum_{v \in \Delta} \lambda_v = 1.$$

The **geometric realisation** of X is the subset $|X|$ of $\mathbf{R}^V$ obtained as the union of the geometric realisations of all the simplices of X.

Remark 2.1.10. (1) For the sake of having a lighter writing, we will often not distinguish an abstract simplicial complex X from its geometric realisation $|X|$, and we will simply call either of them a **simplicial complex**.
(2) The space obtained by looking at the union of all the simplices of dimension at most n of a simplicial complex X is called the **n-skeleton** of X and is denoted $X^{(n)}$.

So far we have not defined any topology nor any metric on our simplicial complexes. We do this now. Note that in this thesis we will mostly focus on simplicial complexes whose simplices are Euclidean, hence the following definition is specific to that case.

Definition 2.1.11. A **piecewise-Euclidean simplicial complex** is a space obtained as follows.
(1) Start with a simplicial complex X with vertex set V and simplex set S.
(2) Choose a set **Shapes(X)** of Euclidean geodesic simplices of finite dimension, where an **Euclidean geodesic simplex** of **dimension** n is the convex hull of $n+1$ affinely independent points in $\mathbf{E}^n$.
(3) For every simplex $\Delta \in S$, choose an affine isomorphism $f_\Delta : \overline{\Delta} \to \Delta$, where $\overline{\Delta}$ is an element of $Shapes(X)$, and require this isomorphism to be such that for every face Δ' of Δ, the map $f_\Delta^{-1} \circ f_{\Delta'}$ is an isometry from $\overline{\Delta'}$ onto its image in $\overline{\Delta}$. Then, define a metric on Δ by pushing the metric coming from $\overline{\Delta}$ through

f_Δ. The resulting space is a simplicial complex in which every simplex has been given an Euclidean metric.

While simplices in a piecewise-Euclidean simplicial complexes come with associated Euclidean metrics, we still have to define a metric on the whole space.

Definition 2.1.12. Let X be a piecewise-Euclidean simplicial complex, and let $x, y \in X$. Then a map $\gamma : [a, b] \to X$ is called a **simplicial curve** if there are $t_0 = a < t_1, < \cdots < t_{n-1} < b = t_n$ such that for every $i \in \{0, \cdots, n-1\}$, the image of the restriction $\gamma_i := \gamma|_{[t_i, t_{i+1}]}$ is contained in a single simplex of X. The **length** of γ is then defined by

$$\ell(\gamma) := \sum_{i=0}^{n-1} \ell(\gamma_i),$$

where $\ell(\gamma_i)$ is the length of the curve γ_i on its given simplex, as computed in Definition 2.1.7. We can then put a pseudo-metric on X by saying that

$$d_X(x, y) := \inf\{\ell(\gamma) \mid \gamma \text{ is a simplicial curve connecting } x \text{ and } y\}.$$

Note that when $Shapes(X)$ is finite, then the above map d is a true metric and the space (X, d) is a complete geodesic space [17, Chapter I.7].

Finally, we would like to bring light on the different types of actions that naturally appear in geometric group theory, as well as on the types of actions that will be the most common in this thesis. Our first definition concerns actions on simplicial complexes:

Definition 2.1.13. Let X and Y be two simplicial complexes. A map $f : X \to Y$ is called a **simplicial map** if the image through f of the vertices of any simplex of X span a simplex of Y. The map f is called a **simplicial isomorphism** if it is simplicial and bijective. The action of a group G on a simplicial complex X is called **simplicial** if for every $g \in G$ the action map $x \mapsto g \cdot x$ is a simplicial isomorphism.

Remark 2.1.14. Let $f : X \to Y$ be a simplicial isomorphism between two piecewise-Euclidean simplicial complexes and suppose that the restriction of f to any simplex of X is an isometry onto its image. Then f is a global isometry from X to Y.

Our next definitions apply to simplicial complexes, although it doesn't cost more to introduce them in a more general setting.

Definition 2.1.15. Let G be a group acting on a topological space X. Then we say that:
(1) G acts on X **by isometries** if X is a metric space and for every $g \in G$ the action map $x \mapsto g \cdot x$ is an isometry;
(2) G acts **properly** on X if for every compact set $K \subseteq X$ the set

$$\{g \in G \mid K \cap (g \cdot K) \neq \varnothing\}$$

is finite;
(3) G acts **cocompactly** on X if there exists a compact set $K \subseteq X$ such that

$$\bigcup_{g \in G} g \cdot K = X.$$

Equivalently, the quotient space X/G is compact.
The action of G on X is called **geometric** if G acts properly and cocompactly by isometries.

Geometric group theory is marked by two predominant kinds of actions, the first kind of action being the geometric actions. When a group G acts geometrically on a proper geodesic metric space X, the group and the space are quasi-isometric (this is known as the Švarc–Milnor lemma). Studying these kind of actions is essential to understand notions such as hyperbolicity, CAT(0)-ness (see Section 2.2) and many other properties related to groups. The second kind of actions that have been intensely studied are actions that are cocompact and by isometries but not necessarily proper. A prime example of this kind of actions is the Bass-Serre theory developed in the 1970's (see [87]). In particular, the actions associated with fundamental groups of graphs of groups and complexes of groups (see Section 2.3) are very often not proper. The group actions involved in this thesis are mostly of this second kind.

The last definitions we would like to introduce are associated with group actions by isometries. As it turns out, these actions are particularly well-behaved when the space under study is a piecewise-Euclidean simplicial complex.

Definition 2.1.16. Consider an action of G by isometries on a metric space X. The **translation length** of an element $g \in G$ is defined as

$$||g|| := \inf\{d_X(x, g \cdot x) \mid x \in X\}.$$

The (potentially empty) set of points $x \in X$ for which the translation length of g is reached is called the **minset** of g, and is denoted by

$$Min(g) := \{x \in X \mid d_X(x, g \cdot x) = ||g||\}.$$

Proposition 2.1.17. ***[17, Chapter II.6]*** *Let G be a group acting by simplicial isometries on a connected piecewise-Euclidean simplicial complex X with $Shapes(X)$ finite. Then for every $g \in G$ the set $Min(g)$ is non-empty.*

Definition 2.1.18. Let G be a group acting by simplicial isometries on a connected piecewise-Euclidean simplicial complex X with $Shapes(X)$ finite, and let $g \in G$. In regards to Proposition 2.1.17, there are two possibilities:
(1) If $||g|| = 0$, then g is called **elliptic**. In that case, g fixes pointwise a non-trivial set of points of X. This set is called the **fixed set** of g and is denoted by $Fix(g)$ (note that $Fix(g) = Min(g)$).
(2) If $||g|| > 0$, then g is called **hyperbolic**. In that case, g admits at least one geodesic called an **axis** of g, that is, a geodesic γ in which we have

$$\forall x \in \gamma, d_X(x, g \cdot x) = ||g||.$$

Note that every axis of g is contained inside of $Min(g)$.

2.2 Curvature

The idea of using curvature to study groups first emerged in the late 19th century from the study of groups acting on spaces that we thought had interesting curvature-like properties. This was notably permitted by the recent work on hyperbolic geometry and soon Riemannian geometry. Although not being formalised before the mid to late 20th century, many noticed that when a group acts nicely (for instance, geometrically) on a space with specific curvature conditions, there is a set of tools that can be used to describe various properties of the group itself. Today the study of groups by means of curvature-like properties has greatly developed, notably when the curvature is non-positive. A lot of different notions of non-positive curvature emerged following Gromov's hyperbolicity condition introduced in the 1980's. In this section, we describe four of these conditions.

Hyperbolicity is a strong group property of negative curvature that inspired most of the conditions of non-positive curvature today. This thesis does not revolve around the hyperbolicity condition as no Artin group (except the free groups) are hyperbolic. Nevertheless, it is still an important notion to mention. Acylindrical hyperbolicity is a generalisation of hyperbolicity that has been proved for many groups, including various Artin groups. This notion will be at the centre of Chapter 3. The CAT(0) property is a strong property of non-positive curvature that is well-suited for geodesic spaces. It is a central notion of this thesis, and notably of Chapter 3 and Chapter 5. Finally, systolicity is a combinatorial analogue of the CAT(0) property, that works usually better when studying high-dimensional simplicial complexes. It will play a central role in Chapter 4.

2.2.1 Hyperbolicity

The theory of hyperbolic groups was introduced by Gromov in 1987 ([45]). His inspirational work highly participated to the growth of geometric group theory and led numerous mathematicians to work on various notions of non-positive curvature for groups. The hyperbolic condition emanates from the wish to formalise the idea that certains groups (seen as metric spaces) behave in a way that shares a lot of similarities with negatively curved spaces coming from classical hyperbolic geometry. We define this notion thereafter.

Definition 2.2.1. Let X be a geodesic metric space and let $\delta \geqslant 0$. Then X is **δ-hyperbolic** if any geodesic triangle $[x, y] \cup [y, z] \cup [z, x]$ is δ-thin, that is, any of the three sides is contained in the union of the δ-neighbourhoods of the other two sides.

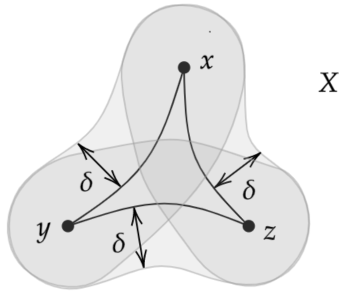


Figure 2.1: A δ-thin triangle.

We thereafter give a few examples of the most basic hyperbolic spaces.

Example 2.2.2. The following spaces are δ-hyperbolic for some $\delta \geqslant 0$:

(1) Bounded spaces: take δ to be at least the diameter of the space.

(2) Trees and real-trees: any geodesic triangle is degenerate, hence these spaces are 0-hyperbolic.

(3) The hyperbolic plane $\mathcal{H}^2$ with its usual metric is $ln(1+\sqrt{2})$-hyperbolic.

For a finitely generated group G, one can define a notion of hyperbolicity from the notion of δ-hyperbolicity of its Cayley graphs. As it turns out, the condition of hyperbolicity for G does not depend on the choice of finite generating set associated to which the Cayley graph corresponds. This comes from the fact that the Cayley graphs of G are all quasi-isometric with each others (see Proposition 2.1.4), along with the following theorem:

Theorem 2.2.3. *Let X and Y be two quasi-isometric metric spaces. If X is δ-hyperbolic for some $\delta \geqslant 0$ then Y is λ-hyperbolic for some $\lambda \geqslant 0$.*

Thus, one can simply define the notion of hyperbolicity as follows:

Definition 2.2.4. Let G be finitely generated group. Then G is **hyperbolic** if there is a finite generating set S such that $Cay_S(G)$ if δ-hyperbolic for some $\delta \geqslant 0$.

Remark 2.2.5. A group G will be called **elementary hyperbolic** if it is virtually cyclic.

We now give some examples of hyperbolic groups.

Theorem 2.2.6. *The following groups are hyperbolic:*

(1) Finite groups: their Cayley graphs are bounded.

(2) More generally, elementary hyperbolic groups are hyperbolic.

(3) Free-groups: their Cayley graphs relatively to the standard generators are trees.

(4) Groups acting geometrically on $\mathcal{H}^2$*, such as cocompact Fuchsian groups.*

(5) Fundamental groups of closed surfaces of negative Euler characteristic.

(6) Groups acting properly discontinuously on locally finite trees.

(7) Certain small cancellation groups such as $C'(1/6)$ *groups.*

(8) Many random groups ([79], [24]). In that sense, "most" groups are hyperbolic.

Thereafter we give some major consequences of being a hyperbolic group.

Theorem 2.2.7. *Let* G *be a (non-elementary) hyperbolic group. Then:*

(1) G *satisfies the Tits alternative, i.e. either it is virtually solvable, or it has non-abelian free subgroups.*

(2) G *is finitely presented and has solvable word problem ([17, Chapter III.*Γ*.2]).*

(3) G *has exponential growth rate.*

(4) G *is biautomatic.*

Even though a lot of groups are hyperbolic, there are many groups that arise naturally which are not hyperbolic. For instance, any group containing a subgroup isomorphic to $\mathbf{Z}^2$ cannot be hyperbolic. However, many groups still behave in a fashion that shares a lot of features with hyperbolic groups. Some of these notions are made explicit in the following sections.

2.2.2 Acylindrical hyperbolicity

The notion of acylindricity goes back to Sela ([86]) and gives conditions on the diameter of fixed-set points of elements associated with group actions on trees. In the more general case of metric spaces, the definition is due to Bowditch ([12]):

Definition 2.2.8. A group G is said to act acylindrically on a space X if for every $R \geqslant 0$, there exist $N > 0$, $L > 0$ such that

$$\forall x, y \in X,\ d(x,y) \geqslant L \Rightarrow |\{g \in G \mid d(x,gx) \leqslant R,\ d(y,gy) \leqslant R\}| \leqslant N.$$

One can see this condition as a kind of properness of the action of G on $X \times X$, minus a "thick diagonal".

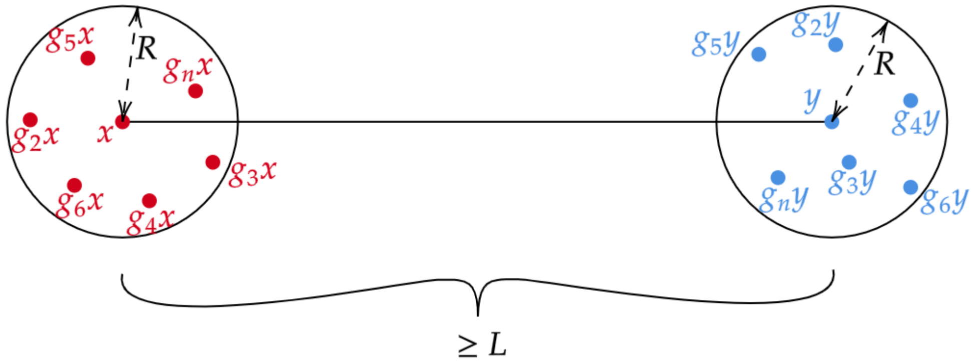


Figure 2.2: The condition of acylindrical hyperbolicity. Points at distance at least L must be such that the size of the intersection of their "R-quasi-stabilisers" is uniformly bounded by N.

The definition of acylindrical hyperbolicity itself is due to Osin ([80]):

Definition 2.2.9. A group G is said to be **acylindrically hyperbolic** if it is not virtually cyclic and has an acylindrical action with unbounded orbits on a hyperbolic space.

While acylindrically hyperbolic groups are not hyperbolic in general, the notion of acylindrical hyperbolicity comes from the idea of a group having "hyperbolic directions". The notion of acylindrical hyperbolicity unifies many previously studied notions. For a start, it generalises the condition of weak acylindricity introduced by Hamenstädt ([50]). It is also related to the existence of weakly contracting elements in the sense of Sisto ([89]). Last but not least, it is a generalisation of the notion of weak proper discontinuity (WPD) introduced by Bestvina and Fujiwara ([10]), that resembles the condition of acylindricity although the action only needs to be acylindrical in the direction of (quasi) axes of hyperbolic elements:

Definition 2.2.10. Let G be a group acting on a geodesic metric space X. We say $h \in G$ is a **WPD element** if

$$\forall \varepsilon, \forall x \in X, \exists M \in \mathbf{N} : |\{g \in G \mid d(x, gx) < \varepsilon, d(h^M(x), gh^M(x)) < \varepsilon\}| < \infty.$$

If every hyperbolic element is WPD then we say G acts on X **weakly properly discontinusouly**.

Merging these different notions, the notion of acylindrical hyperbolicity also merges many classes of groups:

Theorem 2.2.11. *[80]*
(1) Non-elementary hyperbolic groups are acylindrically hyperbolic.
(2) Non-virtually-cyclic relatively hyperbolic groups with proper peripheral subgroups are acylindrically hyperbolic.
(3) All but a finite number of mapping class groups of connected oriented surfaces are acylindrically hyperbolic, and the other ones are finite.
(4) The outer space $Out(F_n)$ is acylindrically hyperbolic for $n \geqslant 2$.
(5) Groups acting properly on a proper CAT(0) space with rank 1 elements are either virtually cyclic or acylindrically hyperbolic.

While being a weaker property than hyperbolicity, acylindrically hyperbolic groups still satisfy a lot of interesting properties:

Theorem 2.2.12. *[80] Let G be an acylindrically hyperbolic group. Then:*
(1) G has finite centre.
(2) For every decomposition $G \cong G_1 \times G_2$ as a direct product one of G_1 or G_2 is finite.
(3) For every decomposition $G = G_1 \cdots G_n$ as a product of subgroups one of the G_i's must be acylindrically hyperbolic.
(4) G contains non-abelian free normal subgroups.
(5) G is SQ-universal, that is, every countable group embeds as a subgroup in some quotient of G.
(6) Every s-normal subgroup H of G is acylindrically hyperbolic, where $H \leqslant G$ is s-normal in G if $|H \cap gHg^{-1}| = \infty$ for every $g \in G$.

Checking whether an action is acylindrical can be tough, as it essentially comes down to controlling the geodesics between two metric balls. Instead, one usually looks for an "acylindrical direction" in the action, more precisely a WPD element with a strongly contracting orbit, from which one can construct an acylindrical action on a larger space. This can be achieved using a criterion from Bestvina, Bromberg and Fujiwara (see Theorem 2.2.15). We need one definition before introducing this criterion:

Definition 2.2.13. Let X be a geodesic metric space, let Y be a subset of X, and let $B \geqslant 0$. Let now $\pi_Y : X \to Y$ denote the nearest point projection onto Y. Then Y is said to be **B-contracting** if for every point $x \in X \backslash Y$ and for every $k > 0$ such that the ball $B_X(x, k)$ does not intersect Y, we have $diam(\pi_Y(B_X(x, k)) \leqslant B$. An element h of a group G acting on X is said to have **B-contracting orbit** if h acts hyperbolically and the axes of h are B-contracting.

Example 2.2.14. (1) For any element $h \in Isom(\mathbf{H}^2)$ acting hyperbolically on $\mathbf{H}^2$ there is a $B \geqslant 0$ such that h has a B-contracting orbit.
(2) A contrario, no element $h \in Isom(\mathbf{E}^2)$ admits a B-contracting orbit.

Theorem 2.2.15. ***[6, Theorem H]*** *Let G be a group acting on a geodesic metric space X such that $h \in G$ is a hyperbolic WPD element with B-contracting orbit. Then G is either virtually cyclic or acylindrically hyperbolic.*

That approach proposed by the above theorem was followed to prove the acylindrical hyperbolicity of different classes of groups ([9],[47],[72]). Note that Theorem 2.2.15 does not require to make explicit an acylindrical action, nor to act on a hyperbolic space. That said, this condition remains hard to check when the space acted upon is not locally compact. In Chapter 3, we study acylindrical hyperbolicity by means of a criterion from Martin ([67]) that resembles Theorem 2.2.15, but that works well when looking at actions on spaces that are not locally compact (see Theorem 3.4).

2.2.3 The CAT(0) property

In the 1940's, Aleksandrov formulated a condition of curvature for all geodesic spaces that was directly inspired from the notion of curvature in Riemannian geometry. This condition was later formalised by Gromov as the CAT(κ) condition, named in reference to Cartan, Aleksandrov, and Toponogov. While being rather easy to state, the CAT(κ) condition allows to extend to all geodesic spaces the notion of curvature by comparing the triangles of a geodesic space to those of the 2-dimensional Riemannian manifold of constant curvature κ. A particularly interesting case, and one on which this thesis will intensely focus, is that of CAT(0) spaces. We recall this condition in full details thereafter. In this section, we follow parts of [17, Chapter II].

Definition 2.2.16. Let X be a geodesic space and let $T := [x, y] \cup [y, z] \cup [z, x]$ be a geodesic triangle in X. Then a comparison triangle for T is a triangle $\overline{T} := [\overline{x}, \overline{y}] \cup [\overline{y}, \overline{z}] \cup [\overline{z}, \overline{x}]$ in the Euclidean plane $\mathbf{E}^2$ for which we have $d_X(x, y) = d_{\mathbf{E}^2}(\overline{x}, \overline{y})$, $d_X(y, z) = d_{\mathbf{E}^2}(\overline{y}, \overline{z})$ and $d_X(z, x) = d_{\mathbf{E}^2}(\overline{z}, \overline{x})$. Note that any point $p \in T$ naturally corresponds to a unique point $\overline{p} \in \overline{T}$ called the comparison point of p in $\overline{T}$.

The space X is said to be CAT(0) if any geodesic triangle $T := [x, y] \cup [y, z] \cup [z, x]$ is "thinner" than a comparison triangle $\overline{T}$ in $\mathbf{E}^2$, that is, for every pair of points $p, q \in T$, we have $d_X(p, q) \leqslant d_{\mathbf{E}^2}(\overline{p}, \overline{q})$.

Figure 2.3: The CAT(0) condition. The triangle $\overline{T}$ is a comparison triangle for T in $\mathbf{E}^2$.

Remark 2.2.17. An important thing to notice is that CAT(0) spaces are always uniquely geodesic.

Example 2.2.18. The following spaces are CAT(0) spaces:

(1) Universal cover of non-positively curved compact manifolds.

(2) Euclidean buildings.

(3) Simply connected cubical complexes in which links of simplices are flag simplicial complexes.

The following definition and theorem allow to rephrase the CAT(0) condition into a more local condition:

Definition 2.2.19. A metric space X is said to have **curvature $\leqslant$ 0** if it is locally CAT(0), i.e. for every $x \in X$ there is a $k \geqslant 0$ such that the ball $B_X(x, k)$ is a CAT(0) space.

Theorem 2.2.20. *[17, Chapter II.5] Let X be a piecewise-Euclidean simplicial complex with Shapes(X) finite. Then the following are equivalent:*

(1) X is CAT(0);

(2) X is uniquely geodesic;

(3) X has curvature $\leqslant 0$ and contains no isometrically embedded loops;

(4) X has curvature $\leqslant 0$ and is simply connected.

Checking whether an Euclidean simplicial complex is CAT(0) is much easier in dimension 2 than in higher dimension. This is due to Lemma 2.2.25 below. First we need to introduce two very useful notions that encapsulate the notion of neighbourhoods for simplices in a simplicial complex. These notions will be used throughout the thesis.

Definition 2.2.21. Let X be a simplicial complex and let σ be a simplex of X. The **star** of σ in X is the subcomplex $St_X(\sigma)$ defined as the union of all the simplices of X that contain σ. The **link** of σ in X is the subcomplex $Lk_X(\sigma)$ defined as the union of the simplices of $St_X(\sigma)$ that are disjoint from σ.

Let now x be any point of X, and let $\sigma := \mathbf{supp(x)}$ be the smallest simplex of X containing x. Then the **star** $St_X(x)$ of x is the star of σ, and the **link** of x is the subcomplex $Lk_X(x)$ defined as the union of the simplices of $St_X(x)$ that don't contain x.

Definition 2.2.22. Let X be a piecewise-Euclidean 2-dimensional simplicial complex. Then the only links that are not trivial nor discrete are the links of vertices (and more generally points) of X. In that case, the link $Lk_X(x)$ of a point $x \in X$ is a subcomplex of X isomorphic to a graph. While this graph inherits a metric from the ambient space X, we define another metric on $Lk_X(x)$ called the **link metric** or the **angular metric** as follows:

• for every edge σ of $Lk_X(x)$ and every points $p, q \in \sigma$, the distance $d_{Lk_X(x)}(p, q)$ between p and q is defined as the angle $\angle_x([x, p], [x, q])$, where $[x, p]$ and $[x, q]$ are the (unique) geodesics of X connecting x to p and q respectively (note that this angle can be mesure in a single simplex that isometrically embeds into $\mathbf{E}^2$);

• the metric on $Lk_X(x)$ is obtained from gluing the metrics of the simplices of $Lk_X(x)$, as done in Definition 2.1.12.

Definition 2.2.23. Let X be a piecewise-Euclidean 2-dimensional simplicial complex and let x be any point of X. Let also γ and γ' be two geodesics meeting at x. Let $\varepsilon > 0$ be small enough so that the sphere $S_X(x, \varepsilon)$ is contained inside of $St_X(x)$ and so that γ and γ' intersect $S_X(x, \varepsilon)$ at two points p and q respectively. The sphere $S_X(x, \varepsilon)$ can be seen as a graph affinely isomorphic to $Lk_X(x)$, and we push the link metric of $Lk_X(x)$ (see Definition 2.2.22) onto $S_X(x,\varepsilon)$ through that affine isomorphism. We call this metric $d_{S_X(x,\varepsilon)}$. The **angle** $\angle_x(\gamma,\gamma')$ between γ and γ' at x is defined as

$$\angle_x(\gamma,\gamma') := d_{S_X(x,\varepsilon)}(p,q).$$

Definition 2.2.24. Let X be a metric space, and consider the family Ω of all the isometrically embedded (equivalently, non-contractible) loops $\gamma : [a,b] \to X$ in X. Then the **systole** of X is defined by

$$sys(X) := \inf\{\ell(\gamma) \mid \gamma \in \Omega\}.$$

Lemma 2.2.25. ***[17, Chapter II.5]*** *Let X be a piecewise-Euclidean 2-dimensional simply connected simplicial complex with Shapes(X) finite. Then X is CAT(0) if and only if for every vertex $v \in X$, the length of an isometrically embedded loop in $Lk_X(v)$ is at least 2π (in other words, $\mathrm{sys}(Lk_X(v)) \geqslant 2\pi$).*

As with the notion of hyperbolicity, the notion of CAT(0) spaces gives rise to a notion of CAT(0) groups:

Definition 2.2.26. A group G is said to be CAT(0) if it acts geometrically on a CAT(0) space.

Example 2.2.27. The following are CAT(0) groups:
(1) The free abelian groups $\mathbf{Z}^n$ (acting naturally on $\mathbf{R}^n$).
(2) The free groups F_n (acting on their usual Cayley graphs).
(3) Fundamental groups of closed surfaces of non-positive Euler characteristic (acting on their universal cover).

2.2.4 Systolicity

In the early 2000's, people such as Haglund, Januszkiewicz and Świątkowski introduced new notions of non-positive curvature for simplicial complexes ([51], [61]). In 2006, Januszkiewicz and Świątkowski formalised a more general notion of non-positive curvature for simplicial complexes by means of two conditions known as k-largeness and k-systolicity ([62]). Their study can be thought of as an extension of small cancellation theory to higher dimensions. Despite not being equivalent to the CAT(0) condition, systolic spaces (resp. groups) share a lot of features with CAT(0) spaces (resp. groups). Hence why this notion can be seen as a combinatorial analogue of the CAT(0) condition. One of the useful things about this notion is that it does not require the presence of a metric on the space of study. Contrary to the CAT(0) condition which can be very hard to check for spaces of high dimension, the notion of systolicity often behaves nicely in any dimension.

Recall that if γ is a simplicial path in the 1-skeleton of a simplicial complex X, then the simplicial length of γ is simply the number $\ell(\gamma)$ of edges contained in γ. The systole of a simplicial complex X is defined in a similar fashion as in Definition 2.2.24, although the simplicial complex X is not required to hold a metric. The next definitions can be found in ([62]):

Definition 2.2.28. The **systole** of a simplicial complex X is the minimal simplicial length of a non-homotopically-trivial loop in its 1-skeleton $X^{(1)}$. For $k \in \{3, \dots, \infty\}$, we say that a simplicial complex X is **locally k-large** if

$$\mathrm{sys}(Lk_X(\Delta)) \geqslant k$$

for all simplices $\Delta \subseteq X$. We say that X is **k-large** if it is locally k-large and $\mathrm{sys}(X) \geqslant k$. The complex X is **k-systolic** if it is connected, simply-connected and locally k-large. Finally, X is called **systolic** if it is 6-systolic.

Definition 2.2.29. A group G is **k-systolic** if it acts simplicially, properly discontinuously and cocompactly on a k-systolic simplicial complex. It is **systolic** if it is k-systolic for some $k \geqslant 6$.

The notion of systolicity was partially inspired from the wish to answer to a question asked independently by Moussong, Gromov and Bestvina ([46], [74]) who suggested there was a bound on the (cohomological) dimension a hyperbolic Coxeter group could have. This was proved wrong in [62], where the authors gave examples of hyperbolic Coxeter groups with arbitrary high cohomological dimension. They also proved that these Coxeter groups were systolic. In [57], the authors also proved that large-type Artin groups were systolic groups.

The notion of systolicity has many consequences for a space. For instance, systolic spaces have path-fixing properties and fixed-point theorems similar to what can happen in other types of non-positively curved spaces such as CAT(0) spaces. Some of these properties will turn out to be very useful in Chapter 4, where they will be given in full details. For now, we only decide to give the following theorem:

Theorem 2.2.30. ***[62, Theorem 4.1]*** *Any finite dimensional systolic simplicial complex is contractible.*

Systolic and k-systolic groups also behave is a nice way. This is highlighted by the following two theorems, which should convince the reader on the strength of systolic geometry.

Theorem 2.2.31. *Any 7-systolic group is word-hyperbolic.*

Theorem 2.2.32. *Any systolic group is biautomatic.*

Consequences of hyperbolicity are given in Section 2.2.1. Biautomaticity on its own also implies interesting properties. For instance, biautomatic groups have quadratic isoperimetric inequalities, their abelian subgroups are undistorted, and their solvable subgroups are virtually abelian.

2.3 Simple complexes of groups

Let G be a group acting by isometries on a simplicial complex X, and suppose that the action admits a fundamental domain Y that is strict, that is, two points of Y never lie in the same orbit. Then one can reconstruct X only from looking at G, Y, and the stabilisers of the simplices of Y. The data of these stabilisers subgroups can be put together in a system called a complex of groups, that resembles the notion of graphs of groups coming from Bass-Serre theory (although complexes of groups work in higher dimension). We introduce these notions in full details thereafter. For additional information, we refer the reader to [17, Chapter II.12].

Definition 2.3.1. A **simple complex of groups** $G(\mathcal{Q})$ over a **poset** (i.e. partially ordered set) $\mathcal{Q}$ consists of:

(1) For each element $\sigma \in \mathcal{Q}$, a group G_σ called the local group at σ.

(2) For each $\tau < \sigma$, an injective morphism $\psi_{\tau\sigma} : G_\sigma \hookrightarrow G_\tau$ such that

$$\tau < \sigma < \rho \Longrightarrow \psi_{\tau\rho} = \psi_{\tau\sigma}\psi_{\sigma\rho}.$$

A **simple morphism** φ from $G(\mathcal{Q})$ to a group G is a map written as $\varphi : G(\mathcal{Q}) \to G$ that associates to each element $\sigma \in \mathcal{Q}$ a morphism $\varphi_\sigma : G_\sigma \to G$ such that if $\tau < \sigma$ then $\varphi_\sigma = \varphi_\tau\psi_{\tau\sigma}$. The map φ is said to be injective on the local groups if φ_σ is injective for each $\sigma \in \mathcal{Q}$. The **direct limit** of the system $(G_\sigma, \psi_{\tau\sigma})$ is the group $\widehat{G(\mathcal{Q})}$ defined by

$$\widehat{G(\mathcal{Q})} := \left(*_{\sigma\in\mathcal{Q}} G_\sigma\right) \Big/ \left\{\psi_{\tau\sigma}(h) = h, \forall h \in G_\sigma, \forall(\tau,\sigma) : \tau < \sigma\right\}.$$

This group is called the **fundamental group** of the complex of groups $G(\mathcal{Q})$.

Example 2.3.2. (1) A n-dimensional simplex of groups is a complex of groups over the poset of the faces of a simplex of dimension n. More precisely, if Δ is a simplex of dimension n with faces $F_1, \cdots, F_n$, a face of codimension k in Δ can be written in a unique way as an intersection

$$F_I := \bigcap_{i\in I} F_i,$$

where $I \subseteq \{1, \cdots, n\}$ is such that $|I| = k$. To obtain a simplex of groups, associate with every face F_I a local group G_{F_I} and with every inclusion $F_I \subseteq F_J$ an injective morphism $\psi_{F_I F_J} : G_{F_J} \hookrightarrow G_{F_I}$. A triangle of groups is a simplex of groups with $n = 2$ (see Figure 2.4 for an example).

(2) Let Δ be an equilateral triangle in the Euclidean plane $\mathbb{E}^2$, and let a, b and c denote the isometries of the plane defined by doing symmetries along the different edges of Δ. Let now G be the subgroup of $Isom(\mathbb{E}^2)$ generated by a, b and c (note that G acts on $\mathbb{E}^2$ by isometries). Let $\mathcal{Q}$ be the poset of simplices of Δ, and for every simplex $\sigma \in \mathcal{Q}$, let G_σ be the subgroup of G corresponding to the stabiliser of σ. Finally, for two simplices $\tau < \sigma$, let $\psi_{\tau\sigma}$ be the natural inclusion of G_σ into G_τ. Then the system $(G_\sigma, \psi_{\tau\sigma})$ is the simple triangle of groups $G(\mathcal{Q})$ described in Figure 2.4. The fundamental group of $G(\mathcal{Q})$ is precisely the (Coxeter) group G.

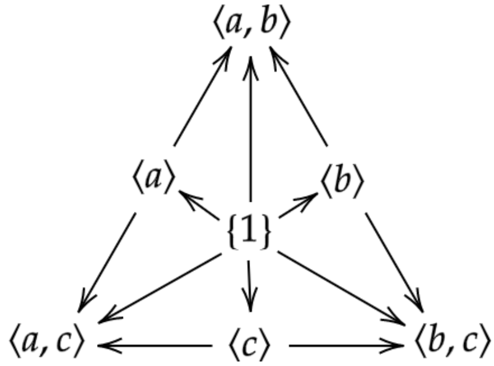


Figure 2.4: Triangle of (Coxeter) groups with fundamental group G. The local group corresponding to the face is $\{1\}$, those corresponding to the edges are the cyclic groups $\langle a\rangle \cong \mathbf{Z}/2\mathbf{Z}$, $\langle b\rangle \cong \mathbf{Z}/2\mathbf{Z}$ and $\langle c\rangle \cong \mathbf{Z}/2\mathbf{Z}$, and those corresponding to the vertices are the dihedral groups $\langle a,b\rangle \cong D_3$, $\langle a,c\rangle \cong D_3$ and $\langle b,c\rangle \cong D_3$. The maps are just the natural inclusions.

Definition 2.3.3. Let X be a simplicial complex and let $\mathcal{P}$ be the poset formed by the simplices of X. Let G be a group acting simplicially on X, and suppose that the fundamental domain Y of this action is **strict**, i.e. such that distinct points of Y always lies in distinct orbits. Let now $\mathcal{Q} := \{\sigma \in \mathcal{P} \mid \sigma \subseteq Y\}$. We can recover a complex of groups $G(\mathcal{Q})$ in the following way :
• To each element $\sigma \in \mathcal{Q}$ corresponds a subgroup G_σ of G which is the stabiliser of σ through the action of G.
• To every inclusion $\tau \subseteq \sigma$ corresponds a map $\psi_{\tau\sigma} : G_\sigma \hookrightarrow G_\tau$ that is the natural inclusion of the corresponding stabilisers.
The complex of groups $G(\mathcal{Q})$ is then defined as

$$G(\mathcal{Q}) := \{(G_\sigma, \psi_{\tau\sigma}) \mid \sigma, \tau \in \mathcal{Q},\ \tau \subseteq \sigma\}.$$

Notice that the inclusions $\varphi_\sigma : G_\sigma \to G$ give a simple morphism $\varphi : G(\mathcal{Q}) \to G$ that is injective on the local groups. A complex of groups $G(\mathcal{Q})$ is said to be **strictly developable** if there exists a simplicial complex X and a simplicial action of G on X with strict fundamental domain some subcomplex Y, such that the complex of groups recovered in the previous way is precisely $G(\mathcal{Q})$.

Definition 2.3.4. Let Y be a simplicial complex and let $\mathcal{Q}$ be the poset of its simplices. Let now $G(\mathcal{Q})$ be a complex of groups and let $\varphi : G(\mathcal{Q}) \to G$ be a simple morphism to some group G, that is injective on the local groups. The

development $D(Y,\varphi)$ of Y along φ is defined by

$$D(Y,\varphi) := {}^{G \times Y}/_{\sim},$$

where $(g,x) \sim (g',x') \iff x = x'$ and $g^{-1}g'$ belongs to the local group of the simplex $supp(x)$. In particular, if $G = \widehat{G(\mathcal{Q})}$, then the space $D(Y,\varphi)$ is called the **universal cover** of $G(\mathcal{Q})$ with fundamental domain Y. This space is always connected and simply connected ([17, Theorem II.12.20]).

We have just seen in Definition 2.3.3 that any strictly developable complex of group admits a simple morphism that is injective on the local groups. Using Definition 2.3.4, one can prove that this is actually an equivalence:

Theorem 2.3.5. *[17, Theorem II.12.18] A complex of groups $G(\mathcal{Q})$ is strictly developable if and only if the natural simple morphism $\varphi : G(\mathcal{Q}) \to \widehat{G(\mathcal{Q})}$ is injective on the local groups.*

Remark 2.3.6. The situation exposed in Theorem 2.3.5 can be synthesised as follows. The space $X := {}^{G \times Y}/_{\sim}$ is a simplicial complex on which G acts in such a way that the stabiliser of a simplex of the form $(1,\sigma)$ is precisely G_σ. The stabiliser of a simplex of the form (g,σ) is $gG_\sigma g^{-1}$.

Example 2.3.7. (1) The triangle of groups described in Figure 2.4 is developable. This comes from the fact that the natural inclusions of the various subgroups into G form a simple morphism $\varphi : G(\mathcal{Q}) \to \widehat{G(\mathcal{Q})}$ that is injective on the local groups.

(2) Consider the poset $\mathcal{Q}$ with 5 elements that is described in Figure 2.5. From this poset we create a complex of groups $G(\mathcal{Q})$ in the following way. The local group corresponding to the central vertex is trivial, the local groups corresponding to the right-most vertex is isomorphic to $\mathbf{Z}^2$, and all the other local groups are isomorphic to $\mathbf{Z}$. The maps coming from the upper and lower vertices to the left-most vertex are the identity maps, and the map coming from the upper (resp. lower) vertex to the right-most vertex is the identity onto the first (resp. the second) standard generator of the $\mathbf{Z}^2$ group. It is not hard to see that the fundamental group of this complex of groups is $\widehat{G(\mathcal{Q})} \cong \mathbf{Z}$ (in the quotient, each of the five generators are identified). In particular, there is no injection from the local group of the right-most vertex into this fundamental group, and thus the map $\varphi : G(\mathcal{Q}) \to \mathbf{Z}$ cannot be simple. This complex of groups is not developable.

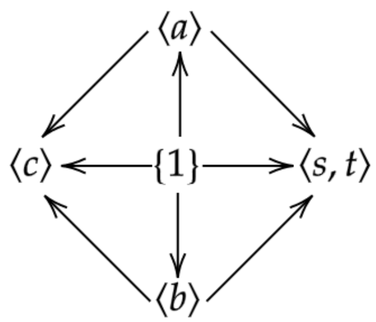


Figure 2.5: Example of a non-developable complex of groups $G(\mathcal{Q})$. The maps are the inclusions described in Example 2.3.7.(2).

2.4 Artin groups

In this section we recall the basic notions surrounding Artin groups, giving more details than in the introduction.

2.4.1 Definitions

We start by recalling the definition of an Artin group:

Definition 2.4.1. Let Γ be a (finite) simplicial graph with vertex set $V(\Gamma)$ and edge set $E(\Gamma)$, and suppose that every edge $e^{ab} \in E(\Gamma)$ is given a coefficient $m_{ab} \in \{2, 3, 4, \cdots\}$. Then Γ defines an **Artin group** A_Γ whose presentation is given by

$$A_\Gamma := \langle\ V(\Gamma)\ |\ \underbrace{aba\cdots}_{m_{ab}\text{ terms}} = \underbrace{bab\cdots}_{m_{ab}\text{ terms}}, \forall e^{ab} \in E(\Gamma)\ \rangle.$$

We set $m_{ab} := \infty$ when the vertices a and b are not connected by an edge. The elements of $V(\Gamma)$ are called the **standard generators** of A_Γ. The **rank** of A_Γ is the cardinality of $V(\Gamma)$, that is, the number of standard generators of A_Γ. The graph Γ also defines a **Coxeter group** W_Γ whose presentation is given by

$$W_\Gamma := \langle\ V(\Gamma)\ |\ s^2 = 1, \forall s \in V(\Gamma),\ \text{and}\ \underbrace{aba\cdots}_{m_{ab}\text{ terms}} = \underbrace{bab\cdots}_{m_{ab}\text{ terms}}, \forall e^{ab} \in E(\Gamma)\ \rangle.$$

The graph Γ is called the **defining graph** of A_Γ and W_Γ.

Remark 2.4.2. There is a natural projection $A_\Gamma \twoheadrightarrow W_\Gamma$ that restricts to the identity on the standard generators. The kernel of that projection is often called the pure Artin group associated with Γ.

Example 2.4.3. Artin groups form a large family of groups, that range from free abelian groups to free groups. Some examples or more complicated Artin groups are given in Figure 2.6.

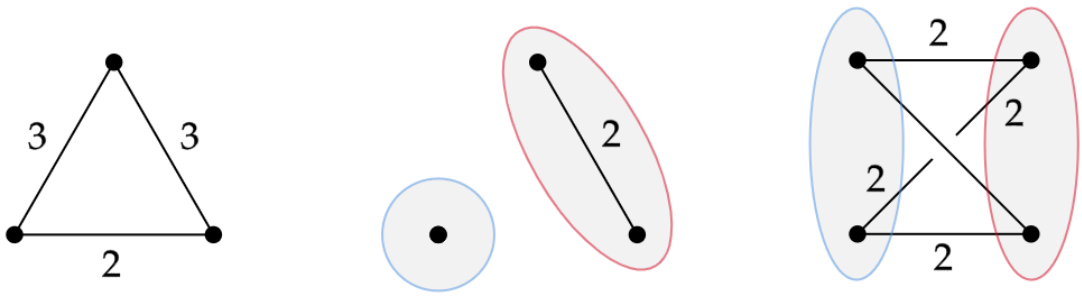


Figure 2.6: Three graphs defining Artin groups. On the left: the graph defining the braid group B_4 on four strands. In the middle: the graph defining an Artin group isomorphic to $\mathbf{Z} * \mathbf{Z}^2$. The decomposition of the graph in its connected components is highlighted. On the right: the graph defining an Artin group isomorphic to $F_2 \times F_2$, where F_2 represents the free group on 2 generators. The decomposition of the graph as a 2-join is highlighted.

One of the most basic tools when working with Artin groups is to consider their parabolic subgroups, which are subgroups corresponding to subgraphs of the defining graph. Before giving an explicit definition of these subgroups, we recall the following very useful theorem of Van der Lek:

Theorem 2.4.4. *[92] Let Γ be a graph defining an Artin group A_Γ, let S be a subset of $V(\Gamma)$, and let Γ' be the subgraph of Γ spanned by the vertices of S. Then the subgroup of A_Γ generated by S is isomorphic to the Artin group $A_{\Gamma'}$.*

Remark 2.4.5. (1) A subgraph of Γ spanned by a subset $S \subseteq V(\Gamma)$ will be called **induced**, or **induced by S**.

(2) For an induced subgraph Γ' of Γ, we will just write $A_{\Gamma'}$ to talk about the subgroup of A_Γ generated by $V(\Gamma')$. Reciprocally, if A_Γ is an Artin group, the notation $A_{\Gamma'}$ will always be used to describe the subgroup of A_Γ generated by $V(\Gamma')$.

Definition 2.4.6. Let A_Γ be an Artin group. A subgroup of A_Γ is called a **standard parabolic subgroup** if it is of the form $A_{\Gamma'}$, where Γ' is an induced subgraph of Γ. A subgroup P of A_Γ is a **parabolic subgroup** of A_Γ if it is conjugated to a standard parabolic subgroup of A_Γ.

Definition 2.4.7. A parabolic subgroup P of A_Γ will be said to be of **type** n if $P = gA_{\Gamma'}g^{-1}$ for some element $g \in A_\Gamma$ and the number of vertices in $V(\Gamma')$ is n. When we care about the precise structure of the subgraph Γ', we will also say the parabolic subgroup P has **type** Γ'.

Remark 2.4.8. If $A_{\Gamma'}$ is a standard parabolic subgroup of type 2 with standard generators $a, b \in V(\Gamma)$, then we will write A_{ab} to talk about $A_{\Gamma'}$. This kind of parabolic subgroup is called a **dihedral Artin** subgroup if $3 \leqslant m_{ab} < \infty$. Similarly if $A_{\Gamma'}$ has type 3 with standard generators $a, b, c \in V(\Gamma)$, then we will write A_{abc} instead of $A_{\Gamma'}$.

2.4.2 Main questions

Despite having a relatively simple presentation, the majority of Artin groups remain quite mysterious in general. In this section we present the main conjectures about Artin groups, as well as where the research stands towards proving them, prior to our work.

A reason why most of the conjectures about Artin groups remain open in general is that the family of Artin groups is so wide that it is hard to create an argument that works for all Artin groups at once. When working on Artin groups, one usually restricts to a specific class of Artin groups, which allows the use of more specific tools.

Consider any defining graph Γ and its associated Artin group A_Γ. On one hand, if the graph Γ has connected components $\Gamma_1, \cdots, \Gamma_k$, then the Artin group A_Γ can be decomposed as a free product $A_\Gamma \cong A_{\Gamma_1} * \cdots * A_{\Gamma_k}$ (see Figure 2.6). On the other hand, if the graph Γ can be decomposed as a join of subgraphs $\Gamma_1, \cdots, \Gamma_k$ such that every edge of the join is labelled by a 2, then A_Γ can be decomposed as a direct product $A_\Gamma \cong A_{\Gamma_1} \times \cdots \times A_{\Gamma_k}$ (see Figure 2.6). When this happens, Γ is a **2-join**, and the Artin group A_Γ is called **reducible**. It is called **irreducible** otherwise. When A_Γ admits such a decomposition as a free product or as a direct product, most of the information regarding A_Γ can be obtained by looking at the subgroups of the form A_{Γ_i} individually. This is why most results about Artin groups assume without loss of generality that the Artin groups are irreducible and have connected defining graphs. Finally, most results also assume that the Artin groups have rank at least 3, as Artin groups of rank $\leqslant 2$ are very well understood already.

The first family of Artin groups to talk about is the family of spherical Artin groups. An Artin group A_Γ is called **spherical** if the associated Coxeter group W_Γ is finite. The family of spherical Artin groups contains the family of braid groups, and can be seen as the "simplest" kind of Artin groups. Although their rank can be arbitrary large, Artin groups of spherical type are very-well understood. This is largely due to the existence of a (Garside) normal form ([38],[37],[35]). Understanding the spherical Artin groups is essential to understand other types of Artin groups. For a general Artin group A_Γ, the combinatorics of the spherical (standard) parabolic subgroups of A_Γ can be used to define the so-called "Deligne complex", which has become an essential tool in the theory of Artin groups (see Section 2.4.3).

The second family we want to talk about is the family of 2-dimensional Artin groups, on which this thesis heavily focuses. By definition, the **dimension** of an Artin group is the maximal rank its spherical parabolic subgroups can have. In particular, the dimension of an Artin group is also the (simplicial) dimension of its associated Deligne complex. It is also conjectured that the dimension of any Artin group equals its cohomological dimension, although this is still open in general. We want to highlight that there is a very pratical way to see directly from its defining graph whether a given Artin group has dimension 2, as is given by the following theorem.

Theorem 2.4.9. *Let A_Γ be an Artin group. Then the following are equivalent:*
(1) The maximal rank of a spherical parabolic subgroup of A_Γ is 2;
(2) The subgroups of A_Γ isomorphic to $\mathbf{Z}^n$ satisfy $n \leqslant 2$;
(3) For every triplet of generators $a, b, c \in V(\Gamma)$, we have

$$\frac{1}{m_{ab}} + \frac{1}{m_{ac}} + \frac{1}{m_{bc}} \leqslant 1.$$

Note the the third point in the above theorem simply says that every 3-cycle in Γ must be labelled with three coefficients whose sum of inverses is no greater than 1. The class of 2-dimensional Artin group has been well-studied over the years. A prime feature of these groups is that their associated Deligne complexes are 2-dimensional, which makes it much easier to study than in higher dimension.

The class of 2-dimensional Artin groups contain other well-studied classes, such as the class of **large-type** Artin groups (every coefficient is at least 3), the class of **extra large-type** Artin groups (every coefficient is at least 4), or even the class of **XXL** Artin groups (every coefficient is at least 5). Many results have been obtained for these classes, both by proving the corresponding groups are non-positively curved, or by using the non-positive curvature properties of group actions to recover information about the groups themselves.

The last family of Artin groups we want to mention is the family of Artin groups of type FC (short for "Flag Complex"). An Artin group A_Γ is said to be of **type FC** if every complete subgraph Γ' of Γ generates an Artin subgroup $A_{\Gamma'}$ of spherical type. Although Artin groups of type FC can be of arbitrary high dimension, their good combinatorial properties make it so that their associated Deligne complex is somewhat well-understood. In particular, there are a lot of geometric tools that can be used to study Artin groups of type FC, and hence much is known about them. This family includes the aforementioned family of spherical Artin groups, as well as the intensely studied class of **right-angled** Artin groups, which are the Artin groups in which the only permitted coefficients are 2 or ∞. However right-angled Artin groups are rather specific within the spectrum of all Artin groups, and people usually study them using tools that can be quite different from all the other families of Artin groups. Finally, we want to highlight that the intersection between the class of 2-dimensional Artin groups and the class of Artin groups of type FC is precisely the class of **triangle-free** Artin groups, i.e. the Artin groups A_Γ where the graph Γ does not contain any 3-cycle.

We now come back to enunciating the mains conjectures about Artin groups. Note that all the conjectures stated below are open in general.

The first conjecture is the most easily-stated and concerns the torsion of elements in Artin groups. An element g of a group G is said to be **torsion** if there is an $n \neq 0$ such that $g^n = 1$. The group G is called **torsion-free** if it contains no non-trivial torsion element.

Conjecture 2.4.10. Every Artin group is torsion-free.

The second conjecture concerns the centres of Artin groups. Spherical Artin groups are known to have non-trivial centres, that are isomorphic to $\mathbf{Z}^n$ for some $n \geqslant 1$. If we assume the Artin groups are also irreducible, their centre become isomorphic to $\mathbf{Z}$ and generated by the so-called "Garside element". The centres of non-spherical Artin groups remain more mysterious.

Conjecture 2.4.11. Every irreducible non-spherical Artin group has trivial centre.

The third conjecture has an algorithmic flavor. A group G with finite generating set S is said to have solvable **word problem** if there exists an algorithm that takes as input any word $w \in F_S$, and tells in a finite time whether w represents the identity in G or not. The group G is said to have solvable **conjugacy problem** if there is an algorithm that can say in a finite time whether any two words $w, v \in F_S$ correspond to conjugated elements of G. Note that solving the conjugacy problem directly solves the word problem.

Conjecture 2.4.12. Every Artin group has solvable word and conjugacy problems.

We now come back to parabolic subgroups. These subgroups are probably the most-studied subgroups of Artin groups. Each of them is itself isomorphic to a smaller Artin group (by Theorem 2.4.4), and they are thought to have a very nice combinatorial behaviour. Many questions can be asked about parabolic subgroups. Can they be defined purely algebraically? Are they conjugacy stable? Are they root stable? A powerful way to obtain many results concerning parabolic subgroups of Artin groups is first to study their intersecting properties. This leads to the following conjecture:

Conjecture 2.4.13. The set of parabolic subgroups of any Artin group is closed under (arbitrary) intersections.

Note that the analogue conjecture for Coxeter groups has been proved to be true in general ([84]).

The next two conjectures have a geometric flavor or non-positive curvature.

Conjecture 2.4.14. Every Artin group is CAT(0).

The CAT(0) property is not the only non-positive curvature property that Artin groups are thought to have. In fact, there has been much progress in the past few years in proving that (some) Artin groups are acylindrically hyperbolic. Irreducible spherical Artin groups have infinite centres, and thus can never be acylindrically hyperbolic. However, their central quotients $A_\Gamma \big/ Z(A_\Gamma)$ are acylindrically hyperbolic ([33]). For non-spherical Artin groups, the following has been conjectured:

Conjecture 2.4.15. Every irreducible non-spherical Artin group is acylindrically hyperbolic.

Note that acylindrically hyperbolic groups have finite centres, and that finite torsion-free subgroups are trivial. Consequently, proving that an Artin group A_Γ is acylindrical hyperbolicity and torsion-free directly proves that it has trivial centre.

We now want to compile briefly the state of the research regarding the above conjectures, priori to the work done in this thesis.

Theorem 2.4.16. *The above conjectures have been proved for the following classes of Artin groups (we only state the results that are maximal):*
Conjecture 2.4.10 (torsion-free-ness): 2-dimensional Artin groups and Artin groups of type FC ([27]).
Conjecture 2.4.11 (trivial centres): 2-dimensional Artin groups, Artin groups of type FC and Artin groups whose defining graphs are not the star of a single vertex ([42], [41], [28]).
Conjecture 2.4.12 (word and conjugacy problems): 2-dimensional Artin groups and Artin groups of type FC ([58], [22]).
Conjecture 2.4.13 (intersections of parabolic subgroups): spherical Artin groups ([23]).
Conjecture 2.4.14 (CAT(0)-ness): right-angled Artin groups, spherical Artin groups of rank 3 ([13]), some 2-dimensional Artin groups ([7], [14]), XXL Artin groups ([48]), the n-strand braid groups for $n \leqslant 6$ ([16], [49]) and 3-dimensional Artin groups of type FC ([5]).
Conjecture 2.4.15 (acylindrical hyperbolicity): Artin groups whose defining graphs

are not joins ([28]), 2-dimensional Artin groups of hyperbolic type ([75]), triangle-free Artin groups ([63]), and Euclidean Artin groups ([19]).

2.4.3 The Deligne complex

The modified Deligne complex, often simply called the Deligne complex, is a combinatorial complex associated with an Artin group A_Γ that as turned out to be extremely useful to understand the group itself. This complex is defined in terms of the combinatorics of the (standard) spherical parabolic subgroups of A_Γ. When the Artin group is 2-dimensional, its Deligne complex has dimension 2, which makes the construction of the complex slightly easier. It is this definition that we will introduce thereafter and use for the rest of the thesis. The definition in the more general case can be found in [27], or equivalently, in Remark 2.4.20 below.

Definition 2.4.17. Let A_Γ be a 2-dimensional Artin group of rank at least 3. In the barycentric subdivision Γ_{bar} of Γ, we denote by v_a the vertex corresponding to a standard generator $a \in V(\Gamma)$, and by v_{ab} the vertex corresponding to an edge of Γ connecting two standard generators a and b. Let now K_Γ be the 2-dimensional complex obtained by coning-off Γ_{bar}. We call the apex of this cone $v_\varnothing$. We define the **type** of a vertex $v \in K_\Gamma$ to be 0 if $v = v_\varnothing$, 1 if $v = v_a$ for some $a \in V(\Gamma)$, and 2 if $v = v_{ab}$ for some $a, b \in V(\Gamma)$. We endow K_Γ with the structure of a complex of groups in the following way. The local groups associated with $v_\varnothing$, v_a and v_{ab} are respectively $\{1\}$, $\langle a \rangle$ and A_{ab}. The natural inclusions of the local groups $\{1\} \subseteq \langle a \rangle \subseteq A_{ab}$ define the maps of the complex of groups. Let $\mathcal{Q}$ be the poset of the standard parabolic subgroups of A_Γ that are spherical, ordered by inclusion. One can easily see that K_Γ is a geometric realisation of $\mathcal{Q}$. Then the simple morphism is the map $\varphi : G(\mathcal{Q}) \to A_\Gamma$ that is given by the natural inclusion of the spherical standard parabolic subgroups into A_Γ. One can easily notice using Definition 2.3.1 that the fundamental group of $G(\mathcal{Q})$ is precisely A_Γ. The development of K_Γ along φ is a 2-dimensional space called the **Deligne complex** associated to A_Γ. We will denote that space by D_Γ. By Theorem 2.4.4, the fundamental group of the complex of groups K_Γ is exactly A_Γ, and hence the Deligne complex of a 2-dimensional Artin group is always connected and simply connected (see Definition 2.3.4).

We briefly name the different subcomplexes of K_Γ. An edge of K_Γ is denoted e_a if it connects $v_\varnothing$ and v_a, e_{ab} if it connects $v_\varnothing$ and v_{ab} and $e_{a,ab}$ if it connects v_a and v_{ab}. A 2-dimensional simplex of K_Γ, also called a **base triangle**, is denoted by T_{ab} if it is spanned by the vertices $v_\varnothing$, v_a and v_{ab}. Note that any translate $g \cdot T_{ab}$ will also be called a base triangle. We now recall the Moussong metric on D_Γ (see [27]). First, we define the angles of every base triangle T_{ab} by:

$$\angle_{v_{ab}}(v_\varnothing, v_a) := \frac{\pi}{2 \cdot m_{ab}}; \qquad \angle_{v_a}(v_\varnothing, v_{ab}) := \frac{\pi}{2}; \qquad \angle_\varnothing(v_a, v_{ab}) := \frac{\pi}{2} - \frac{\pi}{2 \cdot m_{ab}}.$$

Since these angles add up to π, one can choose a Euclidean triangle with the above angles. In particular, every base triangle is Euclidean. Fixing the length of every edge of the form e_s to be 1, one can recover the length of every edge in K_Γ (and thus in D_Γ) using basic trigonometry. The Moussong metric on K_Γ is obtained by gluing the Euclidean metrics coming from every base triangle T_{st} (see Definition 2.1.12). This extends to a metric on D_Γ. Note that both K_Γ and D_Γ are piecewise-Euclidean simplicial complexes (see Section 2.1.2).

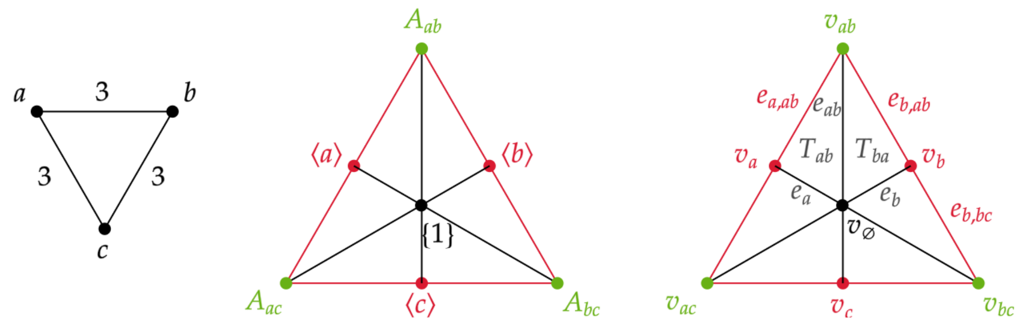


Figure 2.7: <u>On the left:</u> A graph Γ defining a 2-dimensional Artin group A_Γ. <u>In the centre:</u> K_Γ, seen as a complex of groups. <u>On the right:</u> K_Γ, seen as a 2-dimensional subcomplex of D_Γ, along with partial notations of its vertices, edges and faces. The vertices and edges have been given a colour that correspond to the type of their local group (or stabiliser): black for the trivial group, red for an infinite cyclic group, and green for a dihedral Artin group.

Following Definition 2.3.4, the Deligne complex D_Γ can also be described as the space

$$D_\Gamma = A_\Gamma \times K_\Gamma /_\sim,$$

where $(g, x) \sim (g', x') \iff x = x'$ and $g^{-1}g'$ belongs to the local group of the simplex $supp(x)$. The group A_Γ acts naturally on itself via left multiplication, and this induces an action of A_Γ on D_Γ by simplicial morphisms with strict fundamental domain K_Γ.

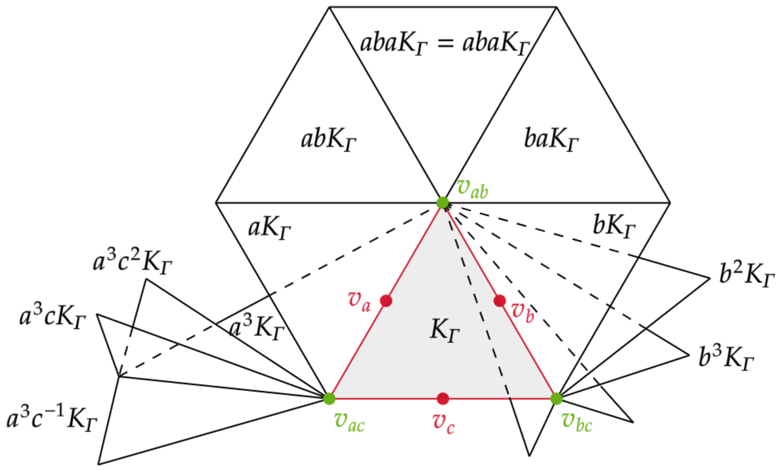


Figure 2.8: Part of the Deligne complex D_Γ associated with the graph Γ from Figure 2.7. For drawing purposes we only drew the edges that have non-trivial stabiliser.

The Deligne complex was first studied in relation with the $K(\pi, 1)$-conjecture for Artin groups, a topological conjecture that is equivalent to the contractibility of the Deligne complex:

Conjecture 2.4.18. For every Artin group A_Γ, the associated Deligne complex D_Γ is contractible.

An important consequence of this conjecture is that it implies for every Artin group A_Γ the existence of a finite-dimensional $K(A_\Gamma, 1)$-space, which also forces the group A_Γ to be torsion-free. A solution that has turned out to be quite fruitful to show Conjecture 2.4.18 for various Artin groups has been to show that their associated Deligne complexes are CAT(0) (see [27]).

We now give several useful remarks regarding the geometric structure of the Deligne complex.

Remark 2.4.19. In light of Definition 2.4.17, the barycentric subdivision Γ_{bar} of Γ can be seen as a subgraph of D_Γ: it is the boundary of the fundamental domain K_Γ. In particular, the edges and vertices of Γ_{bar} can be seen as edges and vertices of K_Γ and thus of D_Γ. They are precisely the edges and vertices whose local groups are the non-trivial spherical standard parabolic subgroups of A_Γ.

Remark 2.4.20. Another pheraps more combinatorial way to look at the Deligne complex is the following. Let A_Γ be any Artin group. Then the Deligne complex D_Γ is the simplicial complex defined as follows:

- The vertex set of D_Γ is the poset of left-cosets of the standard parabolic subgroups of A_Γ that are spherical.
- There is a $(n-1)$-simplex between vertices of D_Γ corresponding to the left-cosets $g_1 A_{\Gamma_1}, \ldots, g_n A_{\Gamma_n}$ whenever there is a sequence of inclusions $g_n A_{\Gamma_n} \subset \cdots \subset g_1 A_{\Gamma_1}$.

Remark 2.4.21. A natural question to ask is what do the links of vertices of D_Γ look like? In light of [17, Construction II.12.24], the link $Lk_{D_\Gamma}(v)$ around a vertex $v \in K_\Gamma$ only depends on the development of the local groups around v. More specifically, the link $Lk_{D_\Gamma}(v)$ is isomorphic to the development $D(Lk_{K_\Gamma}(v), (\psi_v)_{e_\bullet})$ of the link $Lk_{K_\Gamma}(v)$ along the natural inclusion maps $(\psi_v)_{e_\bullet} : G_e \hookrightarrow G_v$, where e is an edge from v to $Lk_{D_\Gamma}(v)$ and $e_\bullet := e \cap Lk_{K_\Gamma}(v)$. In particular, we can give a more precise geometric description of the links of vertices in D_Γ:

- <u>Type 0:</u> $Lk_{D_\Gamma}(v_\varnothing)$ is the development of $Lk_{K_\Gamma}(v_\varnothing)$ over the trivial maps $(\psi_{v_\varnothing})_{e_a} : \{1\} \hookrightarrow \{1\}$ and $(\psi_{v_\varnothing})_{e_{ab}} : \{1\} \hookrightarrow \{1\}$. Notice that $Lk_{K_\Gamma}(v)$ is the graph Γ_{bar} from Remark 2.4.19, and hence $Lk_{D_\Gamma}(v_\varnothing)$ is just the barycentric subdivision of Γ. By construction, the lengths of edges in $Lk_{D_\Gamma}(v_\varnothing)$ are given by

$$\ell(e_{a,ab}) = \angle_{v_\varnothing}(v_a, v_{ab}) = \frac{\pi}{2} - \frac{\pi}{2 \cdot m_{ab}}.$$

- <u>Type 1:</u> $Lk_{D_\Gamma}(v_a)$ is the development of $Lk_{K_\Gamma}(v_a)$ over the maps $(\psi_{v_a})_{e_a} : \{1\} \hookrightarrow \langle a \rangle$ and $(\psi_{v_a})_{e_{a,ab}} : \langle a \rangle \hookrightarrow \langle a \rangle$. It is not hard to see that $Lk_{K_\Gamma}(v_a)$ is just a n_a-pod centered at $v_\varnothing$, where $n_a := |\{b \in V(\Gamma) \backslash \{a\} \mid m_{ab} < \infty\}|$. In particular, $Lk_{D_\Gamma}(v_a)$ is the quotient $Lk_{K_\Gamma}(v_a) \times \langle a \rangle /_\sim$, where $(x, a^n) \sim (y, a^m)$ if and only if either $x = y = v_{ab}$ for some $b \in V(\Gamma)\backslash\{a\}$ with $m_{ab} < \infty$ or $x = y$ and $n = m$. Notice that by construction, every edge e_{ab} has length $\angle_{v_a}(v_\varnothing, v_{ab}) = \pi/2$ in $Lk_{D_\Gamma}(v_a)$.
- <u>Type 2:</u> $Lk_{D_\Gamma}(v_{ab})$ is the development of $Lk_{K_\Gamma}(v_{ab})$ over the three maps $(\psi_{v_{ab}})_{e_{ab}} : \{1\} \hookrightarrow A_{ab}$, $(\psi_{v_{ab}})_{e_{a,ab}} : \langle a \rangle \hookrightarrow A_{ab}$ and $(\psi_{v_{ab}})_{e_{b,ab}} : \langle b \rangle \hookrightarrow A_{ab}$. The link $Lk_{K_\Gamma}(v_{ab})$ is simply a tree T_0 that consists of the two edges e_a and e_b. Consider the Bass-Serre tree T over $T_0 = e_a \cup e_b$ with its associated local groups and maps. In other words, T is the barycentric subdivision of the Bass-Serre tree associated to the splitting $\langle a \rangle * \langle b \rangle$. Then the development of T_0 over the previously described

maps is just the quotient of T by $\langle\langle \underbrace{aba\cdots}_{m_{ab}} = \underbrace{bab\cdots}_{m_{ab}} \rangle\rangle$, because the previous maps inject into

$$A_{ab} \cong F_{ab} \Big/ \langle\langle \underbrace{aba\cdots}_{m_{ab}} = \underbrace{bab\cdots}_{m_{ab}} \rangle\rangle.$$

Notice by construction that the lengths of edges in $Lk_{D_\Gamma}(v_{ab})$ are given by

$$\forall s \in \{a, b\},\ \ell(e_s) = \angle_{v_{ab}}(v_\varnothing, v_s) = \frac{\pi}{2 \cdot m_{ab}}.$$

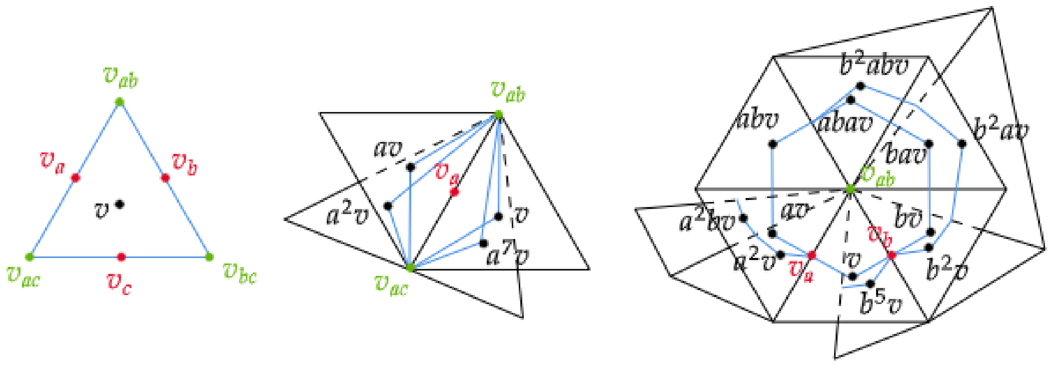


Figure 2.9: Part of the links of the vertices of type 0, 1 and 2 respectively, from left to right. The links are drawn in blue. For drawing purposes, we wrote v instead of $v_\varnothing$.

Using the description of links seen in Remark 2.4.21, Charney and Davis proved the following:

Theorem 2.4.22. ***[27, Proposition 4.4.5]*** *Let A_Γ be a 2-dimensional Artin group of rank at least 3. Then its Deligne complex D_Γ is CAT(0).*

In particular, Conjecture 2.4.18 has been solved by Charney and Davis for all 2-dimensional and FC-type Artin groups ([27]). It has also recently been solved for all affine Artin groups ([81]).

Chapter 3

Acylindrical hyperbolicity

This chapter corresponds to the publication [93]. We thank the anonymous referee for proposing a strategy for improving and making optimal Proposition 3.5.

The goal is this chapter is to study the acylindrical hyperbolicity of 2-dimensional Artin groups. By construction, every reducible Artin group A_Γ decomposes as a direct product of infinite groups and hence can never be acylindrically hyperbolic, by Theorem 2.2.12.(2). Restricting to irreducible Artin groups, it is known that the ones that the ones that are spherical have an infinite cyclic centre, and hence cannot be acylindrically hyperbolic either, by Theorem 2.2.12.(1). However, their central quotients are acylindrically hyperbolic ([33]). It is thus enough to study the 2-dimensional Artin groups that are irreducible and non-spherical. These groups are all conjectured to be acylindrically hyperbolic (see Conjecture 2.4.15).

Recall that within the world of 2-dimensional Artin groups, being non-spherical is equivalent to having rank at least 3. In this chapter, we prove the following result:

Theorem 3.1. *Every irreducible* 2*-dimensional Artin group of rank at least* 3 *is acylindrically hyperbolic.*

For instance, it was not known whether the rather simple following 2-dimensional Artin group was acylindrically hyperbolic:

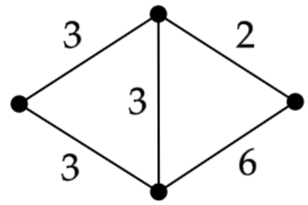


Figure 3.1: Example of an Artin group for which acylindrical hyperbolicity was not previously known.

Note that 2-dimensional Artin groups are torsion-free (see Theorem 2.4.16). In particular, if a 2-dimensional Artin group A_Γ decomposes as a direct product of two non-trivial factors $A_\Gamma = H_1 \times H_2$ then the two factors must be infinite. By Theorem 2.2.12.(2), this imply that A_Γ cannot be acylindrically hyperbolic. Therefore, an immediate corollary of the previous theorem is that decomposability as a direct product can actually be "read" from the graph Γ:

Corollary 3.2. *A 2-dimensional Artin group A_Γ can be decomposed as a non-trivial direct product if and only if it is irreducible (equivalently, Γ is a 2-join).*

Recall that acylindrically hyperbolic groups have finite centres ([80]). Thus it follows from Theorem 3.1 that irreducible 2-dimensional Artin groups of rank at least 3 have finite centres. Since they have no torsion, their centres are actually trivial. This also holds if A_Γ is reducible, as 2-dimensional reducible Artin groups are direct products of free groups, hence have trivial centres. If A_Γ is irreducible and has rank 2 then it is a dihedral Artin group with coefficient at least 3, and $A_\Gamma/Z(A_\Gamma)$ is virtually a free group ([14],[30]), hence acylindrically hyperbolic. Putting together everything that we just discussed, we are able to give a new proof of Conjecture 2.4.11:

Corollary 3.3. *Artin groups of dimension 2 and rank at least 3 have trivial centre. Moreover, all irreducible Artin groups A_Γ of dimension 2 have acylindrically hyperbolic central quotient $A_\Gamma/Z(A_\Gamma)$.*

In general proving acylindrical hyperbolicity can be quite hard. The criterion developed by Bestvina, Bromberg and Fujiwara (see Theorem 2.2.15) allows to construct from a (non-necessarily acylindrical) action on a (non-necessarily hyperbolic) space an acylindrical action on a hyperbolic space. It hence allows to prove that the group under study is either acylindrically hyperbolic or virtually cyclic. This criterion is however not very well-suited for group actions on spaces that are not locally compact, were checking that an element is WPD can be rather tough.

In this chapter, we focus on the action of Artin groups of dimension 2 on their Deligne complexes. Unfortunately, the Deligne complex is not locally compact, which make the use of the aforementioned criterion harder. To bypass that problem we will use a criterion of Martin ([67]) that uses a variant of the WPD condition, generalising to higher dimension a result of [72] for groups acting on trees. We recall this criterion thereafter, in a slightly more specific form:

Theorem 3.4. ***[67, Theorem B]*** *Let X be a CAT(0) simplicial complex, together with an action by simplicial isomorphisms of a group G. Assume that there exists a vertex v of X with stabiliser G_v such that:*
(1) The orbits of G_v on the link $Lk_X(v)$ are unbounded, for the associated angular metric.
(2) G_v is weakly malnormal in G, i.e. there is some $g \in G$ such that $G_v \cap gG_vg^{-1}$ is finite.
Then G is either virtually cyclic or acylindrically hyperbolic.

The proof of Theorem 3.1 has two major steps. First, we show that if A_Γ is not right-angled then there exists a vertex v in the Deligne complex D_Γ associated to the Artin group A_Γ that satisfies Theorem 3.4.(1). Then, we show geometrically that the stabiliser of this vertex is weakly malnormal in A_Γ, satisfying Theorem 3.4.(2). The result then follows from Theorem 3.4

This chapter is organised as follows. In Section 3.1, we study the link of vertices in the Deligne complex and prove the following result:

Proposition 3.5. *Let A_Γ be a 2-dimensional Artin group of rank at least 3 with Deligne complex D_Γ. Suppose that there exists a type 2 vertex $v_{ab} \in D_\Gamma$ whose stabiliser A_{ab} has coefficient $3 \leqslant m_{ab} < \infty$. Then:*
(1) The orbits of A_{ab} on $Lk_{D_\Gamma}(v_{ab})$ are unbounded.
(2) More precisely, the orbits of $\langle g \rangle$ on $Lk_{D_\Gamma}(v_{ab})$ are quasi-isometrically embedded if and only if $g \in A_{ab}$ is not trivial, nor the conjugate of a power of one of the standard generators a or b.

Note that when applied to our specific case, the first hypothesis of Theorem 3.4 is exactly the first result of Proposition 3.5, so that we a priori don't need to prove Proposition 3.5.(2). However, Proposition 3.5.(2) remains interesting on its own, as it has for instance been used by Hagen, Martin and Sisto to prove that extra-large type Artin groups are virtually hierarchically hyperbolic ([53]).

In Section 3.2, we reduce the question of asking whether a dihedral Artin subgroup A_{ab} of A_Γ is weakly malnormal to a geometric question (see Lemma 3.2.1). The existence of weakly malnormal subgroups turns out to be implied by a simple geometric condition on the geodesics in the complex. We are able to show that this condition holds for all irreducible 2-dimensional Artin groups of rank at least 3 (assuming they are not free nor right-angled, see Lemma 3.2.6). In particular, we show that the local group G_v is weakly malnormal in A_Γ, i.e. that G_v satisfies Theorem 3.4.(2). We can then use Theorem 3.4 and prove Theorem 3.1 as an immediate consequence.

3.1 Links of vertices in the Deligne complex

A precise description of the links of vertices in the Deligne complex was given in Remark 2.4.21. Although we got an idea of what these links look like, much remains to be proved, especially for links associated with type 2 vertices of the complex. The goal of this section is to get a better understanding of the links $Lk_{D_\Gamma}(v_{ab})$ of vertices of type 2 in D_Γ, and ultimately to prove Proposition 3.5. Although checking that the first condition of Theorem 3.4 is satisfied is rather easy (see Lemma 3.1.5), proving the second point of Proposition 3.5 will require a much more in-depth study.

3.1.1 Reformulating Proposition 3.3 in terms of syllabic lengths

In this section we reformulate Proposition 3.5 into a more accessible problem (see Proposition 3.1.3). We begin with the following definition, that will be useful throughout all the section:

Definition 3.1.1. Let G be a group with generating set S, and let $\varphi : F_S \twoheadrightarrow G$

be the natural surjection from the free group over S onto G (see Definition 2.1.1).

- Every word $w \in F_S$ can be written uniquely as $w = s_1^{r_1} \cdots s_n^{r_n}$, assuming $s_i \in \mathcal{A}$, $s_i \neq s_{i+1}$ and $r_i \in \mathbf{Z}\backslash\{0\}$. Then the **syllabic length** of w is $\ell_{\mathcal{S}}(w) := n$.
- For every element $g \in G$ we define the **syllabic length** of g as $\ell_{\mathcal{S}}(g) := \min\{\ell_{\mathcal{S}}(w) \mid \varphi(w) = g\}$.

Recall that in the Deligne complex D_Γ associated with an Artin group A_Γ, the stabilisers of vertices of type 2 (ex: v_{ab}) are dihedral Artin groups (ex: A_{ab}). The following lemma makes a connection between the syllabic length of elements $g \in A_{ab}$ and the distances in the link $Lk_{D_\Gamma}(v_{ab})$, according to the angular metric (see Definition 2.2.22).

Lemma 3.1.2. *Let γ be a path in $Lk_{D_\Gamma}(v_{ab})$ joining $v_\varnothing$ and $gv_\varnothing$ for some $g \in A_{ab}$, and suppose that the edges of γ are $e_a,\ a^{n_1}e_a,\ a^{n_1}e_b,\ a^{n_1}b^{n_2}e_b, \cdots,\ a^{n_1}b^{n_2}\cdots x^{n_k}e_x$, in that order, where $x \in \{a, b\}$ and $n_i \in \mathbf{Z}\backslash\{0\}$. Let now $w := a^{n_1}b^{n_2}\cdots x^{n_k}$. Then $\ell(\gamma) = \frac{\pi}{m_{ab}} \cdot \ell_{\mathcal{S}}(w)$. Furthermore, $d_{Lk_{D_\Gamma}(v_{ab})}(v_\varnothing, gv_\varnothing) = \frac{\pi}{m_{ab}} \cdot \ell_{\mathcal{S}}(g)$.*

Proof: First of all, recall that the local group at v_a is $\langle a\rangle$, and hence the set of edges of $Lk_{D_\Gamma}(v_{ab})$ meeting at v_a is $\{a^k e_a \mid k \in \mathbf{Z}\}$. This proves that e_a and $a^{n_1}e_a$ are indeed consecutive to one another, meeting at v_a. Of course, $a^{n_1}e_a$ and $a^{n_1}e_b$ are also consecutive to one another, meeting at $a^{n_1}v_\varnothing$. A similar argument shows that the edges in the statement of the lemma consecutively meet each others. As A_Γ acts by isometries on $Lk_{D_\Gamma}(v_{ab})$, it is clear that the length of every edge of γ is either $\ell(e_a)$ or $\ell(e_b)$, both of which turn out to be equal to $\frac{\pi}{2 \cdot m_{ab}}$. Because the number of edges in γ is precisely $2 \cdot \ell_{\mathcal{S}}(w)$, we get $\ell(\gamma) = \frac{\pi}{m_{ab}} \cdot \ell_{\mathcal{S}}(w)$

Notice that every path γ joining $v_\varnothing$ and $gv_\varnothing$ corresponds to a word w that satisfies $\varphi(w) = g$, where $\varphi : F_{ab} \twoheadrightarrow A_{ab}$ is the natural projection. The distance between $v_\varnothing$ and $gv_\varnothing$ is the length of the shortest of these paths, hence

$$d_{Lk_{D_\Gamma}(v_{ab})}(v_\varnothing, gv_\varnothing) = \min\{\frac{\pi}{m_{ab}} \cdot \ell_{\mathcal{S}}(w) \mid \varphi(w) = g\} = \frac{\pi}{m_{ab}} \cdot \ell_{\mathcal{S}}(g).$$

□

One important consequence of the previous lemma is that we can reformulate Proposition 3.5 in terms of syllabic lengths of elements in the local group of a vertex of type 2. We will prove Proposition 3.5 by proving the following equivalent proposition:

Proposition 3.1.3. *Let A_Γ be a 2-dimensional Artin group of rank at least 3 with Deligne complex D_Γ. Suppose that there exists a vertex $v_{ab} \in D_\Gamma$ whose stabiliser A_{ab} has coefficient $3 \leqslant m_{ab} < \infty$. Then:*
(1) The set $\{\ell_{\mathcal{S}}(g) \mid g \in A_{ab}\}$ is unbounded.
(2) More precisely, the syllabic length $\ell_{\mathcal{S}}(g^n)$ grows linearly in n if and only if $g \in A_{ab}$ is not trivial, nor the conjugate of a power of one of the standard generators a or b.

Remark 3.1.4. Recall that we say that a sequence $\{u_n\}_{n \geqslant 0}$ grows linearly in n if they are constants $B \geqslant A > 0$ and $C \geqslant 0$ such that for any $n \geqslant 0$ we have

$$An - C \leqslant u_n \leqslant Bn + C.$$

The next lemma shows that Proposition 3.1.3.(1), and thus Proposition 3.5.(1), are satisfied. This result will be very useful in the proof of Theorem 3.1. It shows that if A_Γ has a coefficient $m_{ab} \geqslant 3$, then the vertex v_{ab} satisfies the first hypothesis of Theorem 3.4.

Lemma 3.1.5. *Consider an Artin group A_{ab} with coefficient $3 \leqslant m_{ab} \leqslant \infty$. Then*

$$\{\ell_{\mathcal{S}}(g) \mid g \in A_{ab}\} \text{ is unbounded.}$$

Proof: It is known that the quotient $\overline{A}_{ab}$ of A_{ab} by its centre is virtually isomorphic to the free group F_m, for some $m \geqslant 2$ ([14],[30]). In particular, $\overline{A}_{ab}$ is acylindrically hyperbolic. Suppose now that there exists a constant $N \geqslant 0$ such that for every $g \in A_{ab}$, one has $\ell_{\mathcal{S}}(g) < N$, and assume without loss of generality that N is even. This means that $A_{ab} = \langle a \rangle \langle b \rangle \cdots \langle a \rangle \langle b \rangle$ (where the product has N terms). In particular, $\overline{A}_{ab} = \overline{\langle a \rangle}\ \overline{\langle b \rangle} \cdots \overline{\langle a \rangle}\ \overline{\langle b \rangle}$. Using Theorem 2.2.12.(3), we know that one of $\overline{\langle a \rangle}$ or $\overline{\langle b \rangle}$ must be acylindrically hyperbolic, which is impossible, as they are cyclic subgroups of $\overline{A}_{ab}$. Therefore, $\{\ell_{\mathcal{S}}(g), g \in A_{ab}\}$ is unbounded. □

Strategy: The goal of the rest of this section is to understand more those links of the form $Lk_{D_\Gamma}(v_{ab})$, i.e. the links of vertices of type 2 in D_Γ. In particular, we will be able through a more precise analysis of these links to prove Proposition 3.1.3.(2), and thus Proposition 3.5.(2).

We now set for the rest of this section A_{ab} to be a dihedral Artin group with coefficient $3 \leqslant m < \infty$. One can easily see that for any element $g \in A_{ab}$, the syllabic length $\ell_{\mathcal{S}}(g^n)$ is always bounded above by a linear function, such as $\ell(w) \cdot n$ for instance, where w is any word representing g and $\ell(\cdot)$ is the usual length function on words (see Definition 2.1.1). Therefore we will only focus on finding a linear lower bound for $\ell_{\mathcal{S}}(g^n)$.

Our approach is mostly geometric: we study the action of A_{ab} on a graph $\widehat{T}$ (see Definition 3.1.12). In particular, we show that the distance of translation induced by an element $g \in A_{ab}$ gives a lower bound on the syllabic length of g (see Lemma 3.1.13). It then follows immediately that any element $g \in A_{ab}$ that acts hyperbolically on $\widehat{T}$ is such that $\ell_{\mathcal{S}}(g^n)$ admits a linear lower bound in n, giving Proposition 3.1.3.(2) for such elements. It then feels natural to want to determine which elements act hyperbolically on $\widehat{T}$. This will be achieved in Lemma 3.1.14.

It remains to study the elements that do not act hyperbolically on $\widehat{T}$. They all act elliptically and come in two forms: the elements that are conjugate to powers of a standard generator (modulo an element of the centre), and the (non-trivial) elements which admit powers that belong to the centre of A_{ab}. When their "central part" is trivial, the elements g of the first kind are easily shown to satisfy $\ell_{\mathcal{S}}(g^n) \leqslant K_g$ for a constant K_g that does not depend on n. However, the elements of the first kind that don't have a trivial central part and the elements of the second kind have a different behaviour. As will be recalled later, the centre of A_{ab} only contains powers of the Garside element of A_{ab}, which motivates a more in-depth study of the syllabic length of such powers. The method that we use for that last point is more algebraic, and rely on a more direct study of the syllabic length of words, notably using the Garside normal form of elements. As a consequence, we will show that the remaining elliptic elements g are such that the syllabic length $\ell_{\mathcal{S}}(g^n)$ also admits a linear lower bound in n (see Lemma 3.1.18). Alltogether, this will conclude the proof of Proposition 3.1.3.

3.1.2 The action of the local group on $\widehat{T}$.

Our first goal is to define a tree T on which A_{ab} acts nicely with a trivial action of the centre $Z(A_{ab})$. This will be done throughout the next definitions and lemmas. Let us first introduce few notations and recall notions about normal forms and centres in dihedral Artin groups.

Notations: • For two words $u, u' \in F_{ab}$ representing the same element $g \in A_{ab}$, we will simply write $u \equiv u'$ instead of $\varphi(u) = \varphi(u')$. Similarly, for $g \in A_{ab}$, we will write $u \equiv g$ instead of $\varphi(u) = g$.

• We will write $(a, b; k)$ to denote the alternating sequence of the letters a and b, starting with a and of length k, and we will write Δ_a and Δ_b to describe the words $(a, b; m) \in F_{ab}$ and $(b, a; m) \in F_{ab}$ respectively. More explicitely,

$$\Delta_a := \underbrace{aba\cdots}_{m \text{ terms}} \quad \text{and} \quad \Delta_b := \underbrace{bab\cdots}_{m \text{ terms}}.$$

• For a word $u \in F_{ab}$, we denote by $\overline{u}$ the word obtained from u by replacing every a^n by b^n and every b^n by a^n. Moreover, we will denote by $\widetilde{u}$ the element

$$\widetilde{u} := \begin{cases} u & \text{if } m \text{ is even} \\ \overline{u} & \text{if } m \text{ is odd} \end{cases}$$

One can easily notice that for any word $u \in F_{ab}$, we have $\Delta^{\pm 1} \cdot u \equiv \widetilde{u} \cdot \Delta^{\pm 1}$.

Definition 3.1.6. For a dihedral Artin group A_{ab} with coefficient $3 \leqslant m_{ab} < \infty$, the **Garside element** is the element $\Delta \in A_{ab}$ defined by

$$\Delta \equiv \Delta_a \equiv \Delta_b. \quad (*)$$

A strict non-trivial subword of Δ_a or of Δ_b is called an **atom**. It is a standard result ([38], [37], [35]) that for every element $g \in A_{ab}$, there is a word $\text{Gars}(g) \in F_{ab}$ called the **Garside normal form** of g that satisfies $\text{Gars}(g) \equiv g$ and such that one can write

$$\text{Gars}(g) = u_1 \cdots u_n \cdot W,$$

where the u_i are atoms such that the last letter of each u_i matches with the first letter of u_{i+1}, and where $W \equiv \Delta^N$ for some $N \in \mathbf{Z}$ is a product of terms of the form $\Delta_a^{\pm 1}$ and $\Delta_b^{\pm 1}$. This word is not unique, however the atoms of the above decomposition are uniquely defined, and so is N.

At last, we recall that the centre $Z(A_{ab})$ of A_{ab} was described in [18], and takes the form

$$Z(A_{ab}) = \begin{cases} \langle \Delta \rangle \text{ if } m \text{ is even,} \\ \langle \Delta^2 \rangle \text{ if } m \text{ is odd.} \end{cases}$$

We now come back to constructing the desired space. The space that we first define is due to [8, Section 2.1]. One can also recover an equivalent definition by quotient of the space described in [69, Figure 6].

Definition 3.1.7. Let us consider the simplicial complex Y defined by the following (see Figure 3.2):

Vertices: The vertex set of Y is the set of cosets

$$V := A_{ab} \big/ \langle \Delta \rangle = \{g\langle \Delta \rangle \mid g \in A_{ab}\}.$$

A convenient representative for a vertex $g\langle \Delta \rangle$ is the product of the atoms of the Garside normal form of g. This representative is the unique that is in Garside normal form yet does not contain any subword of the form $\Delta_x^{\pm 1}$ for some $x \in \{a, b\}$. We will denote it $g_\bullet$. In this setup, we can see V as the set $\{g_\bullet \mid g \in A_{ab}\}$.

Simplices: For every collection $g_{1\bullet}, \cdots, g_{k\bullet}$ of vertices, the set $\{g_{1\bullet}, \cdots, g_{k\bullet}\}$ spans a k-simplex if and only if for all $i, j \in \{1, \cdots, k\}$, there is an atom x such that $g_{i\bullet} \cdot x = g_{j\bullet}$ or $g_{j\bullet} \cdot x = g_{i\bullet}$. Note that because atoms are subwords of Δ_a or Δ_b, every k-simplex is contained in a maximal m-simplex, where m is the coefficient of A_{ab} (see Figure 3.2).

The group A_{ab} acts naturally on V: if $h \in A_{ab}$ and $g\langle \Delta \rangle \in V$, then $h \cdot g\langle \Delta \rangle := hg\langle \Delta \rangle$. This action extends to a simplicial and cocompact action of A_{ab} on Y, that is transitive on the vertices (see [8]). Note that $Z(A_{ab})$ acts trivially on V, and thus on Y.

Definition 3.1.8. We define a new graph T by the following. The set of vertices of T is the union of two sets: the set V of vertices of Y, and the set V' of maximal simplices of Y (i.e. the m-simplices). Then, we put an edge between a vertex $g_\bullet \in V$ and a vertex $\{g_{1\bullet}, \cdots, g_{m\bullet}\} \in V'$ if and only if $g_\bullet \in \{g_{1\bullet}, \cdots, g_{m\bullet}\}$). Note that T can naturally be seen as a subspace of Y (see Figure 3.2).

Lemma 3.1.9. *The graph T is a tree, and the stabiliser of any edge $e \subseteq T$ is precisely $Z(A_{ab})$.*

Proof: The atoms in the Garside normal form of any element are unique, and this gives Y a structure of tree of m-simplices (see Figure 3.2), where the Y-distance between any vertex $g_\bullet$ and $1_\bullet$ is precisely the number of atoms in $g_\bullet$ (note that the T-distance is twice that amount). The reason m-simplices appear is because once given a non-trivial vertex $g_\bullet$, there are $(m-1)$ different ways one can add an atom on the right side of $g_\bullet$ (assuming this atom starts with a letter that differs from the last letter of $g_\bullet$. In particular, Y retracts on a tree described in Figure 3.2, and that tree is precisely T.

Let now e be any edge of T. Because the action is transitive on the vertex set V, we may as well assume that e contains $1_\bullet$. The other vertex of e corresponds to one of the two simplices $S_a := \{1_\bullet, a_\bullet, \cdots, (a, b; m-1)_\bullet\}$ or $S_b := \{1_\bullet, b_\bullet, \cdots, (b, a; m-1)_\bullet\}$. Let now $g \in A_{ab}$ and suppose that $g \cdot e = e$. Then in particular g fixes $1_\bullet$, so we have $g \cdot \langle\Delta\rangle = \langle\Delta\rangle$, and thus $g \in \langle\Delta\rangle$. If m is even, we are done. If m is odd, it is enough to show that Δ does not fix e, which is clear because it sends S_a onto S_b and vice versa. □

Remark 3.1.10. The valence of a vertex v of T is easy to determine. If $v \in V$, then v belongs to exactly two m-simplices of Y, so the valence of v in T is 2. If $v \in V'$, then v corresponds to a m-simplex of Y, hence is connected to exactly m vertices of Y, and its valence in T is m.

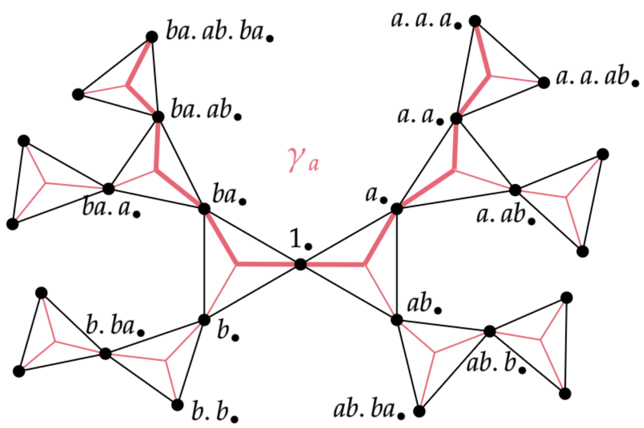


Figure 3.2: Let $m := 3$. <u>In black:</u> Part of the simplicial complex Y with its set of vertices V. <u>In pink:</u> Part of the tree T, that is a deformation retract of Y. The axis $\gamma_a \subseteq T$ is drawn with the thicker line.

Lemma 3.1.11. *The elements of A_{ab} acting elliptically on T are precisely the elements $g \in A_{ab}$ for which there exists an $N \neq 0$ such that $g^N \in Z(A_{ab})$. All the other elements act hyperbolically.*

Proof: Suppose first that $g^N \in Z(A_{ab})$. Then g^N acts trivially on T, so g has finite orbits. In particular these orbits are bounded, so g acts elliptically.

Suppose now that g acts elliptically. Then g fixes a vertex v of T. By Remark 3.1.10, v has at most m neighbours, so $g^{m!}$ fixes the neighbourhood of v. In particular, $g^{m!}$ fixes a vertex $h\langle\Delta\rangle$ of Y (either v, or a vertex in its neighbourhood). The equation $g^{m!} \cdot h\langle\Delta\rangle = h\langle\Delta\rangle$ gives $g^{m!} = h\Delta^K h^{-1}$ for some $K \in \mathbf{Z}$. We obtain $g^{2m!} = h\Delta^{2K}h^{-1} = \Delta^{2K} \in Z(A_{ab})$.

All other elements act hyperbolically because we have a simplicial isometric action on a tree. $\square$

Recall that we are interested in studying the syllabic length of elements of A_{ab} relatively to the standard generators a and b, and in reducing the problem of syllabic lengths to a problem of distances in our space. Unfortunately, one can travel an arbitrary large distance in T using a single syllable, because the generators a and b act hyperbolically on T (see Figure 3.2). To deal with that problem, we decide to cone-off such axes, and their translates:

Definition 3.1.12. Let $s \in \{a, b\}$. We denote by γ_s the axis of s in T, i.e. the bi-infinite geodesic line going through all the vertices of the form $s^k\langle\Delta\rangle$ for all $k \in \mathbf{Z}$. Let us now define a graph $\widehat{T}$ as the cone-off of the tree T along the family of axes $h \cdot \gamma_s$, for all $h \in A_{ab}$ and $s \in \{a, b\}$. More precisely:

- Start with T, and add a new vertex $v_{h,s}$ for every axis of the form $h \cdot \gamma_s$, for all $h \in A_{ab}$ and $s \in \{a, b\}$. We only add one vertex if two axes define the same line, even if they go in opposite directions.
- Connect every vertex $v_{h,s}$ to every vertex of the corresponding axis $h \cdot \gamma_s$.

The following lemma justify the study of the cone-off $\widehat{T}$, as it gives a lower bound on the syllabic length of an element of A_{ab} in terms of distances in $\widehat{T}$.

Lemma 3.1.13. *Let $g \in A_{ab}$. Then*

$$d_{\widehat{T}}(1_\bullet, g \cdot 1_\bullet) \leqslant 2 \cdot \ell_{\mathcal{S}}(g).$$

Proof: The argument is similar to that of Lemma 3.1.2. Let $k := \ell_{\mathcal{S}}(g)$. By the triangle inequality, it is enough to prove that there is a sequence of vertices

$$v_0 := 1_\bullet,\ v_1,\ \cdots,\ v_{k-1},\ v_k := g \cdot 1_\bullet$$

such that $d_{\widehat{T}}(v_i, v_{i+1}) \leqslant 2$. To do so, let $a^{n_1} b^{n_2} \cdots x^{n_k}$ be a word representing g, where $x \in \{a, b\}$ and $n_i \in \mathbf{Z} \backslash \{0\}$, and let

$$g_i := a^{n_1} b^{n_2} \cdots y_i^{n_i},$$

where $y_i \in \{a, b\}$ is the appropriate letter. Consider now the vertices v_i defined by $v_i := g_i \cdot 1_\bullet$, so that $v_0 = 1_\bullet$ and $v_k = g \cdot 1_\bullet$. Because A_{ab} acts on $\widehat{T}$ by isometries, we have for any $0 \leqslant i < k$

$$d_{\widehat{T}}(v_i, v_{i+1}) = d_{\widehat{T}}(g_i \cdot 1_\bullet, g_{i+1} \cdot 1_\bullet) = d_{\widehat{T}}(1_\bullet, y_{i+1}^{n+1} \cdot 1_\bullet).$$

Note that y_{i+1}^{n+1} is just a power of a standard generator $y_{i+1} \in \{a, b\}$, which means $1_\bullet$ and $y_{i+1}^{n+1} \cdot 1_\bullet$ both belong to the axis $\gamma_{y_{i+1}}$. By definition of $\widehat{T}$, such vertices lie within distance 2 of each others. It follows that $d_{\widehat{T}}(v_i, v_{i+1}) \leqslant 2$. □

As explained in the strategy of this section, the previous lemma immediately gives Proposition 3.1.3.(2) for elements of A_{ab} acting hyperbolically on $\widehat{T}$. The goal of the next lemma is to classify these elements:

Lemma 3.1.14. *The elements of A_{ab} acting elliptically on $\widehat{T}$ are precisely the elements $g \in A_{ab}$ satisfy one of the following:*
(1) $g = h \cdot s^N \cdot h^{-1} \cdot W$ for some $N \in \mathbf{Z}$, $s \in \{a, b\}$ and $W \in Z(A_{ab})$;
(2) There exists an $N \neq 0$ such that $g^N \in Z(A_{ab})$.
All the other elements act hyperbolically.

Proof: Let $g \in A_{ab}$. If g satisfies (2), then it already acts elliptically on T by Lemma 3.1.11, so it acts elliptically on $\widehat{T}$ too. If g satisfies (1), it is not hard to see that g fixes the vertex $h \cdot \gamma_s$, hence acts elliptically on $\widehat{T}$. We now suppose that g does not satisfy any of these properties. We already know by Lemma 3.1.11 that g acts hyperbolically on T, with an axis that we call γ_g. We begin by stating the following "small cancellation" claim, which gives the desired result for g. Then, we proceed on proving that g actually satisfies the hypotheses of the claim.

<u>Claim:</u> Suppose that there exists a $K > 0$ such that for every $h \in A_{ab}$ and every $s \in \{a, b\}$, the subtree γ defined by $\gamma := \gamma_g \cap h \cdot \gamma_s$ has diameter at most K. Then g acts hyperbolically on $\widehat{T}$.

Proof of the Claim: Since g acts hyperbolically on T, it is enough to show that there is a constant $C > 0$ such that for all vertices $x, y \in \gamma_g$, we have

$$C \cdot d_T(x, y) \leqslant d_{\widehat{T}}(x, y). \quad (*)$$

Let $\gamma^T_{x,y}$ be the (unique) geodesic connecting x and y in T, and let M be the minimal number of axes of the form $h \cdot \gamma_s$ required to cover all edges of $\gamma^T_{x,y}$ completely. Let also $D := d_{\widehat{T}}(x, y)$. Since every edge has length 1, this means we can reach x from y by using D edges $e_1, \cdots, e_D$ of $\widehat{T}$. Let $x_0, \cdots, x_D$ be the vertices these edges go through (in that order), and let $x_{r_0}, \cdots, x_{r_{D'}}$ be the subset of the above vertices corresponding to those belonging to T (with $r_0 < \cdots < r_{D'}$). Then the vertices x_{r_i} and $x_{r_{i+1}}$ always belong to a common axis. Indeed, if $r_{i+1} - r_i = 1$, then the two vertices are the two endpoints of a common edge of T. On the other hand, if $r_{i+1} - r_i \geqslant 2$, then there is at least one vertex $x_j \in \widehat{T} \backslash T$ that lies between x_{r_i} and $x_{r_{i+1}}$. By definition of $\widehat{T}$, the neighbours of x_j both lie on a common axis. In other words, we must have $r_{i+1} - r_i = 2$, and x_{r_i} and $x_{r_{i+1}}$ belong to a common axis. Let now $\gamma \subseteq T$ be the path obtained by connecting the vertices of the form x_{r_i} through the corresponding axes. Then γ is a subtree of T containing x and y. It is convex, hence must contain the geodesic $\gamma^T_{x,y}$. This means we found a way to cover γ, and thus $\gamma^T_{x,y}$, with $D' \leqslant D$ axes. By definition of M and D, we obtain

$$d_{\widehat{T}}(x, y) \geqslant M. \quad (**)$$

By hypothesis, there is no axis of the form $h \cdot \gamma_s$ that covers a subgraph of $\gamma^T_{x,y}$ of diameter more than K. In particular then, one must use at least $d_T(x, y)/K$ such axes in order to cover $\gamma^T_{x,y}$ completely. This means $M \geqslant d_T(x, y)/K$. We conclude using $(**)$ that $d_{\widehat{T}}(x, y) \geqslant d_T(x, y)/K$, satisying $(*)$. This finishes the proof of the claim.

We now check that the hypothesis of the claim is satisfied. Suppose that no such constant K exists. Then there is an axis $h \cdot \gamma_s$ such that the subtree $\gamma := \gamma_g \cap h \cdot \gamma_s$ has diameter at least $2 \cdot ||g|| + 1$, where $||g||$ is the translation length of g when acting on T. Since γ has diameter at least $2 \cdot ||g|| + 1$, there is an edge $e \subseteq \gamma$ whose distance in T to any of the two enpoints of γ is at least $||g||$. Note that e is a segment of γ_g, so $g \cdot e$ belongs to γ_g as well. By definition, the distance in T between e and $g \cdot e$ is at most $||g||$, which means that $g \cdot e$ belongs to γ as well. In particular, $g \cdot e$ belongs to $h \cdot \gamma_s$. Note that the action of g respects the bipartite structure of T, and thus its translation length $||g||$ is an even number. Note on the other hand that the translation length of $h \cdot s \cdot h^{-1}$ when acting on T is exactly 2 (because its translation length when acting on Y is 1). Since $g \cdot e$ belongs to $h \cdot \gamma_s$, this means there is some constant M such that $g \cdot e$ coincides with $h \cdot s^M \cdot h^{-1} \cdot e$ (actually, $M = \pm ||g||/2$). We get the equation

$$g \cdot e = h \cdot s^M \cdot h^{-1} \cdot e.$$

In particular, the element $g^{-1} \cdot h \cdot s^M \cdot h^{-1}$ stabilises e, hence must belong to $Z(A_{ab})$ by Lemma 3.1.9. We obtain $g = h \cdot s^M \cdot h^{-1} \cdot W$ for some $W \in Z(A_{ab})$. This is absurd by hypothesis. □

3.1.3 The syllabic length of powers of the Garside element

We are now interested in the study of the elements of A_{ab} that act elliptically on $\widehat{T}$, which have been described in Lemma 3.1.14. Our goal will be to give a linear lower bound on the syllabic length of powers of the Garside element (see Lemma 3.1.18). The method is more algebraic, and we decide to briefly recall how one can obtain the Garside normal form of an element $g \in A_{ab}$ (see [71, Section 4] for a similar description).

Algorithm 3.1.15. *Let $g \in A_{ab}$, and let $u \in F_{ab}$ be any word satisfying $u \equiv g$. Then one can obtain* Gars(g) *from u in two steps:*

<u>*Step 1:*</u> *If there is no occurence of a subword of the form $\Delta_x^{\pm 1}$ in u, for some $x \in \{a, b\}$, or if all such occurences appear consecutively on the right-most part of u, go to Step 2. Otherwise, consider the left-most occurence of a $\Delta_x^{\pm 1}$ subword in*

u, and write

$$u = v_1 \cdot \Delta_x^{\pm 1} \cdot v_2,$$

for the appropriate subwords $v_1, v_2 \in F_{ab}$*. Let*

$$u' := v_1 \cdot \widetilde{v_2} \cdot \Delta_x^{\pm 1},$$

and note that $u' \equiv u \equiv g$*. Then replace* u *with* u'*, and proceed through Step 1 again.*

Step 2: At this point, we have a word u *that doesn't contain any subword of the form* $\Delta_x^{\pm 1}$*, except potentially on its right-most part. This means* u *takes the form*

$$u = u_1 \cdots u_n \cdot W,$$

where each u_i *is an atom or the inverse of atom, and* W *is a product of terms of the form* $\Delta_a^{\pm 1}$ *and* $\Delta_b^{\pm 1}$*. Moreover, for every* $1 \leqslant i \leqslant n-1$*, the last letter of* u_i *and the first letter of* u_{i+1} *either have opposite sign, or agree. If there is no negative letter (i.e.* a^{-1} *or* b^{-1}*) in* $u_1 \cdots u_n$*, terminate the algorithm. Otherwise, the word* $u_1 \cdots u_n$ *contains at least one subword that is the inverse of an atom. Locate the left-most subword* u_i *of this form. Without loss of generality,* $u_i = (a^{-1}, b^{-1}; k)$ *for some* $1 \leqslant k < m$ *(if* u_i *starts with* b^{-1} *instead, proceed symmetrically). Write*

$$u = u_1 \cdots u_{i-1} \cdot (a^{-1}, b^{-1}; k) \cdot u_{i+1} \cdots u_n \cdot W,$$

and let

$$u' := u_1 \cdots u_{i-1} \cdot (b, a; m-k) \cdot \widetilde{u_{i+1}} \cdots \widetilde{u_n} \cdot \widetilde{W} \cdot \Delta_x^{-1}$$

for some $x \in \{a, b\}$*. One can check that* $u' \equiv u \equiv g$*. Replace* u *by* u'*, and proceed through Step 2 again.*

Example 3.1.16. Let $m := 3$, and let $u := aba^2b^{-1}a^{-1}baba^2b^4ab$. We denote by u_i the word obtained after the i-th Step of Algorithm 3.1.15. Then:

$$u_1 = ba^{-1}b^{-1}a^2b^3\Delta_a\Delta_b\Delta_a$$
$$u_2 = b^4a^3\Delta_b\Delta_a$$

If we decompose the resulting word according to Definition 3.1.6, we obtain

$$\mathrm{Gars}(u) = b \cdot b \cdot b \cdot ba \cdot a \cdot a \cdot \Delta_b \Delta_a = b^4 a^3 bababa.$$

Lemma 3.1.17. *Let $u \in F_{ab}$ and suppose that $u \equiv \Delta^n$ for some $n \neq 0$. Then u contains a subword of the form $\Delta_x^{\pm 1}$ for some $x \in \{a, b\}$.*

Proof: The proof uses the strategy of Algorithm 3.1.15. Suppose that u does not contain any subword of the form $\Delta_x^{\pm 1}$ with $x \in \{a, b\}$. By definition, when giving u as an input, the first step of Algorithm 3.1.15 is trivial. Starting with the second step of the algorithm, this means we can decompose u in a product of atoms, inverses of atoms, and a power of the Garside element (see Algorithm 3.1.15):

$$u = u_1 \cdots u_k \cdot W$$

When applying the second step of the algorithm until the algorithm terminates, every atom u_i yields an atom u_i' that is either u_i or $\widetilde{u_i}$, and every inverse of an atom u_i yields an atom u_i' that is either u_i^* or $\widetilde{u_i}^*$, where u_i^* is the unique atom such that $u_i = u_i^* \cdot \Delta_x^{-1}$ for some $x \in \{a, b\}$. Note that for every $1 \leqslant i \leqslant k$, u_i is trivial if and only if u_i' is trivial. We obtain the Garside normal form of Δ^n:

$$\mathrm{Gars}(\Delta^n) = u_1' \cdots u_k' \cdot W',$$

for an appropriate W'. Recall that one can find trivial Garside normal forms for Δ^n, such as $\mathrm{Gars}(\Delta^n) = \Delta_x^n$ for $x \in \{a, b\}$. By unicity of the atoms in the decomposition of $\mathrm{Gars}(\Delta^n)$, we obtain that all the u_i' are trivial, and thus so are the u_i. In particular, $u = 1$, which is absurd. □

Lemma 3.1.18. *For any $n \in \mathbf{Z}$, $\ell_{\mathcal{S}}(\Delta^n) \geqslant (m-2) \cdot |n|$.*

Proof: This is clear if $n = 0$. Since $\ell_{\mathcal{S}}(\Delta^n) = \ell_{\mathcal{S}}(\Delta^{-n})$, it is enough to prove that the result holds for $n > 0$. Let $u \in F_{ab}$ be any word representing Δ^n. It is enough to show that

$$\ell_{\mathcal{S}}(u) \geqslant (m-2) \cdot n.$$

We now consider the string of words $u_0, u_1, u_2, \cdots u_\lambda \in F_{ab}$ defined by induction as follows. We first set $u_0 := u$. By Lemma 3.1.17, u_0 contains a subword of the form $\Delta_{x_1}^{\pm 1}$ for some $x_1 \in \{a, b\}$, so we can decompose u_0 as

$$u_0 = u_{0,1} \cdot \Delta_{x_1}^{\pm 1} \cdot u_{0,2},$$

for the appropriate words $u_{0,1}, u_{0,2} \in F_{ab}$. We then set

$$u_1 := u_{0,1} \cdot \widetilde{u_{0,2}}.$$

Note that $u_1 \cdot \Delta_{x_1}^{\pm 1} \equiv u_0$. If u_1 is trivial, set $\lambda = 1$ and stop here. Otherwise, $u_1 \equiv \Delta^{n \pm 1}$ with $n \pm 1 \neq 0$, so we can apply Lemma 3.1.17 again and follow the same construction as above and obtain a word u_2 satisfying $u_2 \cdot \Delta_{x_2}^{\pm 1} = u_1$ for some $x_2 \in \{a, b\}$. As long as $u_i \not\equiv 1$, we continue to construct words u_{i+1} in the fashion described above. The words obtained satisfy $u_{i+1} \cdot \Delta_{x_{i+1}}^{\pm 1} = u_i$ for some $x_{i+1} \in \{a, b\}$. Note that

$$u_i = \underbrace{\overbrace{u_{i,1}}^{k_1 \text{ syl.}} \cdot \overbrace{\Delta_{x_i}^{\pm 1}}^{m \text{ syl.}} \cdot \overbrace{u_{i,2}}^{k_2 \text{ syl.}}}_{\geqslant k_1 + k_2 + m - 2 \text{ syl.}}, \quad \text{and} \quad u_{i+1} = \underbrace{u_{i,1} \cdot \widetilde{u_{i,2}}}_{\leqslant k_1 + k_2 \text{ syl.}},$$

so eventually

$$\ell_{\mathcal{S}}(u_i) \geqslant \ell_{\mathcal{S}}(u_{i+1}) + (m - 2).$$

This means each word u_{i+1} is syllabically shorter than u_i by at least $(m - 2)$ syllables. In particular, this process has to stop after a finite number λ of steps. The final word, u_λ, satisfies

$$u_\lambda \cdot \prod_{i=1}^{\lambda} \Delta_{x_i}^{\pm 1} \equiv u \equiv \Delta^n.$$

In particular then, u_λ represents a power of Δ, but does not contain any subword of the form $\Delta_x^{\pm 1}$ for some $x \in \{a, b\}$. By Lemma 3.1.17, this means u_λ is the trivial word. We obtain

$$\prod_{i=1}^{\lambda} \Delta_{x_i}^{\pm 1} \equiv \Delta^n \Longrightarrow \lambda \geqslant n.$$

Trying to sum up the previous arguments, we have:

(1) For $0 \leqslant i \leqslant \lambda - 1$, each u_{i+1} is syllabically shorter than u_i by at least $(m - 2)$

syllables. In particular, u_λ is syllabically shorter than u by at least $\lambda(m-2)$ syllables.

(2) u_λ is trivial.

(3) $\lambda \geqslant n$.

Altogether, this gives a bound on the syllabic length of u:

$$\ell_{\mathcal{S}}(u) \overset{(1)}{\geqslant} \ell_{\mathcal{S}}(u_\lambda) + \lambda(m-2) \overset{(2)}{=} \lambda(m-2) \overset{(3)}{\geqslant} (m-2) \cdot n.$$

□

We are now able to prove the main Propositions:

Proof of Proposition 3.5.(2) and Proposition 3.1.3.(2): We first recall that the two statements are equivalent, thanks to Lemma 3.1.2. Therefore we will only care on proving Proposition 3.1.3.(2). We divide the proof in four different cases. In all cases except the first one, we will give a linear lower bound of $\ell_{\mathcal{S}}(g^n)$ in terms of n. In all that follows, h is an element of A_{ab}, $s \in \{a, b\}$ is a standard generator, and W is an element of the centre $Z(A_{ab})$.

Case 1: $g = h \cdot s^k \cdot h^{-1}$. Let $M := \ell_{\mathcal{S}}(h) = \ell_{\mathcal{S}}(h^{-1})$. Then for any $n \in \mathbf{Z}$, we have

$$\ell_{\mathcal{S}}(g^n) = \ell_{\mathcal{S}}(h \cdot s^{kn} \cdot h^{-1}) \leqslant \ell_{\mathcal{S}}(h) + \ell_{\mathcal{S}}(s^{kn}) + \ell_{\mathcal{S}}(h^{-1}) = M + 1 + M = 2M + 1.$$

Case 2: $g = h \cdot s^k \cdot h^{-1} \cdot W$ with $W \neq 1$. Then there is a $q \neq 0$ such that $g = h \cdot s^k \cdot h^{-1} \cdot \Delta^q$. Let $g_0 := h \cdot s^k \cdot h^{-1}$, then $g^n = (g_0 \cdot \Delta^q)^n = g_0^n \cdot \Delta^{qn}$. On one hand we know by Case 1 that $\ell_{\mathcal{S}}(g_0^n)$ is uniformly bounded for all $n \geqslant 0$. On the other hand, $\ell_{\mathcal{S}}(\Delta^{qn})$ grows linearly in n, by Lemma 3.1.18. Putting these two facts together shows that $\ell_{\mathcal{S}}(g^n)$ grows linearly as well.

Case 3: $\exists N \neq 0 : g^N \in Z(A_{ab})$. By hypothesis, there is a $q \neq 0$ such that $g^N = \Delta^q$. By Lemma 3.1.18, this means the quantity $\ell_{\mathcal{S}}(g^{Nn})$ grows linearly in n. In particular, the quantity $\ell_{\mathcal{S}}(g^{N \cdot \lfloor \frac{n}{N} \rfloor})$ grows linearly in n as well (for a smaller constant). Note that the difference between $\ell_{\mathcal{S}}(g^n)$ and $\ell_{\mathcal{S}}(g^{N \cdot \lfloor \frac{n}{N} \rfloor})$ is uniformly bounded by the constant $L := \max\{\ell_{\mathcal{S}}(g^i) \mid i = 0, \cdots, N-1\}$. It follows that $\ell_{\mathcal{S}}(g^n)$ also grows linearly in n.

Case 4: We are in none of the previous cases. Then by Lemma 3.1.14, g acts hyperbolically on $\widehat{T}$. In particular, the quantity $d_{\widehat{T}}(1_\bullet, g^n \cdot 1_\bullet)$ grows linearly. We conclude with Lemma 3.1.13. □

3.2 On the geometry of the action

Let A_Γ be a 2-dimensional Artin group of rank at least 3, and let D_Γ be its Deligne complex. Our goal is to show that there exists a vertex $v \in D_\Gamma$, and an element $g \in A_\Gamma$ satisfying the two hypotheses of Theorem 3.4.

3.2.1 The augmented Deligne complex

We have seen in Proposition 3.5.(1) that a strong enough condition for v to satisfy the first hypothesis of Theorem 3.4 is to require that its local group G_v is a dihedral Artin group A_{ab} with coefficient $3 \leqslant m_{ab} < \infty$. When such a vertex v exists, it only remains to show that there exists an element $g \in A_\Gamma$ such that $A_{ab} \cap gA_{ab}g^{-1}$ is finite (i.e. trivial because dihedral Artin groups are torsion-free). Our main geometric tool in order to find such an element is the following lemma:

Lemma 3.2.1. *Let G be a group acting by simplicial isomorphisms on a CAT(0) simplicial complex X of dimension 2. Let $v \in X$, $g \in G$ and denote by G_p the stabiliser of a point $p \in X$. If the unique geodesic γ between v and gv goes through a point with trivial stabiliser, then $G_v \cap G_{gv} = \{1\}$.*

Proof: Any element of $G_v \cap G_{gv}$ fixes v and gv, hence fixes (pointwise) the unique geodesic γ between them. This means that $G_v \cap G_{gv} = G_\gamma$. Let $p \in \gamma$ be a point with trivial stabiliser. Then we have

$$G_v \cap G_{gv} = G_\gamma \subseteq G_p = \{1\}.$$

□

Strategy: The strategy of this section is led by the previous lemma. It is not hard to see that if $v \in D_\Gamma$ is a vertex with stabiliser A_{ab}, then the stabiliser of gv for some $g \in A_\Gamma$ is exactly $gA_{ab}g^{-1}$. Suppose additionally that v satisfies Theorem 3.4.(1), which holds as soon as A_{ab} is large. Our goal will be to construct a geodesic between v and some gv that contains a point with trivial stabiliser.

In the case of the Deligne complex, every point that lies in the interior of a base triangle T_{st} or an edge e_s or e_{st} has trivial stabiliser, hence it is enough to show that γ goes through the interior of such a triangle or edge. In some cases, this will turn out to be quite difficult to prove. However, everything will be more manageable when working in some augmented version of the Deligne complex (see Definition 3.2.3).

The next Proposition will give the structure of the different cases we will encounter:

Proposition 3.2.2. *Let A_Γ be a 2-dimensional Artin group of rank at least 3, and suppose that Γ is connected and that A_Γ is not a right-angled Artin group. Then there exist three distinct generators $a, b, c \in S$ such that $m_{ab} \in \{3, 4, \cdots\}$, $m_{ac} \in \{2, 3, 4, \cdots\}$, $m_{bc} \in \{2, 3, 4, \cdots, \infty\}$ and*

$$\frac{1}{m_{ab}} + \frac{1}{m_{ac}} + \frac{1}{m_{bc}} \leqslant 1,$$

where $\frac{1}{\infty} := 0$. Moreover, we are in exactly one of the following situation:
(1) There is a triplet (a, b, c) as before that satisfies $m_{bc} < \infty$.
(2) There is no triplet (a, b, c) as before with $m_{bc} < \infty$, but there is one that satisfies $m_{bc} = \infty$. Moreover, the graph Γ^{bc} obtained from Γ by adding an edge e^{bc} with coefficient 6 is such that $A_{\Gamma^{bc}}$ has dimension 2.
(3) We are not in the first two situations, and Γ contains a cycle γ with coefficients $(2, 2, 2, n)$ for some $n \geqslant 3$, such that γ is ***full****, in the sense that it does not contain any non-homotopically-trivial strict subcycle.*

Proof : We begin by proving the first statement. Because A_Γ is not right-angled, there is an edge e^{ab} in Γ with coefficient $m_{ab} \in \{3, 4, \cdots\}$. As Γ is connected and has at least 3 vertices, e^{ab} has a neighbouring edge in Γ, say e^{ac}, with coefficient $m_{ac} \in \{2, 3, 4, \cdots\}$. Since A_Γ has dimension 2, the last coefficient $m_{bc} \in \{2, 3, 4, \cdots, \infty\}$ satisfies:

$$\frac{1}{m_{ab}} + \frac{1}{m_{ac}} + \frac{1}{m_{bc}} \leqslant 1.$$

Let's now prove that we are in exactly one of the three cases. The three cases are exclusive by definition, so it is enough to show that if we are not in one of the first two situations, then we must be in the third. To prove this, pick a triplet of the form $m_{ab} \in \{3, 4, \cdots\}$, $m_{ac} \in \{2, 3, 4, \cdots\}$, $m_{bc} = \infty$. By hypothesis, the graph Γ^{bc} obtained from Γ by adding an edge e^{bc} with coefficient 6 is such that $A_{\Gamma^{bc}}$ is not 2-dimensional. This means that there is a generator $d \in S$ such that

$$\frac{1}{6} + \frac{1}{m_{bd}} + \frac{1}{m_{cd}} > 1.$$

This is only possible if $m_{bd} = m_{cd} = 2$. Notice that $m_{ad} = \infty$, otherwise the triplet (a, b, d) would satisfy (1). This means that we have a full cycle $(e^{bd}, e^{cd}, e^{ac}, e^{ab})$ with coefficients $(2, 2, \geqslant 2, \geqslant 3)$ in Γ. If $m_{ac} = 2$, we are done. Suppose that $m_{ac} \geqslant 3$, and add an edge e^{ad} of coefficient 6. Since $A_{\Gamma^{ad}}$ is not 2-dimensional by hypothesis and since

$$\frac{1}{6} + \frac{1}{m_{ab}} + \frac{1}{m_{bd}} \leqslant 1,$$
$$\frac{1}{6} + \frac{1}{m_{ac}} + \frac{1}{m_{cd}} \leqslant 1,$$

then there must be a fifth generator $e \in S$ such that

$$\frac{1}{6} + \frac{1}{m_{ae}} + \frac{1}{m_{de}} > 1.$$

For the same reasons as before, we have $m_{ae} = m_{de} = 2$ and $m_{ce} = \infty$. Hence there is a full cycle $(e^{ae}, e^{de}, e^{cd}, e^{ac})$ with coefficients $(2, 2, 2, \geqslant 3)$ in Γ. □

Recall that our goal in order to prove Theorem 3.1 is to apply Theorem 3.4. For an irreducible 2-dimensional Artin group A_Γ of rank at least 3, it turns out that the Deligne complex D_Γ is exactly the space that we want to act on, at least in the first and third cases or Proposition 3.2.2. Unfortunately, in the second case of Proposition 3.2.2, the space D_Γ is not fit to apply our main geometric tool that is Lemma 3.2.1. The reason, as will be seen later, is that we would like to have three generators $a, b, c \in S$ for which all the triangles T_{ab}, T_{ba}, T_{bc}, T_{cb}, T_{ca}, and T_{ac} belong to D_Γ. This is not the case when $m_{bc} = \infty$. However, notice that in the second case of Proposition 3.2.2, the complex obtained from D_Γ by adding the vertices of the form gv_{bc} and their attached triangles gT_{bc}, gT_{cb} is 2-dimensional by hypothesis. This slightly bigger complex, as defined in the next definition, will be the one to look at when using Lemma 3.2.1 and Theorem

3.4 in that case.

Definition 3.2.3. Let A_Γ be a 2-dimensional Artin group of rank at least 3 with Deligne complex D_Γ and fundamental domain K_Γ. We call the structure of complex of groups on K_Γ inherited from Definition 2.4.17 the **usual** complex of groups associated with K_Γ.

Let now Γ^{st} be the same graph as Γ, except that we add an edge e^{st} with coefficient 6 between s and t if $m_{st} = \infty$. Consider now the 2-dimensional complex $K_{\Gamma^{st}}$ obtained from Definition 2.4.17 for the group $A_{\Gamma^{st}}$. In other words,

$$K_{\Gamma^{st}} := \begin{cases} K_\Gamma \text{ if } m_{st} < \infty \\ K_\Gamma \cup T_{st} \cup T_{ts} \text{ if } m_{st} = \infty, \end{cases}$$

where the angle at v_{st} in T_{st} or T_{ts} is set to be $\frac{\pi}{12}$ if $m_{st} = \infty$. We now want to realise A_Γ as the fundamental group of a complex of groups over $K_{\Gamma^{st}}$. Doing so, we will give $K_{\Gamma^{st}}$ a structure of complex of groups, which may differ from the one coming from Definition 2.4.17. When $m_{st} < \infty$, $K_{\Gamma^{st}} = K_\Gamma$, and we proceed as in Definition 2.4.17: $K_{\Gamma^{st}}$ is simply the usual complex of groups associated with A_Γ. When $m_{st} = \infty$ however, K_Γ is a strict subcomplex of $K_{\Gamma^{st}}$, and we carry the usual complex of groups associated with A_Γ from K_Γ onto the corresponding subcomplex of $K_{\Gamma^{st}}$. We still have to describe the local group at v_{st} and the associated maps. We just set this local group to be the free group F_{st} of rank 2. The associated maps are the obvious morphisms that inject $\{1\}$, $\langle s \rangle$ and $\langle t \rangle$ into F_{st}. We call this structure of complex of group given to $K_{\Gamma^{st}}$ the **augmented** complex of groups associated with A_Γ (relatively to s and t).

If $m_{st} < \infty$, the augmented complex of groups associated with A_Γ coincides with its usual complex of groups. If $m_{st} = \infty$, the augmented complex of groups associated with A_Γ is the same as the one we would get if we took the usual complex of groups associated with $A_{\Gamma^{st}}$, but then replaced the local group A_{st} at v_{st}, that is a dihedral Artin group with coefficient 6, by the free group F_{st}. Note that in both cases, the 2-dimensional complex under the augmented complex of groups associated with A_Γ is $K_{\Gamma^{st}}$. The universal cover D_Γ^{st} of this complex of groups is called the **augmented Deligne complex** of A_Γ (relatively to s and

t). In particular, in the light of Definition 2.4.17, we have

$$D_\Gamma^{st} := A_\Gamma \times K_{\Gamma^{st}} /_\sim ,$$

where $(g, x) \sim (g', x') \iff x = x'$ and $g^{-1}g'$ belongs to the local group of the smallest simplex of $K_{\Gamma^{st}}$ that contains x. The action of A_Γ on itself induces an action of A_Γ on D_Γ^{st} by simplicial morphisms with strict fundamental domain $K_{\Gamma^{st}}$.

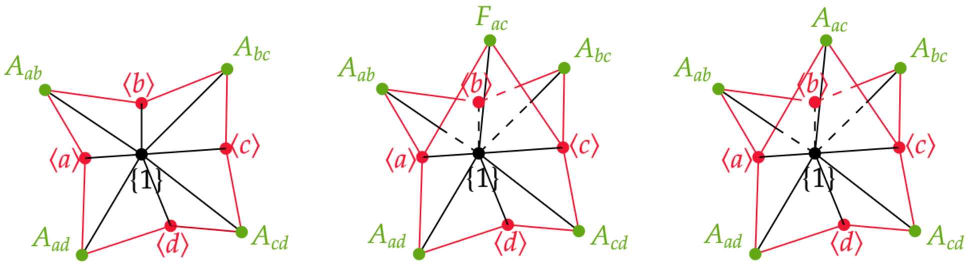


Figure 3.3: Let Γ be a graph defined as a square with vertices a, b, c and d, such that $m_{ac} = \infty$. On the left: The usual complex of groups associated with A_Γ. In the centre: The augmented complex of groups associated with A_Γ relatively to a and c. On the right: The usual complex of groups associated with $A_{\Gamma^{ac}}$.
Note that the first two complexes of groups share the same fundamental groups, and the last two complexes of groups share the same underlying 2-dimensional complex.

Remark 3.2.4. (1) If $m_{st} < \infty$, the augmented Deligne complex D_Γ^{st} and the Deligne complex D_Γ agree.

(2) If $m_{st} = \infty$, then D_Γ^{st} differs from $D_{\Gamma^{st}}$, as the fundamental groups of their associated complexes of groups are not the same: the former is A_Γ while the latter is $A_{\Gamma^{st}}$. In particular, D_Γ^{st} decomposes as a quotient of $A_\Gamma \times K_{\Gamma^{st}}$, while $D_{\Gamma^{st}}$ decomposes as a quotient of $A_{\Gamma^{st}} \times K_{\Gamma^{st}}$. Note however that the fundamental domains of these complexes are the same, as 2-dimensional complexes.

(3) It is important to notice that if a, b, s, t are four (non-necessarily all distinct) generators of A_Γ satisfying $(a, b) \neq (s, t)$ and $m_{ab} < \infty$, then $Lk_{D_\Gamma}(v_{ab}) \cong Lk_{D_\Gamma^{st}}(v_{ab})$ (see Figure 3.3 for instance). In particular, results such as Lemma 3.1.2 or Proposition 3.5.(1) also hold for v_{ab} if we replace D_Γ by D_Γ^{st}.

Lemma 3.2.5. *Let A_Γ be a 2-dimensional Artin group of rank at least 3, and suppose that we are in the second case of Proposition 3.2.2. Then the augmented Deligne complex D_Γ^{bc} of A_Γ is CAT(0).*

Proof: By hypothesis $A_{\Gamma^{bc}}$ has dimension 2, hence its associated Deligne complex $D_{\Gamma^{bc}}$ is CAT(0) (Theorem 2.4.22). We want to show that D_Γ^{bc} is CAT(0). By Lemma 2.2.25, and up to reducing to the fundamental domain, it is enough to show that every vertex $v \in K^{\Gamma^{bc}}$ satisfies

$$\mathrm{sys}(Lk_{D_\Gamma^{bc}}(v)) \geqslant 2\pi. \ (*)$$

Notice that if $v \neq v_{bc}$ then

$$Lk_{D_\Gamma^{bc}}(v) \cong Lk_{D_{\Gamma^{bc}}}(v),$$

and thus $(*)$ follows from the fact that $D_{\Gamma^{bc}}$ is CAT(0), along with Lemma 2.2.25.

If $v = v_{bc}$, then the local group at v is the free group F_{bc} by definition. We can do a similar analysis as the one done in Remark 2.4.21. This time, the maps of the development inject into the free group F_{bc}. Therefore, the link $Lk_{D_\Gamma^{bc}}(v_{bc})$ is isomorphic to the barycentric subdivision of the Bass-Serre tree above the segment of groups with local groups $\langle b\rangle$ and $\langle c\rangle$ on the vertices and $\{1\}$ on the edge. In particular, $Lk_{D_\Gamma^{bc}}(v_{bc})$ is simply-connected, i.e. has infinite systole. □

3.2.2 Finding appropriate weakly malnormal subgroups

We are now ready to prove the following lemma, that shows the existence of appropriate weakly malnormal subgroups of A_Γ, one of the requirements of Theorem 3.4.

Lemma 3.2.6. *Let A_Γ be a 2-dimensional Artin group of rank at least 3, and suppose that Γ is connected and that A_Γ is not a right-angled Artin group. Then there exists an Artin subgroup A_{ab} with coefficient $3 \leqslant m_{ab} < \infty$ and an element $g \in A_\Gamma$ such that $A_{ab} \cap gA_{ab}g^{-1} = \{1\}$.*

Proof: By Proposition 3.2.2, we know that we either have three generators $a, b, c \in V(\Gamma)$ that satisfy exactly one of the following:

(1) $m_{ab}, m_{ac} \in \{3, 4, \cdots\}$ and $m_{bc} \in \{3, 4, \cdots, \infty\}$;

(2) $m_{ac} = 2$, $m_{ab} \in \{3, 4, \cdots\}$ and $m_{bc} \in \{5, 6, \cdots, \infty\}$;

(3) $m_{ac} = 2$, $m_{ab} = m_{bc} = 4$.

Or we have four generators $a, b, c, d \in V(\Gamma)$ satisfying:

(4) The cycle $(e^{bc}, e^{cd}, e^{ad}, e^{ab})$ is full in Γ and has coefficients $(2, 2, 2, n)$ with $n \geqslant 3$.

Let Δ be the abstract complex (see Figure 3.4) defined by:

- In the situations (1), (2) and (3), $\Delta := T_{ab} \cup T_{ba} \cup T_{bc} \cup T_{cb} \cup T_{ca} \cup T_{ac}$.
- In the situation (4), $\Delta := T_{ab} \cup T_{ba} \cup T_{bc} \cup T_{cb} \cup T_{cd} \cup T_{dc} \cup T_{da} \cup T_{ad}$.

Note that in either case, the points in the interior of Δ have trivial stabilisers. Also note that we don't have to look at the augmented Deligne complex in the situations (1) and (2) if $m_{bc} < \infty$, and neither do we in the situations (3) and (4). However in those cases $m_{bc} < \infty$, and hence $D_\Gamma^{bc} = D_\Gamma$, so it will just be convenient to write D_Γ^{bc} to cover all cases.

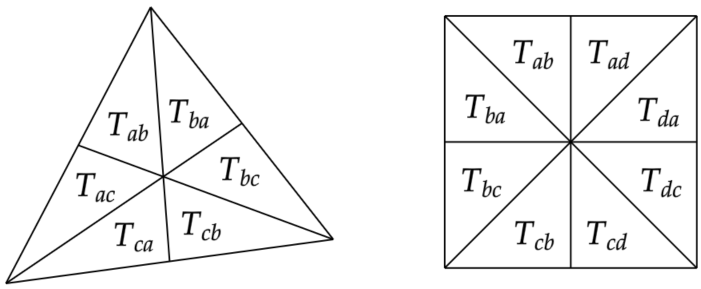


Figure 3.4: Δ in the case (1), (2), (3) (on the left) and (4) (on the right).

Let now P be an abstract complex defined by $P := {P_0}/_{\sim}$, where P_0 is defined depending on the situations given in the beginning of the proof by:

(1) $P_0 := \Delta \sqcup (c\Delta)$.

(2) $P_0 := \Delta \sqcup (c\Delta) \sqcup (cb\Delta) \sqcup (cbc\Delta)$.

(3) $P_0 := \Delta \sqcup (c\Delta) \sqcup (cb\Delta) \sqcup (cbc\Delta) \sqcup (cba\Delta) \sqcup (cbca\Delta) \sqcup (cbcab\Delta) \sqcup (cbcabc\Delta)$.

(4) $P_0 := \Delta \sqcup (c\Delta) \sqcup (d\Delta) \sqcup (cd\Delta)$.

And where $\sim$ corresponds to the gluing shown in Figure 3.5, i.e. P is obtained from P_0 by gluing the different copies of Δ along some of their edges (drawn in blue in Figure 3.5).

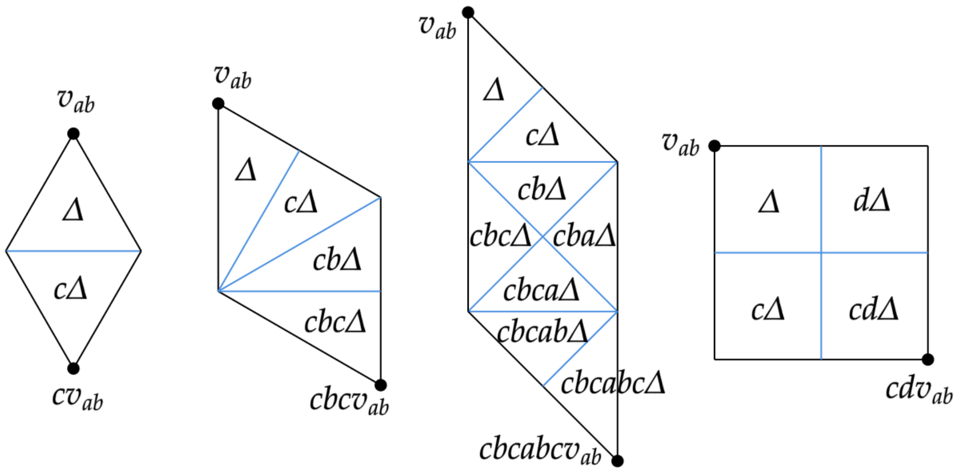


Figure 3.5: Polygon P in the four different cases, from left to right.

A priori, we can't be sure that there are no additional gluings happening in D_Γ^{bc}, so we don't want to look at P as a subcomplex of D_Γ^{bc}, but we want instead look at P through the natural map $f : P \to D_\Gamma^{bc}$ that maps P to D_Γ^{bc}.

<u>Claim 1:</u> P is isometrically embedded in D_Γ^{bc}.

Proof of Claim 1: In the light of [25, Lemma 1.4], it is enough to show that for every $p \in P$, the induced map $f_p : Lk_P(p) \to Lk_{D_\Gamma^{bc}}(p)$ is π-distance preserving, i.e. that

$$\forall x, y \in Lk_P(p),\ d_{Lk_P(p)}(x, y) \geqslant \pi \Rightarrow d_{Lk_{D_\Gamma^{bc}}(p)}(f_p(x), f_p(y)) \geqslant \pi.$$

There are two different situations:

• If $p \in P$ is in the orbit of $v_\varnothing$, then $Lk_{D_\Gamma^{bc}}(p)$ is just the augmented defining graph Γ^{bc} with the appropriate metric (see Definition 3.2.3 and Remark 2.4.21). Notice that, any edge $e^{st} = e_{s,st} \star e_{t,st}$ from s to t in Γ^{bc} has length

$$\ell(e^{st}) = 2 \cdot \angle_\varnothing(v_s, v_{st}) = \pi - \frac{\pi}{m_{st}} \geqslant \frac{\pi}{2},$$

according to the metric on $Lk_{D_\Gamma^{bc}}(p)$. Since $Lk_P(p)$ is simply the full cycle in Γ^{bc} corresponding to the triangle (e^{ab}, e^{ac}, e^{bc}) (in the situations (1), (2) and (3)) or to the square $(e^{bc}, e^{cd}, e^{ad}, e^{ab})$ (in the situation (4)), we can apply [25, Lemma 1.6] and conclude that the map $f_p : Lk_P(p) \hookrightarrow Lk_{D_\Gamma^{bc}}(p)$ is π-distance preserving.

• If $p \in P$ is not in the orbit of $v_\varnothing$, then it is not hard to see from Remark 2.4.21 that every full cycle in $Lk_P(p)$ has length exactly 2π. In particular, the map

f_p must be π-preserving, otherwise we would be able to build an isometrically embedded cycle in $Lk_{D_\Gamma^{bc}}(p)$ of length strictly less than 2π, contradicting the CAT(0)-ness of D_Γ^{bc} (Theorem 2.4.22, Lemma 3.2.5 and Lemma 2.2.25). This finishes the proof of Claim 1.

We can now use [25, Lemma 1.4] and conclude that P is isometrically embedded in X. In particular, geodesics in P project to geodesics in X through f.

Claim 2: $A_{ab} \cap gA_{ab}g^{-1} = \{1\}$ for some $g \in A_\Gamma$.

Proof: Notice that P is CAT(0) by Lemma 2.2.25. In particular, it is uniquely geodesic. Let γ be the geodesic in P defined depending on the situations given in the beginning of the proof by:

(1) γ is the geodesic going from v_{ab} to cv_{ab}.

(2) γ is the geodesic going from v_{ab} to $cbcv_{ab}$.

(3) γ is the geodesic going from v_{ab} to $cbcabcv_{ab}$.

(4) γ is the geodesic going from v_{ab} to cdv_{ab}.

Note that γ is also geodesic in D_Γ^{bc}, by the previous claim. Thanks to Lemma 3.2.1, it is enough to show that γ goes through the interior of some $g_0\Delta$ contained in P. Consider either of the four situations and suppose that it is not the case. Then in particular γ would be contained in the 1-skeleton of P. It is not hard to check, since we know every angle in P by construction, that there must be a vertex v in γ that satisfies $\angle_v^P(\gamma) < \pi$. This is not possible, as γ is a geodesic and P is CAT(0). This finishes the proof of Claim 2, and of the lemma. □

We have worked through everything that was required in order to use our main criterion, that is Theorem 3.4. We can now prove our main Theorem:

Theorem 3.2.7. *Every irreducible* 2*-dimensional Artin group of rank at least* 3 *is acylindrically hyperbolic.*

Proof: Let A_Γ be an irreducible 2-dimensional Artin group of rank at least 3. We can assume that Γ is connected, as otherwise A_Γ splits as a free product $A_{\Gamma_1} * A_{\Gamma_2}$ of infinite groups hence is acylindrically hyperbolic. We can also assume that A_Γ is not a right-angled Artin group, as every irreducible right-angled Artin group that is not cyclic is acylindrically hyperbolic ([80, Section 8]).

Let $a, b, c \in V(\Gamma)$ be the three generators obtained in the proof of Lemma 3.2.6, and consider the action of A_Γ on its augmented Deligne complex D_Γ^{bc}. Note that the latter is CAT(0) by Lemma 3.2.5 and Lemma 2.4.22. Since $m_{ab} \geqslant 3$, we know from Proposition 3.5.(1) and Remark 3.2.4 that the orbits of A_{ab} on $Lk_{D_\Gamma^{bc}}(v_{ab})$ are unbounded. Moreover, we know from Lemma 3.2.6 that there exists an element $g \in A_\Gamma$ such that $A_{ab} \cap gA_{ab}g^{-1} = \{1\}$. Therefore, we can apply Theorem 3.4 and conclude that A_Γ is either virtually cyclic or acylindrically hyperbolic. That A_Γ is not virtually cyclic is clear because it contains $\mathbf{Z}^2$ subgroups (Theorem 2.4.9). $\square$

Chapter 4

Parabolic subgroups

This chapter corresponds to the pre-publication [29], and is a joint work with María Cumplido and Alexandre Martin.

Parabolic subgroups form a natural class of subgroups that has been playing an increasing role in the geometric study of Artin groups in recent years. Hence why understanding their combinatorics has become a topic of interest on its own. Although parabolic subgroups are thought to have a nice combinatorial behaviour, most of the main questions about them remain open in general. In this chapter we consider Artin groups of large-type and prove in that case that the parabolic subgroups do behave nicely. Our main theorem is the following:

Theorem 4.1. *Let A_Γ be a large-type Artin group. Then the intersection of an arbitrary subset of parabolic subgroups of A_Γ is itself a parabolic subgroup. Moreover, the set of parabolic subgroups of A_Γ is a lattice for the inclusion.*

Let A_Γ be a large-type Artin group. Our strategy for studying the parabolic subgroups of A_Γ is geometric. To A_Γ we associate a simplicial complex called its **Artin complex** X_Γ, whose geometry resembles that of the Deligne complex (see Section 2.4.3), except that the Artin complex is constructed from the combinatorics of all strict parabolic subgroups of A_Γ, and not just the spherical ones. In this complex, every strict parabolic subgroup appears as the stabiliser of some simplices, and can thus be studied geometrically. The Artin complex associated with an Artin group can be very high-dimensional, although we are able to understand some of its geometric properties using tools coming from systolic geometry (see Section 2.2.4).

Theorem 4.2. *Let A_Γ be a large-type Artin group of rank at least 3. Then its Artin complex X_Γ is systolic.*

Large-type Artin groups were recently shown to be systolic groups ([57]). However, we emphasise that the systolic geometry appearing here is of a rather different nature. The systolic complex associated to A_Γ considered by Huang-Osajda is essentially a (thickened) Cayley graph of A_Γ for the standard generating set, and as such is quasi-isometric to A_Γ. By contrast, the Artin complex X_Γ studied here is quasi-isometric to the Cayley graph of A_Γ with respect to all its proper parabolic subgroups, and in particular the action of A_Γ on X_Γ is cocompact but far from being proper.

Using Theorem 4.1, we are also able to solve the problem of conjugacy stability for parabolic subgroups. A subgroup H of a group G is **conjugacy stable** if for every pair of elements $g, h \in H$ such that $g = \alpha^{-1}h\alpha$ for some $\alpha \in G$ there is a $\beta \in H$ such that $g = \beta^{-1}h\beta$. We obtain the following result:

Theorem 4.3. *Let $A_{\Gamma'}$ be a standard parabolic subgroup of a large-type Artin group A_Γ. Then $A_{\Gamma'}$ is not conjugacy stable in A_Γ if and only if there exist vertices a and b of Γ' that are connected by an odd-labelled path in Γ and that are not connected by an odd-labelled path in Γ.*

Note that the previous theorem generalises to all parabolic subgroups of large-type Artin groups, as conjugacy stability is preserved under subgroup conjugations. Another application of Theorem 4.1 and of the sytolicity of the Artin complex is that parabolic subgroups are root stable:

Theorem 4.4. *Let A_Γ be a large-type Artin group, let P be a parabolic subgroup of A_Γ, and let $g \in A_\Gamma$. If $g^n \in P$ for some non-zero integer n, then $g \in P$.*

Studying the intersection properties of parabolic subgroups relies on understanding the sets of fixed-points and the normalisers of parabolic subgroups. In particular, a consequence of our work is that we are able to recover the structure of the normaliser of every parabolic subgroup of a large-type Artin group. Although these normalisers had already been studied by preceding authors, our approach allows to recover these results independently and to give an explicit description of these normalisers.

Theorem 4.5. *Let A_Γ be a large-type Artin group and let P be a parabolic subgroup of A_Γ*

- *If type$(P) \geqslant 2$, then $N(P) = P$.*
- *If type$(P) = 1$, then $N(P)$ splits as a direct product of the form $N(P) = P \times F$, where F is a finitely-generated free group. Moreover, there is an explicit description of a basis of F (see Corollary 4.3.17 for details).*

The structure of normalisers of parabolic subgroups in Artin groups of large type had already been investigated by Paris and Godelle, although it is a bit hidden in their papers. In [82], the conjugation of standard parabolic subgroups is described by an algorithm. In particular, we know that the only pairs of different irreducible standard parabolic subgroups that can be conjugated are the spherical ones. In the large case, as all parabolic subgroups are irreducible and the only spherical parabolic subgroups are the dihedral ones, the situation is as follows: $A_{\Gamma'}$ and $A_{\Gamma''}$ are conjugate if and only if $\Gamma' = \Gamma''$ or Γ' and Γ'' are vertices that correspond to standard generators a and b respectively, such that a and b are connected in Γ by an odd-labelled path. Using [42, Definition 4.1, Corollary 4.12], we know that the conjugating elements between two (possibly equal) standard parabolic subgroups $A_{\Gamma'}$ and $A_{\Gamma''}$ must be the product of an element in A_Γ and an element associated to the previous path. If Γ' has type at least 2, such a path does not exists and then $N(A_{\Gamma'}) = A_{\Gamma'}$. If Γ' has type 1, the description of the normaliser is similar to the one given in Corollary 4.3.17. However, the description Godelle gives there is set-theoretic and does not describe the direct product structure.

The structure of the normaliser of cyclic parabolic subgroups for large-type Artin groups (and more generally 2-dimensional Artin groups) had been obtained, albeit under a different name, in [75, Proposition 4.5]. Moreover, a basis of the corresponding free group had been stated as a remark, but without details.

We organise this chapter as follows. In Section 4.1, we introduce the Artin complex of a general Artin group, and show that its local structure is particularly well-behaved, in the sense that the links of simplices are themselves (smaller) Artin complexes. We then use this local structure to prove Theorem 4.2. Section 4.2 exploits the systolic geometry of the Artin complex to prove Theorem 4.1. In

Section 4.3, we study the geometry of fixed-point sets of parabolic subgroups in order to prove Theorem 4.5. Finally, we prove Theorem 4.3 and Theorem 4.4 in Section 4.4.

4.1 Systolicity of the Artin complex

The goal of this section is to introduce our main geometric object, that is the Artin complex associated to an Artin group. Later on, we present some of its basic properties, and we show its systolicity for the case of large-type Artin groups.

Definition 4.1.1. Consider an Artin group A_Γ of rank $n \geqslant 2$, and a simplex S_Γ of dimension $n-1$. We define a simplex of groups over S_Γ as follows. The simplex S_Γ is given a trivial local group. There is a one-to-one correspondence between the standard generators $s_i \in V(\Gamma)$ and the codimension 1 faces of S_Γ, and we denote by Δ_{s_i} these codimension 1 faces. In particular, Δ_{s_i} is given the local group $\langle s_i \rangle$. Changing the codimension, there is a bijection between the strict subsets of $V(\Gamma)$ and the faces of S_Γ. Every face of K of codimension k can be written uniquely as the intersection

$$\Delta_{\Gamma'} := \bigcap_{s_i \in V(\Gamma')} \Delta_{s_i} \text{ for some } \Gamma' \text{ induced strict subgraph of } \Gamma \text{ with } |V(\Gamma')| = k.$$

The face $\Delta_{\Gamma'}$ is then given the local group $A_{\Gamma'}$. The morphism associated to an inclusion of faces $\Delta_{\Gamma''} \subset \Delta_{\Gamma'}$ is the natural inclusion $\psi_{\Gamma'\Gamma''} : A_{\Gamma''} \hookrightarrow A_{\Gamma'}$. Let $\mathcal{Q}$ be the poset of standard parabolic subgroups of A_Γ ordered with natural inclusion. As each $A_{\Gamma'}$ is itself an Artin group, there is a simple morphism $\varphi : G(\mathcal{Q}) \hookrightarrow A_\Gamma$ given by inclusion. The complex $X_\Gamma := D(S_\Gamma, \varphi)$ obtained by development of S_Γ along φ is called the **Artin complex** associated to A_Γ (see Definition 2.3.4).

The action of A_Γ on X_Γ is without inversions and cocompact, with strict fundamental domain a single simplex which is isomorphic to S_Γ. To avoid any confusion, we will from now on denote by $\overline{S_\Gamma}$ the quotient space and by $\overline{\Delta}_{\Gamma'}$ its faces, and we will denote by S_Γ a chosen fundamental domain of X_Γ and by $\Delta_{\Gamma'}$ its faces. For every simplex Δ of X_Γ, there is a unique induced subgraph $\Gamma' \subsetneq \Gamma$ such that Δ is the same orbit as $\Delta_{\Gamma'}$. We say that the simplex Δ is **of type Γ'**.

In light of Definition 2.3.4, the Artin complex X_Γ can also be described by the following:

$$X_\Gamma = A_\Gamma \times K /_\sim ,$$

where $(g,x) \sim (g',x') \iff x = x'$ and $g^{-1}g'$ belongs to the simplex $supp(x)$.

As for the Deligne complex, there is an equivalent definition of the Artin complex in terms of the combinatorics of the parabolic subgroups:

Remark 4.1.2. Consider the following combinatorial complex P_Γ:
• The vertex set of P_Γ is the poset of left-cosets of all the strict standard parabolic subgroups of A_Γ.
• There is a $(n-1)$-simplex between vertices of D_Γ corresponding to the left-cosets $g_1 A_{\Gamma_1}, \ldots, g_n A_{\Gamma_n}$ whenever there is a sequence of inclusions $g_n A_{\Gamma_n} \subset \cdots \subset g_1 A_{\Gamma_1}$.
Then P_Γ is exactly the barycentric subdivision of X_Γ.

Note that the Artin complex resembles to the Deligne complex (see Section 2.4.3), although in the Artin complex we consider all the (strict) parabolic subgroups, and not only the ones that are spherical.

Lemma 4.1.3. *Let A_Γ be an Artin group and let X_Γ be its Artin complex. Then X_Γ is connected. Additionally, if A_Γ has rank at least 3, then X_Γ is simply-connected.*

Proof: This is a consequence of Definition 2.3.4. X_Γ is connected because the Artin group A_Γ is generated by its standard parabolic subgroups. Moreover, if A_Γ has rank at least 3, then A_Γ is the colimit of its strict standard parabolic subgroups, by Theorem 2.4.4, and thus X_Γ is simply-connected. □

Lemma 4.1.4. *Let A_Γ be an Artin group with Artin complex X_Γ. Then the link of a simplex of type Γ' is isomorphic to the Artin complex $X_{\Gamma'}$ associated to the Artin group $A_{\Gamma'}$.*

Proof: By [17, Construction II.12.24], it is possible to describe the link of a simplex in the development of a complex of groups as the development of an appropriate subcomplex of groups (as we did in Remark 2.4.21). We explain below how this applies to X_Γ.

The link of $\overline{\Delta}_{\Gamma'}$ in $\overline{S_\Gamma}$ is a simplex of dimension $|V(\Gamma)'| - 1$, whose poset of faces is isomorphic to the poset of proper subsets of Γ' ordered with the inclusion.

The complex of groups $G(\overline{S_\Gamma})$ induces a complex of groups on the link $Lk_{\overline{S_\Gamma}}(\overline{\Delta_{\Gamma'}})$. Moreover, there is a simple morphism $\varphi_{\Gamma'} : G(Lk_{\overline{S_\Gamma}}(\overline{\Delta_{\Gamma'}})) \to A_{\Gamma'}$ given by the family of homomorphisms

$$(\varphi_{\Gamma'})_{\Gamma''} : A_{\Gamma''} \xrightarrow{\psi_{\Gamma'\Gamma''}} A_{\Gamma'}.$$

It follows from the construction described in [17, Construction II.12.24] that the link of $Lk_{X_\Gamma}(\Delta_{\Gamma'})$ is isomorphic to the development $D(Lk_{\overline{S_\Gamma}}(\overline{\Delta_{\Gamma'}}), \varphi_{S'})$. Note that the induced complex of groups on $Lk_{\overline{S_\Gamma}}(\overline{\Delta_{\Gamma'}})$ is naturally isomorphic to the complex of groups associated to $A_{\Gamma'}$ in Definition 4.1.1. Moreover, the simple morphism $\varphi_{\Gamma'}$ coincides with the simple morphism used in Definition 4.1.1 to define the Artin complex $X_{\Gamma'}$. Putting everything together, it now follows that the link $Lk_{X_\Gamma}(\Delta_{\Gamma'})$ is isomorphic to $X_{\Gamma'}$.

This argument generalises in a straightforward way to any simplex $g\Delta_{\Gamma'}$ of X_Γ of type Γ'. □

We now move towards proving Theorem 4.2, that is, proving that the Artin complex associated with any Artin group of large type is systolic. For more details about systolicity, we refer the reader to Section 2.2.4. The main result we prove about the geometry of the Artin complex is the following:

Theorem 4.1.5. *Let A_Γ be an Artin group of rank at least 3. If all coefficients in A_Γ are at least $k \in \{3, \dots, \infty\}$, then its Artin complex X_Γ is $2k$-systolic. In particular, if A_Γ is of large type, then X_Γ is systolic.*

In order to prove this theorem, we need the following lemma:

Lemma 4.1.6. *Let A_Γ be an Artin group on two generators a, b with coefficient $m_{ab} \in \{3, \dots, \infty\}$ and Artin complex X_Γ. Then $\mathrm{sys}(X_\Gamma) = 2m_{ab}$.*

Proof: If $m_{ab} = \infty$, it follows directly from the definition of the Artin complex that X_Γ is the Bass-Serre tree associated to the splitting $\langle a \rangle * \langle b \rangle$. The result is then immediate. Let us now assume that $m_{ab} < \infty$. Let e be the edge in X_Γ whose vertices x, y correspond to the cosets $\langle a \rangle$ and $\langle b \rangle$. Let γ be a non-backtracking loop in X_Γ. Since X_Γ is a bipartite graph coloured by the cosets of $\langle a \rangle$ and $\langle b \rangle$ respectively, the length of γ is even. Denote by $e_0, e_1, \dots, e_k$ the edges of γ. Since the action of A_Γ on X_Γ is transitive on edges, let us assume that $e_0 = e$. Note that the action of $\langle a \rangle$ is transitive on the set of edges around x, and so is the action of $\langle b \rangle$ on the edges around y. Assume without loss of generality that γ first goes through x, i.e. e_1 and e_0 share the vertex x. Then e_1 must be of the form $a^{r_1}e$, for some $r_1 \in \mathbf{Z}\backslash\{0\}$. Note that the edges e_1 and e_2 then share the vertex $a^{r_1}y$. The action of $a^{r_1}\langle b \rangle a^{-r_1}$ is transitive on the set of edges around $a^{r_1}y$, thus e_2 must of the form $a^{r_1}b^{r_2}e$, for some $r_2 \in \mathbf{Z}\backslash\{0\}$. We continue this process by induction until γ stops. In particular, the final edge e_k is of the form

$$a^{r_1}b^{r_2}\cdots a^{r_{k-1}}b^{r_k}$$

for some $r_1, \ldots, r_k \neq 0$. Since $e_k = e$ as γ is a loop, we get $a^{r_1}b^{r_2}\cdots a^{r_{k-1}}b^{r_k}e = e$. Since $\mathrm{Stab}(e) = \{1\}$, it follows that $a^{r_1}b^{r_2}\cdots a^{r_{k-1}}b^{r_k}$ must be trivial in A_Γ. But it is also a non-trivial word, as γ is not homotopically trivial. By [1, Lemma 6], we must have $k \geqslant 2m_{ab}$. Hence, the combinatorial length of γ is $|\gamma| = k \geqslant 2m_{ab}$. □

Proof of Theorem 4.1.5: We will prove by induction on the number $|V(\Gamma)|$ of generators of the Artin groups A_Γ that their associated Artin complexes X_Γ are $2k$-systolic.

If $|V(\Gamma)| = 3$, we know from Lemma 4.1.3 that X_Γ is connected and simply connected. It only remains to show that for all $g \in A_\Gamma$, for all induced subgraph $\Gamma' \subsetneq \Gamma$, the simplex $g \cdot \Delta_{\Gamma'}$ is such that $Lk_{X_\Gamma}(g \cdot \Delta_{\Gamma'})$ is $2k$-large. If $|V(\Gamma')| = 2$, then the link $Lk_{X_\Gamma}(g \cdot \Delta_{\Gamma'})$ is isomorphic to the Artin complex $X_{\Gamma'}$ associated to the Artin group $A_{\Gamma'}$ (Lemma 4.1.4), and the latter is $2k$-large by Lemma 4.1.6. The cases $|V(\Gamma')| = 0$ or 1 are trivial.

Let us now assume that $|V(\Gamma)| > 3$ and that every Artin complex $A_{\Gamma'}$ with Γ' an induced subgraph of Γ is $2k$-systolic. Again, we know from Lemma 4.1.3 that X_Γ is connected and simply connected, so it only remains to show that for all $g \in A_\Gamma$, for all induced subgraph $\Gamma' \subsetneq \Gamma$, the simplex $g \cdot \Delta_{\Gamma'}$ is such that $Lk_{X_\Gamma}(g \cdot \Delta_{\Gamma'})$ is $2k$-large. If $|V(\Gamma')| \geqslant 2$, then $Lk(g \cdot \Delta_{\Gamma'}, X_\Gamma)$ is isomorphic to the Artin complex $X_{\Gamma'}$ associated to the Artin group $A_{\Gamma'}$ (Lemma 4.1.4). The latter is $2k$-systolic by the induction hypothesis, hence is $2k$-large as well ([62, Proposition 1.4]). Once again, the cases $|V(\Gamma')| = 0$ or 1 are trivial. □

4.2 Intersection of parabolic subgroups

The aim of this section is to use the systolicity of the Artin complex of an Artin group of large type to prove Theorem 4.1. Most of the work will be to prove the Theorem 4.2.2 below.

Definition 4.2.1. Let P_1 and P_2 be two parabolic subgroups of an Artin group A_Γ such that $P_1 \subseteq P_2$. We say that P_1 is a parabolic subgroup of P_2 if $P_1 \subseteq P_2$ is conjugate to an inclusion of standard parabolic subgroups $A_{\Gamma''} \subseteq A_{\Gamma'}$.

Theorem 4.2.2. *Let A_Γ be an Artin group of large-type. Then:*
(1) The intersection of two parabolic subgroups of A_Γ is again a parabolic subgroup of A_Γ.
(2) If P_1 and P_2 are two parabolic subgroups of A_Γ such that $P_1 \subseteq P_2$, then P_1 is a parabolic subgroup of P_2.

Note that the second item in the previous theorem is already a result of [42]. However, we believe the reader may be interested in recovering this result directly from our perspective.

First notice that the Artin complex allows us to understand geometrically the parabolic subgroups of A_Γ, via the following correspondence:

Lemma 4.2.3. *Let A_Γ be an Artin group of rank at least 3 and let X_Γ be its associated Artin complex. Then:*
(1) The strict parabolic subgroups of A_Γ are exactly the stabilisers of simplices of X_Γ.
(2)Let Δ be a simplex of X_Γ. The parabolic subgroups of $Stab_{X_\Gamma}(\Delta)$ are exactly the stabilisers of the simplices that contain Δ.

Proof: By construction, every strict standard parabolic subgroup $A_{\Gamma'}$ is precisely the stabiliser of some simplex $\Delta_{\Gamma'}$ lying on the fundamental domain S_Γ of X_Γ, and vice versa. Moreover, any parabolic subgroup of the form $gA_{\Gamma'}g^{-1}$ is the stabiliser of the simplex $g \cdot \Delta_{\Gamma'}$ for some $g \in A_\Gamma$. To prove the first claim, notice that any simplex of X_Γ can be expressed as $g' \cdot \Delta'$, where Δ' is in S_Γ and $g' \in A_\Gamma$.

Let us now prove the second claim. On the one hand, let P be a parabolic subgroup of $Stab_{X_\Gamma}(\Delta)$. Up to conjugation, we can suppose that Δ lies in S_Γ and that P is the stabiliser of a simplex Δ' that also lies in K. Now notice that, by construction of the fundamental domain, this implies that Δ' contains Δ, as we desired. On the other hand, note that if Δ'' is a simplex that contains Δ, then we can find an element $g \in A_\Gamma$ such that $g \cdot \Delta''$ belongs to S_Γ. Hence $g' Stab_{X_\Gamma}(\Delta'')g'^{-1} \subseteq g' Stab_{X_\Gamma}(\Delta)g'^{-1}$ is an inclusion of standard parabolic subgroups, as we wanted to prove. □

Remark 4.2.4. The previous correspondence is not a bijection between the parabolic subgroups of A_Γ and the simplices of its Artin complex, as two distinct simplices may have the same stabiliser.

Secondly, we mention the following result from systolic geometry that will be used in our proof:

Lemma 4.2.5. *Let G be a group acting without inversions on a systolic complex Y, and let H be a subgroup of G. Suppose that H fixes two vertices v and v' of Y. Then H fixes pointwise every combinatorial geodesic between v and v_0.*

Proof: We prove the result by induction on the combinatorial distance between v and v'. If $d(v, v') = 1$, the result is immediate, as there is a unique edge between v and v'. Suppose by induction that the result is true for vertices at distance at most $n \geqslant 1$, and let v, v' be two vertices of Y at distance $n+1$. Since Y is systolic, it follows from [62, Corollary 7.5] that the combinatorial ball of radius n around v', denoted $B_Y(v', n)$, is a convex subset of Y in the sense of [62, Definition 7.1]. Moreover, by [62, Lemma 7.7], this combinatorial ball intersects the combinatorial ball $B_Y(v, 1)$ along a single simplex. This implies that there exists a simplex Δ of Y containing v, and such that every combinatorial geodesic from v to v' starts with an edge of Δ. In particular, we define Δ' as the simplex of Y spanned by the first edges of all the combinatorial geodesics from v to v'. Since H fixes v and v', H preserves the set of combinatorial geodesics from v to v', and in particular H stabilises Δ'. Since G acts on Y without inversion, it follows that H fixes Δ' pointwise.

Let γ be a combinatorial geodesic from v to v'. By the above, H fixes the first edge e of γ. Let v_1 be the vertex of e distinct from v. We have that H fixes v_1 and v', and these two vertices are at combinatorial distance n. By the induction hypothesis, H fixes pointwise the portion of γ between v_1 and v', and it now follows that H fixes pointwise all of γ. This concludes the induction. □

Proof of Theorem 4.2.2: The two points of the Theorem are trivial if one of the two parabolic subgroups is either the whole group or trivial. So we suppose both parabolic subgroups are proper. We will prove the theorem by induction on the rank n of A_Γ.

If $n = 2$, A_Γ is an Artin group on two generators a, b and there are two cases to consider. If $m_{ab} < \infty$, then A_Γ is a spherical Artin group, so the first point of the theorem follows from [23, Theorem 9.5] and the second point of the theorem follows from [40, Theorem 0.2]. If $m_{ab} = \infty$, then A_Γ is a free group on two generators a, b. Moreover, the proper parabolic subgroups are either trivial or infinite cyclic. Since the action of A_Γ on the Bass-Serre tree associated to the splitting $\langle a \rangle * \langle b \rangle$ has trivial edge stabilisers, it follows that two distinct proper parabolic subgroups intersect trivially. Thus the two points of the theorem follow immediately.

Let us now assume that the result is known for large-type Artin groups of rank $\leqslant n$ with $n \geqslant 2$, and let A_Γ be a large-type Artin group of rank $n + 1$. Let X_Γ be its associated Artin complex.

<u>Claim 1:</u> Let $e_1, \dots, e_k$ be a combinatorial path p in X_Γ. Then there exists a simplex Δ of X_Γ containing the edge e_k such that

$$\bigcap_{1 \leqslant i \leqslant k} Stab_{X_\Gamma}(e_i) = Stab_{X_\Gamma}(\Delta).$$

Proof of Claim 1: We will prove the claim by induction on k. If $k = 1$, p is just the edge e_1 and the proof is trivial. Now suppose that the claim is true for k and let us prove it for $k + 1$. By applying the induction hypothesis to the subpath $e_1, \dots, e_k$, we will then have

$$\bigcap_{1 \leqslant i \leqslant k+1} Stab_{X_\Gamma}(e_i) = Stab_{X_\Gamma}(\Delta') \cap Stab_{X_\Gamma}(e_{k+1}),$$

where Δ' is a simplex containing the edge e_k. Let v be a vertex contained in both e_k and e_{k+1}. By Lemma 4.2.3, this means that both $Stab_{X_\Gamma}(\Delta')$ and $Stab_{X_\Gamma}(e_{k+1})$ are parabolic subgroups of $Stab_{X_\Gamma}(v)$. Also, up to conjugacy, $Stab(v)$ is an Artin group on n generators. Therefore, by the induction hypothesis on n, $Stab_{X_\Gamma}(\Delta') \cap Stab_{X_\Gamma}(e_{k+1})$ is a parabolic subgroup of $Stab(v)$ contained in $Stab_{X_\Gamma}(e_{k+1})$, so it is a parabolic subgroup of $Stab_{X_\Gamma}(e_{k+1})$. Geometrically, $Stab_{X_\Gamma}(\Delta') \cap Stab_{X_\Gamma}(e_{k+1})$ is the stabiliser of some simplex containing e_{k+1}. This finishes the proof of Claim 1.

Claim 2: Let Δ_1 and Δ_2 be two simplices of X_Γ. Then there exists a simplex Δ of X_Γ containing Δ_2 such that $Stab_{X_\Gamma}(\Delta_1) \cap Stab_{X_\Gamma}(\Delta_2) = Stab_{X_\Gamma}(\Delta)$.

Proof of Claim 2: Let Δ' be any simplex of X_Γ and let $V_{\Delta'}$ be the set of vertices of Δ'. As the action of A_Γ on X_Γ is without inversions, we have that $Stab_{X_\Gamma}(\Delta') = \cap_{w \in V_{\Delta'}} Stab(w)$. Define a combinatorial path p that is the concatenation of the three following paths: a combinatorial path p_1 that travels along every vertex in V_{Δ_1}; a combinatorial geodesic p_2 between the endpoint of p_1 and V_{Δ_2}; and a combinatorial path that starts in the endpoint of p_2 and travels along every vertex in V_{Δ_2}. Denote the endpoint of p by v and let E_p be the set of edges of p. Then, using Claim 1 and Lemma 4.2.5, we obtain

$$Stab_{X_\Gamma}(\Delta_1) \cap Stab_{X_\Gamma}(\Delta_2) = \bigcap_{w \in V_{\Delta_1} \cup V_{\Delta_2}} Stab_{X_\Gamma}(w) = \bigcap_{e \in E_p} Stab_{X_\Gamma}(e) = Stab_{X_\Gamma}(\Delta),$$

for some simplex Δ containing v. Now we need to show that Δ contains also Δ_2. Notice that $Stab_{X_\Gamma}(\Delta_2)$ contains $Stab_{X_\Gamma}(\Delta)$ and both $Stab_{X_\Gamma}(\Delta_2)$ and $Stab_{X_\Gamma}(\Delta)$ are parabolic subgroups of $Stab_{X_\Gamma}(v)$. This group is, up to conjugacy, an Artin group on n generators. So by using the induction hypothesis on n, $Stab_{X_\Gamma}(\Delta)$ is a parabolic subgroup of $Stab_{X_\Gamma}(\Delta_2)$, which means that we can choose Δ to contain Δ_2. This finishes the proof of Claim 2.

In particular, note that Claim 2 together with Lemma 4.2.3 implies that the parabolic subgroups of A_Γ are stable under intersection, proving the first point of the Theorem.

Claim 3: For every pair of simplices Δ_1 and Δ_2 of X_Γ such that $Stab_{X_\Gamma}(\Delta_1) \subseteq Stab_{X_\Gamma}(\Delta_2)$, there exists a simplex Δ of X_Γ containing Δ_2 such that $Stab_{X_\Gamma}(\Delta_1) = Stab_{X_\Gamma}(\Delta)$.

Proof of Claim 3: Just notice that $Stab_{X_\Gamma}(\Delta_1) = Stab_{X_\Gamma}(\Delta_1) \cap Stab_{X_\Gamma}(\Delta_2)$, so by Claim 2 there is a simplex Δ containing Δ_2 such that $Stab_{X_\Gamma}(\Delta_1) = Stab_{X_\Gamma}(\Delta)$. This finishes the proof of Claim 3.

We now explain why this claim implies that A_Γ satisfies the second point of the Theorem. Let P_1 and P_2 be two parabolic subgroups of A_Γ such that $P_1 \subseteq P_2$. By Lemma 4.2.3 there are simplices Δ_1 and Δ_2 of A_Γ such that $P_1 = Stab_{X_\Gamma}(\Delta_1)$ and $P_2 = Stab_{X_\Gamma}(\Delta_2)$. By Claim 3, there exists a simplex Δ of X_Γ containing Δ_2 such that $Stab_{X_\Gamma}(\Delta_1) = Stab_{X_\Gamma}(\Delta)$. Again by Lemma 4.2.3, this means that P_1 is a parabolic subgroup of P_2, as we wanted to prove. □

Remark 4.2.6. Notice that the only place where the systolic geometry was used in the previous proof is the argument coming from Lemma 4.2.5 that says that if an element fixes two simplices, then it fixes pointwise a combinatorial path between these simplices. Therefore, a strong enough requirement to prove Theorem 4.2.2 for any Artin group A_Γ is to have this fixing-path condition in its Artin complex X_Γ.

We can generalise some interesting results concerning parabolic subgroups that were previously shown for spherical Artin groups ([23]):

Corollary 4.2.7. *Let A_Γ be an Artin group of large type. Then, an arbitrary intersection of parabolic subgroup of A_Γ is a parabolic subgroup. In particular,*
(1) For a subset $B \subset A_\Gamma$, there is a unique minimal parabolic subgroup of A_Γ (with respect to the inclusion) containing B ;
(2) The set of parabolic subgroups of A_Γ is lattice with respect to the inclusion.

The strategy will be the same standard argument used in [23]. We can find the generalised FC version of the first statement for spherical parabolic subgroups in [73].

Proof of Corollary 4.2.7: Let $\mathcal{P}$ be an arbitrary set of parabolic subgroups of A_Γ and let $Q := \bigcap_{P \in \mathcal{P}} P$. The set Q is contained in every parabolic subgroup in $\mathcal{P}$, so by Theorem 4.2.2, we just need to prove that Q is equivalent to a finite intersection of parabolic subgroups. Notice that every parabolic subgroup is expressed as the conjugate of some standard parabolic subgroup. Since A_Γ is a countable group and standard parabolic subgroups of A_Γ are finitely generated, the set of parabolic subgroups of A_Γ is countable. In particular, $\mathcal{P}$ is countable. Enumerate the elements in $\mathcal{P} = \{P_1, P_2, P_3, \dots\}$ and let

$$Q_m = \bigcap_{1 \leqslant i \leqslant m} P_i.$$

By Theorem 4.2.2, all Q_m's belong to $\mathcal{P}$. As $Q = \cap_{i\in\mathbb{N}} Q_m$, we need to show that the set $\{Q_m \mid m \in \mathbb{N}\}$ is finite.

Let X_Γ be the Artin complex of A_Γ. Notice that we have a descending chain

$$Q_1 \supseteq Q_2 \supseteq Q_3 \supseteq \dots .$$

By doing an induction on the Claim 3 in the proof of Theorem 4.2.2, one can easily see that if $Stab_{X_\Gamma}(\Delta_1) \supsetneq Stab_{X_\Gamma}(\Delta_2) \supsetneq Stab_{X_\Gamma}(\Delta_3) \dots$, the dimension of Δ_i has to be strictly bigger than the dimension of Δ_{i-1}. As the dimension of X_Γ is finite, the chain cannot be infinite. Therefore, Q is the minimal parabolic subgroup on $\mathcal{P}$.

We now prove the two statements of Corollary 4.2.7. To see the first statement, just assume that $\mathcal{P} = \{P \mid B \subset P\}$. For the second statement let P_1 and P_2 be any two parabolic subgroups of A_Γ. We need a maximal parabolic subgroup R_1 contained in P_1 and P_2 and a minimal parabolic subgroup R_2 containing P_1 and P_2. By all the previous discussion, we can set $R_1 = P_1 \cap P_2$ and R_2 is the minimal parabolic subgroup in $\mathcal{P}$ when $\mathcal{P} = \{P \mid P_1 \cup P_2 \subseteq P\}$. $\square$

4.3 Normalisers and fixed-point sets of parabolic subgroups

The goal of this section is to prove Theorem 4.5. In all this section we consider an Artin group A_Γ of rank at least 3. For a parabolic subgroup P of A_Γ, we denote by $Fix(P)$ (or $Fix_{X_\Gamma}(P)$ if we wish to highlight the ambient complex) the fixed-point set of P in X_Γ. Since A_Γ acts on X_Γ without inversions, $Fix(P)$ is a subcomplex of X_Γ. The connection between the normaliser $N(P)$ of a parabolic subgroup P and its fixed-point set $Fix(P)$ is given by the following:

Lemma 4.3.1. *Let P be a parabolic subgroup of A_Γ. Then the normaliser $N(P)$ of P satisfies*

$$N(P) = Stab(Fix(P)).$$

In addition, an element of A_Γ belongs to $N(P)$ if and only if it sends some maximal simplex of $Fix(P)$ to some maximal simplex of $Fix(P)$.

Proof: ($\subseteq$) Let $g \in N(P)$, that is, $gP = Pg$, and let $v \in Fix(P)$. Then

$$P \cdot (g \cdot v) = g \cdot (P \cdot v) = g \cdot v.$$

In particular, $g \cdot v \in Fix(P)$ and thus $g \in Stab(Fix(P))$. ($\supseteq$) Let $g \in Stab(Fix(P))$ and let $\Delta \subseteq Fix(P)$ be a maximal simplex in the sense that $Stab(\Delta) = P$. Then $g \cdot \Delta \in Fix(P)$, thus

$$P \cdot (g \cdot \Delta) = g \cdot \Delta.$$

In particular, gPg^{-1} fixes Δ, hence $gPg^{-1} \subseteq P$. In other words, $g \in N(P)$. □

The key geometric result to prove Theorem 4.5 by means of studying fixed-point sets is the following:

Proposition 4.3.2. *Let A_Γ be a large-type Artin groups, and let P be a parabolic subgroup of A_Γ of type S'.*

- *If $|S'| \geqslant 2$, then $Fix(P)$ is a single simplex.*
- *If $|S'| = 1$, then $Fix(P)$ is a subcomplex whose dual graph is a simplicial tree (see Definition 4.3.8 for the terminology).*

The proof of the above proposition will be split into two cases. We first mention a useful observation that will allow for proofs by induction:

Lemma 4.3.3. *For a simplex Δ of $Fix(P)$ of type Γ'', the link $Lk_{Fix(P)}(\Delta)$ is isomorphic to $Fix_{X_{\Gamma''}}(P)$.*

Proof: We have $Lk_{Fix(P)}(\sigma) = Fix(P) \cap Lk_{X_\Gamma}(\sigma)$. Since $Lk_{X_\Gamma}(\sigma)$ is equivariantly isomorphic to $X_{\Gamma''}$ by Lemma 4.1.4, the previous intersection is thus isomorphic to $Fix_{X_{\Gamma''}}(P)$. □

We start with the case of a parabolic subgroup P of type at least 2.

Lemma 4.3.4. *Let $A_{\Gamma'}$ be a standard parabolic subgroup of type at least 2. Then $Fix(A_{\Gamma'})$ is a single simplex Δ such that $Stab(\Delta) = A_{\Gamma'}$.*

Proof: We begin with the following claim:

Claim: If a subcomplex Y of X_Γ is such that all of its links are simplices or empty, then Y itself is a simplex.

Proof of the Claim: If Y is not a simplex, then it contains a combinatorial path u, v, w that forms a geodesic of X_Γ. The two vertices u, w define two vertices of $Lk_Y(v)$ at distance at least 2 by assumption, hence $Lk_Y(v)$ is not a simplex, which proves the Claim.

Recall from Lemma 4.3.3 that for a simplex Δ of $Fix(P)$ corresponding to a simplex of type Γ'', the link $Lk_{Fix(P)}(\Delta)$ is isomorphic to $Fix_{X_{\Gamma''}}(P)$. If $|V(\Gamma)\backslash V(\Gamma')| = 1$, then $Fix(P)$ must be a single vertex v, as if it weren't, it would follow from the convexity of $Fix(P)$ (Lemma 4.2.5) that P fixes an edge of X_Γ, which is impossible since in that case P is a maximal proper parabolic subgroup of A_Γ. $Fix(A_{\Gamma'})$ being a single simplex now follows by induction on $|V(\Gamma)\backslash V(\Gamma')| \geqslant 1$ by applying the above Claim. The dimension of $Fix(A_{\Gamma'})$ is $|V(\Gamma)\backslash V(\Gamma')| - 1$, so by maximality its stabiliser has to be $A_{\Gamma'}$. □

Corollary 4.3.5. *If P is a parabolic subgroup of A_Γ of type at least 2, then $N(P) = P$.*

Proof: By Lemma 4.3.1 we know that $N(P) = Stab(Fix(P))$. Moreover, we know from Lemma 4.3.4 that there is a simplex Δ in X_Γ such that $Fix(P) = \Delta$ and $Stab(\Delta) = P$. In particular,

$$N(P) = Stab(Fix(P)) = Stab(\Delta) = P.$$

□

We now move to the case of a parabolic subgroup of type 1. We start with the following general observation:

Lemma 4.3.6. *Let P be a parabolic subgroup of A_Γ. Then $Fix(P)$ is contractible.*

The proof of this lemma will rely on the following notion of convexity from [62]:

Definition 4.3.7. A subcomplex Y of a simplicial complex X is **3-convex** if every pair of vertices of Y that are adjacent in X are adjacent in Y, and every combinatorial geodesic of length 2 with endpoints in Y is contained in Y. It is **locally 3-convex** if for every simplex σ of Y, the link $Lk_Y(\sigma)$ is 3-convex in $Lk_X(\sigma)$.

Proof of Lemma 4.3.6: By Lemma 4.2.5, $Fix(P)$ contains every geodesic between two vertices of $Fix(P)$. In particular, it is connected and 3-convex, hence locally 3-convex by [62, Fact 3.3.1]. By [62, Lemma 7.2], $Fix(P)$ is thus contractible. □

It turns out that such fixed-point sets have a very simple geometry. We introduce the following:

Definition 4.3.8. Le P be a parabolic subgroup of type 1 of A_Γ. The **dual graph** T_P of $Fix(P)$ is defined as follows:

- Vertices of T_P correspond to the simplices of $Fix(P)$ of type Γ' with $|V(\Gamma')| = 1$ (called **type 1 vertices**) or of type Γ' with $|V(\Gamma')| = 2$ (called **type 2 vertices**).
- We put an edge between a type 1 vertex Δ and a type 2 vertex Δ' whenever $\Delta' \subset \Delta$.
- Finally, T_P is the subgraph obtained by removing the type 2 vertices that have valence 1.

We think of T_P as a subgraph of the first barycentric subdivision of $Fix(P)$.

Lemma 4.3.9. *The dual graph T_P is a simplicial tree.*

In a nutshell, the proof of Lemma 4.3.9 goes as follows: We construct a sequence of subcomplexes

$$X_0 \supsetneq X_1 \supsetneq \cdots \supsetneq X_k,$$

where X_0 is the first barycentric subdivision of $Fix(P)$ and $X_k = T_P$, and such that for each $0 \leqslant i \leqslant k-1$, X_{i+1} is a deformation retract of X_i. Since X_0 is contractible by Lemma 4.3.6, it will then follow that the graph T_P is also contractible, hence is a tree.

We will need the following standard result from algebraic topology to construct deformation retractions:

Lemma 4.3.10. *Let X be a simplicial complex, and let v be a vertex of X whose link $Lk_X(v)$ is contractible. Then the subcomplex spanned by $X - v$ is a deformation retract of X.*

Proof: Since the star $St_X(v)$ is isomorphic to a cone over $Lk_X(v)$, we first notice that X is obtained from $X - v$ by coning-off the contractible link $Lk_X(v)$. Recall that for a simplicial complex Y and a contractible subcomplex Z, the quotient map $Y \to Y/Z$ obtained by collapsing Z to a point is a homotopy equivalence (see [52, Proposition 0.17]). We thus have the following commutative diagram:

$$\begin{array}{ccc} X - v & \hookrightarrow & X \\ \downarrow & & \downarrow \\ (X-v)/Lk_X(v) & = & X/St_X(v) \end{array}$$

where both vertical arrows are homotopy equivalences since $Lk_X(v)$ and its cone $St_X(v)$ are contractible. Thus, the inclusion $X - v \hookrightarrow X$ is a homotopy equivalence, and it follows from [52, Corollary 0.20] that the subcomplex spanned by $X - v$ is a deformation retract of X. □

Proof of Lemma 4.3.9: Consider the barycentric subdivision $Fix(P)'$ of $Fix(P)$. A vertex v of $Fix(P)'$ corresponds to a simplex of $Fix(P)$; We will call the dimension of the corresponding simplex the *height* of v. For every $0 \leqslant k \leqslant |V(\Gamma)| - 2$, we define the subcomplex X_k of $Fix(P)'$ spanned by the vertices of height at least k. In particular, $X_0 = Fix(P)'$ and $X_{|V(\Gamma)|-2}$ is a subgraph of $Fix(P)'$ containing T_P. We now show that for every $0 \leqslant k \leqslant |V(\Gamma)| - 3$, X_{k+1} is a deformation retract of X_k. Notice that X_k is obtained from X_{k+1} by adding for every vertex v of height k the star $St_{X_k}(v)$, which is isomorphic to a simplicial cone over the link $Lk_{X_k}(v)$. Let v be a vertex of height $0 \leqslant k \leqslant |V(\Gamma)| - 3$. This vertex corresponds to a simplex Δ of $Fix(P)$ of type Γ' for some induced subgraph $\Gamma' \subsetneq \Gamma$ with $|V(\Gamma)'| \geqslant 3$. Note that a vertex of X_k adjacent to v must have height greater than k by construction, hence the link $Lk_{X_k}(v)$ is isomorphic to the first barycentric subdivision of $Lk_{Fix(P)}(\Delta)$. In particular, $Lk_{X_k}(v)$ is isomorphic to the first barycentric subdivision of $Fix_{X_{\Gamma'}}(P)$ by Lemma 4.3.3, and hence is contractible by Lemma 4.3.6. It thus follows from Lemma 4.3.10 that X_{k+1} is a deformation retract of $X_{k+1} \cup St_{X_k}(v)$. Since for two distinct vertices v, v' of height k, the subcomplexes $X_{k+1} \cup St_{X_k}(v)$ and $X_{k+1} \cup St_{X_k}(v')$ intersect along X_{k+1}, we can glue the various deformation retractions into a deformation retraction of

$$X_k = X_{k+1} \cup \bigcup_{\text{height}(v)=k} St_{X_k}(v)$$

onto X_{k+1}. Thus, for every $0 \leqslant k \leqslant |V(\Gamma)| - 3$, X_{k+1} is a deformation retract of X_k. Thus, the graph $X_{|V(\Gamma)|-2}$ is a deformation retract of $X_0 = Fix(P)'$. Since the latter complex is contractible by Lemma 4.3.6, so is the graph $X_{|V(\Gamma)|-2}$, and it follows that $X_{|V(\Gamma)|-2}$ is a tree. Finally, T_P is obtained from $X_{|V(\Gamma)|-2}$ by removing the type 2 vertices that have valence 1. Thus, T_P is a deformation retract of $X_{|V(\Gamma)|-2}$, hence T_P is a tree. □

Note that since $N(P) = Stab(Fix(P))$ by Lemma 4.3.1, $N(P)$ acts on $Fix(P)$, hence on the dual tree T_P. We will use this action to prove the following:

Lemma 4.3.11. *The normaliser $N(P)$ of P splits as a direct product $P \times F$, where F is a finitely generated free group.*

Remark 4.3.12. It can be shown that the tree T_P is $N(P)$-equivariantly isomorphic to the standard tree associated to P as considered in [75, Definition 4.1]. In particular, the proof of Lemma 4.3.11 is essentially the same as the proof of [75, Lemma 4.5]. We however include a proof formulated in our setting for the sake of self-containment.

Since P is a normal subgroup of $N(P)$ acting trivially on T_P by construction of $Fix(P)$, we can look at the induced action of $N(P)/P$ on T_P. We will use this action to completely describe the normaliser $N(P)$. We first need the following result:

Lemma 4.3.13. *For the action of $N(P)/P$ on T_P we have:*

- *Type 1 vertices of T_P have a trivial stabiliser.*
- *Type 2 vertices of T_P have an infinite cyclic stabiliser.*

Proof: We first recall that the centre of a dihedral Artin group A_{ab} with $3 \leqslant m_{ab} < \infty$ is an infinite cyclic subgroup, whose generator is a power of the Garside element Δ_{ab}, as described in Definition 3.1.6.

A type 1 vertex v of T_P corresponds to a maximal simplex of $Fix(P)$. Such a simplex has stabiliser P by construction, hence $Stab_{N(P)/P}(v)$ is trivial. Let v be a type 2 vertex of T_P of type A_{cd}. This vertex corresponds to a simplex with associated coset gA_{cd} for some $g \in A_\Gamma$. It follows from [75, Lemma 4.5] and the structure of the centre of dihedral Artin groups that we have:

- If m_{cd} is even, then

$$Stab_{N(P)/P}(v) = gZ(A_{cd})g^{-1} = \langle g\Delta_{cd}g^{-1}\rangle;$$

- If m_{cd} is odd, then

$$Stab_{N(P)/P}(v) = gZ(A_{cd})g^{-1} = \langle g\Delta_{cd}^2g^{-1}\rangle.$$

□

We are now ready to prove Lemma 4.3.11:

Proof of Lemma 4.3.11: Since two type 1 vertices of T_P corresponding to cosets of the same standard parabolic subgroup are in the same $N(P)$-orbit, hence in the same $N(P)/P$ orbit, it follows that the action of $N(P)/P$ on T_P is cocompact. Thus, $N(P)$ acts cocompactly and without inversion on a simplicial tree. By Lemma 4.3.13 the stabilisers of type 1 vertices are trivial (hence so are the stabilisers of edges) and the stabilisers of type 2 vertices are infinite cyclic. It thus follows from Bass-Serre theory that $N(P)/P$ is a finitely-generated free group, and thus $N(P)$ splits as a direct product $P \times F$, where F is a finitely generated free group. □

We now move towards finding an explicit basis of these normalisers. Finding an explicit basis for the free subgroup appearing in Theorem 4.5 is now a standard application of Bass-Serre theory, which was stated as a remark without further justification in [75, Remark 4.6]. We first start by describing a fundamental domain for the action, as well as the quotient space $T_P/N(P)$.

Definition 4.3.14. Let Γ' be the first barycentric subdivision of Γ. Recall that a vertex of Γ' corresponding to a generator a of A_Γ will is denoted v^a and is said to be of **type 1**, while a vertex of Γ' corresponding to an edge of Γ between generators a and b will be denoted v^{ab} and will be said to be of **type 2**. Let $\Gamma_{a,\text{odd}}$ denote the maximal connected subgraph of Γ that contains the vertex a and only odd-labelled edges. Let Γ_P be the graph obtained from the disjoint union of all the edges of Γ' that contain a vertex of $\Gamma_{a,\text{odd}}$, by the following identification. If such an edge e (e' respectively) of Γ' contains a vertex v (v' respectively) such that v, v' correspond to the same vertex of $\Gamma_{a,\mathrm{odd}}$, then v and v' are identified and define the same vertex of Γ_P.

Some examples of the graph Γ_P are given in Figure 4.1, when the underlying defining graph is a triangle.

Definition 4.3.15. Let e be an edge of Γ_P between a type 1 vertex v^c and a type 2 vertex v^{cd}, for c, d spanning an edge of Γ. We denote by $\widetilde{e}$ the edge of T_P between the vertex $\langle c \rangle$ and the vertex A_{cd}. Choose an orientation of each edge of Γ. For each oriented loop of Γ_P based at v^a, we denote by $e_1, \ldots, e_n$ the oriented sequences of edges of Γ crossed by γ, and we define

$$g_\gamma := \Delta_{e_1}^{\pm 1} \cdots \Delta_{e_n}^{\pm 1},$$

where the sign for each Garside element Δ_{e_i} depends on whether γ follows the orientation of e_i.

We now choose a spanning tree τ of Γ_P, which we think of as being based at v^a. For a vertex v of Γ_P, we denote γ_v the oriented geodesic of τ from v^a to v. Let e be an edge of Γ_P. If e is contained in τ, let v be the vertex of e closest to v^a in τ. If e is not contained in τ, let v be the vertex of e closest to v^a in Γ_P (as Γ_P is bipartite). We denote $g_v := g_{\gamma_v}$, and we set

$$Y_P := \bigcup_{e \subset \Gamma_P} g_v \widetilde{e}.$$

This defines a connected subtree of T_P.

Lemma 4.3.16. *The subtree Y_P is a fundamental domain for the action of $N(P)$ on T_P, and the quotient $T_P/N(P)$ is isomorphic to Γ_P.*

Proof: An edge of T_P corresponds to a pair consisting of a maximal simplex of T_P (of type c for some $c \in V(\Gamma)$) and one of its codimension 1 faces (of type cd for some $d \in V(\Gamma)$ adjacent to c). We thus mention the following useful fact, which is an immediate consequence of Lemma 4.3.1:

Fact: Two edges of T_P in the same A_Γ-orbit are also in the same $N(P)$-orbit. Let us first show that Y_P is a fundamental domain for the action of $N(P)$ (and

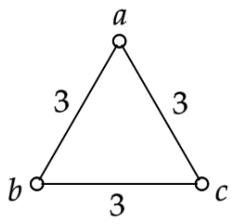
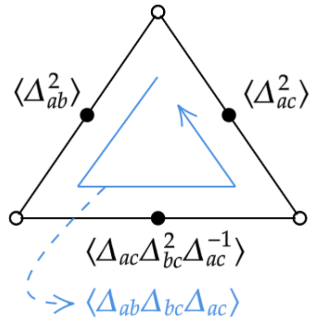
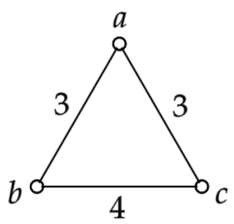
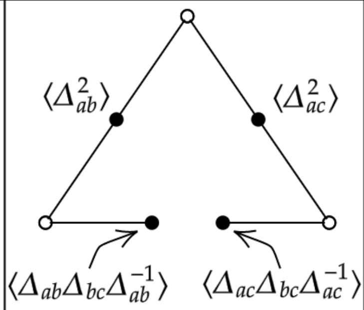
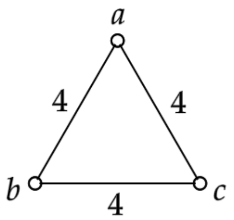
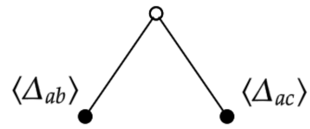
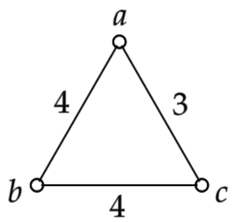
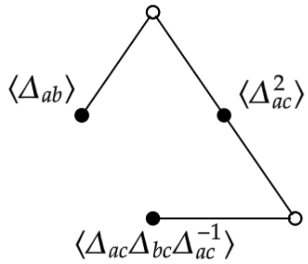
Defining graph Γ	Induced graph of groups on Γ_p	Rank and basis of F, with $N(P) \cong P \times F$
		The rank of F is 4 and a basis of F is given by: $\{\Delta_{ab}^2, \Delta_{ac}^2, \Delta_{ac}\Delta_{bc}^2\Delta_{ac}^{-1}, \Delta_{ab}\Delta_{bc}\Delta_{ac}\}$
		The rank of F is 4 and a basis of F is given by: $\{\Delta_{ab}^2, \Delta_{ac}^2, \Delta_{ab}\Delta_{bc}\Delta_{ab}^{-1}, \Delta_{ac}\Delta_{bc}\Delta_{ac}^{-1}\}$
		The rank of F is 2 and a basis of F is given by: $\{\Delta_{ab}, \Delta_{ac}\}$
		The rank of F is 3 and a basis of F is given by: $\{\Delta_{ab}, \Delta_{ac}^2, \Delta_{ac}\Delta_{bc}\Delta_{ac}^{-1}\}$

Figure 4.1: Examples of computations of normalisers of the parabolic subgroup $P = \langle a \rangle$, for various large-type Artin groups of rank 3. Type 2 vertices of Γ_P are indicated in bold in the second column and come with their infinite cyclic stabilisers. The group element in blue corresponds to the element of a basis of F coming from the fundamental group of Γ_P. Note that the structure of the normaliser for large-type Artin groups of rank 3 depends only on the parity of the labels and not on the labels themselves, so the above cases cover all possible cases.

hence $N(P)/P$ on T_P. The fact that Y_P is connected, hence a subtree of T_P, is a consequence of the construction. By construction of the various edges $\widetilde{e}$, it thus follows that the edges of Y_P are in different A_Γ-orbits, and in particular in different $N(P)$-orbits. Now let e be an edge of T_P. Its type 1 vertex is of type c, for some $c \in V(\Gamma)$ such that $\langle c \rangle$ and $\langle a \rangle$ are conjugated. It thus follows from [82] that $c \in V(\Gamma_{a,odd})$, and it then follows that e is in the A_Γ-orbit, hence the $N(P)$-orbit, of an edge of Y_P. Thus, Y_P is a fundamental domain for the action of $N(P)$ (and hence $N(P)/P$ on T_P. We now want to study the quotient space

$T_P/N(P)$. Let us analyse the action of $N(P)/P$ on T_P at a local level. Let v be a vertex of T_P of type $c \in V(\Gamma)$. By the above remark, we will assume up to to the action of $N(P)$ that this vertex corresponds to the codimension 1 simplex of X_Γ corresponding to $g_v\langle c \rangle$. By construction of T_P, the codimension 1 faces of Δ that correspond to a type 2 vertex of T_P adjacent to v are the simplices corresponding to the parabolic subgroups $g_v A_{cd}$ with d connected to c in Γ. Let v be a vertex of T_P of type A_{cd} where c, d span an edge of Γ. Up to the action of $N(P)$, we will assume that this vertex corresponds to the simplex with associated coset $g_v A_{cd}$. Then it follows from Lemma 4.3.13 that we have:

- If m_{cd} is even, then all the edges of T_P containing v are in the same $\langle \Delta_{cd} \rangle$-orbit.
- If m_{cd} is odd, then there are exactly two $N(P)$-orbits of edges of T_P containing v, corresponding to the $\langle \Delta_{cd}^2 \rangle$-orbits of the maximal simplices of type $\{c\}$ and $\{d\}$ respectively.

The description of the quotient $T_P/N(P)$ now follows from this local description.

□

As mentioned earlier, the fundamental group $N(P)/P$ of this graph of groups over Γ_P is a free group, and by Bass-Serre theory a basis for it is obtained by choosing a generator of each (infinite cyclic) stabiliser of vertex of dihedral type, as well as a family of elements corresponding to a basis of the fundamental group of Γ_P. We now explain how to construct explicitly these elements.

(1) For each vertex v of Y_P of type A_{cd}, a generator of

$$Stab_{N(P)/P}(v) = g_v Z(A_{cd}) g_v^{-1}$$

is given by

$$\begin{cases} g_v \cdot \Delta_{cd}^2 \cdot g_v^{-1} & \text{if } m_{cd} \text{ is odd,} \\ g_v \cdot \Delta_{cd} \cdot g_v^{-1} & \text{otherwise.} \end{cases}$$

(2) A basis of $\pi_1(\Gamma_P)$ is in bijection with the edges of $\Gamma_P - \tau$. Let e be such an edge, joining a type 1 vertex v^c and a type 2 vertex v^{cd}, and let e' be the edge joining v^d and v^{cd}. Then the edges $g_{v^c}\Delta_{cd}^{\pm 1}\widetilde{e}$ and $g_{v^d}\widetilde{e}'$ of Y_P contain two type 2 vertices in the same $N(P)$-orbit, and the geodesic of Y_P between these two vertices project to a loop of Γ_P crossing e exactly once that represents the element

$$g_{v^c} \cdot \Delta_{cd}^{\pm 1} \cdot g_{v^d}^{-1} \in N(P).$$

Note that this element is of the form g_γ, for some combinatorial γ containing e. Thus, a family of elements for point (2) is given by the family of elements g_γ when γ runs over a basis of Γ_P.

We finally obtain the following:

Corollary 4.3.17. *The normaliser $N(P)$ splits as a direct product $N(P) = P \times F$, where F is a finitely-generated free group with a basis given by the following family of elements:*

- *for every vertex v^{cd} of Γ_P, the element*

$$\begin{cases} g_v \cdot \Delta_{cd}^2 \cdot g_v^{-1} & \textit{if } m_{cd} \textit{ is odd,} \\ g_v \cdot \Delta_{cd} \cdot g_v^{-1} & \textit{otherwise.} \end{cases}$$

- *for each combinatorial loop γ based at v_a in a chosen basis of Γ_P, the element g_γ.* □

In Figure 4.1, we give examples for various Artin groups of rank 3 of the normalisers of standard generators.

4.4 Conjugacy stability and root stability

We are now ready to prove Theorem 4.3 and Theorem 4.4. In this section, A_Γ denotes as usual a large-type Artin group of rank at least 3. By Corollary 4.2.7, we can define the following subgroups of A_Γ:

Definition 4.4.1. Let $X \subseteq A_\Gamma$. The minimal parabolic subgroup P_B containing B is called the **parabolic closure** of X.

As is turns out, the parabolic closure of a set $B \subseteq A_\Gamma$ behaves well under conjugacy as illustrated by the following result, which generalises an analogous statement for spherical Artin groups ([23]):

Lemma 4.4.2. *Let $B \subseteq A_\Gamma$ and $\alpha \in A_\Gamma$. Then*

$$P_{\alpha^{-1}B\alpha} = \alpha^{-1}P_B\alpha.$$

In particular, if g and h are conjugate, their parabolic closures correspond to stabilisers of simplices of X_Γ with the same dimension.

Proof: It is obvious that $\alpha^{-1}P_B\alpha$ contains $\alpha^{-1}B\alpha$. We need to prove that this parabolic subgroup is the minimal one containing $\alpha^{-1}B\alpha$. Let Q be any parabolic subgroup containing $\alpha^{-1}B\alpha$. As $\alpha Q\alpha^{-1}$ contains B, $P_B \subseteq \alpha Q \alpha^{-1}$. Therefore, $\alpha^{-1}P_B\alpha \subseteq Q$. □

We are finally able to prove Theorem 4.3, that we restate below for the sake of clarity:

Theorem 4.4.3. *Let $A_{\Gamma'}$ be a standard parabolic subgroup of a large-type Artin group A_Γ. Then $A_{\Gamma'}$ is not conjugacy stable in A_Γ if and only if there exist vertices a and b of Γ' that are connected by an odd-labelled path in Γ and that are not connected by an odd-labelled path in Γ.*

Proof: Let g and h be two elements of $A_{\Gamma'}$ that are conjugated by an element $\alpha \in A_\Gamma$. As $P_g, P_h \subset A_{\Gamma'}$, by Theorem 4.2.2 there must be two induced subgraphs $\Gamma_1, \Gamma_2 \subset \Gamma'$ and $\beta, \beta' \in A_{\Gamma'}$ such that $P_g = \beta^{-1}A_{\Gamma_1}\beta$ and $P_g = \beta'^{-1}A_{\Gamma_2}\beta'$. Since P_g and P_h are conjugate by Lemma 4.4.2, A_{Γ_1} and A_{Γ_2} have to be conjugate. In Section 4.3 we have seen that if A_{Γ_1} has type at least 2 then $\Gamma_1 = \Gamma_2$. Also, if Γ_1 has type 1, then either $\Gamma_1 = \Gamma_2$, or Γ_1 and Γ_2 are vertices of Γ connected by an odd-labelled path in Γ. Thus, there are two possibilities:

(1) Suppose that $P_g = \beta^{-1}A_{\Gamma_1}\beta$ and $P_h = \beta'^{-1}A_{\Gamma_1}\beta'$, with $\Gamma_1 \subseteq \Gamma'$ an induced subgraph and $\beta, \beta' \in A_{\Gamma'}$. Then $(\beta\alpha)^{-1}A_{\Gamma_1}(\beta\alpha) = \beta'^{-1}A_{\Gamma_1}\beta'$ and $\beta\alpha\beta'^{-1}$ normalises A_{Γ_1}. If A_{Γ_1} has rank at least 2, then by Corollary 4.3.5, $N(A_{\Gamma_1}) = A_{\Gamma_1} \subseteq$

$A_{\Gamma'}$, so $\alpha \in A_{\Gamma'}$. If A_{Γ_1} has rank 1 then $g = \beta^{-1}a\beta$ and $h = \beta'^{-1}a\beta'$ for some $a \in V(\Gamma')$, and they are conjugate by $\beta^{-1}\beta' \in A_{\Gamma'}$. (2) Suppose that $g = \gamma^{-1}a^n\gamma$ and $h = \gamma'^{-1}b^n\gamma'$, $\gamma, \gamma' \in A_{\Gamma'}$, where $a, b \in V(\Gamma)$ are connected in Γ by an odd-labelled path. Then, there is an element of A_Γ conjugating a to b. If there is an odd-labelled path in Γ' connecting a to b, then there is an element c in $A_{\Gamma'}$ that conjugates a to b. Thus, $\gamma^{-1}c\gamma'$ conjugates g to h . On the contrary, if there is no such a path in Γ', there is no element in $A_{\Gamma'}$ conjugating a to b. Since the parabolic closures of g and h are respectively $\gamma^{-1}\langle a\rangle\gamma$ and $\gamma'^{-1}\langle b\rangle\gamma'$, by Lemma 4.4.2 there is no element in $A_{\Gamma'}$ conjugating g to h. This is then the only case in which $A_{\Gamma'}$ is not conjugacy stable in A_Γ. □ We also prove Theorem 4.4, that states that the parabolic closure of an element g is stable when taking roots and powers of g. This generalises to large-type Artin groups a result of [23].

Theorem 4.4.4. *Let A_Γ be a large-type Artin group of rank at least 2, and let $g \in A_\Gamma$. Then for every $n \in \mathbf{Z}\backslash\{0\}$ we have $P_g = P_{g^n}$. In particular, if $g^n \in P$ then $g \in P$.*

Before coming to the proof of this Theorem, we first introduce the following lemma. Its result and its proof are analogous to [31, Theorem 7.3].

Lemma 4.4.5. *Let G be a group acting by simplicial automorphisms on a systolic complex X. Suppose that there is a vertex $v \in X$ whose orbit Gv is finite. Then there exists a simplex of X that is invariant under the action of G.*

Proof: The statement of [31, Theorem 7.3] is given for a finite group G. However, their proof only uses the finiteness of G to obtain a finite G-orbit, out of which they construct an invariant simplex. In particular, their proof generalises without any change to the case of an infinite group G with a finite G-orbit. □

Proof of Theorem 4.4.4: We show by induction on the rank $|V(\Gamma)|$ of A_Γ that $P_g = P_{g^n}$. If $|V(\Gamma)| = 2$, A_Γ is a dihedral Artin group. In particular, it is spherical, and the result follows from [23, Corollary 8.3]. Let now $|V(\Gamma)| \geqslant 3$, and suppose that $P_g \neq P_{g^n}$. Since $P_{g^n} \subseteq P_g$, there is a chain of inclusions of the form

$$P_{g^n} \subsetneq P_g \subseteq A_\Gamma.$$

<u>Claim:</u> We have $P_g \subsetneq A_\Gamma$.

Proof of the Claim: Since $P_{g^n} \subsetneq A_\Gamma$, the set $\mathrm{Fix}_{X_\Gamma}(P_{g^n})$ is non-empty. In particular, g^n is elliptic, and thus g has finite orbits, as for every point $v \in \mathrm{Fix}(g^n)$,

$$\langle g \rangle \cdot v = \{v, gv, g^2v, \cdots, g^{n-1}v\}.$$

By Lemma 4.4.5, g must stabilise some simplex Δ in X_Γ. Because the action of A_Γ on X_Γ is without inversions, g must fix Δ pointwise. In other words, $\mathrm{Fix}(g)$ is non-empty, hence $P_g \subsetneq A_\Gamma$. This finishes the proof of the Claim. Now we have

$P_g = hA_{\Gamma'}h^{-1}$ for some $h \in A_\Gamma$ and an induced subgraph $\Gamma' \subsetneq \Gamma$. Notice that

$$h^{-1}P_{g^n}h \subsetneq h^{-1}P_g h = A_{\Gamma'},$$

and thus $P_{h^{-1}g^nh} \subsetneq P_{h^{-1}gh} = A_{\Gamma'}$ by Lemma 4.4.2. As $A_{\Gamma'}$ has stricly lower rank than A_Γ, we can use the induction hypothesis on $X_{\Gamma'}$. This yields $P_{h^{-1}gh} = P_{h^{-1}g^nh}$. In particular, one has $P_g = P_{g^n}$ by Lemma 4.4.2, which is a contradiction.

This proves the main point of the theorem. The last point of the theorem is now immediate. $\square$

Chapter 5

Rigidity and automorphisms

The goal of this chapter is to give a partial answer to the **isomorphism problem** raised in the introduction. Let us recall that an Artin group A_Γ is said to be **free-of-infinity** if $m_{ab} \neq \infty$ for all $a, b \in V(\Gamma)$. In this chapter, we study the rigidity of large-type Artin groups, and more specifically large-type Artin groups that are also free-of-infinity. Our main result is the following:

Theorem 5.1. *The class of large-type free-of-infinity Artin groups is rigid. In other words, if A_Γ and $A_{\Gamma'}$ are two large-type free-of-infinity Artin groups, then A_Γ and $A_{\Gamma'}$ are isomorphic if and only if Γ and Γ' are isomorphic.*

Our work on isomorphisms between large-type free-of-infinity Artin group is closely related with the study of the automorphisms of these Artin groups. In particular, we describe completely the automorphism group and the outer automorphism group of every large-type free-of-infinity Artin group:

Theorem 5.2. *Let A_Γ be a large-type free-of-infinity Artin group of rank at least 3. Then $Aut(A_\Gamma)$ is generated by the conjugations, the graph-induced automorphisms, and the global involution. In particular, $Out(A_\Gamma)$ is finite.*

Note that it is not possible to extend Theorem 5.1 and Theorem 5.2 to all large-type Artin groups, as this bigger family is known to not be rigid and to contain other types of automorphisms (see [32]). In spite of that, we are still able to prove that all large-type Artin groups admit a weaker form of rigidity. The next result we obtain concerns the isomorphisms of large-type Artin groups in general.

Theorem 5.3. *Let A_Γ and $A_{\Gamma'}$ be two large-type Artin groups of rank at least 3. Then any isomorphism $\varphi : A_\Gamma \to A_{\Gamma'}$ induces a bijection between the set of spherical parabolic subgroups of A_Γ and the set of spherical parabolic subgroups of $A_{\Gamma'}$.*

In addition to being a principal tool in the proofs of Theorem 1.12 and Theorem 1.13, the consequences of Theorem 1.14 are various. For a start, it implies that any isomorphism $\varphi : A_\Gamma \to A_{\Gamma'}$ between large-type Artin groups sends the standard generators of A_Γ onto conjugates of standard generators of $A_{\Gamma'}$. When $\Gamma = \Gamma'$, this gives a form a rigidity of the automorphisms of A_Γ, that is in clear contrast with classes such as right-angled Artin groups, in which the automorphism group contains transvections. Another consequence of Theorem 5.3 is that the spherical parabolic subgroups of a large Artin group can be defined in a purely algebraic way, in the sense that they only depend on the abstract group structure and not on a specific choice of defining graph for the group. When the Artin group considered is large but also free-of-infinity, we find a way to "reconstruct" its associated Deligne complex in a purely algebraic manner. We obtain the following result:

Theorem 5.4. *Let A_Γ and $A_{\Gamma'}$ be two large-type free-of-infinity Artin groups of rank at least 3, with Deligne complexes D_Γ and $D_{\Gamma'}$. Then any isomorphism $\varphi : A_\Gamma \to A_{\Gamma'}$ induces a natural simplicial isomorphism $\varphi_* : D_\Gamma \to D_{\Gamma'}$ that can be described explicitly.*

We now bring light on the strategy we use to prove the aforementioned results. The key ingredient into proving Theorem 5.1 and Theorem 5.2 is Theorem 5.4. If we find a way to reconstruct the Deligne complexes of (some) Artin groups with purely algebraic objects, then any isomorphism between these Artin groups will preserve the structure of the algebraic objects, and hence preserve the Deligne complexes themselves. This kind of approach was originally used by Ivanov ([60]) to study the automorphisms of mapping class groups, and has since then been extended to other groups like Higman's group ([68]) or graph products of groups ([39]).

We now consider a large-type Artin group A_Γ. A first step into reconstructing the associated Deligne complex D_Γ is to reconstruct the type 2 vertices of the complex. These vertices are in one-to-one correspondence with the non-free parabolic subgroups of type 2 of A_Γ. We know that these subgroups are dihedral Artin subgroups of A_Γ. However, this is not a strong enough condition to describe them purely algebraically. As it turns out, reconstructing these parabolic subgroups in a purely algebraic manner is made quite complicated by the existence of dihedral Artin subgroups of "exotic" type, which do not correspond to vertices of type 2 in the original Deligne complex. A large part of our work has for goal to find a way to describe these exotic dihedral Artin subgroups explicitly, which then allows us to differentiate them from the dihedral Artin subgroups that correspond to the type 2 vertices of D_Γ. Doing so, we will prove the following:

Theorem 5.5. *Let A_Γ be a large-type Artin group of rank at least 3, and let H be a subgroup of A_Γ that is isomorphic to a dihedral Artin group. Then H is conjugated into one of the following:*
(1) $\langle a, b\rangle$, where $a, b \in V(\Gamma)$ satisfy $m_{ab} < \infty$.
(2) $\langle b, abc\rangle$, where $a, b, c \in V(\Gamma)$ satisfy $m_{ab} = m_{ac} = m_{bc} = 3$.

The next step into reconstructing D_Γ algebraically is to characterise the type 1 vertices of the complex. Unfortunately, the correspondence between the parabolic subgroups of type 2 of A_Γ and the type 2 vertices of D_Γ established at the previous step has no chance to work for type 1 vertices. This is because every parabolic subgroups of type 1 of A_Γ corresponds to infinitely many type 1 vertices of D_Γ, so there is no hope into building a bijection between these subgroups and the type 1 vertices of D_Γ.

When the Artin groups considered are large and free-of-infinity, we find another way to reconstruct the type 1 vertices of D_Γ algebraically. Our strategy involves characterising every type 1 vertex of D_Γ through the (finite) set of type 2 vertices it is connected to. This process comes in very handy, because it allows to immediately state when a type 1 and a type 2 vertices should be connected, which helps reconstructing part of the edges of D_Γ too. At this point, we will already have reconstructed a rather large subcomplex of D_Γ. We will finally be able to reconstruct D_Γ entirely by exploiting the geometry of this subcomplex.

The structure of this chapter is as follows. In Section 5.1 we consider large-type Artin groups. We introduce various algebraic and geometric tools and notions about parabolic subgroups, normalisers, and dihedral Artin subgroups, that will be used through the rest of the chapter. Section 5.2 is dedicated to an in-depth study of the centralisers of hyperbolic elements of A_Γ, and to the action of these centralisers on the minset of the corresponding hyperbolic elements. In this section, we will develop central tools that will be used to study the dihedral Artin subgroups of A_Γ in the next section. In Section 5.3, we describe all the dihedral Artin subgroups of large-type Artin groups explicitly, proving Theorem 5.5. We also find a way to differentiate the dihedral Artin subgroups that correspond to type 2 vertices of D_Γ from those that don't, which ultimately allows to recover Theorem 5.3. In Section 5.4, we suppose that our large-type Artin groups are also free-of-infinity, and we reconstruct the Deligne complex in a purely algebraic manner. Finally in Section 5.5, we use this algebraic description of the Deligne complex to recover Theorem 5.4, Theorem 5.1 and Theorem 5.2.

5.1 Preliminaries.

This section serves as an introduction to many general notions that we will use throughout this chapter. Section 5.1.1 is oriented around the introduction of basic tools about the algebraic structure of the parabolic subgroups and their connection with the geometry of the Deligne complex. In Section 5.1.2 we will talk briefly about dihedral Artin subgroups, introducing some of the material that will be needed in Section 5.2. As explained at the beginning of this chapter, studying the dihedral Artin subgroups is crucial because they appear as stabilisers of vertices in the Deligne complex.

We want to highlight that throughout this chapter, the notation A_Γ will always be used to denote an Artin group whose rank is at least 3.

5.1.1 Parabolic closure, type and normalisers.

In this section we introduce various tools that will be useful throughout the chapter. First of all, we want to introduce a one-dimensional subcomplex of D_Γ that will be a central tool in Sections 5.2, 5.3 and 5.4. This is the goal of the next definition.

Definition 5.1.1. The set of points in D_Γ whose stabiliser is non-trivial is a graph that is the union of all the edges of the form $g \cdot e_{a,ab}$, where $a, b \in V(\Gamma)$ and $g \in A_\Gamma$. It is a strict subset of the 1-skeleton $D_\Gamma^{(1)}$ of D_Γ, that we will call the **essential 1-skeleton** and denote by $D_\Gamma^{(1)-ess}$ (on Figure 2.8, the edges that are drawn are exactly those of the essential 1-skeleton $D_\Gamma^{(1)-ess}$).

Remark 5.1.2. The fact that D_Γ is the union of the translates $g \cdot K_\Gamma$ for all $g \in A_\Gamma$ has two direct consequences:
(1) Since the set of points of $D_\Gamma^{(1)-ess}$ that also belong to the fundamental domain K_Γ is the boundary Γ_{bar} of K_Γ, the graph $D_\Gamma^{(1)-ess}$ is the union of the translates $g \cdot \Gamma_{bar}$, for all $g \in A_\Gamma$.
(2) Since K_Γ is the cone-off of Γ_{bar}, the Deligne complex D_Γ can be obtained from $D_\Gamma^{(1)-ess}$ by coning-off the translates $g \cdot \Gamma_{bar}$, for all $g \in A_\Gamma$.

We now extend the definition of type we introduced for parabolic subgroups (see Definition 2.4.7) and vertices in the Deligne complex (see Definition 2.4.17) to arbitrary sets in A_Γ and arbitrary points in D_Γ:

Definition 5.1.3. For an arbitrary subset $X \subseteq A_\Gamma$, we define the **type** of X to be the type of its parabolic closure P_X. The **type** of a point $p \in D_\Gamma$ is defined to be the type of its stabiliser G_p.

Remark 5.1.4. (1) The definition of type introduced in Definition 5.1.3 is an extension of that given in Definition 2.4.17. In other words, the vertices of type $i \in \{0, 1, 2\}$ from Definition 2.4.17 also have type i relatively to Definition 5.1.3.
(2) The type of a point $p \in D_\Gamma$ always belongs to $\{0, 1, 2\}$. By construction, p has type 2 if and only if it is a type 2 vertex ; it has type 1 if and only if it belongs to $D_\Gamma^{(1)-ess}$ but doesn't have type 2 ; and it has type 0 otherwise.

We recall the following definition, that can be seen as an extension of Definition 2.1.18:

Definition 5.1.5. The **fixed set** of an element $g \in A_\Gamma$ acting on D_Γ is the set

$$Fix(g) := \{p \in D_\Gamma \mid g \cdot p = p\}.$$

The **fixed set** of a subset $X \subseteq A_\Gamma$ is the set

$$Fix(X) := \{p \in D_\Gamma \mid \forall g \in X,\ g \cdot p = p\} = \bigcap_{g \in X} Fix(g).$$

The following lemma will be useful to describe the relation between the type of an element $g \in A_\Gamma$ and its fixed set $Fix(g)$.

Lemma 5.1.6. ***[32, Lemma 8]*** *Let A_Γ be a 2-dimensional Artin group, and let $g \in A_\Gamma$. Then we can classify $Fix(g)$ in the following way:*

- *If $type(g) = 0$, then $g = 1$ and $Fix(g) = D_\Gamma$.*
- *If $type(g) = 1$, then g is elliptic and there are two elements $a \in V(\Gamma')$ and $h \in A_\Gamma$ such that $P_g = h\langle a\rangle h^{-1}$. In particular, $Fix(g)$ is the tree $hFix(a)$.*
- *If $type(g) = 2$, then g is elliptic and there are three elements $a, b \in V(\Gamma')$ and $h \in A_\Gamma$ such that $P_g = hA_{ab}h^{-1}$. In particular, $Fix(g)$ is the vertex hv_{ab}.*
- *If $type(g) \geqslant 3$, then g is hyperbolic and $Fix(g)$ is empty.*

Definition 5.1.7. The tree $hFix(a)$ from Lemma 5.1.6 will be called the **standard tree** associated with $P_g = h\langle a\rangle h^{-1}$.

Lemma 5.1.8. *Let $g \in A_\Gamma$. Then $Fix(g) = Fix(P_g)$.*

Proof: Recall that
$$Fix(P_g) = \bigcap_{h \in P_g} Fix(h).$$
In particular the inclusion $Fix(P_g) \subseteq Fix(g)$ is clear. We prove the other inclusion. We know from Lemma 5.1.6 that $P_{h_1} \subseteq P_{h_2} \Leftrightarrow Fix(h_2) \subseteq Fix(h_1)$. By definition, every element $h \in P_g$ has a parabolic closure satisfying $P_h \subseteq P_g$, which yields $Fix(g) \subseteq Fix(h)$. It follows that
$$Fix(P_g) = \bigcap_{h \in P_g} Fix(h) \supseteq Fix(g).$$
□

Corollary 5.1.9. *Let A_Γ be a large-type Artin group, let $g \in A_\Gamma$, and let $n \neq 0$. Then $type(g) = type(g^n)$ and $Fix(g) = Fix(g^n)$.*

Proof: The first statement is immediate from Theorem 4.4.4. The second statement follows from Lemma 5.1.8 and Theorem 4.4.4:
$$Fix(g) = Fix(P_g) = Fix(P_{g^n}) = Fix(g^n).$$
□

We now introduce a geometric method that allows under mild hypotheses to determine whether two elements of the groups are the same in a very efficient manner. We first need the following definition:

Definition 5.1.10. Consider the morphism $\phi : F_{V(\Gamma)} \to \mathbf{Z}$ sending every generator to 1. Every relation r of A_Γ is in the kernel of ϕ, so the map descends to a quotient map $ht : A_\Gamma \to \mathbf{Z}$. For any element $h \in A_\Gamma$, we call $ht(h)$ the **height** of h.

Lemma 5.1.11. *Let $p \in D_\Gamma$ be a point of type at most 1, and let $h_1, h_2 \in A_\Gamma$ be two elements with same height and satisfying $h_1 \cdot p = h_2 \cdot p$. Then $h_1 = h_2$.*

Proof: First note that $h_1 h_2^{-1} \cdot p = p$ and thus $h_1 h_2^{-1} \in G_p$. In particular, the result is trivial if $type(p) = 0$. So we suppose that $type(p) = 1$, i.e. that there are two elements $s \in V(\Gamma)$ and $g \in A_\Gamma$ such that $G_p = g\langle s \rangle g^{-1}$. Since $h_1 h_2^{-1} \in G_p$, then $h_1 h_2^{-1} = g s^m g^{-1}$ for some $m \in \mathbf{Z}$. On one hand h_1 and h_2 have the same height, so $h_1 h_2^{-1}$ has height 0. On the other hand, the height of $g s^m g^{-1}$ is $1 + m - 1 = m$. This means $m = 0$ and $h_1 h_2^{-1} = 1$. □

We now move towards understanding more normalisers and centralisers of elements of large-type Artin groups, in particular in relation to their type. The following lemma is the analogue of Lemma 4.3.1 for the Deligne complex instead of the Artin complex:

Lemma 5.1.12. *Let A_Γ be a 2-dimensional Artin group, let S be a subset of A_Γ with non-trivial fixed set in D_Γ, and let $N(S)$ denote the normaliser of S in A_Γ. Then*

$$N(S) \subseteq Stab(Fix(S)).$$

Assume additionally that $\exists p \in Fix(S)$ such that $G_p = S$. Then

$$N(S) = Stab(Fix(S)).$$

Proof: ($\subseteq$) Let $g \in N(S)$, that is, $gS = Sg$, and let $p \in Fix(S)$. Then

$$S \cdot (g \cdot p) = g \cdot (S \cdot p) = g \cdot p.$$

In particular, $g \cdot p \in Fix(S)$ and thus $g \in Stab(Fix(S))$.

($\supseteq$) Let $g \in Stab(Fix(S))$ and let $p \in Fix(S)$ be such that $G_p = S$. Then $g \cdot p \in Fix(S)$, i.e.

$$S \cdot (g \cdot p) = g \cdot p.$$

In particular, $g^{-1}Sg$ fixes p, hence $g^{-1}Sg \subseteq G_p = S$. In other words, $g \in N(S)$. □

Lemma 5.1.13. *Let A_Γ be a large-type Artin group, let $g \in A_\Gamma$ be such that $type(g) \leqslant 1$, and let $C(g)$ be the centraliser of g in A_Γ. Then for any $n \neq 0$ we have*

$$N(P_g) = C(g) = C(g^n) = N(P_{g^n}).$$

Proof: The result is trivial if $type(g) = 0$, so we suppose that $type(g) = 1$. The following inclusions are clear:

$$N(P_g) \supseteq C(g) \subseteq C(g^n) \subseteq N(P_{g^n}).$$

We know by Theorem 4.4.4 that $N(P_g) = N(P_{g^n})$, so it is enough to show that $N(P_g) \subseteq C(g)$. The argument is similar to that of Lemma 5.1.11: because $P_g = \langle g \rangle$, any $h \in N(P_g)$ satisfies $h\langle g \rangle h^{-1} = \langle g \rangle$, hence conjugates g to some $hgh^{-1} = g^m$ with $m \in \mathbf{Z}$. It is then easy comparing heights to see that we must have $m = 1$ and thus $hg = gh$. □

We finally state the following useful result:

Proposition 5.1.14. *Let A_Γ be a large-type Artin group with two parabolic subgroups P and P'. If P and P' have the same type and $P \subseteq P'$, then $P = P'$.*

Proof: This follows directly from Theorem 4.2.2.(2), along with the fact that the only parabolic subgroup of P' that has the maximal number of standard generators is P' itself. □

5.1.2 Dihedral Artin subgroups.

We now come to a first study of the dihedral Artin subgroups of a large-type Artin group A_Γ. In this section we introduce some of the notions that will allow us to further study these subgroups in Section 5.2 and Section 5.3. Although dihedral

Artin subgroups have already been talked about in Chapter 3 and Chapter 4, we decide here to recall their exact definition:

Definition 5.1.15. We say that H is a **dihedral Artin subgroup** of A_Γ if there exists an isomorphism f from A_m to H for some $3 \leqslant m < \infty$, where

$$A_m := \langle s', t' \mid \underbrace{s't's' \cdots}_{m \text{ terms}} = \underbrace{t's't' \cdots}_{m \text{ terms}} \rangle.$$

When there is no ambiguity, we will write $s := f(s')$, $t := f(t')$, so that H is the subgroup of A_Γ generated by s and t. For $m' := lcm(m, 2)/2$, the element $z' := (s't')^{m'}$ is generating the centre of A_m (see Definition 3.1.6), and thus the element $z := f(z')$ generates the centre of H.

Let now A_Γ be a large-type Artin group, and let H be an arbitrary dihedral Artin subgroup of A_Γ. The two following lemmas will be useful to describe the type of H.

Lemma 5.1.16. *In H we have $type(z) = type(st) \geqslant 2$.*

Proof: Because $z = (st)^{m'}$, the equality $type(z) = type(st)$ simply comes from Theorem 4.4.4. Suppose now that $type(z) \leqslant 1$. Then $C(z) = C(st)$ by Lemma 5.1.13. Note that every element of H commutes with z, and thus we have $s \in C(z) = C(st)$. In particular then, s commutes with st and hence with t. The elements s and t generate H, so H must be abelian. This is absurd. □

Lemma 5.1.17. *Let $g, h \in A_\Gamma$ be elements satisfying $type(g) = 2$ and $type(h) \geqslant 3$. Then g and h don't commute.*

Proof: If g and h commuted, then h would stabilise the fixed set of g, by Lemma 5.1.12. Because g has type 2, we know from Lemma 5.1.6 that $Fix(g)$ is a single vertex, that h must then fix. This contradicts Lemma 5.1.6, because h has type at least 3. □

Definition 5.1.18. We say that a dihedral Artin subgroup H of A_Γ is **classical** if $type(z) = 2$ and **exotic** if $type(z) \geqslant 3$.

Corollary 5.1.19. *A classical dihedral Artin subgroup can never contain an exotic dihedral Artin subgroup, and vice-versa.*

Proof: This is a consequence of Lemma 5.1.17. Classical dihedral Artin subgroups of A_Γ always contain elements of type 2, but never contain elements of type at least 3, while exotic dihedral Artin subgroup of A_Γ always contain elements of type at least 3, but never contain elements of type 2. The result follows. □

Definition 5.1.20. We say that a dihedral Artin subgroup H of A_Γ is **maximal** if it is not strictly contained in another dihedral Artin subgroup of A_Γ.

Remark 5.1.21. A nice consequence of Corollary 5.1.19 is that it is equivalent to say that a dihedral Artin subgroup is maximal amongst all dihedral subgroups, and to say that it is maximal amongst classical (or exotic) dihedral subgroups.

Our next goal is to classify explicitly all the classical maximal dihedral Artin subgroups of A_Γ (see Corollary 5.1.23). The exotic dihedral Artin subgroups will be studied intensely throughout Section 5.2 and Section 5.3.

Lemma 5.1.22. *Every classical dihedral Artin subgroup H of A_Γ has type 2. This means there are two standard generators $a, b \in V(\Gamma)$ and an element $g \in A_\Gamma$ such that $H \subseteq gA_{ab}g^{-1}$.*

Proof: Because $type(z) = 2$, $P_z = gA_{ab}g^{-1}$ for some generators $a, b \in V(\Gamma)$ and some element $g \in A_\Gamma$. This means that z acts on D_Γ by fixing the vertex gv_{ab}. Because s and z commute, we have

$$z \cdot s \cdot gv_{ab} = s \cdot z \cdot gv_{ab} = s \cdot gv_{ab}.$$

Therefore z fixes $s \cdot gv_{ab}$, so we must have $s \cdot gv_{ab} \in Fix(z)$. By Lemma 5.1.8 $Fix(z) = Fix(P_z) = gv_{ab}$. This means the two vertices gv_{ab} and $s \cdot gv_{ab}$ coincide, i.e. s fixes gv_{ab}. On the other hand, we know from Corollary 5.1.9 that $Fix(z) = Fix(st)$. Since z fixes the vertex gv_{ab}, then st must also fix this vertex. Consequently, both s and st fix gv_{ab}. In particular, $t = s^{-1}(st)$ also fixes gv_{ab}. Since s and t generate H, this means H fixes gv_{ab}, i.e. $H \subseteq gA_{ab}g^{-1}$. □

Corollary 5.1.23. *The set of classical maximal dihedral Artin subgroups of A_Γ is precisely the set of non-free parabolic subgroups of type 2 of A_Γ, i.e. the set*

$$\{gA_{ab}g^{-1} \mid a, b \in V(\Gamma) : m_{ab} < \infty,\ g \in A_\Gamma\}.$$

Proof: ($\supseteq$) Consider a subgroup $H := gA_{ab}g^{-1}$ of A_Γ as described above. It is clear that H is a dihedral Artin subgroup, because $3 \leqslant m_{ab} < \infty$ as A_Γ is large. H is also clearly classical. Let H' be a dihedral subgroup of A_Γ that satisfies $H' \supseteq H$. By Corollary 5.1.19 H' must be classical. By Lemma 5.1.22 then, H and H' both have type 2. Since $H' \supseteq H$, Proposition 5.1.14 gives $H' = H$. This proves that H is maximal.

($\subseteq$) Let H be a classical maximal dihedral Artin subgroup of A_Γ. We know by Lemma 5.1.22 that there are elements $a, b \in V(\Gamma)$ and $g \in A_\Gamma$ such that $H \subseteq gA_{ab}g^{-1}$. Note that $gA_{ab}g^{-1}$ is maximal by the first point. Since H is maximal too, we must have an equality. □

5.2 Centralisers of hyperbolic elements.

Let A_Γ be a large type Artin group and let H be an exotic dihedral Artin subgroup of A_Γ. The centre of H is generated by an element z of type at least 3, i.e. a hyperbolic element. Since $H \subseteq C(z)$, it is relevant in order to understand H to want to understand centralisers of elements like z. The goal of this section is to do exactly that, and ultimately to prove Proposition 5.2.21, in which we describe under mild hypotheses on z the algebraic structure of the centraliser $C(z)$. These hypotheses will always be satisfied for hyperbolic elements that generate centres of exotic dihedral Artin subgroups of A_Γ, so our strategy will apply to these subgroups.

We now briefly explain how we are able to describe these centralisers. Our approach is heavily geometric. If z generates the centre of an exotic dihedral Artin subgroup H, then its type is at least 3. In particular, z acts on D_Γ hyperbolically and its minset $Min(z)$ is non-trivial (see Definition 2.1.16). As it turns out, $Min(z)$ is preserved under the action of $C(z)$ (and hence that of H). Moreover, $Min(z)$ decomposes as the product $\mathcal{T} \times \mathbf{R}$ of a tree with the real line (see Theorem 5.2.1 and Lemma 5.2.3). We will prove that the tree $\mathcal{T}$ has two nice geometric features: it contains an infinite line, and it contains a vertex of valence at least 3 (see Lemma 5.2.5).

For a start, the first feature forces the minset of z to contain a flat plane. Such a situation is only possible if up to conjugation, z belongs to a Artin subgroup

A_{abc} whose coefficients are all 3. In particular then, $Min(z)$ lies inside the Deligne sub-complex $D_{abc} \subseteq D_\Gamma$. The study of $Min(z)$ will then reduce to studying a parabolic subgroup of type 3 of A_Γ (see Lemma 5.2.6). Using the second feature will allow for a precise study of the geometry of $Min(z)$, from which we deduce an explicit algebraic description of $C(z)$ (see Proposition 5.2.21).

5.2.1 Transverse-trees, motivations and first results.

Let A_Γ be an Artin group of large-type, and let $z \in A_\Gamma$ be any element acting hyperbolically on D_Γ (i.e. any element of type at least 3). The goal of this section is to prove the aforementioned Lemma 5.2.5 and Lemma 5.2.6. A nice consequence of these two lemmas will be that if A_Γ is of large type and of hyperbolic type, then A_Γ contains no exotic dihedral Artin subgroup at all. In that case, one can directly move to Section 5.3. However the situation is more complicated when A_Γ is of large-type but not of hyperbolic type (i.e. when Γ contains 3-cycles with coefficients $(3, 3, 3)$). This broader case will be dealt with throughout Section 5.2.

The structure of minsets in a more general setting has been studied by Bridson and Haefliger, so we start by recalling two very useful theorems, that we adapt to our situation:

Theorem 5.2.1. *[17, Chapter II.6] $Min(z)$ is a closed, convex and non-empty subspace of D_Γ (in particular, it is CAT(0)). It is isometric to a direct product $\mathcal{T} \times \boldsymbol{R}$ on which z acts trivially on the first component, and as a translation on the second component. The axes of z are in bijection with the points of $\mathcal{T}$, so that every axis u of z decomposes as $u = \overline{u} \times \boldsymbol{R}$, where $\overline{u}$ is a point of $\mathcal{T}$. In particular, the axes of h are parallel to each others, and their union is precisely $Min(h)$. Furthermore, the centraliser $C(z)$ leaves $Min(z)$ invariant sending axes to axes. It is such that the action of any element $g \in C(z)$ on $Min(z)$ decomposes as an isometry (g_1, g_2) of $\mathcal{T} \times \boldsymbol{R}$, where g_2 is simply a translation. In particular, $C(z)$ preserves $\mathcal{T}$ as well.*

The next Theorem is known as the Flat Strip Theorem:

Theorem 5.2.2. ***[17, Chapter II.2]*** *Let u and v be two parallel geodesic lines in D_Γ. Then their convex hull $c(u, v)$ in D_Γ is isometric to a flat strip $[0, D] \times \boldsymbol{R}$, where D is the distance between u and v.*

We will be able to show later on that under reasonable hypotheses, the set $\mathcal{T}$ is a simplicial tree (see Lemma 5.2.12 and Corollary 5.2.24). For now, and with our current hypotheses, we will only show that $\mathcal{T}$ is a real-tree:

Lemma 5.2.3. *The space $\mathcal{T}$ is a real-tree, i.e. a 0-hyperbolic space.*

Proof: Suppose that $\mathcal{T}$ is not 0-hyperbolic. Then there is a triangle $T \subseteq \mathcal{T}$ that is not a tripod. Since D_Γ is simply-connected and T is not a tripod, one can fill the interior of T with non-trivial 2-dimensional balls. In particular then, $Min(z) = \mathcal{T} \times \mathbf{R} \subseteq D_\Gamma$ must contain 3-dimensional balls. This contradicts the fact that D_Γ is 2-dimensional. □

Definition 5.2.4. We call $\mathcal{T}$ the **transverse-tree** of z in D_Γ.

As explained at the beginning of the section, if z is an element generating the centre of an exotic dihedral Artin group H, then $H \subseteq C(z)$, and Theorem 5.2.1 applies: H acts on $Min(z)$ and on the associated transverse-tree $\mathcal{T}$ in a nice way. In such a situation, $\mathcal{T}$ has nice properties, as made explicit in the statement of the next lemma. Since our main reason for studying the minset of hyperbolic elements is to understand the case of exotic dihedral Artin subgroups, we will throughout the rest of this section assume some of the properties inherited by the transverse-trees associated with such subgroups.

Lemma 5.2.5. *Let H be an exotic dihedral Artin subgroup of A_Γ, and consider the set $Min(z)$ associated with the central element z of H. Then the transverse-tree $\mathcal{T}$ associated with z contains an infinite line and has a vertex of valence at least* 3.

Proof: Let us denote by s and t the standard generators of H (see Definition 5.1.15). Suppose that $\mathcal{T}$ does not contain an infinite line. Then any element that acts preserving $\mathcal{T}$ is elliptic (no element creates an axis in $\mathcal{T}$). Using Theorem 5.2.1, this means any element of $C(z)$ acts elliptically on $\mathcal{T}$. In particular, the elements st and ts act on $\mathcal{T}$ with non-trivial fixed sets. Suppose these fixed sets are disjoint. A classical ping-pong argument shows that the product $(st)\cdot(ts)$ acts hyperbolically on $\mathcal{T}$, which contradicts the fact that every element of $C(z)$ acts elliptically. This means the fixed sets of st and ts intersect non-trivially. Let $\overline{u}$ be a vertex of $\mathcal{T}$ fixed by both st and ts. Then st and ts both act like translations when restricted to u (see Theorem 5.2.1). They have the same direction and the same translation length, because $(st)^{m'} = z = (ts)^{m'}$. In particular, if x is any point of type at most 1 in u, we have $(st) \cdot x = (ts) \cdot x$. Note that st and ts have the same height, so we obtain $st = ts$ by Lemma 5.1.11. This is absurd, and hence $\mathcal{T}$ contains an infinite line.

We now show that $\mathcal{T}$ has a vertex of valence at least 3. Suppose that it doesn't, i.e. every vertex of $\mathcal{T}$ has valence at most 2. Then $\mathcal{T}$ is contained in an infinite line. But $\mathcal{T}$ also contains an infinite line by the previous point, so it must be precisely that line. This means $Min(z) \cong \mathcal{T} \times \mathbf{R}$ is a flat plane. Using Theorem 5.2.1, we know that the elements s and t act on $Min(z) \cong \mathcal{T} \times \mathbf{R}$ like isometries that restrict to translations on the $\mathbf{R}$-component. Depending on whether the action on the $\mathcal{T}$-component is hyperbolic or elliptic (with order 2), each of the elements s or t acts on $Min(z)$ either as a pure translation, or as a (possibly trivial) glide reflection. In any case, the squares s^2 and t^2 act like pure translations on $Min(z)$. In particular, their actions commute. Since there are points in $Min(z)$ with trivial stabilisers, this mean s^2 and t^2 commute as elements of the group, absurd. □

We now move towards the most important result of the beginning of Section 5.2. We show that under mild hypotheses on $\mathcal{T}$, that we recall are satisfied for exotic dihedral Artin groups by Lemma 5.2.5, the study of $Min(z)$ reduces to the study of an Artin subgroup $A_{abc} \subseteq A_\Gamma$ an its associated Deligne subcomplex $D_{abc} \subseteq D_\Gamma$.

Lemma 5.2.6. *Let $z \in A_\Gamma$ be a hyperbolic element and suppose that its transverse-tree $\mathcal{T}$ contains an infinite line. Then up to conjugation of z, there are three generators $a, b, c \in V(\Gamma)$ satisfying $m_{ab} = m_{ac} = m_{bc} = 3$ such that $z \in A_{abc}$. Moreover, the Deligne complex D_{abc} associated with the Artin (sub)group A_{abc} is isometrically embedded into D_Γ, and contains $Min(z)$.*

Proof: By Lemma 5.2.3 $\mathcal{T}$ is a real-tree, that we suppose contains an infinite line L. In particular, $Min(z)$ contains the infinite plane $P := L \times \mathbf{R}$.

Claim 1: Let $g \cdot T_{ab}$ be a base triangle and suppose that there is a point x in the interior of $g \cdot T_{ab}$ that is contained in P. Then $g \cdot T_{ab}$ is contained in P. In particular, P is a union of base triangles.

Proof of Claim 1: Let $y \neq x$ be a point in $g \cdot T_{ab}$, let γ be the geodesic connecting x to y in D_Γ, and let $d := d_{D_\Gamma}(x, y) = \ell(\gamma)$. Because x belongs to the interior of $g \cdot T_{ab}$, there is an $\varepsilon > 0$ such that the ball $B_{D_\Gamma}(x, \varepsilon)$ is a planar disk and is contained inside $g \cdot T_{ab}$ as well. The ball $B_P(x, \varepsilon)$ is also a planar disk, as P is an infinite plane. This means the natural inclusion $B_P(x, \varepsilon) \subseteq B_{D_\Gamma}(x, \varepsilon)$ is an equality. Let $z := \gamma \cap B_P(x, \varepsilon)$. Because P is a flat plane, there is a (unique) geodesic γ' of P that satisfies the following:

$$\gamma' \text{ starts at } x, \text{ passes through } z, \text{ and has length } d. \quad (*)$$

Note that P is a convex subset of $Min(z)$, which itself is convex in D_Γ by Theorem 5.2.1. In particular then, γ' is a geodesic of D_Γ too. It is not hard to see that γ is the unique there is only one geodesic in D_Γ that satisfies $(*)$, and that this geodesic is γ. This means $\gamma = \gamma'$. In particular, $y \in \gamma = \gamma' \subseteq P$. This proves $g \cdot T_{ab} \subseteq p$. The fact that P is a union of base triangles follows. This finishes the proof of Claim 1.

Since $D_\Gamma^{(1)}$ is not dense in D_Γ, there is a point x of type 0 in P that belongs to the interior of a base triangle of the form $g \cdot T_{ab}$, for some elements $a, b \in V(\Gamma)$ and $g \in A_\Gamma$. By Claim 1 then, P contains $g \cdot T_{ab}$. Note that $Min(gzg^{-1}) = gMin(z)$, so up to replacing z with gzg^{-1}, we will suppose that $g = 1$. In particular, P contains T_{ab}, and $v_\varnothing$.

Claim 2: The base triangles containing $v_\varnothing$ in P form a polygon $K := T_{ab} \cup T_{ba} \cup T_{ac} \cup T_{ca} \cup T_{bc} \cup T_{cb}$ that is described in Figure 5.1, for some generators $a, b, c \in V(\Gamma)$ satisfying $m_{ab} = m_{ac} = m_{bc} = 3$.

Proof of Claim 2: P contains $v_\varnothing$, so there is a small enough $\varepsilon > 0$ such that the neighbourhood $B_P(v_\varnothing, \varepsilon)$ is contained in the fundamental domain K_Γ, hence in an union of base triangles of the form T_{st} (in fact, any $\varepsilon \leqslant 1$ works). We consider the angles around $v_\varnothing$ in P, i.e. for each of the above triangle T_{st} we consider the angle

$$\angle_{v_\varnothing}(e_s, e_{st}) := \frac{\pi}{2} - \frac{\pi}{2 \cdot m_{st}}.$$

Because A_Γ is large, every such angle is at least $\frac{\pi}{2} - \frac{\pi}{6} = \frac{\pi}{3}$. On one hand, the minimal length of a non-trivial cycle in the barycentricsubdivision Γ_{bar} of Γ is 6, and thus the link $Lk_P(v_\varnothing)$ contains no non-trivial cycle with strictly less than 6 edges. In particular, there must be at least 6 base triangles around $v_\varnothing$ in P. On the other hand P is an Euclidean plane, hence the sum of all the angles around $v_\varnothing$ in P is exactly 2π. The only possibility is that there are exactly 6 base triangles around $v_\varnothing$ in P, and that the angles are all precisely $\frac{\pi}{3}$. This means the local groups of the type 2 vertices around $v_\varnothing$ in P are all dihedral Artin subgroups with coefficient 3. We obtain the situation described in Figure 5.1. This finishes the proof of Claim 2.

One can easily notice that the polygon K is itself a flat (equilateral) triangle. It is the subcomplex of the fundamental domain K_Γ corresponding to the subgraph of Γ spanned by the vertices a, b and c. The previous reasonning can be applied around any point of P that does not belong to $D_\Gamma^{(1)}$. Consequently, any such point is contained in a flat triangle $K' := g' \cdot (T_{st} \cup T_{ts} \cup T_{sr} \cup T_{rs} \cup T_{tr} \cup T_{rt})$, where $g' \in A_\Gamma$ and $s, t, r \in V(\Gamma)$ are such that $m_{st} = m_{sr} = m_{tr} = 3$. In particular, P is tiled with these "larger" equilateral triangles. We will call such polygons **principal triangles**, to distinguish them from base triangles.

<u>Claim 3:</u> The standard generators s, t and r associated with any principal triangle K' of P are the same standard generators a, b and c as the ones associated with the first principal triangle K. In particular, every principal triangle K' is the g'-translate of K, for some $g' \in A_{abc}$, and the element z belongs to A_{abc}.

<u>Proof of Claim 3:</u> Let $P_0 := K$, and let P_{n+1} be the union of the principal triangles of P that are either in P_n or that share an edge with a principal triangle of P_n. Note that $P = \lim_{n\to\infty} P_n$. We assign a colour to each of the three sides of K. (see Figure 5.1). We extend this system of colour to P by giving to an edge of a principal triangle the colour of its unique translate in K. We show by induction on n that this is well-defined, i.e. that such edges always have a translate in K. The argument is elementary, and relies on completing colours in P_{n+1} from the colours in P_n (see Figure 5.1). If two edges with different colours (say the ones corresponding to distinct generators $s, t \in \{a, b, c\}$) meet at a vertex, then one can find the colour of the 6 edges around that vertex (they will be an alternating sequence of the colours associated with s and t).

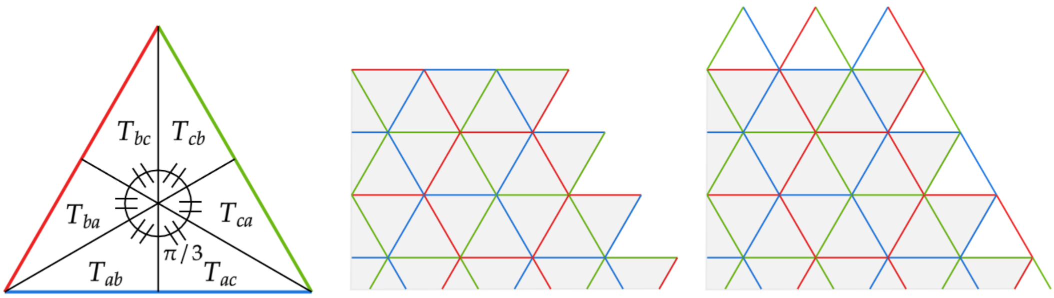


Figure 5.1: On the left: The principal triangle K, which is equal to P_0. In the centre: P_n. On the right: P_{n+1}, with P_n highlighted in gray.

Note that if two principal triangles $g_1 \cdot K$ and $g_2 \cdot K$ share an edge then there is some $s \in \{a, b, c\}$ and $k \neq 0$ such that $g_1 \cdot s^k = g_2$. Starting at K, this shows by induction that any principal triangle K' is actually the g'-translate of K, where g' is a product of powers of a, b and c. In particular then, $g' \in A_{abc}$. Let us now consider $v_\varnothing \in P$. We know that z acts trivially on $\mathcal{T}$. In particular, it acts trivially on L, hence preserves P. This means $z \cdot v_\varnothing \in P$. By the previous argument, we must have $z \in A_{abc}$. This finishes the proof of Claim 3.

Claim 4: D_{abc} is isometrically embedded into D_Γ, and it contains $Min(z)$.

Proof of Claim 4: The first statement is a result of Charney ([25], Lemma 5.1), so we only prove that D_{abc} contains $Min(z)$. The principal triangle K is precisely the intersection $K_\Gamma \cap D_{abc}$, hence belongs to D_{abc}. Since every g'-translate of K belongs to D_{abc} when $g' \in A_{abc}$, the plane P is contained inside of D_{abc} by Claim 3. Let now y be any point of $Min(z)$ that is not in P. Then y projects to a point $\overline{y}$ of $\mathcal{T}$ that is not in L. Because $\mathcal{T}$ is a real-tree, there is a unique geodesic segment L_0 that joins $\overline{y}$ and L in $\mathcal{T}$. They meet at some vertex $\overline{z} \in L$ that cuts L in two pieces $L_1 \cup L_2 = L$. Consider now the union $L' := L_0 \cup L_1$, and consider the half-plane $P' := L' \times \mathbf{R}$. We know the colour of all the edges in P' that belong to the half-plane $P_1 := L_1 \times \mathbf{R} = P' \cap P$. A similar induction process as the one in the proof of Claim 3 allows to extend the system of colour from P_1 to P'. In particular, the same arguments as the ones used in the proof of Claim 3 apply. Consequently, the whole of $Min(z)$ is tiled with principal triangles (or part of principal triangles) that are translates of K through elements of A_{abc}. It follows that $Min(z) \subseteq D_{abc}$. This finishes the proof of Claim 4, and of the Lemma.

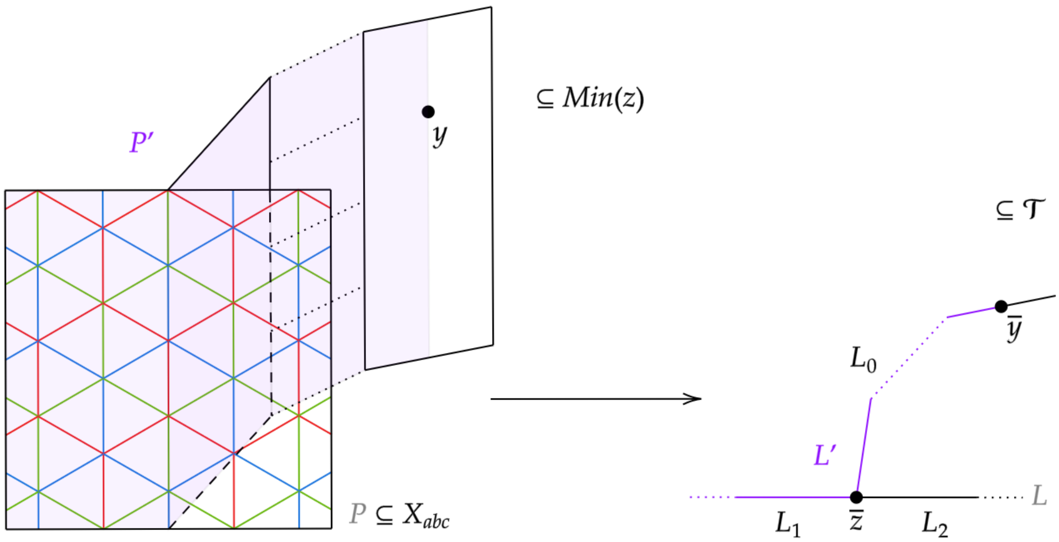


Figure 5.2: Extending the tiling of P to a tiling of $Min(z)$. The left of the picture represents what happens in $Min(z)$, while the right of the picture represents what happens in $\mathcal{T}$. The plane P that we already tiled is in the foreground, while the half-plane P' we want to tile is highligthed in purple. □

Remark 5.2.7. Lemma 5.2.5 along with Lemma 5.2.6 already prove that large Artin groups of hyperbolic type do not have exotic dihedral Artin subgroups.

Lemma 5.2.8. *Let z be a hyperbolic element whose associated transverse-tree contains an infinite line. We know by Lemma 5.2.6 that up to conjugation, $z \in A_{abc}$ for some appropriate standard generators $a, b, c \in V(\Gamma)$. Let x be any point of $Min(z)$, and let g be an element of A_Γ that sends x onto another point of $Min(z)$. Then $g \in A_{abc}$. In particular, $C(z) \subseteq A_{abc}$.*

Proof: First of all, we know by Lemma 5.2.6 that $Min(z) \subseteq D_{abc}$. Let γ be any path in $Min(z)$ connecting x and $g \cdot x$. We use an argument similar to the one used in the proof of Claim 3 of Lemma 5.2.6. Let $x_1, \cdots, x_n$ be the points of type 1 and 2 that γ crosses, in the correct order. Then there is an element $g' = g_1 \cdots g_n$ with $g_i \in G_{x_i}$ that sends x to $g \cdot x$. The local groups G_{x_i}'s are contained in A_{abc} because they are local groups of points of D_{abc}, so eventually $g' \in A_{abc}$. Note that g' and g both send x onto $g \cdot x$. This means there are two elements $h_1 \in G_x$ and $h_2 \in G_{g \cdot x}$ such that $g = h_2 \cdot g' \cdot h_1$. Because x and $g \cdot x$ belong to D_{abc}, the local groups G_x and $G_{g \cdot x}$ are also contained in A_{abc}. Finally, g is a product of three elements of A_{abc}, hence belongs to A_{abc}.

If $g \in C(z)$, then g preserves $Min(z)$ by Theorem 5.2.1, and thus $g \in A_{abc}$ by the previous point. This shows $C(z) \subseteq A_{abc}$. □

5.2.2 The structure of $Min(h)$ and of $\mathcal{T}$.

Let A_Γ be an Artin group of large type, and let $z \in A_\Gamma$ be any element acting hyperbolically on D_Γ. The goal of this section is to study the valence of vertices in the transverse-tree $\mathcal{T}$ associated with z. We suppose for the whole section that $\mathcal{T}$ contains an infinite line (this will always be satisfied when z generates the centre of an exotic dihedral Artin subgroup of A_Γ, by Lemma 5.2.5). In particular then, Lemma 5.2.6 applies, and the situation becomes easier to understand: up to conjugation, $Min(z) \subseteq D_{abc}$, where $a, b, c \in V(\Gamma)$ are three generators satisfying $m_{ab} = m_{ac} = m_{bc} = 3$. As motivated by Lemma 5.2.8, we will then mostly be looking at the action of A_{abc} on D_{abc}, forgetting about the rest of the action of A_Γ on D_Γ (unless specified otherwise). In light of that, the principal triangles in D_{abc} are the translates of the corresponding fundamental domain K (see the proof of Lemma 5.2.6). We will also call the sides of theses principal triangles edges, even though they initially come from the union of two edges of the form $e_{s,st}$ and $e_{s,sr}$. Our main goal is to show the following:

Corollary 5.2.9. *Let u be an axis of z. Then:*
Case 1: $\nexists g \in A_\Gamma \backslash \{1\} : u \subseteq Fix(g)$. *Then $\overline{u}$ has valence at most 2 in $\mathcal{T}$.*
Case 2: $\exists g \in A_\Gamma \backslash \{1\} : u \subseteq Fix(g)$. *Then $\overline{u}$ has infinite valence in $\mathcal{T}$.*

We will prove this result by distinguishing three cases about the structure of axes of z. The result of Corollary 5.2.9 will directly follow from Lemmas 5.2.10, 5.2.17 and 5.2.20. We begin with the following lemma:

Lemma 5.2.10. *Every axis $u \nsubseteq D^{(1)-ess}_{abc}$ of z corresponds to a point $\overline{u}$ whose valence in $\mathcal{T}$ is at most 2.*

Proof: Let us consider an axis $u \nsubseteq D^{(1)-ess}_{abc}$ of z. We want to show that $\overline{u}$ has valence at most 2 in $\mathcal{T}$, i.e. that there is some $\varepsilon > 0$ such that the ball $B_{\mathcal{T}}(\overline{u}, \varepsilon)$ is isometric to an interval of the real line. A direct consequence of Theorem 5.2.1 is that $\forall \varepsilon > 0$, $\forall x \in u$, the ball $B_{\mathcal{T}}(\overline{u}, \varepsilon)$ is isomorphic to the quotient $B_{Min(z)}(x, \varepsilon)/_\sim$, where two points $x, y \in Min(z)$ are equivalent if and only if they belong to a common axis. In particular, it is enough to find some $\varepsilon > 0$ and $x \in u$ for which $B_{Min(z)}(x, \varepsilon)$ is contained in a planar disk. Finally, since $B_{Min(z)}(x, \varepsilon) \subseteq B_{D_{abc}}(x, \varepsilon)$, it is enough to show that $B_{D_{abc}}(x, \varepsilon)$ is a planar disk.

We divide the problem in two cases:

• Suppose first that $u \nsubseteq D^{(1)}_{abc}$. Then u contains a point x that belongs to the interior of a base triangle of the form $g \cdot T_{st}$. It is then clear that there is a small enough $\varepsilon > 0$ such that $B_{D_{abc}}(x, \varepsilon)$ is a planar disk.

• Suppose now that $u \subseteq D^{(1)}_{abc}$. It is not hard to see with a bit of Euclidean geometry that up to symmetry, there are only two kinds of lines in $Min(z)$ that are contained inside $D^{(1)}_{abc}$ (see Figure 5.3). Furthermore, there is only one of these two kinds that does not belong to $D^{(1)-ess}_{abc}$ (the blue line on Figure 5.3). In particular, we directly see that u contains an edge of the form $g \cdot e_{st}$ connecting a vertex of type 0 to a vertex of type 2. Let now $x \in u$ be any point in the interior of this edge. Then there is a small enough $\varepsilon > 0$ such that $B_{D_{abc}}(x, \varepsilon)$ is a planar disk, because $g \cdot e_{st}$ is by construction contained in exactly two base triangles: $g \cdot T_{st}$ and $g \cdot T_{ts}$.

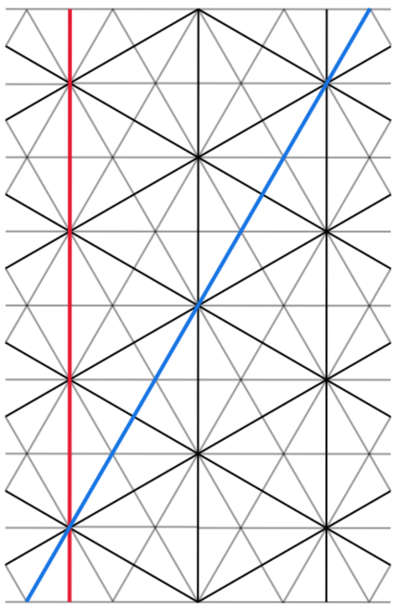


Figure 5.3: The two different types of line that are contained into $D^{(1)}_{abc}$. The red line belongs to $D^{(1)-ess}_{abc}$, while the blue line doesn't. □

One would probably like at this point to be able to see $\mathcal{T}$ as a simplicial tree and not just as a real-tree. While it is indeed true that $\mathcal{T}$ carries a somewhat natural structure of simplicial tree (assuming additional hypotheses on $\mathcal{T}$), it is not that easy to prove. In particular, we don't know at this point whether $\mathcal{T}$ has leaves. As it turns out, we will be able to prove later on that $\mathcal{T}$ does not have any leaf (assuming the same additionnal hypotheses on $\mathcal{T}$). For now, we focus on proving that $\mathcal{T}$ has a "simplicial-like" structure, as described by Lemma 5.2.12. We start by defining the vertices of $\mathcal{T}$:

Definition 5.2.11. We define the set of vertices of $\mathcal{T}$ to be the (possibly empty) set of points $\overline{u}$ whose corresponding axis u is contained inside $D_{abc}^{(1)-ess}$.

Lemma 5.2.12. *If $\overline{u}$ is a vertex of $\mathcal{T}$, the set of vertices of $\mathcal{T}$ is exactly the set of points of $\mathcal{T}$ whose distance to $\overline{u}$ is in $3 \cdot \mathbf{Z}$. In particular, vertices are isolated, and every point of $\mathcal{T}$ of valence at least 3 is a vertex.*

Proof: Let $\overline{u}$ be a vertex of $\mathcal{T}$, and let $\overline{v}$ be any point of $\mathcal{T}$ distinct from $\overline{u}$. Up to using an inductive argument, it is enough to show that if U is the 3-neighbourhood of $\overline{u}$ in $\mathcal{T}$, then the vertices of U that are not $\overline{u}$ are precisely the points of U that are at distance 3 from $\overline{u}$.

By hypothesis $\overline{u}$ is a vertex of $\mathcal{T}$, which means that $u \subseteq D_{abc}^{(1)-ess}$. As stated in the proof of Lemma 5.2.10, this is only possible if u has the form described by the red line in Figure 5.3. Let now $\overline{v} \in U$ be a point distinct from $\overline{u}$. By Theorem 5.2.1, u and v are parallel, so v can be seen as a line in Figure 5.3 that is parallel to u. It is not hard to see that the closest line to u that is parallel to u and belongs to $D_{abc}^{(1)-ess}$ is the vertical black line in the centre of Figure 5.3. With a bit of Euclidean geometry, one can determine that its distance to u is 3. In particular, $\overline{v}$ is a vertex of U distinct from $\overline{u}$ if and only if it is at distance exactly 3 from $\overline{u}$. This shows the desired property, and shows as well that vertices of $\mathcal{T}$ are isolated.

Let now $\overline{u}$ be a point of valence at least 3 in $\mathcal{T}$. By Lemma 5.2.10, the corresponding axis u belongs to $D_{abc}^{(1)-ess}$, which essentially means $\overline{u}$ is a vertex. □

Remark 5.2.13. We say two vertices of $\mathcal{T}$ are **adjacent** if there is no other vertices between them, i.e. if they lie at distance 3 from each others.

Definition 5.2.14. Let $g \cdot K$ and $h \cdot K$ be two principal triangles of D_{abc} that share an edge. Then $g^{-1}h = s^k$ for some standard generator $s \in \{a, b, c\}$ and $k \neq 0$. This defines a **system of arrows** on the principal triangles of D_{abc} in the following way:

(1) Put a single arrow from $g \cdot K$ to $h \cdot K$ whenever $g^{-1}h = s$;

(2) Put a double arrow between $g \cdot \Delta$ and $h \cdot \Delta$ whenever $g^{-1}h = s^k$ with $|k| \geqslant 2$.

Finally, we say a subset of D_{abc} is a **principal hexagon** if it is the union of 6 principal triangles $\{g_i \cdot K\}_{i \in \{1,\cdots,6\}}$ around a common type 2 vertex v of D_{abc} such that $g_i \cdot K$ shares an edge with $g_{i+1} \cdot K$ $[mod\ 6]$.

Lemma 5.2.15. *The system of arrows on a principal hexagon necessarily has one of the two forms described in Figure 5.4:*

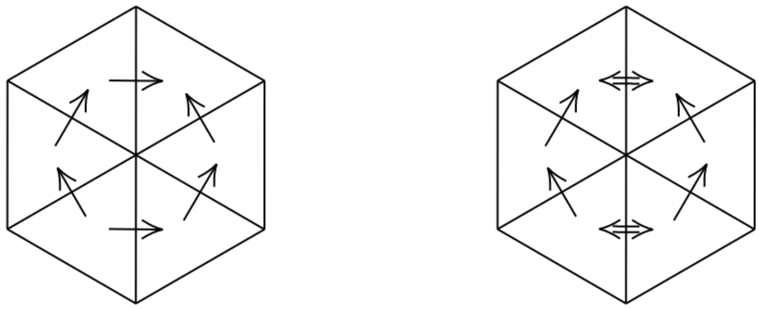


Figure 5.4: The two possible systems of arrows on a principal hexagon, up to symmetries or rotations of the hexagon.

Proof: Consider a principal hexagon obtained as the union of 6 principal triangles $g_i \cdot K$, with $i \in \{1, \cdots, 6\}$. Two adjacent principal triangles $g_i \cdot K$ and $g_{i+1} \cdot K$ $[mod\ 6]$ share an edge, so $g_i^{-1} g_{i+1} = s_i^{k_i}$ for some standard generator $s_i \in V(\Gamma)$. In particular, we have

$$s_1^{k_1} s_2^{k_2} s_3^{k_3} s_4^{k_4} s_5^{k_5} s_6^{k_6} = (g_1^{-1}g_2)(g_2^{-1}g_3)(g_3^{-1}g_4)(g_4^{-1}g_5)(g_5^{-1}g_6)(g_6^{-1}g_1) = 1, \quad (*)$$

where all the k_i are non-zero. Note that the edges between the various principal triangles all meet at a common type 2 vertex of D_{abc}, whose local group is a conjugate of A_{tr} for two standard generators t and r in $\{a, b, c\}$. This means the s_i's are not just any standard generators: they are an alternating sequence of t and r. In particular, $(*)$ becomes

$$t^{k_1} r^{k_2} t^{k_3} r^{k_4} t^{k_5} r^{k_6} = 1.$$

As it turns out, there are very few options on the powers k_i's for such an equality to be possible. These have been classified in [76, Lemma 3.1], and any choice of possible k_i's give rise to one of the two systems of arrows described in Figure 5.4.

□

Remark 5.2.16. One may be able to use Lemma 5.2.15 even if the subset we look at is only part of a principal hexagon. This happens for instance as soon as the centre of the hexagon belongs to the interior of the given subset.

Lemma 5.2.17. *Let u be an axis of z for which we suppose that $u \subseteq D_{abc}^{(1)-ess}$ but there is no element $g \in A_{abc} \backslash \{1\}$ such that $u \subseteq Fix(g)$. Then $\overline{u}$ has valence at most 2 in $\mathcal{T}$.*

Proof: Suppose that $\overline{u} \in \mathcal{T}$ has valence at least 3. We will find a contradiction. Because there is no $g \in A_\Gamma \backslash \{1\}$ such that $u \subseteq Fix(g)$, there exist two consecutive edges e and e' in u that don't have the same stabilisers, i.e. $G_e \neq G_{e'}$. The intersection $v := e \cap e'$ is a vertex of the form $v = h \cdot v_{st}$ for some $s, t \in \{a, b, c\}$ and $h \in A_\Gamma$. By hypothesis, any neighbourhood of $\overline{u}$ in $\mathcal{T}$ contains at least 3 distinct segment meeting at $\overline{u}$. These segments lift to infinite strips in the product $Min(z) = \mathcal{T} \times \mathbf{R}$, and the union of any two of these three strips contains a big enough part of an hexagon of simplices in order to apply Lemma 5.2.15 (see Remark 5.2.16).

We consider (part of) the neighbourhood of v, as described in Figure 5.5. We claim that the only double arrows can appear in this neighbourhood is on edges of u. Indeed, if say the blue half-hexagon had a double arrow between its two upper triangles, then the red and green half-hexagons would have double arrows between their two lower triangles, by Lemma 5.2.15. We then have a contradiction to Lemma 5.2.15 by looking at the hexagon obtained from gluing the red and the green half-hexagons together. From Lemma 5.2.15 again, the two single arrows in the blue half-polygon points towards the same direction. This means that up to replacing s and t by their inverses, we are in the following situation:

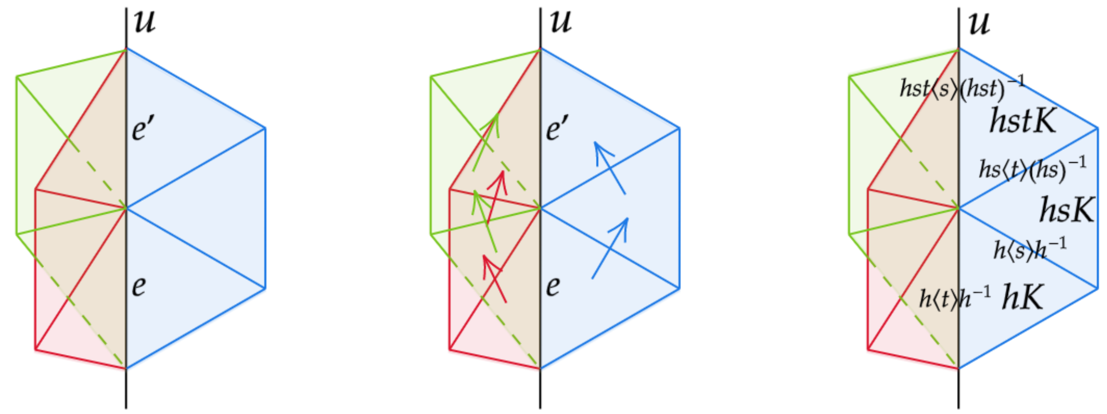


Figure 5.5: On the left: The three half-hexagons around u. In the middle: The only possible system of arrows on the half-hexagons, up to horizontal symmetry. On the right: Some of the simplices around u. The stabilisers of the edges of these simplices can directly be determined from the simplices they belong to.

It is not hard to see that this yields a contradiction, because

$$G_{e'} = (hst) \cdot \langle s \rangle \cdot (hst)^{-1} = h \cdot \langle t \rangle \cdot h^{-1} = G_e.$$

□

Lemma 5.2.18. *Let $z \in A_\Gamma$ be any hyperbolic element, let u be an axis of z, and let $Stab(u)$ be the set of elements of A_Γ that stabilises u. Then:*

If $\nexists g \in A_\Gamma \backslash \{1\} : u \subseteq Fix(g)$, then

$$Stab(u) \cong \langle z_0 \rangle \cong \mathbf{Z},$$

where z_0 acts on u like a non-trivial translation with minimal translation length.

If $\exists g \in A_\Gamma \backslash \{1\} : u \subseteq Fix(g)$, then without loss of generality g is the conjugate of a generator, and

$$Stab(u) \cong \langle g \rangle \times \langle z_0 \rangle \cong \mathbf{Z}^2,$$

where z_0 acts on u like a non-trivial translation with minimal translation length.

Proof: Let $Fix(u)$ be the normal subgroup of $Stab(u)$ consisting of elements of A_Γ that fix u pointwise, and let $\overline{Stab(u)} := Stab(u) \big/ Fix(u)$. It is not hard to see that $Fix(u)$ belongs to the centre of $Stab(u)$. So by construction, $Stab(u)$ can be obtained as a central extension of the following short exact sequence

$$\{1\} \to Fix(u) \to Stab(u) \to \overline{Stab(u)} \to \{1\}. \quad (*)$$

Claim: $\overline{Stab(u)}$ is a discrete subgroup of the group $Isom(u)$ of isometries of u, that consists only of translations.

Proof of the Claim: It is easy to check that $\overline{Stab(u)}$ acts faithfully on u hence is isomorphic to a subgroup of $Isom(u)$. Let $z_0Fix(u) \in \overline{Stab(u)}$. Then $z_0Fix(u)$ acts like a simplicial isometry of the axis u. This already shows $\overline{Stab(u)}$ is a discrete group. To prove that it consists only of translations, we must show that $z_0Fix(u)$, and thus z_0, does not act as a reflection on u. Suppose that z_0 does act like a symmetry on u. Then z_0^2 acts trivially on u. Let $x \in u$ be any point but the central point of the symmetry. Then we have $z_0^2 \in G_x$ but $z_0 \notin G_x$. This contradicts Theorem 4.4.4, and finishes the proof of the Claim.

As a discrete group of translations of the real line, the quotient group $\overline{Stab(u)}$ is isomorphic to $\mathbf{Z}$. It is generated by a shortest possible translation along u, that takes the form $z_0Fix(u)$ for some $z_0 \in Stab(u)$. Let us now come back to the study of $Fix(u)$:

Case 1: $\nexists g \in A_\Gamma \backslash \{1\} : u \subseteq Fix(g)$. We either have $u \nsubseteq D_\Gamma^{(1)-ess}$ or $u \subseteq D_\Gamma^{(1)-ess}$. In the first case, there is an $x \in u$ with trivial local group, and thus $Fix(u) \subseteq Fix(x) = \{1\}$. In the second case, there must be two consecutive edges $e_1, e_2 \subseteq u$ with distinct cyclic local groups. By Theorem 4.1, the intersection of these two local groups is a parabolic subgroup. It is strictly contained inside any of the two cyclic local groups, hence is trivial. Since $Fix(u)$ fixes both edges, it must be trivial too. In both of the cases we obtain $Stab(u) = \overline{Stab(u)} = \langle z_0 \rangle$.

Case 2: $\exists g \in A_\Gamma \backslash \{1\} : u \subseteq Fix(g)$. First note by Lemma 5.1.6 that g has to satisfy $type(g) = 1$. By Corollary 5.1.9, we may as well suppose that g is just a conjugate of a generator. Then $Fix(u)$ has to be cyclic, otherwise we would have edges in u with non-cyclic local group. This means the inclusion $\langle g \rangle \subseteq Fix(u)$ is an equality. Plugging $Fix(u) = \langle g \rangle$ and $\overline{Stab(u)} \cong \mathbf{Z}$ in $(*)$ gives the short exact sequence

$$0 \to \mathbf{Z} \to Stab(u) \to \mathbf{Z} \to 0. \quad (**)$$

By [54, Theorem 3.16], the equivalence classes of possible central extensions for $(**)$ are in one-to-one correspondence with the elements of the cohomology group $H^2(\mathbf{Z}; \mathbf{Z}) = \{1\}$. This means there is only one such extension, and it is the abelian group $\mathbf{Z}^2$. We obtain $Stab(u) = \langle g \rangle \times \langle z_0 \rangle \cong \mathbf{Z}^2$. □

Remark 5.2.19. (1) If the transverse-tree associated with z contains an infinite line, then one can apply Lemma 5.2.8 to any element $g \in Stab(u)$ and any point $x \in u$, and obtain that $g \in A_{abc}$. This shows $Stab(u) \subseteq A_{abc}$.
(2) The choice of z_0 in the above proof is made up to multiplication with an element of $Fix(u)$, i.e. with a power of g.

Lemma 5.2.20. *Let u be an axis of z and suppose that there exists an element $g \in A_\Gamma \backslash \{1\}$ such that $u \subseteq Fix(g)$. Then $\overline{u}$ has infinite valence in $\mathcal{T}$. More precisely, and in the light of Lemma 5.2.18, there is an appropriate choice of $z_0 \in A_\Gamma$ such that we have $Stab(u) \cong \langle g \rangle \times \langle z_0 \rangle \subseteq A_{abc}$ (for appropriate $a, b, c \in V(\Gamma)$) and such that $\langle g \rangle$ acts transitively on the set of edges around $\overline{u}$ and z_0 acts trivially on the set of edges around $\overline{u}$.*

Proof: We first recall that z is supposed to be such that $\mathcal{T}$ contains an infinite line. We are under the hypotheses of Lemma 5.2.6, and there are three standard generators $a, b, c \in V(\Gamma)$ satisfying $m_{ab} = m_{ac} = m_{bc} = 3$ such that $z \in A_{abc}$ and $Min(z) \subseteq D_{abc}$. In particular, D_{abc} is tiled by principal triangles.

Our first goal is to describe $B_{\mathcal{T}}(\overline{u}, \varepsilon)$. Most ideas are similar to the arguments used in the proof of Lemma 5.2.10. However, we will here use slightly more specific tools, as defined thereafter. For any $x \in D_{abc}$, any subset $Y \subseteq D_{abc}$ and any $\varepsilon > 0$, we define the **principal ball** $B^{pr}_Y(x, \varepsilon)$ to be the intersection between the ball $B_Y(x, \varepsilon)$ and the set of all principal triangles of D_{abc} that contain x. For any given $x \in u$, there is always a small enough ε such that the two balls agree. Recall that any principal triangle of $Min(z)$ projects to a segment of length exactly 3 in $\mathcal{T}$. Following the arguments used in the proof of Lemma 5.2.10, for any point $x \in u$ and any $\varepsilon \leqslant 3$, we have

$$B_{\mathcal{T}}(\overline{u}, \varepsilon) \cong B^{pr}_{Min(z)}(x, \varepsilon)/\sim. \;(*)$$

Because $u \subseteq Fix(g)$, we know from Lemma 5.2.18 that we can assume without loss of generality that g is the conjugate of a generator and that $Fix(u) \cong \langle g \rangle \cong \mathbf{Z}$. Consider a type 1 point x in u, whose local group G_x is precisely $\langle g \rangle$. In D_{abc}, the action of the stabiliser of an edge on the set of principal triangles containing that edge is transitive on the set of principal triangles containing that edge. This means the principal ball $B^{pr}_{D_{abc}}(x, 3)$ is the union of principal triangles $\{D_i\}_{i \in \mathbf{Z}}$, around u, for which we have $g \cdot D_i = D_{i+1}$ (see Figure 5.6).

Claim: $B^{pr}_{Min(z)}(x, 3) = B^{pr}_{D_{abc}}(x, 3)$.

Proof of the Claim: The inclusion "$\subseteq$" is trivial, so we show the other inclusion. To do so, consider the 3-neighbourhood of $\overline{u}$ in $\mathcal{T}$. Since $\mathcal{T}$ connected with infinite diameter, the neighbourhood $B^{pr}_{\mathcal{T}}(\overline{u}, 3)$ contains at least one segment of length 3, that lifts to a strip of width 3 around u. Therefore we can assume that D_0 is contained in $B_{Min(z)}(x, 3)$. Let now v be any axis of z going through D_0 but distinct from u (see Figure 5.6). On one hand, the line $g^i \cdot v$ is an axis of $g^i z g^{-i}$. On the other hand, the elements g and z commute by Lemma 5.2.18, and thus $g^i z g^{-i} = z$. This means $g^i \cdot v$ is just another axis of z, hence belongs to $Min(z)$. Because v intersects D_0, the axis $g^i \cdot v$ intersects D_i. Since this argument works for any axis v of h going through D_0, the conjugation by g_i send the union of such axes to another union of axes of h. The first union contains D_0, while the second contains D_i. This proves we have $D_i \subseteq Min(z)$. The argument works for any $i \in \mathbf{Z}$, so the principal triangles $\{D_i\}_{i \in \mathbf{Z}}$ all belong to $Min(z)$, and thus $B^{pr}_{D_{abc}}(x, 3) \subseteq B^{pr}_{Min(z)}(x, 3)$. This finishes the proof of the Claim.

Using $(*)$, the above Claim, and the description of $B^{pr}_{D_{abc}}(x, 3)$, we see that $B_{\mathcal{T}}(\overline{u}, 3)$ is a tree whose segments incoming from $\overline{u}$ form a set of edges $\{e_i\}_{i \in \mathbf{Z}}$ that satisfies $g \cdot e_i = e_{i+1}$. It only remains to show that z_0 can be chosen such that it fixes e_i pointwise, for all $i \in \mathbf{Z}$. Let B_i be the strip corresponding to the lift $e_i \times \mathbf{R}$ of the edge e_i to $Min(z)$ (see Figure 5.6), and let e be the common edge of the principal triangle D_i. As z_0 stabilises u, the edge $z_0 \cdot e$ also belongs to u. In particular, $z_0 \cdot D_0$ intersects u along that edge, which means $z_0 \cdot D_0$ belongs to one of the strips, say B_k. Up to replacing z_0 by $z_0 \cdot g^{-k}$ in the light of Remark 5.2.19.(2), we can assume that $k = 0$. This means that $z_0 \cdot D_0 \subseteq B_0$. Taking the quotient yields $z_0 \cdot e_0 = e_0$, and lifting again gives $z_0 \cdot B_0 = B_0$. This also implies

$$z_0 \cdot B_i = z_0 \cdot (g^i \cdot B_0) = g^i \cdot (z_0 \cdot B_0) = g^i \cdot B_0 = B_i.$$

In particular, $z_0 \cdot e_i = e_i$. Since z_0 preserves each e_i and fixes $\overline{u}$, it must fix each e_i pointwise. $\square$

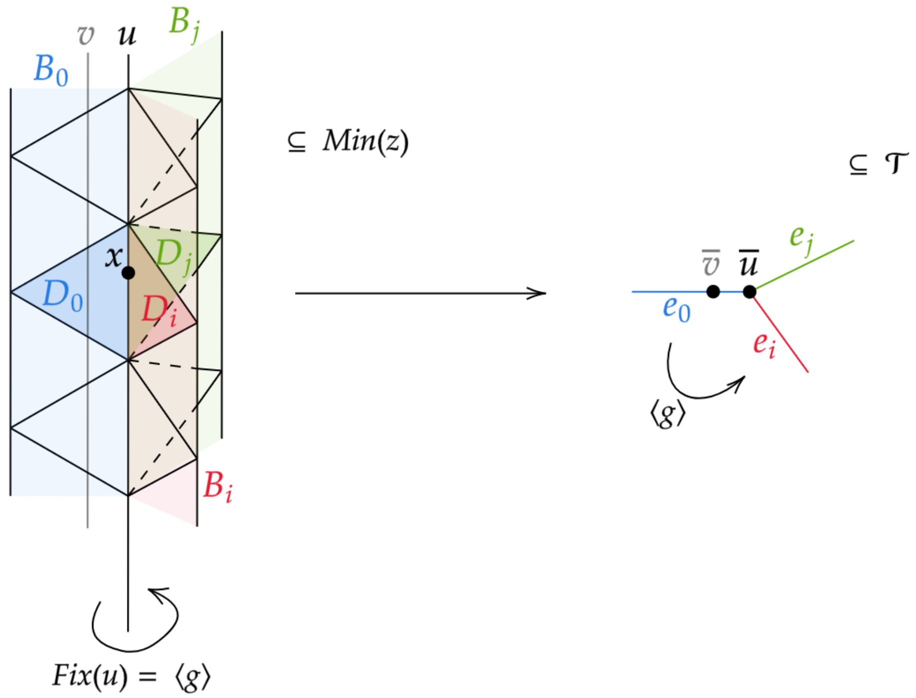


Figure 5.6: The geometric representation of the arguments used in the proof of Lemma 5.2.20. The left of the picture represents what happens in $Min(z)$, while the right of the picture represents what happens in $\mathcal{T}$.

5.2.3 Algebraic description of centralisers.

Let A_Γ be an Artin group of large type, and let $z \in A_\Gamma$ be any hyperbolic element. We suppose as in the previous section that the transverse-tree $\mathcal{T}$ associated with z contains an infinite line, but we now also suppose that it contains a vertex with valence at least 3 (note that it must then have infinite valence, by Corollary 5.2.9). The goal of this section is to use that second hypothesis for an even more precise study. As it turns out, the structure of z, $C(z)$ and $\mathcal{T}$ under these two hypotheses is very rigid. Our goal is to prove the following:

Proposition 5.2.21. *Suppose that $\mathcal{T}$ contains an infinite line and has a vertex of valence at least 3. Then up to conjugation, $z := (abcabc)^n$ for some $n \neq 0$. Moreover, if we set $z_0 := abcabc$, then there is a short exact sequence of the form*

$$\{1\} \to \langle z_0 \rangle \to C(z) \to \overline{C(z)} \to \{1\}, \quad (*)$$

where

$$\overline{C(z)} := C(z) \big/ \langle z_0 \rangle \cong \langle b \rangle * \left(\langle abc \rangle \big/ \langle z_0 \rangle \right) \cong \mathbf{Z} * (\mathbf{Z}/2\mathbf{Z}).$$

In particular, $C(z)$ is a central extension defined by $(*)$. Moreover, $\mathcal{T}$ is isometric to the Bass-Serre tree above the natural segment of groups described by the free product $\mathbf{Z} * \left(\mathbf{Z}/2\mathbf{Z}\right)$.

We recall that $\mathcal{T}$ is supposed throughout this section to contain an infinite line and a vertex of infinite valence. We begin with the following Lemma:

Lemma 5.2.22. *Up to permutation of the elements of the set $\{a, b, c\}$, and up to conjugation by an element of A_{abc}, the element z is given by $z = (abcabc)^n$ for some $n \in \mathbf{Z}\backslash\{0\}$. Moreover if $\overline{u}$ is any vertex of $\mathcal{T}$ with infinite valence, then the corresponding element $z_0 \in Stab(u)$ from Lemma 5.2.20 is $z_0 := abcabc$.*

Proof: Because $\mathcal{T}$ contains a vertex of infinite valence, there must be an axis u of z such that $u \subseteq Fix(g)$ for some element $g \in A_\Gamma\backslash\{1\}$, by Corollary 5.2.9. The element g has type 1, and belongs to A_{abc} by Remark 5.2.19.(1). In particular, up to conjugation by an element of A_{abc}, we can assume that g is a standard generator of A_{abc}, say b for instance. This means $u \subseteq Fix(b)$.

Recall by Lemma 5.2.20 that $\overline{u}$ has infinitely many adjacent vertices in $\mathcal{T}$, so we let $\overline{u}_1$ and $\overline{u}_2$ be two distinct such vertices (see Figure 5.7). Since $\mathcal{T}$ contains an infinite line, one of these two vertices admits at least one other neighbouring vertex, that we call $\overline{u_3}$ (see Figure 5.7). By Lemma 5.2.20, the elements of A_Γ that fix u pointwise form a subgroup $Fix(u) = \langle b \rangle$ that acts transitively on the set of edges around $\overline{u}$. In particular, the convex hull $c(u_1, u)$ is the image of the convex hull $c(u, u_2)$ under an element b^k, with $k \neq 0$. Since $\overline{u}$ has infinite valence, we can assume without loss of generality that u_1 has been chosen so that $|k| \geqslant 2$. In particular, there are double arrows along u, as described in Figure 5.7.

Note that any principal hexagon that splits in two half-hexagons around u carries a system of arrows whose single arrows all point towards the same direction (see Figure 5.7). This is due to Lemma 5.2.15.

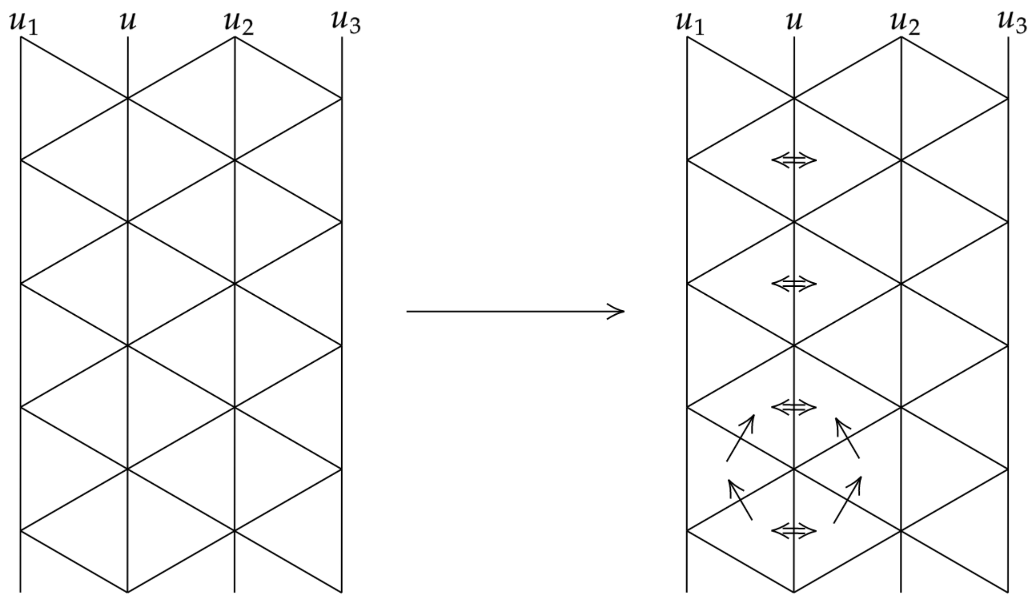


Figure 5.7: The axes u_1, u, u_2 and u_3 of z, along with a partial system of arrows around u.

Claim: The arrows between the principal triangles of the convex hull $c(u, u_2)$ are single arrows and they all point towards the same direction.

Proof of Claim: Suppose that we have in $c(u, u_2)$ arrows that don't point towards the same direction. We will show in the following steps that this yields a contradiction. The different steps refer to Figure 5.8.

Step 1: In order to respect the assumption and the previous statement, there must be two consecutive hexagons around u whose single arrows don't point towards the same direction. So without loss of generality, there are two single arrows pointing towards each others (the blue arrows), say into a principal triangle $h \cdot K$.

Step 2: Use Lemma 5.2.15 to complete the hexagon as drawn (red arrows). Note that the horizontal arrows could be double arrows (in which case every horizontal arrow crossing u_2 must be a doulbe arrow as well), but this doesn't change anything on the rest of the argument.

Step 3: Use Lemma 5.2.15 again to complete the hexagons as drawn (orange arrows).

Step 4: Proceed by induction repeating Step 2 and Step 3 to complete every other hexagon and determine every arrow in the interior of $c(u_1, u_3)$.

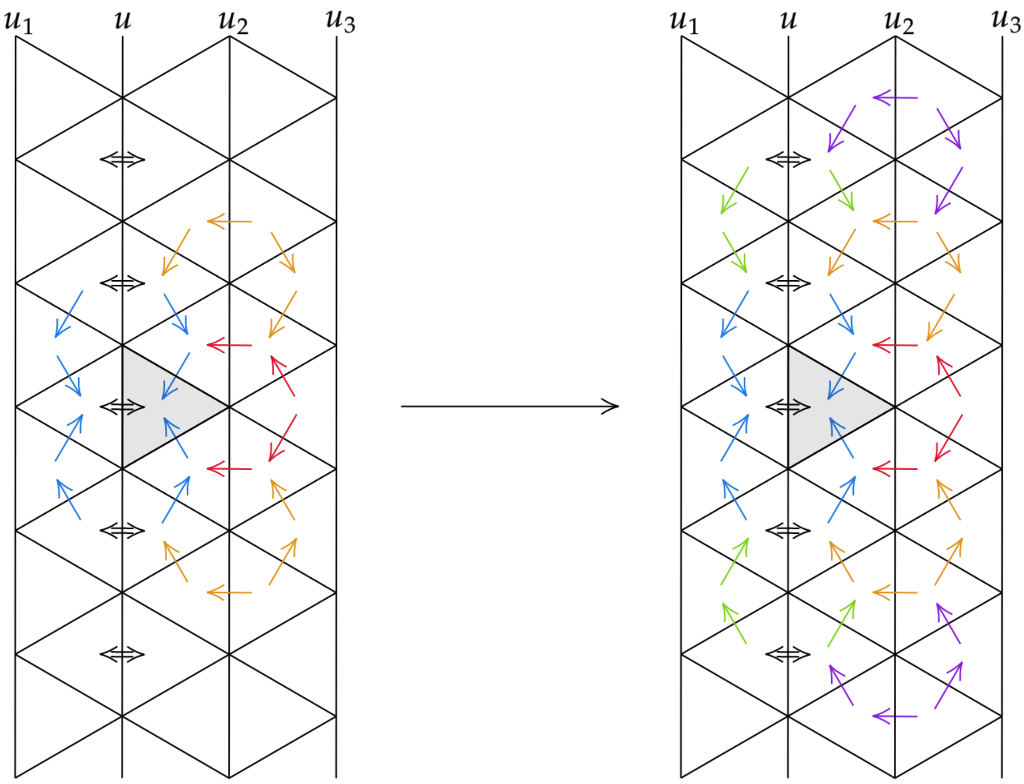


Figure 5.8: The proof of the Claim. The principal triangle $h \cdot K$ is highlighted in grey.
On the left: Step 1 (blue arrows), Step 2 (red arrows) and Step 3 (orange arrows).
On the right: The first iteration of Step 4 (green arrows and then purple arrows). We can apply Step 4 infinitely many time and determine all the arrows in the interior of $c(u_1, u_3)$.

The system of arrows in $c(u_2, u)$ then takes the following form: every arrow above $h \cdot K$ points downwards, and every arrow below $h \cdot K$ points upwards. In particular then, the simplex $h \cdot K$ is the only simplex of $c(u, u_2)$ that has two arrows pointing inside. However, such a property should be inherited by $z \cdot (h \cdot K)$ too, which contradicts uniqueness. This yields a contradiction to the assumption made at the beginning of the proof of the Claim, which eventually proves the Claim.

Let now e_0 be the edge that corresponds to the intersection of K with u. Note that $e_0 \subseteq Fix(b)$, so there is a b^r-translate of K that is contained in $c(u, u_2)$, for some $r \in \mathbf{Z}$. By a similar argument as the one of the claim in the proof of Lemma 5.2.20 we know that the translate $b^{-r} \cdot c(u, u_2)$ is contained in $Min(z)$ as well. So up to applying b^{-r}, we can suppose that K itself is contained in $c(u, u_2)$. By the above Claim, all the arrows in $c(u, u_2)$ are single arrows pointing towards the same direction. We colour every edge of $c(u, u_2)$ so that two edges share the same colour if and only if they are in the same orbit. It is then easy to see from Figure

5.9 that the other edges $\{e_k\}_{k\in\mathbf{Z}}$ of u that are in the orbit of e_0 take the form $(abcabc)^k \cdot e_0$:

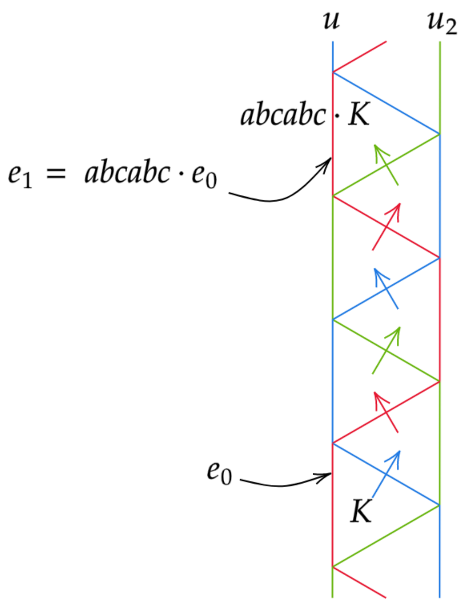


Figure 5.9: The edges of u in the orbit of e_0 can be obtained from e_0 by applying a power of $(abcabc)$. In particular, the elements $(abcabc)^{\pm 1}$ send an edge of u of a given colour to the closest edges in u with the same colour.

The element $abcabc$ acts on u as a translation of minimal length, and hence we can set $z_0 := abcabc$, in the light of Lemma 5.2.18. It remains to show that z_0 acts trivially on the set of edges around $\overline{u}$ in $\mathcal{T}$. This follows from the fact that it preserves the strip described in Figure 5.9 and preserves u. It must then fix one of the edges around $\overline{u}$ in $\mathcal{T}$, and thus all edges around $\overline{u}$, by Lemma 5.2.20. Finally, $z = b^m \cdot (abcabc)^n$ for some $m, n \in \mathbf{Z}$ with $n \neq 0$, by Lemma 5.2.18. Note that z acts trivially on the set of edges around $\overline{u}$, but any $b^m \cdot (abcabc)^n$ with non-trivial m doesn't. This forces $m = 0$, and thus $z = (abcabc)^n$. □

Corollary 5.2.23. *The orbit of any vertex $\overline{u}$ of infinite valence under the action of $C(z)$ is precisely the set of vertices of $\mathcal{T}$. In particular, every vertex of $\mathcal{T}$ has infinite valence.*

Proof: Let u be an axis of z and let $g \in C(z)$. First of all, if $u \subseteq D_{abc}^{(1)-ess}$ then $g \cdot u \subseteq D_{abc}^{(1)-ess}$, because the action is simplicial. In the quotient space $\mathcal{T}$, this means g sends vertices of $\mathcal{T}$ to other vertices of $\mathcal{T}$. It is not hard to see from Figure 5.9 that the element abc sends the axis u onto one of its neighbours $\overline{v}_i$ (it acts as a vertical symmetry of the strip described in Figure 5.9). Moreover, we know by Lemma 5.2.20 that $\overline{v}_i$ is in the orbit of all the other neighbours $\overline{v}_j$ of $\overline{u}$, for $j \in \mathbf{Z}$. This proves that every vertex that is adjacent to $\overline{u}$ is in the orbit of $\overline{u}$. In particular, these vertices have infinite valences, so we can repeat the above process inductively. This yields the desired result. □

Corollary 5.2.24. *$\mathcal{T}$ has no leaf, hence is a simplicial tree (with edge length 3) on which $C(z)$ acts simplicially.*

Proof: Suppose that $\overline{u}$ is a leaf of $\mathcal{T}$. It is easy to see using Lemma 5.2.12 that there is a unique vertex $\overline{v} \in \mathcal{T}$ that is the closest to $\overline{u}$, and that the distance between the two points is bounded by 3. Note that $\overline{v}$ has infinite valence by Corollary 5.2.23. If the distance between $\overline{u}$ and $\overline{v}$ was strictly less than 3, we would obtain a contradiction with Lemma 5.2.20, so this distance must be precisely 3. By Lemma 5.2.12 then, $\overline{u}$ must also be a vertex. It has infinite valence by Corollary 5.2.23, hence cannot be a leaf, by Lemma 5.2.20. $C(z)$ acts simplicially on $\mathcal{T}$ because it preserves its set of vertices, by Corollary 5.2.23.

We now have everything we need in order to prove Proposition 5.2.21, which we do now.

Proof of Proposition 5.2.21: The first statement comes from Lemma 5.2.22, to which we refer for the following arguments. Let u be the axis of h that is contained in $Fix(b)$, let $w := abc \cdot u$ and let v the axis of z that is equidistant from u and w:

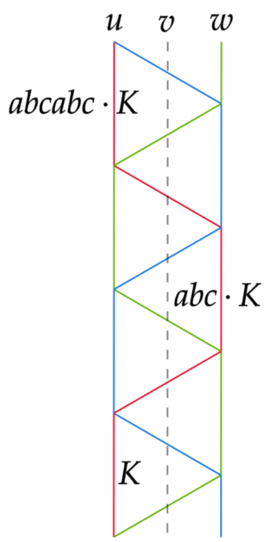


Figure 5.10: The convex hull $c(u, w)$, with the principal triangle $K \subseteq K_\Gamma$.

We say that a segment of $\mathcal{T}$ is a half-edge if its length is half that of an edge of $\mathcal{T}$ and if one of its endpoints is a vertex of $\mathcal{T}$. Let now $\gamma \subseteq \mathcal{T}$ be the half-edge $[\overline{u}, \overline{v}]$. We first prove the following:

Claim: (1) All the half-edges of $\mathcal{T}$ are in the same $C(z)$-orbit.
(2) The element $b\langle z_0\rangle \in \overline{C(z)}$ acts on $\mathcal{T}$ with fixed point $\overline{u}$. Moreover, $\overline{u}$ has infinite valence and $\langle b\rangle\langle z_0\rangle$ acts transitively on the set of edges around $\overline{u}$.
(3) The element $abc\langle z_0\rangle \in \overline{C(h)}$ acts on $\mathcal{T}$ with fixed point $\overline{v}$. Moreover, $\overline{v}$ has valence 2 and $\langle abc\rangle\langle z_0\rangle$ acts transitively on the set of edges around $\overline{v}$.
(4) Any element of $C(z)$ that fixes γ belongs to $\langle z_0\rangle$.

Proof of the Claim: (1) Consider two half-edges γ_1 and γ_2. We know from Corollary 5.2.23 that the vertices of $\mathcal{T}$ are all in the same $C(z)$-orbit. So up to action of $C(z)$ we can assume that γ_1 and γ_2 both contain the vertex $\overline{u}$. Now the action of $\langle b\rangle \subseteq C(z)$ is transitive on the half-edges around $\overline{u}$ (see Lemma 5.2.20), so γ_1 and γ_2 are in the same orbit.
(2) We know that $\overline{u}$ has infinite valence, by Corollary 5.2.9. The element $b\langle z_0\rangle$ preserves u and fixes $\overline{u}$, because $u \subseteq Fix(b)$ and $z_0 \in Stab(u)$. The subgroup $\langle b\rangle\langle z_0\rangle$ acts transitively on the set of vertices around $\overline{u}$, by Lemma 5.2.20.
(3) We know that $\overline{v}$ has valence at most 2, by Corollary 5.2.9. This valence must actually be exactly 2, because $\mathcal{T}$ contains the segment $[\overline{u}, \overline{w}]$ around $\overline{v}$. On one hand, $\langle z_0\rangle$ acts trivially on the set of edges around $\overline{u}$, by Lemma 5.2.20. On the other hand, it is easy to see that the element abc sends u onto w, and reciprocally. In particular, abc preserves v and fixes $\overline{v}$. Together, this means $\langle abc\rangle\langle z_0\rangle$ fixes $\overline{v}$ and acts transitively on the two edges around $\overline{v}$.
(4) Let $g \in C(z)$ and suppose that g fixes γ pointwise. Then g preserves u. Using Lemma 5.2.20 and the fact that g acts trivially on a non-trivial part of an edge around $\overline{u}$, the only possibility is that g is a power of z_0.

We come back to proving the main statement. The half-edge γ is a strict fundamental domain of the action of $\overline{C(z)}$ on $\mathcal{T}$, by (1). Moreover, the various stabilisers under the action of $\overline{C(z)}$ are $\langle b\rangle\langle z_0\rangle$ for the vertex $\overline{u}$, $\langle abc\rangle\langle z_0\rangle$ for the vertex $\overline{v}$, and $\langle z_0\rangle$ for the half-edge γ, by (2), (3) and (4). Note that in $\overline{C(z)}$, these stabilisers are isomorphic to $\mathbf{Z}$ for $\overline{u}$, $\left(\mathbf{Z}/2\mathbf{Z}\right)$ for $\overline{v}$ and $\{1\}$ for γ. The result follows using classical Bass-Serre theory. $\square$

Remark 5.2.25. One can directly see from Proposition 5.2.21 that $C(z)$ does not depend on the value of n. In particular, $C((abcabc)^n) = C(abcabc)$.

5.3 Classifying the dihedral Artin subgroups.

The goal of the present section is to prove Theorem 5.3, which we will do after having proved Theorem 5.5. Besides being interesting on its own, Theorem 5.3 has important consequences, as will be seen in Section 5.4 and Section 5.5.

The strategy in order to prove Theorem 5.3 is to describe the spherical parabolic subgroups of any large-type Artin group A_Γ in a "purely algebraic" manner, i.e. in a way that is preserved under isomorphisms. Large-type Artin groups are 2-dimensional, so their spherical parabolic subgroups are either dihedral Artin subgroups, or infinite cyclic subgroups. Clearly all infinite cyclic subgroups are not parabolic. Pheraps more surprisingly, A_Γ also contains dihedral Artin subgroups that are not parabolic subgroups, in general. In other words, some exotic dihedral Artin subgroups described in Definition 5.1.18 do exist, as soon as A_Γ is not of hyperbolic type. What we would like to do is to be able to differentiate the classical dihedral Artin subgroups from these exotic dihedral Artin subgroups by a criterion that is purely algebraic.

Note that the classical dihedral Artin subgroups that we are interested into are always maximal, as ensured by Corollary 5.1.23. So we will only care to differentiate between classical and exotic dihedral Artin subgroups of A_Γ amongst those that are maximal. Any exotic dihedral Artin subgroup H of A_Γ is contained in the centraliser of a hyperbolic element z generating its centre. These centralisers have being intensely studied throughout Section 5.2. In particular, we were able to give exact presentations of such centralisers (see Proposition 5.2.21). Showing that these centralisers are themselves exotic maximal dihedral Artin subgroups will directly imply that no other exotic maximal dihedral Artin subgroup exists, giving a precise classification of all exotic maximal dihedral Artin subgroups (see Theorem 5.5). This is the goal of Section 5.3.1.

The goal of Section 5.3.2 is to describe an algebraic property that is always satisfied for exotic maximal dihedral Artin subgroups but is never satisfied for classical maximal dihedral Artin subgroups, allowing us to differientiate the two kind of maximal dihedral Artin subgroups purely algebraically (see Corollary 5.3.12). We will then prove Theorem 5.3.

5.3.1 Maximality and presentation.

Let A_Γ be an Artin group of large-type. The centre of any exotic dihedral Artin subgroup H of A_Γ is generated by an element $z \in A_\Gamma$ for which $H \subseteq C(z)$. We saw in Section 5.2 that in this situation the element z takes the form $z = (abcabc)^n$ where $a, b, c \in V(\Gamma)$ satisfy $m_{ab} = m_{ac} = m_{bc} = 3$ and $n \neq 0$. We also describe in Proposition 5.2.21 the way $C(z)$ can be obtained as a central extension.

Let us now come back to a more general case, and consider three standard generators $V(\Gamma)$ satisfying $m_{ac} = m_{ac} = m_{bc} = 3$. We start with the following lemma:

Lemma 5.3.1. *$C(abcabc) = \langle b, abc \rangle$.*

Proof: It is not hard to check that $\langle b, abc \rangle \subseteq C(abcabc)$, so we prove the other inclusion. Let $g \in C(abcabc)$, and let u be the axis of $abcabc$ that belongs to $Fix(b)$ (see Section 5.2.3). By Theorem 5.2.1 the line $g \cdot u$ is also an axis of $abcabc$ (which corresponds to a vertex in the associated transverse-tree). By Proposition 5.2.21 then, there is an element $w \in \langle b, abc \rangle$ such that $w \cdot u = g \cdot u$. It follows that w and g must agree, up to an element $h \in Stab(u)$. By Lemma 5.2.18 and Section 5.2.3, $Stab(u)$ decomposes as a product $Stab(u) \cong \langle b \rangle \times \langle abcabc \rangle \subseteq \langle b, abc \rangle$. Finally, g is a product of two elements of $\langle b, abc \rangle$, hence belongs to $\langle b, abc \rangle$ as well. ☐

Remark 5.3.2. Let $H := \langle s, t \rangle$ be the subgroup of A_Γ generated by

$$s := b^{-1} \text{ and } t := b \cdot abc.$$

If we let $z := abcabc$, then we have $z = stst = tsts$. Moreover, we know from Lemma 5.3.1 that

$$H = \langle b, abc \rangle = C(abcabc) = C(z).$$

We want to show two things:

(1) H really defines a dihedral Artin subgroup of A_Γ. This is the goal of Lemma 5.3.3.

(2) H is maximal. This will be done in Lemma 5.3.4.

Lemma 5.3.3. *H is a dihedral Artin subgroup of A_Γ.*

Proof: Recall that $H = C(z)$, and consider the short exact sequence

$$\{1\} \to \mathbf{Z} \to C(z) \to \mathbf{Z} * \left(\mathbf{Z}/2\mathbf{Z}\right) \to \{1\}, \quad (*)$$

coming from Proposition 5.2.21 and defining the central extension $C(z)$. By [54, Theorem 3.16], the equivalence classes of central extensions of the form $(*)$ are in one-to-one correspondence with elements of the cohomology group

$$H^2(\mathbf{Z} * \left(\mathbf{Z}/2\mathbf{Z}\right); \mathbf{Z}) \;\cong\; H^2(\mathbf{Z}; \mathbf{Z}) \oplus H^2(\left(\mathbf{Z}/2\mathbf{Z}\right); \mathbf{Z}) \;\cong\; \left(\mathbf{Z}/2\mathbf{Z}\right).$$

It follows there are, up to isomorphism, exactly two distinct central extensions satisfying $(*)$, one of which is $C(z)$. These two groups are the following:

$$\left(\mathbf{Z} * \left(\mathbf{Z}/2\mathbf{Z}\right)\right) \times \mathbf{Z} \text{ and } A_4,$$

where A_4 is the dihedral Artin group with coefficient 4. Indeed, the direct product is cleary a fitting central extension, while A_4 is a fitting extension by [14, Lemma 1]. The first group has torsion while the second doesn't. In particular then $C(z)$ must be isomorphic to the second group, i.e. A_4. □

Lemma 5.3.4. *H is maximal amongst the dihedral Artin subgroups of A_Γ.*

Proof: We know from Lemma 5.3.3 that H is an exotic dihedral Artin subgroup of A_Γ. Let H' be a dihedral Artin subgroup of A_Γ satisfying $H' \supseteq H$. Our goal is to show that $H' = H$. We know by Corollary 5.1.19 that H' must also be an exotic subgroup with centre generated by an element z'. We have the following:

$$abc \in C(z) = H \subseteq H' \subseteq C(z'). \quad (*)$$

In particular, the element abc commutes with z', which means z' preserves $Min(abc)$ by Theorem 5.2.1.

<u>Claim:</u> $Min(abc)$ is a single axis, described by the line v in Figure 5.10.

Proof of the Claim: We already know from the proof of Proposition 5.2.21 that the line v of Figure 5.10 is an axis of abc. If v' is another axis of abc, then

v and v' are parallel, and the convex hull $c(v, v')$ is a union of axes of abc (see Theorem 5.2.1). In particular then, there is an axis v'' distinct from v that is arbitrary close to v, say at distance $\varepsilon \leqslant 1$. This axis must belongs to the convex hull $c(u, w)$ described in Figure 5.10. However the element abc acts on this convex hull as a glide reflection around v, whose minset must then only be the central line v. This gives a contradiction, which finishes the proof of the Claim.

Recall that z' preserves $Min(abc) = u$. In particular then, Lemma 5.2.18 applies: $z' \in Stab(u) = \langle z_0 \rangle$, where z_0 is a shortest translation preserving v. It is not hard to notice that abc is such a shortest translation, i.e. that z' is actually a power of abc. Now the element z' described by Lemma 5.2.22 has height $6n$ for some $n \in \mathbf{Z}\backslash\{0\}$. Comparing with the heights of powers of abc, this means we must have $z' = (abc)^{2n} = (abcabc)^n$. Finally, using Remark 5.2.25 we obtain $C(z') = C((abcabc)^n) = C(abcabc) = C(z)$. Together with $(*)$, this shows $H = H'$, as wanted. □

Corollary 5.3.5. *The exotic maximal dihedral subgroups of A_Γ are exactly the subgroups that are conjugated to centralisers of the form*

$$C(z) = \langle b, abc \rangle,$$

where $z = abcabc$ for some generators $a, b, c \in V(\Gamma)$ satisfying $m_{ab} = m_{ac} = m_{bc} = 3$.

Proof: That such a centraliser $C(z)$ is dihedral and maximal follows from Lemma 5.3.3 and Lemma 5.3.4. For the converse, Lemma 5.2.5 along with Lemma 5.2.22 show that the centre of any exotic dihedral subgroup H of A_Γ is generated by an element of the form $z = (abcabc)^n$ for some $n \neq 0$ and some $a, b, c \in V(\Gamma)$ satisfying $m_{ab} = m_{ac} = m_{bc} = 3$. In particular then, $H \subseteq C(z) = C(abcabc)$ by Remark 5.2.25. The centraliser $C(abcabc)$ is dihedral and maximal by Lemma 5.3.3 and Lemma 5.3.4, and thus maximality of H shows that $H = C(abcabc)$. □

We can now put together the various results we proved to recover Theorem 4.5:

Theorem 5.3.6. *Let A_Γ be a large-type Artin group of rank at least 3, and let H be a dihedral Artin subgroup of A_Γ. Then H is conjugated into one of the following:*

(1) $\langle a, b \rangle$, where $a, b \in V(\Gamma)$ satisfy $m_{ab} < \infty$.

(2) $\langle b, abc \rangle$, where $a, b, c \in V(\Gamma)$ satisfy $m_{ab} = m_{ac} = m_{bc} = 3$.

Proof: Let H be a dihedral Artin subgroup of A_Γ. We only need to look at what happens when H is maximal. Now H is either classical or exotic, and a direct use of Lemma 5.1.22 and Corollary 5.3.5 finishes the proof of the theorem. □

5.3.2 Algebraic differentiation of dihedral Artin subgroups.

In Section 5.3.1 we have been able to describe precisely all the maximal exotic dihedral Artin subgroups of A_Γ. By Corollary 5.3.5, they are the centralisers of the form $C(z) = \langle b, abc \rangle$ for appropriate generators. We would like to be able to differentiate these subgroups from the classical maximal dihedral Artin subgroups with a purely algebraic condition, i.e. a condition that is preserved under isomorphisms. The goal of this section is to do precisely that. The next definition introduces the algebraic notion that will allow us to make such a differentiation. As a consequence, we will be able to prove that spherical parabolic subgroups of a large-type Artin group can be defined purely algebraically and are preserved under isomorphisms to other large-type Artin groups (see Theorem 5.3.13).

Definition 5.3.7. A maximal dihedral Artin subgroup H_1 of A_Γ has **isolated intersections** if there exists a maximal dihedral Artin subgroup $H_2 \leqslant A_\Gamma$ distinct from H_1 such that there is no other maximal dihedral Artin subgroup $H_3 \leqslant A_\Gamma$ distinct from H_1 and H_2 for which

$$H_1 \cap H_2 \subseteq H_3.$$

Remark 5.3.8. The notion of being a dihedral Artin subgroup, the notions of intersection or inclusion, and the notion of maximality are all preserved under isomorphisms. In particular, being a maximal dihedral Artin subgroup with no isolated intersections is preserved through isomorphisms as well.

Our goal is to show that the maximal dihedral Artin subgroups of A_Γ with isolated intersection are exactly those that are exotic.

Lemma 5.3.9. *Let H_1 be an exotic maximal Artin subgroup of A_Γ. Then H_1 has isolated intersections.*

We begin by proving the following lemma:

Lemma 5.3.10. *Let $h \in A_\Gamma$ be a hyperbolic element and suppose that no axis of h is contained in $D_\Gamma^{(1)-ess}$, and that the transverse-tree $\mathcal{T}$ of h contains an infinite line. Then $Min(h)$ is a plane that consists of all the lines of D_Γ parallel to u. In particular, this applies to the element $h := babc$.*

Proof: We first prove the general statement. By Lemma 5.2.10, every point of $\mathcal{T}$ has valence 2. It follows that $\mathcal{T}$ is an infinite line, and that $Min(h)$ is a flat plane. Suppose that there is a line w in D_Γ that is parallel to an axis u of h, yet doesn't belong to $Min(h)$. By Theorem 5.2.2, there is a flat strip that connects u to w. Let now v be the line in this strip that cuts the strip into two thinner strips: the strip $c(u, v)$ that belongs to $Min(h)$ and the strip $c(v, w)$ that intersects $Min(h)$ only along v. Since $Min(h)$ is a plane, there must then be at least 3 distinct non-overlapping flat strips meeting at v: one one each side of v in $Min(h)$, and the strip $c(v, w)$. In particular then, for any $\varepsilon > 0$ and any point $x \in v$, the neighbourhood $B_{D_\Gamma}(x, \varepsilon)$ is never just a flat disk. Because $v \subsetneq D_\Gamma^{(1)-ess}$, this contradicts the arguments given in the proof of Lemma 5.2.10. This means no such line w exists, i.e. all lines parallel to u are in $Min(h)$.

To check that this applies to the element $h := babc$ is rather elementary. To picture the situation, an axis of h is described by the blue line in Figure 5.3, call this axis u. The element $z := abcabc$ commutes with h by Lemma 5.3.1, hence acts on $Min(h)$ and on the transverse-tree $\mathcal{T}$ associated with h. It is not hard to check that the action of $abcabc$ on $\mathcal{T}$ is hyperbolic, proving that $\mathcal{T}$ contains an infinite line. In particular, $Min(h)$ is a flat plane. Any other axis of h is parallel to u, and it is not hard using Theorem 5.2.2 the tiling of $Min(h)$ that such a line can never belong to $D_\Gamma^{(1)-ess}$ (see Figure 5.3). □

Proof of Lemma 5.3.9: By Corollary 5.3.5, we can suppose up to conjugation that $H_1 = \langle b, abc \rangle$, where $a, b, c \in V(\Gamma)$ satisfy $m_{ab} = m_{ac} = m_{bc} = 3$. Let us now define another exotic maximal dihedral Artin subgroup H_2 of A_Γ by

$H_2 := \langle a, bac \rangle$, and note that H_2 is distinct from H_1. It is enough to prove that if H_3 is a exotic maximal dihedral Artin subgroup of A_Γ such that $H_1 \cap H_2 \subseteq H_3$, then $H_3 = H_1$ or $H_3 = H_2$.

Let $h := babc = abac$, and note that $h \in H_1 \cap H_2 \subseteq H_3$. We know by Lemma 5.3.10 that $P := Min(h)$ is a plane. Note that the exact structure of this plane is not hard to determine, and is described in Figure 5.11. We first want to show that if z_3 is an element generating the centre of H_3, then P is contained in $Min(z_3)$. To do so, note that $h \in H_3 = C(z_3)$, by Corollary 5.3.5. In particular, h acts on the transverse-tree $\mathcal{T}_3$ of z_3, by Theorem 5.2.1. It is clear that the direction of h and that of z_3 are not the same, simply because the axes of z_3 are parallel to lines in $D_\Gamma^{(1)-ess}$ when the axes of h aren't. In particular then, h must act on $\mathcal{T}_3$ hyperbolically, with an axis that we call γ_3. Consider now the plane $P' := \gamma_3 \times \mathbf{R} \subseteq Min(z_3)$. To prove that P is contained in $Min(z_3)$ is then a consequence of the following:

<u>Claim:</u> $P = P'$.

Proof of the Claim: We first show that h preserves both P and P'. On one hand, h preserves $P = Min(h)$ by definition. On the other hand, Theorem 5.2.1 tells us that the action by isometry of h on $\mathcal{T}_3 \times \mathbf{R}$ decomposes as a couple (h_1, h_2) where h_1 corresponds to the action by isometry of h on $\mathcal{T}_3$, and h_2 corresponds to a translation of the $\mathbf{R}$ component. The action of h_1 restricts to an action on γ_3, and thus the action of h restricts to an action on $\gamma_3 \times \mathbf{R} = P'$.

We now prove that P and P' intersect. Suppose that P and P' are disjoint, and let $M \times M'$ be the subset of $P \times P'$ of couple of points (x, y) minimising the distance between P and P'. Let now $(x, y) \in M \times M'$. Since P and P' are preserved by the action of h, the couple $(h \cdot x, h \cdot y)$ belongs to $P \times P'$. Because the action is via isometries, distance between $h \cdot x$ and $h \cdot y$ is the same as that between x and y. In particular, it is minimising as well, and $(h \cdot x, h \cdot y) \in M \times M'$. Repeating this process shows that M and M' respectively contain the lines ℓ and ℓ' respectively defined by the orbits of x and y under $\langle h \rangle$. Note that because they respectively belong to M and M', the lines ℓ and ℓ' are at constant distance from each others, i.e. they are parallel. Now ℓ is an axis of h, and ℓ' is a line that is parallel to ℓ. By Corollary 5.3.10 then, ℓ' must be an axis of h as well. This means $\ell' \subseteq P$, absurd. So P and P' must intersect.

Consider now a point $x \in P \cap P'$. Because h preserves both P and P', the element $h \cdot x$ belongs to $P \cap P'$ too. In particular, the line ℓ defined by the orbit of x under $\langle h \rangle$ belongs to both P and P'. Now P' can be covered by lines that are all parallel to ℓ. In particular, any such line must belong to P, by Lemma 5.3.10. This shows $P' \subseteq P$. Since the two sets are infinite planes, we obtain $P = P'$, which finishes the proof of the Claim.

We just proved that the plane P described in Figure 5.11 is included inside of $Min(z_3)$. We want to determine the possible values of z_3, by looking at its action on this plane. We have at least two useful pieces of information:
(a) The element z_3 acts trivially on $\mathcal{T}_3$, hence preserves the strips in $Min(z_3)$ that follow the direction of z_3. We know from Section 5.2.3 that these strips live along infinite lines of $D_\Gamma^{(1)-ess}$. So the principal triangle K (labelled by "1" on Figure 5.11) must be sent by z_3 to another principal triangle $z_3 \cdot K$ that also belongs to that strip, i.e. that can be obtained from K by following a line of $D_\Gamma^{(1)-ess}$. Looking at Figure 5.11, there are only 6 possible strips along which z_3 can move K, i.e. 6 possible directions for the action of z_3 on P. They are highlighted in blue in Figure 5.11.
(b) By Corollary 5.3.5, the element generating the centre of H_3 takes the form $g \cdot strstr \cdot g^{-1}$ for some element $g \in A_\Gamma$ and some generators $s, t, r \in V(\Gamma)$ satisfying $m_{st} = m_{sr} = m_{tr} = 3$. This means z_3 is either this element, or its inverse. In particular, the height of z_3 is $ht(z_3) = \pm 6$. The principal triangles $h \cdot K$ for which $ht(h) = \pm 6$ are highlighted in green in Figure 5.11.

The previous observation implies that the only possibilities for z_3 are:

$$z_3 = (abcabc)^{\pm 1} = (z_1)^{\pm 1} \text{ or } z_3 = (bacbac)^{\pm 1} = (z_2)^{\pm 1}.$$

We obtain

$$H_3 = C(z_3) = C(z_1) = H_1 \text{ or } H_3 = C(z_3) = C(z_2) = H_2.$$

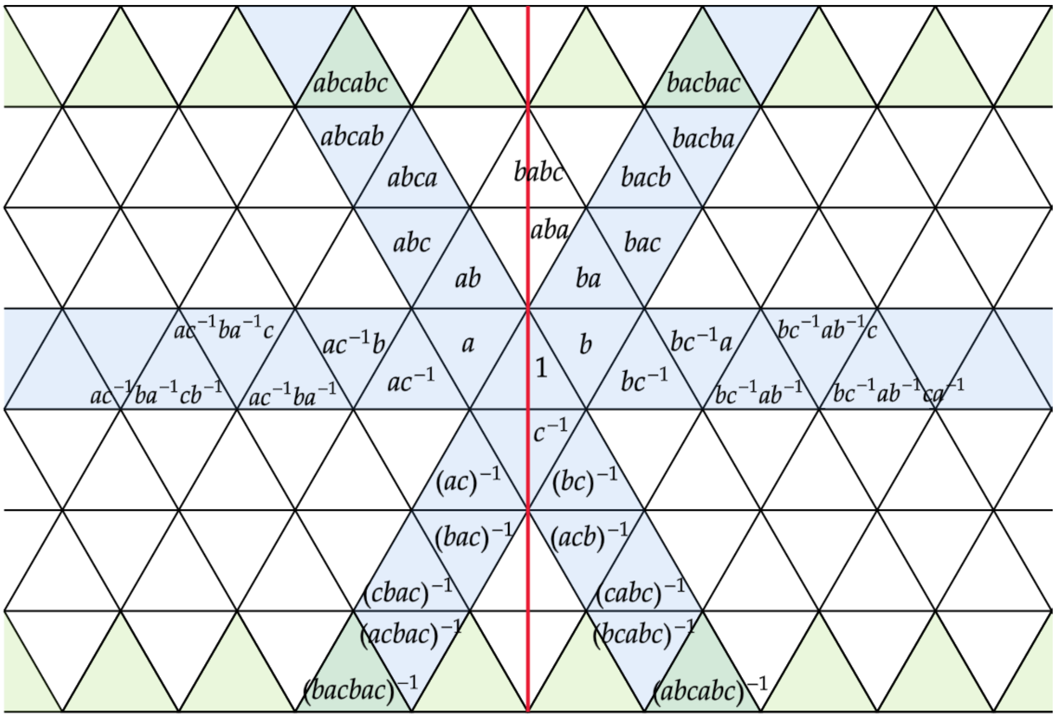


Figure 5.11: A precise description of (some of) the principal triangles of P. For drawing purposes, we only wrote "g" when talking about a principal triangle of the form $g \cdot K$. In blue are highlighted the principal triangles of P that satisfy the condition (a). In green are highlighted the principal triangles $g \cdot K$ of P that satisfy condition (b), i.e. for which $ht(g) = \pm 6$. The axis of $babc$ is highlighted in red. □

Lemma 5.3.11. *Let H_1 be a classical maximal dihedral Artin subgroup of A_Γ. Then H_1 does not have isolated intersections.*

Proof: We know by Lemma 5.1.22 that there are standard generators $a, b \in V(\Gamma)$ such that up to conjugation, $H_1 \subseteq A_{ab}$. Let now H_2 be any maximal dihedral Artin subgroup of A_Γ distinct from H_1 but intersecting H_1 non-trivially. We need to show that there is a maximal dihedral Artin subgroup H_3 of A_Γ distinct from H_1 and H_2, for which $H_1 \cap H_2 \subseteq H_3$.

<u>Claim 1:</u> Any non-trivial element in $H_1 \cap H_2$ has type 1.

Proof of Claim 1: Let $h \in H_1 \cap H_2$ be a non-trivial element. Every element of H_1 has type at most 2 because H_1 is classical, so we only have to show that $type(h) \neq 2$. Suppose the opposite, i.e. that $type(h) = 2$. Then H_2 must be classical, by Corollary 5.1.19. The parabolic closure P_h has type 2 and is contained inside both H_1 and H_2. Since H_1 and H_2 also have type 2, we can use Proposition 5.1.14 to obtain $H_1 = P_h = H_2$, a contradiction. This finishes the proof of Claim 1.

<u>Claim 2:</u> $H_1 \cap H_2$ is cyclic.

Proof of Claim 2: If H_2 is classical, then any element $g \in H_1 \cap H_2$ fixes the fixed sets of H_1 and of H_2. These fixed sets are type 2 vertices, by Lemma 5.1.6, and they are distinct because H_1 and H_2 are distinct. Because the action is by isometries, the element g must also fix (pointwise) the geodesic between these two vertices. Such a geodesic contains a point p of type at most 1, and this point is fixed by any $g \in H_1 \cap H_2$. In particular, $H_1 \cap H_2$ is contained in the stabiliser of p. This stabiliser is cyclic, so we get the desired result.
Let now H_2 be exotic, and suppose that $H_1 \cap H_2$ is not cyclic. Let z_2 be an element generating the centre of H_2, and let $g, g' \in H_1 \cap H_2$. The elements g and g' have type 1 by Claim 1. In particular, they both act elliptically on the transverse-tree $\mathcal{T}_2$ associated with z_2. If the fixed sets of g and g' on $\mathcal{T}_2$ are disjoints, a classical ping-pong argument shows that the product gg' acts hyperbolically on $\mathcal{T}_2$, hence must have type 3. Since $gg' \in H_1 \cap H_2$, we get a contradiction to Claim 1. This means g and g' fix a common point $\overline{u}$ of $\mathcal{T}_2$. In particular, g and g' both belong to the subgroup $Stab(u)$ described in Lemma 5.2.18. They are of type 1, so they must both be powers of the element generating $Fix(u)$. In particular, g and g' belong to a common cyclic group. This finishes the proof of Claim 2.

Look now at the intersection $H_1 \cap H_2$, and let g be an element generating this intersection. Because $type(g) = 1$, we know that $Fix(g)$ is a standard tree in D_Γ, by Lemma 5.1.6. There are infinitely many type 2 vertices on $Fix(g)$. Their associated local groups are maximal dihedral Artin subgroups of A_Γ by Corollary 5.1.23. They are all distinct yet contain $\langle g \rangle$. It follows there is a maximal dihedral Artin subgroup H_3 of A_Γ distinct from both H_1 and H_2 such that $\langle g \rangle = H_1 \cap H_2 \subseteq H_3$. □

Corollary 5.3.12. *Let H be a maximal dihedral Artin subgroup of A_Γ. Then*

$$H \text{ is classical} \Longleftrightarrow H \text{ does not have isolated intersection .}$$

Proof: This directly follows from Lemma 5.3.9 and Lemma 5.3.11. □

We would like to note the important consequences of Corollary 5.3.12. While being a classical maximal dihedral Artin subgroup of A_Γ depends on the type of the elements in the subgroup and thus on the presentation of the group itself, not having isolated intersections is defined purely algebraically and hence preserved by isomorphisms, as emphasised in Remark 5.3.8. These two properties however agree, by Corollary 5.3.12. By Corollary 5.1.23, this means the set of non-free parabolic subgroups of type 2 of A_Γ can be described purely algebraically, and is preserved under isomorphisms. We are now able to prove the main result of Section 5.3, that is, Theorem 5.3

Theorem 5.3.13. *Let A_Γ and $A_{\Gamma'}$ be two large-type Artin groups of rank at least 3. Then any isomorphism $\varphi : A_\Gamma \to A_{\Gamma'}$ induces a bijection between the set of spherical parabolic subgroups of A_Γ and the set of spherical parabolic subgroups of $A_{\Gamma'}$.*

Proof: A direct consequence of the discussion preceding Theorem 5.3.13 is that φ induces a bijection between the set of non-free parabolic subgroups of type 2 of A_Γ and that of $A_{\Gamma'}$. We want to prove that this also holds for the parabolic subgroups of type 1. To do so, we first prove the following.

Claim: The set of parabolic subgroups of type 1 of A_Γ (resp. of $A_{\Gamma'}$) coincides with the set of proper non-trivial intersections of non-free parabolic subgroups of type 2 of A_Γ (resp. of $A_{\Gamma'}$).

Proof of Claim: ($\supseteq$) By Theorem 4.1, the intersection of non-free parabolic subgroups of type 2 of A_Γ is always a parabolic subgroup. If such an intersection is proper and non-trivial, the resulting parabolic subgroup is always of type 1 (use Proposition 5.1.14).

($\subseteq$) Consider a parabolic subgroup H of type 1 of A_Γ. Then $H = h\langle a\rangle h^{-1}$ for some $a \in V(\Gamma)$ and some $h \in A_\Gamma$. By Lemma 5.1.6, $Fix(H)$ is the standard tree $hFix(a)$. Let v and v' be two distinct type 2 vertices of $hFix(a)$. The local groups G_v and $G_{v'}$ are parabolic subgroups of type 2 of A_Γ. They are not free and they are distinct, because their fixed sets are non-empty and disjoint (see Lemma 5.1.6). By Theorem 4.1, their intersection $G_v \cap G_{v'}$ is also a parabolic subgroup of A_Γ. It is strictly contained into G_v and $G_{v'}$ but it is not trivial, so it is a parabolic subgroup of type 1 (use Proposition 5.1.14). The inclusion $H \subseteq G_v \cap G_{v'}$ along with by Proposition 5.1.14 finally gives $H = G_v \cap G_{v'}$. This finishes the proof of the Claim.

The fact that φ induces a bijection between the set of parabolic subgroups of type 1 of A_Γ and that of $A_{\Gamma'}$ is now a direct consequence from the fact that it induces a bijection between the non-free parabolic subgroups of type 2, from the above Claim, and from the fact that being a proper non-trivial intersection is preserved under isomorphisms. Finally, every spherical parabolic subgroup of A_Γ (resp. of $A_{\Gamma'}$) is either a non-free parabolic subgroup of type 2 or a parabolic subgroup of type 1, because A_Γ (resp. $A_{\Gamma'}$) is large hence 2-dimensional. This proves the main statement of the Theorem. $\square$

Corollary 5.3.14. *Let A_Γ and $A_{\Gamma'}$ be two large-type Artin groups of rank at least 3, and suppose that there is an isomorphism $\varphi : A_\Gamma \rightarrow A_{\Gamma'}$. Then for every generator $s \in V(\Gamma)$ there exists a generator $t \in V(\Gamma')$ and an element $g \in A_{\Gamma'}$ such that $\varphi(s) = gt^{\pm 1}g^{-1}$.*

Proof: We know by Theorem 5.3.13 that φ sends the parabolic subgroups of type 1 of A_Γ onto parabolic subgroups of type 1 of $A_{\Gamma'}$. This means $\varphi(\langle s \rangle) = g\langle t \rangle g^{-1}$ for an appropriate $t \in V(\Gamma')$ and $g \in A_{\Gamma'}$. In particular, φ sends any generator of $\langle s \rangle$ to a generator of $g\langle t \rangle g^{-1}$. The result follows. $\square$

Remark 5.3.15. A direct consequence of Corollary 5.3.14 when $A_\Gamma = A_{\Gamma'}$ is that the automorphism group $Aut(A_\Gamma)$ does not contain any transvection.

5.4 Reconstructing the Deligne complex algebraically.

The parabolic subgroups of an Artin group A_Γ do not purely depend on the group itself, but heavily depend on the prescribed set of standard generators of the group. In particular, the Deligne complex D_Γ associated with A_Γ also heavily depends on this set of standard generators. In Section 5.3, we saw that the set of non-free parabolic subgroups of type 2 of A_Γ can be defined with a purely algebraic condition, that does not depend on this set of standard generators (see Theorem 5.3.13). Geometrically, this means one can define the type 2 vertices of D_Γ purely algebraically. The goal of the present section is to extend this construction to the whole complex D_Γ, reconstructing the other vertices, the edges and the simplices of the complex in a purely algebraic way.

Even for the seemingly simplest objects, like the type 1 vertices of D_Γ, the above problem remains complicated. For instance, the correspondence that exists between the type 2 vertices of D_Γ and the non-free parabolic subgroups of type 2 of A_Γ has no analogue for type 1 vertices. Indeed, a parabolic subgroup of type 1 of A_Γ corresponds to a standard tree in D_Γ. This tree contains infinitely many edges, and there is no obvious way to differentiate algebraically two type 1 edges of this tree, because they have the same stabiliser.

In this section, we will require not only that A_Γ is of large-type, but also that its defining graph is complete. In other words, we require that every pair of distinct standard generators $a, b \in V(\Gamma)$ has a coefficient $3 \leqslant m_{ab} < \infty$. Such large-type Artin groups are said to also be free-of-infinity. We start by explaining the notations that we will use throughout the section:

Strategy and notation: As previously mentioned, the strategy of this section is to reconstruct the different vertices, edges and simplices of D_Γ in a purely algebraic way. Our strategy can be divided in four steps. At each step, the goal will be to introduce a set of algebraic objects that "corresponds" to a set of geometric objects of D_Γ. These various correspondences will be made explicit through maps that will be bijections, graph isomorphisms or simplicial isomorphisms, depending on the context. We sum up the various notations that will be used in the following table:

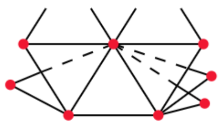
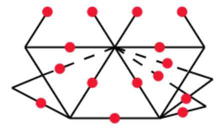
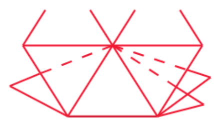
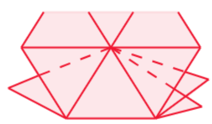
Geometric object	Definition	Algebraic equivalent	Associated map	Picture
V_2	The set of type 2 vertices of X_Γ.	$\mathcal{D}_{V_2}$	$f_{V_2} : \mathcal{D}_{V_2} \to V_2$	
V_1	The set of type 1 vertices of X_Γ.	$\mathcal{D}_{V_1}$	$f_{V_1} : \mathcal{D}_{V_1} \to V_1$	
$D_\Gamma^{(1)-ess}$	See Definition 5.1.1	$\mathcal{D}_1$	$F_1 : \mathcal{D}_1 \to D_\Gamma^{(1)-ess}$	
D_Γ	See Definition 2.4.17	$\mathcal{D}_\Gamma$	$F : \mathcal{D}_\Gamma \to D_\Gamma$	

Figure 5.12: Notations used to describe the various geometric and algebraic objects that will be used in Section 5.4.

5.4.1 Reconstructing $D_\Gamma^{(1)-ess}$.

This first section covers the first three steps of the algebraic reconstruction of D_Γ. The first step will be to build an algebraic equivalent of the set V_2 of type 2 vertices of D_Γ. This is a direct consequence of the results obtained at the end of Section 5.3. The second step will be to build an algebraic equivalent of the set V_1 of type 1 of D_Γ. Finally, the third step will be to describe when the algebraic objects corresponding to the type 2 vertices should be "adjacent" to the algebraic objects corresponding to the type 1 vertices. This will allow us to reconstruct $D_\Gamma^{(1)-ess}$ algebraically.

We let in this section A_Γ be any large-type free-of-infinity Artin group. We start with the following definition:

Definition 5.4.1. We define $\mathcal{D}_{V_2}$ to be the set of classical maximal dihedral Artin subgroups of A_Γ.

Note that $\mathcal{D}_{V_2}$ can equivalently be defined as the set of non-free parabolic subgroups of type 2 of A_Γ, by Corollary 5.1.23. Following the work done in Section 5.3, we know that the elements of $\mathcal{D}_{V_2}$ are precisely the maximal dihedral Artin subgroups of A_Γ that have no isolated intersection (see Definition 5.3.7 and Corollary 5.3.12). In particular, $\mathcal{D}_{V_2}$ can be defined purely algebraically from A_Γ (see Remark 5.3.8).

Lemma 5.4.2. *The map $f_{V_2} : \mathcal{D}_{V_2} \to V_2$ defined as follows is a bijection:*
(1) For every subgroup $H \in \mathcal{D}_{V_2}$, $f_{V_2}(H)$ is the fixed set $Fix(H)$;
(2) For every vertex $v \in V_2$, $f_{V_2}^{-1}(v)$ is the local group G_v.

Proof: This directly follows from Lemma 5.1.6. □

We now come to the harder part of Section 5.4.1: reconstructing the type 1 vertices of D_Γ algebraically. We start with the following definition:

Definition 5.4.3. A couple of subgroups $(H_1, H_2) \in \mathcal{D}_{V_2} \times \mathcal{D}_{V_2}$ is said to have the **adjacency property** if there exists a subgroup $H_3 \in \mathcal{D}_{V_2}$ such that we have

$$(A1)\ H_i \cap H_j \neq \{1\},\ \forall i, j \in \{1, 2, 3\};$$

$$(A2)\ \bigcap_{i=1}^{3} H_i = \{1\}.$$

Definition 5.4.3 really is geometric in essence, and the goal of the next lemma is to highlight that.

Lemma 5.4.4. *A couple (H_1, H_2) has the adjacency property relatively to a third subgroup H_3 if and only if the following hold:*
(1) The three H_i's are distinct subgroups.
(2) The three intersections $(H_i \cap H_j)$'s are parabolic subgroups of type 1, and are distinct. Equivalently, the sets $Fix(H_i \cap H_j)$ are distinct standard trees.
(3) The standard trees $Fix(H_i \cap H_j)$'s intersect each others 2-by-2, but the triple-intersection is trivial.

Proof: ($\Rightarrow$) Suppose that (H_1, H_2) has the adjacency property relatively to a third subgroup H_3. Let $i, j, k \in \{1, 2, 3\}$ be distinct, and suppose that $H_i = H_j$. Then

$$\{1\} \overset{(A2)}{=} H_i \cap H_j \cap H_k = H_i \cap H_k \overset{(A1)}{\neq} \{1\},$$

a contradiction. This proves (1).

In particular, any intersection $H_i \cap H_j$ is a proper non-trivial intersection of parabolic subgroups of type 2 of A_Γ, hence is a parabolic subgroup of type 1 of A_Γ (we use the Claim in the proof of Theorem 5.3.13). It follows that each $Fix(H_i \cap H_j)$ is a standard tree. This proves (2).

Finally, on one hand the three standard trees intersect each others 2-by-2, as for instance the intersection of $Fix(H_i \cap H_j)$ and $Fix(H_i \cap H_k)$ is the vertex $Fix(H_i)$. On the other hand, the intersection of the three standard trees is the intersection of all the 2-by-2 intersections. It is trivial because the three vertices $Fix(H_i)$, $Fix(H_j)$ and $Fix(H_k)$ are distinct, as their corresponding subgroups are. This proves (3).

($\Leftarrow$) Suppose that the three subgroups $H_1, H_2, H_3 \in \mathcal{D}_{V_2}$ satisfy the properties (1), (2) and (3) of the lemma. The fact that all the intersections $(H_i \cap H_j)$'s are parabolic subgroups of type 1 directly implies $(A1)$.

The subgroups $H_i \cap H_j$ and $H_i \cap H_k$ are parabolic subgroups of type 1 of A_Γ, so there intersection is a parabolic subgroup of A_Γ as well, by Theorem 4.1. By Proposition 5.1.14, this intersection cannot be a parabolic subgroup of type 1 of A_Γ, because $H_i \cap H_j$ and $H_i \cap H_k$ are distinct. So it must be trivial. This imples $(A2)$. □

Proposition 5.4.5. *Consider two subgroups $H_1, H_2 \in \mathcal{D}_{V_2}$. Then the following are equivalent:*

(1) The two type 2 vertices v_1, v_2 of D_Γ defined by $v_i := f_{V_2}(H_i)$ are at combinatorial distance 2 in $D_\Gamma^{(1)-ess}$.

(2) The couple (H_1, H_2) satisfies the adjacency property.

Note that the minimal combinatorial distance one can have between two type 2 vertices of $D_\Gamma^{(1)-ess}$ is 2, so the previous proposition gives an algebraic description of when two type 2 vertices of D_Γ are "as close as possible". In order to prove the proposition, we will need the following theorem, which is also known as the combinatorial Gauss-Bonnet formula:

Theorem 5.4.6. ***[70, Theorem 4.6]*** *Let M be a 2-dimensional subcomplex of D_Γ obtained as the union of finitely many polygons. Let M_0 denote the set of type 2 vertices that belong to M, and let M_2 denote the set of polygons whose union is exactly M. A corner of a vertex $v \in M_0$ is a polygon of M in which v is contained, and a corner of a polygon f is a vertex at which two edges of f meet. Let us also define*

$$\forall v \in int(M_0),\ curv(v) := 2\pi - \left(\sum_{c \in Corners(v)} \angle_v(c) \right),$$

$$\forall v \in \partial M_0,\ curv(v) := \pi - \left(\sum_{c \in Corners(v)} \angle_v(c) \right),$$

$$\forall f \in M_2,\ curv(f) := 2\pi - \left(\sum_{c \in Corners(f)} (\pi - \angle_c(f)) \right).$$

Then we have

$$\sum_{f \in M_2} curv(f) + \sum_{v \in M_0} curv(v) = 2\pi.$$

Lemma 5.4.7. *Let x be a vertex of type 1 in D_Γ, i.e. $x = g \cdot v_a$ for some $g \in A_\Gamma$ and $a \in V(\Gamma)$. We recall that Γ_{bar} can be seen as the boundary of the fundamental domain K_Γ of the action of A_Γ on D_Γ, as explained in Remark 2.4.19.*

Then the star $St_{D_\Gamma^{(1)-ess}}(x)$ of x in $D_\Gamma^{(1)-ess}$ is the g-translate of the star $St_{\Gamma_{bar}}(x)$ of x in Γ_{bar}, and takes the form of a n-pod for some $n \geqslant 1$. It is contained in the standard tree $Fix(G_x)$, and in any translate of the fundamental domain that contains x.

Proof: First notice that $St_{D_\Gamma^{(1)-ess}}(x) = St_{D_\Gamma}(x) \cap D_\Gamma^{(1)-ess}$. By [17, Construction II.12.24], the structure of $St_{D_\Gamma}(x)$ can be described as the development of a sub-complex of groups that only depends on the local groups around x. Intersecting with $D_\Gamma^{(1)-ess}$ means further restricting to the local groups around x that contain G_x. These local groups are the g-conjugates of the local groups around v_a, so $St_{D_\Gamma^{(1)-ess}}(x)$ is the g-translate of $St_{\Gamma_{bar}}(v_a)$, which is easily seen to be a n-pod, where n is the number of edges attached to v_a in Γ_{bar} (equivalently, in Γ).

The inclusion $St_{D_\Gamma^{(1)-ess}}(x) \subseteq Fix(G_x)$ comes from the fact that every local group in the star contains G_x. Moreover, $St_{D_\Gamma^{(1)-ess}}(v_a) \subseteq K_\Gamma$ and thus $St_{D_\Gamma^{(1)-ess}}(x) \subseteq h \cdot K_\Gamma$ for every $h \in A_\Gamma$ for which $x \in h \cdot K_\Gamma$. □

Proof of Proposition 5.4.5: [(1) ⇒ (2)]: The vertices v_1 and v_2 are at combinatorial distance 2 from each others, so there is a type 1 vertex x_{12} that is adjacent to both v_1 and v_2. Let us first suppose that x_{12} belongs to K_Γ. By Lemma 5.4.7, K_Γ contains the star $St_{D_\Gamma^{(1)-ess}}(x_{12})$, and this star is the simplicial neighbourhood of x_{12} in Γ_{bar}. In particular then, v_1 and v_2 are distinct vertices of Γ_{bar} that are adjacent to x_{12}. Because Γ is complete, the path joining v_1, x and v_2 can be completed into a cycle $\gamma := (v_1, x_{12}, v_2, x_{23}, v_3, x_{31})$ of length 6 in Γ_{bar}, where the v_i's are type 2 vertices and the x_{ij}'s are type 1 vertices. Let now $H_3 := f_{V_2}^{-1}(v_3)$. All that's left to do is to check that the couple (H_1, H_2) satisfies the adjacency property, with respect to the third group H_3. This directly follow from Lemma 5.4.4: the H_i's are distinct subgroups, the sets $Fix(H_i \cap H_j)$'s are distinct standard trees as they contain the type 1 vertex x_{ij} and no other type 1 vertex of γ, and the trees $Fix(H_i \cap H_j)$'s intersects 2-by-2 along distinct type 2 vertices, hence the triple intersection is trivial.

If x_{12} does not belong to K_Γ, then $x_{12} = g \cdot \overline{x}_{12}$, where $\overline{x}_{12}$ is a type 1 vertex of K_Γ. Proceeding as before on $\overline{x}_{12}$ yields groups H_i for $i \in \{1, 2, 3\}$. Then one can recover an analogous reasoning for x_{12}, using the groups gH_ig^{-1} instead, for $i \in \{1, 2, 3\}$.

[(2) ⇒ (1)]: Let (H_1, H_2) have the adjacency property relatively to a third subgroup H_3, and let $v_i := f_{V_2}(H_i)$ for $i \in \{1, 2, 3\}$. We suppose that the following Claim holds:

Claim: Let v_1, v_2 and v_3 be three distinct type 2 vertices of D_Γ, and suppose that the three geodesics connecting the vertices are contained in distinct standard trees that intersect 2-by-2 but whose triple intersection is empty. Then the triangle formed by these three geodesics is contained in a single fundamental domain $g \cdot K_\Gamma$. In particular, the vertices are at combinatorial distance 2 from each others.

The Claim clearly gives us the desired result, but we still need to show that the hypotheses of the Claim are satisfied. This is a direct consequence of Lemma 5.4.4: the three v_i's are distinct, and the three geodesics of the form γ_{ij} connecting v_i and v_j are contained into the standard trees $Fix(H_i \cap H_j)$. The three γ_{ij}'s intersect 2-by-2, but the triple intersection is empty, by Lemma 5.4.4 again. We now check that the Claim holds:

Proof of the Claim: Let T be the geodesic triangle connecting v_1, v_2 and v_3 and let $M := T \cup int(T)$. We want to prove that M is contained into a single fundamental domain $g \cdot K_\Gamma$. To do so we suppose that it is not the case, and we will exhibit a contraction. We want to apply the Gauss-Bonnet formula on M. By construction, M is a combinatorial subcomplex of D_Γ whose simplices are base triangles of the form $g \cdot T_{st}$. To make the use of the Gauss-Bonnet formula easier, we decide to see M with a coarser combinatorial structure: the one obtained by removing every edge of type 0 and every vertex of type 0 in M. Note that the boundary of M is a union of edges of type 1 of D_Γ, so M is still a subcomplex of D_Γ with this new combinatorial structure. It is a union of polygons of D_Γ whose boundaries are contained in $D_\Gamma^{(1)-ess}$. By Theorem 5.4.6, we have

$$\sum_{\text{faces } f \text{ in } M} curv(f) + \sum_{\text{type 2 vertices in } M} curv(v) = 2\pi. \quad (*)$$

We rewrite this is a manner that is easier to deal with. Let M_2^i be the set of polygons in M that don't contain any element of $\{v_1, v_2, v_3\}$, M_2^c be the set of polygons in M that contain at least one of v_1, v_2 or v_3, M_0^i be the set of type 2 vertices in $int(M)$, M_0^b be the set of type 2 vertices of $\partial M \backslash \{v_1, v_2, v_3\}$, and M_0^c be the set $\{v_1, v_2, v_3\}$ of corners of M. Then:

- Let $C_2^i := \sum_{f \in M_2^i} curv(f)$. Consider a polygon $f \in M_2$, and let m_c be the coefficient of the local group of a corner c of f. Then

$$curv(f) = 2\pi - \left(\sum_{c \in Corners(f)} (\pi - \frac{\pi}{m_c}) \right).$$

Note that $m_c \geqslant 3$ for all $c \in Corners(f)$, so eventually $\pi - \frac{\pi}{m_c} \geqslant \frac{2\pi}{3}$. In particular, f has at least 3 corners, so we obtain

$$curv(f) \leqslant 2\pi - 3 \cdot (\frac{2\pi}{3}) = 0.$$

It follows that $C_2^i \leqslant 0$ as well. Note that as soon as one polygon has at least 4 edges, or as soon as the coefficient of one of the local groups is at least 4, we have $curv(f) < 0$ and thus $C_2^i < 0$.

- Let $C_0^i := \sum_{v \in M_0^i} curv(v)$. Because D_Γ is CAT(0), the systole of the link of any vertex v in D_Γ is at least 2π. In particular, if $v \in M_0^i$, the systole of the link of v in M is at least 2π. It follows that the sum of the angles around v in M is at least 2π. In particular, $curv(v) \leqslant 0$ and thus $C_0^i \leqslant 0$.
- Let $C_0^b := \sum_{v \in M_0^b} curv(v)$. Any $v \in M_0^b$ belongs to a side of T that is a geodesic, so its angle with M must satisfy $\angle_v M \geqslant \pi$. It follows that $curv(v) = \pi - \angle_v M \leqslant 0$, and thus $C_0^b \leqslant 0$ as well.
- Let $C_0^c := \sum_{v_i \in M_0^c} curv(v_i)$ and let $C_2^c = \sum_{f \in M_2^c} curv(f)$. Any corner v_i of $T = \partial M$ belongs to $\lambda_i \geqslant 1$ polygons of M. By construction of the Deligne complex, the angle $\angle_{v_i} M$ is precisely $\lambda_i \cdot \frac{\pi}{m_i}$, where $m_i \geqslant 3$ is the coefficient of H_i. Each of the λ_i polygons f of M containing v_i is such that

$$\begin{aligned} curv(f) &= 2\pi - (\pi - \angle_{v_i}(f)) - \left(\sum_{c \in Corners(f) \setminus \{v_i\}} (\pi - \frac{\pi}{m_c}) \right) \\ &\overset{(**)}{\leqslant} 2\pi - (\pi - \frac{\pi}{m_i}) - 2 \cdot (\pi - \frac{\pi}{3}) \\ &\leqslant \frac{\pi}{m_i} - \frac{\pi}{3}. \end{aligned}$$

The inequality $(**)$ comes from the fact that f has at least 2 other corners than v_i, and that the angle at any corner of f is at most $\pi/3$, because every local group has coefficient at least 3. Note that if f has at least 4 edges then we obtain a strict inequality $curv(f) < \frac{\pi}{m_i} - \frac{\pi}{3}$. Summing everything, we obtain

$$\begin{aligned} C_0^c + C_2^c &= \sum_{v_i \in M_0^c} curv(v_i) + \sum_{f \in M_2^c} curv(f) \\ &\overset{(***)}{\leqslant} \sum_{i \in \{1,2,3\}} (\pi - \lambda_i \cdot \frac{\pi}{m_i}) + \sum_{i \in \{1,2,3\}} \lambda_i \cdot (\frac{\pi}{m_i} - \frac{\pi}{3}) \\ &= 3\pi - \sum_{i \in \{1,2,3\}} \lambda_i \cdot \frac{\pi}{3} \leqslant 2\pi. \end{aligned}$$

Note that it is easy to check that the inequality $(***)$ holds no matter if the polygons containing the v_i's are distinct or if there are polygons of M that contain several of the v_i's. We now notice two things. The first is that as soon as one of the v_i's is contained inside two distinct polygons of M, then $\lambda_i \geqslant 2$ and $C_0^c + C_2^c < 2\pi$. The second is that if a polygon containing one of the v_i's has at least 4 edges, then $curv(f) < \pi/m_i - \pi/3$ and thus $C_0^c + C_2^c < 2\pi$ as well.

With this setting, the equation $(*)$ becomes:

$$C_2^i + C_0^i + C_0^b + (C_0^c + C_2^c) = 2\pi.$$

Note that this equation can hold only if the four terms on the left-hand side are maximal, i.e.:

- $C_2^i = 0$. In particular, every polygon in M_2^i is a triangle, whose corners have local groups with coefficient exactly 3.
- $C_0^i = 0$. In particular, the sum of the angles around any vertex of M_0^i is exactly 2π.
- $C_0^b = 0$, i.e. the angles along the sides of T are exactly π.
- $C_0^c + C_2^c = 2\pi$. In particular, each of the v_i's is contained in a single polygon of M, which is always a triangle.

By hypothesis M does not contain a single polygon, and it is not hard to see that in that case there must be polygons in M that do not contain any of the v_i's (in other words, M_2^i is non-trivial). The first of the above four points implies that every polygon in M_2^i is a flat equilateral triangle. Since the angles along the sides of T are exactly π and since the sum of the angles around any vertex of M_0^i

is 2π, the whole subcomplex M_2^i is actually flat. Let us now consider a triangle $f \in M_2^c$, and let f' be the (unique) polygon in M_2^i that is adjacent to f in the sense that f and f' share an edge (see Figure 5.13). Note that f' is a flat triangle, whose corners have local groups with coefficient 3. We can now use an argument similar to the one used in the proof of Claim 3 of Lemma 5.2.6 to determine the coefficients of the local groups of the corners of f (this is done by propagating a system of colours from f' to f - see Figure 5.13). In particular, the coefficients of the local groups around f must also be 3, which forces f to be an equilateral triangle as well. By applying the same argument to the other polygons of M_2^c, this shows that the whole of M is actually flat, i.e. isometrically embedded into a flat plane.

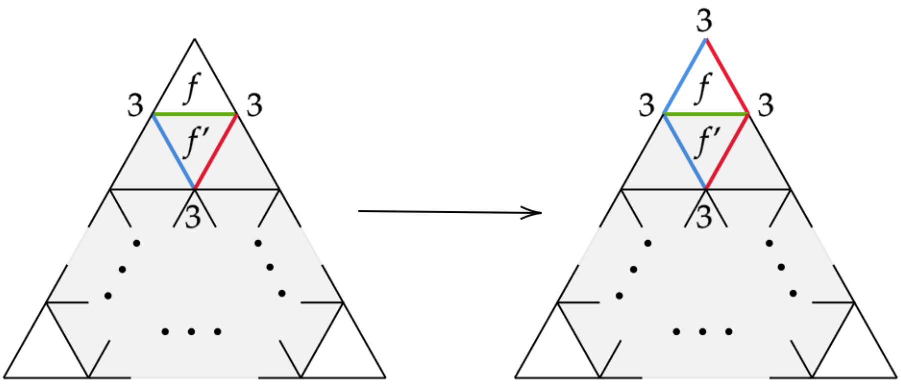


Figure 5.13: Showing that triangles of M_2^c are also equilateral and Euclidean. The simplices that belong to M_2^i are highlighted in grey. They are already known to be equilateral and Euclidean. The edges of f' are drawn with colours corresponding to the edges in K_Γ they are translates of. These colours extend to f, and we can recover the coefficient of the vertex groups of f.

We now put a system of arrows on M (see Definition 5.2.14). Consider a side γ of M. By hypothesis, γ belongs a standard tree $Fix(g\langle s\rangle g^{-1})$ for some $s \in V(\Gamma)$ and some $g \in A_\Gamma$, and $g\langle s\rangle g^{-1}$ acts transitively on the set of strips around γ. Thus we can assume that we have double arrows on γ, as drawn on Figure 5.14. In particular then, all the arrows along γ must be simple arrows, by Lemma 5.2.15. We now proceed to determine all the arrows in M:

Step 1: Put double arrows on the sides of M.

Step 2: The arrow between the two topmost triangles of M must be simple by Lemma 5.2.15. We suppose without loss of generality that it is pointing down.

Step 3: Use Lemma 5.2.15 to complete the hexagons around this first arrow. We obtain two new arrows in M.

Step 4: Use Lemma 5.2.15 on these two arrows and complete an hexagon of M.

Step 5: Proceed by induction using 5.2.15 to determine every arrow in M.

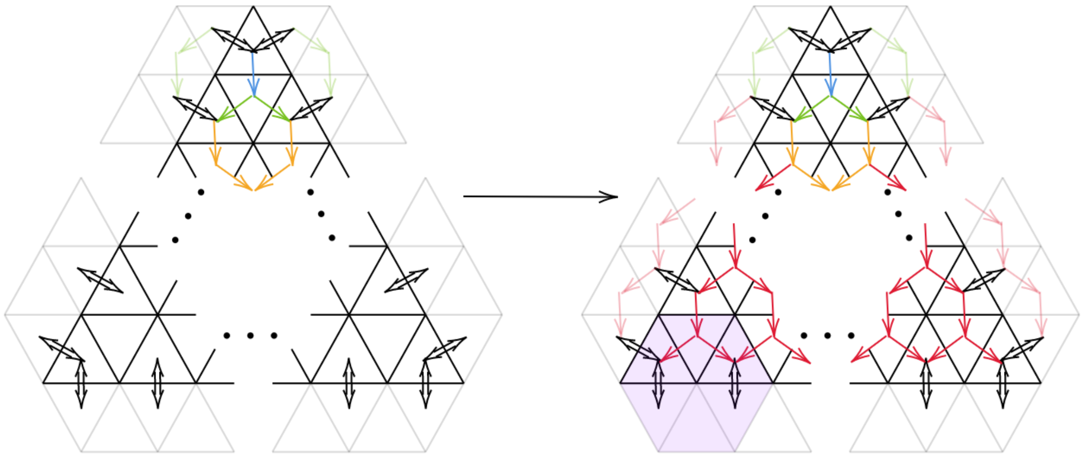


Figure 5.14: Putting a system of arrows on M. On the left: Step 1 (black arrows), Step 2 (blue arrow), Step 3 (green arrows) and Step 4 (orange arrows). On the right: Step 5 (the induction process, red arrows). The purple hexagon gives a contradiction to M being more than one triangle. The simplices and arrows not contained in M are drawn with ligther colours.

Finally, we can see that the system of arrows of any of the hexagons along the bottommost side of M contains two simple arrows pointing away from each other and pointing towards double arrows (see Figure 5.14). This gives a contradiction to Lemma 5.2.15. It follows that M contains a single triangle. In particular, the vertices v_1, v_2 and v_3 are at combinatorial distance 2 from each other. □

We are now able to define explicitely the algebraic analogue of the type 1 vertices of D_Γ:

Definition 5.4.8. Let us consider the poset $\mathcal{P}_f(\mathcal{D}_{V_2})$ of finite sets of distinct elements of $\mathcal{D}_{V_2}$, ordered by the inclusion. We now define $\mathcal{D}_{V_1}$ to be the subset of $\mathcal{P}_f(\mathcal{D}_{V_2})$ of sets $\{H_1, \cdots, H_k\}$ satisfying the following:

(P1) Any subset $\{H_i, H_j\} \subseteq \{H_1, \cdots, H_k\}$ is such that (H_i, H_j) satisfies the adjacency property;

(P2) $\bigcap_{i=1}^{k} H_i \neq \{1\}$;

(P3) $\{H_1, \cdots, H_k\}$ is maximal in $\mathcal{P}_f(\mathcal{D}_{V_2})$ with these properties.

As is was the case for the adjacency property, there is also a geometric meaning behind Definition 5.4.8. While we managed to reconstruct the type 2 vertices of D_Γ directly from the classical maximal dihedral Artin subgroups of A_Γ, we reconstruct a type 1 vertex x of D_Γ from the sets of type 2 vertices of D_Γ that are adjacent to x. This is made more precise thereafter:

Proposition 5.4.9. *The map $f_{V_1} : \mathcal{D}_{V_1} \to V_1$ defined by the following is well-defined and is a bijection:*
(1) For every element $\{H_1, \cdots, H_k\} \in \mathcal{D}_{V_1}$, $f_{V_1}(\{H_1, \cdots, H_k\})$ is the unique vertex $x \in V_1$ that is adjacent to $v_i := f_{V_2}(H_i)$ for every $H_i \in \{H_1, \cdots, H_k\}$.
(2) For every vertex $x \in V_1$, $f_{V_1}^{-1}(x)$ is the set $\{H_1, \cdots, H_k\} \in \mathcal{D}_{V_1}$ of all the subgroups for which $v_i := f_{V_2}(H_i)$ is adjacent to x.

Proof: We first show that the two maps are well-defined. Then, checking that the composition of the two maps gives the identity is straightforward.

f_{V_1} is well-defined: Let $\{H_1, \cdots, H_k\} \in \mathcal{D}_{V_1}$. The intersection $H_1 \cap \cdots \cap H_k$ is an intersection of parabolic subgroups of type 2 of A_Γ, hence is also a parabolic subgroup, by Theorem 4.1. It is proper in any H_i and non-trivial by definition, so it is a parabolic subgroup of type 1 of A_Γ. The corresponding fixed set $T := Fix(H_1 \cap \cdots \cap H_k)$ is a standard tree on which all the vertices $v_i := f_{V_2}(H_i)$ lie. The convex hull C of all the v_i's in T is a subtree of T. By hypothesis, any couple (H_i, H_j) satisfies the adjacency property. Using Proposition 5.4.5, this means the combinatorial distance between any two of the vertices defining the boundary of C is 2, so C has combinatorial diameter 2. As a tree with diameter 2, C contains exactly one vertex that is not a leaf of C, and this vertex must have type 1.

$f_{V_1}^{-1}$ is well-defined: Let now $x \in V_1$, let $\{v_1, \cdots, v_k\}$ be the set of all the type 2 vertices that are adjacent to x, and set $H_i := f_{V_2}^{-1}(v_i)$. We want to check that $\{H_1, \cdots, H_k\} \in \mathcal{D}_{V_1}$, i.e. that the properties (P1), (P2) and (P3) of Definition 5.4.8 are satisfied. First of all, we know that the combinatorial neighbourhood of x is an n-pod that belongs to $Fix(G_x)$, by Lemma 5.4.7. In particular, all the v_i's lie on the standard tree $Fix(G_x)$, which means that G_x is contained in every H_i. This proves (P2).

Proving (P1) is straightforward if we use Proposition 5.4.5: the v_i's are distinct but they are all connected to a common vertex x, so the combinatorial distance between two distinct v_i's is exactly 2.

At last, if $\{H_1, \cdots, H_k\}$ was not maximal, there would be some H_{k+1} such that $\{H_1, \cdots, H_{k+1}\}$ satisfies (P1) and (P2) of Definition 5.4.8. The vertex $v_{k+1} :=$

$f_{V_2}(H_{k+1})$ lies on $Fix(G_x)$ (use (P2)) and is at distance 2 from all the other v_i's (use (P1)), but is not adjacent to x by hypothesis. This means one can connect v_1 and v_2 through $Fix(G_x)$ but without going through the star of x in $Fix(G_x)$. This contradicts $Fix(G_x)$ being a tree. Therefore $\{H_1, \cdots, H_k\}$ is maximal, proving (P3). □

Remark 5.4.10. Let $H \in \mathcal{D}_{V_2}$, $\{H_1, \cdots, H_k\} \in \mathcal{D}_{V_1}$, and let $v := f_{V_2}(H)$, $x := f_{V_1}(\{H_1, \cdots, H_k\})$. Then one can easily deduce from the proof of Proposition 5.4.9 that v and x are adjacent if and only if $H \in \{H_1, \cdots, H_k\}$.

We have now reconstructed the algebraic analogue of the type 2 vertices and the type 1 vertices of D_Γ (see Lemma 5.4.2 and Proposition 5.4.9). To reconstruct the whole of $D_\Gamma^{(1)-ess}$, we only have left to describe when an element of $\mathcal{D}_{V_2}$ and an element of $\mathcal{D}_{V_1}$ should be adjacent. Our method directly follows from Remark 5.4.10:

Definition 5.4.11. We define a graph $\mathcal{D}_1$ by the following:
(1) The vertex set of $\mathcal{D}_1$ is the set $\mathcal{D}_{V_2} \sqcup \mathcal{D}_{V_1}$;
(2) We draw an edge between $H \in \mathcal{D}_{V_2}$ and $\{H_1, \cdots, H_k\} \in \mathcal{D}_{V_1}$ if and only if $H \in \{H_1, \cdots, H_k\}$.

Proposition 5.4.12. *The bijections f_{V_2} and f_{V_1} can be extended into a graph isomorphism $F_1 : \mathcal{D}_1 \to D_\Gamma^{(1)-ess}$.*

Proof: Let $f_{V_2} \sqcup f_{V_1} : \mathcal{D}_{V_2} \sqcup \mathcal{D}_{V_1} \to V_2 \sqcup V_1$. Then $f_{V_2} \sqcup f_{V_1}$ is a bijection by Lemma 5.4.2 and Proposition 5.4.9. We only need to show that two elements of $\mathcal{D}_{V_2} \sqcup \mathcal{D}_{V_1}$ are adjacent if and only if their images through $f_{V_2} \sqcup f_{V_1}$ are adjacent. Notice that

$$\begin{aligned} & H \in \mathcal{D}_{V_2} \text{ and } \{H_1, \cdots, H_k\} \in \mathcal{D}_{V_1} \text{ are adjacent in } \mathcal{D}_1 \\ \overset{(5.4.11)}{\iff} & H \in \{H_1, \cdots, H_k\} \\ \overset{(5.4.10)}{\iff} & f_{V_2}(H) \text{ and } f_{V_1}(\{H_1, \cdots, H_k\}) \text{ are adjacent in } D_\Gamma^{(1)-ess}. \end{aligned}$$

5.4.2 Reconstructing D_Γ.

We saw in the previous section how to reconstruct the graph $D_\Gamma^{(1)-ess}$ in a purely algebraic way. In the current section we will reconstruct the whole of D_Γ algebraically. We suppose throughout this section that A_Γ is a large-type free-of-infinity Artin group.

Definition 5.4.13. A subgraph G of $\mathcal{D}_1$ or of $D_\Gamma^{(1)-ess}$ is called **characteristic** if it is isomorphic to Γ_{bar}, as non-labelled graphs. Then we let $\mathcal{CS}$ be the set of characteristic subgraphs of $\mathcal{D}_1$.

Lemma 5.4.14. *The set of characteristic subgraphs of D_Γ is precisely the set $\{g \cdot \Gamma_{bar} \mid g \in A_\Gamma\}$. In particular, $\mathcal{CS} = \{F_1^{-1}(g \cdot \Gamma_{bar}) \mid g \in A_\Gamma\}$.*

Proof: We focus on proving the first statement, as the second statement directly follows from the first one and the use of Proposition 5.4.12. It is clear that every translate $g \cdot \Gamma_{bar}$ is a characteristic graph, so we only have to show the converse. We first claim the following:

Claim: Any cycle $\gamma \subseteq D_\Gamma^{(1)-ess}$ of length 6 is contained in a single g-translate of the fundamental domain K_Γ.

Proof of the Claim: Recall that $D_\Gamma^{(1)-ess}$ is a bipartite graph with partition sets V_2 and V_1. Consequently $\gamma = (x_1, v_{12}, x_2, v_{23}, x_3, v_{31})$, where the x_i's are type 1 vertices and the v_{ij}'s are type 2 vertices of D_Γ. Consider now the three sub-geodesics $c_1 := (v_{31}, x_1, v_{12})$, $c_2 := (v_{12}, x_2, v_{23})$ and $c_3 := (v_{23}, x_3, v_{31})$, whose union is γ. Each geodesic c_i is contained in the star $St_{D_\Gamma^{(1)-ess}}(x_i)$, which we know by Lemma 5.4.7 is itself included in the standard tree $Fix(G_x)$. Also note that the three corresponding standard trees are distinct, or the fact that γ is a cycle of length 6 would contradict either the convexity of the standard trees, or the fact that they are uniquely geodesic. The three geodesics intersect 2-by-2, but their triple intersection is empty. We can now use the Claim in the proof of Proposition 5.4.5, and recover that γ must be contained in a single translate $g \cdot K_\Gamma$. This finishes the proof of the Claim.

We now come back to our main problem. Let G be a characteristic subgraph. We want to show that G is contained in a single translate $g \cdot \Gamma_{bar}$ for some $g \in A_\Gamma$. First note that because G is isomorphic to Γ_{bar}, the 6-cycles in G correspond to the barycentric subdivisions of the 3-cycles in Γ. In particular, if γ_0 is any 6-cycle in G and e is any edge in G, there exists a finite string of 6-cycles $\gamma_0, \cdots \gamma_n$ such that e belongs to γ_n and such that γ_i, γ_{i+1} share exactly two edges (whose union corresponds to a single edge of Γ). We know by the Claim that each γ_i is contained in a single translate $g_i \cdot K_\Gamma$. We want to show that all the g_i's are the same element. To do so, we show that for every $0 \leqslant i < n$ we have $g_i = g_{i+1}$.

Let $M_i := \gamma_i \cup int(\gamma_i)$. We know by the Claim that $M_i \subseteq g_i \cdot K_\Gamma$ for some $g_i \in A_\Gamma$. The two cycles γ_0 and γ_1 share two edges, whose union corresponds to a single edge of Γ. This means M_i and M_{i+1} share two edges of $D_\Gamma^{(1)-ess}$ (see Figure 5.15). The convex hull of these two edges belongs to a single translate $g \cdot K_\Gamma$, yet belongs to both $g_i \cdot K_\Gamma$ and $g_{i+1} \cdot K_\Gamma$. This proves $g_i = g_{i+1}$. In particular, the edge e belongs to $g \cdot K_\Gamma$. As this works for every edge e of G, we obtain $G \subseteq g \cdot K_\Gamma$.

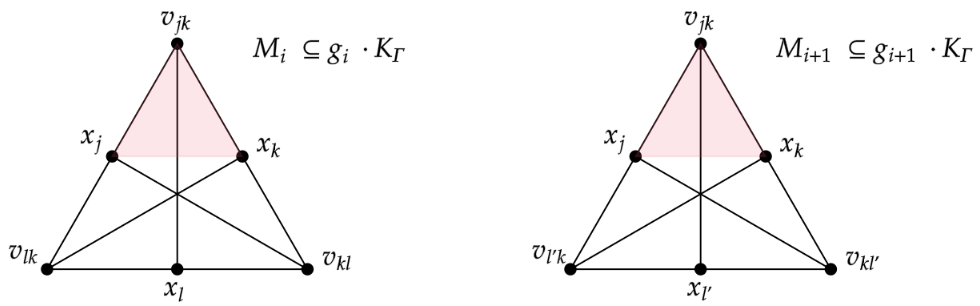


Figure 5.15: The combinatorial subcomplexes M_i (on the left) and M_{i+1} (on the right). Note that M_i and M_{i+1} share three vertices: x_j, v_{jk} and x_k. In particular, they share the convex hull of these vertices, that is highlighted in light red.

Finally, G is contained in the intersection $D_\Gamma^{(1)-ess} \cap g \cdot K_\Gamma = g \cdot \Gamma_{bar}$. But G is isomorphic to Γ_{bar}, so the previous inclusion is actually an equality, i.e. $G = g \cdot \Gamma_{bar}$. □

Definition 5.4.15. Let $\mathcal{D}_\Gamma$ be the 2-dimensional combinatorial complex defined by starting with $\mathcal{D}_1$, and then coning-off every characteristic graph of $\mathcal{D}_1$. The complex $\mathcal{D}_\Gamma$ is called the **algebraic Deligne complex** associated with A_Γ.

Proposition 5.4.16. *The graph isomorphism F_1 from Proposition 5.4.12 can be extended to a simplicial isomorphism $F : \mathcal{D}_\Gamma \to D_\Gamma$.*

Proof: We already know that the map F_1 of Proposition 5.4.12 gives a graph isomorphism between $\mathcal{D}_1$ and $D_\Gamma^{(1)-ess}$. The result now follows from the fact that $\mathcal{D}_\Gamma$ and D_Γ can respectively be obtained from $\mathcal{D}_1$ and $D_\Gamma^{(1)-ess}$ by coning-off their characteristic subgraphs:

- For $\mathcal{D}_\Gamma$, this is simply the definition of the complex;
- For D_Γ, this follows from Lemma 5.4.14 and Remark 5.1.2. □

5.5 Rigidity and Automorphism groups.

Consider a large-type free-of-infinity Artin group A_Γ. In Section 5.4 we introduced various algebraic objects and proved that the Deligne complex D_Γ associated with A_Γ can be reconstructed in a purely algebraic way. This has many consequences for the group. First, it means that large-type free-of-infinity Artin groups that are isomorphic to A_Γ have essentially the same Deligne complexes (Theorem 3.4). We will use this to recover Theorem 5.1. Then, it means the automorphism group $Aut(A_\Gamma)$ acts on D_Γ. In particular, we will see that this action can be used to describe $Aut(A_\Gamma)$ explicitly, which will give Theorem 5.2.

Notation: We know that the Deligne complex D_Γ and the algebraic Deligne complex $\mathcal{D}_\Gamma$ associated with A_Γ are combinatorially isomorphic, by Proposition 5.4.16. To make the notation lighter, we will throughout this section slightly abuse the notation and identify D_Γ with $\mathcal{D}_\Gamma$, without caring about the simplicial isomorphism F.

5.5.1 Rigidity and action of $Aut(A_\Gamma)$ on the Deligne complex.

The main consequence of Section 5.4, and more specifically of Proposition 5.4.16, is Theorem 5.4:

Theorem 5.5.1. *Let A_Γ and $A_{\Gamma'}$ be two large-type free-of-infinity Artin groups of rank at least 3, with respective algebraic Deligne complexes $\mathcal{D}_\Gamma$ and $\mathcal{D}_{\Gamma'}$ (see Definition 5.4.15). Then any isomorphism $\varphi : A_\Gamma \to A_{\Gamma'}$ induces a natural simplicial isomorphism $\varphi_* : \mathcal{D}_\Gamma \to \mathcal{D}_{\Gamma'}$, that can be described explicitely as follows:*
- *For an element $H \in \mathcal{D}^\Gamma_{V_2}$, $\varphi_*(H)$ is simply the subgroup $\varphi(H)$.*
- *For a set $\{H_1, \cdots, H_k\} \in \mathcal{D}^\Gamma_{V_1}$, $\varphi_*(\{H_1, \cdots, H_k\})$ is the set $\{\varphi(H_1), \cdots, \varphi(H_k)\}$;*
- *For an edge e of $\mathcal{D}^\Gamma_1$ connecting H to $\{H_1, \cdots, H_k\}$, $\varphi_*(e)$ is the edge of $\mathcal{D}^{\Gamma'}_1$ connecting $\varphi_*(H)$ to $\varphi_*(\{H_1, \cdots, H_k\})$.*
- *For a simplex f of $\mathcal{D}^\Gamma_1$ connecting H, $\{H_1, \cdots, H_k\}$ and a vertex of type 0 corresponding to the apex of a cone over a characteristic graph G, $\varphi_*(f)$ is the simplex of $\mathcal{D}^{\Gamma'}_1$ connecting $\varphi_*(H)$, $\varphi_*(\{H_1, \cdots, H_k\})$, and the vertex of type 0 corresponding to the apex of the cone over the characteristic graph $\varphi_*(G)$.*

Proof: The result directly follows from the definition of $\mathcal{D}_\Gamma$, that was constructed using algebraic tools that are all preserved under isomorphisms. For the sake of clarity, we give a more detailed proof thereafter. We do that step by step, referring the reader to the different notions introduced in the making of $\mathcal{D}_\Gamma$:

(1) The type 2 vertices (see Definition 5.4.1): $\mathcal{D}^\Gamma_{V_2}$ is the set of non-spherical parabolic subgroups of type 2 of A_Γ. We already know from Theorem 5.3.13 that $\varphi_*(\mathcal{D}^\Gamma_{V_2}) = \mathcal{D}^{\Gamma'}_{V_2}$.

(2) The type 1 vertices (see Definition 5.4.8): $\mathcal{D}^\Gamma_{V_1}$ is the set of finite subsets of $\mathcal{D}^\Gamma_{V_2}$ that satisfy the three conditions (P1), (P2) and (P3). The first condition (P1) is phrased in terms of the adjacency property (see Definition 5.4.3), which is defined in terms of the existence of a subgroup that satisfy two properties (A1) and (A2). These properties are expressed in terms of intersections of the subgroups involved. In particular, one can easily check that the adjacency property for a couple $(H_1, H_2) \in \mathcal{D}^\Gamma_{V_2} \times \mathcal{D}^\Gamma_{V_2}$ is satisfied if and only if the adjacency property for $(\varphi(H_1), \varphi(H_2))$ is satisfied in $\mathcal{D}^{\Gamma'}_{V_2} \times \mathcal{D}^{\Gamma'}_{V_2}$. The property (P2) is defined in terms of a condition of an intersection of subgroups, which is preserved under isomorphisms. The property (P3) is a property of maximality, which is also preserved under isomorphisms. Altogether, we obtain $\varphi_*(\mathcal{D}^\Gamma_{V_1}) = \mathcal{D}^{\Gamma'}_{V_1}$.

(3) The essential 1-skeleton (see Definition 5.4.11): The vertices of $\mathcal{D}^\Gamma_1$ are the type 2 and type 1 vertices previously described. The edges of $\mathcal{D}^\Gamma_1$ are defined as pairs $(H, \{H_1, \cdots, H_k\}) \in \mathcal{D}^\Gamma_{V_2} \times \mathcal{D}^\Gamma_{V_1}$ satisfying $H \in \{H_1, \cdots, H_k\}$. This property of inclusion is obviously preserved under isomorphisms, and thus we have $\varphi_*(\mathcal{D}^\Gamma_1) = \mathcal{D}^{\Gamma'}_1$.

(4) The Deligne complex (see Definition 5.4.15): The simplices of $\mathcal{D}_\Gamma$ can be seen as triplets $(H, \{H_1, \cdots, H_k\}, G) \in \mathcal{D}^\Gamma_{V_2} \times \mathcal{D}^\Gamma_{V_1} \times \mathcal{CS}^\Gamma$ satisfying $H \in \{H_1, \cdots, H_k\}$ and $H, \{H_1, \cdots, H_k\} \in G$. We know by point (3) that $\varphi_*(\mathcal{D}^\Gamma_1) = \mathcal{D}^{\Gamma'}_1$. We first check that for any characteristic graph G of $\mathcal{D}^\Gamma_1$, the graph $\varphi_*(G)$ is also a characteristic graph of $\mathcal{D}^{\Gamma'}_1$. To do so, note that G is by definition the barycentric subdivision of a complete graph on n vertices, where n is the rank of A_Γ. Since φ_* induces an isomorphism of $\mathcal{D}^\Gamma_1$ onto $\mathcal{D}^{\Gamma'}_1$, the graph $\varphi_*(G)$ is also the barycentric subdivision of a complete graph on n vertices. Note that $A_{\Gamma'}$ and A_Γ are isomor-

phic, so $A_{\Gamma'}$ must have rank n as well. It follows that $\varphi_*(G)$ is a characteristic graph of $\mathcal{D}_1^{\Gamma'}$. Note that the fact that $H, \{H_1, \cdots, H_k\} \in G$ along with the previous isomorphism immediately implies that $\varphi_*(H), \varphi_*(\{H_1, \cdots, H_k\}) \in \varphi_*(G)$, and thus φ_* also sends the set of simplices of D_Γ onto the set of simplices of $D_{\Gamma'}$. Two adjacent simplices in $\mathcal{D}_\Gamma$ share two vertices, and it is not hard to check that φ_* sends these vertices onto adjacent vertices of $\mathcal{D}_{\Gamma'}$, and thus sends the simplices onto adjacent simplices. It follows that $\varphi_*(\mathcal{D}_\Gamma) = \mathcal{D}_{\Gamma'}$. □

Remark 5.5.2. A direct consequence of Theorem 5.5.1 and Proposition 5.4.16 is that every isomorphism $\varphi : A_\Gamma \to A_{\Gamma'}$ between large-type free-of-infinity Artin groups yields an isomorphism between the Deligne complexes D_Γ and $D_{\Gamma'}$.

Corollary 5.5.3. *The automorphism group $Aut(A_\Gamma)$ acts naturally and combinatorially on $\mathcal{D}_\Gamma$ and thus on D_Γ.*

Proof: This is a direct consequence of Theorem 5.5.1: any automorphism $\varphi \in Aut(A_\Gamma)$ induces a natural combinatorial automorphism of $\mathcal{D}_\Gamma$, and thus of D_Γ.
□

Remark 5.5.4. The action of an automorphism $\varphi \in Aut(A_\Gamma)$ on $\mathcal{D}_\Gamma$ is entirely determined by its action on the set of type 2 vertices of the complex. This is because every simplex of $\mathcal{D}_\Gamma$, whether it is a type 1 vertex, an edge, or a 2-dimensional simplex, is defined algebraically from the set of type 2 vertices of the complex.

A strong consequence of Theorem 5.5.1 is that we can solve the isomorphism problem for large-type free-of-infinity Artin groups (this is Theorem 5.1).

Theorem 5.5.5. *Let A_Γ and $A_{\Gamma'}$ be two large-type free-of-infinity Artin groups. Then A_Γ and $A_{\Gamma'}$ are isomorphic as groups if and only if Γ and Γ' are isomorphic as labelled graphs.*

Proof: First note that if A_Γ has rank 2, then it has a non-trivial centre (see Definition 3.1.6). In particular, $A_{\Gamma'}$ must also have non-trivial centre, which means it also has rank 2 (use Corollary 3.3). It follows that A_Γ and $A_{\Gamma'}$ are both dihedral Artin groups. Because isomorphic dihedral Artin groups always have the same coefficients (see [83, Theorem 1.1]), the graphs Γ and Γ' must be isomorphic as labelled graphs.

Consider an isomorphism $\varphi : A_\Gamma \to A_{\Gamma'}$. By Theorem 5.5.1, φ induces a simplicial isomorphism $\varphi_* : \mathcal{D}_\Gamma \to \mathcal{D}_{\Gamma'}$ that sends the characteristic subgraphs of $\mathcal{D}_1^\Gamma$ onto the characteristic subgraphs of $\mathcal{D}_1^{\Gamma'}$. In particular then, any characteristic subgraph G of $\mathcal{D}_1^\Gamma$ is sent to a characteristic subgraph $\varphi_*(G)$ of $\mathcal{D}_1^{\Gamma'}$. We state that the isomorphism $\varphi_* : G \to \varphi_*(G)$ is label-preserving. Indeed, every type 2 vertex in G corresponds to a classical maximal dihedral Artin subgroup H of A_Γ with coefficient say m, and the corresponding type 2 vertex in G' corresponds to the dihedral subgroup $\varphi(H)$ that also has coefficient m (once again, isomorphic dihedral Artin groups always have the same coefficients).

By Lemma 5.4.14, there are two elements $g_1 \in A_\Gamma$ and $g_2 \in A_{\Gamma'}$ such that $G = g \cdot \Gamma_{bar}$ and $\varphi_*(G) = g' \cdot \Gamma'_{bar}$. Let $\psi_1 : \Gamma_{bar} \to G$ and $\psi_2 : \Gamma'_{bar} \to \varphi_*(G)$ be the isomorphism defined by the action of g and g' respectively. It is clear that ψ and ψ' are label-preserving. We obtain a string of label-preserving isomorphisms

$$\Gamma_{bar} \overset{\psi_1}{\cong} G \overset{\varphi_*}{\cong} \varphi_*(G) \overset{\psi_2^{-1}}{\cong} \Gamma'_{bar},$$

which finishes the proof of the Theorem. □

5.5.2 Computing the automorphism groups.

Let A_Γ be any large-type free-of-infinity Artin group. This section is dedicated to computing explicitly the automorphism group and the outer automorphism group of A_Γ.

Lemma 5.5.6. *The group $Inn(A_\Gamma)$ of inner automorphisms of A_Γ acts on D_Γ in a natural way: every inner automorphism $\varphi_g : h \mapsto ghg^{-1}$ acts on D_Γ like the element g. Moreover $Inn(A_\Gamma) \cong A_\Gamma$.*

Proof: We begin by proving the first statement. By Remark 5.5.4, it is enough to check that this holds when we restrict the action to type 2 vertices of D_Γ. Let $g \in A_\Gamma$, and let $v \in V_2$ be a type 2 vertex of D_Γ. Then

$$\varphi_g \cdot v := (F \circ \varphi_g \circ F^{-1})(v) = F(\varphi_g(G_v)) = F(gG_vg^{-1}) = F(G_{g \cdot v}) = g \cdot v.$$

The fact that $Inn(A_\Gamma) \cong A_\Gamma$ is a consequence of A_Γ having trivial centre (see Corollary 3.3). □

Lemma 5.5.7. *Let ι be the automorphism of A_Γ defined by $\iota(s) := s^{-1}$ for every generator $s \in V(\Gamma)$, and let $\varphi \in Aut(A_\Gamma)$ be any automorphism. Then one of φ or $\varphi \circ \iota$ is height-preserving.*

Proof: By Corollary 5.5.3 the automorphism φ acts combinatorially on D_Γ. In particular, it sends the vertex $v_\varnothing$ onto the vertex $g \cdot v_\varnothing$ for some element $g \in A_\Gamma$. Using Lemma 5.5.6, the automorphism $\varphi_{g^{-1}} \circ \varphi$ fixes $v_\varnothing$. Since inner automorphisms preserve height, we can suppose up to post-composing by $\varphi_{g^{-1}}$ that φ fixes $v_\varnothing$. In particular, φ preserves Γ_{bar} and thus sends the set of type 1 vertices of K_Γ onto itself. Looking at the action of φ on $\mathcal{D}_\Gamma$, this means φ sends any standard parabolic subgroup of type 1 of A_Γ onto a similar subgroup. Consequently, every standard generator must be sent by φ onto an element that generates such a subgroup, i.e. that has height 1 or -1. There are three possibilities:
(1) $ht(\varphi(s)) = 1, \forall s \in V(\Gamma)$: Then φ is heigh-preserving.
(2) $ht(\varphi(s)) = -1, \forall s \in V(\Gamma)$: Then $\varphi \circ \iota$ is height-preserving.
(3) $\exists s, t \in V(\Gamma) : ht(\varphi(s)) = 1$ and $ht(\varphi(t)) = -1$: This means there are generators $a, b \in V(\Gamma)$ such that $\varphi(s) = a$ and $\varphi(t) = b^{-1}$. Because A_Γ is free-of-infinity, the generators s and t, as well as the generators a and b, generate dihedral Artin subgroups of A_Γ. Note that $\varphi(A_{st}) = \langle \varphi(s), \varphi(t) \rangle = \langle a, b^{-1} \rangle = A_{ab}$. Because φ is an isomorphism we must have $m_{st} = m_{ab}$ (use [83, Theorem 1.1]). Applying φ on both sides of the relation $sts \cdots = tst \cdots$ yields

$$ab^{-1}a \cdots = b^{-1}ab^{-1} \cdots .$$

Note that if we put everything on the same side, we obtain a word with $2m_{st} = 2m_{ab}$ syllables, that is trivial in A_{ab}. The words of length $2m_{ab}$ that are trivial in A_{ab} have been classified in [76, Lemma 3.1], and the word we obtained does not fit this classification, which yields a contradiction. □

Definition 5.5.8. Let $Aut(\Gamma)$ be the group of label-preserving graph automorphism of Γ. We say that an isomorphism $\varphi \in Aut(A_\Gamma)$ is **graph-induced** if there exists a graph automorphism $\phi \in Aut(\Gamma)$ such that $\varphi_*(\Gamma_{bar}) = \phi(\Gamma_{bar})$. We denote by $Aut_{GI}(A_\Gamma)$ the subgroup of $Aut(A_\Gamma)$ consisting of the graph-induced automorphisms.

Remark 5.5.9. Notice that with this definition of graph-induced, the graph-induced automorphisms capture the automorphisms of the group coming from the graph automorphisms, but also the automorphisms coming from the global involution ι.

Lemma 5.5.10. *The map $\mathcal{F} : Aut_{GI}(A_\Gamma) \to Aut(\Gamma) \times \{id, \iota\}$ defined by the following is a group isomorphism:*
Any $\varphi \in Aut_{GI}(A_\Gamma)$ induces an automorphism of Γ_{bar} and thus of Γ. This isomorphism defines the first component of $\mathcal{F}(\varphi)$. The second component of $\mathcal{F}(\varphi)$ is id if φ is height-preserving, and ι otherwise.

Proof: It is easy to check that $\mathcal{F}$ defines a morphism, so we show that it defines a bijection by describing its inverse map. Let $\phi \in Aut(\Gamma) \times \{id, \iota\}$. Then for any standard generator $s \in V(\Gamma)$, the automorphism ϕ sends the vertex v_s corresponding to s onto the vertex $\phi(v_s)$ corresponding to a standard generator that we note s_ϕ. Define φ_ϕ as the (unique) automorphism of A_Γ that sends every standard generator s onto the standard generator s_ϕ. Note that when acting on D_Γ, φ_ϕ restricts to an automorphism of Γ_{bar} that corresponds to the automorphism φ of Γ. For $\varepsilon \in \{0, 1\}$ we let $\mathcal{F}^{-1}((\phi, \iota^\varepsilon)) := \varphi_\phi \circ \iota^\varepsilon$. It is clear that $\varphi_\phi \circ \iota^\varepsilon$ is graph-induced, and it is easy to check that composing $\mathcal{F}^{-1}$ with $\mathcal{F}$ on either side yields the identity. □

We are also able to recover a full description of the automorphism group of large-type free-of-infinity Artin groups, i.e. Theorem 5.2.

Theorem 5.5.11. *Let A_Γ be a large-type free-of-infinity Artin group of rank at least 3. Then we have $Aut(A_\Gamma) \cong A_\Gamma \rtimes (Aut(\Gamma) \times (\mathbf{Z}/2\mathbf{Z}))$ and $Out(A_\Gamma) \cong Aut(\Gamma) \times (\mathbf{Z}/2\mathbf{Z})$.*

Proof: Let $\varphi \in Aut(A_\Gamma)$. The same argument as the one in the proof of Lemma 5.5.7 shows that up to post-composing with an inner automorphism, we may as well assume that φ preserves Γ_{bar}, i.e. that φ is graph-induced. This means

$$Aut(A_\Gamma) \cong Inn(A_\Gamma) \rtimes Aut_{GI}(A_\Gamma),$$

Using Lemma 5.5.6 and Lemma 5.5.10, we obtain

$$Aut(A_\Gamma) \cong A_\Gamma \rtimes (Aut(\Gamma) \times \{id, \iota\}) \cong A_\Gamma \rtimes (Aut(\Gamma) \times (\mathbf{Z}/2\mathbf{Z})).$$

In particular, we have

$$Out(A_\Gamma) \cong Aut(\Gamma) \times (\mathbf{Z}/2\mathbf{Z}).$$

□

Chapter 6

Futures prospects

In this chapter we address several of the questions and problems raised by this thesis, along with potential strategies of resolution.

Problem 6.1. Solve the problem of acylindrical hyperbolicity for other types of Artin groups.

Our solution to Problem 6.1 for 2-dimensional Artin groups (see Theorem 3.1) uses a criterion from [67] (see Theorem 3.4) that relies on the CAT(0)-ness of the space, as well as the local "link condition", and a more global condition of weak malnormality. In [27] Charney and Davis proved that the Deligne complexes associated with Artin groups of type FC are CAT(0). It is thus natural to ask whether this criterion could be used for Artin groups of type FC. The arguments I used in Chapter 3 to prove that the link condition holds are not specific to dimension 2 and it is likely that this condition to also satisfied for Artin groups of type FC. Then it would only remain to show the existence of appropriate weakly malnormal subgroups to prove acylindrical hyperbolicity.

Another possible line of enquiry into acylindrical hyperbolicity is to study the action of Artin groups on suitable CAT(0) cube-complexes. This strategy was used in [28], where the authors looked at actions of Artin groups on what they called the "clique-cube complex". This allowed them to prove acylindrical hyperbolicity for a lot of Artin groups.

Problem 6.2. Solve the problem of rigidity and describe $Aut(A_\Gamma)$ and $Out(A_\Gamma)$ for all large-type Artin groups or all 2-dimensional Artin groups.

Solving these two problems is quite ambitious, but we firmly believe that progress towards solving them can be made. Our solution of Problem 6.2 for large-type free-of-infinity Artin groups relied on reconstructing the Deligne complex in a purely algebraic way, which allowed to build a good action of $Aut(A_\Gamma)$ on the complex (see Corollary 5.5.3). We believe that the results obtained in Theorem 5.1 and Theorem 5.2 can be extended to all large-type Artin groups whose defining graphs have no separating edges or vertices. If this holds true, it would provide an optimal statement of rigidity amongst large-type Artin groups, as the presence of separating edges and vertices allows for diagram twists which give rise to non-isomorphic graphs defining isomorphic Artin groups. It would also provide a maximal subclass of large-type Artin groups whose automorphism groups do not contain "edge twist" isomorphisms (see [32]).

Another approach to study Problem 6.2 lies in our weaker result of rigidity that applies to all large-type Artin groups. Indeed, Theorem 5.3 implies that the "most complicated" vertices of the Deligne complex (i.e. those of type 2) can be reconstructed purely algebraically. It also implies that automorphisms of the group send standard trees onto each others. This allows to reconstruct a large portion of what is known as the coned-off Deligne complex, an extension of the Deligne complex obtained by coning-off these standard trees. This complex has already been studied by various authors in the literature ([53], [75]).

In addition to the previous lines of enquiries, we believe that a large portion of the ideas used in Section 5.2 and Section 5.3 are applicable to 2-dimensional Artin groups. To be more specific, proving Theorem 5.3 involved a profound study of the dihedral Artin subgroups, in which we had to deal with "flats" in the Deligne complex. The flats in the case of dimension 2 are classified (they correspond to the possible triangular tiling of the Euclidean plane), and we already know that most of the arguments we used in Section 5.2 and Section 5.3 will generalise to dimension 2. Hence there is hope to generalise Theorem 5.3 to 2-dimensional Artin groups.

Problem 6.3. Classify (some) Artin groups up to quasi-isometries.

This problem is a natural extension the work we did in Chapter 5. In light of that, a first goal could be to solve Problem 6.3 for large-type free-of-infinity

Artin groups. A successful approach to study this problem of quasi-isometric rigidity is to study the flats and the quasi-flats associated with the group actions. This strategy was notably used to study right-angled Artin groups and some 2-dimensional Artin groups ([66], [55], [56]).

Another perhaps less direct line of investigation for studying quasi-isometries between Artin groups is to prove that they are hierarchically hyperbolic. This notion was introduced a few years ago by Behrstock, Hagen and Sisto in [2]. The idea is to describe the coarse geometry of a group or a space through a "coordinate system" that projects onto various hyperbolic spaces. Hierarchical hyperbolicity gives a strong geometric control over the space, and implies many results that are not true for weaker forms of non-positive curvature. As it turns out, if one knows a group is hierarchically hyperbolic, there is a specific strategy that can be used to study the quasi-isometric rigidity of the given group (see [4], where it was used for mapping class groups). So far, hierarchical hyperbolicity has been proved for braid groups, right-angled Artin groups and extra-large Artin groups ([2], [3], [53]).

Question 6.4. Does the Artin complex carry non-positive curvature properties for classes of Artin groups other than large-type?

Our study of the Artin complex in Chapter 4 raises many questions. For instance, one might wonder whether the Artin complex can be used to study the parabolic subgroups of other classes of Artin groups. Our main tool in studying this complex is the fact that it is systolic (Theorem 5.2), which only happens when the Artin group is large. For most of our results however, we don't need systolicity itself, but rather geometric properties that are consequences of systolicity, such as combinatorial paths being fixed (see Lemma 4.2.5) or simplices being preserved (see Lemma 4.4.5). These properties or similar properties are known to be consequences of other non-positively curved properties.

A partial answer to Question 6.4 has been given by [11], who recently extended part of our results to some Artin groups of dimension 2 by giving the Artin complex a structure of systolic-by-function complex. A possible line of enquiry to extend this result further would be to study metric systolicity, a refinement of the notion of systolicity that would give a more precise study of the complex.

Question 6.5. Can we describe the automorphisms of (classes of) Coxeter groups following the study of the automorphisms of large-type Artin groups?

Most of the tools we use in Chapter 5 are geometric, and rely on a precise study of the Deligne complex. Coxeter groups have a natural analogue to the Deligne complex, called the Davis complex. This complex shares a lot of similarities with the Deligne complex, although it is usually easier to study, as it is locally finite. The strategy of studying the rigidity of Artin and Coxeter groups (almost) simultaneously was used successfully in [32], hence we believe there is hope to apply at least part of our arguments to study Coxeter groups.

Question 6.6. Are large-type Artin groups Hopfian? Are they co-Hopfian?

Recall that a group is said to be Hopfian (resp. co-Hopfian) if every epimorphism (resp. monomorphism) of the group is always an automorphism. The two properties, although being interesting on their own, are consequences of residual finiteness. Residual finiteness is not known for Artin groups in general, but the question has brought interest in the past few years. It is natural after having classified the automorphisms of large-type free-of-infinity Artin groups (see Theorem 5.2) to ask whether our ideas can also be used to study epimorphisms and monomorphisms, hence potentially answering Question 6.6 for (some) large-type Artin groups.

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