Dissipative Solutions To The Stochastic Euler Equations

by

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Abstract

We study the three-dimensional incompressible Euler system subject to stochastic forcing. We develop a concept of dissipative martingale solutions, where the nonlinear terms are described by general Young measures. We construct these solutions as the vanishing limit of solutions to the corresponding stochastic Navier-Stokes equations. This requires a refined stochastic compactness method incorporating the generalised Young measures. As a main novelty, our solutions satisfy a form of the energy inequality which gives rise to the weak-strong uniqueness result (pathwise and in law). A dissipative solution coincides (pathwise and in law) with a strong solution as soon as the later exists.

Furthermore, we extend our results to the compressible Euler system. Here we introduce the concept of *stochastic measure-valued solutions* to the compressible Euler system describing the motion of a temperature-dependent inviscid fluid subject to stochastic forcing, where the nonlinear terms are described by defect measures. These solutions are weak in the probabilistic sense (probability space is not given a 'priori', but part of the solution) and analytical sense (derivatives only exists in the sense distributions). In particular, we show that existence and weak-strong principle (i.e. a weak measure-valued solution coincides with a strong solution provided the later exists), hold true provided they satisfy some form of energy balance. Finally, we show the existence of Markov selection to the associated martingale problem.

Dedication

I dedicate this thesis to God Almighty, a God who fulfils his promises.

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Chapter 1

1.1 Introduction

In *Fluid mechanics* we study the motion of fluids (liquids, gases, and plasmas) and the forces producing the motion. In general we classify these into two categories: *fluid static* (i.e. the study of fluid at rest) and *fluid dynamics* (the study of effect of forces on the fluid motion). Fluid dynamics is fairly an active field of research partly due to its wide range of applications, for instance, in *aerodynamics, meteorology and biology*. On the other hand, the research interest is mathematically motivated as many problems in fluid dynamics are partly solved or unsolved. The thesis seeks to address the later part of research interest.

The dynamics of liquids, gases and plasmas in general are modelled by a system of partial differential equations (PDEs) describing the balance of mass, momentum and energy in the fluid flow. The PDEs approach is *purely* deterministic, and normally used for laminar flows. However, laminar flows are typically unstable, see [7] for more details. Furthermore, many fluid flows in general are turbulent in nature. In 1941, Kolmogorov [71] proposed that small noise prevalent in nature is magnified by the instabilities in the flow and one should consider the velocity in turbulent flow to be a stochastic process. Later on, this led to an increase in the use of stochastic perturbed systems of PDEs to understand turbulent phenomenon (numerical, empirical and physical uncertainties) of fluid flows at high Reynolds number, we refer the reader to [46, 81, 88], and references therein. On this account we study stochastic partial differential equations (SPDEs) in fluid dynamics. In particular, we study the stochastic Euler system of partial differential equations, a classical method of modelling fluid motion. We achieve our goal by narrowing our focus to classical Euler system, Navier-Stokes system and an approximate system of SPDEs that converges to Euler system. We postpone rigorous study of these problems to subsequent sections.

1.1.1 Incompressible stochastic Euler system

We study the stochastic Euler equations describing the motion of an incompressible inviscid fluid in the three-dimensional physical domain $\mathbb{T}^3 \subset \mathbb{R}^3$. To circumvent problems associated with the physical boundaries, we impose periodic boundary conditions, the physical domain $\mathbb{T}^3$ can be identified with a flat torus

$$\mathbb{T}^3 = ([0,1]|_{0,1})^3.$$

Let $Q_T = (0,T) \times \mathbb{T}^3$ be a space-time periodic cylinder, the fluid flow is described by the velocity field $\mathbf{u} : Q_T \to \mathbb{R}^3$, and the pressure $p : Q_T \to \mathbb{R}$ and the system of equations reads

$$\begin{cases} \mathrm{d}\mathbf{u} = -\mathrm{div}(\mathbf{u} \otimes \mathbf{u})\,\mathrm{d}t - \nabla p\,\mathrm{d}t + \phi \mathrm{d}W & \text{in } Q_T, \\ \quad \mathrm{div}\,\mathbf{u} = 0 & \text{in } Q_T, \end{cases} \tag{1.1.1}$$

subject to periodic boundary conditions for $\mathbf{u}$. The first line of equation (1.1.1) is driven by a stochastic force, that is, the cylindrical Wiener process W while ϕ is a Hilbert-Schmidt operator, we refer the reader to Section 3.1.1 for more details.

In a general context, the study of stochastically perturbed equations of motion is motivated as follows, firstly, the reason is as indicated earlier on, that is, modelling perturbations (numerical, empirical, and physical uncertainties) and thermodynamic fluctuations present in fluid flows; in particular, turbulence. Secondly, to circumvent the issue of deterministically ill-posed problems, researchers adopted the use of stochastic perturbation with hope it will provide a regularising effect to the underlying systems. And indeed, recently, the results by Flandoli and Luo [53] showed that a noise of transport type improves the vorticity blow-up control in the Navier-Stokes.

Based on the current research results of three-dimensional stochastic Euler system, smooth solutions to (1.1.1) are only known to exist locally in time, we refer the reader to [69, 57, 78] and references therein for more details. The life span of these solutions is given an a.s. positive stopping time. To begin with, we note that the well-posedness of the incompressible stochastic Euler system in a two-dimensional setting is well understood, see [6, 24, 28, 70] for more details. However, in the three-dimensional case, the existence and uniqueness of global strong solutions is a major open problem. We further note that in the deterministic incompressible Euler system a series of counter examples concerning the uniqueness of solutions have been established. These solutions are called *wild solutions* and they are constructed by using convex integration, a method introduced by De Lellis and Székelyhidi [39, 38]. According to the results presented in [11], introducing stochastic forcing does not seem to change the situation in the context of wild solutions. In regards to our particular system (1.1.1) we refer the reader to recent results of Hofmanová et al [62].

In the context of examples of wild solutions, one may expect to observe singularities in the long-run and non-uniqueness. A natural approach to make sense of such scenarios is the concept of measure-valued solutions as introduced by Di Perna and Majda [42, 43]. These solutions are constructed via compactness arguments and the nonlinearities are described by generalised Young measures, see Section 2.1.5. A generalised Young measure $\mathscr{V} = (\nu_{t,x}, \nu^{\infty}_{t,x}, \lambda)$ consists of an oscillation measure $\nu_{t,x}$ (parametrised probability measure), a concentration angle $\nu^{\infty}_{t,x}$ (parametrised probability measure), and a concentration measure λ (a non-negative Radon measure). The convective term can be re-written as a space time distribution using a generalised measure as follows

$$\mathrm{d}\langle \nu_{t,x}, \xi \otimes \xi \rangle \, \mathrm{d}x\mathrm{d}t + \mathrm{d}\langle \nu^{\infty}_{t,x} \xi \otimes \xi \rangle \, \mathrm{d}\lambda(t,x),$$

where ξ is a dummy variable and

$$\langle \nu_{t,x}, \xi \otimes \xi \rangle := \int_{\mathbb{R}^3} \xi \otimes \xi \, \mathrm{d}\nu_{t,x}(\xi), \quad \langle \nu^{\infty}_{t,x}, \xi \otimes \xi \rangle := \int_{\mathbb{R}^3} \xi \otimes \xi \, \mathrm{d}\nu^{\infty}_{t,x}(\xi).$$

In essence this is the only available framework that allows us to obtain (for any given initial datum) the long-time existence of solutions to the system (1.1.1) while complying with the basic physical principles such as the dissipation of energy (the existence of weak solutions for any initial datum, which violate the energy inequality, has been shown in [89]). In view of the results in [21], the energy inequality implies a weak-strong uniqueness principle for measured-valued solutions in a deterministic setting, that is, a measure-valued solution to Euler system coincides with strong solution as soon as the later exists.

The study of measure-valued solutions is further motivated by applications, firstly, due to results of Brenier et al [21] we view measure-valued solutions as possibly the largest class in which the family of smooth (classical) solutions is stable. Specifically, the weak (measure-valued)-strong uniqueness principle holds, and solutions emanating from numerical schemes can be shown to converge to a measure-valued solution while the convergence to a weak solution is either not known or computational expensive, see Fjordholm et al [52] and references therein for more details.

The results on measure-valued solutions discussed so far apply to the deterministic case. On the other hand there is a strong interest to study measure-valued solutions to the three-dimensional stochastic Euler equations (1.1.1) in order to grasp its long-term dynamics. The first results in this direction were established by Kim in [68]. He showed the existence of martingale solutions to (1.1.1) where the equations of motion are understood in the measure-valued sense. These solutions are weak in the probabilistic sense (that is, the underlying probability space as well as the driving Wiener process are not a priori given but become an integral part of the solution) and analytical sense (derivatives are understood in the sense of distributions). Such a concept is standard for stochastic evolutionary systems when uniqueness is not available. This approach is representative for finite dimensional systems and has also been applied to various stochastic partial differential equations, in particular in fluid mechanics, see [20, 23, 29, 36, 56] and reference therein for more details. The main draw-back of the solutions constructed in [68] is that they only satisfy a form of energy estimate in expectation with an unspecified constant C on the right-hand side instead of an energy inequality as in the deterministic case. Such energy estimate is not sufficient to conclude the weak–strong uniqueness principle holds, a requirement one needs for any reasonable notion of generalised solution, cf [76].

Main results of the incompressible stochastic Euler system

The first phase of the thesis consists of three main results established and published in [18] for an incompressible stochastic Euler system. We also draw the reader's attention to results of [60, 62] that appeared after ours for more insights on properties of martingale solutions to Euler equations. We proceed to give an overview of our results and postpone their rigorous study to later sections.

(1) **Existence of dissipative solutions to (1.1.1) on a torus $\mathbb{T}^3$.**

The aim of these results is to close the gap in [68] and develop a concept of measure-valued martingale solutions to (1.1.1) which satisfy a suitable energy inequality. A concise statement of these main results is given in Theorem 3.1.9. These solutions are weak in the probabilistic sense (that is, the underlying probability space as well as the driving Wiener process are not a priori given but become an integral part of the solution) and analytical sense (derivatives are understood in the sense of distributions). In addition, these solutions are called *dissipative* and our energy inequality can be described using the notion of generalised measures. Let $\mathscr{V} = (\nu_{t,x}, \nu^{\infty}_{t,x}, \lambda)$ be a generalised Young measure associated with the solution, then the kinetic energy (here $\mathscr{L}^1$ denotes the one-dimensional Lebesgue measure)

$$E_t = \frac{1}{2}\int_{\mathbb{T}^3} \langle \nu_{t,x}, |\xi|^2\rangle \,\mathrm{d}x + \frac{1}{2}\lambda_t(\mathbb{T}^3),\, \lambda = \lambda_t \otimes \mathscr{L}^1,$$

satisfies

$$E_{t^+} \leq E_{s^-} + \frac{1}{2}\int_s^t \|\phi\|^2_{L_2}\,\mathrm{d}\tau + \int_s^t \mathbf{u}\cdot\phi\,\mathrm{d}x\,\mathrm{d}W, \quad E_{0^-} = \frac{1}{2}\int_{\mathbb{T}^3} |\mathbf{u}(0)|\,\mathrm{d}x,$$

$\mathbb{P}$-a.s for any $0 \leq s < t$, we refer the reader to Definition 3.1.2 for more details. In the deterministic case the energy is non-increasing and non-negative such that the left- and right-sided limits E_{t^-} and E_{t^+} exist for any t.

In the stochastic case one has instead that the difference between the energy and a continuous function is nonincreasing and that both are pathwise bounded such that the same conclusion holds, see Remark 3.1.3. Nevertheless, some care is required to implement this idea within the stochastic compactness method, see Section 3.1.3.

The Euler equations are linked via a vanishing viscosity limit to the Navier-Stokes equations. The main idea of the proof is to approximate the system (1.1.1) by an incompressible stochastic Navier-Stokes system (3.1.22) (see Section 3.1.2) in a periodic domain $\mathbb{T}^3$. We then establish uniform bounds for the random variables solving the incompressible stochastic Navier-Stokes system and then apply stochastic compactness arguments to the underlying probability laws of these random variables instead of the random variables themselves. We face several difficulties due to the fact that (1.1.1) is infinite-dimensional and, in particular, due to the non-separability of the space of generalised Young measures. We overcome these difficulties in our compactness arguments by using the general Jakubowski–Skorokhod theorem 2.1.21 instead of the more familiar Skorokhod's representation Theorem for Polish spaces.

(2) **Pathwise Weak-strong uniqueness.**

In this case we use the energy inequality introduced above as a tool to establish weak-strong uniqueness property of (1.1.1). The statement of the results is made precise in Theorem 3.1.13. The idea is to assume strong and weak solutions live on the same probability space. In particular, the pathwise approach proves that a dissipative martingale solution agrees with the strong solution if both exist on the same probability space. These results are reminiscent to the deterministic analysis in [21]. At this stage it is crucial that the energy inequality discussed above holds for any time t in order to work with stopping times.

(3) **Weak-strong uniqueness in law.**

The statement of the results is Theorem 3.1.14. In general a more realistic assumption is that the probability spaces on which strong and weak solutions exists are distinct. In this case we show that the probability laws of the weak and the strong solution coincide. This is based on the classical Yamada–Watanabe argument, where a product probability space is constructed. The introduction of the product probability space reduces the weak–strong uniqueness in law problem to the pathwise weak-strong uniqueness already obtained.

Outline of the incompressible stochastic Euler system

Phase one of the thesis is organised as follows: we start off with a general preliminary Section 2.1 and move on mathematical tools used to study the incompressible fluids in Section 3.1.1. In section 3.1.2, we prove existence of martingale solutions to the stochastic incompressible Navier-Stokes. Later on, we dedicate Sections 3.1.3 and 3.1.4, to showing the existence of measure-valued martingale solutions to the stochastic incompressible Euler system using stochastic compactness arguments and proving the weak (measure-valued)-strong uniqueness principle pathwise and in law, respectively.

1.1.2 Complete stochastic Euler system

The study of the compressible fluid models is motivated by the drawbacks encountered in incompressible fluid models. To begin with, we note that the incompressible fluid model provides a good approximation for a variety of engineering applications with slow flows. However, the pressure for these models is not a thermodynamic state variable and as such the fluctuations in density may be neglected. To counter these issues we study the compressible fluid model, a model which incorporates the physical principles (that is, $2^n d$ law of thermodynamics).

To achieve a better representation of physical phenomena on the concept of stochastic measure-valued solutions to the Euler systems we extend the scope to the study of compressible fluids. In this thesis we consider the *complete stochastic Euler System* describing the motion of a temperature dependent compressible inviscid fluid flow driven by stochastic forcing. The fluid model is described by means of three basic state variables: the mass density $\rho = \rho(t,x)$, the velocity field $\mathbf{u} = \mathbf{u}(t,x)$, and the (absolute) temperature $\vartheta = \vartheta(t,x)$, where t is the time, $x \in \mathbb{T}^3$ is the space variable in periodic domain (Eulerian coordinate system). The time evolution of the fluid flow is governed by a system of partial differential equations (mathematical formulations of the physical principles) given by

$$\begin{aligned} \mathrm{d}\rho + \operatorname{div}(\rho\mathbf{u})\,\mathrm{d}t &= 0 \quad \text{in}\, Q, \\ \mathrm{d}(\rho\mathbf{u}) + \operatorname{div}(\rho\mathbf{u}\otimes\mathbf{u})\,\mathrm{d}t + \nabla p(\rho,\vartheta)\,\mathrm{d}t &= \rho\phi\,\mathrm{d}W \quad \text{in}\, Q, \end{aligned} \tag{1.1.2}$$

$$\mathrm{d}\left(\frac{1}{2}\rho|\mathbf{u}|^2+\rho e(\rho,\vartheta)\right) = -\mathrm{div}\left[\left(\frac{1}{2}\rho|\mathbf{u}|^2+\rho e(\rho,\vartheta)+p(\rho,\vartheta)\right)\mathbf{u}\right]\mathrm{d}t + \frac{1}{2}\|\sqrt{\rho}\phi\|_{L_2}^2\,\mathrm{d}t+\rho\phi\cdot\mathbf{u}\mathrm{d}W,$$

satisfying: the balance of mass, momentum, total energy, respectively. Here, $p(\rho,\vartheta)$ denotes pressure, the driving force is represented by a cylindrical Wiener process W, and ϕ is a Hilbert-Schmidt operator, see Section 4.1.3 for details. For completeness, the system (4.1.1) is supplemented by a set of constitutive relations characterising the physical principles of a compressible inviscid fluid. In particular, we assume that the pressure $p(\rho,\vartheta)$ and the internal energy $e=e(\rho,\vartheta)$ satisfy the caloric equation of state

$$p=(\gamma-1)\rho e, \tag{1.1.3}$$

where $\gamma>1$ is the adiabatic constant. In addition, we suppose that the absolute temperature ϑ satisfies the Boyle-Mariotte thermal equation of state:

$$p=\rho\vartheta \quad \text{yielding} \quad e=c_v\vartheta, c_v=\frac{1}{\gamma-1}. \tag{1.1.4}$$

Finally, we suppose that the pressure $p=p(\rho,\vartheta)$, the specific internal energy $e=e(\rho,\vartheta)$, and the specific entropy $s=s(\rho,\vartheta)$ are interrelated through Gibbs' relation

$$\vartheta Ds(\rho,\vartheta)=De(\rho,\vartheta)+p(\rho,\vartheta)D\left(\frac{1}{\rho}\right). \tag{1.1.5}$$

If p,e,s satisfy (1.1.5), in context of any *smooth* solutions to (4.1.1), the Second law of thermodynamics is enforced through the entropy balance equation

$$\mathrm{d}(\rho s(\rho,\vartheta))+\mathrm{div}_x(\rho s(\rho,\vartheta)\mathbf{u})\,\mathrm{d}t=0, \tag{1.1.6}$$

where $s(\rho,\vartheta)$ denotes the (specific) entropy and is of the form

$$s(\rho,\vartheta)=\log(\vartheta^{c_v})-\log(\rho). \tag{1.1.7}$$

For weak solutions, the equality in (1.1.6) no longer holds, the entropy balance is given as an inequality, for more details see [4]. In view of (1.1.4), the state variables ρ,ϑ trivially imply the thermodynamics stability given as

$$\partial_\rho p(\rho,\vartheta) > 0, \quad \partial_\vartheta p(\rho,\vartheta) > 0 \quad \text{for all}\, \rho,\vartheta > 0. \tag{1.1.8}$$

Finally, the initial state of fluid emanates from random initial data

$$\rho(0,\cdot) = \rho_0, \quad \vartheta(0,\cdot) = \vartheta_0, \quad \mathbf{u}(0,\cdot) = \mathbf{u}_0, \tag{1.1.9}$$

subject to initial law Λ. For physical relevant solutions, the problem is augmented by the total energy balance

$$\mathrm{d}\int_{\mathbb{T}^3}\left[\frac{1}{2}\rho|\mathbf{u}|^2+\rho e\right]\mathrm{d}x = \int_{\mathbb{T}^3}\rho\phi\cdot\mathbf{u}\,\mathrm{d}x\mathrm{d}W + \frac{1}{2}\|\sqrt{\rho}\phi\|_{L_2}^2\,\mathrm{d}t. \tag{1.1.10}$$

The strong solutions of the system (1.1.2) satisfy (1.1.10) as a consequence of (1.1.4), but in weak solutions it has to be added in the definition.

In the deterministic case, the concept of measure-valued solutions was extended to compressible fluid dynamics by Neustupa [80], Kröner and Zajackowski [73], and revisited recently by Breit et al [10], Feireisl et. al. in [47, 48] and references therein, where they developed the concept of *dissipative measure-valued solutions*. Moreover, the deterministic counterpart of the Cauchy problem (1.1.2) has been extensively studied, and it is well-known that its solutions exist only for a finite time after which singularities may develop no matter how smooth or small the initial data are. Consequently, the concept of weak (distributional) solutions is sought to study global-in-time behavior of the system (1.1.2). Furthermore, the weak solutions may not be uniquely determined by their initial data. Hence, an admissibility criteria condition must be imposed to select physically relevant solutions. In addition, more recently, the results of DeLellis, Székelyhidi and their collaborators [33, 32, 38] show the existence and non-uniqueness of weak solutions to Euler system via the method of convex integration. In particular, non-uniqueness was established for weak solutions satisfying the standard entropy admissibilty criteria, and to be precise, the deterministic compressible Euler system is ill-posed, see Chiodaroli et al [34] and references therein for more details.

In the stochastic context, existence of global-in-time weak solutions for (1.1.2) were shown in [31] using convex integration, and they identified a large class of initial data for which the *complete* Euler system is ill-posed; that is, there exist infinitely many global in time solutions. Our goal in this part of the thesis is to show the existence of *dissipative* measure-valued solutions to a stochastically driven *complete* Euler system (1.1.2) and properties of solutions. In particular, these measure-valued solutions satisfy the admissibility criterion; they conserve the total energy and satisfy an appropriate form of the entropy (entropy admissibility criterion), and they exist global-in-time for any finite initial data. The concept of measure-valued solutions to fluid model systems driven by stochastic forcing is fairly a new subject area of research. To the best of our knowledge, the study of stochastic measure-valued solutions to the complete Euler system governing the motion of an inviscid, temperature dependent, and compressible fluid subject to stochastic forcing is still an open question. Hence, this is a first attempt to characterise the concept of measure-valued solutions to the full stochastic Euler system. Recently, Hofmanová et al [60] established existence results for *compressible barotropic* Euler system. We expect that the baratropic system can be recovered as a singular limit of our dissipative solutions to (2.2). This is, however, subject to future research.

Main results of the complete stochastic Euler system

(1) **Existence of dissipative measure-valued solutions**

Here we prove the existence of martingale measure-valued solutions to the *complete* Euler system (1.1.2) following the strategy in [68, 18, 60]. The precise statement of these results is Theorem 4.1.6. These solutions are weak, in the analytical sense (derivatives only exists in the sense of distributions) and in the probabilistic sense (the probability space is not a given priori, but an integral part of the solution). The proof outline is as follows. Similarly to the incompressible case, we start off with an approximate system with high order diffusion, see Section 4.1.5, and show existence of solutions to the original problem in the limit via stochastic compactness arguments based on Jakubowski's variant of the Skorokhod representation theorem [64]. The latter is needed due to the complicated *path space* which arises because of the presence of measures describing the oscillations and concentrations in the nonlinearities of the Euler system.

(2) **Weak-strong uniqueness pathwise**

Here the statement of results is Theorem 4.1.7. The idea is to produce compressible results analogous to the incompressible case (pathwise). In reminiscent of the results in [16], we deduce the relative entropy inequality (see Section 4.1.10); a tool that allows us to establish the weak(measure-valued)-strong uniqueness principle, that is, a dissipative measure-valued solution coincides with the strong solution as soon as the later exists. The concept of stochastic weak-strong uniqueness pathwise is analogous to the deterministic counterpart results, see Feiresl-Brezina [47] for more details.

(3) **Strong Markov selection**

Although we do not expect solutions to be unique, there is some hope to select solutions which are in a sense continuous with respect to the initial data. This is the Markov property; the memoryless property of the stochastic process, the probability law of the future only depends on the current state of the process, but it is independent from the past, see the monograph by Stroock and Varadhan [86] for a thorough exposition. Our work shows the stochastic analog results of [10], that is, the existence of Markov selection to the associated martingale problem following the presentations in [12, 54, 58]. At first sight, the overall proof outline is rather similar, however, we encountered several challenges in this thesis. A major challenge originates in the use of defect measures. The defect measures are an equivalence class in time and not stochastic processes in the classical sense. Therefore, it is not clear as to how one applies the Markov selection. To solve this issue we introduce auxiliary continuous stochastic variables $[\mathscr{S}, \mathbf{R}]$ such that

$$\mathscr{S} = \int_0^{\cdot} S\,\mathrm{d}s, \quad \mathbf{R} = \int_0^{\cdot} (\mathscr{R}_{\mathrm{conv}}, \mathscr{R}_{\mathrm{press}}, \mathscr{V}_{t,x})\,\mathrm{d}s,$$

and this allows us to show Markov selection for $[\rho, \mathbf{m}, \mathscr{S}, \mathbf{R}]$, see Section 4.1.11. Here, $\mathscr{S}$ and $\mathbf{R}$ are the defect measures related to entropy balance and momentum equation, respectively. Note, such an approach is reminiscent of [12], where a similar idea was used for the velocity field. It is important to note that, different to [12, 54, 58], we obtain a strong Markov selection. This is due to the energy equality which is a feature of the system (1.1.2) and is not known to hold for the problems studied in [12, 54, 58]. However, a first result on strong Markov selection has been obtained recently by Hofmanová-Zhu-Zhu [62]. In [62] they study the incompressible stochastic Euler equations and obtain the energy equality by introducing a defect measure for the energy which is included in the Markov selection.

Outline of complete stochastic Euler system

Phase two of the thesis is organised as follows: we start off with mathematical framework and main results in Section 4.1.2. In Section 4.1.8, we show existence of martingale solutions for the approximate system. In Section 4.1.9, we prove the existence of measure-valued martingale solutions using stochastic compactness arguments. Finally, Section 4.1.10 is dedicated to showing the weak (measure-valued)-strong uniqueness principle, while Section 4.1.11 is dedicated to the Markov selection.

1.2 Notations

- Ω -sample space
- $\mathbb{P}$ -probability measure
- $\mathscr{F}$-sigma algebra
- $\mathscr{N}$-Null-set
- $(\mathscr{F}_t)_{t\geq 0}$-filtration
- c, C-generic constant that may differ line to line.
- $\langle \cdot \rangle_{\mathbb{T}^3}$-inner product in L^2
- $\langle\langle \cdot \rangle\rangle$-quadratic variation
- $\sim^d$ coincide in distribution
- w.r.t - with respect to
- d -dimensional
- $\sum_{k=1} = \sum_{k=1}^{\infty}$
- $C_b(X)$- the space of bounded continuous functions.
- $\tau_n \wedge \cdot$ - the minimum stopping time

- $\mathbb{T}^3 \subset \mathbb{R}^3$- periodic domain
- $Q_T = (0,T) \times \mathbb{T}^3$- space-time periodic cylinder, (In some cases we use $Q = Q_T$).
- $L^\infty_{w*}(\cdot)$- weak* convergence in L^∞
- ξ- dummy variable
- $a \lesssim b$- $a < cb$ for some arbitrary constant $c > 0$

Chapter 2

2.1 Preliminaries

In this chapter we start off by recalling standard definitions, lemmas and theorems which will be needed in order to study deterministic partial differential equations (PDEs) and stochastic partial differential equations (SPDEs) theory in an effort to make the content of the thesis self contained. Firstly, we consider the general concepts in the study of PDEs theory. In particular, Lebesgue spaces, Bochner spaces and Sobolev spaces including some inequalities associated with the spaces, respectively. Finally, we move onto probability theory, an essential component of studying SPDEs theory. To be specific, we consider the concepts of stochastics in infinite dimensions and tools of compactness in probabilistic settings.

2.1.1 Function spaces

In general the analysis of PDEs is concerned with the ill-posedness and well-posedness properties of the solutions. In order to establish these properties we often face a lot of difficulties especially when one is choosing a space where these solutions may exists. To overcome these situations a lot of research has been done on the study of function spaces. In this section we define several function spaces relevant to the study of PDEs theory, however, we must stress out that the list of material covered here is not exhaustive. We refer the reader to [1, 90] and references therein for further exposition of function spaces and relevant inequalities in the theory of PDEs.

Definition 2.1.1. Let (X,Σ,μ) be a measure space and $p \in [1,\infty)$. Let

$$L^p(X,\mu) := \{u : X \to \overline{\mathbb{R}};\, u \text{ defined } \mu - a.e. \text{ with } \|u\|_X < \infty\}$$

and

$$\|u\|_{L^p(X)} := \left(\int_X |u|^p \mathrm{d}\mu(x)\right)^{\frac{1}{p}} \in [0,\infty].$$

The space $(L^p(X;\mu), \|.\|_{L^p(X)})$ is called Lebesgue space of functions on X which are p-integrable w.r.t the measure μ.

Throughout the thesis we shall adopt $L^p(X)$ as notation for Lebesgue spaces with Lebesgue measure $\mathscr{L}$.

Lemma 2.1.1 (Interpolation in Lebesgue spaces). *Let* $1 \le p,q \le \infty$ *and* $r \in (p,q)$. *Define* θ *by*

$$\frac{1}{r} = \frac{\theta}{p} + \frac{1-\theta}{q}.$$

Then $L^p(X) \cap L^q(X) \subset L^r(X)$ *and we have*

$$\|f\|_{L^r(X)} \le \|f\|^{\theta}_{L^p(X)} \|f\|^{1-\theta}_{L^q(X)} \quad \forall f \in L^p(X) \cap L^q(X).$$

Lemma 2.1.2 (Hölder's inequality). *Let* (X, Σ, μ) *be a measure space. Assume* $1 \le p,q \le \infty$, *and* $\frac{1}{p} + \frac{1}{q} = 1$. *If* $u \in L^P(X)$ *and* $v \in L^q(X)$. *Then* $uv \in L^1(X,\mu)$ *and it holds*

$$\int_X |uv| \, d\mu(x) \le \|u\|_{L^p(X)} \|v\|_{L^q(X)},$$

Bochner space

Let $T > 0$ and X be a real Banach space with norm $\|.\|_X$. We consider the mapping

$$u : [0,T] \to X.$$

Definition 2.1.2. The mapping $u : [0,T] \to X$ is called simple if it has the form

$$u(t) = \sum_{i=1}^{N} \chi_{E_i}(t) x_i \quad t \in [0,T], \; N \in \mathbb{N},$$

where $\cup_i E_i = [0,T], E_i \cap E_j = \emptyset$ for $i \neq j$ and $x_i \in X$ for $i = 1...,N$.

A function $\mathbf{u} : [0,T] \to X$ is called Bochner measurable if and only if there is a sequence u_n of simple functions such that

$$u_n(t) \to u(t) \text{ in } X, \tag{2.1.1}$$

for a.e. t. The function $\mathbf{u} : [0,T] \to X$ is called Bochner integrable if and only if there is a sequence (u_n) of simple functions such that (2.1.1) holds and

$$\int_0^T \|u_n(t) - u(t)\|_X \, \mathrm{d}t \to 0, \text{ as } n \to \infty.$$

The Bochner integral (see [90]) is defined as

$$\int_0^T u(t) \, \mathrm{d}t = \lim_n \int_0^T u_n(t) \, \mathrm{d}t = \lim_n \sum_{k=1}^{N(n)} \mathscr{L}^1(E_k^n) x_k^n.$$

Definition 2.1.3. For $T > 0$ and $1 \le p < \infty$, the space $L^p(0,T;X)$ is the set of all Bochner measurable functions $\mathbf{u} : [0,T] \to X$ such that

$$\|u\|_{L^p(0,T;X)} := \left(\int_0^T \|u\|_X^p \mathrm{d}t \right)^{\frac{1}{p}} < \infty,$$

and the space $L^\infty(0,T;X)$ is the set of all Bochner measurable functions such that

$$\|u\|_{L^\infty(0,T;X)} = \inf_{\mathscr{L}^1(A)=0} \sup_{(0,T)\setminus A} \|u\|_X.$$

Lemma 2.1.3. *The space $L^p(0,T;X), p \in [1,\infty]$, is a Banach space together with the norm $\|.\|_{L^p(0,T;X)}$.*

Lemma 2.1.4. *Let X be Banach space and $1 \le p < \infty$. Let G be dense in $L^p(0,T;\mathbb{R})$ and X_0 dense in X. Then the set*

$$span\{gx_0; g \in G, x_0 \in X_0\}$$

is dense in $L^p(0,T;X)$.

For $u \in L^1(0,T;X)$ we consider the distribution

$$C_0^\infty(0,T) \ni \varphi \mapsto \int_0^T u(t)\varphi'(t) \, \mathrm{d}t \in X.$$

Let Y be a Banach space with $X \hookrightarrow Y$ continuously. If there is $v \in L^1(0,T;Y)$ such that

$$\int_0^T u(t)\varphi'(t)\, \mathrm{d}t = -\int_0^T v(t)\varphi(t)\, \mathrm{d}t \quad \text{for all} \;\; \varphi \in C_0^\infty(0,T)$$

we say that v is the weak derivative of u in Y and write $v = \partial_t u$. We proceed to state the results that provide a compactness criterion for our PDEs system, that is, the Aubin-Lion lemma, see [84, Sect 7.3] for more details.

Lemma 2.1.5 (Aubin-Lions). *Let (X,Y,Z) be a triple of Banach spaces such that the embedding $X \hookrightarrow Y$ is compact and the embedding $Y \hookrightarrow Z$ is continuous. Then the embedding*

$$\{u \in L^p(0,T;X) : \partial_t u \in L^q(0,T;Z)\} \hookrightarrow L^p(0,T;Y)$$

is compact for $1 \leq p,q \leq \infty$.

Sobolev space

In PDE theory, we require spaces containing less smooth functions to prove good analytical estimates of solutions that belong to such spaces. In order to construct these solutions we need to consider the concept of weak partial derivatives. Let $C^\infty(\mathbb{T}^3)$ be the space of infinitely differentiable functions $\phi : \mathbb{T}^3 \to \mathbb{R}$. Suppose we are given a function $u \in C^1(\mathbb{T}^3)$, then for any $\phi \in C^\infty(\mathbb{T}^3)$ integration by parts yields

$$\int_{\mathbb{T}^3} u\phi_{x_i} dx = -\int_{\mathbb{T}^3} u_{x_i}\phi dx \quad \text{for all} \quad \phi \in C^\infty(\mathbb{T}^3), \quad (i = 1,...,n), \tag{2.1.2}$$

as consequence of the periodic boundary conditions. In general for a positive integer k, $u \in C^k(\mathbb{T}^3)$ and $\alpha = (\alpha_1,...,\alpha_n)$ a multiindex of order $|\alpha| = \alpha_1 + \cdots + \alpha_n = k$, we have

$$\int_{\mathbb{T}^3} uD^\alpha\phi dx = (-1)^{|\alpha|}\int_{\mathbb{T}^3} D^\alpha u\phi \,\mathrm{d}x \quad \text{for all} \quad \phi \in C^\infty(\mathbb{T}^3). \tag{2.1.3}$$

This equality follows from

$$D^\alpha\phi = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}\phi,$$

and we can apply (2.1.2) $|\alpha|$ times. Next, assuming (2.1.3) is valid for $u \in C^k(\mathbb{T}^3)$, we examine whether a variant of it might still be true if u is not k times continuously differentiable. Note that the left-hand side of (2.1.3) makes sense if u is locally summable: now given that u is not C^k, the right-hand side form has no obvious meaning. This problem is resolved by seeking a locally summable function v for which (2.1.3) is valid, with v replacing $D^\alpha u$.

Definition 2.1.4 (Weak derivative)**.** Let $u, v \in L^1(\mathbb{T}^3)$ and α a multi-index. Then, v is called α^{th}-weak partial derivative of u if

$$\int_{\mathbb{T}^3} uD^\alpha \phi \, \mathrm{d}x = (-1)^{|\alpha|} \int_{\mathbb{T}^3} v\phi \, \mathrm{d}x \quad \text{for all} \quad \phi \in C^\infty(\mathbb{T}^3). \tag{2.1.4}$$

In this case, if we are given u and if there exist a function v such that (2.1.4) holds for all ϕ, we say that $D^\alpha u = v$ in the weak sense. Now, given the definition of a weak derivative we are now in a position to define a Sobolev space.

Definition 2.1.5. Let $\mathbb{T}^3 \subset \mathbb{R}^3$, fix $1 \leq p \leq \infty$ and let k be a non-negative integer. The Sobolev space $(W^{k,p}(\mathbb{T}^3))$ consists of functions $u \in L^p(\mathbb{T}^3)$ such that for each multi-index α with $|\alpha| \leq k$, the weak derivative $D^\alpha u$ exist and $D^\alpha u \in L^p(\mathbb{T}^3)$. Thus

$$W^{k,p} = \{u \in L^p(\mathbb{T}^3); D^\alpha u \in L^p(\mathbb{T}^3) \text{ for all } |\alpha| \leq k\}.$$

If $u \in W^{k,p}(\mathbb{T}^3)$, its norm is defined by

$$\|u\|_{W^{k,p}(\mathbb{T}^3)} = \left(\sum_{|\alpha| \leq k} \int_{\mathbb{T}^3} |D^\alpha u|^p \, \mathrm{d}x \right)^{\frac{1}{p}}, \quad 1 \leq p < \infty.$$

and

$$\|u\|_{W^{k,\infty}(\mathbb{T}^3)} = \sum_{|\alpha| \leq k} ess \ \sup |D^\alpha u|.$$

In the following we consider special cases of Sobolev spaces, that is, fractional Sobolev spaces. Let H be a separable Banach space with norm $\|\cdot\|_H$. Assume $p > 1, a \in (0,1)$, let $W^{a,p}(0,T;H)$ be a Sobolev space of all $u \in L^p(0,T;H)$ such that

$$\int_0^T \int_0^T \frac{|u(t) - u(s)|^p}{|t-s|^{1+ap}} \, \mathrm{d}t\mathrm{d}s < \infty,$$

endowed with the norm

$$\|u\|^p_{W^{a,p}(0,T;H)} := \int_0^T \|u\|^p \, \mathrm{d}t + \int_0^T \int_0^T \frac{|u(t) - u(s)|^p}{|t-s|^{1+ap}} \, \mathrm{d}t\mathrm{d}s.$$

We proceed to recall the main embedding theorem we shall use in this thesis, we refer the reader to [1, Theorem 5.4] for details.

Theorem 2.1.6 (Sobolev embedding). *Let* $\mathbb{T}^3 \subset \mathbb{R}^d, d = 3$ *, if* $k > l, p < d$ *and* $1 \le p < q < \infty$ *are two real numbers such that*

$$\frac{1}{p} - \frac{k}{d} = \frac{1}{q} - \frac{l}{d}$$

then the embeddings

$$W^{k,p}(\mathbb{T}^3) \subset W^{l,q}(\mathbb{T}^3)$$

are continuous. In the special case when $k = 1$ *and* $l = 0$ *we have that the embeddings*

$$\begin{aligned} W^{1,p}(\mathbb{T}^3) &\hookrightarrow L^{\frac{dp}{d-p}}(\mathbb{T}^3) \text{ if } p < d, \\ W^{1,p}(\mathbb{T}^3) &\hookrightarrow C^{1-\frac{d}{p}}(\mathbb{T}^3) \text{ if } p > d, \end{aligned}$$

are continuous.

We conclude this section by stating some standard inequalities used in PDEs theory.

Lemma 2.1.7 (Poincare inequality). *Let* $\mathbb{T}^3 \subset \mathbb{R}^3$ *be torus domain and* $1 \le p < \infty$*. Then for all functions* $u \in W^{1,p}(\mathbb{T}^3)$ *we have*

$$\|u - (u)_{\mathbb{T}^3}\|_{L^p(\mathbb{T}^3)} \le C \|\nabla u\|_{L^p(\mathbb{T}^3)}, \qquad (u)_{\mathbb{T}^3} = \frac{1}{\mathscr{L}^d(\mathbb{T}^3)} \int_{\mathbb{T}^3} u \, dx,$$

where the constant C *depends only on* p, d *and* $\mathbb{T}^3$.

Theorem 2.1.8 (Gagliardo-Nirenberg-Sobolev inequality). *Assuming* $1 \le p < d = 3$*, then there exists a constant c, depending only on p and d such that*

$$\|u\|_{L^{p^*}(\mathbb{T}^3)} \le c \|Du\|_{L^p(\mathbb{T}^3)}, \tag{2.1.5}$$

where p^* *is the conjugate of p defined by*

$$p^* := \frac{dp}{d-p}, \; p^* > p.$$

2.1.2 Elements of Stochastic analysis

In this section we introduce the elementary stochastic framework used in this thesis. In particular, we gather various definitions and fundamental theorems of stochastic analysis we shall use in analysis of SPDEs in the subsequent sections. For a thorough background on the elementary concepts and fundamental results in this section we refer the reader to [15, 35, 66, 65].

Random variables

To begin with, let the triplet $(\Omega, \mathscr{F}, \mathbb{P})$ denote a probability space with σ-algebra $\mathscr{F}$ on the underlying sample space Ω and a probability measure $\mathbb{P}$. Let $\mathscr{N}$ denote a collection of nullsets, and suppose that $A \in \mathscr{N}$ such that $\mathbb{P}(A) = 0$. The probability space $(\Omega, \mathscr{F}, \mathbb{P})$ is complete if $A \in \mathscr{N}$ and $B \subset A$ imply $B \in \mathscr{F}$(and hence $B \in \mathscr{N}$). A probability space can be made *complete* or *completed* by suitably enlarging the σ-algebra $\mathscr{F}$. Throughout the thesis we shall use the triplet $(\Omega, \mathscr{F}, \mathbb{P})$ to denote a *complete probability* space with σ-algebra $\mathscr{F}$ and a probability measure $\mathbb{P}$. Furthermore, we denote by $([0,1], \overline{\mathscr{B}[0,1]}, \mathfrak{L})$ the standard probability space where $\overline{\mathscr{B}}$ denotes the completion of $\mathscr{B}$, and $\mathfrak{L}$ the Lebesgue measure. A filtration on $(\Omega, \mathscr{F}, \mathbb{P})$ is a collection of non-decreasing family of sub-σ-algebras of $\mathscr{F}$ such that $\mathscr{F}_s \subseteq \mathscr{F}_t$ for all $s < t$. Here, t is understood as a time variable and $\mathscr{F}_t$ contains the history of events until time t. A probability space $(\Omega, \mathscr{F}, \mathbb{P})$ endowed with filtration $(\mathscr{F}_t)_{t\geq 0}$ is called a *stochastic basis* or filtered probability space, and we denote the stochastic basis by $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$.

Definition 2.1.6. A filtration $(\mathscr{F}_t)_{t\geq 0}$ is called right-continuous, if

$$\mathscr{F}_t = \bigcap_{\varepsilon>0} \mathscr{F}_{t+\varepsilon} \quad \forall t \geq 0$$

and left-continuous, if

$$\mathscr{F}_t = \bigcup_{s<t} \mathscr{F}_s \quad \forall t > 0.$$

In addition, in the analysis of subsequent section we will always require our filtration $(\mathscr{F}_t)_{t\geq 0}$ to satisfy the "*usual conditions* ", that is, *complete* and *right-continuous*, specifically,

$$\left\{\mathscr{N} \in \mathscr{F}; \mathbb{P}(\mathscr{N}) = 0\right\} \subset \mathscr{F}_0 \quad \mathscr{F}_t = \bigcap_{\varepsilon > 0} \mathscr{F}_{t+\varepsilon} \quad \forall t \geq 0.$$

We proceed to state definitions and properties related to random variables.

Definition 2.1.7. Let $(X, \mathscr{A})$ be a measurable space. An X-valued *random variable* is a measurable mapping $\mathbf{V} : (\Omega, \mathscr{F}) \to (X, \mathscr{A})$. We denote by $\sigma(\mathbf{V})$ the smallest σ-field with respect to which $\mathbf{V}$ is measurable. Specifically,

$$\sigma(\mathbf{V}) = \left\{\{\omega \in \Omega; \mathbf{V}(\omega) \in A\}; A \in \mathscr{A}\right\}.$$

and $\sigma(\mathbf{V}) \subset \mathscr{F}$.

Definition 2.1.8. The Law $\mathscr{L}$ of an X-valued random variable $\mathbf{V}$, denoted by $\mathscr{L}[\mathbf{V}]$, is a probability measure defined by

$$\mathscr{L}[\mathbf{V}](B) = \mathbb{P}\{\omega \in \Omega : \mathbf{V}(\omega) \in B\}, \quad B \in \mathscr{B}(X).$$

Definition 2.1.9. Let $(X, \mathscr{A})$ be a measurable space. We say that two X-valued random variables $\mathbf{V}$ and $\tilde{\mathbf{V}}$ are *equal in law*, if $\mathscr{L}[\mathbf{V}]$ and $\mathscr{L}[\tilde{\mathbf{V}}]$ coincide.

In our analysis we need to assert some topological assumptions on the state space X, though we should note they may vary depending on application. In that regard, we assume X is a topological space equipped with a Borel σ-field. Moreover, it is essential that the topology on X is completely determined by the family of continuous functions, and to be precise we use *Tikhonov spaces*. We proceed to give a formal definition of Tikhonov spaces.

Definition 2.1.10. A topological space (X, τ) is a Tikhonov space provided it is both completely regular and Hausdorff. To be precise, the following hold

- (X, τ) is a topological space;
- for any points $x_1 \neq x_2 \in X$, there are disjoint open sets containing the two points, respectively;
- for any point $x \in X$ and closed subsets $\mathscr{W} \subset X$ such that $x \notin \mathscr{W}$, there exists a continuous function $f : X \to [0,1]$ such that $f(x) = 0$ and $f(y) = 1$ for all $y \in \mathscr{W}$

Remark 2.1.1. In this thesis we are concerned with the delicate structure of the incompressible Navier–Stokes and Euler systems, and compressible Euler system, our naturally spaces for analysis of these systems will be Banach spaces equipped with weak topology or duals of Banach spaces with weak-$*$ topology (i.e.measures)

We extend the concept of equality of laws to *Tikhonov spaces*.

Definition 2.1.11. Let X be a Tikhonov topological space equipped with the Borel σ-field. Let $\mathbf{V}$ and $\tilde{\mathbf{V}}$ be X-valued random variables, then $\mathscr{L}[\mathbf{V}] = \mathscr{L}[\tilde{\mathbf{V}}]$, if

$$\mathbb{E}[f(\mathbf{V})] = \mathbb{E}[f(\tilde{\mathbf{V}})],$$

holds true for all $f \in C_b(X)$.

In the analysis of our SPDEs we are interested in the asymptotic behaviour of our system. Although we encounter function spaces in the thesis, we shall only limit ourselves to convergences of *sequences* as opposed to *nets*. We proceed to state definitions related to the convergences of sequences.

Definition 2.1.12. Let X be a Banach space endowed with the norm $\|\cdot\|_X$ and let $p \in [1,\infty)$. Then a family of X-valued random variables $\mathbf{V}_n, n \in \mathbb{N}$ on the probability space $(\Omega, \mathscr{F}, \mathbb{P})$ with values in the Banach space $(X, \|\cdot\|_X)$ converges in p-moment (converges in L^p to $\mathbf{V}$), that is, $\mathbf{V}_n \to \mathbf{V}$ in $L^p(\Omega; X)$ provided

$$\lim_{n\to\infty} \mathbb{E}\left[\|\mathbf{V}_n - \mathbf{V}\|_X^p\right] = 0.$$

Definition 2.1.13. Let X be a topological space equipped with the Borel σ-field and let $\mathbf{V}$ and $\mathbf{V}_n, n \in \mathbb{N}$, be X-valued random valued variables on $(\Omega, \mathscr{F}, \mathbb{P})$. We say that a sequence of random variables $\mathbf{V}_n$ converges to $\mathbf{V}$ almost surely, provided

$$\mathbb{P}\left(\omega \in \Omega; \lim_{n\to\infty} \mathbf{V}_n(\omega) = \mathbf{V}(\omega)\right) = 1.$$

Here we note that Definition 2.1.13 is an analogy of almost everywhere results in measure theory. Similarly, the probabilistic analogue of convergence in measure is given by the following definition.

Definition 2.1.14. Let (X, τ) be a locally convex topological space endowed with a family of semi-norms $(d_y)_{y\in Y}$, Y is an indexing set. A sequence of X-valued random variables $(\mathbf{V}_n)_{n\in N}$ defined on the probability space $(\Omega,\mathscr{F},\mathbb{P})$ converges in probability to an X-valued random variable $\mathbf{V}$, denoted $\mathbf{V}_n \to^{\mathbb{P}} \mathbf{V}$ for all $\varepsilon > 0$ and $y \in Y$ provided

$$\lim_{n\to\infty} \mathbb{P}\left\{ \omega \in \Omega : d_y(\mathbf{V}_n(\omega) - \mathbf{V}(\omega)) > \varepsilon \right\} = 0.$$

We conclude the discussion on convergence of random variables by stating the convergence in law results.

Definition 2.1.15. Let X be a Tikhonov topological space equipped with the Borel σ-field and let $\mathbf{V}_n, n \in \mathbb{N}, \mathbf{V}$, be X-valued random variables defined on $(\Omega_n, \mathscr{F}_n, \mathbb{P}_n), n \in \mathbb{N}$ and $(\Omega,\mathscr{F},\mathbb{P})$, respectively. We say that the sequence of random variables $\mathbf{V}_n$ converges to $\mathbf{V}$ in law, if the law $\mathscr{L}[\mathbf{V}_n]$ converges to $\mathscr{L}[\mathbf{V}]$ weakly-* in the sense of probability measures on X, that is,

$$\lim_{n\to\infty} \mathbb{E}[f(\mathbf{V}_n)] = \mathbb{E}[f(\mathbf{V})],$$

holds true for all $f \in C_b(X)$.

Stochastic processes

Let (X,τ) be a topological space endowed with a Borel σ-field. An X-valued stochastic process is a set of random variables $\mathbf{V} = (\mathbf{V}_t)_{t\geq 0}$ on the measurable space $(\Omega,\mathscr{F})$ with values in $(X,\mathscr{B}(X))$, where $\mathscr{B}(X)$ is the Borel σ-algebra. A stochastic process $\mathbf{V}$ can be understood as a function of t and $\omega \in \Omega$, and the mapping of $t \mapsto \mathbf{V}_t(\omega)$ is called the *path or trajectory* of $\mathbf{V}_t$. In the following we state definitions associated with stochastic processes, and properties we shall use in subsequent sections.

Definition 2.1.16. A stochastic process $\mathbf{V}$ is called measurable, if the mapping

$$(t,\omega) \mapsto \mathbf{V}_t(\omega) : ([0,\infty)\times\Omega, \mathscr{B}([0,\infty))\otimes\mathscr{F}) \to (X,\mathscr{B}(X))$$

is measurable.

Definition 2.1.17. Let $(\mathscr{F}_t)_{t\geq 0}$ be a filtration on $(\Omega,\mathscr{F})$. An X-valued stochastic process $\mathbf{V} = \{\mathbf{V}(t); t \in [0,\infty)\}$ is called $(\mathscr{F}_t)_{t\geq 0}$-adapted, provided that, the mapping

$$\omega \mapsto \mathbf{V}_t(\omega) : (\Omega,\mathscr{F}_t) \to (X,\mathscr{B}(X))$$

is measurable for all $t \geq 0$.

Remark 2.1.2. It follows immediately from Definition 2.1.7 that a stochastic process $\mathbf{V}$ is always adapted to its $\mathbb{P}$-augmented canonical filtration, given by

$$\sigma_t[\mathbf{V}] := \bigcap_{s>t} \sigma(\sigma(\mathbf{V}(r); 0 \leq r \leq s) \cup \{\mathscr{N} \in \mathscr{F}; \mathbb{P}(\mathscr{N}) = 0\}), t \geq 0.$$

Definition 2.1.18. Let $(\mathscr{F}_t)_{t\geq 0}$ be a filtration on $(\Omega, \mathscr{F})$. An X-valued stochastic process $\mathbf{V} = \{\mathbf{V}(t); t \in [0,\infty)\}$ is called $(\mathscr{F}_t)$ progressively measurable, if the mapping

$$(s,\omega) \mapsto \mathbf{V}_s(\omega) : ([0,t] \times \Omega, \mathscr{B}([0,\infty)) \otimes \mathscr{F}_t) \to (X, \mathscr{B}(X))$$

is measurable for all $t \geq 0$.

Consequently, it follows immediately that a $(\mathscr{F}_t)$-progressively measurable stochastic process is measurable and $(\mathscr{F}_t)$-adapted. However, the converse is not always true.

Definition 2.1.19. Let $\mathbf{M} = (\mathbf{M}_t)_{t\geq 0}$ be an $(\mathscr{F}_t)_{t\geq 0}-$adapted $\mathbb{R}$-valued stochastic process on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ with $\mathbb{E}[|\mathbf{M}_t|] < \infty$ $\forall t \geq 0$. $\mathbf{M}$ is called a sub-martingale if we have that $\mathbb{P}$-a.s. $\mathbb{E}[\mathbf{M}_t|\mathscr{F}_s] \geq \mathbf{M}_s$ for all $0 \leq s \leq t < \infty$, and super-martingale if we have for all $0 \leq s \leq t < \infty$ that $\mathbb{P}$-a.s. $\mathbb{E}[\mathbf{M}_t|\mathscr{F}_s] \leq \mathbf{M}_s$. Then $\mathbf{M}$ is called a martingale if it is a sub-martingale and a super-martingale i.e. $\mathbb{E}[\mathbf{M}_t|\mathscr{F}_s] = \mathbf{M}_s$.

Definition 2.1.20. Let A be an adapted real-valued stochastic process. A is called increasing if we have for $\mathbb{P}$-a.e. $\omega \in \Omega$

(i) $A_0 = 0$;

(ii) $t \mapsto A_t(\omega)$ is increasing and right-continuous;

(iii) $\mathbb{E}[A_t] < \infty$ for all $t \in [0,\infty)$.

An increasing $\mathbb{R}$-valued stochastic process is called integrable if

$$\mathbb{E}[A_\infty] < \infty, \quad \text{where } A_\infty(\omega) = \lim_{t\to\infty} A_t(\omega) \quad \text{for } \omega \in \Omega.$$

Theorem 2.1.9 (Doob-Meyer decomposition, [66], (Thm. 4.10, p.24))**.** *Let Y be a non-negative sub-martingale with a.s. continuous trajectories. There is a continuous trajectories martingale M and a $\mathbb{P}$-a.s. increasing continuous and adapted process A such* *that*

$$Y_t = M_t + A_t,$$

where the decomposition is unique.

The following theorem is a consequence of Theorem 2.1.9, see [66, Section 1.4] for more details.

Theorem 2.1.10. *Let* $\mathbf{V}$ *be a continuous* L^2*-integrable real-vauled* $(\mathscr{F}_t)$*-martingale, that is,* $\mathbb{E}[\mathbf{V}_t^2] < \infty$*, for all* $t \geq 0$*. Then there exists a unique stochastic process* $\langle\langle \mathbf{V} \rangle\rangle$ *such that:*

(a) $\langle\langle \mathbf{V} \rangle\rangle$ *is* $(\mathscr{F}_t)$*-adapted and has* $\mathbb{P}$*-a.s. non-decreasing trajectories;*

(b) $\langle\langle \mathbf{V} \rangle\rangle(0) = 0$ $\mathbb{P}$*-a.s.;*

(c) $\mathbf{V}^2 - \langle\langle \mathbf{V} \rangle\rangle$ *is a continuous* $(\mathscr{F}_t)$*-martingale.*

Definition 2.1.21. The stochastic process $\langle\langle \mathbf{V} \rangle\rangle$ constructed in Theorem 2.1.10 is called the quadratic variation of $\mathbf{V}$.

Definition 2.1.22. Let $\mathbf{V}, \tilde{\mathbf{V}}$ be stochastic process satisfying the assumptions of Theorem 2.1.10. The process

$$\langle\langle \mathbf{V}, \tilde{\mathbf{V}} \rangle\rangle := \frac{1}{4}\Big(\langle\langle \mathbf{V} + \tilde{\mathbf{V}} \rangle\rangle - \langle\langle \mathbf{V} - \tilde{\mathbf{V}} \rangle\rangle \Big),$$

is called the *cross variation* of $\mathbf{V}, \tilde{\mathbf{V}}$.

To generalise the concept of martingales we need to introduce the notion of local martingales. We first need to introduce the notion of stopping time.

Definition 2.1.23. Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a measurable space endowed with a filtration $(\mathscr{F}_t)_{t \geq 0}$. We say a random variable $\tau : \Omega \to [0, \infty]$ is an $(\mathscr{F}_t)$ *stopping time*, if the event $\{\tau \leq t\}$ belongs to the σ-field $\mathscr{F}_t$ for any $t \in [0, \infty)$.

Definition 2.1.24. Let $\mathbf{V}$ be an X-valued $(\mathscr{F}_t)$-adapted stochastic process. Then $\mathbf{V}$ is an $(\mathscr{F}_t)$-local martingale if there exists an increasing sequence of stopping times $(\tau_n)_{n \in \mathbb{N}}$, $\tau_n \uparrow \infty$ a.s., such that the stopped process $\mathbf{V}_{\tau_n} = \mathbf{V}(\tau_n \wedge \cdot)$ is an $(\mathscr{F}_t)$-martingale for all $n \in \mathbb{N}$.

In accordance with the aim of the thesis we define a Wiener process, an example that is both a continuous-time stochastic process and a martingale, the process plays a key role in our analysis.

Definition 2.1.25. An $\mathbb{R}^n$-valued stochastic process W is called an $(\mathscr{F}_t)$-Wiener process, provided:

(1) W is $(\mathscr{F}_t)$-adapted ;

(2) $W(0) = 0$ $\mathbb{P}$-a.s.;

(3) W has continuous trajectories : $t \mapsto W(t)$ is continuous $\mathbb{P}$-a.s.;

(4) W has independent increments: $W(t) - W(s)$ is independent of $\mathscr{F}_s$ for all $0 \leq s \leq t < \infty$;

(5) W has Gaussian increments: $W(t) - W(s)$ is normally distributed with mean 0 and variance $(t-s)\mathbb{I}$ for all $0 \leq s \leq t < \infty$, i.e $N(0,(t-s)\mathbb{I})$.

For completeness we present an infinite-dimensional generalisation of Wiener process. We accomplish this by introducing the notion of cylindrical Wiener process, and we note that this will be the natural choice for the analysis of the driving force of the Euler and Navier-Stokes systems.

Definition 2.1.26. Let $\mathscr{U}$ be a separable Hilbert space with a complete orthonormal system $(e_k)_{k\in\mathbb{N}}$ and let $(\beta_k)_{k\in\mathbb{N}}$ be a sequence of mutually independent real-valued $(\mathscr{F}_t)$-Wiener processes. The stochastic process W given by the formal expansion

$$W(t) = \sum_{k=1}^{\infty} e_k \beta_k(t)$$

is called cylindrical $(\mathscr{F}_t)$-Wiener process.

2.1.3 Random distributions

Throughout the thesis most random variables we encounter are of the form $\mathbf{V} : \Omega \to L^1(Q_T, \mathbb{R}^3)$. However, such an object is not a stochastic process in the classical sense as it is only defined a.e in time. To remedy such scenarios we use the notion of classical equivalence stochastic processes, that is, *random distributions*, as introduced in [15, Chap. 2.2] to which we refer the reader for more details. The use of random distributions is essential in handling time dependent variables such as the velocity field $\mathbf{u}$ and temperature ϑ, since in general these quantities do not poses well defined instantaneous values at *any* time t.

Definition 2.1.27. Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a complete probability space and $N \in \mathbb{N}$. A mapping

$$\mathbf{V} : \Omega \to (C_c^\infty(Q_T, \mathbb{R}^3))'$$

is called *random distribution* if $\langle \mathbf{V}, \varphi \rangle : \Omega \to \mathbb{R}$ is a measurable function for any $\varphi \in C_c^\infty(Q_T, \mathbb{R}^3)$.

The properties associated with random variables as stated above continue to hold in *random distributions*. In particular, the concept of *progressive measurability*. Here we consider the σ-field of all progressively measurable sets in $\Omega \times [0,T]$ associated to $(\mathscr{F}_t)_{t\geq 0}$. To be precise, $A \subset \Omega \times [0,T]$ belongs to the progressively measurable σ-field provided the stochastic process $(\omega, t) \mapsto \mathbf{1}_A(\omega, t)$ is $(\mathscr{F}_t)_{t\geq 0}$-progressively measurable. We denote by $L^1_{\text{prog}}(\Omega \times [0,T])$ the Lebesgue space of functions that are measurable with respect to the σ-field of $(\mathscr{F}_t)_{t\geq 0}$-progressively measurable sets in $\Omega \times [0,T]$ and we denote by μ_{prog} the measure $\mathbb{P} \otimes \mathscr{L}^1_{[0,T]}$ restricted to the progressively measurable σ-field.

Definition 2.1.28. Let $\mathbf{V}$ be a random distribution in the sense of Definition 2.1.27.

- We say that $\mathbf{V}$ is adapted to $(\mathscr{F}_t)$ if $\langle \mathbf{V}, \varphi \rangle$ is $(\mathscr{F}_t)_{t\geq 0}$-measurable for any $\varphi \in C_c^\infty(Q_T, \mathbb{R}^3)$.
- We say that $\mathbf{V}$ is $(\mathscr{F}_t)_{t\geq 0}$-progressively measurable if $\langle \mathbf{V}, \varphi \rangle \in L^1_{\text{prog}}(\Omega \times [0,T])$ for any $\varphi \in C_c^\infty(Q_T, \mathbb{R}^3)$.

Remark 2.1.3. The property of progressive measurability in both random variables and *random distributions* coincides as long as the distribution defines a stochastic process, see [15, Chp 2, Lemma 2.2.18] for more details.

Finally, a family of σ-fields $(\sigma_t[\mathbf{V}])_{t\geq 0}$ given as

$$\sigma_t[\mathbf{V}] := \bigcap_{s>t} \sigma \left(\bigcup_{\varphi \in C_c^\infty(Q_s;\mathbb{R}^N)} \{\langle \mathbf{V}, \varphi \rangle < 1\} \cup \{\mathscr{N} \in \mathscr{F}, \mathbb{P}(\mathscr{N}) = 0\} \right) \tag{2.1.6}$$

is called the *history* of $\mathbf{V}$. In fact, any random distribution is adapted to its history.

2.1.4 Stochastic Itô integral

In line with the aim of producing a self contained material we proceed to present a standard interpretation of the stochastic Itô integral with respect to the cylindrical Wiener process. We refer the reader to [35, 83] for a detailed study of this material. Let $(W_t)_{t\geq 0}$ be a cylindrical Wiener process (3.1.1) defined on a separable Hilbert space $\mathscr{U}$. Let $\phi = \{\phi(t); t\in[0,\infty)\}$ be a stochastic process taking its values in the space of bounded Hilbert-Schmidt linear operators from $\mathscr{U}$ to a separable space $L^2(\mathbb{T}^3)$, denoted by $L_2(\mathscr{U}, L^2(\mathbb{T}^3))$. The aim is to make sense of the functional

$$\int_0^t \phi(s)\, \mathrm{d}W_s.$$

The functional above defines an $L^2(\mathbb{T}^3)$-valued martingale, see Definition 2.1.19. Recalling Definition (2.1.26), we construct the integral as the sum of stochastic integrals with respect to the real-valued Wiener processes

$$\int_0^t \phi(s)\, \mathrm{d}W_s = \sum_{k=1}^{\infty} \int_0^t \phi(s)e_k\, \mathrm{d}\beta_k(s). \tag{2.1.7}$$

Since ϕ takes values in the space of Hilbert-Smith operators, the right hand side of (2.1.7) converges in a suitable sense in $L^2(\mathbb{T}^3)$ which implies it is an $L^2(\mathbb{T}^3)$-valued martingale. To be precise, the following theorem holds, see [35, Section 4.2].

Theorem 2.1.11. *Let ϕ be an $(\mathscr{F}_t)$-progressively measurable stochastic process such that*

$$\mathbb{E}\int_0^t \|\phi(s)\|^2_{L_2(\mathscr{U}, L^2(\mathbb{T}^3))}\, \mathrm{d}t < \infty. \tag{2.1.8}$$

Then the stochastic Itô integral (2.1.7) is well-defined continuous $L^2(\mathbb{T}^3)$-valued square integrable $(\mathscr{F}_t)$-martingale.

We proceed to state more results related to the Itô integral.

Proposition 2.1.12 (Itô Isometry)**.** *Let ϕ be an elementary $(\mathscr{F}_t)$-adapted stochastic process. Then the stochastic integral (2.1.7) defines a continuous $L^2(\mathbb{T}^3)$-valued square integrable $(\mathscr{F}_t)$-martingale and the following holds true:*

$$\mathbb{E}\left\|\int_0^t \phi(s)\, \mathrm{d}W_s\right\|^2_{L^2(\mathbb{T}^3)} = \mathbb{E}\int_0^t \|\phi(s)\|^2_{L_2(\mathscr{U}, L^2(\mathbb{T}^3))}\, \mathrm{d}s, \tag{2.1.9}$$

for all $t \geq 0$, provided (2.1.8) is satisfied.

In fact, the Itô isometry remains valid for more general integrands satisfying (2.1.8). As the next step, we recall the so-called *Burgholder-Davis-Gundy inequality*, a generalisation of 2.1.9.

Lemma 2.1.13 (Burkholder-Davis-Gundy inequality)**.** *Let X be a separable Hilbert space. Then for any $p \in (0,\infty)$, there exists a constant $C_p > 0$ such that for any $(\mathscr{F}_t)$-progressively measurable stochastic process $\phi(s)$ satisfying (2.1.8), the following inequality*

$$\mathbb{E} \sup_{t\in(0,T)} \left\| \int_0^t \phi(s) \mathrm{d}W(s) \right\|^p \leq C_p \mathbb{E} \left(\int_0^T \|\phi(s)\|^2_{L_2(\mathscr{U},L^2(\mathbb{T}^3))} \right)^{\frac{p}{2}}$$

holds.

In the following we collect elementary results for computing bounds and taking limits in the stochastic terms. To begin with, we consider limits, see [36, Lemma 2.1].

Lemma 2.1.14. *Let $(W^n)_{n\in\mathbb{N}}$ be a sequence of cylindrical Wiener processes over $\mathscr{U}$ with respect to the filtration $(\mathscr{F}_t^n)_{t\geq 0}$. Assume that (Ψ^n) is a sequence of progressively $(\mathscr{F}_t^n)_{t\geq 0}$-measurable processes such that*

$$\Psi^n \in L^2(0,T;L_2(\mathscr{U},L^2(\mathbb{T}^3))).$$

Suppose there is a cylindrical $(\mathscr{F}_t)_{t\geq 0}$- Wiener process W and

$$\Psi \in L^2(0,T;L_2(\mathscr{U},L^2(\mathbb{T}^3))),$$

progressively $(\mathscr{F}_t)_{t\geq 0}$-measurable, such that

$$W^n \to W \qquad in \quad C^0([0,T],\mathscr{U}_0),$$

$$\Psi^n \to \Psi \qquad in \quad L^2(0,T;L_2(\mathscr{U},L^2(\mathbb{T}^3))),$$

in probability. Then we have

$$\int_0^\cdot \Psi^n \, d\mathbf{W}^n \to \int_0^\cdot \Psi \, d\mathbf{W} \text{ in } L^2(0,T;L^2(\mathbb{T}^3)),$$

in probability.

Note the results of Theorem 2.1.8 are an immediate application of Lemma 2.1.14. To conclude limits results we recall the following theorem.

Theorem 2.1.15 (Vitali's Convergence theorem)**.** *Let* $(X_n)_{n\in\mathbb{N}}, X \in (\Omega, \mathscr{F}, \mathbb{P})$ *be a sequence of integrable random variables and an integrable random variable, respectively. Assume* $\sup_n \mathbb{E}|X_n|^p < \infty$*, for some* $p \in (1,\infty)$ *and* $X_n \to X$ *a.s.. Then* $X_n \to X$ *in* L^1 $\mathbb{P}$*-a.s such that*

$$\mathbb{E}|X_n - X| \to 0,$$

$$\mathbb{E}(X_n) \to \mathbb{E}(X).$$

In computing stochastic bounds, we recall results in [56, Lemma 2.1] given below.

Lemma 2.1.16. *Let* $\phi \in L^P(\Omega, \mathscr{F}, \mathbb{P}; L^p(0,T; L_2(\mathscr{U}, L^2(\mathbb{T}^3))))$ $(p \geq 2)$ *be progressively* $(\mathscr{F}_t)_{t\geq 0}$*- measurable and* W *a cylindrical* $(\mathscr{F}_t)_{t\geq 0}$*- Wiener process as in (3.1.1). Then the following holds for any* $\alpha \in (0, 1/2)$

$$\mathbb{E}\left[\left\|\int_0^{\cdot} \phi \, dW_s\right\|^p_{W^{\alpha,p}(0,T;L^2(\mathbb{T}^3))} dt\right] \leq c(\alpha,p)\mathbb{E}\left[\int_0^T \|\phi\|^p_{L_2(\mathscr{U},L^2(\mathbb{T}^3))} \, dt\right].$$

Stochastic analysis in finite dimensions

For $N \in \mathbb{N}$, we consider an $\mathbb{R}^N$-valued process $(\mathbf{X}_t)_{t\in[0,T]}$ on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ with filtration $(\mathscr{F}_t)_{t\geq 0}$ such that

$$\begin{cases} \mathrm{d}\mathbf{X}_t = \mu(t,\mathbf{X}) \, \mathrm{d}t + \Sigma(t,\mathbf{X}) \, \mathrm{d}\mathbf{W}_t, \\ \mathbf{X}(0) = \mathbf{X}_0. \end{cases} \tag{2.1.10}$$

In this case, $\mathbf{W}$ is a standard $(\mathbb{R}^M, M \in \mathbb{N})$-valued Wiener process with respect to the $(\mathscr{F}_t)_{t\geq 0}$, and subject to some initial datum $\mathbf{X}_0 \in L^2(\Omega, \mathscr{F}_0, \mathbb{P})$. Furthermore, the functions

$$\begin{aligned} \mu &: [0,T] \times \mathbb{R}^N \to \mathbb{R}^N, \\ \Sigma &: [0,T] \times \mathbb{R}^N \to \mathbb{R}^{N\times M}, \end{aligned}$$

are continuous in $\mathbf{X} \in \mathbb{R}^N$ for every $t \in [0,T], \omega \in \Omega$. In the classical literature μ and Σ are both assumed to be Lipschitz continuous which is too strong for our application later. Hence, we consider the case with weaker assumptions in [83, Thm. 3.1.1]:

1. We assume that the following integrability condition on μ holds for all $R < \infty$,

$$\int_0^T \sup_{|\mathbf{X}|\leq R} |\mu(t,\mathbf{X})|^2 \,\mathrm{d}t < \infty.$$

2. μ is weakly coercive, i.e. for all $(t,\mathbf{X}) \in [0,T] \times \mathbb{R}^N$ we have that

$$\mu(t,\mathbf{X}) \cdot \mathbf{X} \leq c, \quad \text{for some } c \geq 0.$$

3. μ is locally weakly monotone, i.e. for all $t \in [0,T]$ and all $\mathbf{X},\mathbf{Y} \in \mathbb{R}^N$ with $|\mathbf{X}|,|\mathbf{Y}| \leq R$ the following holds

$$(\mu(t,\mathbf{X})) - (\mu(t,\mathbf{Y})) : (\mathbf{X}-\mathbf{Y}) \leq c(R)|\mathbf{X}-\mathbf{Y}|^2.$$

4. Σ is Lipschitz continuous, i.e. for all $t \in [0,T]$ and all $\mathbf{X},\mathbf{Y} \in \mathbb{R}^N$ the following holds

$$|\Sigma(t,\mathbf{X}) - \Sigma(t,\mathbf{Y})| \leq c|\mathbf{X}-\mathbf{Y}|^2.$$

Theorem 2.1.17. *Let μ and Σ satisfy assumptions (1-4). Assume we have a given probability space $(\Omega,\mathscr{F},\mathbb{P})$ with filtration $(\mathscr{F}_t)_{t\geq 0}$, an initial datum $\mathbf{X}_0 \in L^2(\Omega,\mathscr{F}_0,\mathbb{P})$ and an $(\mathscr{F}_t)_{t\geq 0}$-Wiener process $\mathbf{W}$. Then there is a unique $(\mathscr{F}_t)_{t\geq 0}$-adapted process $\mathbf{X}$ satisfying*

$$\mathbf{X}(t) = \mathbf{X} + \int_0^t \mu(s,\mathbf{X}(s))\, ds + \int_0^t \Sigma(s,\mathbf{X}(s))\, d\mathbf{W}_s, \ \mathbb{P}\text{-a.s.},$$

for every $t \in [0,T]$. The trajectories of $\mathbf{X}$ are $\mathbb{P}$-a.s. continuous and we have

$$\mathbb{E}\left[\sup_{t\in(0,T)} |\mathbf{X}_t|^2\right] < \infty.$$

Itô Formula

Lastly, we state auxiliary results; a tool we shall use in the compressible fluids, that is, an infinitesimal variant of Itô's lemma, see [15, Theorem A.4.1].

Lemma 2.1.18. *Let q be a stochastic process on $(\Omega,\mathscr{F},(\mathscr{F}_t)_{t\geq 0},\mathbb{P})$ such that for some*

$\alpha \in \mathbb{R}$,

$$q \in C_{\text{weak}}([0,T];W^{-\alpha,p}(\mathbb{T}^3)) \cap L^\infty(0,\infty;L^1(\mathbb{T}^3)) \quad \mathbb{P}\text{-}a.s,$$

$$\mathbb{E}\left[\sup_{t\in[0,T]} \|q\|^p_{L^1(\mathbb{T}^3)}\right] < \infty \quad \textit{for all } 1 \le p < \infty, \tag{2.1.11}$$

$$\mathrm{d}q = D^d_t q\,\mathrm{d}t + \mathbb{D}^s_t q\,\mathrm{d}W, \tag{2.1.12}$$

where D^d_t, $\mathbb{D}^d_t$ are progressively measurable with

$$D^d_t q \in L^p(\Omega;L^2(0,T;W^{-\alpha,k}(\mathbb{T}^3))), \qquad \mathbb{D}_t q \in L^p(\Omega;L^2(0,T;L_2(\mathscr{U};W^{-m,2}(\mathbb{T}^3)))),$$
$$\sum_{k\ge1}\int_0^T \|\mathbb{D}^s_t q(e_k)\|^2_1\,\mathrm{d}t \in L^p(\Omega) \quad 1 \le p < \infty, \tag{2.1.13}$$

for some $k > 1$ and some $m \in \mathbb{N}$.

Let w be a stochastic process on $(\Omega,\mathscr{F},(\mathscr{F}_t)_{t\ge0},\mathbb{P})$ satisfying

$$w \in C([0,T];W^{\alpha,k'} \cap C(\mathbb{T}^3) \quad \mathbb{P}\text{-}a.s.,$$

$$\mathbb{E}\left[\sup_{t\in[0,T]} \|w\|^p_{W^{\alpha,k'}\cap C(\mathbb{T}^3)}\right] < \infty, \quad 1 \le p < \infty, \tag{2.1.14}$$

$$dw = D^d_t w + \mathbb{D}^s_t w\mathrm{d}W \tag{2.1.15}$$

where D^d_t, $\mathbb{D}^d_t$ are progressively measurable with

$$D^d_t w \in L^p(\Omega;L^2(0,T;W^{\alpha,k'} \cap C(\mathbb{T}^3)), \qquad \mathbb{D}_t w \in L^p(\Omega;L^2(0,T;L_2(\mathscr{U};W^{m,2}(\mathbb{T}^3)))),$$
$$\sum_{k\ge1}\int_0^T \|\mathbb{D}^s_t q(e_k)\|^2_{W^{\alpha,k'}\cap C(\mathbb{T}^3)}\,\mathrm{d}t \in L^p(\Omega) \quad 1 \le p < \infty. \tag{2.1.16}$$

Let Q be $[\alpha+2]$-continuously differentiable function satisfying

$$\mathbb{E}\left[\sup_{t\in[0,T]} \|Q^{(j)}(w)\|^p_{W^{\alpha,k'}\cap C(\mathbb{T}^3)}\right] < \infty \quad j = 0,1,2, \quad 1 \le p < \infty. \tag{2.1.17}$$

Then

$$\mathrm{d}\left(\int_{\mathbb{T}^3} qQ(w)\,\mathrm{d}x\right) = \left(\int_{\mathbb{T}^3}\left[q\left(Q'(w)D^d_t w + \frac{1}{2}\sum_{k\ge1} Q''(w)|\mathbb{D}^s_t(e_k)|^2\right)\right]\mathrm{d}x\right.$$

$$+\left\langle Q(w), D_s^d q \right\rangle\Big)\,\mathrm{d}t$$
$$+\left(\sum_{\geq 1}\int_{\mathbb{T}^3} \mathbb{D}_t^s q(e_k)\mathbb{D}_t^q w(e_k)\,\mathrm{d}x\right)\mathrm{d}t + \mathrm{d}\mathbb{M}, \tag{2.1.18}$$

where

$$\mathbb{M} = \sum_{k\geq 1}\int_0^t \int_{\mathbb{T}^3} [qQ'(w)\mathbb{D}_t^s w(e_k) + Q(r)\mathbb{D}_t^s q(e_k)]\,\mathrm{d}x\mathrm{d}W_k. \tag{2.1.19}$$

Tools for compactness

The general approach to prove existence in PDEs is as follows: heuristically, one normally constructs a sequence of approximations with the aim of showing that they converge, and to show convergence it is necessary to first establish bounds and take limits. However, in our case we have nonlinear terms in our systems. In such situations bounds alone are not sufficient to pass to the limits, and a standard approach to overcome this issue is the use of compactness arguments. We should note that when dealing with randomness extra care is essential. To be precise, for a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ no topological structure is assumed on Ω and therefore the usual tools based on compact embedding can only be applied to the time and space variable. A general approach used by analysts to account for these problems is the adoption of Prokhorov's and Skorokhod's theorems, we refer the reader to see Billingsley [5, Thm 5.1, 5.2] and Dudley [44, Thm 11.7.2], respectively.

In the context of the problems considered in this thesis we shall summarise the compactness framework material as follows. Let (X,τ) be a topological space, we denote by $\mathscr{B}(X)$ the smallest σ-field on (X,τ) which contains all open sets, that is, topological σ-field. Let $\mathbf{V}$ be an X-valued random variable, see Definition 2.1.7. Then the probability law μ of $\mathbf{V}$ on (X,τ) is given by $\mu = \mathbb{P}\circ\mathbf{V}^{-1}$ i.e. $(\mu(A) = \mathbb{P}(\mathbf{V}\in A)$ for $A \in \mathscr{B}(\mathbf{V}))$. In the following we collect the tools of compactness.

Definition 2.1.29 (Tightness)**.** A family $(\mu_\alpha)_{\alpha\in\mathscr{I}}$ of probability laws on topologically space $(V, \mathscr{B}(V))$ is called tight if for every $\varepsilon > 0$ there is a compact subset $K \subset V$ such that

$$\mu_\alpha(K) \geq 1-\varepsilon$$

for every $\alpha \in \mathscr{I}$.

Lemma 2.1.19 (Prokhorov; [63], Thm 2.6)**.** *Let $(\mu_\alpha)_{\alpha\in\mathscr{I}}$ be a family of probability laws on a Polish space (V,ρ). The family $(\mu_\alpha)_{\alpha\in\mathscr{I}}$ is tight if and only if it is relatively weakly compact.*

Theorem 2.1.20 (Skorokhod representation; [63], Thm 2.7)**.** *Let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence of probability laws on a complete separable metric space (X,τ) such that $\mu_n \to \mu$ weakly in the sense of measures as $n\to\infty$. Then there is a probability space $(\tilde{\Omega},\tilde{\mathscr{F}},\tilde{\mathbb{P}})$ and random variables $(\tilde{\mathbf{V}}_n)_{n\in\mathbb{N}},\tilde{\mathbf{V}} : (\tilde{\Omega},\tilde{\mathscr{F}},\tilde{\mathbb{P}}) \to (X,\mathscr{B}(X))$ such that*

- *The laws of $\tilde{\mathbf{V}}_n$ and $\tilde{\mathbf{V}}$ under $\tilde{\mathbb{P}}$ coincide with μ_n and μ respectively, $n\in\mathbb{N}$.*
- *we have $\tilde{\mathbb{P}}$ a.s. that $\tilde{\mathbf{V}}_n \to \tilde{\mathbf{V}}$ as $n\to\infty$.*

We note Theorem 2.1.20 is only applicable to metric spaces. Consequently, Theorem 2.1.20 is not sufficient to account for our preferred choice of space, that is, Banach spaces with the weak topology. We solve this problem by using a generalisation of Theorem 2.1.20. Firstly we need the following definition.

Definition 2.1.30 (Quasi-Polish space)**.** Let $(\mathscr{X},\tau)$ be a topological space such that there exists a countable family

$$\{f_n : \mathscr{X} \to [-1,1]; n\in\mathbb{N}\}$$

of continuous functions that separates points of $\mathscr{X}$. Then $(\mathscr{X},\tau,(f_n)_{n\in\mathbb{N}})$ is called a quasi-Polish space.

We are now ready to state the generalisation of Theorem 2.1.20, a result we shall use for our compactness arguments.

Theorem 2.1.21 (Jakubowski-Skorokhod representation theorem)**.** *Let (X,τ) be quasi-Polish space (sub-Polish) and let $\mathscr{G}$ be the σ-field generated by $\{f_n; n\in\mathbb{N}\}$. If $(\mu_n)_{n\in\mathbb{N}}$ is a tight sequence of probability measures on $(V,\mathscr{G})$, then there exists a subsequence (μ_{n_k}) and X-valued Borel measurable random variables $(\mathbf{V}_k)_{k\in\mathbb{N}}$ and $\mathbf{V}$ defined on the standard space $([0,1],\overline{\mathscr{B}[0,1]},\mathfrak{L})$, such that μ_{n_k} is the law of $\mathbf{V}_k$ and $\mathbf{V}_k(\omega)$ converges to $\mathbf{V}(\omega)$ in X for every $\omega\in[0,1]$. Moreover, the law of $\mathbf{V}$ is a Radon measure.*

Lemma 2.1.22. *Let $(\Omega,\mathscr{F},\mathbb{P})$ be a probability space and X,Y be two random variables with values in a Polish space (S,d_s). Assume that $\mathbb{E}(h(Y)X)=0$ for all continuous and bounded functions $h : S\to\mathbb{R}$ then, $\mathbb{E}(X|\sigma(Y))=0$*

Proof. Let B be a measurable subset of S. Then there is a sequence of smooth function

(h_n) such that

$$h_n(s) \to I_B(s) \quad \forall s \in S.$$

Where $I_B(s)$ is the indicator function of B. We get by monotone convergence that $\mathbb{E}(h_n(s)) \to \mathbb{E}(I_B(s))$ and then by (choosing h_n appropriately) we have that,

$$\mathbb{E}(I_B(Y)X) = \lim_{n\to\infty} \mathbb{E}(h_n(Y)X) = 0$$

$$i.e.\ \mathbb{E}(I_B(Y)X) = 0 \quad \forall B \in \mathscr{B}(s).$$

Let $U \in \sigma(Y)$, that is, there exist $B \in \mathscr{B}(s)$ such that $U = Y^{-1}(B)$. This implies that

$$\mathbb{E}(I_U X) = \mathbb{E}(I_{Y^{-1}(B)}X) = \mathbb{E}(I_B(Y)X) = 0.$$

Therefore it follows that $\mathbb{E}(I_U X) = 0$, $\forall\ U \in \sigma(Y)$. Then the conditional expectation is given by

$$\mathbb{E}(X|\sigma(Y)) = 0.$$

□

2.1.5 Generalised Young Measures

The premise of this thesis is set on understanding solutions of the Euler system with oscillations and concentrations phenomena. Central to our approach is the concept of a generalised Young measure, a measure that measures both oscillations and concentrations. We adopt a formulation of a generalised Young measure as introduced in [2], see [72, 79] for more details on Young measures in general. Let $\mathscr{M}$ denote the set of Radon measures, we denote by $\mathscr{M}^+$ the set of non-negative Radon measures and by $\mathscr{P}$ the set of probability measures. Let $Q_T = (0,T) \times \mathbb{T}^3$ be a space-time cylinder. We introduce the following definition of a generalised measure.

Definition 2.1.31. A quantity $\mathscr{V} = (\nu_{t,x}, \nu^\infty_{t,x}, \lambda)$ is called a generalised Young measure provided

(a) $\nu_{t,x} \in L^\infty_{w*}(Q_T; \mathscr{P}(\mathbb{R}^3))$ is a parametrised probability measure on $\mathbb{R}^3$;

(b) $\lambda \in \mathscr{M}^+(\overline{Q}_T)$ is a non-negative measure;

(c) $\nu^\infty_{t,x} \in L^\infty_{w*}(Q_T,\lambda;\mathscr{P}(\mathbb{S}^2))$ is a parametrised probability measure on $\mathbb{S}^2$;

(d) We have $\int_{Q_T} \langle \nu_{t,x}, |\xi|^2\rangle \,\mathrm{d}x\mathrm{d}t < \infty$.

We denote by $Y_2(Q_T)$ the space of all generalised Young measures.

We note, any Radon measure $\mu \in \mathscr{M}(Q_T)$ can be represented by a generalised Young measure by setting $\mathscr{V} = (\delta_{\mu^a(t,x)}, \frac{\mathrm{d}\mu^s}{\mathrm{d}|\mu^s|}, |\mu^s|)$, where $\mu = \mu^a \mathrm{d}\mathfrak{L}^n + \mu^s$ is the Radon-Nikodým decomposition of μ. In the present work we consider the Carathéodory functions $f : Q_T \times \mathbb{R}^3 \to \mathbb{R}$ such that the recession function

$$f^\infty(t,x,\xi) := \lim_{s\to\infty} \frac{f(t,x,s\xi)}{s^2}$$

is well-defined and continuous on $\overline{Q}_T \times \mathbb{S}^2$(which implies that f grows at most quadratically in ξ), and we denote the space of all such functions by $\mathscr{G}_2(Q_T)$. In the following, let ξ denote a corresponding dummy-variable. We say a sequence $\{\mathscr{V}^n\} = \{(\nu^n_{t,x}, \nu^{\infty,n}_{t,x}, \lambda^n)\}$ converges weakly* in $Y_2(Q_T)$ to some $\mathscr{V} = (\nu_{t,x}, \nu^\infty_{t,x}, \lambda) \in Y_2(Q_T)$ provided

$$\langle \nu^n_{t,x}, f(\xi)\rangle \mathrm{d}x\mathrm{d}t + \langle \nu^{\infty,n}_{t,x}, f^\infty(\xi)\rangle \mathrm{d}\lambda^n \rightharpoonup^* \langle \nu_{t,x}, f(\xi)\rangle \mathrm{d}x\mathrm{d}t + \langle \nu^\infty_{t,x}, f^\infty(\xi)\rangle \mathrm{d}\lambda \quad \text{in } \mathscr{M}(Q_T)$$

for all $f \in \mathscr{G}_2(Q_T)$, that is

$$\begin{aligned} &\int_{Q_T} \varphi \langle \nu^n_{t,x}, f(\xi)\rangle \mathrm{d}x\mathrm{d}t + \int_{Q_T} \varphi \langle \nu^{\infty,n}_{t,x}, f^\infty(\xi)\rangle \mathrm{d}\lambda^n \\ \to &\int_{Q_T} \varphi \langle \nu_{t,x}, f(\xi)\rangle \mathrm{d}x\mathrm{d}t + \int_{Q_T} \varphi \langle \nu^\infty_{t,x}, f^\infty(\xi)\rangle \mathrm{d}\lambda \end{aligned}$$

for all $\varphi \in C(\overline{Q}_T)$. The space of $\mathscr{G}_2(Q_T)$ is a separable Banach space together with the norm

$$\|f\|_{\mathscr{G}_2(Q_T)} := \sup_{(t,x)\in Q_T, \xi\in B_1(0)} (1-|\xi|) \left| f\left(t,x,\frac{\xi}{1-|\xi|}\right)\right|$$

and $Y_2(Q_T,\mathbb{R}^n)$ is a subspace of its dual. Consequently, $Y_2(Q_T,\mathbb{R}^n)$ together with the weak* convergence introduced above is a quasi-Polish space, see Definition 2.1.30. Specifically, separable Banach spaces endowed with the weak topology and dual spaces of separable Banach spaces are quasi-Polish spaces. In applications, we are interested in long-time behaviour of systems, as such we define

$$Y_2^{\text{loc}}(Q_\infty) = \{\mathscr{V} : \mathscr{V} \in Y_2(Q_T) \forall T > 0\}.$$

Noting that the topology of $Y_2^{\text{loc}}(Q_\infty)$ is generated by the topologies on $Y_2(Q_T)$ in the sense that

$$\mathscr{V}^n \rightharpoonup^* \mathscr{V} \quad \text{in} \quad Y_2^{\text{loc}}(Q_\infty) \Longleftrightarrow \mathscr{V}^n \rightharpoonup^* \mathscr{V} \quad \text{in} \quad Y_2(Q_T) \quad \forall T > 0,$$

it follows that $Y_2^{\text{loc}}(Q_\infty)$ is a quasi-Polish space. We embed $L^2(Q_T)$ into $Y_2(Q_T)$ via the inclusion

$$L^2(Q_T) \ni \mathbf{u} \mapsto (\delta_{u(t,x)}, 0, 0) \in Y_2(Q_T).$$

By the Alaoglu-Bourbaki theorem, for any $M > 0$ there is a compact subset $\mathscr{K}_M$ of $\mathscr{G}_2(Q_T)$ such that

$$\{(\delta_{u(t,x)}, 0, 0) \in Y_2(Q_T) : \|\mathbf{u}\|_{L^2(Q_T)} \leq M\} \subset \mathscr{K}_M. \tag{2.1.20}$$

Since $Y_2(Q_T)$ is weak* closed in $\mathscr{G}_2(Q_T)$ we conclude that $\mathscr{K}_M \cap Y_2(Q_T)$ is compact, where clearly

$$\{(\delta_{u(t,x)}, 0, 0) \in Y_2(Q_T) : \|\mathbf{u}\|_{L^2(Q_T)} \leq M\} \subset \mathscr{K}_M \cap Y_2(Q_T).$$

Finally, it is also important to identify a generalised Young measure with a space-time distribution, that is, for $\mathscr{V} = (\nu_{t,x}, \nu_{t,x}^\infty, \lambda) \in Y_2(Q_T)$ we define

$$\begin{aligned} C_c^\infty(Q_T \times \mathbb{R}^3)^2 \ni (\psi, \varphi) \mapsto & \int_{Q_T} \int_{R^3} \psi(t,x,\xi) \mathrm{d}\nu_{t,x}(\xi) \, \mathrm{d}x \mathrm{d}t \\ & + \int_{Q_T} \int_{\mathbb{R}^3} \varphi(t,x,\xi) \, \mathrm{d}\nu_{t,x}^\infty(\xi) \, \mathrm{d}\lambda(t,x). \end{aligned} \tag{2.1.21}$$

In the following sections we shall study the probability laws on $Y_2(Q_T)$, and we need to make sense of the required σ-field. A suitable candidate is the σ-algebra generated by the functions $\{f_n\}$ in Definition 2.1.30, that is, we set

$$\mathscr{B}_Y := \sigma\left(\bigcup_{n=1}^{\infty} \sigma(f_n)\right). \tag{2.1.22}$$

Chapter 3

3.1 Dissipative solutions to the incompressible stochastic Euler equations

Our goal in this section and its subsequent sections is to formalise the concept of dissipative measure-valued solutions to the incompressible stochastic Euler equations and show their existence. These solutions are weak in the analytical and probabilistic sense. The key idea of the proof hinges on the link between the Euler system and the Navier-Stokes system (in fact, Euler equations are linked via a vanishing viscosity limit to the Navier–Stokes equations). We shall proceed as follows. Firstly, we show the existence of martingale solutions to the incompressible Navier-Stokes system. Finally, we show that martingale solutions to the incompressible Navier-Stokes system converge to dissipative measure-valued martingale solutions of the Euler system as viscosity vanishes. Our strategy is reminiscent to that used in the analogous results of the deterministic case, see [42].

3.1.1 Stochastic Analysis

Here we collect the mathematical framework relevant to the incompressible case. Firstly, we present a thorough outline of the stochastic force (i.e. the noise term) used in our incompressible fluids models. We refer the reader to Section 2.1.4 for brief discussion, and to [35] for more details on the elements of stochastic calculus in infinite dimensions. Let $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$ be a complete stochastic basis with a probability measure $\mathbb{P}$ on $(\Omega, \mathscr{F})$ and right-continuous filtration $(\mathscr{F}_t)_{t\geq 0}$. Let $\mathscr{U}$ be a separable Hilbert space with orthonormal basis $(e_k)_{k\in\mathbb{N}}$. We denote by $L_2(\mathscr{U}, L^2(\mathbb{T}^3))$ the set of Hilbert-Schmidt operators from $\mathscr{U}$ to $L^2(\mathbb{T}^3)$. The stochastic process W is a cylindrical Wiener process $W = (W_t)_{t\geq 0}$ in $\mathscr{U}$, and is of the form

$$W(s) = \sum_{k\in\mathbb{N}} e_k \beta_k(s), \tag{3.1.1}$$

where $(\beta_k)_{k\in\mathbb{N}}$ is a sequence of independent real-valued Wiener processes relative to $(\mathscr{F}_t)_{t\geq 0}$. We select the most natural space $\mathscr{U} = L^2(\mathbb{T}^3)$, and we identify a precise definition of the diffusion coefficient by asserting the following assumptions on ϕ: for every $x \in L^2(\mathbb{T}^3)$ the mapping $\phi : \mathscr{U} \to L^2(\mathbb{T}^3)$ is defined by

$$\phi(e_k) = \phi_k.$$

Since ϕ_k is a Hilbert-Schmidt operator, it follows that

$$\sum_{k\geq 1} \|\phi(e_k)\|^2_{L^2(\mathbb{T}^3)} < \infty. \tag{3.1.2}$$

For $\phi \in L^2(\Omega, \mathscr{F}, \mathbb{P}; L^2(0,T; L_2(\mathscr{U}, L^2(\mathbb{T}^3)))$, the stochastic integral

$$\int_0^t \phi \,\mathrm{d}W = \sum_{k\geq 1} \int_0^t \phi(e_k)\,\mathrm{d}\beta_k,$$

where ϕ is progressively measurable, defines a $\mathbb{P}$-almost surely continuous $L^2(\mathbb{T}^3)$ valued $(\mathscr{F}_t)_{t\geq 0}$-martingale. Furthermore, we can multiply with test functions since

$$\int_{\mathbb{T}^3} \left(\int_0^\tau \phi \,\mathrm{d}W \cdot \varphi \right) \mathrm{d}x = \sum_{k\geq 1} \int_0^\tau \left(\int_{\mathbb{T}^3} \phi_k \cdot \varphi \,\mathrm{d}x \right) \mathrm{d}\beta_k, \quad \varphi \in L^2(\mathbb{T}^3), \tag{3.1.3}$$

is well defined in $L^2(\Omega, \mathscr{F}, \mathbb{P}; C[0,T])$.

We define the auxiliary space $\mathscr{U}_0$ with $\mathscr{U} \subset \mathscr{U}_0$ as

$$\begin{aligned} \mathscr{U}_0 : &= \left\{ u = \sum_k \alpha_k e_k : \sum_k \frac{\alpha_k^2}{k^2} < \infty \right\}, \\ \|u\|^2_{\mathscr{U}_0} : &= \sum_k^\infty \frac{\alpha_k^2}{k^2},\ u = \sum_k \alpha_k e_k, \end{aligned} \tag{3.1.4}$$

thus the embedding $\mathscr{U} \hookrightarrow \mathscr{U}_0$ is Hilbert-Schmidt and the trajectories of W belong $\mathbb{P}$-a.s. to the class $C([0,T]; \mathscr{U}_0)$ (see [35]).

We proceed to introduce the following variant of infinite dimensional Itô-formula [16, Lemma 3.1].

Lemma 3.1.1. *Let* $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$ *be a stochastic basis and* W *a cylindrical* $(\mathscr{F}_t)$*-*

Wiener process and $p \in [1,\infty]$. Let $\mathbf{w}^1, \mathbf{w}^2$ be $(\mathscr{F}_t)$-progressively measurable processes with $\mathbf{w}^1 \in C_w([0,T]; L^2_{\mathrm{div}}(\mathbb{T}^3))$, $\mathbf{w}^2 \in C([0,T]; L^2_{\mathrm{div}}(\mathbb{T}^3))$ and $\mathbf{w}^2 \in L^1(0,T; C^1(\mathbb{T}^3))$ a.s. such that

$$\mathbf{w}^1, \mathbf{w}^2 \in L^2_w(\Omega, L^\infty(0,T; L^2(\mathbb{T}^3))).$$

Suppose that

$$\begin{aligned} \lambda_t \in L^1_{w^*}(\Omega; L^\infty_{w^*}(0,T; \mathscr{M}^+(\mathbb{T}^3))), \quad \mathbf{H}^1 \in L^1_{w^*}(\Omega; L^\infty(0,T; L^1(\mathbb{T}^3))), \\ \mathbf{G}^1 \in L^1_{w^*}(\Omega; L^\infty(Q_T, \lambda_t \otimes \mathscr{L}^1))), \quad \Phi^1 \in L^2(\Omega; L_2(\mathscr{U}; L^2(\mathbb{T}^3)))), \end{aligned}$$

are all $\mathscr{F}_t$-progressively measurable, such that

$$\begin{aligned} \int_{\mathbb{T}^3} \mathbf{w}^1(t) \cdot \varphi \, \mathrm{d}x = \int_{\mathbb{T}^3} \mathbf{w}^1(0) \cdot \varphi \, \mathrm{d}x + \int_0^t \int_{\mathbb{T}^3} \mathbf{H}^1 : \nabla \varphi \, \mathrm{d}x \, \mathrm{d}s \\ + \int_{(0,T)\times\mathbb{T}^3} \mathbf{G}^1 : \nabla \varphi \, \mathrm{d}\lambda + \int_{\mathbb{T}^3} \varphi \cdot \int_0^t \Phi^1 \, \mathrm{d}W \, \mathrm{d}x \end{aligned} \tag{3.1.5}$$

for all $\varphi \in C^\infty_{\mathrm{div}}(\mathbb{T}^3)$.
Suppose further that

$$\mathbf{h}^2 \in L^1_{w^*}(\Omega; L^\infty(Q_T)), \quad \Phi^2 \in L^2(\Omega; L_2(\mathscr{U}; L^2(\mathbb{T}^3))),$$

are $\mathscr{F}_t$-progressively measurable, such that

$$\int_{\mathbb{T}^3} \mathbf{w}^2(t) \cdot \varphi \, \mathrm{d}x = \int_{\mathbb{T}^3} \mathbf{w}^2(0) \cdot \varphi \, \mathrm{d}x + \int_0^t \int_{\mathbb{T}^3} \mathbf{h}^2 \cdot \varphi \, \mathrm{d}x \, \mathrm{d}s + \int_{\mathbb{T}^3} \varphi \cdot \int_0^t \Phi^2 \, \mathrm{d}W \, \mathrm{d}x \tag{3.1.6}$$

for all $\varphi \in C^\infty_{\mathrm{div}}(\mathbb{T}^3)$. Then for all $t \geq 0$ $\mathbb{P}$-a.s we have

$$\int_{\mathbb{T}^3} \mathbf{w}^1(t) \cdot \mathbf{w}^2(t) \, \mathrm{d}x = \int_{\mathbb{T}^3} \mathbf{w}^1(0) \cdot \mathbf{w}^2(0) \, \mathrm{d}x + \int_0^t \int_{\mathbb{T}^3} \mathbf{H}^1 : \nabla \mathbf{w}^2 \, \mathrm{d}x \, \mathrm{d}s$$

$$
\begin{aligned}
&+\int_{(0,T)\times\mathbb{T}^3}\mathbf{G}^1:\nabla\mathbf{w}^2\,\mathrm{d}\lambda+\int_{\mathbb{T}^3}\mathbf{w}^2\cdot\int_0^t\Phi^1\,\mathrm{d}W\,\mathrm{d}x\\
&+\int_0^t\int_{\mathbb{T}^3}\mathbf{h}^2\cdot\mathbf{w}^1\,\mathrm{d}x\,\mathrm{d}s+\int_{\mathbb{T}^3}\mathbf{w}^1\cdot\int_0^t\Phi^2\,\mathrm{d}W\,\mathrm{d}x\\
&+\sum_{k\geq 1}\int_0^t\int_{\mathbb{T}^3}\Phi^1e_k\,\Phi^2e_k\,\mathrm{d}x\mathrm{d}t
\end{aligned}
\tag{3.1.7}
$$

Proof. In order to justify the application of Itô's formula to the process $t\mapsto\int_{\mathbb{T}^3}\mathbf{w}^1(t)\cdot\mathbf{w}^2(t)\,\mathrm{d}x$ we have to perform some regularisation in equation (3.1.5) using mollification in space with parameter $m>0$. For $\varphi\in L^2_{\mathrm{div}}(\mathbb{T}^3)$ we have $\varphi_m\in C^\infty_{\mathrm{div}}(\mathbb{T}^3)$ and

$$
\begin{aligned}
&\|\varphi_m\|_{W^{k,p}_x}\leq c(m)\|\varphi\|_{L^2_x}\quad\forall k\in\mathbb{N}_0,\ p\in[1,\infty],\\
&\|\varphi_m\|_{W^{k,p}_x}\leq\|\varphi\|_{W^{k,2}_x}\quad\forall k\in\mathbb{N}_0,\ p\in[1,\infty],
\end{aligned}
\tag{3.1.8}
$$

provided $\varphi\in L^p(\mathbb{T}^3)$ or $\varphi\in W^{k,p}(\mathbb{T}^3)$, respectively. Moreover,

$$
\varphi_m\to\varphi\quad\text{in}\quad W^{k,p}(\mathbb{T}^3)\quad\forall k\in\mathbb{N}_0,\ p\in[1,\infty),\tag{3.1.9}
$$

$$
\varphi_m\to\varphi\quad\text{in}\quad C^k(\mathbb{T}^3)\quad\forall k\in\mathbb{N}_0,\tag{3.1.10}
$$

as $m\to 0$ provided $\varphi\in W^{k,p}(\mathbb{T}^3)$ or $C^k(\mathbb{T}^3)$, respectively. Finally, the operator $(\cdot)_m$ commutes with derivatives. Inserting $(\varphi)_m$ in (3.1.5) yields

$$
\begin{aligned}
\int_{\mathbb{T}^3}\mathbf{w}^1_m(t)\cdot\varphi\,\mathrm{d}x=&\int_{\mathbb{T}^3}\mathbf{w}^1_m(0)\cdot\varphi\,\mathrm{d}x+\int_0^t\int_{\mathbb{T}^3}\mathbf{H}^1:\nabla(\varphi)_m\,\mathrm{d}x\,\mathrm{d}s\\
&+\int_{(0,t)\times\mathbb{T}^3}\mathbf{G}^1:\nabla(\varphi)_m\,\mathrm{d}\lambda_\sigma\,\mathrm{d}\sigma+\int_{\mathbb{T}^3}\varphi\cdot\int_0^t\Phi^1_m\,\mathrm{d}W\,\mathrm{d}x
\end{aligned}
$$

where Φ^1_m is given by $\Phi^1_me_k=(\Phi^1e_k)_m$ for $k\in\mathbb{N}$. Let $m>0$ be fixed, using (3.1.8) we obtain

$$
\begin{aligned}
\left|\int_0^T\int_{\mathbb{T}^3}\mathbf{H}^1:\nabla(\varphi)_m\,\mathrm{d}x\,\mathrm{d}s\right|&\leq\sup_{0\leq t\leq T}\int_{\mathbb{T}^3}|\mathbf{H}^1|\,\mathrm{d}x\int_0^T\|\nabla(\varphi)_m\|_{L^\infty_x}\,\mathrm{d}\sigma\\
&\leq c(m)\sup_{0\leq t\leq T}\int_{\mathbb{T}^3}|\mathbf{H}^1|\,\mathrm{d}x\int_0^T\|\varphi\|_{L^2_x}\,\mathrm{d}\sigma
\end{aligned}
$$

$\mathbb{P}$-a.s. as well as

$$
\left|\int_{(0,T)\times\mathbb{T}^3}\mathbf{G}^1:\nabla(\varphi)_m\,\mathrm{d}\lambda\right|\leq\sup_{Q_T}|\mathbf{G}^1|\int_0^T\int_{\mathbb{T}^3}|\nabla(\varphi)_m|\,\mathrm{d}\lambda_t\,\mathrm{d}t
$$

$$\leq \sup_{Q_T}|\mathbf{G}^1| \sup_{0\leq t\leq T} \lambda_t(\mathbb{T}^3) \int_0^T \|\nabla(\varphi)_m\|_{L^\infty_x} \mathrm{d}t$$
$$\leq c(m) \sup_{Q_T}|\mathbf{G}^1| \sup_{0\leq t\leq T} \lambda_t(\mathbb{T}^3) \int_0^T \|\varphi\|_{L^2_x} \mathrm{d}t.$$

Therefore, the deterministic parts in the equation for $\mathbf{w}^1_m$ are functionals on L^2. Consequently, we can apply Itô's formula on the Hilbert space $L^2_{\mathrm{div}}(\mathbb{T}^3)$ (see [37, Prop. A.1] and [17, Prop. C.0.1]) to the process $t \mapsto \int_{\mathbb{T}^3} \mathbf{w}^1_m(t)\cdot\mathbf{w}^2(t)\,\mathrm{d}x$ to obtain

$$\begin{aligned}\int_{\mathbb{T}^3} \mathbf{w}^1_m(t)\cdot\mathbf{w}^2(t)\,\mathrm{d}x = &\int_{\mathbb{T}^3} \mathbf{w}^1_m(0)\cdot\mathbf{w}^2(0)\,\mathrm{d}x + \int_0^t\int_{\mathbb{T}^3} \mathbf{H}^1 : (\nabla\mathbf{w}^2)_m\,\mathrm{d}x\,\mathrm{d}s\\ &+ \int_{(0,t)\times\mathbb{T}^3} \mathbf{G}^1 : (\nabla\mathbf{w}^2)_m\,\mathrm{d}\lambda_\sigma\,\mathrm{d}\sigma + \int_{\mathbb{T}^3}\int_0^t \mathbf{w}^2\cdot\Phi^1_m\,\mathrm{d}W\,\mathrm{d}x\\ &+ \int_0^t\int_{\mathbb{T}^3} \mathbf{h}^2\cdot\mathbf{w}^1_m\,\mathrm{d}x\,\mathrm{d}s + \int_{\mathbb{T}^3}\int_0^t \mathbf{w}^1_m\cdot\Phi^2\,\mathrm{d}W\,\mathrm{d}x\\ &+ \sum_{k\geq 1}\int_0^t\int_{\mathbb{T}^3} \Phi^1_m e_k\,\Phi^2 e_k\,\mathrm{d}x\mathrm{d}t.\end{aligned}$$

The desired formulation (3.1.7) follows from taking the limit $m\to 0$. finally, we show how individual terms converge to their limits:

$$\lim_{m\to 0}\int_{\mathbb{T}^3} \mathbf{w}^1_m(0)\cdot\mathbf{w}^2(0)\,\mathrm{d}x \to \int_{\mathbb{T}^3} \mathbf{w}^1(0)\cdot\mathbf{w}^2(0)\,\mathrm{d}x, \tag{3.1.11}$$

the convergence above follows from the following observation:

$$\begin{aligned}&\left|\int_{\mathbb{T}^3} \mathbf{w}^1_m(0)\cdot\mathbf{w}^2(0)\,\mathrm{d}x - \int_{\mathbb{T}^3} \mathbf{w}^1(0)\cdot\mathbf{w}^2(0)\,\mathrm{d}x\right|\\ &\leq \int_{\mathbb{T}^3} |\mathbf{w}^2(0)||\mathbf{w}^1_m(0)-\mathbf{w}^1(0)|\,\mathrm{d}x\\ &\leq \|\mathbf{w}^2(0)\|_2 \underbrace{\|\mathbf{w}^1_m(0)-\mathbf{w}^1(0)\|_2}_{I_m} \text{ by Hölder inequality.}\end{aligned}$$

thus $I_m\to 0$ as $m\to 0$ by assumptions on $\mathbf{w}^1$ so that (3.1.11) holds. To show that

$$\lim_{m\to 0}\int_0^t\int_{\mathbb{T}^3} \mathbf{H}^1 : (\nabla\mathbf{w}^2)_m\,\mathrm{d}x \to \int_0^t\int_{\mathbb{T}^3} \mathbf{H}^1 : \nabla\mathbf{w}^2\,\mathrm{d}x, \tag{3.1.12}$$

we consider,

$$\left|\int_0^t\int_{\mathbb{T}^3} \mathbf{H}^1 : (\nabla\mathbf{w}^2)_m\,\mathrm{d}x - \int_0^t\int_{\mathbb{T}^3} \mathbf{H}^1 : \nabla\mathbf{w}^2\,\mathrm{d}x\right|$$

$$\leq \int_0^t \int_{\mathbb{T}^3} |\mathbf{H}^1||(\nabla \mathbf{w}^2)_m - \nabla \mathbf{w}^2|\,\mathrm{d}x$$
$$\leq \sup_{0\leq t\leq T} \int_{\mathbb{T}^3} |\mathbf{H}^1| \int_0^T \underbrace{\|(\nabla \mathbf{w}^2)_m - \nabla \mathbf{w}^2\|_{L_x^\infty}}_{II_m} \mathrm{d}t,$$

the last line above follows from Hölder's inequality. We also note that $II_m \to 0$ for a.e t as $m \to 0$ by assumptions on $\mathbf{w}^2$, and $\|\nabla \mathbf{w}_m^2\|_{L_x^\infty} \leq \|\nabla \mathbf{w}^2\|_{L_x^\infty}$. Therefore, $II_m \leq 2\|\nabla \mathbf{w}^2\|_{L_x^\infty}$. Then by dominated convergence we deduce that

$$\int_0^T \|(\nabla \mathbf{w}^2)_m - \nabla \mathbf{w}^2\|_{L_x^\infty}\,\mathrm{d}t \to 0, \quad \text{as } m \to 0,$$

so that (3.1.12) holds. To show the limit

$$\lim_{m\to 0} \int_{(0,t)\times\mathbb{T}^3} \mathbf{G}^1 : (\nabla \mathbf{w}^2)_m \,\mathrm{d}\lambda_\sigma\,\mathrm{d}\sigma \to \int_{(0,t)\times\mathbb{T}^3} \mathbf{G}^1 : (\nabla \mathbf{w}^2)\,\mathrm{d}\lambda_\sigma\,\mathrm{d}\sigma, \tag{3.1.13}$$

holds we proceed as follows,

$$\left|\int_Q \mathbf{G}^1 : (\nabla \mathbf{w}^2)_m \,\mathrm{d}\lambda_\sigma\,\mathrm{d}\sigma - \int_Q \mathbf{G}^1 : (\nabla \mathbf{w}^2)\,\mathrm{d}\lambda_\sigma\,\mathrm{d}\sigma\right|$$
$$\leq \int_{Q_T} |\mathbf{G}^1||(\nabla \mathbf{w}^2)_m - (\nabla \mathbf{w}^2)|\,\mathrm{d}\lambda_\sigma\,\mathrm{d}\sigma$$
$$\leq \sup_{Q_T} |\mathbf{G}^1| \int_0^T \int_{\mathbb{T}^3} |(\nabla \mathbf{w}^2)_m - (\nabla \mathbf{w}^2)|\,\mathrm{d}\lambda_\sigma\,\mathrm{d}\sigma$$
$$\leq \sup_{Q_T} |\mathbf{G}^1| \sup_{0\leq t\leq T} \lambda_t(\mathbb{T}^3) \int_0^T \underbrace{\|(\nabla \mathbf{w}^2)_m - (\nabla \mathbf{w}^2)\|_{L_x^\infty}}_{III_m} \mathrm{d}\sigma,$$

then results of (3.1.13) follows by similar closing arguments shown for (3.1.12). To show the limit

$$\lim_{m\to 0} \int_{\mathbb{T}^3} \int_0^t \mathbf{w}^2 \cdot \Phi_m^1 \,\mathrm{d}W\,\mathrm{d}x \to \int_{\mathbb{T}^3} \int_0^t \mathbf{w}^2 \cdot \Phi^1 \,\mathrm{d}W\,\mathrm{d}x, \tag{3.1.14}$$

we use Burkholder-Davis-Gundy inequality to obtain

$$\mathbb{E}\left(\left|\int_0^t \int_{\mathbb{T}^3} \mathbf{w}^2 \cdot (\Phi_m^1 - \Phi^1)\,\mathrm{d}x\mathrm{d}W\right|\right)$$
$$\leq \mathbb{E}\left(\sup_t \left|\int_0^t \int_{\mathbb{T}^3} \mathbf{w}^2 \cdot (\Phi_m^1 - \Phi^1)\,\mathrm{d}x\mathrm{d}W\right|\right)$$
$$\leq c\,\mathbb{E}\left(\left\langle\left\langle \int_0^\cdot \int_{\mathbb{T}^3} \mathbf{w}^2 \cdot (\Phi_m^1 - \Phi^1)\,\mathrm{d}x\mathrm{d}W \right\rangle\right\rangle\right)^{1/2}$$

$$
\begin{aligned}
&= c\,\mathbb{E}\left(\sum_{k\geq 1}\left\langle\left\langle\int_0^{\cdot}\int_{\mathbb{T}^3}\mathbf{w}^2\cdot(\Phi^1_m-\Phi^1)e_k\,\mathrm{d}x\mathrm{d}W_k(t)\right\rangle\right\rangle\right)^{1/2}\\
&= c\,\mathbb{E}\left(\sum_{k\geq 1}\int_0^T\left[\int_{\mathbb{T}^3}\mathbf{w}^2\cdot(\Phi^1_m-\Phi^1)e_k\right]^2\mathrm{d}t\right)^{1/2}\\
&\leq c\,\mathbb{E}\left(\sum_{k\geq 1}\int_0^T\|\mathbf{w}^2\|_2^2\|(\Phi^1_m-\Phi^1)e_k\|_2^2\mathrm{d}t\right)^{1/2} \quad \text{by Hölder inequality}\\
&\leq c\,\mathbb{E}\left(\sup_{0\leq t\leq T}\|\mathbf{w}^2\|_2^2\sum_{k\geq 1}\|(\Phi^1_m-\Phi^1)e_k\|_2^2\right)^{1/2}.\\
&= c\,\underbrace{\|\Phi^1_m-\Phi^1\|_{L_2(\mathscr{U},L^2)}}_{IV_m}\mathbb{E}\sup\|\mathbf{w}^2\|_{L^2}
\end{aligned}
$$

The limit is satisfied when $IV_m \to 0$ as $m \to 0$. To show this, we prove that

$$
\limsup_{m\to 0}\|\Phi^1_m-\Phi^1\|_{L_2(\mathscr{U},L^2)}=\limsup_{m\to 0}\sum_{k\geq 1}\|(\Phi^1 e_k)_m-(\Phi^1 e_k)\|_{L^2}^2<\varepsilon.
$$

for some arbitrary $\varepsilon > 0$. Since Φ^1 is a Hilbert-Schmidt operator, it follows that

$$
\limsup_{m\to 0}\sum_{k\geq 1}\|\Phi^1_m\|_{L^2}^2\leq\sum_{k\geq 1}\|\Phi^1 e_k\|_{L^2}^2<\infty.
$$

Therefore, there exists an $N = N_\varepsilon \in \mathbb{N}$ such that

$$
\sum_{k\geq N_\varepsilon}\|\Phi^1 e_k\|_{L^2}^2<\varepsilon/2.
$$

Thus, for a fixed ε,

$$
\begin{aligned}
\limsup_{m\to 0}\sum_{k\geq 1}\|(\Phi^1 e_k)_m-(\Phi^1 e_k)\|_{L^2}^2 \quad &\leq \quad \underbrace{\limsup_{m\to 0}\sum_{k=1}^{N_\varepsilon-1}\|(\Phi^1 e_k)_m-(\Phi^1 e_k)\|_{L^2}^2}_{0 \text{ as } m\to 0}\\
&+\limsup_{m\to 0}\sum_{k\geq N_\varepsilon}\|(\Phi^1 e_k)_m-(\Phi^1 e_k)\|_{L^2}^2.
\end{aligned}
$$

And now using (3.1.8)-(3.1.9) we deduce that

$$
\limsup_{m\to 0}\sum_{k\geq N_\varepsilon}\|(\Phi^1 e_k)_m-(\Phi^1 e_k)\|_{L^2}^2\leq 2\sum_{k\geq N_\varepsilon}\|\Phi^1 e_k\|_{L^2}^2<\varepsilon.
$$

Since ε was arbitrary chosen, then $IV_m \to 0$ when $\varepsilon \to 0$. Hence convergence in $L^1(\Omega,\cdot)$, therefore, taking a subsequence we obtain a.s convergence in the stochastic integral so that (3.1.14) holds. To show the limit

$$\lim_{m\to 0}\int_0^t\int_{\mathbb{T}^3}\mathbf{h}^2:\mathbf{w}_m^1\,\mathrm{d}x\mathrm{d}s\to\int_0^t\int_{\mathbb{T}^3}\mathbf{h}^2:\mathbf{w}^1\mathrm{d}x\mathrm{d}s. \tag{3.1.15}$$

we observe that

$$\begin{aligned}\left|\int_0^t\int_{\mathbb{T}^3}\mathbf{h}^2:(\mathbf{w}_m^1-\mathbf{w}^1)\,\mathrm{d}x\mathrm{d}s\right| &\leq \int_0^t\int_{\mathbb{T}^3}|\mathbf{h}^2||(\mathbf{w}_m^1-\mathbf{w}^1)|\,\mathrm{d}x\mathrm{d}s\\ &\leq \int_0^t\|\mathbf{h}^2\|_2\|\mathbf{w}_m^1-\mathbf{w}^1\|_2\,\mathrm{d}s.\end{aligned}$$

In view of assumptions on $\mathbf{w}^1$, we use dominated convergence to pass to the limit $m\to 0$, hence (3.1.15) holds. The limit

$$\lim_{m\to 0}\int_{\mathbb{T}^3}\int_0^t\mathbf{w}^1\cdot\Phi_m^2\,\mathrm{d}W\,\mathrm{d}x\to\int_{\mathbb{T}^3}\int_0^t\mathbf{w}^1\cdot\Phi^2\,\mathrm{d}W\,\mathrm{d}x, \tag{3.1.16}$$

follows from using Burgholder-Davis-Gundy inequality to obtain

$$\begin{aligned}&\mathbb{E}\left(\left|\int_0^t\int_{\mathbb{T}^3}\Phi^2\cdot(\mathbf{w}_m^1-\mathbf{w}^1)\,\mathrm{d}x\mathrm{d}W\right|\right)\\ &\leq c\,\mathbb{E}\left(\sup_t\left|\int_0^t\int_{\mathbb{T}^3}\Phi^2\cdot(\mathbf{w}_m^1-\mathbf{w}^1)\,\mathrm{d}x\mathrm{d}W\right|\right)\\ &\leq c\,\mathbb{E}\left(\left\langle\left\langle\int_0^\cdot\int_{\mathbb{T}^3}\Phi^2\cdot(\mathbf{w}_m^1-\mathbf{w}^1)\,\mathrm{d}x\mathrm{d}W\right\rangle\right\rangle\right)^{1/2}\\ &= c\,\mathbb{E}\left(\sum_{k\geq 1}\left\langle\left\langle\int_0^\cdot\int_{\mathbb{T}^3}\Phi^2 e_k\cdot(\mathbf{w}_m^1-\mathbf{w}^1)\,\mathrm{d}x\mathrm{d}W_k(t)\right\rangle\right\rangle\right)^{1/2}\\ &= c\,\mathbb{E}\left(\sum_{k\geq 1}\int_0^T\left[\int_{\mathbb{T}^3}\Phi^2 e_k\cdot(\mathbf{w}_m^1-\mathbf{w}^1)\right]^2\mathrm{d}t\right)^{1/2}\\ &\leq c\,\mathbb{E}\left(\sum_{k\geq 1}\int_0^T\|\Phi^2 e_k\|_2^2\underbrace{\|\mathbf{w}_m^1-\mathbf{w}^1\|_2^2}_{V_m}\mathrm{d}t\right)^{1/2}\quad\text{by Hölder inequality}\end{aligned}$$

since $\mathbf{w}^1\in L^2_w(\Omega,L^\infty(0,T;L^2(\mathbb{T}^3)))$ by assumption, then $V_m\leq 2\|\mathbf{w}^1\|$. Therefore using dominated convergence we deduce that

$$\mathbb{E}\left[\int_0^T\|\mathbf{w}_m^1-\mathbf{w}^1\|_{L^2}\,\mathrm{d}t\right]\to 0,\quad \text{as}\, m\to 0$$

such that (3.1.16) follows. Finally, to show that

$$\lim_{m\to 0}\sum_{k\geq 1}\int_0^t\int_{\mathbb{T}^3}\Phi_m^1 e_k\Phi^2 e_k\,\mathrm{d}x\mathrm{d}t \to \sum_{k\geq 1}\int_0^t\int_{\mathbb{T}^3}\Phi^1 e_k\Phi^2 e_k\,\mathrm{d}x\mathrm{d}t \tag{3.1.17}$$

we consider the following,

$$\left|\sum_{k\geq 1}\int_0^t\int_{\mathbb{T}^3}\Phi^2 e_k\cdot(\Phi_m^1 e_k-\Phi^1 e_k)\,\mathrm{d}x\mathrm{d}t\right| \leq \sup_{0\leq t\leq T}\sum_{k\geq 1}\|\Phi^2 e_k\|_2\underbrace{\|(\Phi_m^1 e_k-\Phi^1 e_k)\|_2}_{VI_m}$$

now using the arguments in (3.1.14) for Φ^1, we deduce that (3.1.17) holds. Hence combining (3.1.11)-(3.1.17), and using (3.1.9) together with the assumptions on $\mathbf{w}^1$ and $\mathbf{w}^2$ we see that all terms converge to their corresponding counterparts and (3.1.7) follows. □

In the following we present a finite dimensional version of [15, Chapter 2; Theorem 2.9.1]

Proposition 3.1.2. *Let U be a random distribution such that $U\in L^1_{loc}([0,\infty))$ $\mathbb{P}$-a.s. Suppose that there is a bounded continuous function b and a collection of random distributions $\mathbb{G}=(G_k)_{k\in\mathbb{N}}$ such that $\mathbb{P}$-a.s.*

$$\sum_{k=1}^{\infty}|G_k|^2\in L^1_{loc}([0,\infty)).$$

Let U_0 be an $\mathfrak{F}_0$-measurable random variable, and $W=(W_k)_{k=1}^{\infty}$ be a collection of real-valued independent Brownian motions. Suppose that the filtration

$$\mathfrak{F}_t=\sigma\Big(\sigma(U_0,\mathbf{r}_tU,\mathbf{r}_tW,\mathbf{r}_t\mathbb{G})\Big),\ t\geq 0,$$

is non-anticipative with respect to W. Let $\tilde{U}_0$ be another random distribution and $\tilde{W}=(\tilde{W}_k)_{k=1}^{\infty}$ another stochastic process and random distributions $\tilde{\mathbb{G}}=(\tilde{G}_k)_{k\in\mathbb{N}}$, such their joint laws coincide, namely,

$$\mathscr{L}[U_0,U,W,\mathbb{G}]=\mathscr{L}[\tilde{U}_0,\tilde{U},\tilde{W},\tilde{\mathbb{G}}]\ \textit{or}\ [U_0,U,W,\mathbb{G}]\overset{d}{\sim}[\tilde{U}_0,\tilde{U},\tilde{W},\tilde{\mathbb{G}}].$$

Then $\tilde{W}$ is a collection of real-valued independent Wiener processes, the filtration

$$\tilde{\mathfrak{F}}_t=\sigma\Big(\sigma(\tilde{U}_0,\mathbf{r}_t\tilde{U},\mathbf{r}_t\tilde{W},\mathbf{r}_t\tilde{\mathbb{G}})\Big),\ t\geq 0,$$

is non-anticipative with respect to $\tilde{W}$, $\tilde{U}_0$ *is* $\tilde{\mathfrak{F}}_0$*-measurable, and*

$$\begin{aligned} &\mathscr{L}\left[\int_0^\infty [\partial_t \psi U + b(U)\psi]\, \mathrm{d}t + \int_0^\infty \sum_{k=1}^\infty \psi G_k \mathrm{d}W_k + \psi(0)U_0\right] \\ &= \mathscr{L}\left[\int_0^\infty \left[\partial_t \psi \tilde{U} + b(\tilde{U})\psi\right]\, \mathrm{d}t + \int_0^\infty \sum_{k=1}^\infty \psi \tilde{G}_k \mathrm{d}\tilde{W}_k + \psi(0)\tilde{U}_0\right] \end{aligned} \tag{3.1.18}$$

for any deterministic $\psi \in C_c^\infty([0,\infty))$.

Proof. We first regularise the problem, and then take limits in different stages to obtain the desired form. Fix the test function ψ in (3.1.18), and consider the convolution kernels

$$[\mathbf{U}]_{t,m} = \theta_m^t(\cdot - m) * \mathbf{U}, \qquad [\tilde{\mathbf{U}}]_{t,m} = \theta_m^t(\cdot - m) * \tilde{\mathbf{U}} \quad m > 0.$$

In addition, we define $\mathbf{U}$ as $\mathbf{U}_0$ for $t < 0$ such that

$$[\mathbf{U}]_{t,m} = [\mathbf{U}_0]_m, \qquad [\tilde{\mathbf{U}}]_{t,m} = [\tilde{\mathbf{U}}_0]_m \qquad \text{for } t \leq 0.$$

Furthermore, we note that $[\mathbf{U}]_{t,m}$ is adapted to

$$\mathfrak{F}_t = \sigma\Big(\sigma(\mathbf{r}_t U, \mathbf{r}_t W, \mathbf{r}_t G_k)\Big),\ t \geq 0,$$

similarly $[\tilde{\mathbf{U}}]_{t,m}$ is adapted to

$$\tilde{\mathfrak{F}}_t = \sigma\Big(\sigma(\mathbf{r}_t \tilde{U}, \mathbf{r}_t \tilde{W}, \mathbf{r}_t \tilde{G}_k)\Big),\ t \geq 0.$$

Finally, we replace G_k by a convolution kernel $G_{k,m}$ such that

$$G_{k,\rho} = \theta_m^t(\cdot - m) * G_k$$

that is, a regularisation in time.

Let $\Delta_J = \frac{T}{J}$ and set

$$t_0 = 0, \qquad t_{j+1} = t_j + \Delta_j, \quad j = 0, \ldots, J-1$$

Step 1:

We begin by re-writing the stochastic term of (3.1.18) using the Reimann-Stieltjes integration form

$$\begin{aligned}
&\mathscr{L}\Bigg[\int_0^\infty [\partial_t \psi [U]_{t,m} + b([U]_{t,m})\psi]\,\mathrm{d}t \\
&+\sum_{k=1}^{K}\left(\sum_{j=1}^{J-1}\psi(t_j)G_{k,m}(t_j)(W_k(t_{j+1})-W_k(t_j))\right)+\psi(0)U_0\Bigg] \\
&=\mathscr{L}\Bigg[\int_0^\infty \left[\partial_t \psi [\tilde{U}]_{t,m} + b([\tilde{U}]_{t,m})\psi\right]\mathrm{d}t \\
&+\sum_{k=1}^{K}\left(\sum_{j=1}^{J-1}\psi(t_j)\tilde{G}_{k,m}(t_j)(\tilde{W}_k(t_{j+1})-\tilde{W}_k(t_j))\right)+\psi(0)U_0\Bigg]
\end{aligned} \tag{3.1.19}$$

Here, we observe that the terms in (3.1.19) are continuous on

$$C_{\text{loc}}([0,\infty)\times C_{\text{loc}}([0,\infty),\mathbb{R}^k)\times C_{\text{loc}}([0,\infty),\mathbb{R}^k).$$

Since the Riemann sums in the stochastic integral converge in probability to their limits, we let $J\to\infty$ so that

$$\begin{aligned}
&\mathscr{L}\left[\int_0^\infty [\partial_t \psi [U]_{t,m} + b([U]_{t,m})\psi]\,\mathrm{d}t + \sum_{k=1}^{K}\int_0^\infty \psi G_{k,m}\mathrm{d}W_k + \psi(0)U_0\right] \\
&=\mathscr{L}\left[\int_0^\infty \left[\partial_t \psi [\tilde{U}]_{t,m} + b([\tilde{U}]_{t,m})\psi\right]\mathrm{d}t + \sum_{k=1}^{K}\int_0^\infty \psi \tilde{G}_{k,m}\mathrm{d}W_k + \psi(0)U_0\right].
\end{aligned} \tag{3.1.20}$$

Step 2:

At this stage we want $m\to 0$ in (3.1.20). Since $\mathbf{U}\in L^1([0,\infty))$ $\mathbb{P}$-a.s we have

$$[\mathbf{U}]_{t,m}\to\mathbf{U}\quad\text{in } L^1_{\text{loc}}([0,\infty))\,\text{as}\, m\to 0\,\mathbb{P}\text{-a.s.}$$

consequently, the following holds

$$[\tilde{\mathbf{U}}]_{t,m}\to\tilde{\mathbf{U}}\quad\text{in } L^1_{\text{loc}}([0,\infty))\,\text{as}\, m\to 0\,\mathbb{P}\text{-a.s.},$$

Finally, we have

$$G_{k,m}\to G_k\quad\text{in } L^1_{\text{loc}}([0,\infty))$$

by definition of convolution kernels. Since $b(\cdot)$ is a bounded and continuous function, then in view of lemma 2.1.14 we take the limit $m \to 0$ in (3.1.20) to obtain

$$\begin{aligned} &\mathscr{L}\left[\int_0^\infty [\partial_t \psi U + b(U)\psi]\,\mathrm{d}t + \sum_{k=1}^K \int_0^\infty \psi G_k \mathrm{d}W_k + \psi(0)U_0\right] \\ &= \mathscr{L}\left[\int_0^\infty [\partial_t \psi \tilde{U} + b(\tilde{U})\psi]\,\mathrm{d}t + \sum_{k=1}^K \int_0^\infty \psi \tilde{G}_k \mathrm{d}W_k + \psi(0)U_0\right]. \end{aligned} \tag{3.1.21}$$

Step 3:

Finally, in view of lemma 2.1.14 we take the limit $k \to \infty$ in (3.1.21) to obtain (3.1.18) as required. □

3.1.2 Stochastic Navier-Stokes equations

In this section we prove the existence of martingale solutions to the system of incompressible Navier-Stokes. Let $\mathbb{T}^3$ be a three-dimensional torus, let $T > 0$, and set $Q = (0,T) \times \mathbb{T}^3$. The incompressible stochastic Navier-Stokes system with viscosity $\varepsilon > 0$ governing the time evolution of velocity field $\mathbf{u}$ and pressure p of fluids reads

$$\begin{cases} \mathrm{d}\mathbf{u} = \varepsilon \Delta \mathbf{u}\,\mathrm{d}t - \mathrm{div}(\mathbf{u} \otimes \mathbf{u})\,\mathrm{d}t - \nabla p\,\mathrm{d}t + \phi\,\mathrm{d}W & \text{in } Q, \\ \mathrm{div}\,\mathbf{u} = 0 & \text{in } Q. \end{cases} \tag{3.1.22}$$

Here the system models the conservation of momentum and and balance of mass for fluid flows with high Reynolds number $(1/\varepsilon)$. In (3.1.22), we assume that the density is constant and set it to $\rho = 1$. Finally, the driving stochastic force W is the cylindrical Wiener process defined on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and the noise coefficient term ϕ is a Hilbert-Schmidt operator, see Section 3.1.1.

Our solution will be weak in the analytical sense (derivatives are understood as distributions) and stochastic sense (underlying probability space is not a priori given but becomes an integral part of the solution). We consider martingale solutions beginning with an initial law defined on

$$L^2_{\mathrm{div}}(\mathbb{T}^3) := \overline{C^\infty_{\mathrm{div}}(\mathbb{T}^3)}^{L^2(\mathbb{T}^3)}.$$

Accordingly, the weak formulation of (3.1.22) yields the desired format that is necessary for weak solutions in the analytical sense. Testing (3.1.22) by $\varphi \in \mathscr{C}^\infty_{\mathrm{div}}(\mathbb{T}^3, \mathbb{R}^d)$ we obtain

$$\begin{aligned}\left\langle \int_0^t \mathrm{d}\mathbf{u}, \varphi \right\rangle_{\mathbb{T}^3} &= \left\langle \int_0^t \varepsilon \Delta \mathbf{u}\, \mathrm{d}s, \varphi \right\rangle_{\mathbb{T}^3} + \left\langle \int_0^t -\mathrm{div}(\mathbf{u} \otimes \mathbf{u})\, \mathrm{d}s, \varphi \right\rangle_{\mathbb{T}^3} \\ &+ \left\langle \int_0^t -\nabla p\, \mathrm{d}s, \varphi \right\rangle_{\mathbb{T}^3} + \left\langle \int_0^t \phi\, \mathrm{d}W_s, \varphi \right\rangle_{\mathbb{T}^3}, \end{aligned} \tag{3.1.23}$$

where $\langle .,. \rangle_{\mathbb{T}^3}$ denotes the $L^2(\mathbb{T}^3)$ inner product. Using Green's identity we obtain

$$\begin{aligned}\left\langle \int_0^t \varepsilon \Delta \mathbf{u} \mathrm{d}s, \varphi \right\rangle_{\mathbb{T}^3} &= \varepsilon \int_0^t \int_{\mathbb{T}^3} \Delta \mathbf{u} \cdot \varphi\, \mathrm{d}x \mathrm{d}s \\ &= -\varepsilon \int_0^t \int_{\mathbb{T}^3} \nabla \mathbf{u} : \nabla \varphi\, \mathrm{d}x \mathrm{d}s. \end{aligned}$$

Now on the account of $\mathrm{div}(p\varphi) = p\,\mathrm{div}\varphi + \nabla p \cdot \varphi = \nabla p \cdot \varphi$ and Divergence Theorem, one obtains

$$\left\langle \int_0^t -\nabla p\, \mathrm{d}s, \varphi \right\rangle_{\mathbb{T}^3} = -\int_{\mathbb{T}^3} \int_0^t \mathrm{div}(p\varphi)\, \mathrm{d}x \mathrm{d}s = 0,$$

therefore, the pressure term vanishes. Using similar arguments we simplify the convective term to

$$\left\langle \int_0^t -\mathrm{div}(\mathbf{u} \otimes \mathbf{u}) \mathrm{d}s, \varphi \right\rangle_{\mathbb{T}^3} = \int_0^t \int_{\mathbb{T}^3} \mathbf{u} \otimes \mathbf{u} : \nabla \varphi\, \mathrm{d}x \mathrm{d}s.$$

Finally, combining these terms yields the weak formulation of (3.1.22) and it reads

$$\begin{aligned}\int_{\mathbb{T}^3} \mathbf{u} \cdot \varphi\, \mathrm{d}x &= \int_{\mathbb{T}^3} \mathbf{u}_0 \cdot \varphi\, \mathrm{d}x + \int_0^t \int_{\mathbb{T}^3} \mathbf{u} \otimes \mathbf{u} : \nabla \varphi\, \mathrm{d}x \mathrm{d}s \\ &+ \varepsilon \int_0^t \int_{\mathbb{T}^3} \nabla \mathbf{u} : \nabla \varphi\, \mathrm{d}x \mathrm{d}s + \int_{\mathbb{T}^3} \int_0^t \phi\, \mathrm{d}W_s\, \mathrm{d}x \cdot \varphi, \end{aligned} \tag{3.1.24}$$

for all $\varphi \in \mathscr{C}^\infty_{\mathrm{div}}(\mathbb{T}^3)$. In the following we give a rigorous definition of a solution to (3.1.22).

Definition 3.1.1 (Solution)**.** Let Λ_0 be a Borel probability measure on $L^2_{\mathrm{div}}(\mathbb{T}^3)$. Then

$$((\Omega, \mathscr{F}, (\mathscr{F}_t)_{t \geq 0}, \mathbb{P}), \mathbf{u}, W)$$

is a weak martingale solution to (3.1.22) with initial datum Λ_0 provided that:

(a) $((\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$ is a stochastic basis with a complete right-continuous filtration,

(b) W is a $(\mathscr{F}_t)_{t\geq 0}$-cylindrical Wiener process,

(c) The velocity field $\mathbf{u}$ is $\mathscr{F}_t$-adapted and satisfies $\mathbb{P}$-a.s

$$\mathbf{u} \in C_{\mathrm{loc}}([0,\infty), W_{div}^{-2,2}(\mathbb{T}^3)) \cap C_{w,\mathrm{loc}}([0,\infty); L^2_{\mathrm{div}}(\mathbb{T}^3)) \cap L^2_{\mathrm{loc}}(0,\infty; W^{1,2}_{\mathrm{div}}(\mathbb{T}^3))$$

(d) $\Lambda_0 = \mathbb{P} \circ \mathbf{u}(0)^{-1}$ (that is $\mathbb{P}(\mathbf{u} \in B) = \Lambda_0(B)$ for all $B \in \mathscr{B}(L^2_{\mathrm{div}}(\mathbb{T}^3))$),

(e) For all $\varphi \in C^\infty_{\mathrm{div}}(\mathbb{T}^3)$ and all $t \in [0,T]$ we have

$$\begin{aligned}\int_{\mathbb{T}^3} \mathbf{u}\cdot\varphi \,\mathrm{d}x &= \int_{\mathbb{T}^3} \mathbf{u}_0\cdot\varphi \,\mathrm{d}x + \int_0^t\int_{\mathbb{T}^3} \mathbf{u}\otimes\mathbf{u} : \nabla\varphi \,\mathrm{d}x\mathrm{d}s \\ &\quad -\varepsilon\int_0^t\int_{\mathbb{T}^3} \nabla\mathbf{u} : \nabla\varphi \,\mathrm{d}x\mathrm{d}s + \int_{\mathbb{T}^3}\int_0^t \phi \,\mathrm{dW}_s\cdot\varphi \,\mathrm{d}x,\end{aligned}$$

$\mathbb{P}$-a.s.

(f) The following energy inequality holds true:

$$E_t + \varepsilon\int_s^t\int_{\mathbb{T}^3} |\nabla\mathbf{u}|^2 \,\mathrm{d}x\mathrm{d}t \leq E_s + \frac{1}{2}\int_s^t \|\phi\|^2_{L_2(\mathscr{U}, L^2(\mathbb{T}^3))}\mathrm{d}t + \int_s^t\int_{\mathbb{T}^3} \mathbf{u}\cdot\phi \,\mathrm{d}x\,\mathrm{d}W, \tag{3.1.25}$$

$\mathbb{P}$-a.s for a.a $s \geq 0$ (including $s = 0$) and all $t \geq s$, where $E_t = \frac{1}{2}\int_{\mathbb{T}^3} |\mathbf{u}|^2 \,\mathrm{d}x$.

Definition 3.1.1 is standard in the theory of stochastic Navier-Stokes equations and can be found in a similar form, for instance in [56, 54]. The energy inequality in (f) is reminiscent of the recent results for compressible fluids in [15]. We note, the energy inequality (f) must be included in the definition as it is an open problem if weak solutions satisfy the energy inequality. The formal computation of the energy inequality is a simple application of Itô formula to the functional $t \mapsto \frac{1}{2}\int_{\mathbb{T}^3} |\mathbf{u}|^2 \,\mathrm{d}x$ as follows.

We apply Itô's formula to the functional

$$f(\mathbf{u}) = \frac{1}{2}\int_{\mathbb{T}^3} |\mathbf{u}|^2 \,\mathrm{d}x,$$

with the following observations

$$\partial_{\mathbf{u}} f(\mathbf{u}) = \langle \mathbf{u}, \cdot \rangle_{\mathbb{T}^3}, \quad \partial^2_{\mathbf{u}} f(\mathbf{u}) = \langle \cdot, \cdot \rangle_{\mathbb{T}^3},$$

$$\mathrm{d} f(\mathbf{u}) = \partial_{\mathbf{u}} f(\mathbf{u}) \cdot \mathrm{d}\mathbf{u} + \frac{1}{2}\partial^2_{\mathbf{u}} \cdot \mathrm{d}\langle\langle \mathbf{u} \rangle\rangle_t.$$

On the account of the balance of momentum equation in (3.1.22) we deduce

$$\mathrm{d}\int_{\mathbb{T}^3} \frac{1}{2}|\mathbf{u}|^2 \,\mathrm{d}x = \varepsilon \int_{\mathbb{T}^3} \mathbf{u} \cdot \Delta\mathbf{u} \,\mathrm{d}x\mathrm{d}t - \int_{\mathbb{T}^3} \mathbf{u} \cdot \operatorname{div}(\mathbf{u} \otimes \mathbf{u}) \,\mathrm{d}x\mathrm{d}t - \int_{\mathbb{T}^3} \nabla p \cdot \mathbf{u} \,\mathrm{d}x\mathrm{d}t$$
$$\frac{1}{2}\sum_{k \geq 1} \int_{\mathbb{T}^3} |\phi|^2 \,\mathrm{d}x\mathrm{d}t + \int_{\mathbb{T}^3} \mathbf{u} \cdot \phi \,\,\mathrm{d}x\mathrm{d}W_s.$$

Consequently, using Green's identity and the divergence free property we observe that the convective and pressure terms vanish yielding

$$\mathrm{d}\int_{\mathbb{T}^3} \frac{1}{2}|\mathbf{u}|^2 \,\mathrm{d}x = \varepsilon \int_{\mathbb{T}^3} |\nabla\mathbf{u}|^2 \,\mathrm{d}x\mathrm{d}t + \frac{1}{2}\sum_{k \geq 1} \int_{\mathbb{T}^3} |\phi|^2 \,\mathrm{d}x\mathrm{d}t + \int_{\mathbb{T}^3} \mathbf{u} \cdot \phi \,\,\mathrm{d}x\mathrm{d}W_s.$$

Finally, integrating in time we deduce the energy inequality

$$\int_{\mathbb{T}^3} \frac{1}{2}|\mathbf{u}|^2 \,\mathrm{d}x + \varepsilon \int_0^t \int_{\mathbb{T}^3} |\nabla\mathbf{u}|^2 \,\mathrm{d}x\mathrm{d}t \leq \int_{\mathbb{T}^3} \frac{1}{2}|\mathbf{u}_0|^2 \,\mathrm{d}x + \frac{1}{2}\sum_{k \geq 1} \int_0^t \int_{\mathbb{T}^3} |\phi_k|^2 \,\mathrm{d}x\mathrm{d}t$$
$$+ \int_{\mathbb{T}^3} \int_0^t \mathbf{u} \cdot \phi \,\,\mathrm{d}x\mathrm{d}W_s.$$

Accordingly, we proceed to state the existence of martingale solutions to the system (3.1.22).

Theorem 3.1.3 (Existence)**.** *Assume that (3.1.2) holds and we have*

$$\int_{L^2_{\mathrm{div}}(\mathbb{T}^3)} \|\mathbf{v}\|^{\beta}_{L^2(\mathbb{T}^3)} d\Lambda_0(\mathbf{v}) < \infty, \tag{3.1.26}$$

for some $\beta > 2$. Then there is a weak martingale solution to (3.1.22) in the sense of Definition 3.1.1.

In order to prove Theorem 3.1.3, we adopt key ideas outlined by Flandoli and Gatarek [56]. However, in contrast to results of [56] we use an elementary approach from [27] to identify the stochastic integral after the limit procedure. The proof follows a presentation by [17].

Approximated system

The main idea is comprised of seeking a solution by separating space and time via Galerkin approximation. We approximate the space $W^{l,2}_{\text{div}}(\mathbb{T}^3)$ by a finite dimensional subspace. Then, the approximated equations are solved by standard arguments from the theory of stochastic differential equations. We obtain a sequence of approximated solutions $\mathbf{u}_N, N \in \mathbb{N}$. Then we show that $\mathbf{u}_N$ converges in some sense to a limit function $\mathbf{u}$, a solution to the weak formulation.

Theorem 3.1.4 (The Approximated System [77])**.** *Let $l \in \mathbb{N}$ and $\mathbb{T}^3$ be a three-dimensional periodic domain. Then, there is a sequence $(\lambda_k) \subset \mathbb{R}$ and a sequence of functions $(\mathbf{w}_k) \subset W^{l,2}_{\text{div}}(\mathbb{T}^3)$ such that*

1. *$\mathbf{w}_k$ is an eigenvector to the eigenvalue λ_k of the Stokes operator in the sense that:*

$$\langle \mathbf{w}_k, \varphi \rangle_{W^{l,2}} = \lambda_k \int_{\mathbb{T}^3} \mathbf{w}_k \varphi \, dx \quad \text{for all} \quad \varphi \in W^{l,2}_{\text{div}}(\mathbb{T}^3),$$

2.
$$\int_{\mathbb{T}^3} \mathbf{w}_k \mathbf{w}_m \, dx = \delta_{km} \text{ for all } k, m \in \mathbb{N},$$

3.
$$1 \leq \lambda_1 \leq \lambda_2 \dots \text{ and } \lambda_k \to \infty,$$

4.
$$\left\langle \frac{\mathbf{w}_k}{\lambda_k}, \frac{\mathbf{w}_m}{\lambda_m} \right\rangle_{W^{l,2}} = \delta_{km} \text{ for all } k, m \in \mathbb{N},$$

5.
$$(\mathbf{w}_k) \text{ is a basis of } W^{l,2}_{\text{div}}(\mathbb{T}^3).$$

We choose $l > 1 + d/2$ such that $W^{l,2}_{\text{div}}(\mathbb{T}^3) \hookrightarrow W^{1,\infty}(\mathbb{T}^3)$ by Sobolev embedding Theorem 2.1.6. Letting $\mathbf{u}_0$ to be an $\mathscr{F}_0$-measurable random variable with values in $L^2_{\text{div}}(\mathbb{T}^3)$ subject to the law Λ_0 (the existence of $\mathbf{u}_0$ follows from Skorokhod theorem 2.1.20), we seek an approximated solution $\mathbf{u}_N$ of the form

$$\mathbf{u}_N = \sum_{k=1}^{N} c_k^N \mathbf{w}_k = \mathbf{C}_N \cdot \omega_N, \quad \omega_N = (\mathbf{w}_1, \dots, \mathbf{w}_k), \tag{3.1.27}$$

where $\mathbf{C}_N = c_k^N : \Omega \times (0,T) \to \mathbb{R}^N$. Thus for $k = 1,\dots,N$ we seek a solution for the system

$$\begin{aligned}
\int_{\mathbb{T}^3} \mathrm{d}\mathbf{u}_N \cdot \mathbf{w}_k \,\mathrm{d}x &= -\varepsilon \int_{\mathbb{T}^3} \nabla \mathbf{u}_N : \nabla \mathbf{w}_k \,\mathrm{d}x\mathrm{d}t + \int_{\mathbb{T}^3} \mathbf{u}_N \otimes \mathbf{u}_N : \nabla \mathbf{w}_k \,\mathrm{d}x\mathrm{d}t \\
&\quad + \int_{\mathbb{T}^3} \phi \,\mathrm{d}W_s^N \cdot \mathbf{w}_k \,\mathrm{d}x, \\
&\quad \mathbf{u}_N(0) = \mathscr{P}_N \mathbf{u}_0,
\end{aligned} \tag{3.1.28}$$

where $\mathscr{P}_N$ is the orthogonal projection $\mathscr{P}_N : L^2_{\mathrm{div}}(\mathbb{T}^3) \to H_N := \mathrm{span}\{\mathbf{w}_1, \dots, \mathbf{w}_k\}$, that is,

$$\mathscr{P}_N \mathbf{v} = \sum_{k=1}^{N} \langle \mathbf{v}, \mathbf{w}_k \rangle_{\mathbb{T}^3} \mathbf{w}_k,$$

and (3.1.28) holds $\mathbb{P}$-a.s for all $t \in [0,T]$. Furthermore, we rewrite the Wiener process as follows

$$W^N(s) = \sum_{k=1}^{N} e_k \beta_k(s) = \mathbf{E}_N \cdot \beta^N(s), \quad \mathbf{E}_N = (e_1, \dots, e_N).$$

Now we re-formulate (3.1.28) using (3.1.27), orthogonality property in Theorem 3.1.4 to deduce

$$\begin{aligned}
\int_{\mathbb{T}^3} \frac{\mathrm{d}\mathbf{u}_N}{\mathrm{d}t} \cdot \mathbf{w}_k \,\mathrm{d}x &= -\varepsilon \int_{\mathbb{T}^3} \nabla \mathbf{u}_N : \nabla \mathbf{w}_k \,\mathrm{d}x + \int_{\mathbb{T}^3} \mathbf{u}_N \otimes \mathbf{u}_N : \nabla \mathbf{w}_k \,\mathrm{d}x \\
&\quad + \int_{\mathbb{T}^3} \phi \,\frac{\mathrm{d}W_s^N}{\mathrm{d}t} \cdot \mathbf{w}_k \,\mathrm{d}x, \\
\sum_{k=1}^{N} \frac{\mathrm{d}c_k^N}{\mathrm{d}t} \int_{\mathbb{T}^3} \mathbf{w}_k \cdot \mathbf{w}_k \,\mathrm{d}x &= -\varepsilon \sum_{k=1}^{N} c_k^N \int_{\mathbb{T}^3} \nabla \mathbf{w}_k : \nabla \mathbf{w}_k \,\mathrm{d}x + \sum_{k,l=l}^{N} c_k^N c_l^N \int_{\mathbb{T}^3} \mathbf{w}_k \otimes \mathbf{w}_l : \nabla \mathbf{w}_k \,\mathrm{d}x \\
&\quad + \int_{\mathbb{T}^3} \phi \,\frac{\mathrm{d}W_s^N}{\mathrm{d}t} \cdot \mathbf{w}_k \,\mathrm{d}x, \\
\sum_{k=1}^{N} \frac{\mathrm{d}c_k^N}{\mathrm{d}t} &= -\varepsilon \sum_{k=1}^{N} c_k^N \int_{\mathbb{T}^3} \nabla \mathbf{w}_k : \nabla \mathbf{w}_k \,\mathrm{d}x + \sum_{k,l=1}^{N} c_k^N c_l^N \int_{\mathbb{T}^3} \mathbf{w}_k \otimes \mathbf{w}_l : \nabla \mathbf{w}_k \,\mathrm{d}x \\
&\quad + \int_{\mathbb{T}^3} \phi \,\frac{\mathrm{d}W_s^N}{\mathrm{d}t} \cdot \mathbf{w}_k \,\mathrm{d}x.
\end{aligned}$$

Solving (3.1.28) is equivalent to solving the system

$$\begin{cases} \mathrm{d}\mathbf{C}_N = [\mu(\mathbf{C}_N)] \,\mathrm{d}t + \Sigma \,\mathrm{d}\beta_t^N, \\ \mathbf{C}_N(0) = \mathbf{C}_0, \end{cases} \tag{3.1.29}$$

where

$$\mu(\mathbf{C}_N) = \left(-\int_{\mathbb{T}^3} \mathbf{C}_N \cdot \varepsilon \nabla \omega_N : \nabla \mathbf{w}_k \, \mathrm{d}x + \int_{\mathbb{T}^3} (\mathbf{C}_N \cdot \omega_N) \otimes (\mathbf{C}_N \cdot \omega_N) : \nabla w_k \, \mathrm{d}x\right)_{k=1}^{N},$$

$$\Sigma = \left(\int_{\mathbb{T}^3} \phi_l \cdot \mathbf{w}_k \mathrm{d}x\right)_{k,l=1}^{N},$$

$$\mathbf{C}_0 = \left(\langle \mathbf{u}_0, \mathbf{w}_k \rangle_{L^2(\mathbb{T}^3)}\right)_{k=1}^{N}.$$

Finally, noting that (3.1.29) is indeed a system of stochastic differential equations (SDEs) we show that it has a *unique* solution by checking that the assumptions of Theorem 2.1.17 are satisfied. We observe

$$\begin{aligned}
&(\mu(\mathbf{C}_N) - \mu(\tilde{\mathbf{C}}_N)) \cdot (\mathbf{C}_N - \tilde{\mathbf{C}}_N) \\
&= -\varepsilon \sum_{k=1}^{N} \int_{\mathbb{T}^3} (c_k^N \nabla \mathbf{w}_k - \tilde{c}_k^N \nabla \mathbf{w}_k) : \nabla \mathbf{w}_k (c_k^N - \tilde{c}_k^N) \, \mathrm{d}x \\
&+ \sum_{k,l=1}^{N} \int_{\mathbb{T}^3} (c_k^N \mathbf{w}_k \otimes c_l^N \mathbf{w}_k - \tilde{c}_k^N \mathbf{w}_k \otimes \tilde{c}_k^N \mathbf{w}_k) : \nabla \mathbf{w}_k (c_k^N - \tilde{c}_k^N) \, \mathrm{d}x.
\end{aligned}$$

By using (3.1.27), the right term of the system simplifies to

$$\begin{aligned}
&-\varepsilon \int_{\mathbb{T}^3} |\nabla \mathbf{u}_N - \nabla \tilde{\mathbf{u}}_N|^2 \, \mathrm{d}x + \int_{\mathbb{T}^3} (\mathbf{u}_N \otimes \mathbf{u}_N - \tilde{\mathbf{u}}_N \otimes \tilde{\mathbf{u}}_N) : (\nabla \mathbf{u}_N - \nabla \tilde{\mathbf{u}}_N) \, \mathrm{d}x \\
&\leq \int_{\mathbb{T}^3} (\mathbf{u}_N \otimes \mathbf{u}_N - \tilde{\mathbf{u}}_N \otimes \tilde{\mathbf{u}}_N) : (\nabla \mathbf{u}_N - \nabla \tilde{\mathbf{u}}_N) \, \mathrm{d}x \\
&\leq C(R,N) |\mathbf{C}_N - \tilde{\mathbf{C}}_N|^2.
\end{aligned}$$

In the last line above, we assume that $|\mathbf{C}_N| \leq R$ and $|\tilde{\mathbf{C}}_N| \leq R$ for some arbitrary R, and use the boundedness of $|\mathbf{w}_k| \leq c$ and $|\nabla \mathbf{w}_k| \leq c$ to deduce the weak monotonicity property

$$(\mu(\mathbf{C}_N) - \mu(\tilde{\mathbf{C}}_N)) \cdot (\mathbf{C}_N - \tilde{\mathbf{C}}_N) \leq C(R,N) |\mathbf{C}_N - \tilde{\mathbf{C}}_N|^2.$$

The Lipschitz continuity property of Σ follows from (3.1.2). In particular, for fixed $N \in \mathbb{N}$ our Σ does not depend on the solution, it is simply a constant matrix. The weak coercivity assumption follows from observing that the term $\int_{\mathbb{T}^3} \mathbf{u}_N \otimes \mathbf{u}_N : \nabla \mathbf{u}_N \, \mathrm{d}x = 0$ so that

$$\mu(t, \mathbf{C}_N) \cdot \mathbf{C}_N = -\varepsilon \int_{\mathbb{T}^3} |\nabla \mathbf{u}_N|^2 \, \mathrm{d}x \leq 0.$$

Thus, the system (3.1.29) satisfies the properties of Theorem 2.1.17. Therefore, (3.1.29) can be solved using standard SDEs theory. Applying the theory of SDEs to (3.1.29) we obtain a unique strong solution $\mathbf{C}_N$ with $\mathbb{P}$-a.s. continuous trajectories. To pass to the limit $N \to \infty$ we derive the following estimates.

Theorem 3.1.5 (A Priori Estimate)**.** *Assume (3.1.2) holds and*

$$\int_{L^2_{div}(\mathbb{T}^3)} \|\mathbf{v}\|^2_{L^2(\mathbb{T}^3)} d\Lambda_0(\mathbf{v}) < \infty.$$

Then the following holds uniformly in N,

$$\mathbb{E}\left[\sup_{t\in(0,T)} \int_{\mathbb{T}^3} |\mathbf{u}_N|^2 dx + \varepsilon \int_Q |\nabla \mathbf{u}_N|^2 dxdt\right] \leq c\left(1 + \int_{L^2_{div}(\mathbb{T}^3)} \|\mathbf{v}\|^2_{L^2(\mathbb{T}^3)} d\Lambda_0(\mathbf{v})\right). \quad (3.1.30)$$

Proof. Let $\mathbf{C}$ be the Itô process given by

$$\mathrm{d}\mathbf{C} = \mu(\mathbf{C})\mathrm{d}t + \Sigma \mathrm{d}W_t,$$

with the abbreviations

$$\mu(\mathbf{C}) = -\varepsilon \int_{\mathbb{T}^3} \mathbf{C}\cdot\nabla\omega_N : \nabla \mathbf{w}_k \,\mathrm{d}x + \int_{\mathbb{T}^3} \mathbf{u}_N \otimes \mathbf{u}_N : \nabla \mathbf{w}_k \,\mathrm{d}x,$$

$$\Sigma = \sum_{k=1}^{N} \int_{\mathbb{T}^3} \phi_k \cdot \mathbf{w}_k \,\mathrm{d}x.$$

Define a function $g(\mathbf{C}) \in C^2([0,\infty)\times\mathbb{R})$ such that $g(\mathbf{C})$ is still an Itô process. We set the function $g(\mathbf{C}) := \frac{1}{2}|\mathbf{C}|^2$ so that $\mathrm{d}g(\mathbf{C}) = \mathrm{d}\frac{1}{2}|\mathbf{C}|^2$. Then applying Ito's formula, one obtains

$$\begin{aligned} \mathrm{d}\frac{1}{2}|\mathbf{C}|^2 &= \mathbf{C}^T \mathrm{d}\mathbf{C} + \frac{1}{2} D^2 g(\mathbf{C})\mathrm{d}\langle\langle \mathbf{C}\rangle\rangle_t \\ &= \mathbf{C}^T \mu(\mathbf{C})\mathrm{d}t + \mathbf{C}^T \Sigma \mathrm{d}W_t + \frac{1}{2} D^2 g(\mathbf{C})\mathrm{d}\langle\langle \mathbf{C}\rangle\rangle_t. \end{aligned} \quad (3.1.31)$$

Integrating (3.1.31) with respect to time

$$\frac{1}{2}|\mathbf{C}(t)|^2 = \frac{1}{2}|\mathbf{C}(0)|^2 + \int_0^t \mathbf{C}^T \mu(\mathbf{C})\mathrm{d}t + \int_0^t \mathbf{C}^T \Sigma \mathrm{d}W_t + \frac{1}{2}\int_0^t D^2 g(\mathbf{C})\mathrm{d}\langle\langle \mathbf{C}\rangle\rangle_t. \quad (3.1.32)$$

Using the approximation

$$\mathbf{u}_N = \sum_{k=1}^{N} c_N^k(t)\mathbf{w}_k,$$

and property (2) from Theorem 3.1.4 we observe that

$$
\begin{aligned}
|\mathbf{C}_N(t)|^2 &= \sum_{k=1}^{N} |c_k^N(t)|^2 \\
&= \sum_{k,l=1}^{N} c_k^N(t) c_l^N(t) \langle \mathbf{w}_k, \mathbf{w}_l \rangle_{\mathbb{T}^3} \\
&= \left\langle \sum_{k=1}^{N} c_k^N(t)\mathbf{w}_k, \sum_{l=1}^{N} c_l^N(t)\mathbf{w}_k \right\rangle_{\mathbb{T}^3} \\
&= \int_{\mathbb{T}^3} \left| \sum_{k=1}^{N} c_k^N(t)\mathbf{w}_k \right|^2 \mathrm{d}x \\
&= \|\mathbf{u}_N(t)\|^2_{L^2(\mathbb{T}^3)}.
\end{aligned}
$$

Finally, we note the following

$$
\mathrm{d}\mathbf{u}_N = \sum_{k=1}^{N} \mathrm{d}c_N^k(t)\mathbf{w}_k,
$$

and for $D^2 g(\mathbf{C}) = I \in \mathbb{R}^{N\times N}$

$$
\int_0^t D^2 g(\mathbf{C}) \mathrm{d}\langle\langle \mathbf{C} \rangle\rangle_t = \int_0^t \sum_{k=1}^{N} \left(\int_{\mathbb{T}^3} \phi_k, \mathbf{w}_k \right)^2 \mathrm{d}s,
$$

Since $\int_{\mathbb{T}^3} \mathbf{u}_N \otimes \mathbf{u}_N : \nabla \mathbf{u}_N \mathrm{d}x = 0$ we deduce

$$
\begin{aligned}
\frac{1}{2}\|\mathbf{u}_N(t)\|^2_{L^2(\mathbb{T}^3)} &= \frac{1}{2}\|\mathscr{P}_N \mathbf{u}_0\|^2_{L^2(\mathbb{T}^3)} - \varepsilon \int_{\mathbb{T}^3}\int_0^t |\nabla \mathbf{u}_N|^2 \,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{T}^3}\int_0^t \mathbf{u}_N \cdot \phi \,\mathrm{d}W_s^N \,\mathrm{d}x \\
&\quad + \frac{1}{2}\int_0^t \sum_{k=1}^{N} \left(\int_{\mathbb{T}^3} \phi_k \mathbf{w}_k \,\mathrm{d}x \right)^2 \mathrm{d}s. \qquad (3.1.33)
\end{aligned}
$$

The estimate follows from taking the supremum in time and building expectations i.e.,

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{t\in(0,T)} \int_{\mathbb{T}^3} |\mathbf{u}_N|^2 \mathrm{d}x + \varepsilon \int_Q |\nabla \mathbf{u}_N|^2 \mathrm{d}x\mathrm{d}t \right] \\
&\leq c \left(\mathbb{E}\left[\|\mathbf{u}_0\|^2_{L^2(0)} \right] + \mathbb{E}\left[\sup_{t\in(0,T)} Y_1(t) \right] + \mathbb{E}\left[Y_2(T) \right] \right),
\end{aligned}
$$

where

$$
Y_1(t) := \int_{\mathbb{T}^3}\int_0^t \mathbf{u}_N \cdot \phi \,\mathrm{d}W_s^N \,\mathrm{d}x,
$$

$$Y_2(T) : \quad = \quad \int_0^t \sum_{l=1}^N \|\phi_l\|^2_{L^2(\mathbb{T}^3)} \, \mathrm{d}s.$$

In view of (3.1.2), we see that

$$\begin{aligned}
\mathbb{E}[Y_2(t)] &= \mathbb{E}\left[\sum_{l=1}^N \int_0^t \int_{\mathbb{T}^3} |\phi_l|^2 \, \mathrm{d}x\mathrm{d}s\right] \\
&\leq \mathbb{E}\left[\sum_{l=1}^\infty \int_0^t \int_{\mathbb{T}^3} |\phi_l|^2 \, \mathrm{d}x\mathrm{d}s\right] \text{ by truncation} \\
&= \mathbb{E}\left[\int_0^t \|\phi\|^2_{L_2(\mathscr{U}, L^2(\mathbb{T}^3))} \, \mathrm{d}s\right] < \infty.
\end{aligned}$$

Furthermore, as consequence of Burkholder-Davis-Gundy inequality Lemma 2.1.13 and (3.1.2) we deduce the estimate

$$\begin{aligned}
\mathbb{E}\left[\sup_{t\in(0,T)} |Y_1(t)|\right] &= \mathbb{E}\left[\sup_{t\in(0,T)} \left|\int_0^t \int_{\mathbb{T}^3} \mathbf{u}_N \cdot \phi \, \mathrm{d}x \, \mathrm{d}W_s^N\right|\right] \\
&= \mathbb{E}\left[\sup_{t\in(0,T)} \left|\sum_{k=1}^N \int_0^t \int_{\mathbb{T}^3} \mathbf{u}_N \cdot \phi_k \, \mathrm{d}x \, \mathrm{d}\beta_k(s)\right|\right] \\
&\leq c\, \mathbb{E}\left[\int_0^T \sum_{k=1}^N \left(\int_{\mathbb{T}^3} \mathbf{u}_N \cdot \phi_k \, \mathrm{d}x\right)^2 \mathrm{d}t\right]^{1/2} \\
&\leq c\, \mathbb{E}\left[\int_0^T \left(\sum_{k=1}^N \int_{\mathbb{T}^3} |\mathbf{u}_N|^2 \, \mathrm{d}x \int_{\mathbb{T}^3} |\phi_k|^2 \, \mathrm{d}x\right) \mathrm{d}t\right]^{1/2} \\
&\leq c\mathbb{E}\left[\int_0^T \int_{\mathbb{T}^3} |\mathbf{u}_N|^2 \mathrm{d}x \, \mathrm{d}t\right]^{1/2} \\
&\leq \delta\mathbb{E}\left[\sup_{t\in(0,T)} \int_{\mathbb{T}^3} |\mathbf{u}_N|^2 \, \mathrm{d}x\right] + c(\delta, T).
\end{aligned}$$

Here, we used (3.1.2), Hölder's inequality, and Young's inequality. Now combining the bounds above we infer

$$\mathbb{E}\left[\sup_{t\in(0,T)} \int_{\mathbb{T}^3} |\mathbf{u}_N|^2 \mathrm{d}x + \varepsilon \int_Q |\nabla \mathbf{u}_N|^2 \mathrm{d}x\mathrm{d}t\right] \leq c\, \mathbb{E}\left[\int_{\mathbb{T}^3} |\mathbf{u}_0|^2 \, \mathrm{d}x + \int_0^t \|\phi\|^2_{L_2(\mathscr{U}, L^2(\mathbb{T}^3))} \, \mathrm{d}s\right]$$

$$+\delta\mathbb{E}\left[\sup_{t\in(0,T)} \int_{\mathbb{T}^3} |\mathbf{u}_N|^2 \, \mathrm{d}x\right] + c(\delta, T).$$

The claim follows from choosing a sufficiently small δ and using the initial law $\Lambda_0 = \mathbb{P} \circ \mathbf{u}_0^{-1}$. □

In general for some $\beta > 2$, we have the following:

Corollary 3.1.6. *Let the assumptions of Theorem 3.1.30 be satisfied and in addition*

$$\int_{L^2_{div}(\mathbb{T}^3)} \|\mathbf{v}\|^{\beta}_{L^2(\mathbb{T}^3)} d\Lambda_0(\mathbf{v}) < \infty.$$

Then, the following holds

$$\mathbb{E}\left[\sup_{t\in(0,T)} \int_{\mathbb{T}^3} |\mathbf{u}_N|^2 dx + \varepsilon \int_Q |\nabla \mathbf{u}_N|^2 dxdt\right]^{\beta/2} \leq c_\beta \left(1 + \mathbb{E}\left[\int_{L^2_{\rm div}(\mathbb{T}^3)} \|\mathbf{v}\|^2_{L^2(\mathbb{T}^3)} d\Lambda_0(\mathbf{v})\right]^{\beta/2}\right).$$

A similar proof to one in Theorem 3.1.30 yields the estimate, but for such case one should use $\beta/2$ power to (3.1.33) before taking expectations.

Remark 3.1.1. Some remarks are in order, the formulation (3.1.33) is the energy equality in the Galerkin approximation.

Compactness

To gain compactness we need to pass to the limit in the nonlinear convective term. Accordingly, in view of (3.1.28) the balance of momentum reads

$$\begin{aligned} \int \mathbf{u}_N(t)\cdot\varphi \mathrm{d}x &= \int_{\mathbb{T}^3} \mathbf{u}_N(t)\cdot \mathscr{P}^l_N\varphi \ \mathrm{d}x \\ &= \int_{\mathbb{T}^3} \mathbf{u}_0\cdot\mathscr{P}^l_N\varphi + \int_0^t\int_{\mathbb{T}^3} \mathbf{H}_N : \nabla\mathscr{P}^l_N\varphi \ dxds \\ &\quad + \int_0^t\int_{\mathbb{T}^3} \phi \ \mathrm{d}W^N_s \mathrm{d}x, \\ \mathbf{H}_N &= -\varepsilon\nabla\mathbf{u}_N + \mathbf{u}_N\otimes\mathbf{u}_N, \end{aligned}$$

for all $\varphi \in C^\infty_{\rm div}(\mathbb{T}^3)$, where $\mathscr{P}^l_N$ is the projection into the space $\mathscr{X}_N$ with respect to $W^{l,2}_{\rm div}(\mathbb{T}^3)$ inner product. Given the a 'priori estimates' in Theorem 3.1.30 and Corollary 3.1.6 for $\beta > 2$, we deduce that

$$\mathbf{H}_N \in L^1(\Omega, L^{p_0}(0,T;L^1(\mathbb{T}^3))), \tag{3.1.34}$$

for some $p_0 > 1$ uniformly in N (provided $\beta > 2$), a claim we make precise in Section 3.1.3.

We derive bounds of the system by computing the deterministic and stochastic part bounds separately. For deterministic part we consider the functional

$$\mathscr{H}_N(t,\varphi) := \int_0^t \int_{\mathbb{T}^3} \mathbf{H}_N : \nabla \mathscr{P}_N^l \varphi \, \mathrm{d}x\mathrm{d}s, \quad \varphi \in \mathscr{C}^\infty_{\mathrm{div}}(\mathbb{T}^3).$$

By (3.1.34) and the Sobolev embedding for $W^{\tilde{l},p_0}(\mathbb{T}^3) \hookrightarrow W_0^{l,2}(\mathbb{T}^3)$ for $\tilde{l} \geq l + d(\frac{1}{p_0} - \frac{1}{2})$ we have the estimate

$$\mathbb{E}\left[\|\mathscr{H}_N\|_{W^{1,p_0}((0,T);W^{-l,2}_{\mathrm{div}}(\mathbb{T}^3))} \right] \leq C,$$

where C is a constant. This follows from observing that $\mathscr{H}_N \in L^{p_0}(0,T;W^{-l,2}_{\mathrm{div}}(\mathbb{T}^3))$ and $\partial t \mathscr{H}_N \in L^{p_0}(0,T;W^{-l,2}_{\mathrm{div}}(\mathbb{T}^3))$ uniformly in N, then $\mathscr{H} \in W^{1,p_0}(0,T;W^{-l,2}_{\mathrm{div}}(\mathbb{T}^3))$.

Proof. We have that,

$$\partial t \mathscr{H}_N(t,\varphi) = \int_{\mathbb{T}^3} \mathbf{H}_N : \nabla \mathscr{P}_N^l \varphi \, \mathrm{d}x \quad \varphi \in W^{l,2}_{\mathrm{div}}(\mathbb{T}^3).$$

Then applying Holder's inequality yields,

$$\begin{aligned} \|\partial t \mathscr{H}\|_{W^{-l,2}_{\mathrm{div}}(\mathbb{T}^3)} &= \sup_{\|\varphi\|_{W^{l,2}_{\mathrm{div}}}=1} \int_{\mathbb{T}^3} \mathbf{H}_N : \nabla \mathscr{P}_N^l \varphi \, \mathrm{d}x \\ &\leq \sup_{\|\varphi\|_{W^{l,2}_{\mathrm{div}}}=1} \|\mathbf{H}_N\|_1 \|\nabla \mathscr{P}_N^l \varphi\|_\infty, \end{aligned}$$

Now using the Sobolev inequality with respect to the embedding $W^{l,2} \hookrightarrow W^{1,\infty}$ and noting that $\mathscr{P}_N^l$ is continuous in $W^{l,2}(\mathbb{T}^3)$ we infer

$$\begin{aligned} \|\partial t \mathscr{H}\|_{W^{-l,p_0}_{\mathrm{div}}(\mathbb{T}^3)} &\leq \sup_{\|\varphi\|_{W^{l,2}_{\mathrm{div}}}=1} \|\mathbf{H}_N\|_1 \|\mathscr{P}_N^l \varphi\|_{W^{1,\infty}} \\ &\leq c \sup_{\|\varphi\|_{W^{l,2}_{\mathrm{div}}}=1} \|\mathbf{H}_N\|_1 \|\mathscr{P}_N^l \varphi\|_{W^{l,2}} \\ &\leq c \sup_{\|\varphi\|_{W^{l,2}_{\mathrm{div}}}=1} \|\mathbf{H}_N\|_1 \|\varphi\|_{W^{l,2}} \\ &= c \sup_{\|\varphi\|_{W^{l,2}_{\mathrm{div}}}=1} \|\mathbf{H}_N\|_1. \end{aligned}$$

Integrating in time and taking expectations yields

$$\mathbb{E}\left[\int_0^T \|\partial_t \mathscr{H}\|_{W_{\mathrm{div}}^{-l,2}}^{p_0}\,\mathrm{d}t\right] \;\leq\; \mathbb{E}\left[\int_0^T \|\mathbf{H}_N\|_1^{p_0}\,\mathrm{d}t\right] \leq C,$$

as a consequence of (3.1.34). Finally, the claim follows from applying Poincaré's inequality 2.1.7.

□

To establish bounds associated with the stochastic part we apply Lemma 2.1.16, a result derived in [56] to estimate the noise term for all $\alpha \in (0,1)$. Using Lemma 2.1.16 and (3.1.2), and for any $\alpha < 1/2$ and $p = 2$ we deduce the estimate

$$\begin{aligned}\mathbb{E}\left[\left\|\int_0^{\cdot} \phi\,\mathrm{d}W_s\right\|_{W^{\alpha,2}(0,T;L^2(\mathbb{T}^3))}^2 \mathrm{d}t\right] &\leq c\mathbb{E}\left[\int_0^{\cdot} \|\phi\|_{L_2(\mathscr{U},L^2(\mathbb{T}^3))}^2\,\mathrm{d}t\right]\\ &\leq c(p,\phi,T).\end{aligned}$$

Finally, combining the stochastic and deterministic bounds yields

$$\mathbb{E}\left[\|\mathbf{u}_N\|_{W^{\alpha,p_0}(0,T;W_{\mathrm{div}}^{-l,p_0}(\mathbb{T}^3))}\right] \leq C. \tag{3.1.35}$$

Now noting that

$$\begin{aligned}W^{\alpha,p_0}((0,T);W_{\mathrm{div}}^{-l,p_0}(\mathbb{T}^3)) \cap L^2(0,T;W_{\mathrm{div}}^{1,2}(\mathbb{T}^3)) \cap L^\infty(0,T;L_{\mathrm{div}}^2(\mathbb{T}^3))\\ \hookrightarrow\!\hookrightarrow L^r(0,T;L_{\mathrm{div}}^r(\mathbb{T}^3)),\end{aligned} \tag{3.1.36}$$

compactly for all $r < 10/3$. The compactness of $\mathbf{u}_N$ follows from applying results established in [56] as stated in following theorem.

Theorem 3.1.7. *Let (X,Y,Z) be a triple of separable and reflexive Banach spaces such that the embedding $X \hookrightarrow Y$ is compact and the embedding $Y \hookrightarrow Z$ is continuous. Then the embedding*

$$L^p(0,T;X) \cap W^{\alpha,p}(0,T;Z) \hookrightarrow L^p(0,T;Y),$$

is compact for $1 < p < \infty$ and $0 < \alpha < 1$.

Let $X := W^{1,2}_{\text{div}}(\mathbb{T}^3)), Y := L^2_{\text{div}}(\mathbb{T}^3))$ and $Z := W^{-l,2}_{\text{div}}(\mathbb{T}^3))$. We are interested in compactness on the space $L^r(0,T;L^r_{\text{div}}\mathbb{T}^3)$. This can be shown through interpolation arguments. That is, let $Y_{\alpha P_0} :=$ (3.1.36) and consider the space

$$X_{\alpha p_0} := W^{\alpha, p_0}((0,T);W^{-l,p_0}_{\text{div}}(\mathbb{T}^3)) \cap L^{p_0}(0,T;W^{1,2}_{\text{div}}(\mathbb{T}^3)) \hookrightarrow\hookrightarrow L^{p_0}(0,T;L^2_{\text{div}}(\mathbb{T}^3)).$$

Now, suppose $(\mathbf{u}_m)_{m\in\mathbb{N}} \subset Y_{\alpha p_0}$ is a bounded sequence. Then $(\mathbf{u}_m)_{m\in\mathbb{N}}$ is a bounded sequence in $X_{\alpha p_0}$. Therefore, there exists a sub-sequence $\mathbf{u}_{m_k}$ such that

$$\mathbf{u}_{m_k} \to \mathbf{u} \text{ in } L^{p_0}(0,T;L^2_{\text{div}}(\mathbb{T}^3)).$$

In the periodic domain $\mathbb{T}^3 \subset \mathbb{R}^3$, we have

$$L^2(0,T;W^{1,2}(\mathbb{T}^3)) \cap L^\infty(0,T;L^2_{\text{div}}(\mathbb{T}^3)) \hookrightarrow L^{\frac{10}{3}}(0,T;L^{\frac{10}{3}}_{\text{div}}(\mathbb{T}^3)) \approx L^{\frac{10}{3}}(Q).$$

For $p_0 \leq 2$ we have

$$\mathbf{u}_{m_k} \to \mathbf{u} \text{ in } L^{p_0}(0,T;L^{p_0}(\mathbb{T}^3)) \approx L^{p_0}(Q).$$

In view of Lemma 2.1.1 we see that

$$\begin{aligned} \|\mathbf{u}_{m_k} - \mathbf{u}\|_{L^r(Q)} &\leq \underbrace{\|\mathbf{u}_{m_k} - \mathbf{u}\|^{\theta}_{L^{p_0}(Q)}}_{\to 0} \|\mathbf{u}_{m_k} - \mathbf{u}\|^{1-\theta}_{L^{\frac{10}{3}}(Q)} \\ &\leq 0. \end{aligned}$$

Given the compact embedding above, we consider the path space

$$V := L^r(0,T;L^r(\mathbb{T}^3)) \otimes C([0,T],\mathscr{U}_0) \otimes L^2_{\text{div}}(\mathbb{T}^3),$$

with the following laws:

$$\begin{cases} \mu_{\mathbf{u}_N} \ \text{ is the law of } \ \mathbf{u}_N \text{ on } L^r(0,T;L^r(\mathbb{T}^3)), \\ \mu_W \ \text{ is the law of } \ W \text{ on } C([0,T],\mathscr{U}_0), \text{ where } \mathscr{U}_0 \ \text{ is defined in (3.1.4)}, \\ \mu_N \ \text{ is the joint law of } \mathbf{u}_N, W, \mathbf{u}_0 \text{ on } V. \end{cases}$$

Considering a ball B_R in the space

$$W^{\alpha,p_0}(0,T;W_{\mathrm{div}}^{-l,2}(\mathbb{T}^3))\cap L^2(0,T;W_{0,\mathrm{div}}^{1,2}(\mathbb{T}^3))\cap L^\infty(0,T;L_{div}^2(\mathbb{T}^3)),$$

we seek its complement B_R^C such that applying Theorem 3.1.30, (3.1.35), and Markov's inequality yields

$$\begin{aligned}\mu_{\mathbf{u}_N}(B_R^C) &= \mathbb{P}\left(\|\mathbf{u}_N\|_{W^{\alpha,p_0}(W_{\mathrm{div}}^{-l,2})}+\|\mathbf{u}_N\|_{L^2(W^{1,2})}+\|\mathbf{u}_N\|_{L^\infty(L^2)}\geq R\right)\\ &\leq \frac{\mathbb{E}}{R}\left[\|\mathbf{u}_N\|_{W^{\alpha,p_0}(W_{\mathrm{div}}^{-l,2})}+\|\mathbf{u}_N\|_{L^2(W^{1,2})}+\|\mathbf{u}_N\|_{L^\infty(L^2)}\right]\\ &\leq \frac{c}{R}.\end{aligned}$$

Therefore, for any $\gamma>0$ there is $R=R(\gamma)$ such that

$$\mu_{\mathbf{u}_N}(B_R)\geq 1-\frac{\gamma}{3},$$

that is, the family of probability laws $\mu_{\mathbf{u}_N}$ is tight by Definition 2.1.29. The law of μ_W is a Radon measure on the Polish space $C([0,T],\mathscr{U}_0)$, and therefore it is tight. This implies that there exists a compact set $C_\gamma\subset C([0,T],\mathscr{U}_0)$ so that $\mu_W(C_\gamma)\geq 1-\gamma/3$. Furthermore, arguing similarly we note there exists a compact subset of $L_{\mathrm{div}}^2(\mathbb{T}^3)$ such that its measure Λ_0 is smaller than $1-\gamma/3$. Consequently, there exists a compact subset $V_\gamma\subset V$ such that $\mu_N(V_\gamma)\geq 1-\gamma$. Thus, $(\mu_N)_{N\in\mathbb{N}}$ is tight by Definition 2.1.29 in the same space. On the account of Lemma 2.1.19, $(\mu_N)_{N\in\mathbb{N}}$ is relatively weakly compact, and as a result we have a weakly convergent sub-sequence with limit μ. Applying Skorokhod's representation theorem 2.1.20 we infer that there exists a probability space $(\tilde{\Omega},\tilde{\mathscr{F}},\tilde{\mathbb{P}})$, a sequence $(\tilde{\mathbf{u}}_N,\tilde{W}^N,\tilde{\mathbf{u}}_{0,N})$ and $(\tilde{\mathbf{u}},\tilde{W},\tilde{\mathbf{u}}_0)$ on $(\tilde{\Omega},\tilde{\mathscr{F}},\tilde{\mathbb{P}})$ both with values in V such that

- The laws of $(\tilde{\mathbf{u}}_N,\tilde{W}^N,\tilde{\mathbf{u}}_{0,N})$ and $(\tilde{\mathbf{u}},\tilde{W},\tilde{\mathbf{u}}_0)$ under $\tilde{\mathbb{P}}$ coincide with μ_N and μ respectively.
- We have the convergences

$$\begin{cases}\tilde{\mathbf{u}}_N\to\tilde{\mathbf{u}} & \text{in}\quad L^r(0,T;L^r(\mathbb{T}^3)),\\ \tilde{W}^N\to\tilde{W} & \text{in}\quad C([0,T],\mathscr{U}_0),\\ \tilde{\mathbf{u}}_{0,N}\to\tilde{\mathbf{u}}_0 & \text{in}\quad L^2(\mathbb{T}^3),\end{cases}\tag{3.1.37}$$

$\tilde{\mathbb{P}}$-a.s.

Note, before passing to the limit in the new probability space, it is essential to establish accurate measurability of the new random variables. For this, we adapt a filtration to the new probability space. Let $\mathbf{z}_t$ be a restriction operator to the interval $[0,t]$ acting on various path spaces. For instance, suppose A is a path space of $L^r(0,T;L^r(\mathbb{T}^3))$ or $C([0,T],\mathscr{U}_0)$ and $t \in [0,T]$, we define

$$\mathbf{z}_t : A \to A|_{[0,t]}, f \to f|_{[0,t]}, \tag{3.1.38}$$

where $\mathbf{z}_t$ is a continuous mapping. Now let $(\tilde{\mathscr{F}}_t^N)_{t\geq 0}$ and $(\tilde{\mathscr{F}}_t)_{t\geq 0}$ denote the $\tilde{\mathbb{P}}$-augmented canonical filtration of the process $(\tilde{\mathbf{u}}_N, \tilde{W}^N)$ and $(\tilde{\mathbf{u}}, \tilde{W})$ respectively, i.e.

$$\tilde{\mathscr{F}}_t = \sigma\left(\sigma(\mathbf{z}_t\tilde{\mathbf{u}}, \mathbf{z}_t\tilde{W}) \cup \{\mathscr{N} \in \tilde{\mathscr{F}};\, \tilde{\mathbb{P}}(\mathscr{N}) = 0\}\right), \quad t \in [0,T].$$

$$\tilde{\mathscr{F}}_t^N = \sigma\left(\sigma(\mathbf{z}_t\tilde{\mathbf{u}}_N, \mathbf{z}_t\tilde{W}^N) \cup \{\mathscr{N} \in \tilde{\mathscr{F}};\, \tilde{\mathbb{P}}(\mathscr{N}) = 0\}\right), \quad t \in [0,T].$$

Here, σ is the smallest σ-algebra of the space. Using (3.1.38) in the new probability space ensures that the stochastic processes are adapted so that we can define stochastic integrals.

The system on the new probability space

In view of Theorem 2.1.20, distribution laws of new probability space coincide with those of old probability space. Consequently, noting that the approximated system holds on the old probability space, it follows that the same is true on the new probability space. To show this, we use the elementary method covered in [27] which has been generalised for various purposes. The underlying aim is to identify corresponding martingale quadratic variation and cross variation.

To begin with, we observe that $\tilde{W}^N$ and W have the same law. As a result, we have a collection of mutually independent real-valued $(\tilde{\mathscr{F}}_t^N)_{t\geq 0}$- Wiener processes $(\tilde{\beta}_k^N)_{k=1,\dots,N}$ such that $\tilde{W}^N = \sum_{k=1} \tilde{\beta}_k^N e_k$. To be precise, we have a collection of mutually independent real-valued $(\tilde{\mathscr{F}}_t)_{t\geq 0}$- Wiener processes $(\tilde{\beta}_k)_{k=1,\dots,N}$ so that $\tilde{W} = \sum_{k=1} \tilde{\beta}_k e_k$. We consider continuous functionals to compute quadratic and cross variations. The idea lies in following observation;

Lemma 3.1.8. *Let* $(\Omega, \mathscr{F}, \mathbb{P})$ *be a probability space and* V *a Polish space with Borelian* σ*- algebra* $\mathscr{B}(V)$*. Let* X, Y *be two random variables on* $(\Omega, \mathscr{F}, \mathbb{P})$ *with values in* V *such that*

$$X : (\Omega, \mathscr{F}) \to (V, \mathscr{B}(V)) \quad \text{and } Y : (\Omega, \mathscr{F}) \to (V, \mathscr{B}(V))$$

are measurable. Let $f : V \to S$ *be measurable function with* S *a Polish space, and we claim that if* X *and* Y *have the same distribution laws then*

$$X \sim^d Y \implies f(X) \sim^d f(Y).$$

Proof. Note that $X \sim^d Y$ means

$$\mathbb{P}(X \in A) = \mathbb{P}(Y \in A) \quad \forall A \in \mathscr{B}(V).$$

$$\begin{aligned} \text{And, } \mathbb{P}(f(X) \in \eta) &= \mathbb{P}(X \in f^{-1}(\eta)) \quad \text{for } \eta \in \mathscr{B}(V), \text{ where } f^{-1} \text{ is the pre-image.} \\ &= \mathbb{P}(Y \in f^{-1}(\eta)) \text{ since } f^{-1}(\eta) \in \mathscr{B}(V) \\ &= \mathbb{P}(f(Y) \in \eta). \end{aligned}$$

Therefore, this implies $f(X) \sim^d f(Y)$ as required. □

We proceed to define for all $t \in [0,t]$ and $\varphi \in C^{\infty}_{c,\mathrm{div}}(\mathbb{T}^3)$ the functionals

$$\begin{aligned} \Phi(\mathbf{v}, \mathbf{v}_0)_t = &\int_{\mathbb{T}^3} \mathbf{v}(t) \cdot \varphi \, \mathrm{d}x - \int_{\mathbb{T}^3} \mathbf{v}_0 \cdot \varphi \, \mathrm{d}x + \underbrace{\int_{\mathbb{T}^3} \int_0^t \mathbf{v} \otimes \mathbf{v} : \nabla \mathscr{P}^l_N \varphi \, \mathrm{d}x\mathrm{d}s}_{I(\mathbf{v})} \\ &+ \varepsilon \int_{\mathbb{T}^3} \int_0^t \nabla \mathbf{v} : \nabla \mathscr{P}^l_N \varphi \, \mathrm{d}x\mathrm{d}s, \end{aligned}$$

$$\Psi_t = \sum_{k=1}^{N} \int_0^t \left(\int_{\mathbb{T}^3} \phi_k \cdot \mathscr{P}^l_N \varphi \, \mathrm{d}x \right)^2 \mathrm{d}s,$$

$$(\Psi_k)_t = \int_0^t \int_{\mathbb{T}^3} \phi_k \cdot \mathscr{P}^l_N \varphi \, \mathrm{d}x\mathrm{d}s.$$

Remark 3.1.2. The functionals

$$(\tilde{\mathbf{u}}_N, \tilde{\mathbf{u}}_0) \to \Phi(\tilde{\mathbf{u}}_N, \tilde{\mathbf{u}}_0)_t, \qquad \Psi_t, \qquad (\Psi_k)_t,$$

are measurable on the the subspace of pathspace where the joint law of $(\mathbf{u}_N, \mathbf{u}_0)$ is supported.

We denote by $\Phi(\mathbf{u}_N, \mathbf{u}_0)_{s,t}$ the increment $\Phi(\mathbf{u}_N, \mathbf{u}_0)_t - \Phi(\mathbf{u}_N, \mathbf{u}_0)_s$ and similarly for $(\Psi)_{s,t}$ and $(\Psi_k)_{s,t}$. Completeness of proof follows from showing that the process $\Phi(\tilde{\mathbf{u}}_N)$ is an $\tilde{\mathscr{F}}_t^N$-martingale such that its corresponding quadratic and cross variations satisfy

$$\langle\langle \Phi(\tilde{\mathbf{u}}_N, \tilde{\mathbf{u}_0})\rangle\rangle = \Psi, \qquad \langle\langle \Phi(\tilde{\mathbf{u}}_N, \tilde{\mathbf{u}_0}), \tilde{\beta}_k\rangle\rangle = \Psi_k, \tag{3.1.39}$$

respectively. In particular, in view of (3.1.39) we see that

$$\begin{aligned} &\left\langle\left\langle \Phi(\tilde{\mathbf{u}}_N, \tilde{\mathbf{u}_0}) - \int_0^{\cdot} \int_{\mathbb{T}^3} \phi \, \mathrm{d}\tilde{W}^N \cdot \mathscr{P}_N^l \varphi \, \mathrm{d}x \right\rangle\right\rangle \qquad (3.1.40) \\ &= \langle\langle \Phi(\tilde{\mathbf{u}}_N, \tilde{\mathbf{u}_0})\rangle\rangle \\ &+ \left\langle\left\langle \int_0^{\cdot} \int_{\mathbb{T}^3} \phi \cdot \mathscr{P}_N^l \varphi \, \mathrm{d}x \, \mathrm{d}\tilde{W}^N \right\rangle\right\rangle \\ &- 2\sum_{k=1}^N \left\langle\left\langle \Phi(\tilde{\mathbf{u}}_N, \tilde{\mathbf{u}_0}), \int_0^{\cdot} \int_{\mathbb{T}^3} \phi \cdot \mathscr{P}_N^l \varphi \, \mathrm{d}x \, \mathrm{d}\tilde{\beta}_k \right\rangle\right\rangle \\ &= \Psi + \Psi \\ &- 2\sum_{k=1}^N \int_0^{\cdot} \int_{\mathbb{T}^3} \phi \cdot \mathscr{P}_N^l \varphi \, \mathrm{d}x \, \mathrm{d}\langle\langle \Phi(\tilde{\mathbf{u}}_N, \tilde{\mathbf{u}_0}), \tilde{\beta}_k\rangle\rangle_s \\ &= 0, \end{aligned} \tag{3.1.41}$$

which implies the desired equation on the new probability space. Next we proceed to verify (3.1.39). On the account of uniform estimates above (3.1.30) the mapping

$$(\tilde{\mathbf{u}}_N, \tilde{\mathbf{u}}_0) \mapsto \Phi(\tilde{\mathbf{u}}_N, \tilde{\mathbf{u}}_0),$$

is well defined and measurable on the subspace of pathspace where the joint law $(\tilde{\mathbf{u}}_N, \tilde{\mathbf{u}}_0)$ is supported. Accordingly, we show continuity for less obvious terms $I(\mathbf{v})$ and note that other terms can be handled similarly. For the convective term, we assume that $\mathbf{v}_N \to \mathbf{v}$ in $L^r(0,T;L^r(\mathbb{T}^3))$ and P_N^l is continuous on $W^{l,2}$ with norm $\|P_N^l \varphi\| \leq 1$ so that

$$\begin{aligned} |I(\mathbf{v}_N) - I(\mathbf{v})| &= \left| \int_{\mathbb{T}^3} \int_0^t \mathbf{v}_N \otimes \mathbf{v}_N : \nabla \mathscr{P}_N^l \varphi \, \mathrm{d}x\mathrm{d}s - \int_{\mathbb{T}^3} \int_0^t \mathbf{v} \otimes \mathbf{v} : \nabla \mathscr{P}_N^l \varphi \, \mathrm{d}x\mathrm{d}s \right| \\ &\leq \int_0^t \int_{\mathbb{T}^3} \left| \underbrace{(\mathbf{v}_N \otimes \mathbf{v}_N - \mathbf{v} \otimes \mathbf{v})}_{a \to 0} : \nabla \mathscr{P}_N^l \varphi \right| \mathrm{d}x\mathrm{d}s. \end{aligned}$$

This follows from rewriting

$$\|\mathbf{v}_N \otimes \mathbf{v}_N - \mathbf{v} \otimes \mathbf{v}\|_{L^1(Q)},$$

as follows

$$\begin{aligned}
\|\mathbf{v}_N \otimes \mathbf{v}_N - \mathbf{v} \otimes \mathbf{v}\|_1 &= \|\mathbf{v}_N \otimes \mathbf{v}_N - \mathbf{v} \otimes \mathbf{v}_N + \mathbf{v} \otimes \mathbf{v}_N - \mathbf{v} \otimes \mathbf{v}\|_{L^1(Q)} \\
&= \|(\mathbf{v}_N - \mathbf{v}) \otimes \mathbf{v}_N - \mathbf{v} \otimes (\mathbf{v} - \mathbf{v}_N)\|_{L^1(Q)} \\
&\leq \|\mathbf{v}_N - \mathbf{v}\|_2 \|\mathbf{v}_N\|_2 + \|\mathbf{v}\|_2 \|\mathbf{v} - \mathbf{v}_N\|_2 \\
&\to 0, \text{ by assumption.}
\end{aligned}$$

Thus, the functionals are well defined $\Phi(\mathbf{v}_N) \to \Phi(\mathbf{v})$ and the mapping $\Phi : \mathbf{v} \to \mathbb{R}$ is continuous. Consequently, the following random variables have the same law

$$\Phi(\mathbf{u}_N, \mathbf{u}_0) \sim^d \Phi(\tilde{\mathbf{u}}_N, \tilde{\mathbf{u}}_0).$$

Now we fix times $s, t \in [0, T]$, where $s < t$ and define a continuous function h such that

$$h : V|_{[0,s]} \to [0, 1].$$

The process

$$\Phi(\mathbf{u}_N, \mathbf{u}_0) = \int_0^t \int_{\mathbb{T}^3} \phi \, \mathrm{d}W_s^N \cdot \mathscr{P}_N^l \varphi \, \mathrm{d}x = \sum_{k=1}^{N} \int_0^t \int_{\mathbb{T}^3} \phi_k \cdot \mathscr{P}_N^l \varphi \, \mathrm{d}x \, \mathrm{d}\beta_k,$$

is a square integrable $(\mathscr{F}_t)_{t \geq 0}$-martingale. Using Theorem 2.1.9 we deduce that

$$[\Phi(\mathbf{u}_N, \mathbf{u}_0)]^2 - \Psi, \qquad \Phi(\mathbf{u}_N, \mathbf{u}_0)\beta_k - \Psi_k,$$

are $(\mathscr{F}_t)_{t \geq 0}$-martingales. Next let $\mathbf{z}_s$ be a restriction of a function to the interval $[0, s]$, then in view of Lemma 2.1.22 and equality of laws we see that

$$\tilde{\mathbb{E}}\left[h(\mathbf{z}_s \tilde{\mathbf{u}}_N, \mathbf{z}_s \tilde{W}^N) \Phi(\tilde{\mathbf{u}}_N, \tilde{\mathbf{u}}_0)_{s,t}\right] = \mathbb{E}[h(\mathbf{z}_s \mathbf{u}_N, \mathbf{z}_s W^N) \Phi(\mathbf{u}_N, \mathbf{u}_0)_{s,t}\Big] = 0,$$

$$\tilde{\mathbb{E}}\left[h(\mathbf{z}_s \tilde{\mathbf{u}}_N, \mathbf{z}_s \tilde{W}^N)[\Phi(\tilde{\mathbf{u}}_N, \tilde{\mathbf{u}}_0)]_{s,t}^2 - (\Psi)_{s,t}\right] = \mathbb{E}\left[h(\mathbf{z}_s \mathbf{u}_N, \mathbf{z}_s W^N)[\Phi(\mathbf{u}_N, \mathbf{u}_0)]_{s,t}^2 - (\Psi)_{s,t}\right] = 0,$$

$$\tilde{\mathbb{E}}\left[h(\mathbf{z}_s\tilde{\mathbf{u}}_N,\mathbf{z}_s\tilde{W}^N)[\Phi(\tilde{\mathbf{u}}_N,\tilde{\mathbf{u}}_0)\tilde{\beta}_k^N]_{s,t}-(\Psi_k)_{s,t}\right]$$
$$=\mathbb{E}\left[h(\mathbf{z}_s\mathbf{u}_N,\mathbf{z}_sW^N)[\Phi(\mathbf{u}_N,\mathbf{u}_0)\beta_k^N]_{s,t}-(\Psi_k)_{s,t}\right]=0.$$

Hence (3.1.39) holds and (3.1.41) follows as a consequence. Moreover, on the new probability space $(\tilde{\Omega},\tilde{\mathscr{F}},\tilde{\mathbb{P}})$ for $(k=1,\dots,N)$ we have the equations

$$\begin{aligned}\int_{\mathbb{T}^3}\mathrm{d}\tilde{\mathbf{u}}_N\cdot\mathbf{w}_k\,\mathrm{d}x &= -\int_{\mathbb{T}^3}\nu\nabla\tilde{\mathbf{u}}_N:\nabla\mathbf{w}_k\,\mathrm{d}x\mathrm{d}t+\int_{\mathbb{T}^3}\tilde{\mathbf{u}}_N\otimes\tilde{\mathbf{u}}_N:\nabla\mathbf{w}_k\,\mathrm{d}x\mathrm{d}t\\ &\quad+\int_{\mathbb{T}^3}\phi\,\mathrm{d}\tilde{W}_s^N\cdot\mathbf{w}_k\,\mathrm{d}x,\\ &\quad\tilde{\mathbf{u}}_N(0)=\mathscr{P}_N^l\tilde{\mathbf{u}}_0. \end{aligned}\tag{3.1.42}$$

Accordingly, we proceed to passage of the limit in new probability space. In view of Theorem 3.1.30 and (3.1.37), we have the convergences

$$\begin{cases}\tilde{\mathbf{u}}_N\rightharpoonup\tilde{\mathbf{u}}\text{ in }L^2(\tilde{\Omega},\tilde{\mathscr{F}},\tilde{\mathbb{P}};L^2(0,T;W^{1,2}_{0,\mathrm{div}}(\mathbb{T}^3))),\\ \tilde{\mathbf{u}}_N\otimes\tilde{\mathbf{u}}_N\to\tilde{\mathbf{u}}_N\otimes\tilde{\mathbf{u}}_N\text{ in }L^{r/2}(\tilde{\Omega},\tilde{\mathscr{F}},\tilde{\mathbb{P}};L^{r/2}(Q)),\\ \tilde{W}^N\to\tilde{W}\text{ in}C([0,T],\mathscr{U}_0)\end{cases}\tag{3.1.43}$$

Furthermore, on the account of Theorem 3.1.30 we note that for $\mathbf{u}\in L^\infty(0,T;L^2(\mathbb{T}^3))$ it holds

$$\tilde{\mathbb{E}}\left[\sup_{t\in(0,T)}\int_{\mathbb{T}^3}|\tilde{\mathbf{u}}(t)|^2\mathrm{d}x\right]<\infty,\text{ a.s.}$$

Now we consider $\theta\in L^2(\tilde{\Omega}\times(0,T))$ and $\varphi\in W^{1,2}_{\mathrm{div}}(\mathbb{T}^3)$ such that

$$\begin{aligned}\tilde{\mathbb{E}}\left[\int_0^T\int_{\mathbb{T}^3}\tilde{\mathbf{u}}(t)\;\theta(t)\;\varphi\;\mathrm{d}x\mathrm{d}t\right] &= \lim_{N\to\infty}\tilde{\mathbb{E}}\left[\int_0^T\int_{\mathbb{T}^3}\tilde{\mathbf{u}}_N(t)\;\theta(t)\;\varphi\;\mathrm{d}x\mathrm{d}t\right]\\ &= \lim_{N\to\infty}\tilde{\mathbb{E}}\left[\int_0^T\int_{\mathbb{T}^3}\tilde{\mathbf{u}}_N(t)\;\theta(t)\;\mathscr{P}_N^l\varphi\;\mathrm{d}x\mathrm{d}t\right]\end{aligned}$$

$$\begin{aligned}=\lim_{N\to\infty}\tilde{\mathbb{E}}\Bigg[\int_0^T\Bigg(&\int_{\mathbb{T}^3}\mathscr{P}_N^l\tilde{\mathbf{u}}_0\;\theta(t)\;\mathscr{P}_N^l\varphi\;\mathrm{d}x-\int_0^t\int_{\mathbb{T}^3}\varepsilon\nabla\tilde{\mathbf{u}}_N\theta(t):\nabla\;\mathscr{P}_N^l\varphi\;\mathrm{d}x\mathrm{d}s\\ &+\int_0^t\int_{\mathbb{T}^3}(\tilde{\mathbf{u}}_N\otimes\tilde{\mathbf{u}}_N)\theta(t):\nabla\;\mathscr{P}_N^l\varphi\;\;\mathrm{d}x\mathrm{d}s+\int_{\mathbb{T}^3}\int_0^t\phi\;\mathrm{d}\tilde{W}_s^N\cdot\;\theta(t)\;\mathscr{P}_N^l\varphi\;\;\mathrm{d}x\Bigg)\,\mathrm{d}t\Bigg].\end{aligned}$$

Since $\mathscr{P}_N^l$ is continuous in $L^2(\mathbb{T}^3)$ we infer that $\mathscr{P}_N^l\varphi\to\varphi$ (by truncation). The convergences of second and third term follow from (3.1.43). For the stochastic limit term, we have the convergences

$$\tilde{W}^N \to \tilde{W} \qquad \text{in } C([0,T],\mathscr{U}_0),$$

in probability. Therefore, passing to the limit yields

$$\begin{aligned}\tilde{\mathbb{E}}\left[\int_0^T \int_{\mathbb{T}^3} \tilde{\mathbf{u}}(t)\ \theta(t)\ \varphi\ \mathrm{d}x\mathrm{d}t\right] =& \tilde{\mathbb{E}}\Bigg[\int_0^T \Bigg(\int_{\mathbb{T}^3} \tilde{\mathbf{u}}_0\ \theta(t)\ \varphi\ \mathrm{d}x \\ &- \int_0^t \int_{\mathbb{T}^3} \varepsilon \nabla \tilde{\mathbf{u}} \theta(t) : \nabla\ \varphi\ \mathrm{d}x\mathrm{d}s \\ &+ \int_0^t \int_{\mathbb{T}^3} (\tilde{\mathbf{u}} \otimes \tilde{\mathbf{u}})\theta(t) : \nabla\ \varphi\ \ \mathrm{d}x\mathrm{d}s \\ &+ \int_{\mathbb{T}^3} \int_0^t \phi\ \mathrm{d}\tilde{W}_s \cdot\ \theta(t)\ \varphi\ \ \mathrm{d}x\Bigg)\ \mathrm{d}t\Bigg].\end{aligned}$$

Upon taking the difference between left and right hand side of the terms above, factoring $\theta(t)$ and applying fundamental lemma of calculus of variations we obtain

$$\begin{aligned}\int_{\mathbb{T}^3} \tilde{\mathbf{u}} \cdot \varphi\ \mathrm{d}x \quad = \quad & \int_{\mathbb{T}^3} \tilde{\mathbf{u}}_0 \cdot \varphi\ \mathrm{d}x - \int_0^t \int_{\mathbb{T}^3} \nu \nabla \tilde{\mathbf{u}} : \nabla \varphi\ \mathrm{d}x\mathrm{d}s + \int_0^t \int_{\mathbb{T}^3} \tilde{\mathbf{u}} \otimes \tilde{\mathbf{u}} : \nabla \varphi\ \mathrm{d}x\mathrm{d}s \\ &+ \int_0^t \int_{\mathbb{T}^3} \phi \cdot \varphi\, \mathrm{d}x\, \mathrm{d}W,\end{aligned}$$

$\tilde{\mathbb{P}}$-a.s. We conclude discussions on the Navier-Stokes system by showing that the energy inequality in the sense of (3.1.25) continues to hold in the new probability space. To begin with, we consider the energy equality in the approximated system of Navier-Stokes which reads

$$\begin{aligned}\frac{1}{2}\|\mathbf{u}_N(t)\|^2_{L^2(\mathbb{T}^3)} \quad = \quad & \frac{1}{2}\|\mathscr{P}_N \mathbf{u}_0\|^2_{L^2(\mathbb{T}^3)} - \varepsilon \int_0^t \int_{\mathbb{T}^3} |\nabla \mathbf{u}_N|^2\ \mathrm{d}x\ \mathrm{d}s + \int_0^t \int_{\mathbb{T}^3} \mathbf{u}_N \cdot \phi\, \mathrm{d}x\, \mathrm{d}W^N_s \\ &+ \frac{1}{2} \int_0^t \sum_{k=1}^N \left(\int_{\mathbb{T}^3} \phi_k \mathbf{w}_k\ \mathrm{d}x\right)^2\ \mathrm{d}s. \end{aligned} \tag{3.1.44}$$

Fixing s we re-write the equality (3.1.44) in the form

$$\begin{aligned}&-\int_s^\infty \partial_t \varphi E^N_t\, \mathrm{d}t - \varphi(s) E^N_s \\ &= \varepsilon \int_s^\infty \int_{\mathbb{T}^3} |\nabla \mathbf{u}_N|^2\ \mathrm{d}x\ \mathrm{d}t + \frac{1}{2}\int_s^\infty \varphi \|\phi\|_{L_2(\mathscr{U}, L^2(\mathbb{T}^3))}\, \mathrm{d}t + \int_s^\infty \varphi \int_{\mathbb{T}^3} \mathbf{u}_N \cdot \phi\, \mathrm{d}x\ \mathrm{d}W^N\end{aligned}$$

$\mathbb{P}$-a.s. for all $\varphi \in C_c^\infty([s,\infty);[0,\infty))$, where $E_t^N = \frac{1}{2}\int_{\mathbb{T}^3} |\mathbf{u}_N(t)|^2 \,\mathrm{d}x$. Applying convergences in (3.1.43), Lemma 2.1.14 and Proposition 3.1.2 yields existence of (3.1.25) on the new probability space $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{\mathbb{P}})$.

3.1.3 Dissipative solutions to Stochastic Euler equation

In this section we aim to show that Navier-Stokes system converges to the measure-valued solutions (dissipative) of the stochastic Euler system in the vanishing viscosity. The stochastic Euler system read as

$$\begin{cases} \mathrm{d}\mathbf{u} = -(\nabla \mathbf{u})\mathbf{u}\,\mathrm{d}t - \nabla p\,\mathrm{d}t + \phi\,\mathrm{d}W & \text{in } Q, \\ \operatorname{div} \mathbf{u} = 0 & \text{in } Q, \end{cases} \tag{3.1.45}$$

Here W is a cylindrical Wiener process as introduced in Sect. 3.1.1. In the following we give a rigorous definition of a dissipative measure-valued solutions to Euler system (3.1.45).

Definition 3.1.2 (Solution)**.** Let Λ be a Borel probability measure on $L^2_{\mathrm{div}}(\mathbb{T}^3)$. Then

$$((\Omega, \mathscr{F}, (\mathscr{F}_t), \mathbb{P}), \mathbf{u}, \mathscr{V}, W)$$

is called a dissipative (*measure-valued*) martingale solution to (3.1.45) with the initial data Λ provided

1. $(\Omega, \mathscr{F}, (\mathscr{F}_t), \mathbb{P})$ is a stochastic basis with a complete right-continuous filtration,
2. W is an $(\mathscr{F}_t)$-cylindrical Wiener process,
3. the velocity $\mathbf{u}$ is $(\mathscr{F}_t)$-adapted and satisfies $\mathbb{P}$-a.s.

$$\mathbf{u} \in C([0,T], W^{-4,2}_{\mathrm{div}}(\mathbb{T}^3)) \cap L^\infty(0,T; L^2_{\mathrm{div}}(\mathbb{T}^3));$$

4. $\mathscr{V} = (\nu_{t,x}, \nu^\infty_{t,x}, \lambda)$ is $(\mathscr{F}_t)$-adapted and $\mathscr{V} \in \mathbf{Y}_2(Q, \mathbb{R}^3)$ $\mathbb{P}$-a.s.
5. $\Lambda = \mathbb{P} \circ (v(0))^{-1}$.

6. for all $\varphi \in C^\infty_{\mathrm{div}}(\mathbb{T}^3)$ and all $t \in [0,T]$ there holds $\mathbb{P}$-a.s.

$$\int_{\mathbb{T}^3} \mathbf{u}(t)\cdot\varphi\,\mathrm{d}x = \int_{\mathbb{T}^3} \mathbf{u}(0)\cdot\varphi\,\mathrm{d}x + \int_0^t \int_{\mathbb{T}^3} \langle \nu_{t,x}, \xi\otimes\xi\rangle : \nabla\varphi\,\mathrm{d}x\,\mathrm{d}s \\ + \int_{(0,t)\times\mathbb{T}^3} \langle \nu^\infty_{t,x}, \xi\otimes\xi\rangle : \nabla\varphi\,\mathrm{d}\lambda \ + \int_{\mathbb{T}^3}\int_0^t \varphi\cdot\phi\,\mathrm{d}W\,\mathrm{d}x. \tag{3.1.46}$$

7. The energy inequality holds in the sense that

$$E_{t^+} \le E_{s^-} + \frac{1}{2}\int_s^t \|\phi\|^2_{L_2(\mathscr{U},L^2(\mathbb{T}^3))}\,\mathrm{d}\sigma + \int_s^t \int_{\mathbb{T}^3} \mathbf{u}\cdot\phi\ \mathrm{d}x\,\mathrm{d}W, \tag{3.1.47}$$

   $\mathbb{P}$-a.s. for all $0 \le s < t$, where $E_t = \frac{1}{2}\int_{\mathbb{T}^3}\langle \nu_{t,x}, |\xi|^2\rangle\,\mathrm{d}x + \frac{1}{2}\lambda_t(\mathbb{T}^3)$ for $t \ge 0$ with $\lambda = \lambda_t \otimes \mathscr{L}^1$ and $E_{0^-} = \frac{1}{2}\int_{\mathbb{T}^3}|\mathbf{u}(0)|^2\,\mathrm{d}x$.

Remark 3.1.3. We augment the energy inequality with the following observations. At first sight, it is not clear why the left- and right-sided limits

$$E_{t^-} = \lim_{\tau\nearrow t} E_\tau, \quad E_{t^+} = \lim_{\tau\searrow t} E_\tau$$

exists in any time-point. To begin with, we only show that

$$E_t \le E_{s^-} + \frac{1}{2}\int_s^t \|\phi\|^2_{L_2(\mathscr{U},L^2(\mathbb{T}^3))}\,\mathrm{d}t + \int_s^t \int_{\mathbb{T}^3} \mathbf{u}\cdot\phi\ \mathrm{d}x\,\mathrm{d}W,$$

$\mathbb{P}$-a.s. for a.a. $0 < s < t$, see (3.1.68). Accordingly, this implies that the mapping

$$t \mapsto E_t - \frac{1}{2}\int_s^t \|\phi\|^2_{L_2(\mathscr{U},L^2(\mathbb{T}^3))}\,\mathrm{d}\sigma - \int_s^t \int_{\mathbb{T}^3} \mathbf{u}\cdot\phi\ \mathrm{d}x\,\mathrm{d}W,$$

is non-increasing. We note that the left- and right-sided limits exist in all points since the mapping is also pathwise bounded. In addition, $\int_0^\cdot \|\phi\|_{L_2(\mathscr{U},L^2(\mathbb{T}^3))}\,\mathrm{d}t$ and $\int_0^\cdot \int_{\mathbb{T}^3} \mathbf{u}\cdot\phi\ \mathrm{d}x\,\mathrm{d}W$ are continuous such that the left-and right-sided limits also exist for E_t. Finally, we obtain $E_{t^+} \le E_{t^-}$, such that there could be energetic sinks but no positive jumps in the energy.

The following theorem is the main result of Section 3.1.3 and it states the existence of dissipative measure-valued martingale solution to (3.1.45) in the sense of Definition 3.1.2.

Theorem 3.1.9 (Existence)**.** *Assuming that (3.1.2) holds and we have*

$$\int_{L^2_{div}(\mathbb{T}^3)} \|\mathbf{v}\|^{\beta}_{L^2(\mathbb{T}^3)} d\Lambda_0(\mathbf{v}) < \infty, \tag{3.1.48}$$

for some $\beta > 2$. *Then there is a dissipative measure-valued solution to (3.1.45) in the sense of Definition 3.1.2.*

Heuristically, for the proof of Theorem 3.1.9 we approximate (3.1.45) by a sequence of solutions to (3.1.22) with vanishing viscosity. Consequently, as a by product of our proof we obtain the following result

Corollary 3.1.10. *Let* Λ *be a given Borel probability measure on* $L^2_{\text{div}}(\mathbb{T}^3)$ *such that*

$$\int_{L^2_{\text{div}}(\mathbb{T}^3)} \|\mathbf{v}\|^{\beta}_{L^2_x} \, \mathrm{d}\Lambda(\mathbf{v}) < \infty,$$

for some $\beta > 2$. *If* $((\Omega^\varepsilon, \mathscr{F}^\varepsilon, (\mathscr{F}^\varepsilon_t), \mathbb{P}^\varepsilon), \mathbf{u}^\varepsilon, W^\varepsilon)$ *is finite energy weak martingale solution to (3.1.22) in the sense of Definition 3.1.1 with initial law* Λ, *then there is a subsequence such that*

$$\mathbf{u}_\varepsilon \to \mathbf{u} \quad \textit{in law on} \quad C_{w,\mathrm{loc}}([0,\infty); L^2_{\text{div}}(\mathbb{T}^3))$$

where $\mathbf{u}$ *is a dissipative solution to (3.1.45) in the sense of Definition 3.1.2.*

Remark 3.1.4. For any $\varepsilon > 0$ Theorem 3.1.3 yields the existence of a martingale solution

$$((\Omega^\varepsilon, \mathscr{F}^\varepsilon, (\mathscr{F}^\varepsilon_t), \mathbb{P}^\varepsilon), \mathbf{u}^\varepsilon, W^\varepsilon)$$

to (3.1.22). Note without loss of generality we can assume that the probability space as well as the Wiener process W^ε do not depend on ε, that is the solution is given by

$$((\Omega, \mathscr{F}, (\mathscr{F}_t), \mathbb{P}), \mathbf{u}^\varepsilon, W).$$

The assertion above follows from noting that for a martingale solution constructed by compactness method based on Skorokhod's theorem we may consider

$$(\Omega^\varepsilon, \mathscr{F}^\varepsilon, \mathbb{P}^\varepsilon) = ([0,1], \mathscr{B}\overline{[0,1]}, \mathfrak{L}^1),$$

we refer the reader to [64] for more details.

We proceed to the proof of the Theorem 3.1.3 which we split in several parts.

A priori estimates

Proposition 3.1.11 (A Priori Estimate). *Assume (3.1.2) holds and*

$$\int_{L^2_{div}(\mathbb{T}^3)} \|\mathbf{v}\|^2_{L^2(\mathbb{T}^3)} d\Lambda(\mathbf{v}) < \infty.$$

Then the following holds uniformly in ε,

$$\mathbb{E}\left[\sup_{t\in(0,T)} \int_{\mathbb{T}^3} |\mathbf{u}_\varepsilon|^2 dx + \varepsilon \int_Q |\nabla \mathbf{u}_\varepsilon|^2 dxdt\right]^p \le C(\Lambda, \phi, p, T), \tag{3.1.49}$$

for some $p = \beta > 2$, where β comes from Theorem 3.1.9.

Proof of Proposition 3.1.11. The estimate follows from taking the supremum in time and building expectations of the energy inequality (3.1.25) as shown below:

$$\begin{aligned}
&\mathbb{E}\left[\frac{1}{2}\sup_{t\in(0,T)} \int_{\mathbb{T}^3} |\mathbf{u}_\varepsilon|^2 \,\mathrm{d}x + \varepsilon \int_0^T \int_{\mathbb{T}^3} |\nabla \mathbf{u}_\varepsilon|^2 \,\mathrm{d}x\mathrm{d}t\right]^p \\
&\le \mathbb{E}\left[\frac{1}{2}\int_{\mathbb{T}^3} |\mathbf{u}_0|^2 \,\mathrm{d}x\right]^p \\
&+ \underbrace{\mathbb{E}\left[\int_0^t \int_{\mathbb{T}^3} \mathbf{u}_\varepsilon \phi \,\mathrm{d}x \,\mathrm{d}W_s\right]^p}_{I(\mathbf{u}_\varepsilon)} + \mathbb{E}\left[\frac{1}{2}\sum_{k\ge 1} \int_0^T \int_{\mathbb{T}^3} |\phi_k|^2 \,\mathrm{d}x\mathrm{d}t\right]^p
\end{aligned} \tag{3.1.50}$$

Applying Bukholder-Davis-Gundy inequality 2.1.13 to $I(\mathbf{u}_\varepsilon)$, and using the notation $\phi_k = \phi e_k$ yields:

$$\begin{aligned}
\mathbb{E}\left[\sup_{t\in(0,T)}\left|\int_0^t \int_{\mathbb{T}^3} \mathbf{u}_\varepsilon \phi \,\mathrm{d}W_s\right|\right]^p &= \mathbb{E}\left[\sup_{t\in(0,T)}\left|\int_0^t \sum_{k\ge 1}\int_{\mathbb{T}^3} \mathbf{u}_\varepsilon \cdot \phi_k \mathrm{d}x \,\mathrm{d}\beta_k(s)\right|\right]^p \\
&\le c\mathbb{E}\left[\left\langle\left\langle \int_0^\cdot \sum_{k\ge 1}\int_{\mathbb{T}^3} \mathbf{u}_\varepsilon \cdot \phi_k \mathrm{d}x \,\mathrm{d}\beta_k(s)\right\rangle\right\rangle_T\right]^{p/2} \\
&\le c\mathbb{E}\left[\sum_{k\ge 1}\int_0^T \left(\int_{\mathbb{T}^3} \mathbf{u}_\varepsilon \cdot \phi_k \mathrm{d}x\right)^2 \mathrm{d}s\right]^{p/2} \\
&\le c\mathbb{E}\left[\int_0^T \left(\int_{\mathbb{T}^3} |\mathbf{u}_\varepsilon|^2 \mathrm{d}x\right) \underbrace{\sum_{k\ge 1}\left(\int_{\mathbb{T}^3} |\phi_k|^2 \mathrm{d}x\right)}_{=\|\phi\|_{L_2(\mathscr{U},L^2)}} \mathrm{d}s\right]^{p/2} \\
&\le c_\phi \mathbb{E}\left[\int_0^T \int_{\mathbb{T}^3} |\mathbf{u}_\varepsilon|^2 \,\mathrm{d}x\mathrm{d}s\right]^{p/2}.
\end{aligned}$$

On the account of Young's inequality, for every $\delta > 0$ we infer

$$\mathbb{E}\left[\sup_{t\in(0,T)}\left|\int_0^t\int_{\mathbb{T}^3}\mathbf{u}_\varepsilon\phi \, \mathrm{d}W_s\right|\right]^p \le \delta\mathbb{E}\left[\sup_{t\in(0,T)}\int_{\mathbb{T}^3}|\mathbf{u}_\varepsilon|^2 \, \mathrm{d}x\mathrm{d}s\right]^p + c(\delta,T).$$

Finally, we obtain

$$\begin{aligned}
&\mathbb{E}\left[\frac{1}{2}\sup_{t\in(0,T)}\int_{\mathbb{T}^3}|\mathbf{u}_\varepsilon|^2 \, \mathrm{d}x + \varepsilon\int_0^T\int_{\mathbb{T}^3}|\nabla\mathbf{u}_\varepsilon|^2 \, \mathrm{d}x\mathrm{d}t\right]^p \qquad (3.1.51)\\
&\le \mathbb{E}\left[\frac{1}{2}\int_{\mathbb{T}^3}|\mathbf{u}_0|^2 \, \mathrm{d}x\right]^p + \delta\mathbb{E}\left[\sup_{t\in(0,T)}\int_{\mathbb{T}^3}|\mathbf{u}_\varepsilon|^2 \, \mathrm{d}x\right]^p \\
&+ c(\delta,T) + \mathbb{E}\left[\frac{1}{2}\sum_{k\ge1}\int_0^T\int_{\mathbb{T}^3}|\phi_k|^2 \, \mathrm{d}x\mathrm{d}t\right]^p.
\end{aligned}$$

The claim follows from using (3.1.2), assumptions on initial data and taking δ small enough. □

In infinite dimensional space, the best we can get is $\mathbf{u}_\varepsilon \rightharpoonup \mathbf{u}$ (weakly) as $\varepsilon \to 0$, but weak convergence is not sufficient to guarantee convergence of non-linear terms.

Compactness

To gain compactness we need to pass to the limit in the convective term. We consider $\varphi \in \mathscr{C}^\infty_{\mathrm{div}}(\mathbb{T}^3)$ and by Definition 3.1.1 we obtain

$$\begin{aligned}
\int_{\mathbb{T}^3}\mathbf{u}_\varepsilon\cdot\varphi \, \mathrm{d}x &= \int_{\mathbb{T}^3}\mathbf{u}_0\cdot\varphi \, \mathrm{d}x + \int_{\mathbb{T}^3}\int_0^t H_\varepsilon : \nabla\varphi \, \mathrm{d}x\mathrm{d}s \\
&+ \int_0^t\int_{\mathbb{T}^3}\phi\cdot\varphi \, \mathrm{d}x \, \mathrm{d}W_s, \qquad (3.1.52)
\end{aligned}$$

$\mathbb{P}$-a.s, where $H_\varepsilon := \mathbf{u}_\varepsilon\otimes\mathbf{u}_\varepsilon + \varepsilon\nabla\mathbf{u}_\varepsilon$. Using the a 'priori estimates' in Theorem 3.1.30 we deduce

$$H_\varepsilon \in L^1(\Omega, L^2(0,T;L^1(\mathbb{T}^3))), \qquad (3.1.53)$$

uniformly in ε. In following arguments we use the short-hand notations $(L^\infty_t, L^1_x) := L^\infty(0,T;L^1(\mathbb{T}^3))$, for convenience. The claim follows from using Proposition 3.1.11 and interpreting $|\mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon|$ as $|\mathbf{u}_\varepsilon|^2$ so that

$$\mathbb{E}\left[\sup_{t\in(0,T)} \int_{\mathbb{T}^3} |\mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon| \,\mathrm{d}x\right] \leq c,$$

which implies that $\mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon$ is bounded in $L^1_{\omega\in\Omega}(L^\infty_t, L^1_x)$. Similarly for $\varepsilon\nabla\mathbf{u}_\varepsilon$, we consider the functional:

$$\mathbb{E}\int_0^t \int_{\mathbb{T}^3} |\sqrt{\varepsilon}\nabla\mathbf{u}_\varepsilon|^2 \,\mathrm{d}x\mathrm{d}s = \varepsilon\mathbb{E}\int_0^t \int_{\mathbb{T}^3} |\nabla\mathbf{u}_\varepsilon|^2 \,\mathrm{d}x\mathrm{d}s \leq c,$$

so that $\varepsilon\nabla v_\varepsilon$ is bounded in $L^2_{\omega\in\Omega}(L^2_t(L^2_x))$, and we obtain the claim in (3.1.53) by taking the intersection of $L^2_{\omega\in\Omega}(L^2_t(L^2_x))$ and $L^1_{\omega\in\Omega}(L^\infty_t, L^1_x)$ spaces. In addition, as a result of (3.1.53) the following bound holds

$$\mathbb{E}\left[\int_0^T \|H_\varepsilon\|_{L^1_x} \,\mathrm{d}t\right] \leq c.$$

To bound the system in (3.1.52), we split it into two parts: the deterministic and stochastic part separately. For the deterministic part, we consider the functional

$$\mathscr{H}_\varepsilon(t,\varphi) := \int_0^t \int_{\mathbb{T}^3} H_\varepsilon : \nabla\varphi \,\mathrm{d}x\mathrm{d}s, \quad \varphi \in \mathscr{C}^\infty_{\mathrm{div}}(\mathbb{T}^3).$$

We observe that

$$\partial_t \mathscr{H}_\varepsilon(t,\varphi) \in L^1(\Omega; L^2(0,T; W^{-3,2}_{\mathrm{div}}(\mathbb{T}^3)),$$

uniformly in ε. By setting $\mathscr{H}(0) = 0$ at $t = 0$, and using Proposition 3.1.11 we deduce the estimate:

$$\mathbb{E}\left[\|\mathscr{H}_\varepsilon\|_{W^{1,2}([0,T];W^{-3,2}_{\mathrm{div}}(\mathbb{T}^3))}\right] \leq c.$$

By the embedding of $W^{1,2}(0,T) \hookrightarrow C^\alpha(0,T)$ for $\alpha := 1 - 1/2$ we obtain the estimate

$$\mathbb{E}\left[\|\mathscr{H}_\varepsilon\|_{C^\alpha([0,T];W^{-3,2}_{\mathrm{div}}(\mathbb{T}^3))}\right] \leq c,$$

where $c > 0$ is a constant. For the stochastic functional we apply Lemma 2.1.16 and relation (3.1.2) to deduce the estimate

$$\mathbb{E}\left\|\int_0^\cdot \phi \,\mathrm{d}W\right\|^p_{C^\beta(L^2(\mathbb{T}^3))} \leq c\,\mathbb{E}\int_0^T \|\phi\|^p_{L_2(\mathscr{U},L^2(\mathbb{T}^3))} \mathrm{d}t \leq c. \tag{3.1.54}$$

Remark 3.1.5. The space $W^{a,p}(0,T;L^2(\mathbb{T}^3))$ is continuously embedded into $C^\beta(0,T;L^2(\mathbb{T}^3))$, for any $\beta \in (0, a-1/p)$ and every $p > 2$.

Finally, combining the stochastic and deterministic terms above and upon taking the intersection of C^α and C^β for all $0 < \beta < \alpha < 1$ we infer that

$$\mathbb{E}\left[\|\mathbf{u}_\varepsilon\|_{C^\beta([0,T];W^{-3,2}_{\mathrm{div}}(\mathbb{T}^3))}\right] \leq c. \tag{3.1.55}$$

uniformly for all ε.

We seek to show tightness of the sequence of approximate solutions using the compact embeddings

$$\begin{aligned}
&C^\beta([0,T];W^{-3,2}_{\mathrm{div}}(\mathbb{T}^3)) \hookrightarrow\hookrightarrow C([0,T];W^{-4,2}_{\mathrm{div}}(\mathbb{T}^3)) \\
&C^\beta([0,T];W^{-3,2}_{\mathrm{div}}(\mathbb{T}^3)) \cap L^\infty(0,T;L^2_{\mathrm{div}}(\mathbb{T}^3)) \hookrightarrow\hookrightarrow C_w([0,T];L^2_{\mathrm{div}}(\mathbb{T}^3))
\end{aligned}$$

Accordingly, for $T > 0$ we consider the path space

$$\mathscr{X}_T = L^2_{\mathrm{div}}(\mathbb{T}^3) \otimes C([0,T];W^{-3,2}_{\mathrm{div}}(\mathbb{T}^3)) \cap C_w([0,T];L^2_{\mathrm{div}}(\mathbb{T}^3)) \otimes \mathbf{Y}_2(Q;\mathbb{R}^d) \otimes C([0,T];\mathscr{U}_0),$$

and introduce a restriction operator which we denote by $\mathbf{r}_t$ for some $t \in (0,T]$. Here $\mathbf{r}_t$ restricts measurable functions (or space-time distributions) defined on $(0,\infty)$ to $(0,T)$ with the following laws:

$$\begin{cases}
\mu_{\mathbf{r}_t \mathbf{u}_\varepsilon} \ \text{ is the law of } \ \mathbf{u}_\varepsilon \text{ on } C([0,T];W^{-3,2}_{0,\mathrm{div}}(\mathbb{T}^3)), \\
\mu_{\mathbf{r}_t W} \ \text{ is the law of } \ W \text{ on } C([0,T],\mathscr{U}_0), \text{ where } \mathscr{U}_0 \ \text{ is defined in (3.1.4)}, \\
\mu_{(\delta_{\mathbf{r}_t \mathbf{u}_\varepsilon},0,0) \in \mathbf{Y}_2(Q;\mathbb{R}^d)} \ \text{ is the law on } \ \mathbf{Y}_2(Q,\mathbb{R}^d),
\end{cases}$$

where $(\delta_{\mathbf{u}_\varepsilon}, 0, 0)$ is a generalised Young measure and is the same as $\mathscr{V}_\varepsilon$. To be precise

$$(\delta_{\mathbf{u}_\varepsilon}, 0, 0) = (\delta_{\mathbf{u}_\varepsilon}, \nu^\varepsilon_{t,x}, 0) = (\delta_{\mathbf{u}_\varepsilon}, 0, \lambda).$$

Denoting by $\mathscr{L}[\mathbf{u}_0, \mathbf{r}_t\mathbf{u}_\varepsilon, \mathbf{r}_t\mathscr{V}_\varepsilon, \mathbf{r}_t W]$ the law on $\mathscr{X}_T$, we observe that tightness on $\mathscr{X}_T$ implies tightness of $\mathscr{L}[\mathbf{u}_0, \mathbf{u}_\varepsilon, \mathscr{V}_\varepsilon, W]$ on $\mathscr{X}$. Fixing $T > 0$, we consider a ball B_R in the space

$$C^{\beta}(0,T;W^{-3,2}_{\mathrm{div}}(\mathbb{T}^3))\cap L^{\infty}(0,T;L^2_{div}(\mathbb{T}^3)),$$

and denote by B_R^C its complement. Applying the bounds in Proposition 3.1.11 and (3.1.55), and using Markov's inequality we deduce

$$\begin{aligned}
\mu_{\mathbf{r}_t\mathbf{u}_\varepsilon}(B_R^C) &= \mathbb{P}\left(\|\mathbf{r}_t\mathbf{u}_\varepsilon\|_{C^{\beta}(W^{-3,2}_{\mathrm{div}})}+\|\mathbf{r}_t\mathbf{u}_\varepsilon\|_{L^{\infty}(L^2)}\geq R\right)\\
&\leq \frac{\mathbb{E}}{R}\left[\|\mathbf{r}_t\mathbf{u}_\varepsilon\|_{C^{\beta}(W^{-l,2}_{\mathrm{div}})}+\|\mathbf{r}_t\mathbf{u}_\varepsilon\|_{L^{\infty}(L^2)}\right]\\
&\leq \frac{C}{R}.
\end{aligned}$$

For any $\gamma>0$ there is $R=R(\gamma)$ such that

$$\mu_{\mathbf{r}_t\mathbf{u}_\varepsilon}(B_R)\geq 1-\frac{\gamma}{3},$$

that is, the family of probability laws $\mu_{\mathbf{r}_t\mathbf{u}_\varepsilon}$ is tight by Definition 2.1.29. The law of μ_W is a Radon measure on the Polish space $C([0,T],\mathscr{U}_0)$, and therefore it is tight. This implies that there exists a compact set $C_\gamma\subset C([0,T],\mathscr{U}_0)$ so that $\mu_W(C_\gamma)>1-\gamma/3$. We set $\mathscr{V}_\varepsilon=(\delta_{\mathbf{u}_\varepsilon},0,0)\in\mathbf{Y}_2(Q,\mathbb{R}^d)$ to be the generalised Young measure associated with $\mathbf{u}_\varepsilon$. Arguing similarly as above we have that for a ball $B_R\in L^{\infty}((0,T);L^2(\mathbb{T}^3))$ we obtain

$$\mu_{\mathbf{r}_t\mathbf{u}_\varepsilon}(B_{R(\gamma)})\geq 1-\frac{\gamma}{3},$$

for some $R=R(\gamma)$. Now recalling (2.1.20) and we observe

$$L^2(Q_T)\ni\mathbf{u}_\varepsilon\mapsto(\delta_{\mathbf{u}_\varepsilon},0,0)\in\mathbf{Y}_2(Q_T).$$

Consequently, the law $\mu_{\mathbf{r}_t\mathscr{V}_\varepsilon}=\mu_{\mathbf{r}_t(\delta_{\nu_\varepsilon},0,0)}$ is tight by Definition 2.1.29 in the same space. Furthermore, the law $\mu_{\mathbf{u}_0}$ is a Radon measure on the Polish space $L^2_{\mathrm{div}}(\mathbb{T}^3)$ and as such $\mu_{\mathbf{u}_0}$ is tight . Accordingly, $\mathscr{L}[\mathbf{u}_0,\mathbf{r}_t\mathbf{u}_\varepsilon,\mathbf{r}_t\mathscr{V}_\varepsilon,\mathbf{r}_tW]$ is tight on $\mathscr{X}_T$. Noting that T was arbitrary chosen we infer that $\mathscr{L}[\mathbf{u}_0,\mathbf{u}_\varepsilon,\mathscr{V}_\varepsilon,W]$ is tight on $\mathscr{X}$. Finally, we apply Lemma 2.1.19 and Jakubowski's version of the representation Theorem 2.1.21, see [64], to deduce the following proposition (we refer to [82],Theorem A.1, for results that combine Prokhorov's and Skorokhod's theorem for quasi-Polish spaces). In particular, one first has to replace the family of random variables indexed by ε by a countable sub-family.

We note that the law of $(\tilde{\mathbf{u}}_0^{\varepsilon_m}, \tilde{\mathbf{u}}_{\varepsilon_m}, \tilde{\mathscr{V}}_{\varepsilon_m}, \tilde{W}_{\varepsilon_m}), m \in \mathbb{N}$ is a sequence of tight measures on $(\mathscr{X}, \mathscr{B}_{\mathscr{X}})$. Consequently, its weak* limit is tight as well and hence Radon.

Proposition 3.1.12. *There exists a nullsequence $(\varepsilon_m)_{m\in\mathbb{N}}$, a complete probability space $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{\mathbb{P}})$ with $(\mathscr{X}, \mathscr{B}_{\mathscr{X}})$-valued random variables*

$$(\tilde{\mathbf{u}}_0^{\varepsilon_m}, \tilde{\mathbf{u}}_{\varepsilon_m}, \tilde{\mathscr{V}}_{\varepsilon_m}, \tilde{W}_{\varepsilon_m}), m \in \mathbb{N}, \text{ and } (\tilde{\mathbf{u}}_0, \tilde{\mathbf{u}}, \tilde{\mathscr{V}}, \tilde{W})$$

such that

(a) *For all $m \in \mathbb{N}$ the law of $(\tilde{\mathbf{u}}_0^{\varepsilon_m}, \tilde{\mathbf{u}}_{\varepsilon_m}, \tilde{\mathscr{V}}_{\varepsilon_m}, \tilde{W}_{\varepsilon_m})$ on $\mathscr{X}$ is given by $\mathscr{L}[\mathbf{u}_0^{\varepsilon_m}, \mathbf{u}_{\varepsilon_m}, \mathscr{V}_{\varepsilon_m}, W_{\varepsilon_m}]$;*

(b) *The law of $(\tilde{\mathbf{u}}_0, \tilde{\mathbf{u}}, \tilde{\mathscr{V}}, \tilde{W})$ is a Radon measure on $\mathscr{X}, \mathscr{B}_{\mathscr{X}}$;*

(c) *$(\tilde{\mathbf{u}}_0^{\varepsilon_m}, \tilde{\mathbf{u}}_{\varepsilon_m}, \tilde{\mathscr{V}}_{\varepsilon_m}, \tilde{W}_{\varepsilon_m})$ converges $\tilde{\mathbb{P}}$-almost surely to $(\tilde{\mathbf{u}}_0, \tilde{\mathbf{u}}, \tilde{\mathscr{V}}, \tilde{W})$ in the topology of $\mathscr{X}$, i.e.*

$$\begin{aligned}
&\tilde{\mathbf{u}}_0^{\varepsilon_m} \to \tilde{\mathbf{u}}_0 \text{ in } \quad L^2(\mathbb{T}^3)\, \tilde{\mathbb{P}}\text{-a.s.};\\
&\tilde{\mathbf{u}}_{\varepsilon_m} \to \tilde{\mathbf{u}} \text{ in } \quad C_{\mathrm{loc}}([0,\infty); W_{\mathrm{div}}^{-4,2}(\mathbb{T}^3))\, \tilde{\mathbb{P}}\text{-a.s.};\\
&\tilde{\mathbf{u}}_{\varepsilon_m} \to \tilde{\mathbf{u}} \text{ in } \quad C_{w,\mathrm{loc}}([0,\infty); L^2_{\mathrm{div}}(\mathbb{T}^3));\\
&\tilde{\mathscr{V}}_{\varepsilon_m} \rightharpoonup \tilde{\mathscr{V}} \text{ in } \quad \mathbf{Y}_2^{\mathrm{loc}}(Q_\infty)\, \tilde{\mathbb{P}}\text{-a.s.}\\
&\tilde{W}_{\varepsilon_m} \to \tilde{W} \text{ in } \quad C([0,\infty); \mathscr{U}_0)\, \tilde{\mathbb{P}}\text{-a.s.}
\end{aligned} \tag{3.1.56}$$

Some remarks are in order, we henceforth adopt the analogy

$$\tilde{\mathbf{u}}_{\varepsilon_m}(t,x) = \langle \tilde{\mathscr{V}}_{t,x}^{\varepsilon_m}, \xi \rangle, \quad \tilde{\mathbf{u}}(t,x) = \langle \tilde{\mathscr{V}}_{t,x}, \xi \rangle\, \mathbb{P}\text{-a.s.}, \tag{3.1.57}$$

where $\tilde{\mathscr{V}}_\varepsilon = (\tilde{\nu}_{t,x}^{\varepsilon_m}, \tilde{\nu}_{t,x}^{\infty,\varepsilon_m}, \tilde{\lambda}_{\varepsilon_m})$ and $\tilde{\mathscr{V}} = (\tilde{\nu}_{t,x}, \tilde{\nu}_{t,x}^\infty, \tilde{\lambda})$. The first part of the assertion in (3.1.57) follows from the observation: for $T > 0$ and $\psi \in C_c^\infty(Q_T)$ we consider a continuous mapping on the path space given by

$$(\mathbf{w}, \mathscr{V}) \mapsto \int_{Q_T} (\mathbf{w} - \langle \nu_{(t,x)}, \xi \rangle) \cdot \psi \,\mathrm{d}x\mathrm{d}t.$$

In view of Proposition 3.1.12 we deduce

$$\int_{Q_T} (\tilde{\mathbf{u}}_{\varepsilon_m} - \langle \tilde{\nu}_{t,x}^{\varepsilon_m}, \xi \rangle) \cdot \psi \,\mathrm{d}x\mathrm{d}t \sim^d \int_{Q_T} (\mathbf{u}_{\varepsilon_m} - \langle \nu_{t,x}^{\varepsilon_m}, \xi \rangle) \cdot \psi \,\mathrm{d}x\mathrm{d}t = 0,$$

for arbitrary ψ and T. Note the second part in the assertion (3.1.57) follows applying Proposition 3.1.12 and passing to the limit $m \to \infty$. Similarly, for any $T > 0$ we can chose $f \in \mathscr{G}_2(Q_T)$ and $\varphi \in C(\overline{Q}_T)$ arbitrary the mappings

$$(\mathbf{w}, \mathscr{V}) \mapsto \int_{Q_T} \varphi \langle \nu_{t,x} - \delta_{\mathbf{w}(t,x)}, f(\xi) \rangle \, \mathrm{d}x\mathrm{d}t + \int_{Q_T} \varphi \langle \nu^{\infty}_{t,x}, f^{\infty}(\xi) \rangle \, \mathrm{d}\lambda,$$

to show that

$$\tilde{\mathscr{V}}_{\varepsilon} = (\tilde{\nu}^{\varepsilon_m}_{t,x}, \tilde{\nu}^{\infty,\varepsilon_m}_{t,x}, \tilde{\lambda}_{\varepsilon_m}) = (\delta_{\tilde{\mathbf{u}}_{\varepsilon_m}(t,x)}, 0, 0) \quad \text{for a.a. } (t,x) \in Q_\infty.$$

Accordingly, to ensure accurate measurability of the new random variables we adapt a filtration to the new probability space. Now, let $(\tilde{\mathscr{F}}^{\varepsilon}_t)_{t \geq 0}$ and $(\tilde{\mathscr{F}}_t)_{t \geq 0}$ denote the $\tilde{\mathbb{P}}$-augmented canonical filtration of the process $(\tilde{\mathbf{u}}^{\varepsilon_m}_0, \tilde{\mathbf{u}}_{\varepsilon}, \tilde{\mathscr{V}}_{\varepsilon_m} \tilde{W}_{\varepsilon_m})$ and $(\tilde{\mathbf{u}}_0, \tilde{\mathbf{u}}, \tilde{\mathscr{V}}, \tilde{W})$, respectively, i.e.

$$\tilde{\mathscr{F}}^{\varepsilon}_t = \sigma\left(\sigma(\tilde{\mathbf{u}}^{\varepsilon_m}_0, \mathbf{r}_t \tilde{\mathbf{u}}_{\varepsilon_m}, \mathbf{r}_t \tilde{\mathscr{V}}_{\varepsilon_m}, \mathbf{r}_t \tilde{W}_{\varepsilon_m}) \cup \{\mathscr{N} \in \tilde{\mathscr{F}};\ \tilde{\mathbb{P}}(\mathscr{N}) = 0\}\right), \quad t \geq 0.$$

$$\tilde{\mathscr{F}}_t = \sigma\left(\sigma(\tilde{\mathbf{u}}_0, \mathbf{r}_t \tilde{\mathbf{u}}, \mathbf{r}_t \tilde{\mathscr{V}}, \mathbf{r}_t \tilde{W}) \cup \{\mathscr{N} \in \tilde{\mathscr{F}};\ \tilde{\mathbb{P}}(\mathscr{N}) = 0\}\right), \quad t \geq 0,$$

where σ_t denotes the history of a random distribution as defined in (2.1.6), and note here we identify generalised Young measures as random distribution in the sense of (2.1.21). The setup above guarantees that the stochastic processes are adapted enabling us to define stochastic integrals.

New Probability Space

Using the elementary method covered in [27], and arguing as in Section 3.1.2 we proceed to show that the approximated equations

$$\begin{aligned}\int_{\mathbb{T}^3} \mathbf{u}(t) \cdot \varphi \, \mathrm{d}x = \int_{\mathbb{T}^3} \mathbf{u}(0) \cdot \varphi \, \mathrm{d}x + \int_0^t \int_{\mathbb{T}^3} \langle \nu_{t,x}, \xi \otimes \xi \rangle : \nabla \varphi \, \mathrm{d}x \, \mathrm{d}s \\ + \int_{(0,t) \times \mathbb{T}^3} \langle \nu^{\infty}_{t,x}, \xi \otimes \xi \rangle : \nabla \varphi \, \mathrm{d}\lambda - \varepsilon_m \int_0^t \int_{\mathbb{T}^3} \mathbf{u}(t) \cdot \Delta \varphi \mathrm{d}x \mathrm{d}s \\ + \int_{\mathbb{T}^3} \int_0^t \varphi \cdot \phi \, \mathrm{d}W \, \mathrm{d}x,\end{aligned}$$

continue to hold on the new probability space. In order to do this for all $t \in [0,T]$ and $\varphi \in C^{\infty}_{\mathrm{div}}(\mathbb{T}^3)$ we consider the following functionals:

$$\begin{aligned}\mathscr{M}^{\varepsilon_m}(\mathbf{u}_0,\mathbf{u},\mathscr{V})_t &= \int_{\mathbb{T}^3}(\mathbf{u}(t)-\mathbf{u}_0)\cdot\varphi\,\mathrm{d}x + \varepsilon_m\int_0^t\int_{\mathbb{T}^3}\mathbf{u}(t)\cdot\Delta\varphi\mathrm{d}x\mathrm{d}s\\ &\quad -\int_0^t\int_{\mathbb{T}^3}\langle\nu_{t,x},\xi\otimes\xi\rangle:\nabla\varphi\,\mathrm{d}x\,\mathrm{d}s - \int_{(0,t)\times\mathbb{T}^3}\langle\nu^{\infty}_{t,x},\xi\otimes\xi\rangle:\nabla\varphi\,\mathrm{d}\lambda\end{aligned}$$

$$\Psi_t = \sum_{k=1}\int_0^t\left(\int_{\mathbb{T}^3}\phi_k\cdot\varphi\,\mathrm{d}x\right)^2\mathrm{d}s, \quad (\Psi_k)_t = \int_0^t\int_{\mathbb{T}^3}\phi_k\cdot\varphi\,\mathrm{d}x\mathrm{d}s.$$

Next, we proceed to show that the functional $\mathscr{M}^{\varepsilon_m}(\mathbf{u}_0,\mathbf{u},\mathscr{V})_t$ is well defined. Accordingly, it suffices to that show that the linear functional is continuous. For terms with Banach spaces it suffices to show that the terms are bounded. Note Proposition 3.1.11 yields the boundedness of first two terms on $\mathscr{M}^{\varepsilon_m}(\mathbf{u},\mathbf{u}_0,\mathscr{V})_t$ in $W^{-4,2}_{\mathrm{div}}(\mathbb{T}^3)$, for instance

$$\begin{aligned}\left|\varepsilon_m\int_0^t\int_{\mathbb{T}^3}\mathbf{u}\cdot\Delta\varphi\,\mathrm{d}x\mathrm{d}s\right| &\le \varepsilon_m\int_0^t\int_{\mathbb{T}^3}|\mathbf{u}\Delta\varphi|\,\mathrm{d}x\mathrm{d}s\\ &\le \varepsilon\int_0^t\|\mathbf{u}\|_{W^{-4,2}(\mathbb{T}^3)}\|\Delta\varphi\|_{W^{4,2}(\mathbb{T}^3)}\,\mathrm{d}s\\ &\le c(\varepsilon_m,\varphi,T)\sup_t\|\mathbf{u}(t)\|_{W^{-4,2}(\mathbb{T}^3)}.\end{aligned}$$

Accordingly, for terms with measures, we refer the reader to the discussion of general Young measures in $\mathbf{Y}_2(Q)$; Section 2.1.5. Indeed, $\mathscr{M}^{\varepsilon_m}(\mathbf{u},\mathbf{u}_0,\mathscr{V})_t$ is continuous on the path space. Now let $\mathscr{M}(\tilde{\mathbf{u}}_0^{\varepsilon_m},\tilde{\mathbf{u}}_{\varepsilon_m},\tilde{\mathscr{V}}_{\varepsilon_m})_{s,t}$ denote the increment $\mathscr{M}(\tilde{\mathbf{u}}_0^{\varepsilon_m},\tilde{\mathbf{u}}_{\varepsilon_m},\tilde{\mathscr{V}}_{\varepsilon})_t - \mathscr{M}(\tilde{\mathbf{u}}_0^{\varepsilon_m},\tilde{\mathbf{u}}_{\varepsilon_m},\tilde{\mathscr{V}}_{\varepsilon_m})_s$ and similarly for $\Psi_{s,t}$ and $(\Psi_k)_{s,t}$. In the new probability space, completeness of proof follows from showing that deterministic part is equivalent to the stochastic part, that is,

$$\mathscr{M}(\tilde{\mathbf{u}}_0,\tilde{\mathbf{u}}_{\varepsilon_m},\tilde{\mathscr{V}}_{\varepsilon_m}) = \int_0^t\int_{\mathbb{T}^3}\phi\cdot\varphi\,\mathrm{d}x\mathrm{d}\tilde{W}_s^{\varepsilon_m}, \tag{3.1.58}$$

and passage to the limit $m\to\infty$. To show (3.1.58) holds, it suffices to identify the corresponding quadratic and cross variations of the $\tilde{\mathscr{F}}_t^{\varepsilon}$-martingale process $\mathscr{M}(\tilde{\mathbf{u}}_0,\tilde{\mathbf{u}}_{\varepsilon_m},\tilde{\mathscr{V}}_{\varepsilon_m})$ as follows:

$$\langle\langle\mathscr{M}(\tilde{\mathbf{u}}_0,\tilde{\mathbf{u}}_{\varepsilon_m},\tilde{\mathscr{V}}_{\varepsilon_m})\rangle\rangle = \Psi_t, \qquad \langle\langle\mathscr{M}(\tilde{\mathbf{u}}_0,\tilde{\mathbf{u}}_{\varepsilon_m},\tilde{\mathscr{V}}_{\varepsilon_m}),\tilde{\beta}_k\rangle\rangle = (\Psi_k)_t, \tag{3.1.59}$$

respectively. In particular, (3.1.59) yields the assertion

$$\left\langle\!\left\langle \mathscr{M}(\tilde{\mathbf{u}}_0, \tilde{\mathbf{u}}_{\varepsilon_m}, \tilde{\mathscr{V}}_{\varepsilon_m}) - \int_0^t \int_{\mathbb{T}^3} \phi \cdot \varphi \, \mathrm{d}x \mathrm{d}\tilde{W}_s^{\varepsilon_m} \right\rangle\!\right\rangle = 0, \tag{3.1.60}$$

which implies the desired equation holds on the new probability space provided (3.1.59) holds. Now we proceed to justify that indeed (3.1.59) holds true. To begin with, on the account of proposition 3.1.8, the mapping

$$(\mathbf{u}_0, \mathbf{u}, \mathscr{V}) \mapsto \mathscr{M}(\mathbf{u}_0, \mathbf{u}, \mathscr{V}),$$

is well-defined and continuous on the path space. Consequently, in view of Lemma 3.1.8 we obtain

$$\mathscr{M}(\mathbf{u}_0^{\varepsilon_m}, \mathbf{u}_{\varepsilon_m}, \mathscr{V}_{\varepsilon_m}) \sim^d \mathscr{M}(\tilde{\mathbf{u}}_0^{\varepsilon_m}, \tilde{\mathbf{u}}_{\varepsilon_m}, \tilde{\mathscr{V}}_{\varepsilon}).$$

Next, we fix times $s, t \in [0, T]$, where $s < t$ and define a continuous function h such that

$$h : V|_{[0,s]} \rightarrow [0, 1].$$

The process

$$\mathscr{M}(\mathbf{u}_0, \mathbf{u}_{\varepsilon_m}, \mathscr{V}_{\varepsilon_m}) = \int_0^t \int_{\mathbb{T}^3} \phi \cdot \varphi \, \mathrm{d}x \mathrm{d}W_s^{\varepsilon_m} = \sum_{k \geq 1} \int_0^t \int_{\mathbb{T}^3} \phi_k \cdot \varphi \, \mathrm{d}x \, \mathrm{d}\beta_k^{\varepsilon_m},$$

is a square integrable $(\mathscr{F}_t)_{t \geq 0}$-martingale, using Theorem 2.1.9 we deduce that

$$[\mathscr{M}(\mathbf{u}_0, \mathbf{u}_{\varepsilon_m}, \mathscr{V}_{\varepsilon_m})]^2 - \Psi_t, \qquad \mathscr{M}(\mathbf{u}_0, \mathbf{u}_{\varepsilon_m}, \mathscr{V}_{\varepsilon_m})\beta_k - (\Psi_k)_t,$$

are $(\mathscr{F}_t)_{t \geq 0}$-martingales. Now let $\mathbf{z}_s$ be a restriction of a function to the interval $[0, s]$. Accordingly, in view of equality laws in Proposition 3.1.12 we infer

$$\begin{aligned} &\tilde{\mathbb{E}}[h(\mathbf{u}_0^{\varepsilon_m}, \mathbf{z}_s \tilde{\mathbf{u}}_{\varepsilon_m}, \mathbf{z}_s \tilde{\mathscr{V}}_{\varepsilon_m}, \mathbf{z}_s \tilde{W} \varepsilon_m) \mathscr{M}(\mathbf{u}_0^{\varepsilon_m}, \tilde{\mathbf{u}}_{\varepsilon_m}, \tilde{\mathscr{V}}_{\varepsilon_m})_{s,t}] \\ &\quad = \mathbb{E}[h(\mathbf{u}_0^{\varepsilon_m}, \mathbf{z}_s \mathbf{u}_{\varepsilon_m}, \mathbf{z}_s \mathscr{V}_{\varepsilon_m}, \mathbf{z}_s W_{\varepsilon_m}) \mathscr{M}(\mathbf{u}_0^{\varepsilon_m}, \mathbf{u}_{\varepsilon_m}, \mathscr{V}_{\varepsilon_m})_{s,t}] = 0, \end{aligned} \tag{3.1.61}$$

$$\tilde{\mathbb{E}}[h(\mathbf{u}_0^{\varepsilon_m}, \mathbf{z}_s \tilde{\mathbf{u}}_{\varepsilon_m}, \mathbf{z}_s \tilde{\mathscr{V}}_{\varepsilon_m}, \mathbf{z}_s \tilde{W} \varepsilon_m)([\mathscr{M}(\mathbf{u}_0^{\varepsilon_m}, \tilde{\mathbf{u}}_{\varepsilon_m}, \tilde{\mathscr{V}}_{\varepsilon_m})]^2 - \Psi)_{s,t}] \tag{3.1.62}$$

$$= \mathbb{E}[h(\mathbf{u}_0^{\varepsilon_m}, \mathbf{z}_s\mathbf{u}_{\varepsilon_m}, \mathbf{z}_s\mathscr{V}_{\varepsilon_m}, \mathbf{z}_s W_{\varepsilon_m})([\mathscr{M}(\mathbf{u}_0^{\varepsilon_m}, \mathbf{u}_{\varepsilon_m}, \mathscr{V}_{\varepsilon_m})]^2 - \Psi)_{s,t}])] = 0,$$

$$\tilde{\mathbb{E}}[h(\mathbf{u}_0^{\varepsilon_m}, \mathbf{z}_s\tilde{\mathbf{u}}_{\varepsilon_m}, \mathbf{z}_s\tilde{\mathscr{V}}_{\varepsilon_m}, \mathbf{z}_s\tilde{W}\varepsilon_m)(\mathscr{M}(\mathbf{u}_0^{\varepsilon_m}, \tilde{\mathbf{u}}_{\varepsilon_m}, \tilde{\mathscr{V}}_{\varepsilon_m})\beta_k - (\Psi_k))_{s,t})] \tag{3.1.63}$$
$$= \mathbb{E}[h(\mathbf{u}_0^{\varepsilon_m}, \mathbf{z}_s\mathbf{u}_{\varepsilon_m}, \mathbf{z}_s\mathscr{V}_{\varepsilon_m}, \mathbf{z}_s W_{\varepsilon_m})(\mathscr{M}(\mathbf{u}_0^{\varepsilon_m}, \mathbf{u}_{\varepsilon_m}, \mathscr{V}_{\varepsilon_m})\beta_k - (\Psi_k))_{s,t}] = 0.$$

Therefore, (3.1.59) holds and (3.1.60) follows as a consequence. To identify the equation of momentum in the sense of (3.1.46) in the new probability space we use the convergences from Proposition 3.1.12 and the higher moments from (3.1.11) to pass to the limit in (3.1.61)–(3.1.63). To be precise, we apply Vitali's convergence Theorem 2.1.15 in (3.1.61)–(3.1.63) and proceed as follows. To begin with, the uniform integrability of the $(\mathscr{F}_t)_{t\geq 0}$-martingale follows from noting that

$$\begin{aligned}\mathbb{E}|(\mathscr{M}(\mathbf{u}_0^{\varepsilon_m}, \tilde{\mathbf{u}}_{\varepsilon_m}, \tilde{\mathscr{V}}_{\varepsilon_m})^2 - \Psi)_{s,t}|^p \quad &\leq \quad c\,\mathbb{E}\mathscr{M}(\mathbf{u}_0^{\varepsilon_m}, \tilde{\mathbf{u}}_{\varepsilon_m}, \tilde{\mathscr{V}}_{\varepsilon_m})_s^{2p} + c\,\mathbb{E}\Psi_s^p \\ &\quad +c\,\mathbb{E}\mathscr{M}(\mathbf{u}_0^{\varepsilon_m}, \tilde{\mathbf{u}}_{\varepsilon_m}, \tilde{\mathscr{V}}_{\varepsilon_m})_t^{2p} + c\,\mathbb{E}\Psi_t^p,\end{aligned}$$

for some $p > 1$. Applying Lemma 2.1.16 and 3.1.58 we deduce

$$\begin{aligned}\mathbb{E}|\mathscr{M}(\mathbf{u}_0^{\varepsilon_m}, \mathbf{u}_{\varepsilon_m}, \mathscr{V}_{\varepsilon_m})|^{2p} \quad &= \quad \mathbb{E}\left|\int_{\mathbb{T}^3}\varphi\int_0^t \phi\,\mathrm{d}W\,\mathrm{d}x\right|^{2p} \\ &= \quad \mathbb{E}\left|\left\langle\varphi, \int_0^t \phi\,\mathrm{d}W\right\rangle_{L^2}\right|^{2p} \\ &\leq \quad \|\varphi\|_2^{2p}\mathbb{E}\left\|\int_0^t \phi\,\mathrm{d}W\right\|_2^{2p} \\ &\leq \quad c\,\mathbb{E}\int_0^T \|\phi\|_{L^2(\mathscr{U},L^2(\mathbb{T}^3))}^{2p}\mathrm{d}s.\end{aligned}$$

Accordingly, we obtain $\mathbb{E}|\mathscr{M}(\mathbf{u}_0^{\varepsilon_m}, \mathbf{u}_{\varepsilon_m}, \mathscr{V}_{\varepsilon_m})|^{2p} < c$, by equality of laws in Proposition 3.1.12 we have $\tilde{\mathbb{E}}|\mathscr{M}(\tilde{\mathbf{u}}_0^{\varepsilon_m}, \tilde{\mathbf{u}}_{\varepsilon_m}, \tilde{\mathscr{V}}_{\varepsilon_m})|^{2p} < c$ such that

$$\tilde{\mathbb{E}}\left|[h(\mathbf{u}_0^{\varepsilon_m}, \mathbf{z}_s\tilde{\mathbf{u}}_{\varepsilon_m}, \mathbf{z}_s\tilde{\mathscr{V}}_{\varepsilon_m}, \mathbf{z}_s\tilde{W}\varepsilon_m)([\mathscr{M}(\mathbf{u}_0^{\varepsilon_m}, \tilde{\mathbf{u}}_{\varepsilon_m}, \tilde{\mathscr{V}}_{\varepsilon_m})]^2 - \Psi)_{s,t}]\right|^{2p} \leq C.$$

Using Theorem 2.1.15 and Proposition 3.1.12 together with continuity of $\mathscr{M}(\mathbf{u}_0^{\varepsilon_m}, \tilde{\mathbf{u}}_{\varepsilon_m}, \tilde{\mathscr{V}}_{\varepsilon_m})$ to pass to the limit $m \to \infty$ we infer

$$\tilde{\mathbb{E}}[h(\mathbf{u}_0^{\varepsilon_m}, \mathbf{z}_s\tilde{\mathbf{u}}_{\varepsilon_m}, \mathbf{z}_s\tilde{\mathscr{V}}_{\varepsilon_m}, \mathbf{z}_s\tilde{W}\varepsilon_m)([\mathscr{M}(\mathbf{u}_0^{\varepsilon_m}, \tilde{\mathbf{u}}_{\varepsilon_m}, \tilde{\mathscr{V}}_{\varepsilon_m})]^2 - \Psi)_{s,t}] \to 0.$$

We note upon taking the limit $m \to \infty$ the term $\varepsilon_m \int_0^t \int_{\mathbb{T}^3} \tilde{\mathbf{u}}_{\varepsilon_m} \cdot \Delta\varphi \,\mathrm{d}x\mathrm{d}s$ vanishes. To show this, we consider the following:

$$\mathbb{E}\left[\varepsilon_m \int_0^t \int_{\mathbb{T}^3} \tilde{\mathbf{u}}_\varepsilon \cdot \Delta\varphi \,\mathrm{d}x\mathrm{d}s\right] \quad \leq \quad \varepsilon_m c(\varphi) \underbrace{\mathbb{E}\left(\|\tilde{\mathbf{u}}_\varepsilon\|_{L^2(Q)}\right)}_{\leq c},$$

since $\varepsilon_m \to 0$ as $m \to \infty$, taking the limit yields

$$\begin{aligned}\int_{\mathbb{T}^3} \tilde{\mathbf{u}}(t) \cdot \varphi \,\mathrm{d}x = \int_{\mathbb{T}^3} \tilde{\mathbf{u}}(0) \cdot \varphi \,\mathrm{d}x + \int_0^t \int_{\mathbb{T}^3} \langle \tilde{\mathbf{v}}_{t,x}, \xi \otimes \xi \rangle : \nabla\varphi \,\mathrm{d}x\,\mathrm{d}s \\ + \int_{(0,t)\times\mathbb{T}^3} \langle \tilde{\mathbf{v}}^\infty_{t,x}, \xi \otimes \xi \rangle : \nabla\varphi \,\mathrm{d}\tilde{\lambda} + \int_{\mathbb{T}^3} \int_0^t \varphi \cdot \phi \,\mathrm{d}\tilde{W} \,\mathrm{d}x \end{aligned} \tag{3.1.64}$$

$\mathbb{P}$-a.s, as desired.

Finally, we aim to show the energy inequality in the sense of (3.1.47) continues to hold in the new probability space. We begin by introducing the abbreviations

$$\mathfrak{M}_t^{\varepsilon_m} = \int_0^t \mathbf{u} \cdot \phi \,\mathrm{d}x\mathrm{d}W^{\varepsilon_m} \quad \tilde{\mathfrak{M}}_t^{\varepsilon_m} = \int_0^t \tilde{\mathbf{u}} \cdot \phi \,\mathrm{d}x\mathrm{d}\tilde{W}^{\varepsilon_m},$$

for the stochastic integrals. Accordingly, in the Navier-Stokes system (3.1.22) (i.e. the original probability space) for $E_t^{\varepsilon_m} = \frac{1}{2}\int_{\mathbb{T}^3} |\mathbf{u}_{\varepsilon_m}|^2 \,\mathrm{d}x$ we infer

$$E_t^{\varepsilon_m} \leq E_s^{\varepsilon_m} + \frac{1}{2}\int_s^t \|\phi\|^2_{L_2(\mathscr{U},L^2(\mathbb{T}^3))} \,\mathrm{d}\sigma + \mathfrak{M}_t^{\varepsilon_m} - \mathfrak{M}_s^{\varepsilon_m}$$

for a.a. s (including $s = 0$) and all $t \geq s$. Fixing s we re-write the formulation above in the form

$$\begin{aligned} &-\int_s^\infty \partial_t \varphi E_t \,\mathrm{d}t - \varphi(s) E_s^{\varepsilon_m} \\ &\leq \frac{1}{2}\int_s^\infty \varphi \|\phi\|_{L_2(\mathscr{U},L^2(\mathbb{T}^3))} \,\mathrm{d}t + \int_s^\infty \varphi \int_{\mathbb{T}^3} \mathbf{u}_{\varepsilon_m} \cdot \phi \,\mathrm{d}x \,\mathrm{d}W_{\varepsilon_m} \end{aligned}$$

$\mathbb{P}$-a.s. for all $\varphi \in C_c^\infty([s,\infty);[0,\infty))$. On the account of Proposition 3.1.2 and 3.1.12 the energy inequality continues to hold in the new probability space and we obtain

$$\tilde{E}_t^{\varepsilon_m} \leq \tilde{E}_s^{\varepsilon_m} + \frac{1}{2}\int_s^t \|\phi\|^2_{L_2(\mathscr{U},L^2(\mathbb{T}^3))} \,\mathrm{d}\sigma + \tilde{\mathfrak{M}}_t^{\varepsilon_m} - \tilde{\mathfrak{M}}_s^{\varepsilon_m}$$

for a.a. s (including $s=0$) and all $t \geq s$. We note averaging in t and s yields

$$\begin{aligned}\frac{1}{r}\int_{t-r}^{t}\tilde{E}_k^{\varepsilon_m}\,\mathrm{d}k \quad \leq \quad & \frac{1}{r}\int_{t-r}^{t}\tilde{E}_\tau^{\varepsilon_m}\,\mathrm{d}\tau + \frac{1}{r}\int_{s-r}^{s}\int_{t-r}^{t}\left(\frac{1}{2}\int_\tau^t \|\phi\|^2_{L_2(\mathscr{U},L^2(\mathbb{T}^3))}\,\mathrm{d}\sigma\right)\mathrm{d}k\,\mathrm{d}\tau \\ & + \frac{1}{r}\int_{s-r}^{s}\int_{t-r}^{t}(\tilde{\mathfrak{M}}_t^{\varepsilon_m} - \tilde{\mathfrak{M}}_\tau^{\varepsilon_m})\,\mathrm{d}k\,\mathrm{d}\tau \qquad (3.1.65)\end{aligned}$$

provided $s>0$ and $r<\min\{s,t-s\}$. The setup above allows us to show our aim. We proceed to show the energy inequality holds by considering two scenarios: the case when $s>0$ and the case $s=0$. In the case of $s>0$, we first pass the limit in m and then in r. We note the additional integrals in (3.1.65) guarantee that our energy is continuous on the path space. Consequently, applying Proposition 3.1.12 yields the $\mathbb{P}$-a.s. expected convergences of energy terms. For instance in the noise term to show that as $m\to\infty$ we obtain

$$\tilde{\mathfrak{M}}_t^{\varepsilon_m} \to \tilde{\mathfrak{M}}_t := \int_0^t \tilde{\mathbf{u}}\cdot\phi\,\mathrm{d}x\mathrm{d}\tilde{W} \text{ in } L^2_{\mathrm{loc}}([0,\infty)), \qquad (3.1.66)$$

in probability the convergence follows from applying Lemma 2.1.14. In order to apply Lemma 2.1.14 we use $(3.1.56)_5$ for convergence of Wiener process, and in addition we seek

$$\int_{\mathbb{T}^3}\mathbf{u}_{\varepsilon_m}\cdot\phi\,\mathrm{d}x \to \int_{\mathbb{T}^3}\mathbf{u}\cdot\phi\,\mathrm{d}x \quad \text{in} \quad L^2_{\mathrm{loc}}([0,\infty);L_2(\mathscr{U},\mathbb{T}^3)) \qquad (3.1.67)$$

in probability. Indeed, (3.1.67) holds $\mathbb{P}$-a.s. on the account of $(3.1.56)_3$, that is,

$$\int_{\mathbb{T}^3}\mathbf{u}_{\varepsilon_m}\cdot\phi\,\mathrm{d}x \to \int_{\mathbb{T}^3}\mathbf{u}\cdot\phi\,\mathrm{d}x \quad \text{in} \quad L_2(\mathscr{U},\mathbb{T}^3))$$

for all $t\geq 0$. Consequently, using Proposition (3.1.11) yields convergence in

$$L^2(\tilde{\Omega};L_2(\mathscr{U};\mathbb{T}^3))$$

using higher moments. Therefore, applying Proposition (3.1.11) yields (3.1.67) (to be precise, we have an $L^2(\tilde{\Omega})$-convergence). Finally, passing to the limit in (3.1.65) (first in m and then in r) yields

$$\tilde{E}_t \leq \tilde{E}_s + \frac{1}{2}\int_s^t \|\phi\|^2_{L_2(\mathscr{U},L^2(\mathbb{T}^3))}\,\mathrm{d}t + \tilde{\mathfrak{M}}_t - \tilde{\mathfrak{M}}_s \qquad (3.1.68)$$

provided t,s are Lebesgue points of $\tilde{E}_t = \frac{1}{2}\langle \nu_{t,x}, |\xi|^2\rangle\,\mathrm{d}x + \frac{1}{2}\tilde{\lambda}_t(\mathbb{T}^3)$. Note the conclusion in (3.1.68) uses the fact that $\frac{1}{r}\tilde{E}\tilde{\lambda}((t-r,t)\times\mathbb{T}^3)$ stays bounded in r by (3.1.65). Accordingly, this shows that $\tilde{\lambda}=\tilde{\lambda}_t\otimes\mathfrak{L}^1$ with $\lambda_t\in L^\infty_{w*}(0,T;\mathscr{M}^+(\mathbb{T}^3))$ $\tilde{\mathbb{P}}$-a.s. On the account of (3.1.68) the functional

$$t\mapsto\tilde{E}_t-\frac{1}{2}\int_s^t\|\phi\|^2_{L_2(\mathscr{U},L^2(\mathbb{T}^3))}\,\mathrm{d}\sigma-\tilde{\mathfrak{M}}_t$$

is non-increasing. Upon recalling Proposition 3.1.11 the above functional is pathwise bounded, consequently, left- and right-sided limits exist for all points. In addition,

$$\int_s^t\|\phi\|^2_{L_2(\mathscr{U},L^2(\mathbb{T}^3))}\,\mathrm{d}\sigma$$

and $\tilde{\mathfrak{M}}_t$ are continuous, which implies that left- and right-sided limits also exists for $\tilde{E}_t$. Finally, we approximate arbitrary t and s by Lebesgue points and use (3.1.68) to deduce

$$\tilde{E}_{t^+}\le\tilde{E}_{s^-}+\frac{1}{2}\int_s^T\|\phi\|^2_{L_2(\mathscr{U},L^2(\mathbb{T}^3))}\,\mathrm{d}t+\tilde{\mathfrak{M}}_t-\tilde{\mathfrak{M}}_s \tag{3.1.69}$$

$\tilde{\mathbb{P}}$-a.s. for all $t>s>0$. In the case $s=0$ we argue similarly as shown above for (3.1.65), but we exclude averaging in s. Accordingly, we deduce

$$\frac{1}{r}\int_{t-r}^t\tilde{E}_k^{\varepsilon_m}\,\mathrm{d}k\le\tilde{E}_0^{\varepsilon_m}+\frac{1}{2r}\int_{t-r}^t\int_0^r\|\phi\|^2_{L_2(\mathscr{U},L^2(\mathbb{T}^3))}\,\mathrm{d}\sigma\,\mathrm{d}k+\frac{1}{r}\int_{t-r}^t\tilde{\mathfrak{M}}_k^{\varepsilon_m}\,\mathrm{d}k$$

$\tilde{\mathbb{P}}$-a.s. provided $r<t$. Applying Proposition 3.1.12, and using $E_0^{\varepsilon_m}=\frac{1}{2}\int_{\mathbb{T}^3}|\mathbf{u}_0^{\varepsilon_m}|^2\,\mathrm{d}x$ and (3.1.66) we infer

$$\tilde{E}_{t^+}\le\tilde{E}_{s^-}+\frac{1}{2}\int_0^t\|\phi\|^2_{L_2(\mathscr{U},L^2(\mathbb{T}^3))}\,\mathrm{d}k+\tilde{\mathfrak{M}}_t \tag{3.1.70}$$

$\tilde{\mathbb{P}}$-a.s for all $t>0$. In conclusion, combining (3.1.69) and (3.1.70) guarantees the existence of energy inequality in the new probability space. Accordingly, this completes the proof of Theorem 3.1.9.

3.1.4 Weak-Strong Uniqueness

One of the fundamental concepts (i.e weak-strong principle) we seek to address in this thesis is the relation between the dissipative solution in the sense of Definition 3.1.2 and a strong solution to (3.1.45). This concept is reminiscent to the results of [16] on the stochastic compressible Navier-Stokes system. In general a strong solution to the stochastic Euler system is known to exist at least for a short time. To begin with, we state the definition of a strong solution.

Definition 3.1.3. Let $(\Omega, \mathscr{F}, (\mathscr{F}_t), \mathbb{P})$ be a stochastic basis with a complete right-continuous filtration, let W be an $(\mathscr{F}_t)$-cylindrical Wiener process. A random variable $\mathbf{u}$ and a stopping time τ is called a (local) strong solution system to (3.1.45) provided

(a) the process $t \mapsto \mathbf{u}(t\wedge\tau, \cdot) \in L^2(\mathbb{T}^3)$ is $(\mathscr{F}_t)$-adapted,

$$\mathbf{u}(t\wedge\tau,\cdot), \nabla\mathbf{u}(t\wedge\tau,\cdot) \in C_{\mathrm{loc}}([0,\infty)\times\mathbb{T}^3)$$

a.s. and

$$\mathbb{E}\left[\sup_{0\le t\le T} \|\mathbf{u}(\cdot\wedge\tau)\|^2_{L^2_x} + \int_0^{T\wedge\tau} \|\nabla\mathbf{u}\|_{L^\infty_x}\right]^p < \infty \text{ for all } 1 \le p < \infty;$$

(b) for all $\varphi \in C^\infty_{\mathrm{div}}(\mathbb{T}^3)$ and all $t \ge 0$ there holds $\mathbb{P}$-a.s.

$$\int_{\mathbb{T}^3} \mathbf{u}(t\wedge\tau)\cdot\varphi \,\mathrm{d}x = \int_{\mathbb{T}^3} \mathbf{u}(0)\cdot\varphi\,\mathrm{d}x - \int_0^{t\wedge\tau}\int_{\mathbb{T}^3} (\nabla\mathbf{u})\mathbf{u}\cdot\varphi\,\mathrm{d}x\,\mathrm{d}s + \int_{\mathbb{T}^3}\int_0^{t\wedge\tau} \varphi\cdot\phi\,\mathrm{d}W\,\mathrm{d}x.$$

(c) we have $\mathrm{div}\mathbf{u}(t\wedge\tau,\cdot) = 0$ $\mathbb{P}$-a.s.

Remark 3.1.6. The energy inequality of strong solutions to the system (3.1.45) follows from applying Itô's formula to $f(u) = \frac{1}{2}\int_{\mathbb{T}^3} |\mathbf{u}|^2\,\mathrm{d}x$ (in the Hilbert space version for $L^2_{\mathrm{div}}(\mathbb{T}^3)$) and satisfies

$$\frac{1}{2}\int_{\mathbb{T}^3} |\mathbf{u}|^2 \mathrm{d}x = \frac{1}{2}\int |\mathbf{u}_0|^2\,\mathrm{d}x + \int_0^t \phi\cdot\mathbf{u}\,\mathrm{d}x\mathrm{d}W_s + \frac{1}{2}\int_0^t \|\phi\|^2_{L_2(\mathscr{U}, L^2(\mathbb{T}^3))}\mathrm{d}s \tag{3.1.71}$$

for all $t \in [0,\tau]$ $\mathbb{P}$-a.s.

The results on the existence of local-in-time strong solutions to (3.1.45) system (under slip boundary conditions and not periodic setting) were established in [57, Theorem 4.3] under certain assumptions imposed on the noise coefficient ϕ.

Pathwise Weak-strong Uniqueness

To begin with, we consider the case when a dissipative solution and the strong solution are defined on the same probability space. The weak-strong uniqueness principle is given by the following theorem.

Theorem 3.1.13 (Pathwise Weak-Strong Uniqueness)**.** *The pathwise weak-strong uniqueness holds true for the stochastic Euler equations* (3.1.45) *in the following sense: let*

$$((\Omega, \mathscr{F}, (\mathscr{F}_t), \mathbb{P}), \mathbf{u}, \mathscr{V}, W)$$

be a dissipative martingale solution to (3.1.45) *in the sense of Definition 3.1.2 and let* $\mathbf{v}$ *and a stopping time* τ *be a strong solution of the same problem in the sense of Definition 3.1.3 defined on the same stochastic basis with the same Wiener process and with the same initial data (that is,* $\mathbf{v}(0,\cdot) = \mathbf{u}(0,\cdot)$ $\mathbb{P}$*-a.s.). Then we have for a.a* (t,x) *that* $\mathbf{u}(t\wedge\tau, x) = \mathbf{v}(t\wedge\tau, x)$ *and*

$$(\nu_{t\wedge\tau,x}, \nu^{\infty}_{t\wedge\tau,x}, \lambda) = (\delta_{\mathbf{u}(t\wedge\tau,x)}, 0, 0)$$

a.s.

We proceed to prove Theorem 3.1.13.

Proof. To begin with, we introduce the stopping time

$$\tau_M = \inf\left\{t \in (0,\tau) | \quad \|\nabla \mathbf{v}(t,\cdot)\|_{L^\infty_x} > M\right\}, M > 0,$$

and define $\tau_M = \tau$ if $\{\cdots\} = \emptyset$. On the account of Definition 3.1.3 we note that

$$\mathbb{E}[\sup_{t\in[0,\tau]} \|\mathbf{v}(t,\cdot)\|_{L^\infty_x}] < \infty,$$

consequently we have

$$\mathbb{P}[\tau_M < \tau] \leq \mathbb{P}\left[\sup_{t\in[0,\tau]} \|\nabla\mathbf{v}(t,\cdot)\|_{L^\infty_x} \geq M\right] \leq \frac{1}{M}\mathbb{E}\left[\sup_{t\in[0,\tau]} \|\nabla\mathbf{v}(t,\cdot)\|_{L^\infty_x}\right] \to 0$$

as $M \to \infty$ by Tschebyscheff's inequality. Accordingly, we have

$$\tau_M \to \tau \quad \text{in probability.}$$

Therefore, it suffices to show the claim in $(0, \tau_M)$ for a fixed M. Assuming the existence of strong solutions to (3.1.45) in the sense of Definition 3.1.3 we consider the functional

$$F(t) = \frac{1}{2}\int_{\mathbb{T}^3} \langle \nu_{t,x}, |\xi - \mathbf{v}|^2\rangle \,\mathrm{d}x + \frac{1}{2}\lambda_t(\mathbb{T}^3), \tag{3.1.72}$$

defined for a.a. $t < \tau$ where $\langle \nu_{t,x}, |\xi - \mathbf{v}|^2\rangle = \langle \nu_{t,x}, |\xi|^2\rangle - 2\mathbf{v}\langle \nu_{t,x}, \xi\rangle + |\mathbf{v}|^2$. $F(t)$-functional acts as tool that allows us to compare two solutions. Now noting that $\mathbf{u} = \langle \nu_{t,x}, \xi\rangle$, for convenience we re-write (3.1.72) as follows

$$\begin{aligned} F(t) &= \frac{1}{2}\left(\int_{\mathbb{T}^3}\langle \nu_{t,x}, |\xi|^2\rangle + \lambda_t(\mathbb{T}^3) - 2\int_{\mathbb{T}^3}\mathbf{vu} + \int_{\mathbb{T}^3}|\mathbf{v}|^2\,\mathrm{d}x\right) \\ &= E(t) + \frac{1}{2}\int_{\mathbb{T}^3}|\mathbf{v}|^2\,\mathrm{d}x - \int_{\mathbb{T}^3}\mathbf{vu}\,\mathrm{d}x. \end{aligned}$$

In addition, the notion of (3.1.72) can be extended to any $t < \tau$ by setting

$$F(t) = E(t^+) + \frac{1}{2}\int_{\mathbb{T}^3}|\mathbf{v}|^2\,\mathrm{d}x - \int_{\mathbb{T}^3}\mathbf{vu}\,\mathrm{d}x,$$

by observing that $\mathbf{u}$ and $\mathbf{v}(\cdot\wedge\tau)$ belong to $C_w([0,T];L^2(\mathbb{T}^3))$. Computing the expectation of $F(t\wedge\tau)$, and using (3.1.47) and (3.1.71) yields

$$\begin{aligned} &\mathbb{E}[F(t\wedge\tau_M)] \qquad (3.1.73) \\ &= \mathbb{E}[E(t\wedge\tau_M)] + \frac{1}{2}\mathbb{E}\int_{\mathbb{T}^3}|\mathbf{v}(t\wedge\tau_M)|^2\mathrm{d}x - \mathbb{E}\int_{\mathbb{T}^3}\mathbf{u}(t\wedge\tau_M)\mathbf{v}(t\wedge\tau_M)\mathrm{d}x \\ &\le \mathbb{E}\left(\int_{\mathbb{T}^3}|\mathbf{v}(0)|^2\,\mathrm{d}x + \int_0^{t\wedge\tau_M}\|\phi\|^2_{L_2(\mathscr{U},L^2(\mathbb{T}^3))}\,\mathrm{d}s\right) - \underbrace{\mathbb{E}\int_{\mathbb{T}^3}\mathbf{u}(t\wedge\tau_M)\mathbf{v}(t\wedge\tau_M)\mathrm{d}x}_{A}, \end{aligned}$$

where $\mathbf{u}(0) = \mathbf{v}(0)$. Now using Lemma 3.1.1 the above term A is re-written in the form

$$\begin{aligned} A &:= \int_{\mathbb{T}^3}\mathbf{u}(t\wedge\tau_M)\mathbf{v}(t\wedge\tau_M)\mathrm{d}x \\ &= \int_{\mathbb{T}^3}\mathbf{u}(0)\mathbf{v}(0)\mathrm{d}x + \underbrace{\int_0^{t\wedge\tau_M}\int_{\mathbb{T}^3}\langle \nu_{t,x}, \xi\otimes\xi\rangle : \nabla\mathbf{v}\,\mathrm{d}x\,\mathrm{d}s}_{A_I} \\ &\quad + \underbrace{\int_0^{t\wedge\tau_m}\int_{\mathbb{T}^3}\langle \nu^\infty_{t,x}, \theta\otimes\theta\rangle : \nabla\mathbf{v}\,\mathrm{d}\lambda}_{A_{II}} \\ &\quad + \int_{\mathbb{T}^3}\mathbf{v}\int_0^{t\wedge\tau_M}\phi^1\mathrm{d}W\mathrm{d}x + \underbrace{\int_0^{t\wedge\tau_M}\int_{\mathbb{T}^3}\mathrm{div}(\mathbf{v}\otimes\mathbf{v}) : \mathbf{u}\,\mathrm{d}x\,\mathrm{d}s}_{A_{III}} \\ &\quad + \int_{\mathbb{T}^3}\mathbf{u}\int_0^{t\wedge\tau_M}\phi^2\mathrm{d}W\mathrm{d}x + \sum_{k\ge1}\int_0^t\int_{\mathbb{T}^3}\phi^1 e_k\phi^2 e_k\,\mathrm{d}x\,\mathrm{d}s + \mathbb{M}_{t\wedge\tau_M}, \end{aligned}$$

where

$$\mathbb{M}_{t\wedge\tau_M} = \sum_{k\geq 1}\int_0^{t\wedge\tau_M}\int_{\mathbb{T}^3}\left[\mathbf{u}\phi^2 e_k + \mathbf{v}\phi^1 e_k\right]\mathrm{d}x\,\mathrm{d}W_k.$$

We note the stochastic terms of A vanish upon computing $\mathbb{E}[A]$. By assumption $\phi^1 = \phi^2 = \phi$ (since our solutions lie in same path with same noise), consequently,

$$\sum_{k\geq 1}\int_0^{t\wedge\tau_M}\int_{\mathbb{T}^3}\phi^1 e_k\phi^2 e_k\,\mathrm{d}x\,\mathrm{d}s = \int_0^{t\wedge\tau_M}\|\phi\|^2_{L_2(\mathscr{U},L^2(\mathbb{T}^3))}\,\mathrm{d}s.$$

Since $\mathbf{u}(0)\mathbf{v}(0) = |\mathbf{v}(0)|^2$ by assumption, replacing A in (3.1.73) by the new formulation above yields

$$\begin{aligned}\mathbb{E}[F(t\wedge\tau_M)] &\leq -\mathbb{E}(A_I + A_{II} + A_{III})\\ &\leq \mathbb{E}(|A_I + A_{II} + A_{III}|).\end{aligned}\tag{3.1.74}$$

Using standard identities for the nonlinear term together with the properties: $\operatorname{div}\mathbf{v} = 0$ and $\operatorname{div}\mathbf{u} = 0$ we infer

$$A_I + A_{III} = \int_0^{t\wedge\tau_M}\int_{\mathbb{T}^3}\langle\nu_{t,x},(\xi-\mathbf{v})\otimes(\xi-\mathbf{v})\rangle : \nabla\mathbf{v}\,\mathrm{d}x\,\mathrm{d}s.\tag{3.1.75}$$

To verify (3.1.75) we consider following expansion

$$\begin{aligned}A_I + A_{III} &= \int_0^{t\wedge\tau_M}\int_{\mathbb{T}^3}\langle\nu_{t,x},(\xi-\mathbf{v})\otimes(\xi-\mathbf{v})\rangle : \nabla\mathbf{v}\,\mathrm{d}x\,\mathrm{d}s\\
&= \int_0^{t\wedge\tau_M}\int_{\mathbb{T}^3}\langle\nu_{t,x},\xi\otimes\xi\rangle : \nabla\mathbf{v}\,\mathrm{d}x\,\mathrm{d}s - \int_0^{t\wedge\tau_M}\int_{\mathbb{T}^3}\langle\nu_{t,x},\xi\otimes\mathbf{v}\rangle : \nabla\mathbf{v}\,\mathrm{d}x\,\mathrm{d}s\\
&\quad - \int_0^{t\wedge\tau_M}\int_{\mathbb{T}^3}\langle\nu_{t,x},\mathbf{v}\otimes\xi\rangle : \nabla\mathbf{v}\,\mathrm{d}x\,\mathrm{d}s + \int_0^{t\wedge\tau_M}\int_{\mathbb{T}^3}\mathbf{v}\otimes\mathbf{v} : \nabla\mathbf{v}\,\mathrm{d}x\,\mathrm{d}s\\
&= \int_0^{t\wedge\tau_M}\int_{\mathbb{T}^3}\langle\nu_{t,x},\xi\otimes\xi\rangle : \nabla\mathbf{v}\,\mathrm{d}x\,\mathrm{d}s - \int_0^{t\wedge\tau_M}\int_{\mathbb{T}^3}\mathbf{u}\otimes\mathbf{v} : \nabla\mathbf{v}\,\mathrm{d}x\,\mathrm{d}s\\
&\quad - \int_0^{t\wedge\tau_M}\int_{\mathbb{T}^3}\mathbf{v}\otimes\mathbf{u} : \nabla\mathbf{v}\,\mathrm{d}x\,\mathrm{d}s + \int_0^{t\wedge\tau_M}\int_{\mathbb{T}^3}\mathbf{v}\otimes\mathbf{v} : \nabla\mathbf{v}\,\mathrm{d}x\,\mathrm{d}s.\end{aligned}$$

Using integration by parts, the integrals $\int_0^{t\wedge\tau_M}\int_{\mathbb{T}^3}\mathbf{v}\otimes\mathbf{v} : \nabla\mathbf{v}\,\mathrm{d}x\,\mathrm{d}s$ and $\int_0^{t\wedge\tau_M}\int_{\mathbb{T}^3}\mathbf{v}\otimes\mathbf{u} : \nabla\mathbf{v}\,\mathrm{d}x\,\mathrm{d}s$ vanish to yield the desired form of $A_I + A_{III}$. Next, we compute estimates for A_I, A_{II}, A_{III} and take expectations.

$$A_I + A_{II} + A_{III} = \int_0^{t\wedge\tau_M}\int_{\mathbb{T}^3}\langle\nu_{t,x},(\xi-\mathbf{v})\otimes(\xi-\mathbf{v})\rangle : \nabla\mathbf{v}\,\mathrm{d}x\,\mathrm{d}s$$

$$
\begin{aligned}
&+\int_{(0,t\wedge\tau_M)\times\mathbb{T}^3}\langle \nu^\infty,\theta\otimes\theta\rangle:\nabla\mathbf{v}\,\lambda_s(\mathrm{d}x)\mathrm{d}s\\
\le\;& \int_0^{t\wedge\tau_M}\int_{\mathbb{T}^3}\langle \nu_{t,x},|\xi-\mathbf{v}|^2\rangle:\nabla\mathbf{v}\,\mathrm{d}x\,\mathrm{d}s\\
&+\int_{(0,t)\times\mathbb{T}^3}\langle \nu^\infty,\theta\otimes\theta\rangle:\nabla\mathbf{v}\,\lambda_s(\mathrm{d}x)\mathrm{d}s\\
\le\;& c\int_0^{t\wedge\tau_M}F(s)\|\nabla\mathbf{v}\|_\infty\mathrm{d}s. \qquad (3.1.76)
\end{aligned}
$$

Therefore, in view of (3.1.74) taking the expectation of (3.1.76) yields

$$
\mathbb{E}[F(t\wedge\tau_M)]\le c\,\mathbb{E}\int_0^{t\wedge\tau_M}F(s)\|\nabla\mathbf{v}\|_\infty\mathrm{d}s, \qquad (3.1.77)
$$

for some constant $c>0$. Finally, by using Gronwall's lemma, the inequality in (3.1.77) implies that $\mathbb{E}[F(t\wedge\tau_M)]=0$ for a.e. t as required. This completes the claim of Theorem 3.1.13 by definition of the functional F. □

Remark 3.1.7. Setting $\tau_M=T$ to be deterministic in the proof of Theorem 3.1.13 yields $\mathbf{u}=\mathbf{v}$ and $\mathscr{V}=(\delta_{\mathbf{u}},0,0)$ $\mathbb{P}$-a.s., that is

$$
\mathbb{P}\left(\left\{\mathbf{u}(t,x)=\mathbf{v}(t,x)\quad\text{for a.a. }(t,x)\in Q_T\right\}\right)=1,
$$

$$
\mathbb{P}\left(\left\{(\nu_{t,x},\nu^\infty_{t,x},\lambda)=(\delta_{\mathbf{u}(t,x)},0,0)\right\}\right)=1.
$$

Weak-Strong Uniqueness in Law

Finally we consider the case when a dissipative solution and a strong solution are defined on distinct probability spaces. Accordingly, we give a precise notion for this case in the following theorem.

Theorem 3.1.14. *The weak-strong uniqueness in law holds true for the stochastic Euler system (3.1.45) in the following sense: Let*

$$
\left[(\Omega^1,\mathscr{F}^1,(\mathscr{F}^1_t)_{t\ge0},\mathbb{P}^1),\mathbf{u}^1,\mathscr{V}^1,W^1\right]
$$

be a dissipative measure-valued martingale solution to (3.1.45) in the sense of Definition 3.1.2 and

$$
\left[(\Omega^2,\mathscr{F}^2,(\mathscr{F}^2_t)_{t\ge0},\mathbb{P}^2),\mathbf{u}^2,\mathscr{V}^2,W^2\right]
$$

be a strong solution of the same problem in the sense of Definition 3.1.3(with $\tau_M = T$ deterministic) such that

$$\Lambda = \mathbb{P}^1 \circ (\mathbf{u}^1(0))^{-1} = \mathbb{P}^2 \circ (\mathbf{u}^2(0))^{-1},$$

then

$$\mathbb{P}^1 \circ (\mathscr{V}^1, \mathbf{u}^1)^{-1} = \mathbb{P}^2 \circ (\mathscr{V}^2, \mathbf{u}^2)^{-1}. \tag{3.1.78}$$

The results in Theorem 3.1.14 are reminiscent to the classical work of Yamada-Watanabe on stochastic differential equations as presented in [66]. However, the application of these results to (3.1.45) is not a straight forward process due to difficulties caused by working with infinite-dimensional spaces and the non-separability of the space of generalised Young measures. We prove Theorem 3.1.14 as follows.

Proof. We set

$$\mathbf{v}^j(t) = \mathbf{u}^j(t) - \mathbf{u}^j(0), \quad j = \{1,2\}, \quad 0 \le t < \infty,$$

and we consider the topological space

$$\Theta = L^2_{\mathrm{div}}(\mathbb{T}^3) \times C((0,T); W^{-4,2}_{\mathrm{div}}(\mathbb{T}^3)) \cap C_w(L^2_{\mathrm{div}}(\mathbb{T}^3)) \times \mathbf{Y}(Q;\mathbb{R}^d) \times C([0,T],\mathscr{U}_0),$$

and we denote by $\mathscr{B}(\Theta)$ the σ-field generated by Θ, that is,

$$\begin{aligned}\mathscr{B}(\Theta) =& \mathscr{B}(L^2_{\mathrm{div}}(\mathbb{T}^3)) \times \mathscr{B}(C((0,T); W^{-4,2}_{\mathrm{div}}(\mathbb{T}^3)) \cap C_w(L^2_{\mathrm{div}}(\mathbb{T}^3))) \\ & \times \mathscr{B}(\mathbf{Y}(Q;\mathbb{R}^d)) \times \mathscr{B}(C([0,T],\mathscr{U}_0)).\end{aligned}$$

Given the j-solution consists of $[\mathbf{u}^j_0, \mathbf{v}^j, \mathscr{V}^j, W^j]$, the probability law of $\mathscr{L}[\mathbf{u}^j_0, \mathbf{v}^j, \mathscr{V}^j, W^j]$ on $(\Theta, \mathscr{B}(\Theta))$ is denoted by μ^j (recall that $\mathscr{V}^2 = (\delta_{\mathbf{u}^2}, 0, 0)$ for strong solution) such that the j^{th}-joint law is of the form

$$\mu^j(\mathfrak{A}) = \mathbb{P}^j([\mathbf{u}^j_0, \mathbf{v}^j, \mathscr{V}^j, W^j] \in \mathfrak{A}) \quad \mathfrak{A} \in \mathscr{B}(\Theta), j = 1,2, \tag{3.1.79}$$

where $\mathbb{P}^j$ is a probability measure on the space $(\Theta, \mathscr{B}(\Theta))$. Let $\theta = (\tilde{\mathbf{u}}_0, \tilde{\mathbf{v}}, \tilde{\mathscr{V}}, \tilde{W})$ denote the generic element of Θ. The marginal of each $\mathbb{P}^j$ on $\tilde{\mathbf{u}}_0$-coordinate is the measure Λ, the marginal on the $\tilde{W}$-coordinate is Wiener measure $\mathbb{P}_*$ on the space $C([0,T],\mathscr{U}_0)$. Therefore, the distribution law of the pair $(\tilde{\mathbf{u}}_0, \tilde{W})$ is the product measure $\Lambda \times \mathbb{P}_*$ since $(\mathbf{u}^j_0)$ is

$\mathscr{F}_0^{(j)}$-measurable and $\tilde{W}^{(j)}$ is independent of $\mathscr{F}_0^{(j)}$. In addition, under $\mathbb{P}^j$ the initial law value of $\mathbf{v}$-coordinate is zero a.s.

Assuming that the two solutions exist possibly in two distinct probability spaces we construct a product probability space. In the new probability space we use the same canonical elements $[\mathbf{u}_0^j, \mathbf{v}^j, \mathscr{V}^j, W^j], j = \{1,2\}$ and we preserve the joint distributions of solutions. The construction of the product probability space relies on the concept of regular conditional probabilities. In the following we outline the approach to our particular case. Accordingly, let $(\Omega, \mathscr{Y}, P)$ be a probability space, Where Ω is a Hausdorff topological space and $\mathscr{Y}$ is countable generated σ-field. Then P is called regular if for all A we have

$$P(A) = \sup\{P(K) : K \subset A \,\text{is compact}, K \in \mathscr{Y}\}, \forall A \in \mathscr{Y},$$

Moreover, $(\Omega, \mathscr{Y}, P)$ is a radon space (see [Thm 2.1, [75]]). The given assumptions on $(\Omega, \mathscr{Y}, P)$ guarantee existence of a regular conditional probability for P, see e.g. [59, Introduction]. We proceed to show similar results hold in our case. Since $C([0,T], \mathscr{U}_0) \times C((0,T); W_{\text{div}}^{-4,2}(\mathbb{T}^3)) \times C_w(L^2_{\text{div}}(\mathbb{T}^3))$ and $L^2(\mathbb{T}^3)$ spaces are quasi-Polish and Banach respectively; we infer that they are Hausdorff. In addition, as both $\mathscr{B}(C([0,T], \mathscr{U}_0))$ and $\mathscr{B}(C((0,T); W_{\text{div}}^{-4,2}(\mathbb{T}^3))$ are Polish spaces they are countably generated. And as for the space $\mathscr{B}(C_w(L^2_{\text{div}}(\mathbb{T}^3)))$, we refer the reader to [26, Section 4] and references therein. Finally, $\mathscr{B}(\mathbf{Y}(Q; R^d))$ is countably generated for each $n \in \mathbb{N}$ because the function f_n from Definition 2.1.30 range in Polish space $[-1,1]$ and are continuous. Therefore, $(\Theta, \mathscr{B}(\Theta))$ is a radon space, consequently, there exists a regular conditional probability

$$Q_j(\tilde{\mathbf{u}}_0, \tilde{W}, \mathfrak{A}) : L^2(\mathbb{T}^3) \times C([0,T], \mathscr{U}_0) \times \mathscr{B}_{\mathbf{u}} \otimes \mathscr{B}_{\mathbf{Y}} \to [0,1]$$

such that

(i) For each $(\tilde{\mathbf{u}}_0, \tilde{W}) \in L^2(\mathbb{T}^3) \times C([0,T], \mathscr{U}_0)$ we have

$$Q_j(\mathbf{w}, B, \cdot) : \mathscr{B}(C((0,T); W_{\text{div}}^{-4,2}(\mathbb{T}^3)) \cap C_w(L^2_{\text{div}}(\mathbb{T}^3)) \times \mathbf{Y}_2(Q_T); \mathscr{B}_{\mathbf{u}} \times \mathscr{B}_{\mathbf{Y}}) \to [0,1]$$

is a probability measure;

(ii) The mapping $(\tilde{\mathbf{u}}_0, \tilde{W}) \mapsto Q_j(\tilde{\mathbf{u}}_0, \tilde{W}, \mathfrak{A})$ is $\mathscr{B}(L^2(\mathbb{T}^3)) \times \mathscr{B}(C([0,T], \mathscr{U}_0))$ measurable for $\mathfrak{A} \in \mathscr{B}_u \otimes \mathscr{B}_Y$;

(iii) We have that

$$\mu^j(G\times\mathfrak{A}) = \int_G Q_j(u_0,w;\mathfrak{A})\Lambda(\mathrm{d}(\tilde{\mathbf{u}}_0))\mathbb{P}_*(\mathrm{d}\tilde{W}),\ \mathfrak{A}\in\mathscr{B}_{\mathbf{u}}\times\mathscr{B}_{\mathbf{Y}},$$

for all $G\in\mathscr{B}(L^2_{\mathrm{div}}(\mathbb{T}^3))\times\mathscr{B}(C([0,T],\mathscr{U}_0)))$.

Finally, we proceed to construct a product probability space. Let $(\tilde{\Omega},\tilde{\mathscr{F}})$ be a measurable space, where

$$\tilde{\Omega} = \Theta\times\mathbf{Y}_2(Q;\mathbb{R}^d)\times C((0,T);W^{-4,2}_{\mathrm{div}}(\mathbb{T}^3))\cap C_w(L^2_{\mathrm{div}}(\mathbb{T}^3)),$$

and $\tilde{\mathscr{F}}$ is the completion σ-field on $\tilde{\Omega}$

$$\mathscr{B}(\Theta)\times\mathscr{B}(\mathbf{Y}(Q;\mathbb{R}^d))\times\mathscr{B}(C((0,T);W^{-4,2}_{\mathrm{div}}(\mathbb{T}^3))\cap C_w(L^2_{\mathrm{div}}(\mathbb{T}^3))),$$

under the probability measure

$$\tilde{\mathbb{P}}(\mathrm{d}\omega) = Q_1(\tilde{\mathbf{u}}_0,\tilde{W};\mathrm{d}(\tilde{\mathscr{V}}^1,\tilde{\mathbf{v}}^1))Q_2(\tilde{\mathbf{u}}_0,\tilde{W};\mathrm{d}(\tilde{\mathscr{V}}^2,\tilde{\mathbf{v}}^2))\Lambda(\mathrm{d}(\tilde{\mathbf{u}}_0))\tilde{\mathbb{P}}_*(\mathrm{d}\tilde{W}). \tag{3.1.80}$$

On the account (3.1.79), property (iii) and (3.1.80) we infer

$$\begin{aligned}
\tilde{\mathbb{P}}(G\times\mathfrak{A}_1\times\mathfrak{A}_2) &= \int_{G\times\mathfrak{A}_1\times\mathfrak{A}_2} Q_1(\tilde{\mathbf{u}}_0,\tilde{W};\mathrm{d}(\tilde{\mathscr{V}}^1,\tilde{\mathbf{v}}^1))Q_2(\tilde{\mathbf{u}}_0,\tilde{W};\mathrm{d}(\tilde{\mathscr{V}}^2,\tilde{\mathbf{v}}^2))\Lambda(\mathrm{d}(\tilde{\mathbf{u}}_0))\tilde{\mathbb{P}}_*(\mathrm{d}\tilde{W})\\
&= \int_G Q_1(\tilde{\mathbf{u}}_0,\tilde{W};\mathfrak{A}_1)Q_2(\tilde{\mathbf{u}}_0,\tilde{W};\mathfrak{A}_2)\Lambda(\mathrm{d}(\tilde{\mathbf{u}}_0))\tilde{\mathbb{P}}_*(\mathrm{d}\tilde{W}).\\
&= \int_G Q_j(\tilde{\mathbf{u}}_0,\tilde{W};\mathfrak{A})\Lambda(\mathrm{d}(v_0))\tilde{\mathbb{P}}_*(\mathrm{d}w)\\
&= \mu_j(G\times\mathfrak{A}),
\end{aligned} \tag{3.1.81}$$

for $\mathfrak{A}_1,\mathfrak{A}_2\in\mathscr{B}_{\mathbf{u}}\otimes\mathscr{B}_Y$ and $G\in\mathscr{B}(L^2(\mathbb{T}^3))\otimes\mathscr{B}(C([0,T],\mathscr{U}_0))$. Consequently, in the space $(\tilde{\Omega},\tilde{\mathscr{F}},\tilde{\mathbb{P}})$ for $j=1,2$ we obtain

$$\tilde{\mathbb{P}}(\{\tilde{\omega}=[\tilde{\mathbf{u}}_0,\tilde{W},\tilde{\mathbf{v}}^1,\tilde{\mathscr{V}}^1,\tilde{\mathbf{v}}^2,\tilde{\mathscr{V}}^2]\in\tilde{\Omega}:(\tilde{\mathbf{u}}_0,\tilde{W},\tilde{\mathbf{v}}^j,\tilde{\mathscr{V}}^j)\in\mathfrak{A}\}) = \mu_j(\mathfrak{A}),\mathfrak{A}\in\mathscr{B}(\Theta).$$

Similarly we let $\omega=(\tilde{\mathbf{u}}_0,\tilde{\mathbf{v}}^1,\tilde{\mathscr{V}}^1,\tilde{\mathbf{v}}^2,\tilde{\mathscr{V}}^2,\tilde{W})$ denote a generic element of $\tilde{\Omega}$. Accordingly, the law of $[\tilde{\mathbf{u}}_0,\tilde{\mathbf{v}}^1,\tilde{\mathscr{V}}^1,\tilde{\mathbf{v}}^2,\tilde{\mathscr{V}}^2,\tilde{W}]$ under $\tilde{\mathbb{P}}$ coincides with the law of $[\mathbf{u}_0^j,\mathbf{v}^j,\mathscr{V}^j,W^j]$ under $\mathbb{P}_j$ in the original space. Consequently, the law of $(\tilde{\mathbf{u}}_0+\tilde{\mathbf{v}}^j,\tilde{\mathscr{V}}^j,\tilde{W})$ under $\tilde{\mathbb{P}}$ coincides with the law of $(\mathbf{v}^j, \mathscr{V}^j, W^j)$ under $\mathbb{P}_j$. In conclusion, we infer that $(\tilde{\mathbf{u}}_0 + \tilde{\mathbf{v}}^j, \tilde{\mathscr{V}}^j, \tilde{W})$ solves (3.1.45) for $j = 1,2$, that is,

$$\left[(\tilde{\Omega}, \tilde{\mathscr{F}}, (\tilde{\mathscr{F}}_t)_{t\geq 0}, \tilde{\mathbb{P}}), \tilde{\mathbf{u}}_0 + \tilde{\mathbf{v}}^j, \tilde{\mathscr{V}}^j, \tilde{W}\right]$$

is a dissipative martingale solution to (3.1.45) in the sense of Definition 3.1.2 for $j = 1$, while $j = 2$ is the strong solution $\mathbf{v}^2 + \mathbf{u}_0$. Arguing as in (3.1.57), the assertion

$$\tilde{\mathbf{v}}^j(t,x) + \tilde{\mathbf{u}}_0(x) = \langle \tilde{\nu}_{t,x}, \xi \rangle \quad \text{for a.a. } (t,x) \in Q_T$$

holds $\mathbb{P}$-a.s., where $\tilde{\mathscr{V}}^j = (\tilde{\nu}^j_{t,x}, \tilde{\nu}^{j,\infty}_{t,x}, \tilde{\lambda})$. Next we aim to verify the concluding statement above and we argue as in Section 3.1.2. To begin with, we ensure correct measurabilities in the new probability space $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{\mathbb{P}})$ by endowing it with a filtration that satisfies usual conditions i.e.,

$$\tilde{\mathscr{F}}^j_t = \sigma\Big(\sigma(\tilde{\mathbf{u}}_0, \mathbf{r}_t\tilde{W}, \mathbf{r}_t\tilde{\mathbf{v}}^j, \mathbf{r}_t\tilde{\mathscr{V}}^j) \cup \sigma_t[\tilde{\mathscr{V}}^j] \cup \{\mathscr{N} \in \tilde{\mathscr{F}}; \tilde{\mathbb{P}}(\mathscr{N}) = 0\}\Big), j = 1,2,$$

$$\tilde{\mathscr{F}}_t = \sigma\Big(\sigma(\tilde{\mathbf{u}}_0, \mathbf{r}_t\tilde{W}, \mathbf{r}_t\tilde{\mathbf{v}}^1, \mathbf{r}_t\tilde{\mathbf{v}}^2) \cup \sigma_t[\tilde{\mathscr{V}}^1] \cup \sigma_t[\tilde{\mathscr{V}}^2] \cup \{\mathscr{N} \in \tilde{\mathscr{F}}; \tilde{\mathbb{P}}(\mathscr{N}) = 0\}\Big),$$

where σ_t denotes the history of a random distribution, and the generalised Young measures are viewed in the sense of random distributions as in Definition 2.1.27. Accordingly, for all $t \in [0,T]$ and $\varphi \in C^\infty_{\mathrm{div}}(\mathbb{T}^3)$ we consider the following functionals:

$$\begin{aligned}\mathscr{M}(\mathbf{w}, \mathscr{V})_t &= \int_{\mathbb{T}^3} \mathbf{w}(t)\cdot\varphi\,\mathrm{d}x - \int_0^t \int_{\mathbb{T}^3} \langle \nu_{t,x}, \xi\otimes\xi \rangle : \nabla\varphi\,\mathrm{d}x\,\mathrm{d}s \\ &\quad - \int_{(0,t)\times\mathbb{T}^3} \langle \nu^\infty_{t,x}, \xi\otimes\xi \rangle : \nabla\varphi\,\mathrm{d}\lambda\end{aligned}$$

$$\Psi_t = \sum_{k=1} \int_0^t \left(\int_{\mathbb{T}^3} \phi_k \cdot \varphi\ \mathrm{d}x \right)^2 \mathrm{d}s,$$

$$(\Psi_k)_t = \int_0^t \int_{\mathbb{T}^3} \phi_k \cdot \varphi\ \mathrm{d}x\mathrm{d}s.$$

Arguing similarly as in (3.1.3), the functional $\mathscr{M}(\mathbf{w}, \mathscr{V})_t$ is well-defined. Now let $\mathscr{M}(\tilde{\mathbf{v}}^j + \tilde{\mathbf{u}}_0, \tilde{\mathscr{V}}^j)_{s,t}$ denote the increment $\mathscr{M}(\tilde{\mathbf{v}}^j + \tilde{\mathbf{u}}_0, \mathscr{V}^j)_t - \mathscr{M}(\tilde{\mathbf{v}}^j + \tilde{\mathbf{u}}_0, \tilde{\mathscr{V}}^j)_s$ and similarly for $\Psi_{s,t}$ and $(\Psi_k)_{s,t}$. To show that (3.1.45) continues to hold in the new probability space it suffices to show

$$\mathscr{M}(\tilde{\mathbf{v}}^j + \tilde{\mathbf{u}}_0, \tilde{\mathscr{V}}^j) = \int_0^t \int_{\mathbb{T}^3} \phi \cdot \varphi\ \mathrm{d}x\mathrm{d}\tilde{W}^j_s, \tag{3.1.82}$$

that is, the deterministic part is equivalent to the stochastic part. We verify (3.1.82) by computing its corresponding quadratic and cross variations of the $\mathscr{F}_t^j$-martingale process $\mathscr{M}(\tilde{\mathbf{v}}^j+\tilde{\mathbf{u}}_0,\tilde{\mathscr{V}}^j)$ as follows:

$$\langle\langle\mathscr{M}(\tilde{\mathbf{v}}^j+\tilde{\mathbf{u}}_0,\tilde{\mathscr{V}}^j)\rangle\rangle=\Psi_t,\qquad \langle\langle\mathscr{M}(\tilde{\mathbf{v}}^j+\tilde{\mathbf{u}}_0,\tilde{\mathscr{V}}^j),\beta_k^j\rangle\rangle=(\Psi_k)_t, \tag{3.1.83}$$

respectively. Indeed, in case of (3.1.83) we have

$$\left\langle\left\langle\mathscr{M}(\tilde{\mathbf{v}}^j+\tilde{\mathbf{u}}_0,\tilde{\mathscr{V}}^j)-\int_0^t\int_{\mathbb{T}^3}\phi\cdot\varphi\,\mathrm{d}x\mathrm{d}\tilde{W}_s^j\right\rangle\right\rangle=0. \tag{3.1.84}$$

Therefore, $(\tilde{\mathbf{v}}^j+\tilde{\mathbf{u}}_0,\tilde{\mathscr{V}}^j,\tilde{W})$ solves the system (3.1.45) on the new probability space provided (3.1.83) holds. We proceed to show that $\mathscr{M}(\tilde{\mathbf{v}}^j+\tilde{\mathbf{u}}_0,\tilde{\mathscr{V}}^j)$ is indeed a martingale. Since the mapping

$$(\mathbf{v}^j,\mathscr{V}^j)\mapsto\mathscr{M}(\mathbf{v}^j,\mathscr{V}^j)$$

is well-defined and continuous we infer

$$\mathscr{M}(\mathbf{v}^j,\mathscr{V}^j)\sim^d\mathscr{M}(\tilde{\mathbf{v}}^j+\tilde{\mathbf{u}}_0,\tilde{\mathscr{V}}^j)$$

Next we we fix $s,t\in[0,T]$, with $s<t$ and consider a continuous function h such that

$$h:V|_{[0,s]}\to[0,1].$$

Since the process

$$\mathscr{M}(\mathbf{v}^j,\mathscr{V}^j)=\int_0^t\int_{\mathbb{T}^3}\phi\cdot\varphi\,\mathrm{d}x\mathrm{d}W_s^j$$

is a square integrable $(\mathscr{G}_t^j)_{\geq 0}$-martingale, we deduce that

$$[\mathscr{M}(\mathbf{v}^j,\mathscr{V}^j)]^2-\Psi_t,\qquad \mathscr{M}(\mathbf{v}^j,\mathscr{V}^j)\beta_k^j-(\Psi_k)_t,$$

are $(\mathscr{F}_t^j)_{t\geq 0}$-martingales. Let $\mathbf{r}_t$ be a restriction function to the interval $[0,t]$, then in view of Proposition 3.1.12 we infer

$$\mathbb{E}^{\mu_j}[h(\mathbf{r}_t\mathbf{v}^j,\mathbf{r}_t\mathscr{V}^j,\mathbf{r}_tW^j)\mathscr{M}(\mathbf{v}^j,\mathscr{V}^j)_{s,t}]=$$

$$\mathbb{E}^{\tilde{\mathbb{P}}}[h(\mathbf{r}_t[\tilde{\mathbf{v}}^j+\tilde{\mathbf{u}}_0],\mathbf{r}_t\tilde{\mathscr{V}}^j,\mathbf{r}_t\tilde{W}^j)\mathscr{M}(\tilde{\mathbf{v}}^j+\tilde{\mathbf{u}}_0,\tilde{\mathscr{V}}^j))_{s,t}]=0,$$

$$\mathbb{E}^{\mu_j}[h(\mathbf{r}_t v^j, \mathbf{r}_t W^j, \mathbf{r}_t \mathscr{V}^j)([\mathscr{M}(\mathbf{v}^j, \mathscr{V}^j)]^2 - \Psi)_{s,t}]$$

$$= \mathbb{E}^{\tilde{\mathbb{P}}}[h(\mathbf{r}_t[\tilde{\mathbf{v}}^j + \tilde{\mathbf{u}}_0], \mathbf{r}_t \tilde{\mathscr{V}}^j, \mathbf{r}_t \tilde{W}^j)([\mathscr{M}(\tilde{\mathbf{v}}^j + \tilde{\mathbf{u}}_0, \tilde{\mathscr{V}}^j)]^2 - \Psi)_{s,t}])] = 0,$$

$$\mathbb{E}^{\mu_j}[h(\mathbf{r}_t v^j, \mathbf{r}_t W^j, \mathbf{r}_t \mathscr{V}^j)(\mathscr{M}(v^j, \mathscr{V}^j)\beta_k^j - (\Psi_k))_{s,t})]$$

$$= \mathbb{E}^{\tilde{\mathbb{P}}}[h(\mathbf{r}_t[\tilde{\mathbf{v}}^j + \tilde{\mathbf{u}}_0], \mathbf{r}_t \tilde{\mathscr{V}}^j, \mathbf{r}_t \tilde{W}^j)(\mathscr{M}(\tilde{\mathbf{v}}^j + \tilde{\mathbf{u}}_0, \tilde{\mathscr{V}}^j)\beta_k^j - (\Psi_k))_{s,t}] = 0.$$

Accordingly, we have shown (3.1.83) and (3.1.84) hold. Consequently, both solutions satisfy the momentum equation in the sense of Definition 3.1.2 driven by the stochastic Wiener process $\tilde{W}$. Furthermore, using Proposition 3.1.12 and arguing as in Section 3.1.3 (the energy arguments) we infer that the energy inequality continues to hold on the product probability space.

Finally, to show that pathwise uniqueness implies uniqueness in the sense of probability law we apply results in Theorem 3.1.13. We note this holds provided that $\tilde{\mathbf{u}}^2 = \tilde{\mathbf{v}}^2 + \tilde{\mathbf{u}}_0$ is strong solution. We assign $\tilde{\mathbf{u}}^2$ a strong solution based on following argument. We recall that on the original space $(\Omega^2, \mathscr{F}^2, \mathbb{P}^2)$ the strong solution $\mathbf{u}^2$ is supported on $C([0,T]; C^1(\mathbb{T}^3))$ and we have the assertion $\mathscr{V}^2 = (\delta_{\mathbf{u}^2(t,x)}, 0, 0)$ $\mathbb{P}^2$-a.s. Moreover, the embedding

$$C([0,T]; C^1(\mathbb{T}^3)) \hookrightarrow C([0,T]; W^{-4,2}(\mathbb{T}^3))$$

is continuous and dense such that

$$C([0,T]; C^1(\mathbb{T}^3)) \in \mathscr{B}(C([0,T]; W^{-4,2}(\mathbb{T}^3))) \subset \mathscr{B}_{\mathbf{u}},$$

we refer the reader to [82, Corrollary A.2] for more details. Therefore, we infer that

$$\mu_2(C([0,T]; C^1(\mathbb{T}^3))) = \mathbb{P}_2(\mathbf{v} \in C([0,T]; C^1(\mathbb{T}^3))) = 1,$$

and $\tilde{\mathbf{u}}^2$ is a strong solution to the system (3.1.45) in the sense of Definition 3.1.3 (with $\tau_M = T$) on $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{\mathbb{P}})$. On the account of Theorem 3.1.13 we infer

$$\tilde{\mathbb{P}}\left(\left\{(\tilde{\mathbf{u}}^1, \tilde{\mathscr{V}}^1) = (\tilde{\mathbf{v}}^2 + \tilde{\mathbf{u}}_0, \tilde{\mathscr{V}}^2) = (\tilde{\mathbf{v}}^2 + \tilde{\mathbf{u}}_0, (\delta_{\tilde{\mathbf{u}}}, 0, 0))\right\}\right) = 1.$$

Consequently, we conclude

$$\begin{aligned}\mu_1[(\tilde{\mathbf{u}}_0^1,\tilde{\mathbf{v}}^1,\tilde{\mathscr{V}}^1,\tilde{W})\in\mathfrak{A}] &= \tilde{\mathbb{P}}(\omega\in\Omega;(\tilde{\mathbf{u}}_0^1,\tilde{\mathbf{v}}^1,\tilde{\mathscr{V}}^1,\tilde{W})\in\mathfrak{A})\\ &= \tilde{\mathbb{P}}(\tilde{\omega}\in\tilde{\Omega};(\tilde{\mathbf{u}}_0,\tilde{\mathbf{v}}^2,\tilde{W},\tilde{\mathscr{V}}^2)\in\mathfrak{A})\\ &= \mu_2[(\tilde{\mathbf{u}}_0,\tilde{\mathbf{v}}^2,\tilde{W},\tilde{\mathscr{V}}^2)\in\mathfrak{A}]\quad \mathfrak{A}\in\mathscr{B}(\Theta).\end{aligned}$$

The proof is complete. □

Chapter 4

4.1 Compressible Fluids

4.1.1 Introduction

In this section we consider the *complete stochastic Euler System*. The system models an ideal fluid which is temperature dependent, compressible, inviscid and driven by stochastic forcing. In particular, the fluid model is described by means of three basic state variables: the mass density $\rho = \rho(t,x)$, the velocity field $\mathbf{u} = \mathbf{u}(t,x)$, and the (absolute) temperature $\vartheta = \vartheta(t,x)$, where t is the time, $x \in \mathbb{T}^3$ is a space variable in periodic domain (Eulerian coordinate system). We study the global-in-time evolution of the fluid model flow governed by a system of partial differential equations (mathematical formulations of the physical principles) given by

$$\begin{aligned} \mathrm{d}\rho + \mathrm{div}(\rho\mathbf{u})\,\mathrm{d}t &= 0 \quad \text{in}\, Q, \\ \mathrm{d}(\rho\mathbf{u}) + \mathrm{div}(\rho\mathbf{u}\otimes\mathbf{u})\,\mathrm{d}t + \nabla p(\rho,\vartheta)\,\mathrm{d}t &= \rho\phi\,\mathrm{d}W \quad \text{in}\, Q, \\ \mathrm{d}\left(\frac{1}{2}\rho|\mathbf{u}|^2 + \rho e(\rho,\vartheta)\right) &= -\mathrm{div}\left[\left(\frac{1}{2}\rho|\mathbf{u}|^2 + \rho e(\rho,\vartheta) + p(\rho,\vartheta)\right)\mathbf{u}\right]\mathrm{d}t \\ &\quad + \frac{1}{2}\|\sqrt{\rho}\phi\|_{L_2}^2\,\mathrm{d}t + \rho\phi\cdot\mathbf{u}\mathrm{d}W, \end{aligned} \tag{4.1.1}$$

satisfying: the balance of mass, momentum and total energy, respectively. In (4.1.1), $p(\rho,\vartheta)$ denotes pressure, W denotes the driving force given by a cylindrical Wiener process, and ϕ is a Hilbert-Schmidt operator, see Section 4.1.3 for details. In order to model a physical relevant (i.e. realistic) fluid we endow the system (4.1.1) by a set of constitutive relations characterising the physical principles of a compressible inviscid fluid. Accordingly, we assume that the pressure $p(\rho,\vartheta)$ and the internal energy $e =$

$e(\rho,\vartheta)$ satisfy the caloric equation of state

$$p = (\gamma - 1)\rho e, \tag{4.1.2}$$

where $\gamma > 1$ is the adiabatic constant. Next we assert that the absolute temperature ϑ satisfies the Boyle-Mariotte thermal equation of state:

$$p = \rho\vartheta \quad \text{yielding} \quad e = c_v\vartheta, c_v = \frac{1}{\gamma - 1}. \tag{4.1.3}$$

Finally, we assume that the pressure $p = p(\rho,\vartheta)$, the specific internal energy $e = e(\rho,\vartheta)$, and the specific entropy $s = s(\rho,\vartheta)$ are interrelated through Gibbs' relation

$$\vartheta Ds(\rho,\vartheta) = De(\rho,\vartheta) + p(\rho,\vartheta)D\left(\frac{1}{\rho}\right), \tag{4.1.4}$$

where D is a derivative. At this stage it is essential to note that if p, e and s satisfy (4.1.4), in context of any *smooth* solutions to (4.1.1), the Second law of thermodynamics is enforced through the entropy balance equation

$$\mathrm{d}(\rho s(\rho,\vartheta)) + \mathrm{div}_x(\rho s(\rho,\vartheta)\mathbf{u})\,\mathrm{d}t = 0, \tag{4.1.5}$$

where $s(\rho,\vartheta)$ denotes the (specific) entropy and is of the form

$$s(\rho,\vartheta) = \log(\vartheta^{c_v}) - \log(\rho). \tag{4.1.6}$$

The premise of this section is to study the weak solutions of (4.1.1). In the context of weak solutions, the equality in (4.1.5) no longer holds. In this case the balance of entropy is given as an inequality, we refer the reader to [4] for more details. The assumptions in (4.1.3) for the state variables ρ,ϑ trivially imply thermodynamics stability

$$\partial_\rho p(\rho,\vartheta) > 0, \quad \partial_\vartheta p(\rho,\vartheta) > 0 \quad \text{for all}\, \rho,\vartheta > 0. \tag{4.1.7}$$

One of the difficulties we face in fluid models is the analysis of fluid's interaction with prescribed boundaries. To circumvent such problems from physical boundaries we assign periodic boundary conditions to our system (4.1.1),that is, the physical domain $\mathbb{T}^3$ can be identified with a flat torus

$$\mathbb{T}^3 = ([0,1]|_{0,1})^3.$$

Finally, the initial state of fluid emanates from random initial data

$$\rho(0,\cdot) = \rho_0, \quad \vartheta(0,\cdot) = \vartheta_0, \quad \mathbf{u}(0,\cdot) = \mathbf{u}_0, \tag{4.1.8}$$

subject to initial law Λ. To ensure our solutions are physical relevant we augment the problem by the total energy balance of the form

$$\mathrm{d}\int_{\mathbb{T}^3}\left[\frac{1}{2}\rho|\mathbf{u}|^2+\rho e\right]\mathrm{d}x = \frac{1}{2}\|\sqrt{\rho}\phi\|_{L_2}^2\,\mathrm{d}t + \int_{\mathbb{T}^3}\rho\phi\cdot\mathbf{u}\,\mathrm{d}x\mathrm{d}W. \tag{4.1.9}$$

We note the strong solutions of the complete Euler system (4.1.1) satisfy the energy equality (4.1.9), but in weak solutions it has to be added in the definition.

4.1.2 Mathematical framework and main results

In addition to general materials on random variables in Section 2.1 we present the probability framework for Markov selection and stochastic framework suitable for solving (4.1.1). We conclude the section by stating the main results of the complete Euler system.

Probability framework

Let (X,τ) be a topological space. We denote by $\mathscr{B}(X)$ the σ-algebra of Borel subsets of X. Let $\mathscr{P}$ be a Borel measure on X, the symbol $\overline{\mathscr{B}(X)}$ denotes the σ-algebra of all Borel subsets of X augmented by all zero measure sets. Let $\mathrm{Prob}[X]$ denote the set of all Borel probability measures on a topological space X. Furthermore, let $([0,1],\overline{\mathscr{B}[0,1]},\mathscr{L})$ denote the standard probability space, where $\mathscr{L}$ is one dimensional Lebesgue measure.

Trajectory/Path spaces

Let (X,d_X) be a Polish space. For $t>0$ we introduce the path spaces

$$\Omega_X^{[0,T]} = C([0,T];X), \qquad \Omega_X^{[T,\infty)} = C([T,\infty);X), \qquad \Omega_X^{[0,\infty)} = C([0,\infty);X),$$

the path spaces are Polish as long as X is Polish, and we denote by $\mathscr{B}_T = \mathscr{B}(\Omega_X^{[0,T]})$ the Borel σ-algebra. Then, for $\omega \in \Omega_X^{[0,T]}$ we define a time shift operator

$$\Phi_\tau : \Omega_X^{[t,\infty)} \to \Omega_X^{[T+\tau,\infty)}, \qquad \Phi_\tau[\omega]_s = \omega_{s-\tau}, \, s \geq T+\tau,$$

where Φ_τ is an isometry from $\Phi_\tau : \Omega_X^{[T,\infty)}$ to $\Omega_X^{[T+\tau,\infty)}$. For a Borel measure ν on $\Omega_X^{[t,\infty)}$, the time shift $\Phi_{-\tau}[\nu]$ is a Borel measure on the space $\Omega_X^{[T-\tau,\infty)}$ given by

$$\Phi_{-\tau}[\nu](B) = \nu(\Phi_\tau(B)), \qquad B \in \mathscr{B}(\Omega_X^{[T-\tau,\infty)}).$$

We proceed to recall the results of Stroock and Varadhan [86] we need for our analysis. In view of [86, Theorem 1.1.6] we obtain disintegration results, that is, existence of regular conditional probability law.

Theorem 4.1.1 (Disintegration)**.** *Let X be a polish space. Given* $\mathscr{P} \in Prob[\Omega_X^{[0,\infty)}]$*, let* $T > 0$ *be a finite* $\mathscr{B}_t$*-stopping time. Then there exists a unique family of probability measures*

$$\mathscr{P}|_{\mathscr{B}_T}^{\tilde{\omega}} \in Prob[\Omega_X^{[T,\infty)}] \; for \; \mathscr{P}\text{-a.a.}\tilde{\omega}$$

such that the mapping

$$\Omega_X^{[0,\infty)} \ni \tilde{\omega} \mapsto \mathscr{P}|_{\mathscr{B}_T}^{\tilde{\omega}} \in Prob[\Omega_X^{[T,\infty)}]$$

is $\mathscr{B}_T$*-measurable and the following properties hold*

(a) For $\omega \in \Omega_X^{[T,\infty)}$ *we have* $\mathscr{P}|_{\mathscr{B}_T}^{\tilde{\omega}}$*-a.s.*

$$\omega(T) = \tilde{\omega}(T);$$

(b) For any Borel set $A \subset \Omega_X^{[0,T]}$ *and any Borel set* $B \subset \Omega_X^{[T,\infty)}$*,*

$$\mathscr{P}(\omega|_{[0,T]} \in A, \omega|_{[T,\infty)} \in B) = \int_{\tilde{\omega}|_{[0,T]} \in A} \mathscr{P}|_{\mathscr{B}_T}^{\tilde{\omega}}(B) \, \mathrm{d}\mathscr{P}(\tilde{\omega}).$$

Note, a conditional probability corresponds to disintegration of probability measure with respect to a σ-field. Accordingly, *reconstruction* can be understood as the opposite process of disintegration, that is, some sort of "gluing together" procedure. On the account of results established in [86, Lemma 6.1.1] and [86, Theorem 6.1.2] we have the following results on reconstruction.

Theorem 4.1.2 (Reconstruction)**.** *Let X be a Polish space. Let $\mathscr{P} \in Prob[\Omega_X^{[0,\infty)}]$ and T be a finite $\mathscr{B}_T$-stopping time. Suppose that Q_ω is a family of probability measures, such that*

$$\Omega_X^{[0,\infty)} \ni \omega \mapsto Q_\omega \in \ Prob[\Omega_X^{[T,\infty)}],$$

is $\mathscr{B}_T$-measurable. Then there exists a unique probability measure $\mathscr{P} \otimes_T Q$ such that :

(a) For any Borel set $A \in \Omega_X^{[0,T]}$ we have

$$(\mathscr{P} \otimes_T Q)(A) = \mathscr{P}(A);$$

(b) For $\tilde{\omega} \in \Omega$ we have $\mathscr{P}$-a.s.

$$(\mathscr{P} \otimes_T Q)|_{\mathscr{B}_T}^{\tilde{\omega}} = Q_{\tilde{\omega}}$$

Markov processes

We proceed to study Markov process following the abstract framework in [12] and references therein. Assuming (X, d_X) and (F, d_F) are two Polish spaces, let the embedding $F \hookrightarrow X$ be continuous and dense. Moreover, let Y be a Borel subset of F. Since (Y, d_F) is not necessarily a complete *space*, it may happen that the embedding $Y \hookrightarrow X$ is not dense. A family of probability measures $\{\mathscr{P}_y\}_{y\in Y}$ on $\Omega_X^{[0,\infty)}$ is called Markovian if we have for $y \in Y$ that

$$\mathscr{P}_{\omega(\tau)} = \Phi_{-\tau}\mathscr{P}_y|_{\mathscr{B}_\tau}^{\omega} - \text{a.a.}\ \omega \in \Omega_X^{[0,\infty)} \text{ and all } \tau \geq 0.$$

Next we define probability measures with support only on certain subset of a Polish space.

Definition 4.1.1. Let Y be a Borel subset of F and let $\mathscr{P} \in \text{Prob}[\Omega_X^{[0,\infty)}]$. A family of probability measures $\mathscr{P}$ is concentrated on the paths with values in Y if there is some $A \in \mathscr{B}(\Omega_X^{[0,\infty)})$ such that $\mathscr{P}(A) = 1$ and $A \subset \{\omega \in \Omega_X^{[0,\infty)} : \omega(\tau) \in Y \, \forall \tau \geq 0\}$. We write $\mathscr{P} \in \text{Prob}_Y[\Omega_X^{[0,\infty)}]$.

We generalise the classical Markov property (*to a situation where it only holds for a.e time-point*) as follows:

Definition 4.1.2 (Almost Sure Markov property)**.** Let $y \mapsto \mathscr{P}_y$ be a measurable map defined on a measurable subset $Y \subset F$ with values in $\mathrm{Prob}_Y[\Omega_X^{[0,\infty)}]$. The family $\{\mathscr{P}_y\}_{y\in Y}$ has the almost sure Markov property if for each $y \in Y$ there is a set $\mathfrak{Z} \subset (0,\infty)$ with zero Lebesgue measure such that

$$\mathscr{P}_{\omega(\tau)} = \Phi_{-\tau}\mathscr{P}_y|^{\omega}_{\mathscr{B}_\tau} \quad \text{for } \mathscr{P}_y - \text{a.a.}\omega \in \Omega_X^{[0,\infty)}$$

and all $\tau \notin \mathfrak{Z}$

Finally, based on the link between disintegration and reconstruction as observed by [74], we have the following definition.

Definition 4.1.3 (Almost sure pre-Markov family)**.** Let Y be a Borel subset of F. Let $\mathscr{C}$: $Y \to \mathrm{Comp}(\mathrm{Prob}[\Omega_X^{[0,\infty)}] \cap \mathrm{Prob}_Y[\Omega_X^{[0,\infty)}])$ be a measurable map, where $\mathrm{Comp}(\cdot)$ denotes the family of all compact subsets. The family $\{\mathscr{C}(y)\}_{y\in Y}$ is almost surely pre-Markov if for each $y \in Y$ and $\mathscr{P} \in \mathscr{C}(y)$ there is a set $\mathfrak{Z} \subset (0,\infty)$ with zero Lebesgue measure such that the following holds for all $\tau \notin \mathfrak{Z}$:

(a) The disintegration property holds, that is, we have

$$\Phi_{-\tau}\mathscr{P}|^{\omega}_{\mathscr{B}_\tau} \in \mathscr{C}(\omega(\tau)) \quad \text{for } \mathscr{P}\text{-a.a. } \omega \in \Omega_X^{[0,\infty)};$$

(b) The reconstruction property holds, that is, for each $\mathscr{B}_T$-measurable map $\omega \mapsto Q_\omega$: $\Omega_X^{[0,\infty)} \to \mathrm{Prob}(\Omega_X^{[\tau,\infty)})$ with

$$\Phi_{-\tau}Q_\omega \in \mathscr{C}(\omega(\tau)) \quad \text{for } \mathscr{P}\text{-a.a. } \omega \in \Omega_X^{[0,\infty)}$$

we have $\mathscr{P} \otimes_\tau Q \in \mathscr{C}(y)$.

Note Definition 4.1.3 is motivated by results in [54, 58]. We conclude our probability framework by stating the following results.

Theorem 4.1.3 (Markov Selection)**.** *Let Y be a Borel subset of F. Let $\{\mathscr{C}(y)\}_{y\in Y}$ be an almost sure pre-Markov family (as defined in Definition 4.1.3) with nonempty convex values. Then there is a measurable map $y \mapsto \mathscr{P}_y$ defined on Y with values in* $\mathrm{Prob}_Y[\Omega_X^{[0,\infty)}]$ *such that $\mathscr{P}_y \in \mathscr{C}(y)$ for all $y \in Y$ and $\{\mathscr{P}_y\}_{y\in Y}$ has the almost sure Markov property (as defined in Definition 4.1.2)*

The following proposition is proved in [54].

Proposition 4.1.4 ([54], Proposition B.1)**.** *Let α and β be two real-valued continuous and $(\mathscr{B}_t)$-adapted stochastic processes on Ω such that $\alpha,\beta : [0,\infty)\times\Omega \to \mathbb{R}$ and let $t_0 \geq 0$. Then for $\mathscr{P} \in \mathrm{Prob}[\Omega]$ the following conditions are equivalent:*

(a) $(\alpha_t)_{t\geq 0}$ *is a* $((\mathscr{B}_t)_{t\geq 0},\mathscr{P})$*-square integrable martingale with quadratic variation* $(\beta_t)_{t\geq 0}$

(b) For $\mathscr{P}$*-a.a.* $\omega \in \Omega$ *the stochastic process* $(\alpha_t)_{t\geq t_0}$ *is a* $((\mathscr{B}_t)_{t\geq t_0},\mathscr{P}^{\omega}_{\mathscr{B}_{t_0}})$*-square integrable martingale with quadratic variation* $(\beta_t)_{t\geq t_0}$ *and we have* $\mathbb{E}^{\mathscr{P}}[\mathbb{E}^{\mathscr{P}|^{\cdot}_{\mathscr{B}_{t_0}}}[\beta_t]] < \infty$ *for all* $t \geq t_0$.

4.1.3 Stochastic analysis

Let $(\Omega,\mathscr{F},(\mathscr{F}_t)_{t\geq 0},\mathbb{P})$ be a complete stochastic basis with a probability measure $\mathbb{P}$ on $(\Omega,\mathscr{F})$ and a right-continuous filtration $(\mathscr{F}_t)_{t\geq 0}$. Let $\mathscr{U}$ be a separable Hilbert space with an orthonormal basis $(e_k)_{k\in\mathbb{N}}$. We denote by $L_2(\mathscr{U},L^2(\mathbb{T}^3))$ the set of Hilbert-Schmidt operators from $\mathscr{U}$ to $L^2(\mathbb{T}^3)$. The stochastic process W is a cylindrical Wiener process $W = (W_t)_{t\geq 0}$ in $\mathscr{U}$, and is of the form

$$W(s) = \sum_{k\in\mathbb{N}} e_k\beta_k(s), \tag{4.1.10}$$

where $(\beta_k)_{k\in\mathbb{N}}$ is a sequence of independent real-valued Wiener process relative to $(\mathscr{F}_t)_{t\geq 0}$. To identify the precise definition of the diffusion coefficient, set $\mathscr{U} = \ell^2$ and consider $\rho \in L^\gamma(\mathbb{T}^3)$, $\rho > 0$, then the mapping $\phi \in L_2(\mathscr{U},L^2(\mathbb{T}^3))$, that is, $\phi : \mathscr{U} \to L^2(\mathbb{T}^3)$ is defined as follows

$$\phi(e_k) = \phi_k.$$

We suppose that ϕ is a Hilbert-Schmidt operator such that

$$\sum_{k\geq 1} \|\phi(e_k)\|^2_{L^\infty(\mathbb{T}^3)} < \infty, \tag{4.1.11}$$

in particular we have $\phi \in L_2(\mathscr{U}, L^\infty(\mathbb{T}^3))$. Consequently, since ϕ is bounded we deduce

$$\|\sqrt{\rho}\phi\|^2_{L_2(\mathscr{U},L^2(\mathbb{T}^3))} \lesssim c(\phi)(\|\rho\|_{L^1(\mathbb{T}^3)}). \tag{4.1.12}$$

In this setting $\rho\phi \in L_2(\mathscr{U}, L^1(\mathbb{T}^3))$. Now arguing similarly as in [20], we expect the momentum equation to be satisfied only in the sense of distributions and consider the embedding $L^1 \hookrightarrow W^{-k,2}(\mathbb{T}^3)$ (which holds provided $k > 3/2$), and we interpret the stochastic integral as a process in the Banach space $W^{-k,2}(\mathbb{T}^3)$, $k > 3/2$. Then the stochastic integral

$$\int_0^\tau \rho\phi \,\mathrm{d}W = \sum_{k\geq 1} \int_0^\tau \rho\phi(e_k)\,\mathrm{d}\beta_k,$$

takes values in the Banach space $C([0,T]; W^{-k,2}(\mathbb{T}^3))$ in the sense that

$$\int_{\mathbb{T}^3} \left(\int_0^\tau \rho\phi \,\mathrm{d}W \cdot \varphi \right) \mathrm{d}x = \sum_{k\geq 1} \int_0^\tau \left(\int_{\mathbb{T}^3} \rho\phi(e_k)\cdot\varphi \,\mathrm{d}x \right) \mathrm{d}\beta_k, \quad \varphi \in W^{k,2}(\mathbb{T}^3), k > \frac{3}{2}. \tag{4.1.13}$$

Finally, we define the auxiliary space $\mathscr{U}_0$ with $\mathscr{U} \subset \mathscr{U}_0$ as

$$\begin{aligned} \mathscr{U}_0 : &= \left\{ e = \sum_k \alpha_k e_k : \sum_k \frac{\alpha_k^2}{k^2} < \infty \right\}, \\ \|e\|^2_{\mathscr{U}_0} : &= \sum_k^\infty \frac{\alpha_k^2}{k^2}, \; e = \sum_k \alpha_k e_k, \end{aligned} \tag{4.1.14}$$

so that the embedding $\mathscr{U} \hookrightarrow \mathscr{U}_0$ is Hilbert-Schmidt, and the trajectories of W belong $\mathbb{P}$-a.s. to the class $C([0,T]; \mathscr{U}_0)$ (see [35]).

4.1.4 Strong solutions

The concept of weak(measure-valued)-strong uniqueness principle for dissipative solutions requires existence of strong solutions. These solutions are strong in the probabilistic and PDE sense, at least locally in time. In particular, the Euler system (4.1.1) will be satisfied pointwise (not only in the sense of distributions) on the given stochastic basis associated to the cylindrical Wiener process W.

Definition 4.1.4 (Strong Solution)**.** Let $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$ be a complete stochastic basis with a right-continuous filtration, let W be an $(\mathscr{F}_t)_{t\geq 0}$-cylindrical Wiener process. The triplet $[r, \Theta, \mathbf{U}]$ and a stopping time $\mathfrak{t}$ is called a (local) strong solution to the system (4.1.1) provided:

- the density $r > 0$ $\mathbb{P}$-a.s., $t \mapsto r(t \wedge \mathfrak{t}, \cdot) \in W^{3,2}(\mathbb{T}^3)$ is $(\mathscr{F}_t)_{t\geq 0}$-adapted,

$$\mathbb{E}\left[\sup_{t\in[0,T]} \|r(t\wedge \mathfrak{t},\cdot)\|^p_{W^{3,2}}\right] < \infty \quad \text{for all } 1 \leq p < \infty;$$

- the temperature $\Theta > 0$ $\mathbb{P}$-a.s., $t \mapsto \Theta(t \wedge \mathfrak{t}, \cdot) \in W^{3,2}(\mathbb{T}^3)$ is $(\mathscr{F}_t)_{t\geq 0}$-adapted,

$$\mathbb{E}\left[\sup_{t\in[0,T]} \|\Theta(t\wedge \mathfrak{t},\cdot)\|^p_{W^{3,2}}\right] < \infty \quad \text{for all } 1 \leq p < \infty;$$

- the velocity $t \mapsto \mathbf{U}(t \wedge \mathfrak{t}, \cdot) \in W^{3,2}(\mathbb{T}^3)$is $(\mathscr{F}_t)_{t\geq 0}$-adapted,

$$\mathbb{E}\left[\sup_{t\in[0,T]} \|\mathbf{U}(t\wedge \mathfrak{t},\cdot)\|^p_{W^{3,2}}\right] < \infty \quad \text{for all } 1 \leq p < \infty;$$

- for all $t \in [0,T]$ there holds $\mathbb{P}$-a.s.

$$r(t\wedge\mathfrak{t}) = \rho(0) - \int_0^{t\wedge\mathfrak{t}} \operatorname{div}_x(r\mathbf{U})\,\mathrm{d}t,$$

$$(r\mathbf{U})(t\wedge\mathfrak{t}) = (r\mathbf{U})(0) - \int_0^{t\wedge} \operatorname{div}(r\mathbf{U}\otimes\mathbf{U})\,\mathrm{d}t - \int_0^{t\wedge\mathfrak{t}} \nabla_x p(r,\Theta)\,\mathrm{d}t + \int_0^{t\wedge\mathfrak{t}} r\phi\,\mathrm{d}W,$$

$$(rs(r,\Theta))(t\wedge\mathfrak{t}) = (rs(r,\Theta))(0) - \int_0^{t\wedge\mathfrak{t}} \operatorname{div}_x(rs(r,\Theta)\mathbf{U})\,\mathrm{d}t,$$

where s is the total entropy given by (4.1.6).

Remark 4.1.1. We expect a blow up in finite time for strong solutions as in the deterministic case [85].

4.1.5 The approximate system

To begin, we introduce a cut-off function

$$\chi \in C^\infty(\mathbb{R}), \chi(z) = \begin{cases} 1\,\text{for}\,z \leq 0, \\ \chi'(z) \leq 0 \text{ for } 0 < z < 1, \\ \chi(z) = 0 \text{ for } z \geq 1, \end{cases}$$

together with the operator

$$\phi_\varepsilon = \chi\left(|\mathbf{v}| - \frac{1}{\varepsilon}\right)\phi, \quad \varepsilon > 0. \tag{4.1.15}$$

The operator with a cut-off expression is needed for exponential estimates, see Section 4.1.8 for details. Let $Q = (0,T) \times \mathbb{T}^3$ be a periodic space-time cylinder. We consider a stochastic variant of a system introduced in [73], and further refined in [10]. That is, the *complete* Euler system (4.1.1) is approximated by:

$$\begin{cases} \mathrm{d}\rho + \mathrm{div}(\rho\mathbf{u})\mathrm{d}t = 0 & \text{in } Q, \\ \mathrm{d}\rho\mathbf{u} + \mathrm{div}(\rho\mathbf{u}\otimes\mathbf{u})\mathrm{d}t + \nabla_x p(\rho,s)\,\mathrm{d}t = \varepsilon\mathscr{L}\mathbf{u}\,\mathrm{d}t + \rho\phi_\varepsilon\mathrm{d}W & \text{in } Q, \\ \mathrm{d}\rho s + \mathrm{div}(\rho s\mathbf{u})\,\mathrm{d}t \geq 0 & \text{in } Q, \end{cases} \tag{4.1.16}$$

with initial conditions

$$\rho(0.\cdot) = \rho_{0,\varepsilon}\,\mathbf{u}(0,\cdot) = \mathbf{u}_{0,\varepsilon},\, s(0,\cdot) = s_{0,\varepsilon}.$$

Here, the unknown fields are: the fluid density $\rho = \rho(t,x)$, the velocity $\mathbf{u} = \mathbf{u}(t,x)$ and the total entropy ($S = \rho s$). We denote by $\mathscr{L}$, the suitable 'viscosity' operator.

Let $W^{3,2}(\mathbb{T}^3)$ be a separable Hilbert space complemented with $((;))$, a scalar product on $W^{3,2}(\mathbb{T}^3)$, i.e.

$$((\mathbf{v};\mathbf{w})) = \sum_{|\alpha|=3}\int_{\mathbb{T}^3}\nabla_x^\alpha\mathbf{v}\cdot\nabla_x^\alpha\mathbf{w}\,\mathrm{d}x + \int_{\mathbb{T}^3}\mathbf{v}\cdot\mathbf{w}\,\mathrm{d}x,\ \mathbf{v},\mathbf{w}\in W^{3,2}(\mathbb{T}^3).$$

In reference to [67], we consider a self-adjoint operator $\mathscr{L}$ on $W^{3,2}(\mathbb{T}^3)$ associated with the bilinear form $((;))$ given by

$$\mathscr{L}\mathbf{u} = \Delta^3\mathbf{u} - \mathbf{u} = \sum_{|\alpha|=3}(\nabla_x^\alpha)\nabla_x^\alpha\mathbf{u} - \mathbf{u}.$$

In view of the viscosity operator $\mathscr{L}$ considered, the weak formulation associated with the momentum equation in (4.1.16) reads

$$\left[\int_{\mathbb{T}^3} \rho \mathbf{u} \cdot \varphi \,\mathrm{d}x\right]_{t=0}^{t=\tau} = \int_0^T \int_{\mathbb{T}^3} \left[\rho \mathbf{u} \otimes \mathbf{u} : \nabla_x \varphi + \rho^\gamma \exp\left(\frac{s}{c_v}\right) \mathrm{div}\varphi\right] \mathrm{d}x\mathrm{d}t$$
$$-\varepsilon \int_0^T ((\mathbf{u};\varphi))\,\mathrm{d}t + \int_0^T \int_{\mathbb{T}^3} \rho \phi \cdot \varphi \,\mathrm{d}x\mathrm{d}W,$$

for any $\tau > 0$, and any $\varphi \in W^{3,2}(\mathbb{T}^3)$. The continuity equation and total entropy in (4.1.16) are solved strongly, while the momentum equation is solved in the weak sense. We expect the approximate system (4.1.16) to have stochastically strong solutions, but in the present work martingale solutions are sufficient for our purposes. In the following we state the existence theorem of martingale solutions to the approximate system (4.1.16).

Theorem 4.1.5. *Assume (4.1.11) holds. Let Λ_ε be a Borel probability measure on $L^\gamma(\mathbb{T}^3) \times L^\gamma(\mathbb{T}^3) \times L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^3)$ such that*

$$\Lambda_\varepsilon \left\{ (\rho, S, \mathbf{m}) \in L^\gamma(\mathbb{T}^3) \times L^\gamma(\mathbb{T}^3) \times L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^3) : 0 < \underline{\rho} < \rho < \overline{\rho}, 0 < \underline{\vartheta} < \vartheta < \overline{\vartheta} \right\} = 1,$$

where $\underline{\vartheta}, \overline{\vartheta}, \underline{\rho}, \overline{\rho}$ are deterministic constants. Moreover, the moment estimate

$$\int_{L^\gamma(\mathbb{T}^3)\times L^\gamma(\mathbb{T}^3)\times L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^3)} \left\| \frac{1}{2}\frac{|\mathbf{m}|^2}{\rho} + c_v \rho^\gamma \exp\left(\frac{S}{c_v \rho}\right) \right\|_{L^1(\mathbb{T}^3)}^p \mathrm{d}\Lambda_\varepsilon < \infty,$$

holds for all $p \geq 1$. Then there exists a martingale solution to the approximate problem (4.1.16) subject to initial law Λ_ε.

4.1.6 Measure-valued solutions

In order to introduce the concept of stochastic measure-valued martingale solutions, we reformulate the *complete* Euler system using the variables $\mathbf{m} = \rho \mathbf{u}$ and $S = \rho s$ so that (4.1.1) reads

$$\mathrm{d}\rho + \mathrm{div}_x \mathbf{m}\,\mathrm{d}t = 0 \tag{4.1.17}$$
$$\mathrm{d}\mathbf{m} + \mathrm{div}_x \left(\frac{\mathbf{m}\otimes\mathbf{m}}{\rho}\right) \mathrm{d}t + \nabla_x p(\rho, s)\,\mathrm{d}t = \rho\phi \mathrm{d}W \text{ in } Q, \tag{4.1.18}$$
$$\mathrm{d}S + \mathrm{div}_x \left(\frac{S\mathbf{m}}{\rho}\right) \mathrm{d}t \geq 0 \quad \text{in } Q. \tag{4.1.19}$$

We note that, in general, the admissibility criterion for physically possible solutions is the energy equality and it is the only *tool* of establishing a *priori* bounds. However, the 'a *priori*' bounds deduced do not guarantee strong convergences of nonlinear terms $\frac{\mathbf{m}\otimes\mathbf{m}}{\rho}, p(\rho,s) \in L^1(\mathbb{T}^3)$ due to the presence of *oscillations* and *concentrations*. Given such a scenario, we adopt the characterisation of (nonlinear terms) in the weak formulation as combination of Young measures and defect measures.

- Young measures are probability measures on the phase space, they capture the oscillations of the solution.
- Defect measures are measures on physical space-time and they account for the 'blow up' type collapse due to possible concentration points.

We are now ready to introduce the concept of measure-valued martingale solutions to the *complete* stochastic Euler system (4.1.17)-(4.1.19). From here onward, we denote by $\mathscr{M}^+$ the space of non-negative radon measures, and we denote by A the space of "dummy variables":

$$A = \left\{ [\rho', \mathbf{m}', S'] \middle| \rho' \geq 0, \mathbf{m}' \in \mathbb{R}^3, S' \in \mathbb{R} \right\} \tag{4.1.20}$$

Let $\mathscr{P}(A)$ denote the space of probability measures on A.

Definition 4.1.5 (Dissipative measure-valued martingale solution)**.** Let Λ be a Borel probability measure on $L^\gamma \times L^{\frac{2\gamma}{\gamma+1}} \times L^\gamma$ and $\phi \in L_2(\mathscr{U}; L^2(\mathbb{T}^3))$. Then

$$((\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P}), \rho, \mathbf{m}, S, \mathscr{R}_{\text{conv}}, \mathscr{R}_{\text{press}}, \mathscr{V}_{t,x}, W)$$

is called a dissipative measure-valued solution to (4.1.17)-(4.1.19) with initial law Λ, provided[1]:

(a) $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$ is a stochastic basis with complete right-continuous filtration;

(b) W is a $(\mathscr{F}_t)_{t\geq 0}$-cylindrical Wiener process;

(c) The density ρ is $(\mathscr{F}_t)_{t\geq 0}$-adapted and satisfies $\mathbb{P}$-a.s.

$$\rho \in C_{\text{loc}}([0,\infty), W^{-4,2}(\mathbb{T}^3)) \cap L^\infty_{\text{loc}}(0,\infty; L^\gamma(\mathbb{T}^3));$$

[1]Some of our variables are not stochastic processes in the classical sense and we use their adaptedness in the sense of random distributions as introduced in Section 2.1.3, see [15, Chap. 2.2] for more details.

(d) The momentum $\mathbf{m}$ is $(\mathscr{F}_t)_{t\geq 0}$-adapted and satisfies $\mathbb{P}$-a.s.

$$\mathbf{m} \in C_{\mathrm{loc}}([0,\infty), W^{-4,2}(\mathbb{T}^3)) \cap L^{\infty}_{\mathrm{loc}}(0,\infty; L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^3));$$

(e) The total entropy S is $(\mathscr{F}_t)_{t\geq 0}$-adapted and satisfies $\mathbb{P}$-a.s.

$$S \in L^{\infty}([0,\infty), L^{\gamma}(\mathbb{T}^3)) \cap BV_{w,\mathrm{loc}}(0,\infty; W^{-l,2}(\mathbb{T}^3)), l > \frac{5}{2};$$

(f) The parametrised measures $(\mathscr{R}_{\mathrm{conv}}, \mathscr{R}_{\mathrm{press}}, \mathscr{V})$ are $(\mathscr{F}_t)_{t\geq 0}$-progressively measurable and satisfy $\mathbb{P}$-a.s.

$$t \mapsto \mathscr{R}_{\mathrm{conv}}(t) \in L^{\infty}_{\text{weak-}(*)}(0,\infty; \mathscr{M}^+(\mathbb{T}^3, \mathbb{R}^{3\times 3})); \tag{4.1.21}$$

$$t \mapsto \mathscr{R}_{\mathrm{press}}(t) \in L^{\infty}_{\text{weak-}(*)}(0,\infty; \mathscr{M}^+(\mathbb{T}^3, \mathbb{R})); \tag{4.1.22}$$

$$(t,x) \mapsto \mathscr{V}_{t,x} \in L^{\infty}_{\text{weak-}(*)}(Q; \mathscr{P}(A)); \tag{4.1.23}$$

(g) $\Lambda = \mathbb{P} \circ (\rho(0), \mathbf{m}(0), S0)^{-1}$;

(h) For all $\varphi \in C^{\infty}(\mathbb{T}^3)$ and all $\tau > 0$ there holds $\mathbb{P}$-a.s.

$$\left[\int_{\mathbb{T}^3} \rho\varphi \, \mathrm{d}x\right]_{t=0}^{\tau=0} = \int_0^{\tau}\int_{\mathbb{T}^3} \mathbf{m}\cdot\nabla\varphi \, \mathrm{d}x\mathrm{d}t; \tag{4.1.24}$$

(i) For all $\varphi \in C^{\infty}(\mathbb{T}^3)$ and all $\tau > 0$ there holds $\mathbb{P}$-a.s.

$$\begin{aligned}\left[\int_{\mathbb{T}^3} \mathbf{m}\cdot\varphi\right]_{t=0}^{t=\tau} &= \int_0^{\tau}\int_{\mathbb{T}^3}\left[\frac{\mathbf{m}\otimes\mathbf{m}}{\rho} : \nabla\varphi + \rho\exp\left(\frac{S}{c_v\rho}\mathrm{div}\varphi\right)\right]\mathrm{d}x\mathrm{d}t \\ &\quad + \int_0^{\tau} \nabla\varphi : \mathrm{d}\mathscr{R}_{\mathrm{conv}}\mathrm{d}t + \int_0^{\tau}\int_{\mathbb{T}^3} \mathrm{div}\varphi \, \mathrm{d}\mathscr{R}_{\mathrm{press}}\mathrm{d}t \\ &\quad + \int_0^{\tau} \varphi\cdot\rho\phi \, \mathrm{d}x\mathrm{d}W;\end{aligned} \tag{4.1.25}$$

(j) The total entropy holds in the sense that

$$\int_0^{\tau}\int_{\mathbb{T}^3}\left[\langle\mathscr{V}_{t,x}; Z(S')\rangle\partial_t\varphi + \langle\mathscr{V}_{t,x}, Z(S')\frac{\mathbf{m}'}{\rho'}\rangle\cdot\nabla\varphi\right]\mathrm{d}x\mathrm{d}t \leq \left[\int_{\mathbb{T}^3}\langle\mathscr{V}_{t,x}; Z(S')\rangle\varphi \, \mathrm{d}x\right]_{t=0}^{t=\tau}, \tag{4.1.26}$$

for any $\varphi \in C^1([0,\infty)\times\mathbb{T}^3), \varphi \geq 0, \mathbb{P}$-a.s.,and any, $Z \in BC(\mathbb{R})$ non-decreasing.

(k) The total energy satisfies

$$\mathbf{E}_t = \mathbf{E}_s + \frac{1}{2}\int_s^t \|\sqrt{\rho}\phi\|^2_{L_2(\mathscr{U};L^2(\mathbb{T}^3))}\,\mathrm{d}\sigma + \int_s^t\int_{\mathbb{T}^3} \mathbf{m}\cdot\phi\,\mathrm{d}x\mathrm{d}W, \tag{4.1.27}$$

$\mathbb{P}$-a.s. for a.a. $0 \le s < t$, where

$$\mathbf{E}_t = \int_{\mathbb{T}^3}\left[\frac{1}{2}\frac{|\mathbf{m}|^2}{\rho} + c_v\rho^\gamma \exp\left(\frac{S}{c_v\rho}\right)\right]\mathrm{d}x + \frac{1}{2}\int_{\mathbb{T}^3}\mathrm{d}\mathrm{tr}\mathscr{R}_{\mathrm{conv}}(t) + c_v\int_{\mathbb{T}^3}\mathrm{d}\mathrm{tr}\mathscr{R}_{\mathrm{press}}(t)$$

for $t > 0$ and

$$E_0 = \int_{\mathbb{T}^3}\left[\frac{1}{2}\frac{|\mathbf{m}_0|^2}{\rho_0} + c_v\rho_0^\gamma \exp\left(\frac{S_0}{c_v\rho_0}\right)\right]\mathrm{d}x.$$

Remark 4.1.2. The use of cut-off function Z in (4.1.26) is inspired by Chen and Frid [30].

4.1.7 Main results

We proceed to state the second main results of the thesis. The existence of dissipative measure-valued martingale solutions is given in the following theorem.

Theorem 4.1.6. *Assuming (4.1.11) holds. Let Λ be a Borel probability measure on $L^\gamma(\mathbb{T}^3)\times L^\gamma(\mathbb{T}^3)\times L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^3)$ such that*

$$\Lambda\left\{(\rho,S,\mathbf{m})\in L^\gamma(\mathbb{T}^3)\times L^\gamma(\mathbb{T}^3)\times L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^3) : 0<\underline{\rho}<\rho<\overline{\rho}, 0<\underline{\vartheta}<\vartheta<\overline{\vartheta}\right\} = 1,$$

where $\underline{\vartheta},\overline{\vartheta},\underline{\rho},\overline{\rho}$ are deterministic constants. Moreover, the moment estimate

$$\int_{L^\gamma(\mathbb{T}^3)\times L^\gamma(\mathbb{T}^3)\times L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^3)}\left\|\frac{1}{2}\frac{|\mathbf{m}|^2}{\rho} + c_v\rho^\gamma\exp\left(\frac{S}{c_v\rho}\right)\right\|^p_{L^1(\mathbb{T}^3)}\mathrm{d}\Lambda < \infty,$$

holds for all $p \ge 1$. Then there exists a dissipative measure-valued martingale solution to the complete stochastic Euler system (4.1.17)-(4.1.19) in the sense of Definition 4.1.5 subject to initial law Λ.

Furthermore, in view of (4.1.2) the following condition (i.e. *purely technical hypothesis*) is satisfied

$$|p(\rho,\vartheta)| \lesssim (1+\rho+\rho e(\rho,\vartheta)+\vartheta|s(\rho,\vartheta)|), \tag{4.1.28}$$

and we shall use it to establish bounds. The use of (4.1.28) will be made clear in later sections when deriving bounds. In addition to existence results we establish the following weak (measure-valued)-strong uniqueness principle:

Theorem 4.1.7. *The pathwise weak-strong uniqueness holds true for the system (4.1.17)-(4.1.19) in the following sense. Let the thermodynamics functions* $e = e(\rho, \vartheta), s = s(\rho, \vartheta)$, *and* $p = p(\rho, \vartheta)$ *satisfy the Gibbs' relation (4.1.4), and the technical hypothesis (4.1.28). let*

$$[((\Omega, \mathscr{F}, (\mathscr{F})_{t\geq 0}, \mathbb{P}), \rho, \mathbf{m}, S, \mathscr{R}_{\text{conv}}, \mathscr{R}_{\text{press}}, \mathscr{V}_{t,x}, W)$$

be a dissipative measure-valued martingale solution to (4.1.17)-(4.1.19) in the sense of Definition (4.1.5), let the triplet $[r, \Theta, \mathbf{U}]$ *and a stopping time* $\mathfrak{t}$ *be a strong solution in the sense of Definition 4.1.4 of the same problem; defined on the stochastic basis with the same Wiener process and with initial data*

$$\rho(0,\cdot) = r(0,\cdot), \quad \mathbf{u}(0,\cdot) = \mathbf{U}(0,\cdot), \qquad \vartheta(0,\cdot) = \Theta(0,\cdot) \quad \mathbb{P}\text{-}a.s. \tag{4.1.29}$$

Then

$$[\rho, \vartheta, \mathbf{u}](\cdot \wedge \mathfrak{t}) \equiv [r, \Theta, \mathbf{U}](\cdot \wedge \mathfrak{t}),$$

and

$$\mathscr{R}_{\text{conv}} = \mathscr{R}_{\text{press}} = 0,$$

$\mathbb{P}$*-a.s., and for any* $(t,x) \in (0,T) \times \mathbb{T}^3$

$$\mathscr{V}_{t\wedge\mathfrak{t},x} = \delta_{r,\mathbf{U},s(r,\Theta)},$$

$\mathbb{P}$*-a.s.*

4.1.8 Basic approximate problem

We devote this section of the thesis to the sketch proof of Theorem 4.1.5, that is, existence of martingale solutions to (4.1.16). The sketch proof of the theorem follows from the ideas presented in [13] to which we refer to for further details. We construct these solutions via a multi-level approximation scheme. The idea here is to start with a finite dimensional approximation of Galerkin type. However, as a consequence of maximum principle (usually incompatible with Galerkin type approximation) this can only be applied to the momentum equation since we need the density ρ and temperature ϑ to be positive at the first level of approximation. Adopting the approximation scheme introduced in [50] and adapted to the stochastic setting in [13] with appropriate adjustments, we regularize our system as follows.

Let Δ be the Laplace operator defined on the periodic domain $\mathbb{T}^3$. Let $\{\mathbf{w}_n\}_{n\geq 1}$ be the orthonormal system of the associated eigenfunctions. We consider the Galerkin method given by the family of finite-dimensional spaces;

$$H_m = \left(\mathrm{span}\left\{\mathbf{w}_\mathrm{n} \;\middle|\; \mathrm{n} \leq \mathrm{m}\right\}\right)^3, m = 1,2,\ldots$$

endowed with the Hilbert structure of the Lebesgue space $L^2(Q,\mathbb{R}^3)$. Let

$$\Pi_m : L^2(Q,\mathbb{R}^3) \to H_m,$$

be the associated L^2-orthogonal projection, and we have $W^{2,2}(\mathbb{T}^3,\mathbb{R}^3) \hookrightarrow\hookrightarrow C(\mathbb{T}^3,\mathbb{R}^3)$. Indeed,

$$\|\Pi_m[\mathbf{f}]\|_{L^\infty(\mathbb{T}^3)} \lesssim \|\Pi_m[\mathbf{f}]\|_{W^{2,2}(\mathbb{T}^3)} \lesssim \|\mathbf{f}\|_{W^{2,2}(\mathbb{T}^3)}, \tag{4.1.30}$$

where the associated embedding constants are independent of m. Furthermore, since H_m is finite dimensional, all norms are equivalent on H_m for any fixed m- (a property that will be frequently used at the first level of approximation). Finally, we introduce the operator

$$[\mathbf{v}]_R = \chi(\|\mathbf{v}\|_{H_m} - R)\mathbf{v},$$

defined for $\mathbf{v} \in H_m$. Let $Q = (0,T) \times \mathbb{T}^3$ be the space-time cylinder, we seek to solve the basic approximate system:

$$\mathrm{d}\rho + \mathrm{div}(\rho[\mathbf{u}]_R)\,\mathrm{d}t = 0, \tag{4.1.31}$$

$$\begin{aligned} \mathrm{d}\Pi_m[\rho\mathbf{u}] + \Pi_m[\mathrm{div}(\rho[\mathbf{u}]_R \otimes \mathbf{u})]\,\mathrm{d}t + \Pi_m\Big[\chi(\|\mathbf{u}\|_{H_m} - R)\nabla\left(p(\rho,\vartheta)\right)\Big] \\ = \Pi_m\Big[\varepsilon\mathscr{L}\mathbf{u}\Big]\,\mathrm{d}t + \rho\Pi_m[(\phi_\varepsilon)]\,\mathrm{d}W, \end{aligned} \tag{4.1.32}$$

$$\mathrm{d}S + [\mathrm{div}(S[\mathbf{u}]_R)]\,\mathrm{d}t = 0, \tag{4.1.33}$$

subject to initial law Λ, prescribed with random initial data

$$\rho(0,\cdot) = \rho_0 \in C^{2+\nu}(\mathbb{T}^3),\, \rho_0 > 0, \vartheta(0,\cdot) = \vartheta_0,\, \vartheta_0 \in C^{2+\nu}(\mathbb{T}^3),\, \vartheta_0 > 0,$$
$$\mathbf{u}(0,\cdot) = \mathbf{u}_0 \in H_m. \tag{4.1.34}$$

In our basic approximate system (4.1.31)-(4.1.33), equations (4.1.31) and (4.1.33) are deterministic, that is, they can be solved pathwise, and equation (4.1.32) involves stochastic integration. The Galerkin projection applied above reduces the problem to a variant of ordinary stochastic differential equation. We solve the system (4.1.31)-(4.1.34) using an iteration scheme.

Iteration Scheme

We construct solutions to (4.1.31)-(4.1.34) using a modification of the Cauchy collocation method. Thus, fixing a time step $h > 0$ we set

$$\rho(t,\cdot) = \rho_0, \quad \vartheta(t,\cdot) = \vartheta_0, \quad \mathbf{u}(t,\cdot) = \mathbf{u}_0, \quad \text{for } t \le 0, \tag{4.1.35}$$

and define recursively, for $t \in [nh,(n+1)h)$

$$\mathrm{d}\rho + \operatorname{div}(\rho[\mathbf{u}(nh,\cdot)]_R)\,\mathrm{d}t = 0, \quad \rho(nh,\cdot) = \rho(nh-,\cdot), \tag{4.1.36}$$

$$\mathrm{d}S + [\operatorname{div}(S[\mathbf{u}(nh,\cdot)]_R)]\,\mathrm{d}t = 0, \quad \vartheta(nh,\cdot) = \vartheta(nh-,\cdot), \tag{4.1.37}$$

Here, the unknown quantities ρ, ϑ are uniquely deduced from the deterministic equations (4.1.36) and (4.1.37) in terms of $\mathbf{u}$ and initial data. Now given ρ, ϑ we solve

$$\mathrm{d}\Pi_m[\rho\mathbf{u}] + \Pi_m\left[\operatorname{div}\Big(\rho[\mathbf{u}(nh,\cdot)]_R \otimes \mathbf{u}(nh,\cdot)\Big)\right]\mathrm{d}t + \Pi_m\left[\chi(\|\mathbf{u}(nh,\cdot)\|_{H_m} - R)\nabla\left(p(\rho,\vartheta)\right)\right]\mathrm{d}t$$

$$= \Pi_m\left[\varepsilon\mathscr{L}\mathbf{u}\right]\mathrm{d}t + \Pi_m[\rho(\phi_\varepsilon)]\,\mathrm{d}W, \quad t \in [nh,(n+1)h), \quad \mathbf{u}(nh,\cdot) = \mathbf{u}(nh-). \tag{4.1.38}$$

To solve (4.1.38), it is convenient to reformulate the system using d$\mathbf{u}$. We observe that

$$\mathrm{d}\Pi_m(\rho\mathbf{u}) = \Pi_m(\mathrm{d}\rho\mathbf{u}) + \Pi_m(\rho\,\mathrm{d}\mathbf{u}) = \Pi_m(\partial_t\rho\mathbf{u}) + \Pi_m(\rho\,\mathrm{d}\mathbf{u}).$$

We adopt the linear mapping $\mathscr{M}[\rho]$,

$$\mathscr{M}[\rho] : H_m \to H_m, \quad \mathscr{M}[\rho](\mathbf{u}) = \Pi_m(\rho\mathbf{u}),$$

or, equivalently,

$$\int_Q \mathscr{M}[\rho]\mathbf{u}\cdot\varphi \,\mathrm{d}x \equiv \int_Q \rho\mathbf{u}\cdot\varphi\,\mathrm{d}x \quad \text{for all } \varphi \in H_m,$$

and its properties as introduced in ([51], section 2.2). To be specific, using maximum principle, we take ρ to be bounded from below away from zero so that the operator $\mathscr{M}[\rho]$ is invertible. Then we reformulate the relation in (4.1.38) to obtain

$$\begin{aligned}
\mathbf{u}(t) - \mathbf{u}(nh-) + \mathscr{M}^{-1}[\rho(t)]\int_{nh}^{t} \Pi_m\left[\operatorname{div}\Big(\rho[\mathbf{u}(nh,\cdot)]_R \otimes \mathbf{u}(nh,\cdot)\Big)\right]\mathrm{d}t \\
+ \mathscr{M}^{-1}[\rho(t)]\int_{nh}^{t} \Pi_m\Big[\chi(\|\mathbf{u}(nh,\cdot)\|_{H_m} - R)\nabla\big(p(\rho,\vartheta)\big)\Big]\mathrm{d}t \\
= \mathscr{M}^{-1}[\rho(t)]\int_{nh}^{t} \Pi_m\Big[\varepsilon\mathscr{L}\mathbf{u}\Big]\mathrm{d}t \\
+ \mathscr{M}^{-1}[\rho(t)]\int_{nh}^{t} \rho\Pi_m[(\phi_\varepsilon)]\,\mathrm{d}W, \quad nh < t < (n+1)h.
\end{aligned} \tag{4.1.39}$$

The constructed iteration scheme (4.1.36)-(4.1.38) gives a unique solution $[\rho, \vartheta, \mathbf{u}]$ for any initial data (4.1.35), and the variables ρ, ϑ and $\mathbf{u}$ are continuous in time $\mathbb{P}$-a.s. Indeed, we find solution ρ and ϑ such that

$$\rho \in C([0,T]; C^{2+\nu}(\mathbb{T}^3)), \vartheta \in C([0,T]; C^{2+\nu}(\mathbb{T}^3)) \text{ a.s.}$$

by applying standard results (see, e.g [87]) pathwise. Finally, given ρ and ϑ we can find the velocity

$$\mathbf{u} \in C([0,T]; H_m), \mathbb{P}\text{-a.s.}$$

solving (4.1.38) recursively.

The limit for vanishing time step

The solution $[\rho, \vartheta, \mathbf{u}]$ provided by the iteration scheme (4.1.36)-(4.1.38) exists for any time step h. Next, we show that as $h \to 0$ the iteration scheme yields the basic approximate system (4.1.31)-(4.1.33). This essentially follows from establishing uniform bounds for (4.1.36)-(4.1.38) independent of h following the arguments presented in [13, Section

3.2]. In particular, we assume the initial data satisfies the bounds

$$0<\underline{\rho}\leq\rho_0,\|\rho_0\|_{C^{2+\nu}(\mathbb{T}^3)}\leq\overline{\rho},\quad 0<\underline{\vartheta}\leq\vartheta_0,\|\vartheta_0\|_{C^{2+\nu}(\mathbb{T}^3)}\leq\overline{\vartheta},$$

for deterministic constants $\underline{\rho}$ and $\overline{\rho}$ with $\nu>0$. Taking into account the standard results on compressible transport equations and that $[\mathbf{u}]_R$ is bounded in any Sobolev space in terms of R, and the equivalence of norms we have:

- A priori bound for density ρ is given by

$$\operatorname{ess\,sup}(\|\rho(t,\cdot)\|_C^{2+\nu}+\|\partial_t\rho(t,\cdot)\|_{C^\nu}+\|\rho^{-1}(t)\|_{C(\overline{Q})}\lesssim c(m,R,T,\underline{\rho},\overline{\rho}),\ \mathbb{P}\text{-a.s} \tag{4.1.40}$$

  uniformly in h for deterministic constants $\underline{\rho}$ and $\overline{\rho}$ with $\nu>0$.

- Similarly, a priori bound for total entropy is

$$\operatorname{ess\,sup}(\|S(t,\cdot)\|_C^{2+\nu}+\|\partial_t S(t,\cdot)\|_{C^\nu}+\|S^{-1}(t)\|_{C(\overline{Q})}\lesssim c(m,R,T,\underline{S},\overline{S}), \tag{4.1.41}$$

  $\mathbb{P}$-a.s., where the same bound of ϑ follows immediately from using $S=\rho(\log\vartheta^{c_v}-\log(\rho))$, for deterministic constants $\underline{\vartheta}$ and $\overline{\vartheta}$ uniform in h.

- Following the lines in [13] (Section 3.2), that is, establishing bounds for relation (4.1.38), we use a test function $\varphi\in H_m$ and take a supremum over φ, pass to expectations and apply Burkholder-Davis-Gundy inequality to control the noise term. Finally, applying Gronwall's lemma we deduce the estimate

$$\mathbb{E}\left[\sup_{\tau\in[0,T]}\|\Pi_m[\rho\mathbf{u}](\tau,\cdot)\|_{H_m}^r+\varepsilon\sup_{\tau\in[0,T]}\|\mathbf{u}(\tau,\cdot)\|_{H_m}^r\right]\lesssim c(r,T)\mathbb{E}[1+\|\mathbf{u}_0\|_{H_m}^r],\quad r>1. \tag{4.1.42}$$

  To pass to the limit $h\to 0$ in the momentum equation (4.1.38) we require the uniform bound (4.1.42) and compactness on the velocities in the space $C([0,T],H_m)$. Furthermore, we need to control the difference $(\mathbf{u}-\mathbf{u}(nh,\cdot))$ uniformly in time. Similarly, following closely the presentations in [13] with appropriate modifications to our particular case we infer

$$\mathbb{E}\left[\|\mathbf{u}\|_{C^\beta([0,T];H_m)}\right]\lesssim\mathbb{E}\left[\|\mathbf{u}_0\|_{H_m}^r+1\right],\quad r>2,\beta\in\left(0,\frac{1}{2}-\frac{1}{r}\right), \tag{4.1.43}$$

  uniformly in h. The 'a priori' bounds (4.1.40)-(4.1.43) are sufficient to take the limit $h \to 0$ in the iteration scheme (4.1.36)-(4.1.38).

We consider the joint law of the basic state variables $[\rho, \vartheta, \mathbf{u}, W]$ in the pathspace

$$\mathfrak{G} \equiv C^l([0,T];C^{2+l}(\mathbb{T}^3)) \times C^l([0,T];C^{2+l}(\mathbb{T}^3)) \times C^l([0,T];H_m) \times C([0,T];\mathscr{U}_0), \quad l \in (0,\overline{\nu}), \tag{4.1.44}$$

where $\overline{\nu}$ is the minimum Hölder exponent in (4.1.43). Now let $[\rho_h, \vartheta_h, \mathbf{u}_h, W]$ be the unique solution to the iteration scheme (4.1.36)-(4.1.38), with initial data being $\mathscr{F}_0$ measurable and satisfying

$$0 < \underline{\rho} \leq \rho_0, \|\rho_0\|_{C^l([0,T];C^{2+l}(\mathbb{T}^3))} \leq \overline{\rho}, \quad 0 < \underline{\vartheta} \leq \vartheta_0, \|\vartheta_0\|_{C^l([0,T];C^{2+l}(\mathbb{T}^3))} \leq \overline{\vartheta}, \tag{4.1.45}$$

as well as

$$\mathbb{E}\left[\|\mathbf{u}_0\|^r_{H_m}\right] \leq \overline{u} \quad \text{for some } r > 2. \tag{4.1.46}$$

$\mathbb{P}$-a.s., Denote by $\mathscr{L}[\rho_h, \vartheta_h, \mathbf{u}_h, W]$ the joint law of $[\rho_h, \vartheta_h, \mathbf{u}_h, W]$ on $\mathfrak{G}$, and by

$$\mathscr{L}[\rho_h], \mathscr{L}[\vartheta_h], \mathscr{L}[\mathbf{u}_h] \quad \text{and} \, \mathscr{L}[W]$$

the corresponding marginals, respectively. As a consequence of bounds established (4.1.40)-(4.1.43), the joint law $\mathscr{L}[\rho_h, \vartheta_h, \mathbf{u}_h, W]$ is *tight* on the Quasi-Polish space $\mathfrak{G}$. By applying Jakubowski-Skorokhod's representation Theorem 2.1.21 we get a new probability space with new random variables, a.s convergence of new variables on the pathspace (*w.l.o.g* we keep the same notation). Performing the limit $h \to 0$ in the new probablity space yields

$$\partial_t \rho + \operatorname{div}(\rho[\mathbf{u}]_R) = 0, \tag{4.1.47}$$

$$\mathrm{d}\Pi_m[\rho \mathbf{u}] + \Pi_m[\operatorname{div}(\rho[\mathbf{u}]_R \otimes \mathbf{u})]\,\mathrm{d}t + \Pi_m\Big[\chi(\|\mathbf{u}\|_{H_m} - R)\nabla(p(\rho, \vartheta))\Big] = \Pi\Big[\varepsilon \mathscr{L}\mathbf{u}\Big]\,\mathrm{d}t + \rho \Pi_m[(\phi_\varepsilon)]\,\mathrm{d}W, \tag{4.1.48}$$

$$\partial_t S + \operatorname{div}(S[\mathbf{u}]_R) = 0. \tag{4.1.49}$$

The system (4.1.47)-(4.1.49) is still depended on R and m. Now our goal is to perform the limits $R \to \infty$ and $m \to \infty$, respectively. The procedure is similar to the above discussion for the limit $h \to 0$. To proceed as discussed above, we start off by deducing uniform bounds enforced by random initial data and the energy balance.

Energy balance

A solution to the approximate system (4.1.31)-(4.1.33) satisfies a variant of energy balance. Derivation of total energy balance to the system consist of testing (4.1.32) with the test function $\mathbf{u}$ and integrating by parts the resultant formulation. Observe that the scalar product

$$\int_{\mathbb{T}^3} \Pi_m(\rho\mathbf{u}) \cdot \mathbf{u} \mathrm{d}x = \int_{\mathbb{T}^3} \rho|\mathbf{u}|^2 \mathrm{d}x$$

and the linear mapping $\mathscr{M}$ yields

$$\int_{\mathbb{T}^3} \mathscr{M}^{-1}[\rho]\Pi_m[\mathbf{u}] \cdot \Pi_m[\rho\mathbf{u}]\,\mathrm{d}x = \int_{\mathbb{T}^3} \rho\mathscr{M}^{-1}[\rho]\Pi_m[\mathbf{u}] \cdot \mathbf{u}\,\mathrm{d}x = \int_{\mathbb{T}^3} \mathbf{u} \cdot \mathbf{u}\,\mathrm{d}x.$$

Now we are ready to derive the total energy balance, for this, we consider the following proposition.

Proposition 4.1.8. *let $[\rho, \vartheta, \mathbf{u}, W]$ be a martingale solution of the basic approximate system (4.1.31)-(4.1.33). Then the following total energy balance equations*

$$\mathrm{d}\int_{\mathbb{T}^3} \left[\frac{1}{2}\rho|\mathbf{u}|^2 + \rho e\right] \mathrm{d}x + \varepsilon((\mathbf{u},\mathbf{u}))\,\mathrm{d}t = \frac{1}{2}\sum_{k=1}^{\infty} \int_{\mathbb{T}^3} \rho|\Pi_m[\phi_\varepsilon e_k]|^2 \,\mathrm{d}x\mathrm{d}t + \int_{\mathbb{T}^3} \rho\Pi_m[\phi_\varepsilon] \cdot \mathbf{u}\,\mathrm{d}x\mathrm{d}W. \tag{4.1.50}$$

holds $\mathbb{P}$-a.s.

Proof. Applying Itô's formula to the functional

$$f(\rho, \rho\mathbf{u}) = \frac{1}{2}\int_{\mathbb{T}^3} \rho|\mathbf{u}|^2 \,\mathrm{d}x = \frac{1}{2}\int_{\mathbb{T}^3} \frac{|\mathbf{m}|^2}{\rho}\,\mathrm{d}x,$$

from (4.1.32) we obtain,

$$\begin{aligned} \mathrm{d}\int_{\mathbb{T}^3} \frac{1}{2}\rho|\mathbf{u}|^2 \,\mathrm{d}x \quad &= \quad -\int_{\mathbb{T}^3} \left[\mathrm{div}(\rho[\mathbf{u}]_R \otimes \mathbf{u}) + \chi(\|\mathbf{u}\|_{H_m} - R)\nabla_x p(\rho, \vartheta)\right] \cdot \mathbf{u}\,\mathrm{d}x\mathrm{d}t \\ &+ \int_{\mathbb{T}^3} \varepsilon\mathscr{L}\mathbf{u} \cdot \mathbf{u}\,\mathrm{d}x\mathrm{d}t - \frac{1}{2}\int_{\mathbb{T}^3} |\mathbf{u}|^2 \mathrm{d}\rho\,\mathrm{d}x \\ &+ \frac{1}{2}\sum_{k=1}^{\infty} \int_{\mathbb{T}^3} \rho|\Pi_m[(\phi_\varepsilon)e_k]|^2 \,\mathrm{d}x\mathrm{d}t + \int_{\mathbb{T}^3} \rho\phi_\varepsilon \cdot \mathbf{u}\,\mathrm{d}x\mathrm{d}W. \end{aligned} \tag{4.1.51}$$

Furthermore, from the continuity equation (4.1.31), we deduce that;

$$\frac{1}{2}\int_{\mathbb{T}^3}|\mathbf{u}|^2 \mathrm{d}\rho \mathrm{d}x = -\frac{1}{2}\int_{\mathbb{T}^3}\operatorname{div}(\rho[\mathbf{u}]_R)\cdot|\mathbf{u}|^2\,\mathrm{d}x\mathrm{d}t,$$

such that the integral with convective term simplifies to

$$\int_{\mathbb{T}^3}\operatorname{div}(\rho[\mathbf{u}]_R\otimes\mathbf{u})\cdot\mathbf{u}\,\mathrm{d}x = \frac{1}{2}\int_{\mathbb{T}^3}\operatorname{div}(\rho[\mathbf{u}]_R)\cdot|\mathbf{u}|^2\,\mathrm{d}x,$$

and

$$\int_{\mathbb{T}^3}\chi(\|\mathbf{u}\|_{H_m}-R)\nabla_x p(\rho,\vartheta)\cdot\mathbf{u}\,\mathrm{d}x \quad = \quad -\int_{\mathbb{T}^3}p(\rho,\vartheta)\operatorname{div}[\mathbf{u}]_R\,\mathrm{d}x.$$

In view of the above observations, (4.1.51) reduces to

$$\begin{aligned}\mathrm{d}\int_{\mathbb{T}^3}\frac{1}{2}\rho|\mathbf{u}|^2\,\mathrm{d}x+\varepsilon((\mathbf{u};\mathbf{u})) \quad &= \quad \int_{\mathbb{T}^3}p(\rho,\vartheta)\operatorname{div}[\mathbf{u}]_R\,\mathrm{d}x\mathrm{d}t \\ &+\frac{1}{2}\sum_{k=1}^{\infty}\int_{\mathbb{T}^3}\rho|\Pi_m[(\phi_\varepsilon)_{e_k}]|^2\,\mathrm{d}x\mathrm{d}t+\int_{\mathbb{T}^3}\rho\phi_\varepsilon\cdot\mathbf{u}\,\mathrm{d}x\mathrm{d}W.\end{aligned} \tag{4.1.52}$$

Finally, re-writing the entropy equation as an expression of internal energy using Gibb's relation (4.1.4) the followings holds

$$\int_{\mathbb{T}^3}p(\rho,s)\operatorname{div}[\mathbf{u}]_R\,\mathrm{d}x = -\mathrm{d}\int_{\mathbb{T}^3}\rho e\,\mathrm{d}x.$$

Consequently, re-writing (4.1.52) yields

$$\mathrm{d}\int_{\mathbb{T}^3}\frac{1}{2}\rho|\mathbf{u}|^2+\rho e\,\mathrm{d}x+\varepsilon((\mathbf{u};\mathbf{u}))\,\mathrm{d}t \quad = \quad \frac{1}{2}\sum_{k=1}^{\infty}\int_{\mathbb{T}^3}\rho|\Pi_m[(\phi_\varepsilon)_{e_k}]|^2\,\mathrm{d}x\mathrm{d}t+\int_{\mathbb{T}^3}\rho\phi_\varepsilon\cdot\mathbf{u}\,\mathrm{d}x\mathrm{d}W,$$

as required. □

Uniform Bounds

Keeping $\varepsilon > 0$ fixed, we derive bounds independent of the parameters R and m. We note, the projections Π_m are bounded by (4.1.30). In view of (4.1.50), we deduce the estimate

$$\int_{\mathbb{T}^3} \left[\frac{1}{2}\rho|\mathbf{u}|^2 + \rho e\right] \mathrm{d}x + \varepsilon \int_0^T ((\mathbf{u},\mathbf{u}))\,\mathrm{d}t \lesssim \left(\mathbf{E}_0 + c(T,\phi_\varepsilon,\overline{\rho}) + M_t\right), \tag{4.1.53}$$

where

$$\mathbf{E}_0 = \int_{\mathbb{T}^3} \left[\frac{1}{2}\rho_0|\mathbf{u}_0|^2 + \rho_0 e_0\right] \mathrm{d}x, \quad M_t = \int_0^T \int_{\mathbb{T}^3} \rho \Pi_m[\phi_\varepsilon]\cdot \mathbf{u}\,\mathrm{d}x\mathrm{d}W.$$

Furthermore, taking the exponential of (4.1.53) and the expectation of the resultant exponent formulation we obtain

$$\mathbb{E}\left[\exp\left(\lambda \mathbf{E}_t + \lambda \int_0^T ((\mathbf{u},\mathbf{u}))\,\mathrm{d}t\right)\right] \le c\,\mathbb{E}\left[\exp\left(\lambda M_t\right)\right] \lesssim c(\lambda) \quad \forall \lambda > 0, \tag{4.1.54}$$

$\mathbb{P}$-a.s, the bound follows from applying exponential version of Burkholder-Davis-Gundy inequality to the r.h.s of (4.1.54), and using $\chi(|\mathbf{u}| - \frac{1}{\varepsilon})\mathbf{u} \le 1/\varepsilon$ to deduce

$$\begin{aligned}
\langle\langle M_t \rangle\rangle &= \sum_k \int_0^T \left(\int_{\mathbb{T}^3} \underbrace{\rho \Pi_m(\phi_\varepsilon)_{e_k}\cdot \mathbf{u}}_{=\rho\Pi_m\phi e_k \chi(|\mathbf{u}|-\frac{1}{\varepsilon})\mathbf{u}}\ \mathrm{d}x\right)^2 \mathrm{d}t \\
&\le c(\varepsilon)\sum_k \int_0^T \underbrace{\left(\int_{\mathbb{T}^3} \rho\,\mathrm{d}x\right)^2}_{\le c(\overline{\rho})} \|\Pi_m \phi e_k\|^2_{L^\infty_x}\,\mathrm{d}t \\
&\lesssim c(\varepsilon,\phi,\overline{\rho}).
\end{aligned}$$

The last line above follows from boundedness of density (4.1.45) and application of (4.1.30).

Limit $R \to \infty$. We assume the parameter m is fixed. The approximate problem (4.1.31)-(4.1.33) admits a martingale solution $[\rho_R, \vartheta_R, \mathbf{u}_R]$ with initial law Λ for any fixed $R > 0$. To perform the limit $R \to \infty$, we establish compactness of the phase variables and use Jakubowski's variant of the Skorokhod representation Theorem 2.1.21.

Compactness. We recall the standard regularity estimates of compressible transport equations in [87] applied to (4.1.47):

$$\begin{aligned}\|\rho(t,\cdot)\|_{L^\infty(0,T;W^{2,2}(\mathbb{T}^3))} &\lesssim \|\rho_0\|_{W^{2,2}(\mathbb{T}^3)} \exp\left(\int_0^T \|[\mathbf{u}]_R\|_{W^{3,2}}\,\mathrm{d}t\right),\\ &\lesssim \|\rho_0\|_{W^{2,2}(\mathbb{T}^3)} \exp\left(\int_0^T \|\mathbf{u}\|_{W^{3,2}}\,\mathrm{d}t\right),\end{aligned} \tag{4.1.55}$$

and

$$\begin{aligned}\|\nabla\rho(t,\cdot)\|_{L^\infty(0,T;W^{1,2}(\mathbb{T}^3))} &\lesssim \|\rho_0\|_{W^{2,2}(\mathbb{T}^3)} \exp\left(\int_0^T \|[\mathbf{u}]_R\|_{W^{3,2}}\,\mathrm{d}t\right),\\ &\lesssim \|\rho_0\|_{W^{2,2}(\mathbb{T}^3)} \exp\left(\int_0^T \|\mathbf{u}\|_{W^{3,2}}\,\mathrm{d}t\right).\end{aligned} \tag{4.1.56}$$

We control the right-hand side of (4.1.55) and (4.1.56) in expectation by using (4.1.54) to deduce the estimate

$$\mathbb{E}\left[\|\rho\|_{L^\infty(0,T;L^\infty(\mathbb{T}^3))}\right] \lesssim c, \quad \mathbb{E}\left[\|\nabla\rho\|_{L^\infty(0,T;L^6(\mathbb{T}^3))}\right] \lesssim c, \tag{4.1.57}$$

where $c > 0$ is dependent on initial data. In view of (4.1.53), (4.1.56),(4.1.57) and (4.1.47) we deduce that

$$\mathbb{E}\left[\|\partial_t\rho\|_{L^2(0,T;L^\infty(\mathbb{T}^3))}\right] \lesssim c.$$

Finally, we obtain the estimate

$$\mathbb{E}\left[\|\partial_t\rho\|_{L^2(0,T;L^\infty(\mathbb{T}^3)}\right] + \mathbb{E}\left[\|\rho\|_{L^\infty(0,T;L^\infty(\mathbb{T}^3)}\right] \lesssim c, \tag{4.1.58}$$

where $c > 0$ is dependent on initial data. The standard regularity estimate of the total entropy for the variable S follows same arguments as shown for ρ and we obtain the ϑ estimate via the relation $s = \log\vartheta^{c_v} - \log(\rho)$. Consequently, using (4.1.48), (4.1.53) and (4.1.30), the compactness of $\rho\mathbf{u}$ with respect to the time variable follows from the bound

$$\mathbb{E}\left[\|\rho\mathbf{u}\|^r_{C^\alpha([0,T];W^{-3,2}(\mathbb{T}^3))}\right] \lesssim c(r), \tag{4.1.59}$$

for all $0 < \alpha(r) < 1/2$. Accordingly, with established uniform bounds necessary to perform the limit, we proceed as in h-limit. We consider the joint law of the basic state variables $[\rho, S, \mathbf{u}, W]$ in the pathspace

$$\mathfrak{G} \equiv L^2(0,T;W^{1,2}(\mathbb{T}^3)) \times L^2(0,T;W^{1,2}(\mathbb{T}^3)) \times C([0,T];W^{-4,2}(\mathbb{T}^3)) \tag{4.1.60}$$

$$\times L^2(0,T;W^{3,2}(\mathbb{T}^3)) \times C([0,T];\mathscr{U}_0). \tag{4.1.61}$$

Let $[\rho_R, \vartheta_R, \mathbf{m}_R, W]$ be the unique solution to the iteration scheme (4.1.36)-(4.1.38) with respect to initial law Λ and assume

$$0 < \underline{\rho} \leq \rho_0, \|\rho_0\|_{W^{2,2}(\mathbb{T}^3))} \leq \overline{\rho}, \quad 0 < \underline{\vartheta} \leq \vartheta_0, \|\vartheta_0\|_{W^{2,2}(\mathbb{T}^3))} \leq \overline{\vartheta},$$

as well as

$$\mathbb{E}\left[\|\mathbf{u}_0\|_{H_m}^r\right] \leq \bar{u} \quad \text{for some } r > 2, \tag{4.1.62}$$

$\mathbb{P}$-a.s. Arguing similarly as in the h-limit (with obvious modifications): We apply the Jakubowski's-Skorokhod representation Theorem 2.1.21 , see [64] for more details, and create new probability space with new sequence of random variables that are a.s convergent (*w.l.o.g* we keep the same notation). Thus, passing the limit $R \to \infty$ in (4.1.47)-(4.1.49) yields

$$\partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0, \tag{4.1.63}$$

$$\mathrm{d}\Pi_m[\rho \mathbf{u}] + \Pi_m[\operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u})]\,\mathrm{d}t + \Pi_m\left[\nabla\left(p(\rho, \vartheta)\right)\right] \tag{4.1.64}$$

$$= \Pi_m\left[\varepsilon \mathscr{L}\mathbf{u}\right]\mathrm{d}t + \rho \Pi_m[(\phi)]\,\mathrm{d}W,$$

$$\partial_t S + \operatorname{div}(S\mathbf{u}) = 0. \tag{4.1.65}$$

Galerkin Limit

Limit $m \to \infty$. The approximate problem (4.1.63)-(4.1.65) admits a martingale solution $[\rho_m, \vartheta_m, \mathbf{u}_m]$ with initial law Λ for any fixed $m > 0$. We proceed step by step as in the R-limit, that is, following preceding parts, we establish uniform bounds (compactness) and perform the limit. In this case, the density estimate (4.1.58) and similarly the temperature estimate continue to hold for $m \to \infty$, and we can weaken the regularity of initial data in (4.1.60) by considering a sequence of initial laws that lose regularity when $m \to \infty$ (the existence of initial data with given law follows from the Skorokhod Theorem 2.1.20).

Performing the limit $m \to \infty$ yields a martingale solution to the system

$$\begin{aligned}
\mathrm{d}\rho + \operatorname{div}(\rho \mathbf{u})\,\mathrm{d}t &= 0,\\
\mathrm{d}\rho\mathbf{u} + \operatorname{div}(\rho\mathbf{u}] \otimes \mathbf{u})\,\mathrm{d}t + \nabla\left(p(\rho,\vartheta)\right)&\\
&= \varepsilon\mathscr{L}\mathbf{u}\,\mathrm{d}t + \rho\phi_\varepsilon\,\mathrm{d}W,\\
\mathrm{d}S + \operatorname{div}(S\mathbf{u})\,\mathrm{d}t &= 0.
\end{aligned}$$

this completes the proof of Theorem 4.1.5.

4.1.9 Existence results

The proof Theorem 4.1.6 consists of establishing 'a priori bounds' from the energy inequality, compactness arguments in space-time variables, see Subsection 2.1.4 , and application of Jakubowski's-Skorokhod representation Theorem 2.1.21 to deal with Quasi-Polish spaces.

Remark 4.1.3. For any $\varepsilon > 0$ Theorem 4.1.5 yields the existence of martingale solution

$$((\Omega_\varepsilon, \mathscr{F}_\varepsilon, (\mathscr{F}_t^\varepsilon), \mathbb{P}_\varepsilon), \rho_\varepsilon, \mathbf{m}_\varepsilon, S_\varepsilon, W^\varepsilon)$$

to (4.1.16). We can assume without loss of generality that the probability space does not depend on ε (see, e.g [64]), that is, the solution is given by

$$((\Omega, \mathscr{F}, (\mathscr{F}_t^\varepsilon), \mathbb{P}), \rho_\varepsilon, \mathbf{m}_\varepsilon, S_\varepsilon, W^\varepsilon).$$

We are now ready to consider the following proposition of 'a priori bounds'.

Proposition 4.1.9. *Let $p \in [1,\infty)$. Then the functions $\rho, \mathbf{u}$ and s satisfy the following*

$$\begin{aligned}
&\mathbb{E}\left(\sup_{t\in(0,T)}\int_{\mathbb{T}^3}\left[\frac{1}{2}\frac{|\mathbf{m}_\varepsilon|^2}{\rho_\varepsilon} + c_v\rho_\varepsilon^\gamma \exp\left(\frac{S_\varepsilon}{c_v\rho_\varepsilon}\right)\right]\mathrm{d}x + \varepsilon\int_0^T((\mathbf{u}_\varepsilon;\mathbf{u}_\varepsilon))\,\mathrm{d}x\mathrm{d}t\right)^p \qquad (4.1.66)\\
&\le C\left(1 + \mathbb{E}\left[\int_{\mathbb{T}^3}\left(\frac{1}{2}\rho_{0,\varepsilon}|\mathbf{u}_{0,\varepsilon}|^2 + c_v\rho_{0,\varepsilon}^\gamma\exp\left(\frac{S_{0,\varepsilon}}{c_v\rho_{0,\varepsilon}}\right)\right)\mathrm{d}x\right]^p\right) \le C(T,\overline{\rho},\phi,\Lambda),
\end{aligned}$$

uniformly in ε, where Λ is the initial law.

Proof. First, we observe that the energy formulation of the approximate system (4.1.16) is of the form

$$\int_{\mathbb{T}^3}\left[\frac{1}{2}\frac{|\mathbf{m}|^2}{\rho}+\rho e\right]\mathrm{d}x+\varepsilon\int_0^T((\mathbf{u},\mathbf{u}))\,\mathrm{d}t$$
$$=\int_{\mathbb{T}^3}\left[\frac{1}{2}\frac{|\mathbf{m}|^2}{\rho}+\rho_0 e_0\right]\mathrm{d}x+\frac{1}{2}\sum_{k=1}^{\infty}\int_0^T\int_{\mathbb{T}^3}\rho|\phi e_k|^2\,\mathrm{d}x\mathrm{d}t+\int_0^T\int_{\mathbb{T}^3}\rho\phi_\varepsilon\cdot\mathbf{u}\,\mathrm{d}x\mathrm{d}W$$

To show the estimate holds we take the supremum in time first, and complete the proof by taking the expectations. Accordingly, splitting terms and proving them individual in separate steps yields:

- firstly, considering the correction term we deduce

$$\frac{1}{2}\sum_k\int_0^T\int_{\mathbb{T}^3}\rho|\phi_\varepsilon e_k|^2\mathrm{d}x\mathrm{d}t=\frac{1}{2}\sum_k\int_0^T\int_{\mathbb{T}^3}|\sqrt{\rho}\phi_\varepsilon e_k|^2\mathrm{d}x\mathrm{d}t$$
$$\leq\frac{1}{2}\int_0^T\|\sqrt{\rho}\phi\|^2_{L_2(\mathscr{U},L^2(\mathbb{T}^3))}\mathrm{d}t<\infty.$$

  The bound follows from the assumptions of ϕ in (4.1.11) and using

$$\|\rho\|_{L^1_x}=\|\rho_0\|_{L^1_x}\leq\overline{\rho},$$

  for some constant $\overline{\rho}<\infty$.

- Next, the noise term. Here we take supremum in time and build expectations. Furthermore, we use the Burgholder-Davis-Gundy inequality to obtain

$$\mathbb{E}\left(\sup_{t\in(0,T)}\left|\int_0^t\int_{\mathbb{T}^3}\rho\phi\varepsilon\mathbf{u}\,\mathrm{d}x\mathrm{d}W\right|\right)$$
$$=\mathbb{E}\left(\sup_{t\in(0,T)}\left|\sum_k\int_0^t\int_{\mathbb{T}^3}\rho[\phi_\varepsilon]e_k\mathbf{u}\,\mathrm{d}x\mathrm{d}\beta_k\right|\right)$$
$$\leq c\mathbb{E}\left(\int_0^T\sum_k\left[\int_{\mathbb{T}^3}\rho[\phi_\varepsilon]e_k\mathbf{u}\,\mathrm{d}x\right]^2\mathrm{d}t\right)^{1/2}$$
$$\leq c\mathbb{E}\left(\int_0^T\sum_k\|\sqrt{\rho}\phi e_k\|^2_{L_2(\mathscr{U},L^2(\mathbb{T}^3))}\int_{\mathbb{T}^3}|\sqrt{\rho}\mathbf{u}|^2\,\mathrm{d}x\mathrm{d}t\right)^{1/2}$$
$$\leq c(\phi)\mathbb{E}\left(\sup_{t\in(0,T)}\underbrace{\int_{\mathbb{T}^3}\rho\mathrm{d}x}_{\leq\overline{\rho}}\int_{\mathbb{T}^3}\rho|\mathbf{u}|^2\,\mathrm{d}x\right)^{1/2}$$
$$\leq\frac{\delta}{2}\mathbb{E}\left(\sup_{t\in(0,T)}\int_{\mathbb{T}^3}\rho|\mathbf{u}|^2\,\mathrm{d}x\right)+c^2(\phi,\overline{\rho},\delta),$$

where the last line follows from Young's inequality. Now taking delta δ small enough, we can absorb the supremum term from the right.

- We note that expectation on initial data is bounded by assumption i.e.

$$\mathbb{E}\left[\int_{\mathbb{T}^3}\left(\frac{1}{2}\rho_0|\mathbf{u}_0|^2+c_v\rho_0^\gamma \exp\left(\frac{\mathbf{S}_0}{c_v\rho_0}\right)\right)\mathrm{d}x\right]^p<\infty.$$

Hence combining the correction and stochastic terms we deduce (4.1.66).

□

In view of Proposition 4.1.9, we establish the following bounds:
Firstly, we consider $Z\in BC(\mathbb{R})$ such that

$$Z'\geq 0, Z(s)\begin{cases}<0 & \text{for}\, s<s_0,\\ =0 & \text{for}\, s\geq s_0,\end{cases}$$

then the total entropy in (4.1.16) satisfies the minimum principle provided that

$$S_0\geq \rho_0\mathbf{s}_0>-\infty \text{ a.a in } \mathbb{T}^3,$$

we refer the reader to [47] for details. Since the entropy is bounded below, using (4.1.66) we deduce

$$\mathbb{E}\left(\sup_{t\in[0,T]}\int_{\mathbb{T}^3}\rho^\gamma\,\mathrm{d}x\right)\lesssim C(T,\overline{\rho},\phi,\Lambda), \tag{4.1.67}$$

for any $t\in[0,T]$. Now using $\mathbf{m}=\rho\mathbf{u}$, we observe

$$|\mathbf{m}|^{\frac{2\gamma}{\gamma+1}}=|\rho|^{\frac{\gamma}{\gamma+1}}\left|\frac{\mathbf{m}}{\sqrt{\rho}}\right|^{\frac{2\gamma}{\gamma+1}}\lesssim\rho^\gamma+\frac{|\mathbf{m}|^2}{\rho},$$

and we obtain

$$\mathbb{E}\left(\sup_{t\in[0,T]}\int_{\mathbb{T}^3}|\mathbf{m}|^{\frac{2\gamma}{\gamma+1}}\,\mathrm{d}x\right)\lesssim C(T,\overline{\rho},\phi,\Lambda), \tag{4.1.68}$$

for any $t\in[0,T]$. Bounds on the total entropy S. Since $S\geq s_0\rho$, for $S\leq 0$

$$|S|=-S\leq -s_0\rho.$$

If $S \geq 0$, we note

$$\rho^{\gamma}\exp\left(\frac{S}{c_v\rho}\right) = c_v^{-\gamma}\frac{\exp\left(\frac{S}{c_v\rho}\right)}{(\frac{S}{c_v\rho})^{\gamma}}S^{\gamma} \gtrsim S^{\gamma};$$

hence

$$\mathbb{E}\left(\sup_{t\in[0,T]}\int_{\mathbb{T}^3}|S|^{\gamma}\,\mathrm{d}x\right) \lesssim C(T,\overline{\rho},\phi,\Lambda), \tag{4.1.69}$$

for any $t \in [0,T]$. Finally, we derive an estimate for the quantity $S/\sqrt{\rho}$. For $S \leq 0$, we obtain

$$\left|\frac{S}{\sqrt{\rho}}\right| \leq -s_0\sqrt{\rho}.$$

If $S > 0$, we have

$$\rho^{\gamma}\exp\left(\frac{S}{c_v\rho}\right) = \rho^{\gamma}\exp\left(\frac{S}{\sqrt{\rho}}\frac{1}{c_v\sqrt{\rho}}\right) = c_v^{-2\gamma}\frac{\exp\left(\frac{S}{\sqrt{\rho}}\frac{1}{c_v\sqrt{\rho}}\right)}{\left(\frac{S}{\sqrt{\rho}}\frac{1}{c_v\sqrt{\rho}}\right)^{2\gamma}}\left(\frac{S}{\sqrt{\rho}}\right)^{2\gamma} \gtrsim \left(\frac{S}{\sqrt{\rho}}\right)^{2\gamma},$$

and in view of this result we deduce the bound

$$\mathbb{E}\left(\sup_{t\in[0,T]}\int\left|\frac{S}{\sqrt{\rho}}\right|^{2\gamma}\mathrm{d}x\right) \lesssim C(T,\overline{\rho},\phi,\Lambda). \tag{4.1.70}$$

In view of the above bounds (4.1.67)-(4.1.70) and Proposition 4.1.9 we deduce the following (uniform) bounds

$$\sqrt{\varepsilon}\mathbf{u}_{\varepsilon} \in L^p(\Omega;L^2([0,T];W^{3,2}(\mathbb{T}^3))) \tag{4.1.71}$$

$$\rho_{\varepsilon} \in L^p(\Omega;L^{\infty}([0,T];L^{\gamma}(\mathbb{T}^3))), \tag{4.1.72}$$

$$\mathbf{m}_{\varepsilon} \in L^p(\Omega;L^{\infty}([0,T];L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^3))), \tag{4.1.73}$$

$$\frac{\mathbf{m}_{\varepsilon}}{\sqrt{\rho_{\varepsilon}}} \in L^p(\Omega;L^{\infty}([0,T];L^2(\mathbb{T}^3))), \tag{4.1.74}$$

$$\frac{\mathbf{m}_{\varepsilon}\otimes\mathbf{m}_{\varepsilon}}{\rho} \in L^p(\Omega;L^{\infty}([0,T];L^1(\mathbb{T}^3))), \tag{4.1.75}$$

$$S_{\varepsilon} \in L^p(\Omega;L^{\infty}([0,T];L^{\gamma}(\mathbb{T}^3))), \tag{4.1.76}$$

$$\frac{S_\varepsilon}{\sqrt{\rho_\varepsilon}} \quad \in \quad L^p(\Omega;L^\infty([0,T];L^{2\gamma}(\mathbb{T}^3))), \tag{4.1.77}$$

A priori estimates

The bounds established in (4.1.71)-(4.1.77) on their own are not sufficient for us to pass to the limit. Especially, on the nonlinear terms. We solve this problem by introducing the compactness arguments. In particular, we use this procedure in the nonlinear convective and pressure terms. We start off by considering the balance of momentum given by

$$\begin{aligned}\int_{\mathbb{T}^3} \rho\mathbf{u}_\varepsilon\cdot\varphi\,\mathrm{d}x \;&=\; \int_{\mathbb{T}^3}\rho_0\mathbf{u}_0\cdot\varphi\,\mathrm{d}x+\int_0^t\int_{\mathbb{T}^3}\rho_\varepsilon\mathbf{u}_\varepsilon\otimes\mathbf{u}_\varepsilon:\nabla\varphi\,\mathrm{d}x\mathrm{d}s\\ &\quad-\varepsilon\int_0^t\int_{\mathbb{T}^3}\nabla\Delta\mathbf{u}_\varepsilon\cdot\nabla\Delta\varphi\,\mathrm{d}x\mathrm{d}s-\varepsilon\int_0^t\mathbf{u}_\varepsilon\varphi\,\mathrm{d}x\mathrm{d}s\\ &\quad+\int_0^t\int_{\mathbb{T}^3}\rho_\varepsilon^\gamma\exp\left(\frac{S_\varepsilon}{c_v\rho_\varepsilon}\right)\cdot\mathrm{div}\varphi\,\mathrm{d}x\mathrm{d}t+\int_{\mathbb{T}^3}\int_0^t\rho_\varepsilon\phi_\varepsilon dW_s\cdot\varphi\mathrm{d}x,\end{aligned}$$

for all $\varphi\in C^\infty(\mathbb{T}^3)$. We show boundedness of the system by considering deterministic and stochastic parts separately. For the deterministic case, we consider the functional

$$\begin{aligned}\mathscr{H}_\varepsilon(t,\varphi):=&\int_0^t\int_{\mathbb{T}^3}\rho_\varepsilon\mathbf{u}_\varepsilon\otimes\mathbf{u}_\varepsilon:\nabla\varphi\,\mathrm{d}x\mathrm{d}s-\varepsilon\int_0^t((\mathbf{u}_\varepsilon;\varphi))\mathrm{d}s\\ &+\int_0^t\int_{\mathbb{T}^3}\rho_\varepsilon^\gamma\exp\left(\frac{S_\varepsilon}{c_v\rho_\varepsilon}\right)\cdot\mathrm{div}\varphi\,\mathrm{d}x\mathrm{d}s.\end{aligned}$$

We observe that

$$\begin{cases}\partial_t\mathscr{H}_\varepsilon(t,\varphi)\in L^1(\Omega;L^2(0,T;W^{-3,2}(\mathbb{T}^3)),\\ \mathscr{H}_\varepsilon(t,\varphi)\in L^1(\Omega;W^{1,2}(0,T;W^{-3,2}(\mathbb{T}^3)),\end{cases}$$

uniformly in ε. Then we deduce the estimate

$$\mathbb{E}\left[\|\mathscr{H}_\varepsilon\|_{W^{1,2}([0,T];W^{-3,2}_{\mathrm{div}}(\mathbb{T}^3))}\right]\leq C.$$

The stochastic term yields

$$\mathbb{E}\left[\left\|\int_0^\cdot\rho_\varepsilon\phi_\varepsilon\,\mathrm{d}W^\varepsilon\right\|^p_{C^\alpha([0,T];L^2(\mathbb{T}^3))}\right]\leq c\mathbb{E}\left[\int_0^T\|\sqrt{\rho}\phi\|^p_{L_2(\mathscr{U},L^2(\mathbb{T}^3))}\,\mathrm{d}t\right]\leq c(\overline{\rho},p,\phi,T),$$

for all $\alpha\in(1/p,1/2)$ and $p>2$, see [[17], Lemma 9.1.3. b)] or [[61], Lemma 4.6]). Now combining the deterministic and stochastic estimates, and using the embedding $W_t^{1,2} \hookrightarrow C_t^\alpha$ and $L_x^2 \hookrightarrow W_x^{-3,2}$ shows

$$\mathbb{E}\left[\|\rho_\varepsilon \mathbf{u}_\varepsilon\|_{C^\alpha([0,T];W^{-3,2}(\mathbb{T}^3))}\right] \leq c(T), \tag{4.1.78}$$

for all $\alpha < 1/2$. On the regularity of mass continuity we have

$$\int_0^T \int_{\mathbb{T}^3} \partial_t \rho \varphi \,\mathrm{d}x = \int_0^T \int_{\mathbb{T}^3} [\rho \mathbf{u} \nabla_x \varphi] \,\mathrm{d}x\mathrm{d}t,$$

so that

$$\begin{aligned}
\sup_t \|\partial_t \rho\|_{W^{-3,2}} &= \sup_t \sup_{\|\varphi\|_{W^{3,2}}} \int_{\mathbb{T}^3} \rho_\varepsilon \mathbf{u}_\varepsilon \nabla_x \varphi \,\mathrm{d}x \\
&\leq \sup_t \sup_{\|\varphi\|_{W^{3,2}}} \|\rho_\varepsilon \mathbf{u}_\varepsilon\|_1 \|\nabla_x \varphi\|_\infty \\
&\leq \sup_t \sup_{\|\varphi\|_{W^{3,2}}} \|\rho_\varepsilon \mathbf{u}_\varepsilon\|_1 \|\varphi\|_{W^{3,2}} \\
&\leq C \sup_t \|\rho_\varepsilon \mathbf{u}_\varepsilon\|_1.
\end{aligned}$$

Consequently, $\partial_t \rho_\varepsilon \in L^\infty(0,T;W^{-3,2}(\mathbb{T}^3))$ a.s. such that

$$\rho_\varepsilon \in L^1(\Omega; W^{1,\infty}(0,T;W^{-3,2}(\mathbb{T}^3)).$$

Using (4.1.66) and the embedding $W_t^{1,\infty} \hookrightarrow C_t^\alpha$ we infer

$$\mathbb{E}\left[\|\rho_\varepsilon\|_{C^\alpha([0,T];W^{-3,2}(\mathbb{T}^3))}\right] \leq C.$$

Similarly, for the entropy balance we have

$$\int_0^T \int_{\mathbb{T}^3} \partial_t S \varphi \,\mathrm{d}x = \int_0^T \int_{\mathbb{T}^3} \left[S \frac{\mathbf{m}}{\rho} \nabla_x \varphi\right] \mathrm{d}x\mathrm{d}t,$$

and arguing as in the mass continuity case we deduce

$$\mathbb{E}\left[\|S_\varepsilon\|_{C^\alpha([0,T];W^{-3,2}(\mathbb{T}^3))}\right] \leq C.$$

Compactness Argument

Accordingly, we proceed to show tightness of the approximate solutions using the following compact embeddings.

$$C^{\alpha}([0,T];W^{-3,2}(\mathbb{T}^3))\cap L^{\infty}(L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^3)) \hookrightarrow\hookrightarrow C([0,T];W^{-4,2}(\mathbb{T}^3))\cap C_w(L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^3)). \tag{4.1.79}$$

$$C^{\alpha}([0,T];W^{-3,2})\cap L^{\infty}(0,T;L^{\gamma}(\mathbb{T}^3)) \hookrightarrow\hookrightarrow C([0,T];W^{-4,2}(\mathbb{T}^3))\cap C_w(0,T;L^{\gamma}(\mathbb{T}^3)). \tag{4.1.80}$$

We set the spaces:

$$\begin{aligned}
\mathscr{X}_{\rho_0} &:= L^{\gamma}(\mathbb{T}^3), & \mathscr{X}_{\mathbf{m}_0} &= L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^3),\\
\mathscr{X}_{\mathbf{m}} &:= C([0,T];W^{-4,2}(\mathbb{T}^3))\cap C_w(L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^3)), & \mathscr{X}_W &:= C([0,T];\mathscr{U}_0),\\
\mathscr{X}_{\rho} &:= C([0,T];W^{-4,2}(\mathbb{T}^3))\cap C_w([0,T];L^{\gamma}(\mathbb{T}^3)), & \mathscr{X}_{\mathrm{C}} &:= L^{\infty}(0,T;\mathscr{M}^{+}(\mathbb{T}^3,\mathbb{R}^{3\times 3})),\\
\mathscr{X}_{\mathrm{P}} &:= L^{\infty}(0,T;\mathscr{M}^{+}(\mathbb{T}^3,\mathbb{R}), & \mathscr{X}_{\mathbf{U}} &:= L^{2}(0,T;W^{3,2}(\mathbb{T}^3))),\\
\mathscr{X}_{S} &:= C([0,T];W^{-4,2})\cap C_w([0,T];L^{\gamma}(\mathbb{T}^3)), & \mathscr{X}_{S_0} &:= L^{\gamma}(\mathbb{T}^3),\\
\mathscr{X}_{\mathrm{Q}} &:= L^{\infty}_{w*}(Q;\mathscr{P}(A)),
\end{aligned}$$

with respect to weak-* topology for all spaces with $L^{\infty}(\cdot,\mathscr{M}^{\cdot}(\cdot))$. Furthermore, for $T>0$, we choose the product path space

$$\mathfrak{X}_T := \mathscr{X}_{\rho_0}\times\mathscr{X}_{\mathbf{m}_0}\times\mathscr{X}_{\mathbf{S}_0}\times\mathscr{X}_{\rho}\times\mathscr{X}_{\mathbf{m}}\times\mathscr{X}_{S}\times\mathscr{X}_{Q}\times\mathscr{X}_{\text{prss}}\times\mathscr{X}_{\text{conv}}\times\mathscr{X}_{W}, \tag{4.1.81}$$

with the following laws:

$$\begin{cases} \mu_{(\rho\mathbf{u})_\varepsilon} \text{ is the law of } \rho_\varepsilon\mathbf{u}_\varepsilon \text{ on } C([0,T];W^{-4,2}(\mathbb{T}^3))\cap C_w(L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^3)),\\ \mu_{\rho_\varepsilon} \text{ is the law of } \rho_\varepsilon \text{ on } C([0,T];W^{-4,2})\cap C_w(0,T;L^\gamma(\mathbb{T}^3)),\\ \mu_{S_\varepsilon} \text{ is the law of } S_\varepsilon \text{ on } C([0,T];W^{-4,2})\cap C_w(0,T;L^\gamma(\mathbb{T}^3)),\\ \mu_W \text{ is the law of } W \text{ on } C([0,T],\mathscr{U}_0). \end{cases}$$

In addition, let $\mu_{\mathbf{U}_\varepsilon}, \mu_{\mathrm{C}_\varepsilon}, \mu_{\mathrm{P}_\varepsilon}$ and $\mu_{\mathrm{Q}_\varepsilon}$ denote the laws of

$$\mathbf{U}_\varepsilon := \sqrt{\varepsilon}\mathbf{u} \qquad \mathrm{C}_\varepsilon := \rho_\varepsilon\mathbf{u}_\varepsilon\otimes\mathbf{u}_\varepsilon, \qquad \mathrm{P}_\varepsilon := \rho_\varepsilon^\gamma \exp\left(\frac{S_\varepsilon}{c_v\rho_\varepsilon}\right), \qquad Q_\varepsilon := S_\varepsilon\frac{\mathbf{m}_\varepsilon}{\rho_\varepsilon},$$

respectively. Let $\mathbf{r}_T$ be the restriction operator which restricts measurable functions (or space-time distributions) defined on $(0,\infty)$ to $(0,T)$. We denote by

$$\mathscr{L}_T[\rho_0,\rho_0\mathbf{u}_0,S_0,\rho_\varepsilon,\rho_\varepsilon\mathbf{u}_\varepsilon,S_\varepsilon,\mathbf{U}_\varepsilon,P_\varepsilon,C_\varepsilon,Q_\varepsilon,W]$$

the probability law on $\mathfrak{X}_T$. Note, tightness on

$$\mathscr{L}_T[\rho_0,\rho_0\mathbf{u}_0,S_0,\rho_\varepsilon,\rho_\varepsilon\mathbf{u}_\varepsilon,S_\varepsilon,\mathbf{U}_\varepsilon,P_\varepsilon,C_\varepsilon,Q_\varepsilon,W]$$

for any $T>0$ implies tightness of $\mathscr{L}[\rho_0,\rho_0\mathbf{u}_0,S_0,\rho_\varepsilon,\rho_\varepsilon\mathbf{u}_\varepsilon,S_\varepsilon,\mathbf{U}_\varepsilon,P_\varepsilon,C_\varepsilon,Q_\varepsilon,W]$ on $\mathfrak{X} = \cap_T\mathfrak{X}_T$. For $\rho\mathbf{u}$, we fix $T>0$ and consider the ball B_{R_1} in the space

$$C^\alpha([0,T];W^{-3,2}(\mathbb{T}^3))\cap L^\infty(L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^3)).$$

Using the Markov inequality, (4.1.66) and (4.1.78) on the complement $B_{R_1}^C$ we obtain

$$\begin{aligned} \mu_{(\rho\mathbf{u})_\varepsilon}(\mathscr{B}_{R_1}^C) &= \mathbb{P}\left(\|\rho\mathbf{u}_\varepsilon\|_{C^\alpha([0,T];W^{-3,2}(\mathbb{T}^3))}+\|\rho\mathbf{u}_\varepsilon\|_{L^\infty(L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^3))}\geq R\right)\\ &\leq \frac{\mathbb{E}}{R_1}\left(\|\rho\mathbf{u}_\varepsilon\|_{C^\alpha([0,T];W^{-3,2}(\mathbb{T}^3))}+\|\rho\mathbf{u}_\varepsilon\|_{L^\infty(L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^3))}\right)\\ &\leq \frac{C}{R_1}. \end{aligned}$$

Therefore, for a fixed $\eta > 0$ we find $R_1(\eta)$ with

$$\mu_{(\rho\mathbf{u})_\varepsilon}(\mathscr{B}_{R_1}) \geq 1 - \eta.$$

Hence, the law $\mu_{(\rho\mathbf{u})_\varepsilon}$ is tight. Using similar arguments as shown above we infer that the laws: $[\mu_{\rho_\varepsilon}, \mu_{S_\varepsilon}, \mu_{\mathbf{U}_\varepsilon}]$ are tight. We proceed to show the less obvious argument of tightness in measures.

Proposition 4.1.10. *The law μ_{C_ε} is tight.*

Proof. We consider a ball $B_R \in L^\infty(0,T;\mathscr{M}^+(\mathbb{T}^3,\mathbb{R}^{3\times 3}))$ that is relatively compact with respect to weak-* topology. Now taking the complement (B_R^c) of the ball and using Markov-inequality we deduce

$$\begin{aligned}
\mathscr{L}[C_\varepsilon](B_R^c) &= \mathbb{P}\left(\int_0^T \int_{\mathbb{T}^3} \mathrm{d}|C_\varepsilon|\mathrm{d}t > R\right) \\
&= \mathbb{P}\left(\int_0^T \int_{\mathbb{T}^3} \left|\frac{\mathbf{m}_\varepsilon \otimes \mathbf{m}_\varepsilon}{\rho_\varepsilon}\right| \mathrm{d}x\mathrm{d}t > R\right) \\
&\leq \frac{1}{R}\mathbb{E}\left\|\frac{\mathbf{m}_\varepsilon \otimes \mathbf{m}_\varepsilon}{\rho_\varepsilon}\right\|_{L^\infty(0,T;L^1(\mathbb{T}^3))} \leq \frac{C}{R},
\end{aligned}$$

where the last line follows from Proposition 4.1.9. Therefore, for a fixed $\eta > 0$ we find $R(\eta)$ with

$$\mathscr{L}[C_\varepsilon](B_R) \geq 1 - \eta.$$

The proof is complete. □

Similarly, arguing as shown above, the laws: $\mu_{\mathrm{P}_\varepsilon}$ and $\mu_{\mathrm{Q}_\varepsilon}$ are tight. The laws

$$\mu_W, \mu_{\rho_0}, \mu_{\rho_0\mathbf{u}_0} \quad \text{and } \mu_{S_0}$$

are tight since they are Radon measures on the Polish spaces. Therefore, we can infer that the law $\mathscr{L}_T[\rho_0, \rho_0\mathbf{u}_0, S_0, \rho_\varepsilon, \rho_\varepsilon\mathbf{u}_\varepsilon, S_\varepsilon, \mathbf{U}_\varepsilon, P_\varepsilon, C_\varepsilon, Q_\varepsilon, W]$ is a sequence of tight measures on $(\mathfrak{X}_T)$. Consequently, its weak-* limit is tight as well and hence a Radon measure. Since T was arbitrary chosen we deduce that

$$\mathscr{L}[\rho_0, \rho_0\mathbf{u}_0, S_0, \rho_\varepsilon, \rho_\varepsilon\mathbf{u}_\varepsilon, S_\varepsilon, \mathbf{U}_\varepsilon, P_\varepsilon, C_\varepsilon, Q_\varepsilon, W]$$

is tight on $\mathfrak{X}$. In view of the Jakubowksi's version of Skorokhod representation theorem

[64] (see also Brzezniak et al.[22], and [15, Section 2.8] for property d), we have the following proposition.

Proposition 4.1.11. *There exists a nullsequence $(\varepsilon_m)_{m\in\mathbb{N}}$, a complete probability space $(\tilde{\Omega},\tilde{\mathscr{F}},\tilde{\mathbb{P}})$ with $(\mathfrak{X},\mathscr{B}_{\mathfrak{X}})$-valued random variables $(\tilde{\rho}_{0\varepsilon_m},\tilde{\rho}_{0,\varepsilon_m}\tilde{\mathbf{u}}_{0,\varepsilon_m},\tilde{S}_{0,\varepsilon_m},\tilde{\rho}_{\varepsilon_m},\tilde{\rho}_{\varepsilon_m}\tilde{\mathbf{u}}_{\varepsilon_m},\tilde{S}_{\varepsilon_m},\tilde{\mathbf{U}}_{\varepsilon_m},\tilde{P}_{\varepsilon_m},\tilde{C}_{\varepsilon_m},\tilde{Q}_{\varepsilon_m},\tilde{W}_{\varepsilon_m})$, $m\in\mathbb{N}$, and $(\tilde{\rho}_0,\tilde{\rho}_0\tilde{\mathbf{u}}_0,\tilde{S}_0,\tilde{\rho},\tilde{\mathbf{m}},\tilde{S},\tilde{\mathbf{U}},\tilde{P},\tilde{C},\tilde{Q},\tilde{W})$ such that*

(a) For all $m\in\mathbb{N}$ the law of

$$(\tilde{\rho}_{0\varepsilon_m},\tilde{\rho}_{0,\varepsilon_m}\tilde{\mathbf{u}}_{0,\varepsilon_m},\tilde{S}_{0,\varepsilon_m},\tilde{\rho}_{\varepsilon_m},\tilde{\rho}_{\varepsilon_m}\tilde{\mathbf{u}}_{\varepsilon_m},\tilde{S}_{\varepsilon_m},\tilde{\mathbf{U}}_{\varepsilon_m},\tilde{P}_{\varepsilon_m},\tilde{C}_{\varepsilon_m},\tilde{Q}_{\varepsilon_m},\tilde{W}_{\varepsilon_m})$$

on $\mathfrak{X}$ is given by (coincides with) $\mathscr{L}[\rho_0,\rho_0\mathbf{u}_0,S_0,\rho_\varepsilon,\rho_\varepsilon\mathbf{u}_\varepsilon,S_\varepsilon,\mathbf{U}_\varepsilon,P_\varepsilon,C_\varepsilon,Q_\varepsilon,W]$;

(b) The law of

$$(\tilde{\rho}_0,\tilde{\rho}_0\tilde{\mathbf{u}}_0,\tilde{S}_0,\tilde{\rho},\tilde{\mathbf{m}},\tilde{S},\tilde{\mathbf{U}},\tilde{P},\tilde{C},\tilde{Q},\tilde{W})$$

is a Radon measure on $(\mathfrak{X},\mathscr{B}_{\mathfrak{X}})$;

(c) $(\tilde{\rho}_{0\varepsilon_m},\tilde{\rho}_{0,\varepsilon_m}\tilde{\mathbf{u}}_{0,\varepsilon_m},\tilde{S}_{0,\varepsilon_m},\tilde{\rho}_{\varepsilon_m},\tilde{\rho}_{\varepsilon_m}\tilde{\mathbf{u}}_{\varepsilon_m},\tilde{S}_{\varepsilon_m},\tilde{\mathbf{U}}_{\varepsilon_m},\tilde{P}_{\varepsilon_m},\tilde{C}_{\varepsilon_m},\tilde{Q}_{\varepsilon_m},\tilde{W}_{\varepsilon_m})$, $m\in\mathbb{N}$, converges $\tilde{\mathbb{P}}$-almost surely to $(\tilde{\rho}_0,\tilde{\rho}_0\tilde{\mathbf{u}}_0,\tilde{S}_0,\tilde{\rho},\tilde{\mathbf{m}},\tilde{S},\tilde{\mathbf{U}},\tilde{P},\tilde{C},\tilde{Q},\tilde{W})$ in the topology of $\mathfrak{X}$, i.e.

$$\begin{cases}\tilde{\rho}_{0,\varepsilon_m}\to\tilde{\rho}_0 \; in\, L^{\gamma}(\mathbb{T}^3),\\ \tilde{\rho}_{0,\varepsilon_m}\tilde{\mathbf{u}}_{0,\varepsilon_m}\to\tilde{\mathbf{m}}_0 \; in\; L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^3),\\ \tilde{S}_{0,\varepsilon_m}\to\tilde{S}_0 \; in\;\; C([0,T];W^{-4,2})\cap C_w(0,T;L^{\gamma}(\mathbb{T}^3)),\\ \tilde{\rho}_{\varepsilon_m}\to\tilde{\rho} \; in\;\; C([0,T];W^{-4,2})\cap C_w(0,T;L^{\gamma}(\mathbb{T}^3)),\\ \tilde{S}_{\varepsilon_m}\to\tilde{S} \; in\;\; C([0,T];W^{-4,2})\cap C_w(0,T;L^{\gamma}(\mathbb{T}^3)),\\ \tilde{\mathbf{U}}_{\varepsilon_m}\to\tilde{\mathbf{0}} \; in\;\; L^2([0,T];W^{3,2}(\mathbb{T}^3))),\\ \tilde{\rho}_{\varepsilon_m}\tilde{\mathbf{u}}_{\varepsilon_m}\to\tilde{\mathbf{m}} \; in\, C([0,T];W^{-4,2}(\mathbb{T}^3))\cap C_w(L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^3)),\\ \tilde{P}_{\varepsilon_m}\to\overline{\tilde{P}} \; in\, L^{\infty}_{w^*}(0,T;\mathscr{M}^+(\mathbb{T}^3,\mathbb{R}),\\ \tilde{C}_{\varepsilon_m}\to\overline{\tilde{C}} \; in\, L^{\infty}_{w^*}(0,T;\mathscr{M}^+(\mathbb{T}^3,\mathbb{R}^{3\times3}),\\ \tilde{Q}_{\varepsilon_m}\to\overline{\tilde{Q}} \; in\, L^{\infty}_{w^*}(Q;\mathscr{P}(A)),\\ \tilde{W}_{\varepsilon_m}\to\tilde{W} \; in\, C([0,T];\mathscr{U}_0),\end{cases} \tag{4.1.82}$$

$\tilde{\mathbb{P}}$-a.s.

(d) *For any Carathéodory function* $H = H(t,x,\rho,\mathbf{m},S)$ *where* $(t,x) \in (0,T) \times \mathbb{T}^3$, $(\rho,\mathbf{m},S) \in \mathbb{R}^5$, *satisfying for some* $q_1,q_2,q_3 > 0$ *the growth condition*

$$H(t,x,\rho,\mathbf{m},S) \lesssim 1+|\rho|^{q_1}+|\mathbf{m}|^{q_2}+|S|^{q_2}$$

uniformly in (t,x)*, we denote by* $\overline{H(\rho,\mathbf{m},S)}(t,x) = \langle \mathscr{V}_{t,x}, H\rangle$. *Then the following*

$$H(\tilde\rho_{\varepsilon_m},\tilde\rho_{\varepsilon_m}\tilde{\mathbf{u}}_{\varepsilon_m},\tilde S_{\varepsilon_m}) \rightharpoonup \overline{H(\tilde\rho,\tilde{\mathbf{m}},\tilde S)} \quad \text{in } L^k((0,T)\times\mathbb{T}^3)$$

holds $\tilde{\mathbb{P}}$*-a.s. as* $m\to\infty$ *for all* $1 < k \leq \frac{\gamma+1}{q_1} \wedge \frac{2}{q_2}$.

We note Proposition 4.1.11 yields existence of new probability space with new random variables, however, no guarantees of correct measurability. To circumvent this problem we introduce filtration to guarantee adaptedness of new random variables and to ensure that the stochastic integral continues to hold in the new probability space. We simplify notation as follows, set

$$\mathscr{X}_0 := \left[\tilde\rho_0,\tilde{\mathbf{m}}_0,\tilde S_0\right], \mathscr{X} := \left[\tilde\rho,\tilde{\mathbf{m}},\tilde S,\tilde{\mathbf{U}},\right].$$

Let $\tilde{\mathscr{F}}_t$ and $\tilde{\mathscr{F}}_t^{\varepsilon_m}$ be the $\tilde{\mathbb{P}}$-augmented filtration of random variables

$$(\tilde\rho_0,\tilde{\mathbf{m}}_0,\tilde S_0,\tilde\rho,\tilde{\mathbf{m}},\tilde S,\tilde{\mathbf{U}},\tilde P,\tilde C,\tilde W)$$

and $(\tilde\rho_{0\varepsilon_m},\tilde\rho_{0,\varepsilon_m}\tilde{\mathbf{u}}_{0,\varepsilon_m},\tilde{\mathbf{S}}_{0,\varepsilon_m},\tilde\rho_{\varepsilon_m},\tilde\rho_{\varepsilon_m}\tilde{\mathbf{u}}_{\varepsilon_m},\tilde{\mathbf{S}}_{\varepsilon_m},\tilde{\mathbf{U}}_{\varepsilon_m},\tilde P_{\varepsilon_m},\tilde C_{\varepsilon_m},\tilde Q_{\varepsilon_m},\tilde W_{\varepsilon_m})_{m\in\mathbb{N}}$, respectively, i.e.

$$\begin{aligned}
\tilde{\mathscr{F}}_t &= \sigma(\sigma(\mathscr{X}_0,\mathbf{r}_t\mathscr{X},\mathbf{r}_t\tilde W)\cup\sigma_t(\tilde P,\tilde C,\tilde Q)\cup\{\mathscr{N}\in\tilde{\mathscr{F}};\tilde{\mathbb{P}}(\mathscr{N})=0\}), t\geq 0,\\
\tilde{\mathscr{F}}_t^{\varepsilon_m} &= \sigma(\sigma(\mathscr{X}_{0,\varepsilon_m},\mathbf{r}_t\mathscr{X}_{\varepsilon_m},\mathbf{r}_t\tilde W_{\varepsilon_m})\cup\sigma_t(\tilde P_{\varepsilon_m},\tilde C_{\varepsilon_m},\tilde Q_{\varepsilon_m},)\cup\{\mathscr{N}\in\tilde{\mathscr{F}};\tilde{\mathbb{P}}(\mathscr{N})=0\}), t\geq 0.
\end{aligned}$$

Here $\mathbf{r}_t$ denotes the restriction operator to the interval $[0,t]$ on the path space and σ_t[2] denotes the history of a random distribution.

[2] The family of σ-fields $(\sigma_t[\mathbf{V}])_{t\geq 0}$ given as random distribution history of

$$\sigma_t[\mathbf{V}] := \bigcap_{s>t}\sigma\left(\bigcup_{\varphi\in C_c^\infty(Q;\mathbb{R}^3)}\{\langle\mathbf{V},\varphi\rangle<1\}\cup\{N\in\mathscr{F},\mathbb{P}(N)=0\}\right)$$

is called the history of $\mathbf{V}$. In fact, any random distribution is adapted to its history, see 2.1.3.

The new probability space

In this section we will use the elementary method from [27], also used in Section 3.1.2, to show that the approximated equations hold in the new probability space. The essence of this elementary method is to identify the quadratic and cross variations corresponding to the martingale with limit Wiener process obtained via compactness. Now in view of proposition 4.1.11, we note that $\tilde{W}$ has the same law as W. And as result, there exist a collection of independent real-valued $(\tilde{\mathscr{F}}_t)_{t\geq 0}$ - Wiener process $\tilde{\beta}_k^{\varepsilon_m}$ such that $\tilde{W}^N = \sum_k \tilde{\beta}_k^{\varepsilon_m} e_k$. To be specific, there exist a collection of independent real-valued $(\tilde{\mathscr{F}}_t)_{t\geq 0}$ - Wiener process $\tilde{\beta}_k$ such that $\tilde{W} = \sum_k \tilde{\beta}_k e_k$. For all $t \in [0,T]$ and $\varphi \in C_c^\infty(\mathbb{T}^3)$ define the functionals:

$$
\begin{aligned}
\mathscr{M}^{\varepsilon_m}(\rho_0,\mathbf{m}_0,\rho,\mathbf{m},\mathbf{U},C,P)_t \quad = \quad & \int_{\mathbb{T}^3}(\mathbf{m}-\mathbf{m}_0)\cdot\varphi\,\mathrm{d}x - \int_0^t\int_{\mathbb{T}^3}\nabla\varphi\,\mathrm{d}C\mathrm{d}s \\
& \sqrt{\varepsilon_m}\int_0^t\int_{\mathbb{T}^3}\nabla\Delta\mathbf{U}\cdot\nabla\Delta\varphi\,\mathrm{d}x\mathrm{d}s - \sqrt{\varepsilon_m}\int_0^t\mathbf{U}\varphi\,\mathrm{d}x\mathrm{d}s \\
& -\int_0^t\int_{\mathbb{T}^3}\operatorname{div}\varphi\,\mathrm{d}P\mathrm{d}s,
\end{aligned}
$$

$$
\Psi_t = \sum_{k=1}\int_0^t\left(\int_{\mathbb{T}^3}\rho\phi_{e_k}\cdot\varphi\;\mathrm{d}x\right)^2\mathrm{d}s,
$$

$$
(\Psi_k)_t = \int_0^t\int_{\mathbb{T}^3}\rho\phi_{e_k}\cdot\varphi\;\mathrm{d}x\mathrm{d}s.
$$

Now, let $\mathscr{M}^{\varepsilon_m}(\tilde{\rho}_{0,\varepsilon_m},\tilde{\mathbf{m}}_{0,\varepsilon_m},\tilde{\rho}_{\varepsilon_m},\tilde{\mathbf{m}}_{\varepsilon_m},\tilde{\mathbf{U}}_{\varepsilon_m},\tilde{C}_{\varepsilon_m},\tilde{P}_{\varepsilon_m})_{s,t}$ denote the increment $\mathscr{M}^{\varepsilon_m}(\tilde{\rho}_{0,\varepsilon_m},\tilde{\mathbf{m}}_{0,\varepsilon_m},\tilde{\rho}_{\varepsilon_m},\tilde{\mathbf{m}}_{\varepsilon_m},\tilde{\mathbf{U}}_{\varepsilon_m},\tilde{C}_{\varepsilon_m},\tilde{P}_{\varepsilon_m})_t - \mathscr{M}^{\varepsilon_m}(\tilde{\rho}_{0,\varepsilon_m},\tilde{\mathbf{m}}_{0,\varepsilon_m},\tilde{\rho}_{\varepsilon_m},\tilde{\mathbf{m}}_{\varepsilon_m},\tilde{\mathbf{U}}_{\varepsilon_m},\tilde{C}_{\varepsilon_m},\tilde{P}_{\varepsilon_m})_s$ and similarly for $\Psi_{s,t}$ and $(\Psi_k)_{s,t}$. In the new probability space, completeness of proof follows from showing that

$$
\mathscr{M}^{\varepsilon_m}(\tilde{\rho}_{0,\varepsilon_m},\tilde{\mathbf{m}}_{0,\varepsilon_m},\tilde{\rho}_{\varepsilon_m},\tilde{\mathbf{m}}_{\varepsilon_m},\tilde{\mathbf{U}}_{\varepsilon_m},\tilde{C}_{\varepsilon_m},\tilde{P}_{\varepsilon_m})_t = \int_0^t\int_{\mathbb{T}^3}\tilde{\rho}_{\varepsilon_m}\phi\cdot\varphi\;\mathrm{d}x\mathrm{d}\tilde{W}_s^{\varepsilon_m}. \tag{4.1.83}
$$

For (4.1.83) to hold, it suffices to show that $\mathscr{M}^{\varepsilon_m}(\tilde{\rho}_{0,\varepsilon_m},\tilde{\mathbf{m}}_{0,\varepsilon_m},\tilde{\rho}_{\varepsilon_m},\tilde{\mathbf{m}}_{\varepsilon_m},\tilde{\mathbf{U}}_{\varepsilon_m},\tilde{C}_{\varepsilon_m},\tilde{P}_{\varepsilon_m})_t$ is an $(\mathscr{F}_t^{\varepsilon_m})_{t\geq 0}$-martingale process and its corresponding quadratic and cross variations satisfy, respectively,

$$\left\langle\!\!\left\langle \mathscr{M}^{\varepsilon_m}(\tilde{\rho}_{0,\varepsilon_m},\tilde{\mathbf{m}}_{0,\varepsilon_m},\tilde{\rho}_{\varepsilon_m},\tilde{\mathbf{m}}_{\varepsilon_m},\tilde{\mathbf{U}}_{\varepsilon_m},\tilde{C}_{\varepsilon_m},\tilde{P}_{\varepsilon_m}) \right\rangle\!\!\right\rangle = \Psi, \tag{4.1.84}$$

$$\left\langle\!\!\left\langle \mathscr{M}^{\varepsilon_m}(\tilde{\rho}_{0,\varepsilon_m},\tilde{\mathbf{m}}_{0,\varepsilon_m},\tilde{\rho}_{\varepsilon_m},\tilde{\mathbf{m}}_{\varepsilon_m},\tilde{\mathbf{U}}_{\varepsilon_m},\tilde{C}_{\varepsilon_m},\tilde{P}_{\varepsilon_m}),\tilde{\beta}_k \right\rangle\!\!\right\rangle = \Psi_k, \tag{4.1.85}$$

and consequently

$$\left\langle\!\!\left\langle \mathscr{M}^{\varepsilon_m}(\tilde{\rho}_{0,\varepsilon_m},\tilde{\mathbf{m}}_{0,\varepsilon_m},\tilde{\rho}_{\varepsilon_m},\tilde{\mathbf{m}}_{\varepsilon_m},\tilde{\mathbf{U}}_{\varepsilon_m},\tilde{C}_{\varepsilon_m},\tilde{P}_{\varepsilon_m}) - \int_0^t \int_{\mathbb{T}^3} \tilde{\rho}_{\varepsilon_m}\phi\cdot\varphi \,\mathrm{d}x\mathrm{d}\tilde{W}_s^{\varepsilon_m} \right\rangle\!\!\right\rangle = 0, \tag{4.1.86}$$

which implies the desired equation on the new probability space. We note that (4.1.84) and (4.1.85) hold based on the following observation: the mapping

$$(\rho_0,\mathbf{m}_0,\rho,\mathbf{m},\mathbf{U},C,P) \mapsto \mathscr{M}(\rho_0,\mathbf{m}_0,\rho,\mathbf{m},\mathbf{U},C,P)_t$$

is well-defined and continuous on the path space. Using proposition 4.1.11 we infer that

$$\mathscr{M}^{\varepsilon_m}(\rho_{0,\varepsilon_m},\mathbf{m}_{0,\varepsilon_m},\rho_{\varepsilon_m},\mathbf{m}_{\varepsilon_m},\mathbf{U}_{\varepsilon_m},C_{\varepsilon_m},P_{\varepsilon_m}) \sim^d \mathscr{M}^{\varepsilon_m}(\tilde{\rho}_{0,\varepsilon_m},\tilde{\mathbf{m}}_{0,\varepsilon_m},\tilde{\rho}_{\varepsilon_m},\tilde{\mathbf{m}}_{\varepsilon_m},\tilde{\mathbf{U}}_{\varepsilon_m},\tilde{C}_{\varepsilon_m},\tilde{P}_{\varepsilon_m}).$$

Fixing times $s,t \in [0,T]$, with $s<t$ we consider a continuous function h such that

$$h : V|_{[0,s]} \to [0,1].$$

The process

$$\begin{aligned}
\mathscr{M}^{\varepsilon_m}(\rho_{0,\varepsilon_m},\mathbf{m}_{0,\varepsilon_m},\rho_{\varepsilon_m},\mathbf{m}_{\varepsilon_m},\mathbf{U}_{\varepsilon_m},C_{\varepsilon_m},P_{\varepsilon_m}) &= \int_0^t \int_{\mathbb{T}^3} \rho_{\varepsilon_m}\phi\cdot\varphi \,\mathrm{d}x\mathrm{d}W_s^{\varepsilon_m} \\
&= \sum_{k=1} \int_0^t \int_{\mathbb{T}^3} \rho_{\varepsilon_m}\phi_{e_k}\cdot\varphi \,\mathrm{d}x \,\mathrm{d}\beta_k^{\varepsilon_m},
\end{aligned}$$

is a square integrable $(\mathscr{F}_t)_{t\geq 0}$-martingale, consequently, we infer

$$[\mathscr{M}^{\varepsilon_m}(\rho_{0,\varepsilon_m},\mathbf{m}_{0,\varepsilon_m},\rho_{\varepsilon_m},\mathbf{m}_{\varepsilon_m},\mathbf{U}_{\varepsilon_m},C_{\varepsilon_m},P_{\varepsilon_m})]^2 - \Psi,$$

$$\mathscr{M}^{\varepsilon_m}(\rho_{0,\varepsilon_m},\mathbf{m}_{0,\varepsilon_m},\rho_{\varepsilon_m},\mathbf{m}_{\varepsilon_m},\mathbf{U}_{\varepsilon_m},C_{\varepsilon_m},P_{\varepsilon_m})\beta_k - \Psi_k,$$

are $(\mathscr{F}_t)_{t\geq 0}$-martingales. Now we set

$$\mathbf{X} := [\rho_0, \mathbf{m}_0, \rho, \mathbf{m}, \mathbf{U}, C, P], \qquad \mathbf{X}_{\varepsilon_m} := [\rho_{0,\varepsilon_m}, \mathbf{m}_{0,\varepsilon_m}, \rho_{\varepsilon_m}, \mathbf{m}_{\varepsilon_m}, \mathbf{U}_{\varepsilon_m}, C_{\varepsilon_m}, P_{\varepsilon_m}],$$

and

$$\tilde{\mathbf{X}} := [\tilde\rho_0, \tilde{\mathbf{m}}_0, \tilde\rho, \tilde{\mathbf{m}}, \tilde{\mathbf{U}}, \tilde C, \tilde P], \qquad \tilde{\mathbf{X}}_{\varepsilon_m} := [\tilde\rho_{0,\varepsilon_m}, \tilde{\mathbf{m}}_{0,\varepsilon_m}, \tilde\rho_{\varepsilon_m}, \tilde{\mathbf{m}}_{\varepsilon_m}, \tilde{\mathbf{U}}_{\varepsilon_m}, \tilde C_{\varepsilon_m}, \tilde P_{\varepsilon_m}].$$

Let $\mathbf{r}_s$ be a restriction function to the interval $[0,s]$. In view of Proposition 4.1.11 and the equality of laws we obtain:

$$\tilde{\mathbb{E}}\left[h(\mathbf{r}_s\tilde{\mathbf{X}}_{\varepsilon_m}, \mathbf{r}_s\tilde{W}^{\varepsilon_m})\mathscr{M}^{\varepsilon_m}(\tilde{\mathbf{X}}_{\varepsilon_m})_{s,t}\right] = \mathbb{E}\left[h(\mathbf{r}_s\mathbf{X}_{\varepsilon_m}, \mathbf{r}_s W^{\varepsilon_m})\mathscr{M}^{\varepsilon_m}(\mathbf{X}_{\varepsilon_m})_{s,t}\right] = 0 \tag{4.1.87}$$

$$\begin{aligned}&\tilde{\mathbb{E}}\left[h(\mathbf{r}_s\tilde{\mathbf{X}}_{\varepsilon_m}, \mathbf{r}_s\tilde{W}^{\varepsilon_m})([\mathscr{M}^{\varepsilon_m}(\tilde{\mathbf{X}}_{\varepsilon_m})]^2 - \Psi)_{s,t}\right]\\ &= \mathbb{E}\left[h(\mathbf{r}_s\mathbf{X}_{\varepsilon_m}, \mathbf{r}_s W^{\varepsilon_m})([\mathscr{M}^{\varepsilon_m}(\mathbf{X}_{\varepsilon_m})]^2 - \Psi)_{s,t}\right] = 0\end{aligned} \tag{4.1.88}$$

$$\begin{aligned}&\tilde{\mathbb{E}}\Big[h(\mathbf{r}_s\tilde{\mathbf{X}}_{\varepsilon_m}, \mathbf{r}_s\tilde{W}^{\varepsilon_m})(\mathscr{M}^{\varepsilon_m}(\tilde{\mathbf{X}}_{\varepsilon_m})\beta_k - (\Psi_k))_{s,t})\\ &= \mathbb{E}\left[h(\mathbf{r}_s\mathbf{X}_{\varepsilon_m}, \mathbf{r}_s W^{\varepsilon_m})(\mathscr{M}^{\varepsilon_m}(\mathbf{X}_{\varepsilon_m})\beta_k - (\Psi_k))_{s,t})\right] = 0\end{aligned} \tag{4.1.89}$$

Therefore, (4.1.84) and (4.1.85) hold, and consequently, (4.1.86) follows. Thus the momentum formulation:

$$\begin{aligned}\int_{\mathbb{T}^3}(\tilde{\mathbf{m}}_{\varepsilon_m})\cdot\varphi\,\mathrm{d}x \;=\; & \int_{\mathbb{T}^3}(\tilde{\mathbf{m}}_{0,\varepsilon_m})\cdot\varphi\,\mathrm{d}x + \int_0^t\int_{\mathbb{T}^3}\nabla\varphi\,\mathrm{d}\tilde C_{\varepsilon_m}\mathrm{d}s\\ & -\sqrt{\varepsilon_m}\int_0^t\int_{\mathbb{T}^3}\nabla\Delta\tilde{\mathbf{U}}_{\varepsilon_m}\cdot\nabla\Delta\varphi\,\mathrm{d}x\mathrm{d}s - \sqrt{\varepsilon_m}\int_0^t\tilde{\mathbf{U}}_{\varepsilon_m}\varphi\,\mathrm{d}x\mathrm{d}s\\ & + \int_0^t\int_{\mathbb{T}^3}\mathrm{div}\varphi\,\mathrm{d}\tilde P_{\varepsilon_m}\mathrm{d}s + \int_{\mathbb{T}^3}\int_0^t\tilde\rho_{\varepsilon_m}\phi\, d\tilde W_s^{\varepsilon_m}\cdot\varphi\,\mathrm{d}x,\end{aligned}$$

holds $\tilde{\mathbb{P}}$-a.s in new probability space $(\tilde\Omega, \tilde{\mathscr{F}}, \tilde{\mathbb{P}})$. We note that, the terms in the continuity equation and entropy balance are continuous on the path-space and as such, both equations continue to hold on the new probability space $\tilde{\mathbb{P}}$-a.s as well.

Passage to the limit

To identify the limits in the nonlinear terms we first introduce defect measures. For this, we adopt the notion of measures as presented in [14]. In view of Proposition 4.1.11 we have

$$p(\tilde{\rho}_{\varepsilon_m}, \tilde{S}_{\varepsilon_m}) \to \overline{p(\tilde{\rho}, \tilde{S})} \text{ weakly-(*) in } L^\infty(0,T;\mathscr{M}^+(\mathbb{T}^3,\mathbb{R}).$$

Noting that $p(\tilde{\rho}, \tilde{S}) = \tilde{\rho}^\gamma \exp\left(\frac{\tilde{\mathbf{S}}}{c_v \tilde{\rho}}\right)$ is a convex functional, we deduce

$$0 \leq p(\tilde{\rho}, \tilde{S}) \leq \overline{p(\tilde{\rho}, \tilde{S})}, \quad \tilde{\mathscr{R}}_{\text{press}} \equiv \overline{p(\tilde{\rho}, \tilde{S})} - p(\tilde{\rho}, \tilde{S}) \in L^\infty(0,T;\mathscr{M}^+(\mathbb{T}^3,\mathbb{R}).$$

Arguing similarly for the convective term,

$$\frac{\tilde{\mathbf{m}}_{\varepsilon_m} \otimes \tilde{\mathbf{m}}_{\varepsilon_m}}{\tilde{\rho}_{\varepsilon_m}} \to \overline{\frac{\tilde{\mathbf{m}} \otimes \tilde{\mathbf{m}}}{\tilde{\rho}}} \text{ weakly-(*) in } L^\infty(0,T;\mathscr{M}^+(\mathbb{T}^3,\mathbb{R}^{3\times 3}),$$

setting

$$\tilde{\mathscr{R}}_{\text{conv}} \equiv \overline{\frac{\tilde{\mathbf{m}} \otimes \tilde{\mathbf{m}}}{\tilde{\rho}}} - \frac{\tilde{\mathbf{m}} \otimes \tilde{\mathbf{m}}}{\tilde{\rho}},$$

for $\xi \in \mathbb{R}^3$, convexity implies

$$\begin{aligned} \tilde{\mathscr{R}}_{\text{conv}} : (\xi \otimes \xi) &= \lim_{\varepsilon_m \to 0} \left[\frac{\tilde{\mathbf{m}}_{\varepsilon_m} \otimes \tilde{\mathbf{m}}_{\varepsilon_m}}{\rho_{\varepsilon_m}} : (\xi \otimes \xi) \right] - \frac{\tilde{\mathbf{m}} \otimes \tilde{\mathbf{m}}}{\rho} : (\xi \otimes \xi) \\ &= \lim_{\varepsilon_m \to 0} \left[\frac{|\tilde{\mathbf{m}}_{\varepsilon_m} \cdot \xi|^2}{\tilde{\rho}_{\varepsilon_m}} - \frac{|\tilde{\mathbf{m}} \cdot \xi|^2}{\tilde{\rho}} \right] \geq 0, \end{aligned}$$

so that $\tilde{\mathscr{R}}_{\text{conv}} \in L^\infty(0,T;\mathscr{M}^+(\mathbb{T}^3,\mathbb{R}^{3\times 3}))$. In the entropy case we use Proposition 4.1.11 property (d). Specifically, we use the fundamental theorem of Young measures (see [3]) and argue as follows. Let A be defined as in (4.1.20), that is, the state variables space for solving the entropy equations. On the account of Proposition 4.1.11 property (d) we infer that there exists a family of parameterized probability measures

$$(t,x) \mapsto \mathscr{V}_{(t,x) \in ((0,T) \times \mathbb{T}^3)} \in L^\infty((0,T) \times \mathbb{T}^3, \mathscr{P}(A))$$

such that

$$\langle \mathscr{V}_{t,x}, H(\tilde{\rho}, \tilde{\mathbf{m}}, \tilde{S}) \rangle = \overline{H(\tilde{\rho}, \tilde{\mathbf{m}}, \tilde{S})}(\tilde{\omega}, t, x)$$

for any $H \in C_c(A)$ and a.a. $(t,x) \in ((0,T) \times \mathbb{T}^3)$ whenever it holds that

$$H(\tilde{\rho}_{\varepsilon_m}, \tilde{\mathbf{m}}_{\varepsilon_m}, \tilde{S}_{\varepsilon_m}) \to \overline{H(\tilde{\rho}, \tilde{\mathbf{m}}, \tilde{S})}$$

weakly-* in $L^\infty((0,T) \times \mathbb{T}^3, \mathscr{P}(A))$. Here the family of probability measures is called the Young measures associated with the sequence $\{\rho_{\varepsilon_m}, \mathbf{m}_{\varepsilon_m}, S_{\varepsilon_m}\}_{\varepsilon_m > 0}$. Moreover, since $[(t,x) \mapsto \langle \mathscr{V}_{t,x}, H(\tilde{\rho}, \tilde{\mathbf{m}}, \tilde{S}) \rangle]$ is a family of parametrised measures acting on the phase space A, $\overline{H(\tilde{\rho}, \tilde{\mathbf{m}}, \tilde{S})}$ a signed measure, then the difference

$$\overline{H(\tilde{\rho}, \tilde{\mathbf{m}}, \tilde{S})} - [(t,x) \mapsto \langle \mathscr{V}_{t,x}, H(\tilde{\rho}, \tilde{\mathbf{m}}, \tilde{S}) \rangle \in \mathscr{M}^+((0,T) \times \mathbb{T}^3)]$$

vanishes for entropy equations, see [47, 15] for more details. To perform the stochastic limit term we use Lemma 2.1.14. On the account of convergences in Proposition 4.1.11, Lemma 2.1.14 and the higher moments from (4.1.87)-(4.1.89) we can pass to the limit $\varepsilon_m \to 0$ in the momentum equation in (4.1.16) and obtain

$$\begin{aligned}
\tilde{\mathbb{E}} \left[\int_0^T \int_{\mathbb{T}^3} \tilde{\mathbf{m}} \cdot \varphi \, \mathrm{d}x \mathrm{d}t \right] = \tilde{\mathbb{E}} \Bigg[\int_0^T \Bigg(\int_{\mathbb{T}^3} \tilde{\mathbf{m}}_0 \cdot \varphi \, \mathrm{d}x + \int_0^t \int_{\mathbb{T}^3} \overline{\frac{\mathbf{m} \tilde{\otimes} \mathbf{m}}{\rho}} : \nabla \varphi \, \mathrm{d}x \mathrm{d}s \\
+ \int_0^t \int_{\mathbb{T}^3} \overline{\tilde{\rho}^\gamma \exp\left(\frac{\tilde{\mathbf{S}}}{c_v \tilde{\rho}} \right)} \cdot \mathrm{div} \varphi \, \mathrm{d}x \mathrm{d}s \\
+ \int_{\mathbb{T}^3} \int_0^t \tilde{\rho} \phi d\tilde{W}_s \cdot \varphi \, \mathrm{d}x \Bigg) \mathrm{d}t \Bigg].
\end{aligned} \tag{4.1.90}$$

Consequently, the momentum equation in the sense of (4.1.25) follows from rewriting (4.1.90) using defect measures. Similarly, using Proposition 4.1.11 we perform $\varepsilon_m \to 0$ limit in the mass continuity and total entropy to deduce the equivalence of (4.1.24) and (4.1.26) in the new probability space, respectively.

On the Energy inequality

Finally, we consider the energy equality. In the original probability space, the approximate system (4.1.16) has an energy equality of the form

$$E_t^{\varepsilon_m} = E_s^{\varepsilon_m} + \frac{1}{2} \int_s^t \| \sqrt{\rho_{\varepsilon_m}} \phi \|^2_{L_2(\mathscr{U}, L^2(\mathbb{T}^3))} \mathrm{d}\sigma + \int_s^t \int_{\mathbb{T}^3} \mathbf{m}_{\varepsilon_m} \phi \, \mathrm{d}x \mathrm{d}W^{\varepsilon_m},$$

$\mathbb{P}$-a.s for a.a $0 \le s < t$, where

$$E_t^{\varepsilon_m} = \int_{\mathbb{T}^3} \left[\frac{1}{2} \frac{|\mathbf{m}_{\varepsilon_m}|^2}{\rho_{\varepsilon_m}} + c_v \rho_{\varepsilon_m}^{\gamma} \exp\left(\frac{\mathbf{S}_{\varepsilon_m}}{c_v \rho_{\varepsilon_m}} \right) \right] \mathrm{d}x + \varepsilon_m \int_0^T ((\mathbf{u}_{\varepsilon_m}, \mathbf{u}_{\varepsilon_m}))\, \mathrm{d}t,$$

for a.a $t \ge 0$. For any fixed s this is equivalent to

$$-\int_s^\infty \partial_t \varphi E_t^{\varepsilon_m}\, \mathrm{d}t - \varphi(s) E_s^{\varepsilon_m} = \frac{1}{2} \int_s^\infty \varphi \|\sqrt{\rho_{\varepsilon_m}} \phi\|^2_{L_2(\mathscr{U}, L^2(\mathbb{T}^3))}\, \mathrm{d}t + \int_s^\infty \varphi \int_{\mathbb{T}^3} \mathbf{m}_{\varepsilon_m} \cdot \phi \, \mathrm{d}x \mathrm{d}W^{\varepsilon_m},$$

$\mathbb{P}$-a.s for all $\varphi \in C_0^\infty([s,\infty))$. By virtue of Proposition 3.1.2 and Proposition 4.1.11 the energy equality continues to hold in the new probability space and reads

$$\tilde{E}_t^{\varepsilon_m} = \tilde{E}_s^{\varepsilon_m} + \frac{1}{2} \int_s^t \|\sqrt{\tilde{\rho}_{\varepsilon_m}} \phi\|^2_{L_2(\mathscr{U}, L^2(\mathbb{T}^3))}\, \mathrm{d}\sigma + \int_s^t \int_{\mathbb{T}^3} \tilde{\mathbf{m}}_{\varepsilon_m} \phi \, \mathrm{d}x \mathrm{d}\tilde{W}^{\varepsilon_m},$$

$\tilde{\mathbb{P}}$-a.s. for a.a s (including $s = 0$) and all $t \ge s$. Averaging in t and s, and arguing as in Section 3.1.3; the energy arguments, the above expression becomes continuous on the path space. Furthermore, fixing $s = 0$, we use Lemma 2.1.14, the bounds established in Proposition 4.1.11, and higher moments to perform the limit $\varepsilon_m \to 0$ and obtain

$$\tilde{\mathbf{E}}_t \le \tilde{\mathbf{E}}_0 + \frac{1}{2} \int_s^t \|\sqrt{\tilde{\rho}} \phi\|^2_{L_2(\mathscr{U}; L^2(\mathbb{T}^3))}\, \mathrm{d}\sigma + \int_s^t \int_{\mathbb{T}^3} \tilde{\mathbf{m}} \cdot \phi \, \mathrm{d}x \mathrm{d}\tilde{W}, \tag{4.1.91}$$

$\mathbb{P}$-a.s. for a.a. $t \in [0,T]$, where

$$\tilde{\mathbf{E}}_t = \int_{\mathbb{T}^3} \left[\frac{1}{2} \frac{|\tilde{\mathbf{m}}|^2}{\tilde{\rho}} + c_v \tilde{\rho}^{\gamma} \exp\left(\frac{\tilde{S}}{c_v \tilde{\rho}} \right) \right] \mathrm{d}x + \frac{1}{2} \int_{\mathbb{T}^3} \mathrm{d}\mathrm{tr}\mathscr{R}_{\mathrm{conv}}(t) + c_v \int_{\mathbb{T}^3} \mathrm{d}\mathscr{R}_{\mathrm{press}}(t),$$

and

$$\tilde{E}_0 = \int_{\mathbb{T}^3} \left[\frac{1}{2} \frac{|\tilde{\mathbf{m}}_0|^2}{\tilde{\rho}_0} + c_v \tilde{\rho}_0^{\gamma} \exp\left(\frac{S_0}{c_v \rho_0} \right) \right] \mathrm{d}x.$$

Performing the limit $\varepsilon_m \to 0$ yields an energy inequality. Our goal now is to convert (4.1.91) to equality, for this, we argue as in [10]. The entropy balance in the approximate system (4.1.16) holds as an equality. Hence, to convert (4.1.91) to equality, it is sufficient to augment the term contributing to the internal energy ($\mathscr{R}_{\mathrm{press}}(t)$) by $h(t)\mathrm{d}x$ with suitable spatially homogeneous $h \ge 0$. And $\mathscr{R}_{\mathrm{press}}(t)$ acts on $\mathrm{div}_x \varphi$ in a periodic domain $\mathbb{T}^3$, therefore,

$$\int_{\mathbb{T}^3} h(t) \operatorname{div}\varphi \, \mathrm{d}x = 0.$$

Finally, for any s we have

$$-\int_s^\infty \partial_t \varphi \tilde{E}_t \, \mathrm{d}t - \varphi(s)\tilde{E}_s = \frac{1}{2}\int_s^\infty \varphi \|\sqrt{\rho}\phi\|^2_{L_2(\mathscr{U}, L^2(\mathbb{T}^3))} \, \mathrm{d}t + \int_s^\infty \varphi \int_{\mathbb{T}^3} \tilde{\mathbf{m}} \cdot \phi \, \mathrm{d}x \mathrm{d}W,$$

$\mathbb{P}$-a.s for all $\varphi \in C_0^\infty([s,\infty))$.

4.1.10 Weak-strong Uniqueness

In this section we aim to show that the weak-strong principle (i.e. a stochastic measure-valued martingale solution to (4.1.17)-(4.1.19) coincides with a strong solution so long as the later exists) holds. In order to do this, we need to introduce a *relative entropy inequality*; a tool that allows us to compare two solutions. In the following analysis, it is more convenient to express the variable S as $\rho s(\rho, E)$ where $E = \rho e(\rho, \vartheta)$ and to work with new state variables: the density ρ, the momentum $\mathbf{m}$ and the internal energy E, we refer the reader to [47] for more details.

Now following the presentation in [49], we introduce the (thermodynamic potential) *ballistic free energy*

$$H_\Theta(\rho, \vartheta) = \rho e(\rho, \vartheta) - \Theta \rho s(\rho, \vartheta), \tag{4.1.92}$$

introduced by Gibbs and more recently by Erickson [45]. In addition to Lemma 2.1.18, we consider the *relative energy* functional in the context of measure-valued martingale solutions to the complete Euler system given by

$$\begin{aligned}
\mathscr{E}\left(\rho, E, \mathbf{m} \middle| r, \Theta, \mathbf{U}\right) \quad = \quad & \int_{\mathbb{T}^3} \left[\frac{1}{2}\frac{|\mathbf{m}|^2}{\rho} + E\right] \mathrm{d}x + \frac{1}{2}\int_{\mathbb{T}^3} \mathrm{d}\operatorname{tr}[\mathscr{R}_{\mathrm{conv}}] + c_v \int_{\mathbb{T}^3} \mathrm{d}\mathscr{R}_{\mathrm{press}} \\
& - \int_{\mathbb{T}^3} \mathbf{m} \cdot \mathbf{U} \, \mathrm{d}x + \int_{\mathbb{T}^3} \frac{1}{2}\rho \, |\mathbf{U}|^2 \, \mathrm{d}x \\
& - \int_{\mathbb{T}^3} \Theta \rho s(\rho, E) \, \mathrm{d}x - \int_{\mathbb{T}^3} \rho \partial_\rho H_\Theta \, \mathrm{d}x \\
& + \int_{\mathbb{T}^3} \partial_\rho H_\Theta(r, \Theta)(r) - H_\Theta(r, \Theta) \, \mathrm{d}x,
\end{aligned} \tag{4.1.93}$$

where the relative functional (4.1.93) is defined for all $t \in [0, T]$. Now, having stated

Lemma 2.1.18 and the relative energy functional, we are in a position to derive the *relative entropy inequality*.

Proposition 4.1.12 (Relative Entropy Inequality)**.** *Let*

$$((\Omega,\mathscr{F},(\mathscr{F}_t)_{t\geq 0},\mathbb{P}),\rho,\mathbf{m},S,\mathscr{R}_{\text{conv}},\mathscr{R}_{\text{press}},\mathscr{V}_{t,x},W)$$

be a dissipative measure-valued martingale solution to the system (4.1.17)-(4.1.19). Let $(r,\Theta,\mathbf{U})$ be a trio of stochastic processes defined on the same probability space and adapted to the filtration $(\mathscr{F}_t)_{t\geq 0}$ such that

$$\begin{aligned} \mathrm{d}r &= D_t^d r\,\mathrm{d}t, \\ \mathrm{d}\mathbf{U} &= D_t^d \mathbf{U}\,\mathrm{d}t + \mathbb{D}_t^s \mathbf{U}\,\mathrm{d}W, \\ \mathrm{d}\Theta &= D_t^d \Theta\,\mathrm{d}t, \\ \mathrm{d}[\partial_\rho H_\Theta(r,\Theta)] &= D_t^d[\partial_\rho H_\Theta(r,\Theta)]\,\mathrm{d}t, \end{aligned} \tag{4.1.94}$$

and[3]

$$r\in C([0,T];C^1(\mathbb{T}^3)),\quad \Theta\in C([0,T];C^1(\mathbb{T}^3)),\quad \mathbf{U}\in C([0,T];C^1(\mathbb{T}^3)),\quad \mathbb{P}\text{-a.s.},$$

$$\mathbb{E}\left[\sup_{t\in[0,T]}\|r\|^2_{W^{1,q}(\mathbb{T}^3)}\right]^k + \mathbb{E}\left[\sup_{t\in[0,T]}\|\mathbf{U}\|^2_{W^{1,q}(\mathbb{T}^3)}\right]^q \leq c(q),\quad \textit{for all } 2\leq q<\infty,$$

$$0<\underline{r}\leq r(t,x)\leq \overline{r}\quad \mathbb{P}\text{-a.s.},$$

$$\mathbb{E}\left[\sup_{t\in[0,T]}\|\Theta\|^2_{W^{1,q}(\mathbb{T}^3)}\right]^k \leq c(q),\quad \textit{for all } 2\leq q<\infty,$$

$$0<\underline{\Theta}\leq \Theta(t,x)\leq \overline{\Theta}\quad \mathbb{P}\text{-a.s.}.$$

Furthermore, $r,\Theta,\mathbf{U}$, satisfy

$$D^d r, D^d\Theta, D^d\mathbf{U}\in L^q(\Omega;C(0,T;C^1(\mathbb{T}^3)))\qquad \mathbb{D}^s\mathbf{U}\in L^2(\Omega;L^2(0,T;L_2(\mathscr{U};L^2(\mathbb{T}^3))),$$

[3]Note, the moment bound for Θ below implies the same for $S(r,\Theta)$ by (4.1.6) since r and Θ are bounded below and above.

$$\left(\sum_{k\geq 1}|\mathbb{D}^s\mathbf{U}(e_k)|^q\right)^{\frac{1}{q}} \in L^q(\Omega;L^q(0,T;L^q(\mathbb{T}^3))), \tag{4.1.95}$$

respectively. Then the relative entropy inequality:

$$\mathscr{E}\left(\rho,E,\mathbf{m}\middle|r,\Theta,\mathbf{U}\right) \leq \mathscr{E}\left(\rho,E,\mathbf{m}\middle|r,\Theta,\mathbf{U}\right)(0) + \int_0^\tau \mathscr{Q}\left(\rho,E,\mathbf{m}\middle|r,\Theta,\mathbf{U}\right)\mathrm{d}t + \mathbb{M}, \tag{4.1.96}$$

holds $\mathbb{P}$*-a.s for all* $\tau \in (0,T)$*, where*

$$\begin{aligned}
\mathscr{Q}\left(\rho,\vartheta,\mathbf{u}\middle|r,\Theta,\mathbf{U}\right) = &\int_{\mathbb{T}^3}\rho\left(\frac{\mathbf{m}}{\rho}-\mathbf{U}\right)\cdot\nabla_x\mathbf{U}\cdot\left(\mathbf{U}-\frac{\mathbf{m}}{\rho}\right)\mathrm{d}x \\
&+\int_{\mathbb{T}^3}[(D_t^d\mathbf{U}+\mathbf{U}\cdot\nabla_x\mathbf{U})\cdot(\rho\mathbf{U}-\mathbf{m})-p(\rho,\vartheta)\mathrm{div}_x\mathbf{U}]\,\mathrm{d}x \\
&-\int_0^\tau\int_{\mathbb{T}^3}[\langle\mathscr{V}_{t,x};\rho s(\rho,E)\rangle D_t^d\Theta+\langle\mathscr{V}_{t,x};s(\rho,E)\mathbf{m}\rangle\cdot\nabla_x\Theta]\,\mathrm{d}x\mathrm{d}t \\
&+\int_0^T\int_{\mathbb{T}^3}[\rho s(r,\Theta)\partial_t\Theta+\mathbf{m}s(r,\Theta)\cdot\nabla_x\Theta]\,\mathrm{d}x\mathrm{d}t \\
&+\int_{\mathbb{T}^3}\left(\left(1-\frac{\rho}{r}\right)\partial_t p(r,\Theta)-\frac{\mathbf{m}}{r}\cdot\nabla_x p(r,\Theta)\right)\mathrm{d}x., \\
&-\sum_{k\geq 1}\int_{\mathbb{T}^3}\mathbb{D}_t^s\mathbf{U}(e_k)\cdot\rho\phi(e_k)\,\mathrm{d}x-\int_{\mathbb{T}^3}\nabla\mathbf{U}:\mathrm{d}\mathscr{R}_{conv}-\int_{\mathbb{T}^3}\mathrm{div}\mathbf{U}\mathrm{d}\mathscr{R}_{press} \\
&+\frac{1}{2}\|\sqrt{\rho}\phi\|^2_{L_2(\mathscr{U},L^2(\mathbb{T}^3))}+\frac{1}{2}\sum_{k\geq 1}\int_{\mathbb{T}^3}\rho|\mathbb{D}_t^s\mathbf{U}(e_k)|^2\,\mathrm{d}x,
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{M} \;=\; &\int_0^\tau\int_{\mathbb{T}^3}\mathbf{m}\phi\,\mathrm{d}x\mathrm{d}W \\
&-\int_0^t\int_{\mathbb{T}^3}\left[\mathbf{m}\mathbb{D}_t^s\mathbf{U}+\mathbf{U}\rho\phi\right]\mathrm{d}x\mathrm{d}W+\int_0^t\int_{\mathbb{T}^3}\rho\mathbf{U}\cdot\mathbb{D}_t^s\mathbf{U}\,\mathrm{d}x\mathrm{d}W.
\end{aligned}$$

Remark 4.1.4. From here onward we shall use the cut-off function Z and set $Z(s(\rho,E)) = s(\rho,E)$ for convenience, see (4.1.112); a detailed discussion on the properties of Z.

Proof. We observe that the right-hand-side of the formulation (4.1.93) follows from energy inequality. Therefore, using the energy inequality and Lemma 2.1.18, we proceed in several steps as follows:

Step 1: To compute $\mathrm{d}\int_{\mathbb{T}^3}\mathbf{m}\cdot\mathbf{U}\,\mathrm{d}x$ we recall that $q=\mathbf{m}$ satisfies hypotheses (2.1.11) , (2.1.13) with some $k < \infty$. Applying Lemma 2.1.18 we deduce

$$
\begin{aligned}
\mathrm{d}\left(\int_{\mathbb{T}^3} \mathbf{m}\cdot\mathbf{U}\,\mathrm{d}x\right) \quad = \quad & \left(\int_{\mathbb{T}^3}\left[\mathbf{m}\cdot D_t^d\mathbf{U} + \left(\frac{\mathbf{m}\otimes\mathbf{m}}{\rho}\right)\cdot\nabla\mathbf{U} + p(\rho,s)\mathrm{div}\mathbf{U}\right]\mathrm{d}x\right)\mathrm{d}t \\
& + \sum_{k\geq 1}\int_{\mathbb{T}^3}\mathbb{D}_t^s\mathbf{U}(e_k)\cdot\rho\phi(e_k)\,\mathrm{d}x\mathrm{d}t + \int_{\mathbb{T}^3}\nabla\mathbf{U} : \mathrm{d}\mathscr{R}_{\mathrm{conv}}\mathrm{d}t \\
& + \int_{\mathbb{T}^3}\mathrm{div}\mathbf{U}\mathrm{d}\mathscr{R}_{\mathrm{press}}\mathrm{d}t + \mathrm{d}\mathbf{M}_1,
\end{aligned} \tag{4.1.97}
$$

where

$$
M_1 = \int_0^t\int_{\mathbb{T}^3}\left[\mathbf{m}\mathbb{D}_t^s\mathbf{U} + \mathbf{U}\rho\phi\right]\mathrm{d}x\mathrm{d}W.
$$

Similarly to (4.1.97), we compute

$$
\begin{aligned}
\mathrm{d}\left(\int_{\mathbb{T}^3}\frac{1}{2}\rho|\mathbf{U}|^2\,\mathrm{d}x\right) \quad = \quad & \int_{\mathbb{T}^3}\rho\mathbf{u}\cdot\nabla\mathbf{U}\cdot\mathbf{U}\,\mathrm{d}x\mathrm{d}t + \int_{\mathbb{T}^3}\rho\mathbf{U}\cdot D_t^d\mathbf{U}\,\mathrm{d}x\mathrm{d}t \\
& + \frac{1}{2}\sum_{k\geq 1}\rho|\mathbb{D}_t^s\mathbf{U}(e_k)|^2\,\mathrm{d}x\mathrm{d}t + \mathrm{d}M_2,
\end{aligned} \tag{4.1.98}
$$

where

$$
M_2 = \int_0^t\int_{\mathbb{T}^3}\rho\mathbf{U}\cdot\mathbb{D}_t^s\mathbf{U}\,\mathrm{d}x\mathrm{d}W.
$$

Step 2: We set

$$
\begin{aligned}
\mathscr{D}(t) : \quad &= \quad \frac{1}{2}\int_{\mathbb{T}^3}\mathrm{d}\mathrm{tr}[\mathscr{R}_{\mathrm{conv}}] + c_v\int_{\mathbb{T}^3}\mathrm{d}\mathscr{R}_{\mathrm{press}}, \\
\mu : \quad &= \quad \mathrm{tr}[\mathscr{R}_{\mathrm{conv}}] + \mathrm{tr}[\mathbb{I}\mathscr{R}_{\mathrm{press}}].
\end{aligned}
$$

Combining relations (4.1.97) and (4.1.98) with the total energy balance (4.1.27) we obtain

$$
\begin{aligned}
\int_{\mathbb{T}^3}&\left[\frac{1}{2}\rho|\mathbf{u}-\mathbf{U}|^2 + \rho e(\rho,\vartheta)\right]\mathrm{d}x + \mathscr{D}(t) - \int_{\mathbb{T}^3}\left[\frac{1}{2}\rho|\mathbf{u}_0-\mathbf{U}_0|^2 + \rho_0 e(\rho_0,\vartheta_0)\right]\mathrm{d}x \\
&= \int_0^\tau\int_{\mathbb{T}^3}\rho(D_t^d\mathbf{U} + \mathbf{u}\cdot\nabla_x\mathbf{U})\cdot(\mathbf{U}-\mathbf{u})] - p(\rho,\vartheta)\mathrm{div}_x\mathbf{U}]\,\mathrm{d}x\mathrm{d}t \\
&+ \frac{1}{2}\int_0^\tau\|\sqrt{\rho}\phi\|^2_{L_2(\mathscr{U},L^2(\mathbb{T}^3))}\,\mathrm{d}s + \int_0^\tau\int_{\mathbb{T}^3}\mathbf{m}\phi\,\mathrm{d}x\mathrm{d}W - \int_0^\tau\int_{\mathbb{T}^3}\nabla\mathbf{U}\,\mathrm{d}\mu\mathrm{d}t - M_1 + M_2 \\
&- \sum_{k\geq 1}\int_0^\tau\int_{\mathbb{T}^3}\mathbb{D}_t^s\mathbf{U}(e_k)\cdot\rho\phi(e_k)\,\mathrm{d}x\mathrm{d}t + \frac{1}{2}\sum_{k\geq 1}\rho|\mathbb{D}_t^s\mathbf{U}(e_k)|^2\,\mathrm{d}x\mathrm{d}t,
\end{aligned} \tag{4.1.99}
$$

for any $0 \leq \tau \leq T$. Computing $\mathrm{d}\int_{\mathbb{T}^3}\rho\langle\mathscr{V}_{t,x};s\rangle\cdot\Theta\,\mathrm{d}x$ (i.e. testing the entropy balance (4.1.26) with Θ), we recall that $q = \rho\langle\mathscr{V}_{t,x};\mathbf{s}\rangle$ with $\mathbb{D}_t^s q = 0$ satisfies hypotheses (2.1.11), (2.1.13) for some $k < \infty$. In view of Lemma 2.1.18 we obtain

$$\mathrm{d}\left(\int_{\mathbb{T}^3}\langle\mathscr{V}_{t,x};\rho Z(s(\rho,\vartheta))\rangle\cdot\Theta\mathrm{d}x\right)\geq\int_{\mathbb{T}^3}\langle\mathscr{V}_{t,x};Z(s(\rho,\vartheta))\mathbf{m}\rangle\cdot\nabla_x\Theta\,\mathrm{d}x\mathrm{d}t$$
$$+\int_{\mathbb{T}^3}\langle\mathscr{V}_{t,x};Z(s(\rho,\vartheta))\rangle D_t^d\Theta\,\mathrm{d}x\mathrm{d}t, \qquad (4.1.100)$$

where $D_t^d\Theta=\partial_t\Theta$. Using $s(\rho,\vartheta)=Z(s(\rho,\vartheta))$, summing (4.1.99) and (4.1.100) yields

$$\begin{aligned}
&\int_{\mathbb{T}^3}\left[\frac{1}{2}\rho|\mathbf{u}-\mathbf{U}|^2+\rho e(\rho,\vartheta)-\Theta\langle\mathscr{V}_{t,x};\rho s(\rho,\vartheta)\rangle\right](\tau,\cdot)\,\mathrm{d}x+\mathscr{D}(t)\\
&\quad-\int_{\mathbb{T}^3}\left[\frac{1}{2}\rho|\mathbf{u}_0-\mathbf{U}_0|^2+\rho_0 e(\rho_0,\vartheta_0)-\Theta_0\langle\mathscr{V}_{t,x};\rho_0 s(\rho_0,\vartheta_0)\rangle\right]\mathrm{d}x\\
&\quad\leq\int_0^\tau\int_{\mathbb{T}^3}[\rho(D_t^d\mathbf{U}+\mathbf{u}\cdot\nabla_x\mathbf{U})\cdot(\mathbf{U}-\mathbf{u})-p(\rho,\vartheta)\mathrm{div}_x\mathbf{U}]\,\mathrm{d}x\mathrm{d}t \qquad (4.1.101)\\
&\quad-\int_0^\tau\int_{\mathbb{T}^3}[\langle\mathscr{V}_{t,x};\rho s(\rho,\vartheta)\rangle D_t^d\Theta+\langle\mathscr{V}_{t,x};s(\rho,\vartheta)\mathbf{m}\rangle\cdot\nabla_x\Theta]\,\mathrm{d}x\mathrm{d}t\\
&\quad+\frac{1}{2}\int_0^\tau\|\sqrt{\rho}\phi\|^2_{L_2(\mathscr{U},L^2(\mathbb{T}^3))}\,\mathrm{d}t+\int_0^\tau\int_{\mathbb{T}^3}\mathbf{m}\phi\,\mathrm{d}x\mathrm{d}W-\int_0^\tau\int_{\mathbb{T}^3}\nabla\mathbf{U}\,\mathrm{d}\mu\mathrm{d}t-M_1+M_2\\
&\quad-\sum_{k\geq1}\int_0^\tau\int_{\mathbb{T}^3}\mathbb{D}_t^s\mathbf{U}(e_k)\cdot\rho\phi(e_k)\,\mathrm{d}x\mathrm{d}t+\frac{1}{2}\sum_{k\geq1}\rho|\mathbb{D}_t^s\mathbf{U}(e_k)|^2\,\mathrm{d}x\mathrm{d}t.
\end{aligned}$$

In addition, testing the continuity equation (4.1.24) with $\partial_\rho H_\Theta(r,\Theta)$,that is, computing

$$\mathrm{d}\int_{\mathbb{T}^3}\rho\partial_\rho H_\Theta(r,\Theta)\mathrm{d}x$$

we have

$$\mathrm{d}\left(\int_{\mathbb{T}^3}\rho\partial_\rho H_\Theta(r,\Theta)\,\mathrm{d}x\right)=\int_{\mathbb{T}^3}\rho\mathbf{u}\cdot\nabla_x(\partial_\rho H_\Theta(r,\Theta))\,\mathrm{d}x+\int_{\mathbb{T}^3}\rho D_t^d(\partial_\rho H_\Theta(r,\Theta))\,\mathrm{d}x. \qquad (4.1.102)$$

Combining (4.1.101) and (4.1.102) we obtain

$$\begin{aligned}
&\int_{\mathbb{T}^3}\left[\frac{1}{2}\rho|\mathbf{u}-\mathbf{U}|^2+\rho e(\rho,\vartheta)-\Theta\langle\mathscr{V}_{t,x};\rho s(\rho,\vartheta)\rangle-\partial_\rho H_\Theta(r,\Theta)\right](\tau,\cdot)\,\mathrm{d}x+\mathscr{D}(t)\\
&\quad-\int_{\mathbb{T}^3}\left[\frac{1}{2}\rho|\mathbf{u}_0-\mathbf{U}_0|^2+\rho_0 e(\rho_0,\vartheta_0)-\Theta_0\langle\mathscr{V}_{t,x};\rho_0 s(\rho_0,\vartheta_0)\rangle-\rho_0\partial_\rho H_{\Theta(0,\cdot)}(r_0,\Theta_0)\right]\mathrm{d}x\\
&\quad\leq\int_0^\tau\int_{\mathbb{T}^3}[\rho(D_t^d\mathbf{U}+\mathbf{u}\cdot\nabla_x\mathbf{U})\cdot(\mathbf{U}-\mathbf{u})-p(\rho,\vartheta)\mathrm{div}_x\mathbf{U}]\,\mathrm{d}x\mathrm{d}t \qquad (4.1.103)\\
&\quad-\int_0^\tau\int_{\mathbb{T}^3}[\langle\mathscr{V}_{t,x};\rho s(\rho,\vartheta)\rangle D_t^d\Theta+\langle\mathscr{V}_{t,x};s(\rho,\vartheta)\mathbf{m}\rangle\cdot\nabla_x\Theta]\,\mathrm{d}x\mathrm{d}t\\
&\quad-\int_0^\tau\int_{\mathbb{T}^3}[\rho\mathbf{u}\cdot\nabla_x(\partial_\rho H_\Theta(r,\Theta))\,\mathrm{d}x+\rho D_t^d(\partial_\rho H_\Theta(r,\Theta))]\,\mathrm{d}x\mathrm{d}t\\
&\quad+\frac{1}{2}\int_0^\tau\|\sqrt{\rho}\phi\|^2_{L_2(\mathscr{U},L^2(\mathbb{T}^3))}\,\mathrm{d}t+\int_0^\tau\int_{\mathbb{T}^3}\mathbf{m}\phi\,\mathrm{d}x\mathrm{d}W-\int_0^\tau\int_{\mathbb{T}^3}\nabla\mathbf{U}\,\mathrm{d}\mu\mathrm{d}t-M_1+M_2
\end{aligned}$$

$$-\sum_{k\geq 1}\int_0^\tau\int_{\mathbb{T}^3}\mathbb{D}_t^s\mathbf{U}(e_k)\cdot\rho\phi(e_k)\,\mathrm{d}x\mathrm{d}t+\frac{1}{2}\sum_{k\geq 1}\rho|\mathbb{D}_t^s\mathbf{U}(e_k)|^2\,\mathrm{d}x\mathrm{d}t.$$

Using $D_t^d\Theta=\partial_t\Theta$ and $D_t^d(\partial_\rho H_\Theta(r,\Theta))=\partial_t(\partial_\rho H_\Theta(r,\Theta))$, we add

$$\begin{aligned}\int_0^\tau\int_{\mathbb{T}^3}\partial_t(r\partial_\rho H_\Theta(r,\vartheta)-H_\Theta(r,\vartheta))\,\mathrm{d}x\mathrm{d}t &= \int_{\mathbb{T}^3}(r\partial_\rho H_\Theta(r,\vartheta)-H_\Theta(r,\vartheta))\,\mathrm{d}x\\ &\quad -\int_{\mathbb{T}^3}r_0\partial_\rho H_{\Theta_0}(r_0,\vartheta_0)-H_{\Theta_0}(r_0,\vartheta_0))\,\mathrm{d}x,\end{aligned}$$

to both sides of (4.1.103) and obtain

$$\begin{aligned}&\int_{\mathbb{T}^3}\left[\frac{1}{2}\rho|\mathbf{u}-\mathbf{U}|^2+H_\Theta(\rho,\vartheta)-\partial_\rho H_\Theta(r,\Theta)(\rho-r)-H_\Theta(r,\vartheta)\right](\tau,\cdot)\,\mathrm{d}x+\mathscr{D}(t)\\
&\leq\int_{\mathbb{T}^3}\left[\frac{1}{2}\rho|\mathbf{u}_0-\mathbf{U}_0|^2+H_{\Theta_0}(\rho_0,\vartheta_0)-\partial_\rho H_{\Theta_0}(r_0,\Theta_0)(\rho_0-r_0)-H_{\Theta_0}(r_0,\vartheta_0)\right]\mathrm{d}x\\
&+\int_0^\tau\int_{\mathbb{T}^3}[\rho(D_t^d\mathbf{U}+\mathbf{u}\cdot\nabla_x\mathbf{U})\cdot(\mathbf{U}-\mathbf{u})-p(\rho,\vartheta)\mathrm{div}_x\mathbf{U}]\,\mathrm{d}x\mathrm{d}t \qquad (4.1.104)\\
&-\int_0^\tau\int_{\mathbb{T}^3}[\langle\mathscr{V}_{t,x};\rho s(\rho,\vartheta)\rangle D_t^d\Theta+\langle\mathscr{V}_{t,x};s(\rho,\vartheta)\mathbf{m}\rangle\cdot\nabla_x\Theta]\,\mathrm{d}x\mathrm{d}t\\
&-\int_0^\tau\int_{\mathbb{T}^3}[\rho\partial_t(\partial_\rho H_\Theta(r,\Theta))+\rho\mathbf{u}\cdot\nabla_x(\partial_\rho H_\Theta(r,\Theta))]\,\mathrm{d}x\mathrm{d}t\\
&+\int_0^T\int_{\mathbb{T}^3}\partial_t(r\partial_\rho H_\Theta(r,\vartheta)-H_\Theta(r,\vartheta))\,\mathrm{d}x\mathrm{d}t.\\
&+\frac{1}{2}\int_0^\tau\|\sqrt{\rho}\phi\|^2_{L_2(\mathscr{U},L^2(\mathbb{T}^3))}\,\mathrm{d}t+\int_0^\tau\int_{\mathbb{T}^3}\mathbf{m}\phi\,\mathrm{d}x\mathrm{d}W-\int_0^\tau\int_{\mathbb{T}^3}\nabla\mathbf{U}\,\mathrm{d}\mu\mathrm{d}t-M_1+M_2\\
&-\sum_{k\geq 1}\int_0^\tau\int_{\mathbb{T}^3}\mathbb{D}_t^s\mathbf{U}(e_k)\cdot\rho\phi(e_k)\,\mathrm{d}x\mathrm{d}t+\frac{1}{2}\sum_{k\geq 1}\rho|\mathbb{D}_t^s\mathbf{U}(e_k)|^2\,\mathrm{d}x\mathrm{d}t.\end{aligned}$$

Furthermore noting that $H_\Theta(\rho,\vartheta)$ is function of three variables $[(\Theta,\rho,\Theta)]$, by chain rule we deduce

$$\begin{aligned}\partial_y\left(\partial_\rho H_\Theta(r,\Theta)\right) &= -s(r,\Theta)\partial_y\Theta-r\partial_\rho s(r,\Theta)\partial_y\Theta+\partial^2_{\rho,\rho}H_\Theta(r,\Theta)\partial_y\rho\\ &\quad+\partial^2_{\rho,\vartheta}H_\Theta(r,\Theta)\partial_y\Theta \qquad (4.1.105)\end{aligned}$$

for $y=t,x$. And in view of (4.1.105), we rewrite (4.1.104) in the form

$$
\begin{aligned}
&\int_{\mathbb{T}^3}\left[\frac{1}{2}\rho|\mathbf{u}-\mathbf{U}|^2+H_\Theta(\rho,\vartheta)-\partial_\rho H_\Theta(r,\Theta)(\rho-r)-H_\Theta(r,\vartheta)\right](\tau,\cdot)\,\mathrm{d}x+\mathscr{D}(t)\\
&\leq\int_{\mathbb{T}^3}\left[\frac{1}{2}\rho|\mathbf{u}_0-\mathbf{U}_0|^2+H_{\Theta_0}(\rho_0,\vartheta_0)-\partial_\rho H_{\Theta_0}(r_0,\Theta_0)(\rho_0-r_0)-H_{\Theta_0}(r_0,\vartheta_0)\right]\mathrm{d}x\\
&+\int_0^\tau\int_{\mathbb{T}^3}[\rho(D_t^d\mathbf{U}+\mathbf{u}\cdot\nabla_x\mathbf{U})\cdot(\mathbf{U}-\mathbf{u})-p(\rho,\vartheta)\mathrm{div}_x\mathbf{U}]\,\mathrm{d}x\mathrm{d}t \qquad (4.1.106)\\
&-\int_0^\tau\int_{\mathbb{T}^3}[\langle\mathscr{V}_{t,x};\rho s(\rho,\vartheta)\rangle D_t^d\Theta+\langle\mathscr{V}_{t,x};s(\rho,\vartheta)\mathbf{m}\rangle\cdot\nabla_x\Theta]\,\mathrm{d}x\mathrm{d}t\\
&+\int_0^T\int_{\mathbb{T}^3}[\rho s(r,\Theta)\partial_t\Theta+\mathbf{m}s(r,\Theta)\cdot\nabla_x\Theta]\,\mathrm{d}x\mathrm{d}t\\
&+\int_0^T\int_{\mathbb{T}^3}\rho\,(r\partial_\rho s(r,\Theta)\partial_t\Theta+r\partial_\rho s(r,\Theta)\mathbf{u}\cdot\nabla_x\Theta)\,\mathrm{d}x\mathrm{d}t\\
&-\int_0^T\int_{\mathbb{T}^3}\rho\,(\partial^2_{\rho,\rho}H_\Theta(r,\Theta)\partial_t r+\partial^2_{\rho,\vartheta}H_\Theta(r,\Theta)\partial_t\Theta)\,\mathrm{d}x\mathrm{d}t\\
&-\int_0^T\int_{\mathbb{T}^3}\rho\mathbf{u}(\partial^2_{\rho,\rho}H_\Theta(r,\Theta)\nabla_x r+\partial^2_{\rho,\vartheta}H_\Theta(r,\Theta)\nabla_x\Theta)\,\mathrm{d}x\mathrm{d}t\\
&+\int_0^T\int_{\mathbb{T}^3}\partial_t(r\partial_\rho H_\Theta(r,\vartheta)-H_\Theta(r,\vartheta))\,\mathrm{d}x\mathrm{d}t.\\
&+\frac{1}{2}\int_0^\tau\|\sqrt{\rho}\phi\|^2_{L_2(\mathscr{U},L^2(\mathbb{T}^3))}\,\mathrm{d}t+\int_0^\tau\int_{\mathbb{T}^3}\mathbf{m}\phi\,\mathrm{d}x\mathrm{d}W-\int_0^\tau\int_{\mathbb{T}^3}\nabla\mathbf{U}\,\mathrm{d}\mu\mathrm{d}t-M_1+M_2\\
&-\sum_{k\geq1}\int_0^\tau\int_{\mathbb{T}^3}\mathbb{D}_t^s\mathbf{U}(e_k)\cdot\rho\phi(e_k)\,\mathrm{d}x\mathrm{d}t+\frac{1}{2}\sum_{k\geq1}\rho|\mathbb{D}_t^s\mathbf{U}(e_k)|^2\,\mathrm{d}x\mathrm{d}t
\end{aligned}
$$

To further simplify (4.1.106), we use Gibbs' relation to deduce the following identities:

$$\partial^2_{\rho,\rho}H_\Theta(r,\Theta)=\frac{1}{r}\partial_\rho p(r,\Theta),$$

$$r\partial_\rho s(r,\Theta)=-\frac{1}{r}\partial_\vartheta p(r,\Theta), \qquad (4.1.107)$$

$$\partial^2_{\rho,\vartheta}H_\Theta(r,\Theta)=\partial_\rho(\rho(\vartheta-\Theta)\partial_\vartheta s)(r,\Theta)=(\vartheta-\Theta)\partial_\rho(\rho\partial_\vartheta s(\rho,\Theta))(r,\Theta)=0,$$

and

$$r\partial_\rho H_\Theta(r,\Theta)-H_\Theta(r,\Theta)=p(r,\Theta).$$

In view of the identities derived from Gibbs' relation, we observe the following:

$$\rho\mathbf{u}(\partial^2_{\rho,\rho}H_\Theta(r,\Theta)\nabla_x r+\partial^2_{\rho,\vartheta}H_\Theta(r,\Theta)\nabla_x\Theta)=\rho\mathbf{u}\left(\frac{1}{r}\partial_\rho p(r,\Theta)\nabla_x r\right)$$

and

$$\rho r\partial_\rho\mathbf{s}(r,\Theta)\mathbf{u}\cdot\nabla_x\Theta=-\rho\mathbf{u}\left(\frac{1}{r}\partial_\vartheta p(r,\Theta)\nabla_x\Theta\right),$$

combining these terms we obtain

$$\frac{\rho\mathbf{u}}{r}\cdot\nabla_x p(r,\Theta)=\rho\mathbf{u}\left(\frac{1}{r}\partial_\vartheta p(r,\Theta)\nabla_x\Theta\right)+\rho\mathbf{u}\left(\frac{1}{r}\partial_\rho p(r,\Theta)\nabla_x r\right).$$

Arguing similarly for remaining terms using the identities in (4.1.107), we obtain a more concise form

$$\begin{aligned}
&\int_{\mathbb{T}^3}\left[\frac{1}{2}\rho|\mathbf{u}-\mathbf{U}|^2+H_\Theta(\rho,\vartheta)-\partial_\rho H_\Theta(r,\Theta)(\rho-r)-H_\Theta(r,\vartheta)\right](\tau,\cdot)\,\mathrm{d}x\\
&\le\int_{\mathbb{T}^3}\left[\frac{1}{2}\rho|\mathbf{u}_0-\mathbf{U}_0|^2+H_{\Theta_0}(\rho_0,\vartheta_0)-\partial_\rho H_{\Theta_0}(r_0,\Theta_0)(\rho_0-r_0)-H_{\Theta_0}(r_0,\vartheta_0)\right]\mathrm{d}x\\
&+\int_0^\tau\int_{\mathbb{T}^3}\rho(\mathbf{u}-\mathbf{U})\cdot\nabla_x\mathbf{U}\cdot(\mathbf{U}-\mathbf{u})\,\mathrm{d}x\mathrm{d}t\\
&+\int_0^\tau\int_{\mathbb{T}^3}[\rho(D_t^d\mathbf{U}+\mathbf{U}\cdot\nabla_x\mathbf{U})\cdot(\mathbf{U}-\mathbf{u})-p(\rho,\vartheta)\mathrm{div}_x\mathbf{U}]\,\mathrm{d}x\mathrm{d}t \qquad (4.1.108)\\
&-\int_0^\tau\int_{\mathbb{T}^3}[\langle\mathscr{V}_{t,x};\rho s(\rho,\vartheta)\rangle D_t^d\Theta+\langle\mathscr{V}_{t,x};s(\rho,\vartheta)\mathbf{m}\rangle\cdot\nabla_x\Theta]\,\mathrm{d}x\mathrm{d}t\\
&+\int_0^T\int_{\mathbb{T}^3}[\rho s(r,\Theta)\partial_t\Theta+\mathbf{m}s(r,\Theta)\cdot\nabla_x\Theta]\,\mathrm{d}x\mathrm{d}t\\
&+\int_0^T\int_{\mathbb{T}^3}\left(\left(1-\frac{\rho}{r}\right)\partial_t p(r,\Theta)-\frac{\rho}{r}\mathbf{u}\cdot\nabla_x p(r,\Theta)\right)\mathrm{d}x\mathrm{d}t.\\
&+\frac{1}{2}\int_0^\tau\|\sqrt{\rho}\phi\|^2_{L_2(\mathscr{U},L^2(\mathbb{T}^3))}\,\mathrm{d}t+\int_0^\tau\int_{\mathbb{T}^3}\mathbf{m}\phi\,\mathrm{d}x\mathrm{d}W-\int_0^\tau\int_{\mathbb{T}^3}\nabla\mathbf{U}\,\mathrm{d}\mu\mathrm{d}t-M_1+M_2\\
&-\sum_{k\ge1}\int_0^\tau\int_{\mathbb{T}^3}\mathbb{D}_t^s\mathbf{U}(e_k)\cdot\rho\phi(e_k)\,\mathrm{d}x\mathrm{d}t+\frac{1}{2}\sum_{k\ge1}\rho|\mathbb{D}_t^s\mathbf{U}(e_k)|^2\,\mathrm{d}x\mathrm{d}t,
\end{aligned}$$

the proof is complete.

□

Remark 4.1.5. The *relative entropy inequality* is satisfied for any trio $[r,\Theta,\mathbf{U}]$ provided p,e and s satisfy the Gibbs' relation (4.1.4) only, that is, the particular model from (4.1.2) and (4.1.3) is not needed here.

We are now ready to prove Theorem 4.1.7, accordingly, we use Proposition 4.1.12 and a

Gronwall type argument to prove a pathwise weak-strong uniqueness claim as follows.

Proof of the claim. :

Step 1

We begin by introducing a stopping time

$$\tau_M = \inf\left\{ t \in (0,\mathrm{t}) | \quad \|\mathbf{U}(s,\cdot)\|_{W^{1,2}(\mathbb{T}^3)} > M \right\}$$

Since $[r,\Theta,\mathbf{U}]$ is a strong solution,

$$\mathbb{P}\left[\lim_{M\to\infty} \tau_M = \mathrm{t}\right] = 1;$$

therefore, it is enough to show results for a fixed M. Furthermore, $[r,\Theta,\mathbf{U}] \equiv [\rho,\vartheta,\mathbf{u}]$ satisfies an equation of the form (4.1.94), with

$$D_t^d = -\mathbf{U}\cdot\nabla_x\mathbf{U} - \frac{1}{r}\nabla_x p(r,\Theta), \quad \mathbb{D}_t^s\mathbf{U} = \phi, \quad D_t^d r = -\mathrm{div}_x(r\mathbf{U}).$$

Remark 4.1.6. Note that the Itô correction term in (4.1.96) vanishes for our choice of $D_t^d\mathbf{U}$.

Step 2

We proceed to recall assumptions and properties needed to show the pathwise weak-strong uniqueness principle. For $M > 0$, we have

$$\sup_{t\in[0,\tau_M]} \|\nabla\mathbf{U}\|_{L^\infty(\mathbb{T}^3)} \leq c(M). \tag{4.1.109}$$

Since r satisfies the continuity equation and hypothesis (4.1.29), then from maximum and minimum principle we have

$$0 < \underline{r}_M \leq r(t\wedge\mathrm{t}) \leq \overline{r}_M$$

for some deterministic constants $\underline{r}_M, \overline{r}_M$. Similarly, for Θ we have

$$0 < \underline{\Theta}_M \leq \Theta(t\wedge\mathrm{t}) \leq \overline{\Theta}_M.$$

The relative energy (4.1.93) can be re-written as

$$
\begin{aligned}
&\mathscr{E}\left(\rho, E, \mathbf{m} \middle| r, \Theta, \mathbf{U}\right) \\
&= \int_{\mathbb{T}^3} \left(\frac{1}{2}\rho \left| \frac{\mathbf{m}}{\rho} - \mathbf{U} \right|^2 + E - \Theta \rho \mathbf{s}(\rho, E) - \partial_\rho H_\Theta(r,\Theta)(\rho - r) - H_\Theta(r,\Theta) \right) \mathrm{d}x \\
&+ \frac{1}{2} \int_{\mathbb{T}^3} \mathrm{d}\operatorname{tr}\mathscr{R}_{\text{conv}}(t) + c_v \int_{\mathbb{T}^3} \mathrm{d}\mathscr{R}_{\text{press}}(t).
\end{aligned} \tag{4.1.110}
$$

where $E = \rho e(\rho, \vartheta)$. Let $\mathscr{K} \subset (0, \infty)$ be a fixed compact set containing the trajectories

$$
\cup_{t \in [0,\tau_M], x \in \mathbb{T}^3} [r(t,x), \Theta(t,x)]
$$

and let $\tilde{\mathscr{K}}$ denote its image with new phase variables

$$
(\rho, \vartheta) \mapsto [\rho, \rho e(\rho, \vartheta)] : (0,\infty)^2 \to (0,\infty)^2.
$$

We introduce a function $\Phi(\rho, E)$,

$$
\Phi \in C_c^\infty(0,\infty)^2, 0 \le \Phi \le 1, \Phi|_B = 1,
$$

where B is an open neighborhood of $\tilde{K}$ in $(0,\infty)$. Accordingly, given a measurable function $G(\rho, E, \mathbf{m})$, we set

$$
G = G_{\text{ess}} + G_{\text{res}}, \quad G_{\text{ess}} = \Phi(\rho, E) G(\rho, E, \mathbf{m}), \quad G_{\text{res}} = (1 - \Phi(\rho, E)) G(\rho, E, \mathbf{m}). \tag{4.1.111}
$$

Arguing similarly to the works of [47, 50], we assign G_{ess} to the 'essential part' that describes the behaviour of the non-linearity in the non-degenerate area where both ρ and ϑ are bounded below and above. On the other hand, G_{res} accounts for the 'residual part' that captures the behaviour in the singular regime $\rho, \vartheta \to 0$ or/and $\rho, \vartheta \to \infty$.

On the property of Z the cut-off function we argue as follows. Let $Z = Z_{a,b} \in BC(R), -\infty \le$

$a < b \leq \infty$,

$$Z_{a,b}(s) = \begin{cases} a \, \text{for} \, s < a, \\ s \, \text{for} \, s \in [a,b] \\ b \, \text{for} \, s \geq b, \end{cases} \tag{4.1.112}$$

and we fix a, b finite such that

$$[Z_{a,b}(s(\rho,E))]_{\text{ess}} = \Phi(\rho,E) Z_{a,b}(s(\rho,E)) = \Phi(\rho,E) Z(s(\rho,E)) = [s(\rho,E)]_{\text{ess}}.$$

Finally, in view of (4.1.111) we recall the coercivity properties of $\mathscr{E}$ proved in ([50], Chapter 3, Proposition 3.2),

$$\begin{aligned} \mathscr{E}\Big(\rho,E,\mathbf{m}|r,\Theta,\mathbf{U}\Big) \ \gtrsim \ & \int_{\mathbb{T}^3} \left[|\rho - r|^2 + |E - re(r,\Theta)|^2 + \left| \frac{\mathbf{m}}{\rho} - \mathbf{U} \right|^2 \right]_{\text{ess}} \mathrm{d}x \\ & + \int_{\mathbb{T}^3} \left[1 + \rho + \rho|s(\rho,E)| + E + \frac{|\mathbf{m}|}{\rho} \right]_{\text{res}} \mathrm{d}x. \end{aligned} \tag{4.1.113}$$

Step 3:

In view of (4.1.110) and Proposition 4.1.12, we apply the relative entropy inequality (4.1.96) on the time interval $[0, \tau_M]$

$$\begin{aligned} \mathscr{E}\Big(\rho,E,\mathbf{m}\Big|r,\Theta,\mathbf{U}\Big)(t \wedge \tau_M) \leq & \mathscr{E}\Big(\rho,E,\mathbf{m}\Big|r,\Theta,\mathbf{U}\Big)(0) \\ & + \int_0^{\tau \wedge \tau_M} \mathscr{Q}\Big(\rho,E,\mathbf{m}\Big|r,\Theta,\mathbf{U}\Big) \, \mathrm{d}t + \mathbb{M}(t \wedge \tau_M), \end{aligned} \tag{4.1.114}$$

with

$$\begin{aligned} \mathscr{Q}\Big(\rho,E,\mathbf{m}\Big|r,\Theta,\mathbf{U}\Big) = & \int_{\mathbb{T}^3} \rho \left(\frac{\mathbf{m}}{\rho} - \mathbf{U} \right) \cdot \nabla_x \mathbf{U} \cdot \left(\mathbf{U} - \frac{\mathbf{m}}{\rho} \right) \mathrm{d}x \\ & + \int_{\mathbb{T}^3} [(\rho \mathbf{U} - \mathbf{m})(D_t^d \mathbf{U} + \mathbf{U} \cdot \nabla_x \mathbf{U}) - p(\rho,E) \mathrm{div}_x \mathbf{U}] \, \mathrm{d}x \\ & - \int_0^\tau \int_{\mathbb{T}^3} [\langle \mathscr{V}_{t,x}; \rho s(\rho,E) \rangle D_t^d \Theta + \langle \mathscr{V}_{t,x}; s(\rho,E) \mathbf{m} \rangle \cdot \nabla_x \Theta] \, \mathrm{d}x\mathrm{d}t \\ & + \int_0^T \int_{\mathbb{T}^3} [\rho s(r,\Theta) \partial_t \Theta + \mathbf{m} s(r,\Theta) \cdot \nabla_x \Theta] \, \mathrm{d}x\mathrm{d}t \\ & + \int_{\mathbb{T}^3} \left(\left(1 - \frac{\rho}{r} \right) \partial_t p(r,\Theta) - \frac{\mathbf{m}}{r} \cdot \nabla_x p(r,\Theta) \right) \mathrm{d}x \\ & - \int_{\mathbb{T}^3} \nabla \mathbf{U} : \mathrm{d}\mathscr{R}_{\text{conv}} - \int_{\mathbb{T}^3} \mathrm{div} \mathbf{U} \mathrm{d}\mathscr{R}_{\text{press}}. \end{aligned}$$

(4.1.115)

To apply a Gronwall's type argument we first need to show the estimate

$$\mathscr{Q}\left(\rho, E, \mathbf{m} \middle| r, \Theta, \mathbf{U}\right) \lesssim c\,\mathscr{E}\left(\rho, E, \mathbf{m} \middle| r, \Theta, \mathbf{U}\right), \tag{4.1.116}$$

holds for some constant $c > 0$. We proceed by splitting terms and proving estimates separate as follows. In view of (4.1.109), the estimate on defect measures is given by

$$\int_0^{\tau\wedge\tau_m} \int_{\mathbb{T}^3} \nabla_x \mathbf{U} : \mathrm{d}[\mathscr{R}_{\mathrm{conv}} + \mathscr{R}_{\mathrm{press}}\mathbb{I}]\,\mathrm{d}t \lesssim c(M)\frac{1}{2}\int_0^{\tau\wedge\tau_m} \int_{\mathbb{T}^3} \mathrm{d}\mathrm{trace}[\mathscr{R}_{\mathrm{conv}} + \mathscr{R}_{\mathrm{press}}\mathbb{I}]\,\mathrm{d}t. \tag{4.1.117}$$

Similarly, using (4.1.109) we obtain

$$\begin{aligned} \left|\int_{\mathbb{T}^3} \rho\left(\frac{\mathbf{m}}{\rho} - \mathbf{U}\right)\cdot\nabla_x\mathbf{U}\cdot\left(\mathbf{U} - \frac{\mathbf{m}}{\rho}\right)\mathrm{d}x\right| &\leq \int_{\mathbb{T}^3} \rho\left|\frac{\mathbf{m}}{\rho} - \mathbf{U}\right|^2 |\nabla_x\mathbf{U}|\,\mathrm{d}x, \\ &\lesssim c(M)\int_{\mathbb{T}^3}\rho\left|\frac{\mathbf{m}}{\rho} - \mathbf{U}\right|^2\mathrm{d}x. \end{aligned}$$

Furthermore, we observe that

$$(\rho\mathbf{U} - \mathbf{m})(D_t^d\mathbf{U} + \mathbf{U}\cdot\nabla_x\mathbf{U}) - \frac{\mathbf{m}}{r}\cdot\nabla_x p(r,\Theta) = -\frac{\rho\mathbf{U}}{r}\cdot\nabla_x p(r,\Theta).$$

Consequently, (4.1.115) reduces to

$$\begin{aligned} \mathscr{Q}\left(\rho, E, \mathbf{m} \middle| r, \Theta, \mathbf{U}\right) \lesssim &-\int_0^\tau \int_{\mathbb{T}^3} [\langle \mathscr{V}_{t,x}; \rho s(\rho,\vartheta)\rangle \partial_t\Theta + \langle \mathscr{V}_{t,x}; s(\rho,\vartheta)\mathbf{m}\rangle\cdot\nabla_x\Theta]\,\mathrm{d}x\mathrm{d}t \\ &+ \int_0^T \int_{\mathbb{T}^3} [\rho s(r,\Theta)\partial_t\Theta + \mathbf{m}s(r,\Theta)\cdot\nabla_x\Theta]\,\mathrm{d}x\mathrm{d}t \\ &+ \int_{\mathbb{T}^3} [p(r,\Theta)\mathrm{div}_x\mathbf{U} - p(\rho,E)\mathrm{div}_x\mathbf{U}]\,\mathrm{d}x \\ &+ \int_{\mathbb{T}^3}\left((r-\rho)\frac{1}{r}\partial_t p(r,\Theta) - \frac{\rho\mathbf{U}}{r}\cdot\nabla_x p(r,\Theta) - p(r,\Theta)\mathrm{div}_x\mathbf{U}\right)\mathrm{d}x \\ &+ c_1\,\mathscr{E}\left(\rho, E, \mathbf{m} \middle| r, \Theta, \mathbf{U}\right). \end{aligned} \tag{4.1.118}$$

Re-writing terms with entropy s using the notation introduced in (4.1.111) we consider the formulation

$$
\begin{aligned}
&-\int_{\mathbb{T}^3}[\mathbf{m}(\langle\mathscr{V}_{t,x};s(r,\Theta)\rangle-\langle\mathscr{V}_{t,x};s(\rho,E)\rangle)\cdot\nabla_x\Theta]\,\mathrm{d}x\\
&+\int_{\mathbb{T}^3}[\rho(\langle\mathscr{V}_{t,x};s(r,\Theta)\rangle-\langle\mathscr{V}_{t,x};s(\rho,E)\rangle)\partial_t\Theta]\,\mathrm{d}x\\
&=-\int_{\mathbb{T}^3}\left[[\mathbf{m}(\langle\mathscr{V}_{t,x};s(r,\Theta)\rangle-\langle\mathscr{V}_{t,x};s(\rho,E)\rangle)]_{\mathrm{ess}}\cdot\nabla_x\Theta\right]\mathrm{d}x\\
&+\int_{\mathbb{T}^3}\left[[\rho(\langle\mathscr{V}_{t,x};s(r,\Theta)\rangle-\langle\mathscr{V}_{t,x};s(\rho,E)\rangle)]_{\mathrm{ess}}\partial_t\Theta\right]\mathrm{d}x\\
&-\int_{\mathbb{T}^3}([\rho\langle\mathscr{V}_{t,x};s(\rho,E)\rangle]_{\mathrm{res}}\partial_t\Theta+[\langle\mathscr{V}_{t,x};s(\rho,E)\rangle\mathbf{m}]_{\mathrm{res}}\cdot\nabla_x\Theta)\,\mathrm{d}x\\
&+\int_{\mathbb{T}^3}([\rho]_{\mathrm{res}}\langle\mathscr{V}_{t,x};s(r,\Theta)\rangle\partial_t\Theta+[\mathbf{m}]_{\mathrm{res}}\cdot\langle\mathscr{V}_{t,x};s(r,\Theta)\rangle\nabla_x\Theta)\,\mathrm{d}x.
\end{aligned}
\tag{4.1.119}
$$

Here we bound the residual terms in (4.1.119) by applying (4.1.113). In the essential part, we first revert to the use of original variables (ρ,ϑ) and note the following

$$
[s(\rho,\vartheta(\rho,E))-s(r,\Theta)]_{\mathrm{ess}}\approx\partial_\rho s(r,\Theta)[\rho-r]_{\mathrm{ess}}+\partial_\vartheta s(r,\Theta)[\vartheta(\rho,E)-\Theta]_{\mathrm{ess}},
$$

for such case the difference is proportional to

$$
[\rho-r]^2_{\mathrm{ess}}+[E-re(r,\Theta)]^2_{\mathrm{ess}},
$$

is controlled by the left hand side of (4.1.113). Consequently, taking into account the discussions above the inequality (4.1.118) reduces to

$$
\begin{aligned}
&\mathscr{Q}\left(\rho,E,\mathbf{m}\middle|r,\Theta,\mathbf{U}\right)\\
&\lesssim-\int_{\mathbb{T}^3}[\mathbf{m}(\partial_\rho\langle\mathscr{V}_{t,x};s(r,\Theta)\rangle[\rho-r]_{\mathrm{ess}}+\partial_\vartheta\langle\mathscr{V}_{t,x};s(r,\Theta)\rangle[\vartheta(\rho,E)-\Theta]_{\mathrm{ess}})\cdot\nabla_x\Theta]\,\mathrm{d}x\\
&-\int_{\mathbb{T}^3}[\rho(\partial_\rho\langle\mathscr{V}_{t,x};s(r,\Theta)\rangle[\rho-r]_{\mathrm{ess}}+\partial_\vartheta\langle\mathscr{V}_{t,x};s(r,\Theta)\rangle[\vartheta(\rho,E)-\Theta]_{\mathrm{ess}})\partial_t\Theta]\,\mathrm{d}x\\
&+\int_{\mathbb{T}^3}[p(r,\Theta)\mathrm{div}_x\mathbf{U}-p(\rho,E)\mathrm{div}_x\mathbf{U}]\,\mathrm{d}x\\
&+\int_{\mathbb{T}^3}\left((r-\rho)\frac{1}{r}\partial_t p(r,\Theta)-\frac{\rho\mathbf{U}}{r}\cdot\nabla_x p(r,\Theta)-p(r,\Theta)\mathrm{div}_x\mathbf{U}\right)\mathrm{d}x\\
&+c_1\mathscr{E}\left(\rho,E,\mathbf{m}\middle|r,\Theta,\mathbf{U}\right).
\end{aligned}
\tag{4.1.120}
$$

Now using the fact that r and $\mathbf{U}$ satisfy the equation of mass continuity, we deduce the following identity

$$
\begin{aligned}
&(r-\rho)\frac{1}{r}\partial_t p(r,\Theta)+\nabla_x p(r,\Theta)\cdot\mathbf{U}-\frac{\rho}{r}\mathbf{U}\cdot\nabla_x p(r,\Theta)+\operatorname{div}_x\mathbf{U}(p(r,\Theta)-p(\rho,\vartheta))\\
&=\operatorname{div}_x\mathbf{U}\Big(p(r,\Theta)-\partial_\rho p(r,\Theta)(r-\rho)-\partial_\vartheta p(r,\Theta)(\Theta-\vartheta)-p(\rho,\vartheta))\Big)\\
&+r(\rho-r)\partial_\rho s(r,\Theta)\Big(\partial_t\Theta+\mathbf{U}.\nabla_x\Theta\Big)+r(\Theta-\Theta)\partial_\vartheta s(r,\Theta)\Big(\partial_t\Theta+\mathbf{U}.\nabla_x\Theta\Big)
\end{aligned}
$$

Accordingly, the residual part of the above expression is absorbed by left-hand side of (4.1.28). Next applying the identity derived above, we may re-write (4.1.120) taking into account the estimation of (4.1.119) as

$$
\begin{aligned}
&\mathscr{Q}\Big(\rho,E,\mathbf{m}\Big|r,\Theta,\mathbf{U}\Big)\\
&\lesssim\int_{\mathbb{T}^3}\Big(p(r,\Theta)-\partial_\rho p(r,\Theta)(r-\rho)-\partial_\vartheta p(r,\Theta)(\Theta-\vartheta)-p(\rho,\vartheta))\Big)\operatorname{div}_x\mathbf{U}\,\mathrm{d}x\\
&+c_2\mathscr{E}\Big(\rho,E,\mathbf{m}\Big|r,\Theta,\mathbf{U}\Big).\\
&\lesssim\int_{\mathbb{T}^3}[\rho-r]^2_{\text{ess}}+[E-re(r,\Theta)]^2_{\text{ess}}\,\mathrm{d}x\\
&+c_3\mathscr{E}\Big(\rho,E,\mathbf{m}\Big|r,\Theta,\mathbf{U}\Big).
\end{aligned}
\tag{4.1.121}
$$

Finally, in view of (4.1.113), the term

$$
[\rho-r]^2_{\text{ess}}+[E-re(r,\Theta)]^2_{\text{ess}}
$$

is absorbed by the left-hand side to obtain the desired form in (4.1.116).

Step 4

In view of the above estimates, the relative entropy inequality (4.1.114) reduces to

$$
\begin{aligned}
&\mathscr{E}\Big(\rho,E,\mathbf{m}\Big|r,\Theta,\mathbf{U}\Big)(t\wedge\tau_M)\\
&\lesssim\mathscr{E}\Big(\rho,E,\mathbf{m}\Big|r,\Theta,\mathbf{U}\Big)(0)+\int_0^{\tau\wedge\tau_M}c\,\mathscr{E}\Big(\rho,E,\mathbf{m}\Big|r,\Theta,\mathbf{U}\Big)(t)\,\mathrm{d}t+\mathbb{M}(t\wedge\tau_M).
\end{aligned}
\tag{4.1.122}
$$

To conclude we take the expectation in $[t\wedge\tau_M]$ and apply Gronwall's lemma yielding

$$\mathbb{E}\left[\mathscr{E}\left(\rho,E,\mathbf{m}\middle|r,\Theta,\mathbf{U}\right)(t\wedge\tau_M)\right]\le c(M)\mathbb{E}\left[\mathscr{E}\left(\rho,E,\mathbf{m}\middle|r,\Theta,\mathbf{U}\right)(0)\right], \qquad (4.1.123)$$

where

$$\mathbb{E}\left[\mathscr{E}\left(\rho,E,\mathbf{m}\middle|r,\Theta,\mathbf{U}\right)(0)\right]=0$$

by assumptions. Therefore, we observe that

$$\mathbb{E}\left[\mathscr{E}\left(\rho,E,\mathbf{m}\middle|r,\Theta,\mathbf{U}\right)(t\wedge\tau_M)\right]=0$$

for all $t\in(0,T)$, yielding the claim. □

Remark 4.1.7. We note arguing similarly as in the proof of Theorem 3.1.14 for an incompressible Euler system, the weak-strong(measure-valued) uniqueness property in law continues to hold in the compressible Euler system.

4.1.11 Martingale solutions as measures on the space of trajectories

We dedicate this section and subsequent sections to the study of Markov selection for the complete stochastic Euler system (4.1.17)-(4.1.19). To begin with, we address the difficulties one encounters when applying Markov selection on systems with measures. Accordingly, we observe that from the proof of Theorem 4.1.7, the natural filtration associated to a dissipative measure-valued martingale solution in the sense of Definition 4.1.5 is the joint canonical filtration of $[\rho,\mathbf{m},S,\mathscr{R}_{\text{conv}},\mathscr{R}_{\text{press}},\mathscr{V}_{t,x},W]$. However, the canonical processes $[S,\mathscr{R}_{\text{conv}},\mathscr{R}_{\text{press}},\mathscr{V}_{t,x}]$ are class of equivalences in time and not a stochastic processes in the classical sense. Therefore, it is not obvious as to how one should formulate the Markovianity of the system (4.1.17)-(4.1.19). To circumvent this problem, we shall introduce new variables $\mathscr{S},\mathbf{R}$ (time integrals) such that

$$\mathscr{S}=\int_0^{\cdot}S\,\mathrm{d}s,\quad \mathbf{R}=\int_0^{\cdot}\left(\mathscr{R}_{\text{conv}},\mathscr{R}_{\text{press}},\mathscr{V}_{t,x}\right)\mathrm{d}s.$$

Consequently, the notion of new variables allows us to establish the Markov selection for the joint law of $[\rho,\mathbf{m},\mathscr{S},\mathbf{R}]$. In this case, the stochastic process has continuous trajectories and contains all necessary information. The initial data for $[\mathscr{S},\mathbf{R}]$ is superfluous and only needed for technical reasons in the selection process. To study Markov selection, it is desirable to consider the martingale solutions as probability measures $\mathscr{P} \in \text{Prob}\,[\Omega]$ such that

$$\Omega = C_{\text{loc}}([0,\infty); W^{-k,2}(\mathbb{T}^3)),$$

where $k > 3/2$. Adopting the set-up of Section 4.1.2 we set $X = W^{-k,2}(\mathbb{T}^3)$. Accordingly, let $\mathscr{B}$ denote the Borel σ-field on Ω. Let $\xi = (\xi^1, \xi^2, \xi^3, \xi^4)$ denote the canonical process of projections such that

$$\xi = (\xi^1, \xi^2, \xi^3, \xi^4) : \Omega \to \Omega, \quad \xi_t \omega = (\xi^1_t, \xi^2_t, \xi^3_t, \xi^4_t)(\omega) = \omega_t \in W^{-k,2}(\mathbb{T}^3), \text{for any } t \geq 0,$$

where the notation ω_t indicates that our random variable is time dependent. In addition, let $(\mathscr{B}_t)_{t\geq 0}$ be the filtration associated to canonical process given by

$$\mathscr{B}_t := \sigma(\xi|_{[0,t]}), \quad t \geq 0,$$

which coincides with the Borel σ-field on $\Omega^{[0,t]} = ([0,t]; W^{-k,2}(T))$. From here henceforth, we shall consider analysis of the dissipative martingale solutions

$$((\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P}), \rho, \mathbf{m}, S, \mathscr{R}_{\text{conv}}, \mathscr{R}_{\text{press}}, \mathscr{V}_{t,x}, W),$$

in the sense of Definition 4.1.5 as probability laws $\mathscr{P}$, that is,

$$\mathscr{P} = \mathscr{L}\left[\rho, \mathbf{m}, \int_0^{\cdot} S\,\mathrm{d}s, \int_0^{\cdot} (\mathscr{R}_{\text{conv}}, \mathscr{R}_{\text{press}}, \mathscr{V}_{t,x})\,\mathrm{d}s\right] \in \text{Prob}[\Omega].$$

Consequently, we obtain the probability space $(\Omega, \mathscr{B}, (\mathscr{B}_t)_{t\geq 0}, \mathscr{P})$. Furthermore, we introduce the space

$$\begin{aligned} F &= \left\{ [\rho, \mathbf{m}, \mathscr{S}, \mathbf{R}] \in \tilde{F} \,\middle|\, \int_{\mathbb{T}^3} \frac{|\mathbf{m}|^2}{|\rho|} \mathrm{d}x < \infty \right\}, \\ \tilde{F} &= L^\gamma(\mathbb{T}^3) \times L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^3) \times (L^\gamma(\mathbb{T}^3)) \times (W^{-k,2}(\mathbb{T}^3, \cdot))^2 \times (W^{-k,2}(\mathbb{T}^3, A)). \end{aligned}$$

where $A = \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}$. We augment F with the points of the form $(0, \mathbf{0}, \mathscr{S}, \mathbf{R})$ for $\mathscr{S} \in L^\gamma(\mathbb{T}^3)$ and $\mathbf{R} \in W^{-k,2}(\mathbb{T}^3, \mathbb{R}^{15})$. Therefore, F is Polish space with metric

$$\mathrm{d}_F(y,z) = \mathrm{d}_Y((y^1, \mathbf{y}^2, \mathbf{y}^3, \mathbf{y}^4), (z^1, \mathbf{z}^2, \mathbf{z}^3, \mathbf{z}^4)) = \|y - z\|_{\tilde{F}} + \left\| \frac{\mathbf{y}^2}{\sqrt{|y^1|}} - \frac{\mathbf{z}^2}{\sqrt{|z^1|}} \right\|_{L^2_x}. \tag{4.1.124}$$

Moreover, inclusion $F \hookrightarrow X$ is dense. Accordingly, we define a subset

$$Y = \left\{ [\rho, \mathbf{m}, \mathscr{S}, \mathbf{R}] \in X \middle| \rho \not\equiv 0, \rho \geq 0, \int_{\mathbb{T}^3} \frac{|\mathbf{m}|^2}{\rho} \mathrm{d}x < \infty \right\}.$$

We observe that (Y, d_F) is not complete because $\rho \not\equiv 0$, and the inclusion $Y \hookrightarrow X$ is not dense since $\rho \geq 0$. The probability law $\mathscr{P}(t,\cdot)$ continues to hold (supported) in Y, and consequently determines *the set of admissible initial conditions*.

Definition 4.1.6 (Dissipative measure-valued martingale solution)**.** A borel probability measure $\mathscr{P}$ on Ω is called a solution to the martingale problem associated to (4.1.17)-(4.1.19) provided:

(a) it holds

$$\begin{aligned}
&\mathscr{P}\left(\xi^1 \in C_{\mathrm{loc}}[0,\infty); (L^\gamma(\mathbb{T}^3), w)), \xi^1 \geq 0\right) = 1,\\
&\mathscr{P}\left(\xi^2 \in C_{\mathrm{loc}}[0,\infty); (L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^3), w))\right) = 1,\\
&\mathscr{P}\left(\xi^3 \in W^{1,\infty}([0,\infty); L^\gamma(\mathbb{T}^3))\right) = 1,\\
&\mathscr{P}\left(\xi^4 \in W^{1,\infty}_{\mathrm{weak}-(*)}(0,\infty; \mathscr{M}^+(\mathbb{T}^3, \mathbb{R}^{3\times 3}))\right) = 1;\\
&\mathscr{P}\left(\xi^4 \in W^{1,\infty}_{\mathrm{weak}-(*)}(0,\infty; \mathscr{M}^+(\mathbb{T}^3, \mathbb{R}))\right) = 1;\\
&\mathscr{P}\left(\xi^4 \in (W^{1,\infty}_{\mathrm{weak}-(*)}((0,\infty) \times \mathbb{T}^3; \mathbb{P}(\mathbb{R}^5)))\right) = 1;
\end{aligned}$$

(b) the total energy

$$\mathfrak{E} = \int_{\mathbb{T}^3} \left[\frac{1}{2} \frac{|\xi^2|^2}{\xi^1} + c_v (\xi^1)^\gamma \exp\left(\frac{\xi^3}{c_v \xi^1} \right) \right] \mathrm{d}x + \frac{1}{2} \int_{\mathbb{T}^3} \mathrm{d}\mathrm{tr}\xi^4_{\mathrm{conv}}(t) + c_v \int_{\mathbb{T}^3} \mathrm{d}\xi^4_{\mathrm{press}}(t)$$

belongs to the space $L^\infty_{\mathrm{loc}}(0,\infty)$ $\mathscr{P}$-a.s.;

(c) it holds $\mathscr{P}$-a.s.

$$\left[\int_{\mathbb{T}^3} \xi^1_t \psi \right]_{t=0}^{t=\tau} - \int_0^\tau \int_{\mathbb{T}^3} \xi^2_t \cdot \nabla\psi \, \mathrm{d}x \, \mathrm{d}t = 0$$

for any $\psi \in C^1(\mathbb{T}^3)$ and $\tau \geq 0$;

(d) for any $\varphi \in C^1(\mathbb{T}^3, \mathbb{R}^3)$, the stochastic process

$$
\begin{aligned}
\mathscr{M}(\varphi) : [\omega, \tau] \mapsto & \left[\int_{\mathbb{T}^3} \xi_t^2 \cdot \varphi \right]_{t=0}^{t=\tau} \\
& - \int_0^\tau \int_{\mathbb{T}^3} \left[\frac{\xi_t^2 \otimes \xi_t^2}{\xi_t^1} : \nabla \varphi + \xi_t^1 \exp\left(\frac{\xi_t^3}{c_v \xi_t^1} \right) \operatorname{div} \varphi \right] \mathrm{d}x\mathrm{d}t \\
& - \int_0^\tau \nabla \varphi : \mathrm{d}\xi^4_{t\,\mathrm{conv}} \mathrm{d}t - \int_0^\tau \int_{\mathbb{T}^3} \operatorname{div} \varphi \, \mathrm{d}\xi^4_{t\,\mathrm{press}} \mathrm{d}t
\end{aligned}
$$

is a square integrable $((\mathfrak{B}_t)_{t \geq 0}, \mathscr{P})$-martingale with quadratic variation

$$
\frac{1}{2} \int_0^\tau \sum_{k=1}^\infty \left(\int_{\mathbb{T}^3} \xi_t^1 \phi e_k \cdot \varphi \, \mathrm{d}x \right)^2 \mathrm{d}t;
$$

(e) It holds $\mathscr{P}$-a.s.

$$
\begin{aligned}
\int_0^\tau \int_{\mathbb{T}^3} \left[\langle (\xi_\nu^4)_{t,x}; Z(\tilde{S}) \rangle \partial_t \varphi + \langle (\xi_\nu^4)_{t,x}, Z(\tilde{S}) \tilde{\mathbf{m}} / \tilde{\rho} \rangle \cdot \varphi \right] \mathrm{d}x\mathrm{d}t & \\
\leq \left[\int_{\mathbb{T}^3} \langle (\xi_\nu^4)_{t,x}; Z(\tilde{S}) \rangle \varphi \, \mathrm{d}x \right]_{t=0}^{t=\tau} &
\end{aligned}
\tag{4.1.125}
$$

for any $\varphi \in C^1([0, \infty) \times \mathbb{T}^3)$, $\varphi \geq 0$, and any $Z \in BC(\mathbb{R})$ non-decreasing.

(f) The stochastic process

$$
\mathscr{E} : [\omega, \tau] \mapsto \mathfrak{E}_\tau - \mathfrak{E}_0 - \frac{1}{2} \int_0^\sigma \| \sqrt{\xi^1} \phi \|^2_{L_2(\mathscr{U}; L^2(\mathbb{T}^3))} \mathrm{d}\sigma \tag{4.1.126}
$$

is a square integrable $((\mathscr{B}_t)_{t \geq 0}, \mathscr{P})$-martingale with quadratic variation

$$
\frac{1}{2} \int_0^\tau \sum_{k=1}^\infty \left(\int_{\mathbb{T}^3} \xi_t^2 \cdot \phi e_k \, \mathrm{d}x \right)^2 \mathrm{d}t
$$

for $\tau \geq 0$.

In the following we state the relation between Definition 4.1.5 and Definition 4.1.6.

Proposition 4.1.13. *The following statement holds true*

1. *Let* $((\Omega, \mathscr{F}, (\mathscr{F}_t)_{t \geq 0}, \mathbb{P}), \rho, \mathbf{m}, S, \mathscr{R}_{conv}, \mathscr{R}_{press}, \mathscr{V}_{t,x}, W)$ *be a dissipative martingale solution to (4.1.17)-(4.1.19) in the sense of Definition 4.1.5. Then for every* $\mathscr{F}_0$-*measurable random variables* $[\mathscr{S}_0, \mathbf{R}_0]$ *with values in* $\mathscr{S} \in W^{-k,2}(\mathbb{T}^3) \cap L^\gamma(\mathbb{T}^3)$

and $\mathbf{R} \in W^{-k,2}(\mathbb{T}^3, \mathbb{R}^{15})$ *we have that*

$$\mathscr{P} = \mathscr{L}\left[\rho, \mathbf{m} = \rho\mathbf{u}, \mathscr{S}_0 + \int_0^{\cdot} \mathscr{S}\,\mathrm{d}s, \mathbf{R}_0 + \int_0^{\cdot} \mathbf{R}\,\mathrm{d}s\right] \in \mathrm{Prob}[\Omega] \tag{4.1.127}$$

is a solution to the martingale problem associated to (4.1.17)-(4.1.19) in the sense of Definition 4.1.6.

2. *Let* $\mathscr{P}$ *be a solution to the martingale problem associated to (4.1.17)-(4.1.19) in the sense of Definition 4.1.6. Then there exists a dissipative martingale solution*

$$((\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P}), \rho, \mathbf{m}, S, \mathscr{R}_{conv}, \mathscr{R}_{press}, \mathscr{V}_{t,x}, W_1)$$

to the system (4.1.17)-(4.1.19) in the sense of Definition 4.1.5 satisfying properties (a)-(j), furthermore, there exists W_2 *such that*

$$((\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P}), \rho, \mathbf{m}, S, \mathscr{R}_{conv}, \mathscr{R}_{press}, \mathscr{V}_{t,x}, W_2)$$

satisfies Definition 4.1.5 property (k) and an $\mathscr{F}_0$*-measurable random variables* $[\mathscr{S}_0, \mathbf{R}_0]$ *for* $\mathscr{S} \in W^{-k,2}(\mathbb{T}^3) \cap L^{\gamma}(\mathbb{T}^3)$ *and* $\mathbf{R} \in W^{-k,2}(\mathbb{T}^3, \mathbb{R}^{15})$ *such that*

$$\mathscr{P} = \mathscr{L}\left[\rho, \mathbf{m} = \rho\mathbf{u}, \mathscr{S}_0 + \int_0^{\cdot} S\,\mathrm{d}s, \mathbf{R}_0 + \int_0^{\cdot} \mathbf{R}\,\mathrm{d}s\right] \in \mathrm{Prob}[\Omega], \tag{4.1.128}$$

where W_1 *and* W_2 *correspond to Wiener process generated by momentum equation and energy equality, respectively.*

We proceed to present a proof of Proposition 4.1.13 following the arguments presented in [12] with appropriate adjustment to our system (4.1.17)-(4.1.19).

Proof. **Step 1:** *Definition 4.1.5 implies Definition 4.1.6.*
Our aim is to show that the probability law given by (4.1.127) is a solution to the martingale problem associated to (4.1.17)-(4.1.19) in the sense of Definition 4.1.6. To proceed, we let

$$((\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P}), \rho, \mathbf{m}, S, \mathscr{R}_{\mathrm{conv}}, \mathscr{R}_{\mathrm{press}}, \mathscr{V}_{t,x}, W_1)$$

be a dissipative solution martingale solution to Euler system (4.1.17)-(4.1.19) in the sense of Definition 4.1.5 and let $[\mathscr{S}_0, \mathbf{R}_0]$ be arbitrary $\mathscr{F}_0$-measurable random variables with $\mathscr{S} \in W^{-l,2}(\mathbb{T}^3) \cap L^{\gamma}(\mathbb{T}^3)$ and $\mathbf{R} \in W^{-k,2}(\mathbb{T}^3, \mathbb{R}^{15})$. Accordingly, we observe that property (a) of Definition 4.1.6 follows from properties: (c), (d),(e) and (f) in Definition 4.1.5 *(the adaptedness of terms with measures are understood in the sense of random distributions, see Section 2.1)* and the definition of $\mathscr{P}$ as the pushforward measure generated by $[\rho, \mathbf{m}, \mathscr{S}, \mathbf{R}]$.

Similarly, we note that the total energy, mass continuity equation and balance of entropy are measurable functions on the subset Ω with $\mathscr{P}$ a.s.(i.e. the law $\mathscr{P}$ is supported on Ω). Consequently, we obtain that the properties (b), (c) and (e) of Definition 4.1.6 hold. Finally, we proceed to show that properties (d) and (f) of Definition 4.1.6 hold. This follows from noting that the functionals $\mathscr{M}(\varphi)$ and $\mathscr{E}$ are measurable on the subset of Ω where $\mathscr{P}$ is supported. Specifically, we re-write the equation of moment in form

$$\begin{aligned}\left[\int_{\mathbb{T}^3} \mathbf{m}\cdot\varphi\right]_{t=0}^{t=\tau} \quad - \quad & \int_0^\tau \int_{\mathbb{T}^3} \left[\frac{\mathbf{m}\otimes\mathbf{m}}{\rho} : \nabla\varphi + \rho \exp\left(\frac{S}{c_v\rho}\operatorname{div}\varphi\right)\right] \mathrm{d}x\mathrm{d}t \\ & - \int_0^\tau \nabla\varphi : \mathrm{d}\mathscr{R}_{\mathrm{conv}}\mathrm{d}t - \int_0^\tau \int_{\mathbb{T}^3} \operatorname{div}\varphi \, \mathrm{d}\mathscr{R}_{\mathrm{press}}\mathrm{d}t \\ & = \int_0^\tau \varphi\cdot\rho\phi \, \mathrm{d}x\mathrm{d}W, \end{aligned} \tag{4.1.129}$$

$\mathbb{P}$-a.s. for all $\varphi \in C^\infty(\mathbb{T}^3)$ and all $\tau > 0$. We observe that the left hand side of (4.1.129) is a martingale with respect to the canonical filtration generated by $[\rho, \mathbf{m}, \mathscr{S}, \mathbf{R}]$. Consequently, $\mathscr{M}(\varphi)$ is subject to properties of a martingale. To show property (d) of Definition 4.1.6 we argue as follows. Let $\mathbf{V}$ be a stochastic process, we denote by $\mathbf{V}_{t,s}$ the increments $\mathbf{V}_t - \mathbf{V}_s$ for $s \le t$. We consider $\varphi \in C^\infty(\mathbb{T}^3)$ and a continuous function $h : \Omega^{[0,s]} \to [0,1]$ such that

$$\mathbb{E}^{\mathscr{P}}[h(\xi|_{[0,s]})\mathscr{M}(\varphi)_{s,t}] = \mathbb{E}^{\mathbb{P}}[h([\rho, \mathbf{m}, \mathscr{S}, \mathbf{R}]|_{[0,s]})\mathfrak{M}(\varphi)_{s,t}] = 0,$$

where

$$\begin{aligned}\mathfrak{M}(\varphi) \quad = \quad & \left[\int_{\mathbb{T}^3} \mathbf{m}\cdot\varphi\right]_{t=0}^{t=\tau} - \int_0^\tau \int_{\mathbb{T}^3} \left[\frac{\mathbf{m}\otimes\mathbf{m}}{\rho} : \nabla\varphi + \rho \exp\left(\frac{S}{c_v\rho}\operatorname{div}\varphi\right)\right] \mathrm{d}x\mathrm{d}t \\ & - \int_0^\tau \nabla\varphi : \mathrm{d}\mathscr{R}_{\mathrm{conv}}\mathrm{d}t - \int_0^\tau \int_{\mathbb{T}^3} \operatorname{div}\varphi \, \mathrm{d}\mathscr{R}_{\mathrm{press}}\mathrm{d}t. \end{aligned}$$

Furthermore, we deduce

$$\mathbb{E}^{\mathscr{P}}[h(\xi|_{[0,s]})[\mathscr{M}^2(\varphi)]_{s,t} - \mathscr{Q}(\varphi)_{s,t}] = \mathbb{E}^{\mathbb{P}}[h([\rho,\mathbf{m},\mathscr{S},\mathbf{R}]|_{[0,s]})[\mathfrak{M}^2(\varphi)]_{s,t} - \mathfrak{Q}(\varphi)_{s,t}] = 0,$$

where

$$\mathscr{Q}(\varphi) = \int_0^\tau \sum_{k=1}^\infty \left(\int_{\mathbb{T}^3} \xi_t^1 \phi e_k \cdot \varphi \,\mathrm{d}x\right)^2 \mathrm{d}t.$$

$$\mathfrak{Q}(\varphi) = \int_0^\tau \sum_{k=1}^\infty \left(\int_{\mathbb{T}^3} \rho \phi e_k \cdot \varphi \,\mathrm{d}x\right)^2 \mathrm{d}t.$$

Accordingly, we conclude that $\mathscr{M}(\varphi)$ is a $\mathscr{B}_t$-martingale with quadratic variation $\mathscr{Q}(\varphi)$. We observe that property (f) of Definition 4.1.6 follows from arguing as in arguments of property (d) with appropriate adjustments. This completes proof of **Step 1**.

Step 2: *Definition 4.1.6 implies Definition 4.1.5.*

To begin with, let $\mathscr{P} \in \mathrm{Prob}[\Omega]$ be a solution to the martingale problem (4.1.17)-(4.1.19) in the sense of Definition 4.1.6. To complete step two we need to find a stochastic basis $(\mathscr{X}, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathscr{P})$, density ρ, momentum $\mathbf{m}$, entropy S, convective ($\mathscr{R}_{\mathrm{conv}}$) and pressure measures ($\mathscr{R}_{\mathrm{press}}$), respectively, Young measure $\mathscr{V}_{t,x}$ and a cylindrical $(\mathscr{F}_t)$- measurable Wiener process W such that

$$((\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P}), \rho, \mathbf{m}, S, \mathscr{R}_{\mathrm{conv}}, \mathscr{R}_{\mathrm{press}}, \mathscr{V}_{t,x}, W_j), j = 1,2;$$

is a dissipative solution to (4.1.17)-(4.1.19) in the sense of Definition 4.1.5. In the case of $j = 1$ (without labelling) we use momentum equation and argue as follows. Taking into account property (d) of Definition 4.1.6 in conjunction with the standard martingale representation theorem, see [35, Theorem 8.2], we deduce that there exists an extended stochastic basis

$$(\Omega \times \tilde{\Omega}, \mathscr{B} \otimes \tilde{\mathscr{B}}, (\mathscr{B}_t \otimes \tilde{\mathscr{B}}_t)_{t\geq 0}, \mathscr{P} \otimes \tilde{\mathscr{P}}),$$

and an $(\mathscr{B}_t \otimes \tilde{\mathscr{B}}_t)_{t\geq 0}$-measurable cylindrical Wiener process $W_1 = \sum_{k=1}^\infty W_k e_k$ such that

$$\mathfrak{Q}(\varphi) = \sum_{k=1}^\infty \int_0^\tau \left(\int_{\mathbb{T}^3} \rho \phi e_k \cdot \varphi \,\mathrm{d}x\right) \mathrm{d}W_k,$$

where

$$\rho(\omega,\tilde{\omega}) = \xi^1(\omega), \quad \mathbf{m}(\omega,\tilde{\omega}) = \xi^2(\omega), \quad S(\omega,\tilde{\omega}) = \xi^3(\omega), \tag{4.1.130}$$

and

$$\xi^4(\omega) = \Big(\mathscr{R}_{\text{conv}}(\omega,\tilde{\omega}), \mathscr{R}_{\text{press}}(\omega,\tilde{\omega}), \mathscr{V}_{t,x}(\omega,\tilde{\omega})\Big). \tag{4.1.131}$$

Finally, we proceed to account for the noise term W_2 generated by the energy inequality. Similarly to the case above, we apply the standard martingale representation theorem again to the extended space $(\Omega\times\tilde{\Omega})$ and obtain an extended stochastic basis

$$(\Omega\times\tilde{\Omega}\times\overline{\Omega}, \mathscr{B}\otimes\tilde{\mathscr{B}}, (\mathscr{B}_t\otimes\tilde{\mathscr{B}}_t\otimes\overline{\mathscr{B}}_t)_{t\geq 0}, \mathscr{P}\otimes\tilde{\mathscr{P}}\otimes\overline{\mathscr{P}}),$$

and an $(\mathscr{B}_t\otimes\tilde{\mathscr{B}}_t\otimes\overline{\mathscr{B}}_t)_{t\geq 0}$-measurable cylindrical Wiener process $W = \sum_{k=1}^{\infty} W_k e_k$ such that

$$\mathscr{E} = \sum_{k=1}^{\infty}\int_0^{\tau}\int_{\mathbb{T}^3}\xi_t^2\cdot\phi e_k \,\mathrm{d}x\,\mathrm{d}W_k$$

with $\xi_t^1, \xi_t^2, \xi_t^3, \xi_t^4$ satisfying the conditions in (4.1.130)-(4.1.131), and are extended trivially (with constant values w.r.t $(\overline{\Omega})$. Accordingly, we let $(\mathscr{X}, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathscr{P})$ to be the above extended probability space associated to the filtration $(\mathscr{B}_t\otimes\tilde{\mathscr{B}}_t\otimes\overline{\mathscr{B}})_{t\geq 0}$, then

$$((\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P}), \rho, \mathbf{m}, S, \mathscr{R}_{\text{conv}}, \mathscr{R}_{\text{press}}, \mathscr{V}_{t,x}, W_j), j = 1,2;$$

is a dissipative martingale solution to (4.1.17)-(4.1.19) in the sense of Definition 4.1.5. In addition, the following holds

$$\mathscr{P} = \mathscr{L}\left[\rho, \mathbf{m} = \rho\mathbf{u}, \mathscr{S}_0 + \int_0^{\cdot} S\,\mathrm{d}s, \mathbf{R}_0 + \int_0^{\cdot}\mathbf{R}\,\mathrm{d}s\right] \in \mathrm{Prob}[\Omega],$$

with $\mathscr{S}_0(\omega,\tilde{\omega}) = \xi_0^3(\omega)$ and $\mathbf{R} = (\mathscr{R}_{\text{conv}}, \mathscr{R}_{\text{press}}, \mathscr{V}_{t,x})_0(\omega,\tilde{\omega}) \equiv (0,0,0)_0(\omega)$ by definition. To be precise, we conclude that applying the martingale representation theorem twice completes proof of **Step 2**. □

4.1.12 Markov selection

In this section we state and prove the strong Markov selection to the complete stochastic Euler system (4.1.17)-(4.1.19). Let $y \in Y$ be an admissible initial data (condition), we denote by $\mathscr{P}_y$ a solution to the martingale problem associated with (4.1.17)-(4.1.19) starting on y at time $t=0$; that is, the marginal of $\mathscr{P}_y$ at $t=0$ is Λ_y. We start off with the definition of strong Markov family;

Definition 4.1.7. A family $(\mathscr{P}_y)_{y\in Y} \in \mathrm{Prob}[\Omega]$ of probability measures is called a strong Markov selection family provided

(1) For every $A \in \mathscr{B}$, the mapping $y \mapsto \mathscr{P}_y(A)$ is $\mathscr{B}(Y)/\mathscr{B}([0,1])$-measurable.

(2) For every finite $(\mathscr{B}_t)_{t\geq 0}$-stopping time T, every $y \in Y$ and $\mathscr{P}_y$-a.s. $\omega \in \Omega$

$$\mathscr{P}_y|^{\omega}_{\mathscr{B}_T} = \mathscr{P}_y \circ \Phi^{-1}_{-\tau}.$$

Accordingly, a strong Markov family follows from the so-called pre-Markov family via a selection procedure. Finally, we have all we need to state the following theorem.

Theorem 4.1.14. *Assume (4.1.11) and (4.1.12) holds. Then there exists a family $\{\mathscr{P}_y\}_{y\in Y}$ of solutions to the martingale problem associated to (4.1.17)-(4.1.19) in the sense of Definition 4.1.6 with a.s. Markov property (as defined in Definition 4.1.3)*

We set $y = (y^1, \mathbf{y}^2, \mathbf{y}^3, \mathbf{y}^4) \in Y$ and denote by $\mathscr{C}(y)$ the set of probability laws $\mathscr{P}_y \in \mathrm{Prob}[\Omega]$ solving the martingale problem associated to (4.1.17)-(4.1.19) with initial law $[\Lambda_y]$. The proof of Theorem 4.1.14 follows from applying abstract result of Theorem 4.1.3. In particular, we show that the family $\{\mathscr{C}(y)\}_{y\in Y}$ of solutions to the martingale problem satisfies the disintegration and reconstruction properties in Definition 4.1.3.

Lemma 4.1.15. *For each $y = (y^1, \mathbf{y}^2, \mathbf{y}^3, \mathbf{y}^4) \in Y$. The set $\mathscr{C}(y)$ is non-empty and convex. Furthermore, for every $\mathscr{P} \in \mathscr{C}(y)$, the marginal at every time $t \in (0,\infty)$ is supported on Y.*

Proof. Assuming $y \in Y$, application of Theorem 4.1.6 yields existence of a martingale solution to the problem (4.1.17)-(4.1.19) in the sense of Definition 4.1.5. Consequently, by Proposition 4.1.13 we infer that for each $y \in Y$ the set $\mathscr{C}(y)$ is non-empty. For some $\lambda \in (0,1)$, we consider $\mathscr{P}_1, \mathscr{P}_2 \in \mathscr{C}(y)$ such that $\mathscr{P} = \lambda \mathscr{P}_1 + (1-\lambda)\mathscr{P}_2$. Then convexity follows from noting that properties of Definition 4.1.5 involve integration with respect to the elements of $\mathscr{C}(y)$. In view of Definition 4.1.5 property (f) (energy equality), the marginal $\mathscr{P} \in \mathscr{C}(y)$ at every $t \in (0,\infty)$ is supported in Y. □

For compactness we consider the following Lemma.

Lemma 4.1.16. *Let $y \in Y$. Then $\mathscr{C}(y)$ is a compact set and the map $\mathscr{C} : Y \to \mathrm{Comp}(\mathrm{Prob}[\Omega])$ is Borel measurable.*

Proof. The lemma follows from the claim: Let $(y_n = (\rho_n, \mathbf{m}_n, \mathscr{S}_n, \mathbf{R}_n))_{n\in\mathbb{N}} \subset Y$ be a sequence converging in Y to some $(y = (\rho, \mathbf{m}, \mathscr{S}, \mathbf{R}))$ with respect to the metric d_F in (4.1.124). Let $\mathscr{P}_n \in C(y_n), n \in \mathbb{N}$. Then for each $(\mathscr{P}_n)_{n\in N}$, the sequence converges to some $\mathscr{P} \in \mathscr{C}(y)$ weakly in $\mathrm{Prob}[\Omega]$. Since Y is a metric space the measurability of the map $y \mapsto \mathscr{C}(y)$ follows from using [86, Theorem 12.1.8] for the metric space (Y, d_F). Accordingly, the claim is an immediate consequence of Theorem 4.1.6. Consequently, by Proposition 4.1.4 $\mathscr{P}$ is a solution to a martingale problem with initial law Λ. Therefore, $\mathscr{P} \in \mathscr{C}(y)$ as required. □

Finally, we verify that $\mathscr{C}(y)$ has disintegration and reconstruction property in the sense of Definition 4.1.3.

Lemma 4.1.17. *The family $\{\mathscr{C}(y)\}_{y\in Y}$ satisfies the disintegration property of Definition 4.1.3.*

Proof. Fix $y \in Y$, $\mathscr{P} \in \mathscr{C}(y)$ and let T be $\mathscr{B}_t$-stopping time. In view of Theorem 4.1.1, we know there exists a family of probability measures;

$$\Omega \ni \tilde{\omega} \mapsto \mathscr{P}|^{\tilde{\omega}}_{\mathscr{B}_T} \in \mathrm{Prob}[\Omega^{[T,\infty)}]$$

such that

$$\omega(T) = \tilde{\omega}(T),\ \mathscr{P}|^{\tilde{\omega}}_{\mathscr{B}_T}\text{-a.s.}, \qquad \mathscr{P}(\omega|_{[0,T]} \in A, \omega|_{[T,\infty)} \in B) = \int_{\tilde{\omega}|_{[0,T]} \in A} \mathscr{P}|^{\tilde{\omega}}_{\mathscr{B}_T}(B)\, \mathrm{d}\mathscr{P}(\tilde{\omega}), \tag{4.1.132}$$

for any Borel sets: $A \subset \Omega^{[0,T]}$ and $B \subset \Omega^{[T,\infty)}$. Here, we want to show that

$$\Phi_{-\tau}\mathscr{P}|^{\tilde{\omega}}_{\mathscr{B}_T} \in \mathscr{C}(\omega(T)) \quad \text{for } \tilde{\omega} \in \Omega, \mathscr{P}\text{-a.s.}$$

Thus we are seeking an $\mathscr{P}|^{\tilde{\omega}}_{\mathscr{B}_T}$-nullset N outside of which properties (a)-(f) of Definition 4.1.6 holds for $\mathscr{P}|^{\tilde{\omega}}_{\mathscr{B}_T}$. To begin with, set $N_a, \ldots, N_f$ to each of the properties (a)-(f) of Definition 4.1.6, respectively, and let $N = N_a \cup \cdots \cup N_f$. Arguing similarly along the lines of [54, Lemma 4.4] and [12] we have the following observations:

(1) Set

$$H_T = \Big\{ \omega \in \Omega : \omega|_{[0,T]} \in C([0,T];L^\gamma(\mathbb{T}^3)) \times C([0,T];L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^3)) \times W^{1,2}([0,\infty),L^\gamma(\mathbb{T}^3)) \cap BV^2_{w,\mathrm{loc}}(0,\infty;W^{-l,2}(\mathbb{T}^3)) \times (W^{1,\infty}_{\text{weak-}*}(0,\infty;\mathscr{M}^+(\mathbb{T}^3)))^2 \Big\}$$

$$H^T = \Big\{ \omega \in \Omega : \omega|_{[T,\infty)} \in C([0,\infty);L^\gamma(\mathbb{T}^3)) \times C([0,\infty);L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^3)) \times W^{1,2}([0,\infty),L^\gamma(\mathbb{T}^3)) \cap BV^2_{w,\mathrm{loc}}(0,\infty;W^{-l,2}(\mathbb{T}^3)) \times (W^{1,\infty}_{\text{weak-}*}(0,\infty;\mathscr{M}^+(\mathbb{T}^3)))^2 \Big\},$$

in view of property (a) in Definition 4.1.6, for $\mathscr{P}$ we obtain

$$1 = \mathscr{P}(H_T \cap H^T) = \int_{H_T} \mathscr{P}|^{\tilde{\omega}}_{\mathscr{B}_T}(H^T)\,\mathrm{d}\mathscr{P}(\tilde{\omega}),$$

therefore, there is an $\mathscr{P}|^{\tilde{\omega}}_{\mathscr{B}_T}$-nullset N_a such that $\mathscr{P}|^{\tilde{\omega}}_{\mathscr{B}_T}(H^T) = 1$ holds for $\mathscr{P}$-a.a.$\tilde{\omega} \in \Omega$ (i.e. the remaining $\tilde{\omega} \in \Omega$ are contained in nullset N_a).

(2) Similarly, for the total energy property (b) in Definition 4.1.6 we set

$$\mathfrak{H}_T = \{\omega \in \Omega : \mathfrak{E}|_{[0,T] \in L_{\mathrm{loc}}}(0,T)\},$$
$$\mathfrak{H}^T = \{\omega \in \Omega : \mathfrak{E}|_{[T,\infty) \in L_{\mathrm{loc}}}(T,\infty)\}.$$

Since the property (b) holds $\mathscr{P}$ a.s., arguing as in proof for property (a) (i.e. substituting H_T and H^T with $\mathfrak{H}_T$ and $\mathfrak{H}^T$, respectively) we deduce that there holds $\mathscr{P}|^{\tilde{\omega}}_{\mathscr{B}_T}(H^T) = 1$ for $\mathscr{P}$-a.s. ω. Consequently, this yields the nullset (N_b).

(3) For property (c), let $(\psi_n)_{n\in\mathbb{N}}$ be a dense subset of $W^{k,2}(\mathbb{T}^3)$ and fix $n \in \mathbb{N}$. For each $n \in \mathbb{N}$ we assign an $\mathscr{P}$-nullset N^n_c and set $N_c = \bigcup_{n\in\mathbb{N}} N^n_c$. To proceed, we split the continuity equation as follows:

$$\left[\int_{\mathbb{T}^3} \xi^1_t \psi_n\right]_{t=0}^{t=\tau} - \int_0^\tau \int_{\mathbb{T}^3} \xi^2_t \cdot \nabla\psi_n \mathrm{d}x\,\mathrm{d}t = 0, \quad \forall\, 0 \le \tau \le T, \tag{4.1.133}$$

$$\left[\int_{\mathbb{T}^3} \xi^1_t \psi_n\right]_{t=T}^{t=\tau} - \int_0^\tau \int_{\mathbb{T}^3} \xi^2_t \cdot \nabla\psi_n \mathrm{d}x\,\mathrm{d}t = 0, \quad \forall\, T < \tau < \infty, \tag{4.1.134}$$

and consider the sets

$$\mathfrak{A}_T = \{\omega \in \Omega : \omega|_{[0,T]} \text{ satisfies } (4.1.133)\}$$
$$\mathfrak{A}^T = \{\omega \in \Omega : \omega|_{[T,\infty)} \text{ satisfies } (4.1.134)\}.$$

As the property (c) holds for $\mathscr{P}$, arguing similarly as in proof of (a) and (b) yields a nullset N_c^n.

(4) In case of momentum equation (d), let $(\varphi_n)_{n\in\mathbb{N}}$ be a dense subset of $W^{k,2}(\mathbb{T}^3,\mathbb{R}^3)$ and fix $n\in\mathbb{N}$. Similarly, we assign for each $n\in\mathbb{N}$ an $\mathscr{P}$-nullset N_d^n and set $N_d=\bigcup_{n\in\mathbb{N}}N_d^n$. Noting that property (d) holds for $\mathscr{P}$, then $(\mathscr{M}_t(\varphi_n))_{t\geq 0}$ is a $((\mathscr{B}_t)_{t\geq 0},\mathscr{P})$-square integrable martingale with quadratic variation

$$(\mathscr{Q}(\varphi_n))_\tau=\frac{1}{2}\int_0^\tau\sum_{k=1}^{\infty}\left(\int_{\mathbb{T}^3}\xi_t^1\varphi e_k\cdot\varphi\right)^2\mathrm{d}t;$$

As a consequence of Proposition 4.1.4, for $\mathscr{P}$-a.a. $\tilde{\omega}$ we deduce that $(\mathscr{M}_t(\varphi_n))_{t\geq T}$ is a $((\mathscr{B}_t)_{t\geq T},\mathscr{P}|_{\mathscr{B}_T}^{\tilde{\omega}})$-square integrable martingale with quadratic variation $(\mathscr{Q}(\varphi_n))_{t\geq T}$.

(5) In the entropy inequality (e), the arguments coincide with those of proof of (c).

(6) Similarly, for (f) we can argue as in the proof of (d) *to obtain the nullset* N_f^n.

Choosing $N=N_a\cup\cdots\cup N_f$ *completes the proof.*

□

Lemma 4.1.18. *The family* $\{\mathscr{C}(y)\}_{y\in Y}$ *satisfies the reconstruction property of Definition 4.1.3.*

Proof. Fix $y\in Y$, $\mathscr{P}\in\mathscr{C}(y)$ and let T be $\mathscr{B}_t$-stopping time. In view of Theorem 4.1.2, suppose that Q_ω is a family of probability measures such that

$$\Omega\ni\omega\mapsto Q_\omega\in\ \mathrm{Prob}[\Omega^{[T,\infty)}],$$

is $\mathscr{B}_T$-measurable. Then there exists a unique probability measure $\mathscr{P}\otimes_T Q$ such that :

(a) For any Borel set $A\in\Omega^{[0,T]}$ we have

$$(\mathscr{P}\otimes_T Q)(A)=\mathscr{P}(A);$$

(b) For $\tilde{\omega}\in\Omega$ we have $\mathscr{P}$-a.s.

$$(\mathscr{P}\otimes_T Q)|_{\mathscr{B}_T}^{\tilde{\omega}}=Q_{\tilde{\omega}}$$

We aim to prove that for a $Q_\omega : \Omega \to \text{Prob}[\Omega^{[T,\infty)}]$-$\mathscr{B}_T$-measurable map such that there is $N \in \mathscr{B}_T$ with $\mathscr{P}(N) = 0$ and for all $\omega \notin N$ it holds

$$\omega(T) \in Y \quad \text{and} \quad \Phi_{-T} Q_\omega \in \mathscr{C}(\omega(T));$$

then $(\mathscr{P} \otimes_T Q) \in \mathscr{C}(y)$. In order to do this we have to verify properties (a)-(f) in Definition 4.1.6. The proof follows along the lines of [54], Lemma 4.5. Adopting the notation introduced in Lemma 4.1.17, we argue as follows:
Here we note that Q_ω is a regular conditional probability distribution of $(\mathscr{P} \otimes_T Q)$ with respect to $\mathscr{B}_T$.

(1) Since (a) holds for Q_ω we have $Q_\omega(H^T) = 1$ such that

$$\mathscr{P} \otimes_T Q(H_T \cap H^T) = \int_{H_T} Q_\omega[H^T] \, \mathrm{d}\mathscr{P}(\omega) = 1.$$

(2) For properties (b), (c) and (e) of Definition 4.1.6 we argue as in property (a) (Using the notation developed for each property, respectively).

(3) In the case of property (d),we proceed as follows:
Since (d) holds for Q_ω we know that $(\mathscr{M}_t(\varphi_n))_{t \geq T}$ is a $((\mathscr{B}_t)_{t \geq T}, Q_\omega)$-square integrable martingale for all $\varphi \in C^1(\mathbb{T}^3)$. Consequently, by Proposition 4.1.4 we deduce that $(\mathscr{M}_t(\varphi_n))_{t \geq T}$ is a $((\mathscr{B}_t)_{t \geq T}, \mathscr{P} \otimes_T Q)$-square integrable martingale as well. Observing that $\mathscr{P}$ and $\mathscr{P} \otimes_T Q$ coincides on $\mathscr{B}(\Omega^{[0,T]})$ and $(\mathscr{M}_t(\varphi_n))_{0 \leq t \leq T}$ is a $((\mathscr{B}_t)_{0 \leq t \leq T}, \mathscr{P})$-martingale (since $\mathscr{P}$ satisfies property (d)) we infer that $(\mathscr{M}_t(\varphi_n))_{t \geq 0}$ is a $((\mathscr{B}_t)_{t \geq 0}, \mathscr{P} \otimes_T Q)$-martingale.

(4) Property (f) follows by the same argument as in property (d) (with obvious modifications).

□

Chapter 5

5.1 Published papers

D. Breit, T. C. Moyo: Dissipative Solutions to the Stochastic Euler Equations. Journal of Mathematical Fluid Mechanics, 23, 1-23.(2021).
T.C.Moyo: Dissipative solutions and Markov selection to the complete stochastic Euler system, arXiv preprint arXiv:2112.09955 (2021).

Bibliography

[1] R. A. Adams: Sobolev Spaces, Pure Appl. Math., vol. 65, Academic Press, Inc., New York, 1975.

[2] J.J. Alibert, G. Bouchitté: Non-uniform integrability and generalized Young measures. J. Conv. Anal. **4**, 129–147 (1997).

[3] J.M. Ball: A version of the fundamental theorem for Young measures. In Lect. Notes in Physics 344, Springer-Verlag, pages 207–215, 1989.

[4] S. Benzoni-Gavage, D. Serre: Multidimensional hyperbolic partial differential equations, First order systems and applications. *Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, Oxford, 2007.*

[5] P. Billingsley: Convergence of Probability Measures, Wiley Series in Probability and Statistics, Probability and Statistics, John Wiley & Sons, Inc., New York, second edition, 1999. A Wiley-Interscience Publication.

[6] H. Bessaih, F. Flandoli. 2-D Euler equation perturbed by noise. Nonlinear Differ. Equ. Appl. **6**, 35-54, (1999).

[7] B. Birnir: The Kolomogorov-Obukhov Statistical Theory of Turbulence. © Springer Science, Business Media New York 2013, J Nonlinear Sci 23(**4**): 657-688, (2013).

[8] T. Buckmaster, C. De Lellis, L. Jr Székelyhidi, V. Vicol: Onsager's Conjecture for Admissible Weak Solutions. Comm. Pure Appl. Math., 72: 229-274, (2019).

[9] D. Breit: An Introduction to Stochastic Navier–Stokes Equations, Mathematical Aspects of Fluid Flows. Nečas Center Series, 1-51 © 2018 Springer International Publishing AG.

[10] D. Breit, E. Feireisl, M. Hofmanová: Dissipative solutions and semiflow selection for the complete Euler system, Commun. Math. Phys. 376, 1471-1497 (2020).

[11] D. Breit, E. Feireisl, M. Hofmanová: On solvability and ill-posedness of the compressible Euler system subject to stochastic forces. Anal. PDE 13, 371-402. (2020).

[12] D. Breit, E. Feireisl, M Hofmanová: Markov selection for the stochastic compressible Navier-Stokes system, *The Annals of Applied Probability*,(2020) Vol. 30, No.6, 2547-2572.

[13] D. Breit, E. Feireisl: Stochastic Navier–Stokes–Fourier equations. *Indiana Univ. Math. J. 69, 911–975. (2020).*

[14] D. Breit, E. Feireisl, M. Hofmanova: Generalized solutions to models of inviscid fluids, *Discrete and Continuous dynamical systems series B* Volume 25, Number 10, October 2020.

[15] D. Breit, E. Feireisl, M. Hofmanová: Stochastically Forced Compressible Fluid Flows. De Gruyter Series in Applied and Numerical Mathematics. De Gruyter, Berlin (2018).

[16] D. Breit, E. Feireisl, M. Hofmanová: Compressible fluids driven by stochastic forcing: the relative energy inequality and applications. Comm. Math. Phys., 350(2):443-473, 2017.

[17] D. Breit: Existence theory for generalized Newtonian Fluids. Mathematics in Science and Engineering. El-sevier, Academic Press, London (2017).

[18] D. Breit, T. C. Moyo: Dissipative Solutions to the Stochastic Euler Equations. Journal of Mathematical Fluid Mechanics, 23, 1-23.(2021).

[19] D. Breit, E. Feireisl, M. Hofmanová: Stochastically forced compressible fluid flows. *De Gruyter Series in Applied and Numerical Mathematics.* De Gruyter, Berlin/Munich/Boston, (2018).

[20] D. Breit, M. Hofmanová: Stochastic Navier–Stokes equations for compressible fluids. *Indiana Univ. Math. J. 65, 1183-1250, (2016).*

[21] Y. Brenier, C. De Lellis, and L. Székelyhidi Jr. Weak-strong uniqueness for measured-valued solutions. *Comm. Math Phys.,* 305(2):351-361, 2011.

[22] Z. Brzeźniak, E. Hausenblas, P.A. Razafinimandimby: Stochastic Reaction-diffusion Equations Driven by Jump Processes. *Potential Anal*, 49, 131-201, 2018.

[23] Z. Brzeźniak, E. Motyl: Existence of a martingale solution of the stochastic Navier–Stokes equations in unbounded 2D and 3D domains. J. Differ. Equ. **254**, 1627–1685 (2013).

[24] Z. Brzeźniak, S. Peszat: Stochastic two dimensional Euler equations. Ann. Probab. **29**(4), 1796-1832 (2001).

[25] Z. Brzeźniak, S. Peszat: Strong Local and Global Solutions for Stochastic Navier–Stokes Equations, (2000). Infnite Dimensional Stochastic Analysis (Amsterdam, 1999), Verh. Afd. Natuurkd. 1. Reeks. K. Ned. Akad.Wet., Vol. 52. Amsterdam: R. Neth. Acad. Arts Sci., pp. 85-98.

[26] Brzézniak, Z. Ondreját, M. Seidler, J.: Invariant measures for stochastic nonlinear beam and wave equations. J. Di. Equ. 260, 41574179. (2016).

[27] Z. Brzezniak, M. Ondreját: Strong solutions to the stochastic wave equations with values in Riemannian manifolds, J. Funct. Anal. **253**, 449 – 481, (2007).

[28] M. Capiński, N.J. Cutland: Stochastic Euler equations on the torus. Ann. Appl. Probab. **9**, 688-705 (1999).

[29] M. Capiński, N. Cutland: Stochastic Navier–Stokes equations. Acta Appl. Math. **25**(1), 59–85 (1991).

[30] G. Q. Chen and H. Frid: Uniqueness and asymptotic stability of Riemann solutions for the compressible Euler equations. *Trans. Amer. Math. Soc.*, 353(3):1103-1117, 2001.

[31] E. Chiodaroli, E. Feireisl, and F. Flandoli: Ill posedness for the full Euler system driven by multiplicative white noise. Indiana Univ. Math. J. **(70)**(4),1267-1282, (2021).

[32] E. Chiodaroli, C. De Lellis, and O. Kreml. Global ill-posedness of the isentropic system of gas dynamics. *Comm. Pure Appl. Math.,* 68(7):1157-1190, 2015.

[33] E. Chiodaroli: A counterexample to well-posedness of entropy solutions to the compressible Euler system. *J. Hyperbolic Differ. Equ.,* 11(3):493-519, 2014.

[34] E. Chiodaroli, E. Feireisl, O. Kreml: On the weak solutions to the equations of a compressible heat conducting gas. Anna. I. H. Poincaré- An 32(2015)225-243.

[35] G. Da Prato, J. Zabczyk: Stochastic Equations in Infinite Dimensions. Encyclopedia Math. Appl., vol. 44(1992), Cambridge University Press, Cambridge.

[36] A. Debussche, N. Glatt-Holtz, R. Temam: Local martingale and pathwise solutions for an abstract fluids model. Physica D: *Nonlinear Phenomena* **14–15**, 1123–1144, (2011).

[37] A. Debussche, M. Hofmanová, J. Vovelle: Degenerate parabolic stochastic partial differential equations: Quasilinear case. Ann. Probab. **44**(3): 1916-1955, (2016).

[38] C. De Lellis and L. Székelyhidi, Jr. On admissibility criteria for weak solutions of the Euler equations. *Arch. Ration. Mech. Anal.,* 195(1):225-260, 2010.

[39] C. De Lellis and L. Székelyhidi, Jr. The Euler equations as a differential inclusion. *Ann. of Math.*(2), 170(3):1417-1436, 2009.

[40] R. J. Diperna, A. J. Majda: Oscillations and concentrations in weak solutions of the incompressible fluid equations. Commun. Math. Phys. 108 (1987), 667–689.

[41] R. J. Dirpena: Measure-valued solutions to conservation laws. *Arch. Rat. Mech. Anal.,*88:223-270, 1985.

[42] R. J. Diperna, A. J. Majda: Oscillations and concentrations in weak solutions of the incompressible fluid equations. *Commun. Math. Phys. 108(4), 667-689. (1987).*

[43] R. J. Diperna: Measure-valued solutions of conservation law. Arch. Ration. Mech. Anal. **88**, 223-270 (1985).

[44] R. M. Dudley: Real Analysis and Probability, volume 74 of Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, 2002. Revised reprint of the 1989 original.

[45] J. L. Erickson: Introduction to the Thermodynamics of Solids, revised edn. Applied Mathematical Sciences, vol. 131. Springer, New York, 1998.

[46] G. Eyink. Exact results on staitionary turbulence in 2D: Consequences of vorticity conservation. Physica D: Nonlinear Phenomena, 91(1):97-142, 1996.

[47] E. Feireisl, J. Bŕezina: Measured-valued solutions to the complete Euler system. J. Math. Soc. Japan 70(4), 1227-1245, (2018).

[48] E. Feireisl, P. Gwiazda, A. Świerczewska-Gwiazda, E. Wiedemann: Dissipative measure valued solutions to the compressible Navier-Stokes system. *Calc. Var. Partial Differential Equations.,* 55(6): Art No. 141, 2016.

[49] E. Feireisl, A. Novotný: Weak-strong Uniqueness property for the full Navier-Stokes-Fourier System. Arch. Rational Mech. Anal.204(2012) 683-706.

[50] E. Feireisl and A. Novotný: Singular limits in thermodynamics of viscous fluids. Birhauser Verlag, Basel, 2009.

[51] E. Feireisl, A. Novotný, H. Petzeltová, On the existence of globally defined weak solutions to the Navier-Stokes equations, J. Math. Fluid. Mech. 3 (2001) 358-392.

[52] U. K. Fjordholm, R. Käppeli, S. Mishra, and E. Tadmor. Construction of approximate entropy measure valued solutions for hyperbolic systems of conservation Laws. *Foundations Comp Math.,* pages 1-65, 2015.

[53] F. Flandoli, D. Luo. High mode transport improves vorticity blow-up control in 3D Navier-Stokes equations. Probab. Theory Relat. Fields 180, 309-363 (2021).

[54] F. Flandoli, M. Romito: Markov selection for the 3D stochastic Navier-Stokes equations. *Probab. Theory Related Fields* (**140**) (2008) $407-458$.

[55] F. Flandoli: An introduction to 3D stochastic fluid dynamics. In SPDE hydrodynamic: Recent Progress and Prospects. Lecture Notes in Math. (**1942**), 51-150. Springer, Berlin,(2008).

[56] F. Flandoli, D. Gątarek: Martingale and stationary solutions for stochastic Navier-Stokes equations, Prob. Theory Related Fields (**102**) (1995) 367-391.

[57] N. E. Glatt-Holtz and V. C. Vicol: Local and global existence of smooth solutions for the stochastic Euler equations with multiplicative noise. *Ann. Probab.* 42(1), 80-145, (2014).

[58] B. Goldys, M. Röckner, X. Zhang: Martingale selections and Martingale selections for stochastic partial differential equations. *Stochastic Process. Appl* (2009) **119** 1725-1764.

[59] J. Hoffmann-Jorgensen: Existence of conditional probabilities. Math. Scad (**28**), 257-264 (1971).

[60] M. Hofmanová, U. Koley, and U. Sarkar: Measure valued solutions to the stochastic compressible Euler equations and incompressible limits.*Communications in Partial Differential Equations* (2022).

[61] M. Hofmanová: Degenerate parabolic stochastic partial differential equations. Stoch. Process. Appl. $123(12), 4294-4336$ (2013).

[62] M. Hofmanová, R. Zhu, X. Zhu: On Ill- and Well-Posedness of Dissipative Martingale Solutions to Stochastic 3D Euler Equations. *Communications on Pure and Applied Mathematics* (2021).

[63] N. Ikeda, S. Watanabe: Stochastic Differential Equations and Diffusion Processes, 2nd ed. North-Holland Mathematical Library 24. North-Holland, Amsterdam, 1989.

[64] A. Jakubowski: The almost sure Skorokhod representation for subsequences in non-metric spaces. Teor. Veroyatnost. i Primenen 42(1), 209-216(1997); translation in Theory Probab. Appl. 42(1), 167-174 (1998).

[65] O. Kallenberg. Foundations of modern probability. *Probability and its Applications (New York)*. Springer -Verlag, New York, second edition, 2002.

[66] I. Karatzas, S.E. Shreve: Brownian Motion and Stochastic Calculus, volume 113 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, second edition, 1991.

[67] T. Kato, C.Y. Lai: Nonlinear evolution equations and the Euler flow. J. Funct. Anal. 56, 15–28 (1984).

[68] J. U. Kim: Measure valued solutions to the stochastic Euler equations in $\mathbb{R}^d$. *Stoch PDE: Anal Comp.,* 3: 351-369, 2015.

[69] J.U. Kim: Existence of a local smooth solution in probability to the stochastic Euler equations in R^3. J. Funct. Anal. **256**, 3660-3687 (2009).

[70] J.U. Kim: On the stochastic Euler equations in a two-dimensional domain. SIAM J. Math. Annal. **33**(5), 1211-1227 (2002).

[71] A.N. Kolmogorov: The local structure of turbulence in incompressible viscous fluid for very large Reynolds number. Dokl. Akad. Nauk SSSR 30, 9–13 (1941).

[72] J. Kristensen, F. Rindler: Characterization of Generalized Gradient Young Measures Generated by Sequences in $W^{1,1}$ and BV. *Arch. Rational Mech. Anal. 197, 539–598.* (2010).

[73] D. Kröner, W.M. Zajaczkowski: Measure-valued Solutions of the Euler Equations for Ideal Compressible Polytropic Fluids. Mathematical Methods in The Applied Sciences,19 (1996), 235-252.

[74] N.V. Krylov: The selection of a Markov process from a Markov system of processes, and the construction of quasidiffusion processes. IzV. Ross. Akad. Nauk Ser. Mat. (1973) **3**7 691-708.

[75] D. Jr. Leao, M. Fragoso and P. Ruffino: Regular conditional probability, disintegration of probability and Radon spaces. Proyecciones., Vol. 23, No 1, pp. 15-29, May 2004.

[76] P.L. Lions: Mathematical Topics in Fluid Mechanics. Vol. 1. Incompressible Models. Oxford Lecture Series in Mathematics and Its Applications, 3. Oxford University Press, New York (1996).

[77] J. Málek, J. Nečas, M. Rokyta, M. Růžička: Weak and measure valued solutions to evolutionary PDEs. Chapman & Hall, London-WienHeim-New York, 1996.

[78] R Mikulevicius, G. Valiukevicius: On stochastic Euler equation in R^d. Electron. J. Probab. **5**, 1-20 (2000).

[79] S. Müller: Variational models for microstructure and phase transitions. In Calculus of variations and geometric evolution problems (Cetraro, 1996), volume 1713 of Lecture Notes in Math., pages 85-210, Springer, Berlin, 1999.

[80] J. Neustupa: Measure-valued solutions of the Euler and Navier-Stokes equations for compressible baratropic fluids. *Math., Nachr.,* 163:217-227, (1993).

[81] E. Novikov. Functionals and the random-force method in the turbulance theory. Soviet Physics JETP, 20:1290-1294, 1965.

[82] M. Ondreját: Stochastic nonlinear wave equations in local Sobolev spaces. Elect. J. Prob. **15**, 1041-1091, (2010).

[83] C. Prévôt, M. Röckner: A concise course on the stochastic partial differential equations. Lecture Notes in Mathematics, 1905. Springer, Berlin, 2007.

[84] T. Roubíĉek: Nonlinear Partial Differential Equations with Applications (2nd ed.). Basel, Birkhäuser. ISBN 978-3-0348-0512-4, (2013).

[85] J. Smoller: Shock Waves and Reaction–Diffusion Equations. Springer, New York (1967).

[86] D.W. Stroock, S. R. S. Varadhan: Multidimensional Diffusion Process. Grundlehren der Mathematischen Wissenschaften, *Fundamental Principles of mathematical Sciences 233*. Springer, Berlin. MR0532498 (1979).

[87] A. Valli, W. M. Zajackowski, Navier-Stokes Equations for Compressible Fluids: Global Existence and Qualitative properties of the solutions in the general case. Commun. Math. Phys. 103, 259-296, 1986.

[88] M. Vishik, A. Komech, and A. Fursikov: Some mathematical problems of statistical hydromechanics. Uspekhi Mat. Nauk, 34 (5(209)):135-210, 256, 1979.

[89] E. Wiedemann: Existence of weak solutions for the incompressible Euler equations. Ann. Inst. H. Poincaré Anal. Non Linéaire **28**, 727-730 (2011).

[90] K. Yosida: Functional Analysis (6^{th} ed.), Springer, 1980.