BPS COHOMOLOGY FOR 2-CALABI–YAU CATEGORIES

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DECLARATION

I declare that this thesis was composed by myself, that the work contained herein is my own except where explicitly stated otherwise in the text, and that this work has not been submitted for any other degree or processional qualification except as specified.

The main results of this thesis are based on the following preprints which have been submitted for publication in scientific journals.

B. Davison, L. Hennecart, and S. Schlegel Mejia. "BPS Algebras and Generalised Kac-Moody Algebras from 2-Calabi-Yau Categories". In: (Mar. 22, 2023). arXiv: 2303.12592. preprint

B. Davison, L. Hennecart, and S. Schlegel Mejia. "BPS Lie Algebras for Totally Negative 2-Calabi-Yau Categories and Nonabelian Hodge Theory for Stacks". In: (Dec. 15, 2022). arXiv: 2212.07668. preprint

Chapter 2 is based on the following journal article.

S. Schlegel Mejia. "BPS cohomology for rank 2 degree 0 Higgs bundles (and more)". In: *Journal of Algebra* 636 (2023), pp. 666–694

Sebastian Schlegel Mejia

To my teachers

Dan Zeegers, Björn Fütterer, and Richard Pink

ABSTRACT

The integrality conjecture predicts how the refined Donaldson–Thomas invariants of Abelian categories are determined by smaller BPS invariants. In this thesis, we prove the cohomological integrality conjecture for 2-Calabi–Yau categories, that is, we prove that the underlying mixed Hodge structure of the Borel–Moore homology of the moduli stacks of objects is isomorphic to a symmetric algebra generated by BPS cohomology and the $\mathbb{C}^\times$-equivariant cohomology of a point.

2-Calabi–Yau (2CY) categories are homological dimension 2 categories together with bifunctorial pairing on the shifted extension groups. This pairing can roughly be thought of as a symplectic form on the category. The ubiquity of 2CY categories is illustrated by the following list of examples: categories of coherent sheaves on symplectic surfaces, local systems on Riemann surfaces, categories of Higgs bundles, and representations of preprojective algebras of quivers.

The cohomological Hall algebra (CoHA) $\mathcal{A}$ of a 2CY category plays a prominent role. Using the CoHA we identify the BPS algebra as the universal enveloping algebra of a generalised Kac–Moody (GKM) Lie algebra which we define to be the BPS Lie algebra of the 2CY category. Thus we realise the BPS cohomology as the underlying mixed Hodge structure of the GKM Lie algebra; we prove a Poincaré–Birkhoff–Witt-type isomorphism for the CoHA in terms of the BPS Lie algebra and the $\mathbb{C}^\times$-equivariant cohomology of a point.

The explicit description of the BPS Lie algebra as a GKM algebra reveals a core aspect of our result: the BPS Lie algebra is generated in terms of the intersection cohomology of good moduli spaces of objects in the 2CY category. In the case of totally negative 2CY categories (pairs of nonzero objects of the category have negative Euler pairing) we find the BPS Lie algebra to be the free Lie algebra generated by the intersection cohomology.

The main results of the thesis are based on joint work with Ben Davison and Lucien Hennecart.

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CONTENTS

NOTATION AND CONVENTIONS

The natural numbers $\mathbb{N}$ includes zero. Thus, together with addition, $\mathbb{N}$ is a monoid. Unless otherwise specified, we work over the complex numbers $\mathbb{C}$.

We always use cohomological notation for complexes. We turn homological notation H_* into cohomological notation by setting $(H_*)^i = H_{-i}$.

We abuse terminology and call $\mathbb{C}$-linear Abelian categories just Abelian categories.

We write semisimple objects F of an Abelian category $\mathscr{A}$ as direct sums $F = \bigoplus_{i=1}^n F_i^{\oplus m_i}$ for pairwise nonisomorphic simple objects F_i and $m_i > 0$. The F_i are called the simple summands of the semisimple objects and the integer m_i is called the multiplicity of the corresponding summand F_i.

Linear algebra constructions on $\mathbb{Z}$-graded vector spaces V^* are taken in the symmetric monoidal category of $\mathbb{Z}$-graded vector spaces with symmetric monoidal product defined by $(V^* \otimes W^*)^n = \bigoplus_{p+q=n} V^p \otimes W^q$ with symmetriser $(v \otimes w) \mapsto (-1)^{\deg(v)\deg(w)}(w \otimes v)$. Thus signs appear according to the Koszul sign rule. In particular a graded vector space with a Lie bracket is a graded Lie superalgebra.

We sometimes use the language and formalism of ∞-categories. To indicate when we do we use derivatives of the word “homotopy”. For example, to emphasize that we are taking a colimit in the ∞-category sense we say “homotopy colimit”. We use the languages of model categories and ∞-categories interchangably.

Quivers

A quiver Q is a quadruple of data $Q = (Q_0, Q_1, s, t)$ consisting of two (often finite) sets, Q_0 *the set of vertices* and Q_1 *the set of arrows*, and two maps, *the source map* $s\colon Q_1 \to Q_0$ and *the target map* $t\colon Q_1 \to Q_0$.

A *(finite) path* in Q is a finite sequence of arrows in Q such that the source of the $(n+1)$st arrow is the target of the nth arrow. The *path algebra* $\mathbb{C}Q$ is the algebra of linear combinations of paths in Q where the product of two paths is defined by concatination.

A *dimension vector* of a quiver Q is an element of *the dimension monoid* $\mathbb{N}^{Q_0}$ of Q. We denote dimension vectors by underlined roman letters such as $\underline{d} = (d_i)_{i \in Q_0}, \underline{m} = (m_i)_{i \in Q_0}$.

Given a quiver Q there are some natural dimension vectors corresponding to the projective simples in Q. These are the dimension vectors $\underline{e}_i$ defined to be 1 at the vertex i and zero elsewhere.

The *opposite* or *dual* Q^* of a quiver $Q = (Q_0, Q_1, s, t)$ is the quiver $Q^* = (Q_0^*, Q_1^*, s^*, t^*) := (Q_0, Q_1, t, s)$. In words, the opposite quiver is the same quiver but with reversed arrows.

The *double* $\overline{Q}$ of a quiver Q is the quiver $\overline{Q} = (Q_0, Q_1 \sqcup Q_1^*, s \sqcup t, t \sqcup s)$. In words to obtain the double of a quiver for each arrow (including the loops) in the original quiver one adds an arrow in the opposite direction. For every arrow $a \in Q_1$ we denote the dual arrow by $a^* \in Q_1^*$.

The *path algebra* $\mathbb{C}Q$ *of a quiver* Q is the associative algebra generated as a $\mathbb{C}$-vector spaces by finite length paths in Q, including the paths e_i of length zero, where multiplication is defined by concatinating paths.

The *preprojective algebra* Π_Q *of a quiver* Q is the quotient of the path algebra of the doubled quiver by the relation $\sum_{a \in Q_1}[a, a^*]$, in formulas the definition reads

$$\Pi_Q = \mathbb{C}\overline{Q} \Big/ \sum_{a \in Q_1} [a, a^*].$$

A *representation* of a quiver Q is a module of its path algebra. Equivalently, it is the assignment of a vector space V_i to every vertex $i \in Q_0$ and a linear map $\rho_a \colon V_i \to V_j$ for every arrow $i \xrightarrow{a} j$. A *preprojective representation of a quiver* Q is a Π_Q-module. Throughout, there will be notations depending on Abelian categories $\mathscr{A}$, for example $\mathfrak{M}_{\mathscr{A}}$. Whenever we take the Abelian category to be that of preprojective representations we write Π_Q instead of $\mathrm{Rep}(\Pi_Q)$, for example we write $\mathfrak{M}_{\Pi_Q}$ instead of $\mathfrak{M}_{\mathrm{Rep}(\Pi_Q)}$.

Spaces

A *variety* is a reduced separated finite type scheme over $\mathbb{C}$.

An *Artin stack* over the complex numbers is a stack $\mathfrak{X}$ in the big étale topology with a representable diagonal admitting a smooth atlas $X \to \mathfrak{X}$ from an algebraic space X. We use capital Fraktur letters $\mathfrak{M}, \ldots, \mathfrak{X}, \mathfrak{Y}, \mathfrak{Z}$ to denote (Artin) stacks.

We sometimes use derived stacks, which we indicate by prepending and 'R' to the underlying classical object, e. g. $R\mathfrak{X}$ denotes a derived enhancement of a stack $\mathfrak{X}$.

Throughout this thesis we abuse notation and use the same symbol to denote a scheme or stack and its analytification. An *analytic variety* is a reduced and separated complex analytic space.

For an algebraic group G we denote by BG its classifying stack.

Cohomological invariants of varieties are taken with coefficients in the rational numbers $\mathbb{Q}$, or, more generally, in sheaves of $\mathbb{Q}$-vector spaces. Whenever we omit the coeffiecients from (co)homology we mean that we take constant coefficients in $\mathbb{Q}$, for example, $H(X) = H(X; \mathbb{Q})$ or $IH(X) = IH(X; \mathbb{Q}) = IH(X; \mathbb{Q}_X)$.

The symbol $\mathbb{L}$ denotes the 'realisations' of the ' Lefschetz motive' $\mathbb{A}^1$. More concretely, depending on the context, $\mathbb{L}$ denotes the mixed

Hodge structure $\mathrm{H}_c(\mathbb{A}^1, \mathbb{Q})$ or the class $[\mathbb{A}^1]$ in naive Grothendieck rings of varieties or stacks.

Calligraphic fonts

We use different calligraphic fonts. The font $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{E}, \mathcal{F}, \mathcal{G}, \ldots$ is reserved for (complexes of) quasi-coherent sheaves on schemes/stacks/derived stacks. Two exceptions are $\mathcal{H}$, which is used to denote cohomology objects with respect to a t-structure, and $\mathcal{M}, \mathcal{X}, \mathcal{Y}$, which are used to denote the good moduli spaces of Artin stacks

The script font $\mathscr{A}, \mathscr{B}, \mathscr{C}, \mathscr{D}, \ldots$ is reserved for categories, usually dg categories or Abelian categories. The font $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{E}, \mathcal{F}, \mathcal{G}, \mathcal{H}, \ldots$ is reserved for mixed Hodge modules or complexes with constructible cohomology.

Abuse of notation

We often abuse notation and denote different, but related, objects by the same symbol. The author apologises for any confusion.

1

INTRODUCTION

The aim of this thesis is to establish the *cohomological integrality conjecture* for 2-Calabi–Yau Abelian categories. This first chapter is a non-rigorous introduction to the cohomological integrality conjecture from the point of view of counting objects in Abelian categories. .

1.1 BPS INVARIANTS AND BPS COHOMOLOGY

Counting vector spaces over finite fields

We begin by counting (isomorphism classes of) finite dimensional vector spaces over a finite field $\mathbb{F}_q$.

To be precise, we count the number of points in the groupoid of finite dimensional vector spaces. For a groupoid Γ we define *its groupoid count of points* to be

$$|\Gamma| = \sum_{x \in \pi_0(\Gamma)} \frac{1}{|\mathrm{Aut}_\Gamma(x)|} \tag{1.1}$$

This definition is an instance of the principle that tracking stabilisers when considering quotients is good mathematical book-keeping.[1]

There are infinitely many isomorphism classes of finite dimensional vector spaces: one for every dimension. So we keep a separate count for each dimension. The number of dimension n vector spaces over $\mathbb{F}_q$ is

$$|\mathrm{pt}/\mathrm{GL}_n(\mathbb{F}_q)| = \frac{1}{|\mathrm{GL}_n(\mathbb{F}_q)|} \in \mathbb{Q}(q^{1/2}). \tag{1.2}$$

We define the *DT invariant counting* n*-dimensional vector spaces* to be a certain normalisation of this count:

$$\mathrm{DT}(\mathrm{vect}_{\mathbb{F}_q}, n) := \frac{(-q)^{-(-\dim(\mathrm{GL}_n(\mathbb{F}_q)))/2}}{|\mathrm{GL}_n(\mathbb{F}_q)|} = \frac{(-q)^{n^2/2}}{\prod_{i=0}^{n-1}(q^n - q^i)}.$$

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[1] We invite the reader to motivate the definition of (1.1) by considering the orbit groupoid of a finite group acting on a finite set and studying the orbit-stabiliser formula.

As is common in enumerative problems we gather these numbers in a generating function, called the *DT partition function* of $\mathrm{vect}_{\mathbb{F}_q}$,

$$Z_{\mathrm{DT}}(t) := \sum_{n=0}^{\infty} \mathrm{DT}(\mathrm{vect}_{\mathbb{F}_q}, n)t^n = \sum_{n=0}^{\infty} \frac{(-q)^{n^2/2}}{\prod_{i=0}^{n-1}(q^n - q^i)} t^n,$$

which is a power series with coefficients in $\mathbb{Q}(q^{1/2})$. Applying the q-binomial identity one finds the product expansion

$$Z_{\mathrm{DT}}(t) = \prod_{k=0}^{\infty} \frac{1}{(1 - q^k q^{1/2} t)}. \tag{1.3}$$

There is an operation on power series called the *plethystic exponential*

$$\mathrm{Exp}\colon \mathbb{Q}(q^{1/2})[[t]]_+ \longrightarrow \mathbb{Q}(q^{1/2})[[t]]^\times$$

uniquely determined by the condition that it is a continuous group homomorphism (take the additive group structure on $\mathbb{Q}(q^{1/2})[t]_+$) and setting

$$\mathrm{Exp}(at) = \frac{1}{(1-t)^a} \text{ for all } a \in \mathbb{Q}.$$

Thus as a reinterpretation of (1.3) we have

$$Z_{\mathrm{DT}}(t) = \mathrm{Exp}\left(q^{1/2} t \sum_{k=0}^{\infty} q^k\right) = \mathrm{Exp}\left(\frac{-q^{1/2}}{|\mathbb{F}_q^\times|} t\right).$$

Observe that $\mathbb{F}_q^\times$ is the automorphism group of simple objects. Thus, if we ignore the term $-q^{1/2}/|\mathbb{F}_q^\times|$, we could interpret the expression inside of the plethystic exponential as the generating function for the count of simple objects which we call the *BPS partition function* and denote by $Z_{\mathrm{BPS}}(t)$. In this case, the fact that there is but a single simple vector space, and it is one-dimensional, can be read off from $Z_{\mathrm{BPS}}(t) = t$.

Counting vector spaces over the complex numbers

We want to define Donaldson–Thomas invariants without reference to positive characteristic. We do this via geometry. We consider the stack of finite dimensional vector spaces $\mathfrak{M}$. It decomposes

$$\mathfrak{M} = \bigsqcup_{n \geqslant 0} \mathfrak{M}_n$$

into the stacks $\mathfrak{M}_n \simeq \mathrm{pt}/\mathrm{GL}_n$ of n-dimensional vector spaces. The compactly supported cohomology $H_c(\mathfrak{M}_n)$ is the compactly supported GL_n-equivariant cohomology of the point pt and carries a (pure) mixed Hodge structure. Explicitly, we have, as Hodge structures,

$$H_c(\mathrm{pt}/\mathrm{GL}_n) \cong \mathbb{Q}[c_1^\vee, \ldots, c_n^\vee] \tag{1.4}$$

where c_i^{-1} is a $(-i,-i)$-class for $i = 1, \dots, n$. In particular, its weight series in the variable q is

$$w_q(\mathfrak{M}_n) = \frac{1}{\prod_{i=0}^{n-1}(q^n - q^i)}, \tag{1.5}$$

which as a Laurent polynomial in q is equal to the $\mathbb{F}_q$-point count (1.2).

We define the *DT cohomology of n-dimensional vector spaces* to be the the (shifted and Tate twisted) compactly supported cohomology of the stack

$$H_{DT}(\mathfrak{M}_n) = H_c(\mathfrak{M}_n)[n^2](n^2/2) \tag{1.6}$$

and the *DT cohomology of vector spaces* is

$$H_{DT}(\mathfrak{M}) = \bigoplus_{n \geqslant 0} H_{DT}(\mathfrak{M}_n). \tag{1.7}$$

The graded weight series of $H_{DT}(\mathfrak{M})$ is the *refined DT partition function for vector spaces*

$$Z_{DT,q}(t) = \sum_{n \geqslant 0} DT_q(\mathrm{vect}, n) t^n \in \mathbb{Q}[q^{\pm 1/2}][[t]]$$

which, as above, satisfies

$$Z_{DT,q}(t) = \mathrm{Exp}\left(-q^{-1/2} w_q(B\mathbb{C}^\times) t\right). \tag{1.8}$$

so that the *refined BPS partition function* is $Z_{BPS,q}(t) = t$.

From (1.4) it follows that

$$H_{DT}(\mathfrak{M}) \cong \mathrm{Sym}\left(H_{DT}(\mathfrak{M}_1)\right). \tag{1.9}$$

Graded weight series of $\mathbb{N}_{>0}$-graded mixed Hodge structures are elements of the ring $\mathbb{Q}(q^{1/2})[[t]]$. Symmetric algebras of $\mathbb{N}_{>0}$-graded mixed Hodge structures are $\mathbb{N}$-graded mixed Hodge structures with one-dimensional degree zero piece. Thus taking graded weight series of $\mathbb{N}_{>0}$-graded mixed Hodge structures turns the continous symmetric monoidal functor

$$\mathrm{Sym}\colon (\mathrm{MHS}_{\mathbb{N}_{>0}}, \oplus) \longrightarrow (\mathrm{MHS}_{\mathbb{N}}, \otimes)$$

into a continous group homomorphism

$$w_q(\mathrm{Sym})\colon \mathbb{Q}(q^{1/2})[[t]]_+ \longrightarrow \mathbb{Q}(q^{1/2})[[t]]^\times.$$

Moreover, $w_q(\mathrm{Sym})$ satisfies $w_q(\mathrm{Sym})(at) = (1-t)^{-a}$. By the classifying axioms of the plethystic exponential it follows that $\mathrm{Exp} = w_q(\mathrm{Sym})$.

In conclusion, the decategorification by graded weight series of (1.9) is the identity (1.8).

1.1.1 *The cohomological integrality conjecture*

Given an Abelian category $\mathscr{A}$ for which we can form a 'reasonable' moduli stack of objects $\mathfrak{M}_{\mathscr{A}}$ the *cohomological DT invariant* $\mathrm{H}_{\mathrm{DT}}(\mathfrak{M}_{\mathscr{A}})$ of $\mathscr{A}$ should be some cohomological invariant (e. g. a mixed Hodge structure) of the stack $\mathfrak{M}_{\mathscr{A}}$ for which there exists a 'finite' cohomological invariant $\mathrm{BPS}_{\mathscr{A}}$, called *BPS cohomology*, such that

$$\mathrm{H}_{\mathrm{DT}}(\mathfrak{M}_{\mathscr{A}}) \cong \mathrm{Sym}\left(\mathrm{BPS}_{\mathscr{A}} \otimes \mathrm{H}_{\mathrm{DT}}(\mathrm{B}\mathbb{C}^{\times})\right).$$

This general expectation is called the *cohomological integrality conjecture*. Its decategorification to graded weight series

$$\mathrm{Z}_{\mathrm{DT},q}(\mathfrak{M}_{\mathscr{A}})(t) = \mathrm{Exp}\left(\mathrm{Z}_{\mathrm{BPS},q}(\mathfrak{M}_{\mathscr{A}})(t)\mathrm{Z}_{\mathrm{DT}}(\mathrm{B}\mathbb{C}^{\times})\right)$$

is called the *(refined) integrality conjecture.*

Homological dimension one

Suppose $\mathscr{A}$ is a homological dimension one category, i. e. $\mathrm{Ext}^{i}_{\mathscr{A}}(E, E) = 0$ for all $i > 2$.

The tangent space at a point $p \in \mathfrak{M}_{\mathscr{A}}$ corresponding to an object E is given by $\mathrm{Ext}^{1}_{\mathscr{A}}(E, E)$ and the obstruction space by $\mathrm{Ext}^{2}_{\mathscr{A}}(E, E)$. Hence the moduli stacks $\mathfrak{M}_{\mathscr{A}}$ are smooth. Consequently, the shifted ordinary cohomology

$$\mathrm{H}_{\mathrm{DT}}(\mathfrak{M}_{\mathscr{A}}) \coloneqq \mathrm{H}(\mathfrak{M}_{\mathscr{A}}) \otimes \mathbb{L}^{-\dim(\mathfrak{M}_{\mathscr{A}})/2}$$

is the only real choice available[2] for the cohomological DT invariant.

Theorem 1.1 ([MR19; MR15; Mei15]). *For a large class of Abelian categories $\mathscr{A}$ of homological dimension one, for which the moduli stacks of objects admit a good moduli space $\mathfrak{M}_{\mathscr{A}} \to \mathcal{M}_{\mathscr{A}}$, the cohomological integrality conjecture for*

$$\mathrm{H}_{\mathrm{DT}}(\mathfrak{M}_{\mathscr{A}}) = \mathrm{H}(\mathfrak{M}_{\mathscr{A}}) \otimes \mathbb{L}^{-\dim(\mathfrak{M}_{\mathscr{A}}/2)}$$

holds and moreover

$$\mathrm{BPS}_{\mathscr{A}} = \mathrm{IH}(\mathcal{M}_{\mathscr{A}}) \otimes \mathbb{L}^{-\dim(\mathcal{M}_{\mathscr{A}}/2)}.$$

Examples include categories of representations of quivers (without relations), categories of semistable coherent sheaves on curves, or semistable coherent one-dimensional sheaves on del Pezzo surfaces.

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[2] We have $\mathrm{H} \otimes \mathbb{L}^{-\dim(\mathfrak{M}_{\mathscr{A}})/2} \cong (\mathrm{H}_c \otimes \mathbb{L}^{\dim(\mathfrak{M}_{\mathscr{A}})/2})^{\vee} \cong \mathrm{H}^{\mathrm{BM}}\mathbb{L}^{-\dim(\mathfrak{M}_{\mathscr{A}})/2}$

3-Calabi–Yau categories

Roughly, an Abelian category $\mathscr{A}$ is n-Calabi–Yau if there are bifunctorial graded-symmetric perfect pairings

$$\mathrm{Ext}^{*}_{\mathscr{A}}(E, F) \times \mathrm{Ext}^{n-*}_{\mathscr{A}}(F, E) \longrightarrow \mathbb{C}.$$

The prototypical example of n-Calabi–Yau categories is the category of coherent sheaves on a smooth projective Calabi–Yau n-fold X, for which Serre duality combined with an isomorphism $\omega_X \cong \mathcal{O}_X$ is the source of the pairings.

Donaldson–Thomas invariants are most naturally seen as enumerative invariants of 3-Calabi–Yau categories. Indeed, Donaldson–Thomas invariants are originally a count of stable one-dimensional sheaves on Calabi–Yau threefolds [Tho00].

A general theorem [Ben+15] states that the moduli stacks of objects in 3-Calabi–Yau categories are locally modelled by critical loci of functions $f\colon U \to \mathbb{C}$ on a smooth variety. The cohomological DT invariant is then expected to be the cohomology of a certain perverse sheaf ϕ_{DT} called the *DT sheaf* which locally on a critical chart $f\colon U \to \mathbb{C}$ is given by the perverse vanishing cycles sheaf ϕ_f.

A fundamental special case is the category of representations of the Jacobi algebra of a quiver with potential. Its moduli stack of objects $\mathfrak{M}_{Q,W}$ is the critical locus of the trace of the potential $\mathrm{tr}(W)\colon \mathfrak{M}_Q \to \mathbb{C}$, which is a function on the smooth stack $\mathfrak{M}_Q$.

Theorem 1.2 ([DM20]). *For the 3-Calabi–Yau categories of representations of the Jacobi algebra of a quiver with potential (Q, W) the cohomological integrality conjecture for the vanishing cycle cohomology*

$$H_{DT}(\mathfrak{M}_{Q,W}) \cong H_c(\mathfrak{M}_Q, \phi_{\mathrm{tr}(W)} \otimes \mathbb{L}^{-\dim(\mathfrak{M}_Q)/2}) \tag{1.10}$$

holds and moreover

$$\mathrm{BPS}_{Q,W} = \phi_{\mathrm{tr}(W)} \mathrm{IH}(\mathcal{M}_Q).$$

In fact in [DM20] a canonical isomorphism (1.10) is constructed as a Poincaré–Birkhoff–Witt (PBW) type isomorphism coming from an algebra structure and a canonical filtration on $H_{DT}(\mathfrak{M}_{Q,W})$ (c. f. §6.2.2).

The PBW isomorphism for representations of quivers without potential (which is a homological dimension one category) is a special case of Theorem 1.2 by taking $W = 0$.

Two dimensional categories and dimensional reduction

To every homological dimension two category $\mathscr{A}$ one can canonically associate a 3-Calabi–Yau category $\mathscr{G}_3(\mathscr{A})$ called its 3-Calabi–Yau completion.

Kinjo's dimensional reduction isomorphism [Kin22] combined with [BCS22] establishes an isomorphism between the cohomology of the DT sheaf of the 3-Calabi–Yau completion with the Borel–Moore homology of the moduli stack of objects of the two-dimensional category

$$\mathrm{H}_{\mathrm{DT}}(\mathfrak{M}_{\mathscr{G}_3(\mathscr{A})}) \cong \mathrm{H}^{\mathrm{BM}}(\mathfrak{M}_{\mathscr{A}}).$$

Thus the DT invariant for two-dimensional categories should be the Borel–Moore homology of the moduli stack of objects. This is further confirmed by the following two cohomological integrality theorems.

Theorem 1.3 ([Dav22b]). *For categories* $\mathrm{Rep}(\Pi_Q)$ *of representations of preprojective algebras* Π_Q *there exists a BPS cohomology* BPS_{Π_Q} *and a PBW isomorphism*

$$\mathrm{Sym}\left(\mathrm{BPS}_{\Pi_Q} \otimes \mathrm{H}(\mathrm{B}\mathbb{C}^\times)\right) \xrightarrow{\cong} \mathrm{H}^{\mathrm{BM}}(\mathfrak{M}_{\Pi_Q}).$$

Theorem 1.4 ([KK21]). *For categories* $\mathrm{Higgs}^{\mathrm{ss}}(C)$ *of semistable Higgs bundles on a smooth projective curve* C *, there exists a graded mixed Hodge structure* $\mathrm{BPS}_{\mathrm{Higgs}^{\mathrm{ss}}(C)}$ *and a cohomological integrality isomorphism*

$$\mathrm{H}^{\mathrm{BM}}(\mathfrak{M}_{\mathrm{Higgs}^{\mathrm{ss}}(C)}) \cong \mathrm{Sym}\left(\mathrm{BPS}_{\mathrm{Higgs}^{\mathrm{ss}}(C)} \otimes \mathrm{H}(\mathrm{B}\mathbb{C}^\times)\right).$$

Both of the categories $\mathrm{Rep}(\Pi_Q)$ and $\mathrm{Higgs}^{\mathrm{ss}}(C)$ are 2-Calabi–Yau.

1.1.2 *BPS cohomology for 2-Calabi–Yau categories*

The main contribution of this thesis is the proof of a PBW isomorphism for arbitrary 2-Calabi–Yau categories together with an explicit description of the BPS cohomology in terms of the intersection cohomology of the good moduli spaces.

Theorem 1.5 (Corollary 6.33, [DHS23, Corollary 1.8]). *Suppose* $\mathscr{A}$ *is a 2-Calabi–Yau Abelian category with a good moduli theory* $\mathrm{JH}_{\mathscr{A}} : \mathfrak{M}_{\mathscr{A}} \to \mathcal{M}_{\mathscr{A}}$. *Then the Poincaré–Birkhoff–Witt morphism*

$$\mathrm{Sym}\left(\mathrm{BPS}_{\mathscr{A}} \otimes \mathrm{H}(\mathrm{B}\mathbb{C}^\times)\right) \longrightarrow \mathrm{H}^{\mathrm{BM}}(\mathfrak{M}_{\mathscr{A}})$$

is an isomorphism.

An important special case is the case of *totally negative* 2-Calabi–Yau categories, categories for which the Euler form of nonzero objects is negative: $(E, F) < 0$ for $E, F \in \mathscr{A} \setminus \{0\}$.

In this case the BPS Lie algebra is the free Lie algebra generated by the intersection cohomology of the good moduli spaces containing simple objects

$$\mathrm{BPS}_{\mathscr{A}} = \mathrm{Free}_{\mathrm{Lie}}\left(\bigoplus_{\nu \in \Sigma_{\mathscr{A}}} \mathcal{IC}(\mathcal{M}_{\mathscr{A},\nu})\right).$$

1.2 SUMMARY OF THE CHAPTERS

In the previous section we have left out almost all of the important and interesting technical details. These are treated throughout the thesis.

Chapter 2 is a standalone chapter mostly independent of the rest of the thesis. Therein we compute the E-polynomial of the stack of 'rank 2' objects $\mathfrak{M}_{\mathscr{A},2}$ in a totally negative 2-Calabi–Yau category. The ingredients that go into the computation appear later in a fully fledged form in the proof of the PBW theorem, and as such this chapter could be seen as a comfortable warm up for the rest of the thesis. It also contains a fun section on χ-independence phenomena.

Chapter 3 gathers and develops the necessary preliminaries and moduli theory to make sense of the phrase '2-Calabi–Yau Abelian category with a good moduli theory'. It begins by recalling aspects of dg categories and their moduli of objects. The moduli of objects in an Abelian subcategory of a dg category is defined. So are the stacks of short exact sequences and filtrations. The definition of good moduli spaces of Artin stacks is recalled. The categories of preprojective representations and categories of sheaves on smooth projective varieties are running examples used to illustrate the concepts introduced.

The purpose of Chapter 4 is twofold: to introduce the reader to the formalism of the derived category of mixed Hodge modules and to apply this formalism to define the cohomological Hall algebra of a 2-Calabi–Yau category. We explain the modifications of the derived category of mixed Hodge modules needed for the applications in this thesis. The definition of the relative cohomological Hall algebra (CoHA) is given in sufficiently flexible generality to prove the key technical result of compatibility of cohomological Hall algebra products with Ext-quivers.

Chapter 5 is a leisurely introduction to the definition of generalised Kac–Moody (GKM) Lie algebra (objects) associated to monoid schemes. The chapter also contains important properties of the geometry of the good moduli spaces of 2-Calabi–Yau categories.

Chapter 6 is the proof of the PBW theorem for the relative CoHA 2-Calabi–Yau categories. The BPS algebra of a 2-Calabi–Yau category is defined as a subalgebra of the relative CoHA and is shown to be naturally isomorphic to the GKM algebra associated to the good moduli monoid of a 2-Calabi–Yau category. The BPS Lie algebra is defined to be the corresponding GKM Lie algebra which we show satisfies the PBW isomorphism for the relative CoHA.

2

THE MOTIVATING MOTIVIC CALCULATION

> *"You can have all the desire and ache inside you want, but what you really need is a concrete starting point."*
>
> —Haruki Murakami, *Killing Commendatore*

In §2.1 we define mixed Hodge structures and their E-series. We define the E-series of a finite type Artin stack with affine stabilisers via the formalism of Grothendieck rings of stacks.

The heart of this chapter is §2.2. We introduce the problem of computing the E-series of the moduli stack of 'rank 2' objects in a totally negative 2-Calabi–Yau category admitting a good moduli theory. The main takeaway is that the E-series of the stack is determined by the intersection E-polynomials of good moduli spaces. The key tool in the calculations is the use of the stack of short exact sequences.

Throughout the chapter we neglect the disctinction between isomorphisms and equivalences of stacks.

The material in this chapter is based on [Sch23] and benefited greatly from thorough comments by an anonymous referee.

2.1 PRELIMINARIES ON MOTIVIC CALCULATIONS

2.1.1 *Mixed Hodge structures and their E-series*

The cohomology $H(X)$ of a smooth projective variety X over $\mathbb{C}$ is canonically endowed with a Hodge structure. A *weight* k *Hodge structure* on a $\mathbb{Q}$-vector space H is the data of a direct sum decomposition of its complexification

$$H \otimes \mathbb{C} = \bigoplus_{p+q=k} H^{p,q} \tag{2.1}$$

satisfying:

$$\overline{H^{p,q}} = H^{q,p}.$$

A *weight* k *Hodge structure* on a graded vector space H^* is the data of a weight $k+p$ Hodge structures on its pth graded piece H^p for every p.

Alternatively, a weight k Hodge structure is defined as the data of a $\mathbb{Q}$-vector space H and a descending filtration on the complexification $F^* \subseteq H \otimes \mathbb{C}$, called the *Hodge filtration* satisfying $H = F^p H \oplus \overline{F^q H}$

for all $p + q = k + 1$ The filtration is recovered from the direct sum decomposition (2.1) by setting $F^p := \bigoplus_{p' \geqslant p} H^{p',q'}$. The summands of (2.1) are recovered from the filtration by $H^{p,q} = F^p H \cap F^q H$ for $p + q = k$.

As soon as a variety is not smooth or not projective, then we can no longer guarantee the existence of a Hodge structure on its cohomology. Instead, the cohomology carries a *mixed Hodge structure*.

A *(graded) mixed Hodge structure* is the data of

(i) a $\mathbb{Z}$-graded vector space H^* over $\mathbb{Q}$,
(ii) an increasing *weight filtration* W_* on the graded vector space H,
(iii) and a decreasing *Hodge filtration* F^* on the complexfied graded vector space $H_\mathbb{C} = H \otimes_\mathbb{Q} \mathbb{C}$

such that the filtration F on the complexification of each associated graded piece $W_k H / W_{k-1} H$ endows said piece with a weight k Hodge structure.

Given a finite dimensional mixed Hodge structure H we define its *E-polynomial (also called Hodge–Deligne polynomial)* to be the Laurent polynomial recording the dimensions of the (p,q)-pieces of the associated graded of the weight filtration of H:

$$\begin{aligned} E(H; u, v) &:= \sum_{p,q \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} (-1)^{p+q} \dim(\mathrm{Gr}_F^p \mathrm{Gr}_{p+q}^W(H^n)) u^p v^q \\ &\in \mathbb{Z}[u, v, u^{-1}, v^{-1}] \end{aligned} \tag{2.2}$$

Definition 2.1. A mixed Hodge structure (H, W, F) is *bounded above (below)* if $H^n = 0$ for $n \gg 0$ ($n \ll 0$).

A mixed Hodge structure (H, W, F) is *of finite type* if

$$\dim(\mathrm{Gr}_F^p \mathrm{Gr}_{p+q}^W(H^n)) < \infty \text{ for all } p, q, n \in \mathbb{Z}$$

and

$$\mathrm{Gr}_{p+n}^W H^n = 0 \text{ for } p > n.$$

We denote by MHS^- the category of bounded above and of finite type mixed Hodge structures. Similarly, we denote by MHS^+ the category of bounded below and locally finite mixed Hodge structures. Both MHM^- and MHS^+ are tensor categories, the monoidal product makes their Grothendieck groups $K_0(\mathrm{MHS}^\pm)$ into rings.

Example 2.2. The compactly supported cohomology $H_c(\mathfrak{X})$ for all finite type Artin stacks $\mathfrak{X}$ with affine stabilisers are bounded above and of finite type mixed Hodge structures [Dav22a, Lemma 4.6].

Definition 2.3. We define the *E-series* of a bounded above (or below), locally finite, mixed Hodge structure (H, W, F) by the same forumula as in (2.2) but in a larger ring

$$\begin{aligned} E(H; u, v) &:= \sum_{p,q \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} (-1)^{p+q} \dim(\mathrm{Gr}_F^p \mathrm{Gr}_{p+q}^W(H^n)) u^p v^q \\ &\in \mathbb{Z}[[u^{-1}, v^{-1}]][u, v] \qquad (\text{or } \mathbb{Z}[[u, v]][u^{-1}, v^{-1}]). \end{aligned}$$

The E-series defines ring homomorphisms

$$\begin{aligned} E\colon\ K_0(\mathrm{MHS}^-) &\longrightarrow \mathbb{Z}[[u^{-1}, v^{-1}]][u, v] \\ E\colon\ K_0(\mathrm{MHS}^+) &\longrightarrow \mathbb{Z}[[u, v]][u^{-1}, v^{-1}]. \end{aligned}$$

2.1.2 *Grothendieck rings of stacks and varieties*

We recall the definition of the *Grothendieck ring of varieties* $K_0(\mathrm{Var})$ (see for example [Bri12, §2] for a thorough exposition). As an Abelian group $K_0(\mathrm{Var})$ is the quotient of the free Abelian group generated by isomorphism classes $[X]$ of finite type separated schemes over $\mathbb{C}$ by the relations

$$[X] = [Z] + [X \setminus Z] \tag{2.3}$$

for every closed subscheme $Z \subseteq X$. The relations (2.3) are called the *cut-and-paste relations*. We make $K_0(\mathrm{Var})$ into a ring by defining the product of generators to be

$$[X][Y] \coloneqq [X \times Y].$$

For every finite type separated scheme X over $\mathbb{C}$ its compactly supported cohomology $H_c(X)$ is endowed with a mixed Hodge structure by [Del71; Del74]. The *E-polynomial* of a finite type separated scheme X is the E-polynomial of its compactly supported cohomology $E(X) = E(H_c(X))$.

Lemma-Definition 2.4. The E-polynomial of a variety defines a ring morphism called the *E-polynomial*

$$\begin{aligned} E \coloneqq E_{\mathrm{Var}}\colon\ K_0(\mathrm{Var}) &\longrightarrow \mathbb{Z}[u^{\pm 1}, v^{\pm 1}] \\ [X] &\longmapsto E(X) \end{aligned}$$

Proof. We have $E(X \times Y) = E(X)E(Y)$ by the Künneth formula for compactly supported cohomology. Similarly, for a closed subscheme $Z \subseteq X$ we have the long exact sequence in compactly supported cohomology

$$\dots \longrightarrow H^n_c(Z) \longrightarrow H^n_c(X) \longrightarrow H^n_c(X \setminus Z) \longrightarrow \dots$$

which is an exact sequence of mixed Hodge structures. Abbreviating $\mathrm{Gr}^p_{p+q} = \mathrm{Gr}^p_F \mathrm{Gr}^W_{p+q}$ and taking graded pieces induces long exact sequences

$$\dots \to \mathrm{Gr}^p_{p+q} H^n_c(Z) \to \mathrm{Gr}^p_{p+q} H^n_c(X) \to \mathrm{Gr}^p_{p+q} H^n_c(X \setminus Z) \to \dots$$

for each $p, q \in \mathbb{Z}$. Taking Euler characteristics and summing over $p, q \in \mathbb{Z}$ we deduce $E(X) = E(Z) + E(X \setminus Z)$. □

Example 2.5. (i) $q \coloneqq E(\mathbb{A}^1) = uv$

(ii) $E(\mathbb{G}_m) = E(\mathbb{A}^1 \setminus \{0\}) = q - 1$
(iii) $E(\mathbb{P}^1) = E(\mathbb{A}^1 \sqcup \{\infty\}) = q + 1$
(iv) $E(GL_2) = q(q+1)(q-1)^2$
(v) $E(GL_n) = \prod_{i=1}^{n-1}(q^n - q^i)$

There are two equivalent approaches to defining the E-series of a quotient stack. The first uses the Grothendieck ring of stacks. The second uses an algebro-geometric approximation of the Borel construction to define a mixed Hodge structure on the compactly supported cohomology of a quotient stack, see §4.4 and Example 4.35 for details.

The *Grothendieck ring of stacks* $K_0(\mathrm{St})$ is the quotient of the free Abelian group generated by isomorphism classes of finite type Artin stacks with affine stabilisers by the relations

$$[\mathfrak{X}] = [\mathfrak{Z}] + [\mathfrak{X} \setminus \mathfrak{Z}] \tag{2.4}$$

$$[\mathfrak{E}] = [\mathfrak{X} \times \mathbb{A}^r] \tag{2.5}$$

for every closed substack $\mathfrak{Z} \subseteq \mathfrak{X}$ and vector bundle $\mathfrak{E} \to \mathfrak{X}$ of rank r. (See [Bri12, §3] for a thorough exposition on this ring.)

Note that for a rank r vector bundle $E \longrightarrow X$ over a scheme X, the relation (2.5) is automatically satisfied. Indeed, let $X = \bigcup_i U_i$ be a trivialising open cover of E, then using the stratification

$$E = \bigcup_i E|_{U_i} = \bigsqcup_i E|_{U_i \setminus \bigcup_{j \neq i} U_i} \cong \bigsqcup_i \left(U_i \setminus \bigcup_{j\neq i} U_j\right) \times \mathbb{A}^r$$

we have

$$[E] = \sum_i \left[U_i \setminus \bigcup_{j \neq i} U_j \times \mathbb{A}^r\right] = [\mathbb{A}^r] \sum_i \left[U_i \setminus \bigcup_{j\neq i} U_j\right] = [\mathbb{A}^r][X].$$

Thus viewing a scheme as a stack yields a ring morphism

$$\mathfrak{I}' \colon K_0(\mathrm{Var}) \longrightarrow K_0(\mathrm{St}).$$

Remark 2.6. It is not known whether $\mathfrak{I}'$ is injective.

We denote the class of the affine line in $K_0(\mathrm{Var})$ and in $K_0(\mathrm{St})$ by $\mathbb{L}$. We have the following description of $K_0(\mathrm{St})$ in terms of $K_0(\mathrm{Var})$.

Proposition 2.7 ([Eke09, Theorem 1.2]). *The classes* $\mathbb{L}$ *and* $[GL_n]$ *in* $K_0(\mathrm{St})$ *are invertible with inverses* $[B\mathbb{G}_a]$ *and* $[BGL_n]$ *respectively, and the induced ring morphism*

$$\mathfrak{I} \colon K_0(\mathrm{Var})[\mathbb{L}^{-1}, [GL_n]^{-1}] \longrightarrow K_0(\mathrm{St})$$

is an isomorphism.

We use the isomorphism $\mathfrak{I}$ to define the E-series of a finite type Artin stack with affine stabilisers.

Definition 2.8. We define the *E-series of stacks* as the ring morphism

$$E \coloneqq E_{\mathrm{St}} \coloneqq E_{\mathrm{Var}} \circ \mathfrak{I}^{-1} \colon \ K_0(\mathrm{St}) \longrightarrow \mathbb{Z}[[u^{-1}, v^{-1}]][u, v].$$

Example 2.9. Let G be a finite group acting on a variety X. Then $E(X/G) = E(H_c(X)^G)$.

An algebraic group G is *special* if every G-torsor over a scheme is Zariski-locally trivial. For example, the general linear groups GL_n are special, as every locally free sheaf in the étale topology is locally free in the Zariski topology. For special algebraic groups G the class $[G]$ in $K_0(\mathrm{St})$ is invertible with inverse $[BG]$.

Example 2.10. Let G be a special algebraic group acting on a finite type separated scheme X. The E-series of the quotient stack X/G is $E(X/G) = E(X)/E(G)$.

Proof. We calculate in the Grothendieck ring of stacks. Since G is special, the pullback of the projection $X \to X/G$ to any scheme is a Zariski fibration with fibre G. Hence $[X] = [X/G][G]$. By the invertability (which follows from the same argument if we take $X = \mathrm{pt}$) of $[G]$ we obtain the desired forumla. $\square$

λ-ring structures

For the computations in §2.2 it is convenient to use the language of λ-rings. We do not recall the basics of this language and instead refer to the book [Knu73].

The ring $R = \mathbb{Z}[[u^{-1}, v^{-1}]][u, v]$ carries two natural λ-ring structures coming from the isomorphism $R \cong K_0(\mathrm{Vect}^{-}_{\mathbb{Z}^2})$ where $\mathrm{Vect}^{-}_{\mathbb{Z}^2}$ is the symmetric monoidal Abelian category of $\mathbb{Z}^2$-graded, bounded above vector spaces with finite dimensional graded pieces. The first is the λ-ring structure $\lambda(t) = \sum_{n \geqslant 0} \lambda^n t^n$ induced by taking alternating powers $\Lambda^n V$ for $V \in \mathrm{Ob}(\mathrm{Vect}^{-}_{\mathbb{Z}^2})$. The second is the *symmetric power* λ-ring structure $\sigma(t) = \sum_{n \geqslant 0} \sigma^n t^n$ induced by taking symmetric powers $\mathrm{Sym}^n V$ for $V \in \mathrm{Ob}(\mathrm{Vect}^{-}_{\mathbb{Z}^2})$. The λ-ring structures λ and σ in R are opposite.

In the calculation in §2.2 we only use σ^2 and λ^2, which are explicitly given for all $f \in R = \mathbb{Z}[[u^{-1}, v^{-1}]][u, v]$ by

$$\sigma^2(f) = \frac{1}{2}(f(u, v)^2 + f(u^2, v^2)) \quad \text{and}$$
$$\lambda^2(f) = \frac{1}{2}(f(u, v)^2 - f(u^2, v^2)).$$

The ring $K_0(\mathrm{St})$ admits a pre-λ-ring structure

$$\mathrm{Sym}([\mathfrak{X}])(t) = \sum_{n \geqslant 0} \mathrm{Sym}^n([\mathfrak{X}]) t^n$$

given by the (stacky) symmetric powers $\mathrm{Sym}^n([\mathfrak{X}]) = [\mathrm{Sym}^n(\mathfrak{X})]$. The opposite pre-λ-ring structure $\Lambda([\mathfrak{X}])(t) = \sum_{n \geqslant 0} \Lambda^n([\mathfrak{X}]) t^n$ is given by

$\Lambda([\mathfrak{X}])(t) = (\mathrm{Sym}([\mathfrak{X}])(-t))^{-1}$. We think of $\Lambda^n([\mathfrak{X}])$ as the class of the nth alternating power of $[\mathfrak{X}]$. Indeed, for all varieties X we have the equality

$$E(\Lambda^n([X])) = E(\Lambda^n H_c(X)).$$

The E-series ring homomorphism of Definition 2.8 is a homomorphism of pre-λ-rings, because $\mathfrak{I}$ is an isomorphism of pre-λ-rings (see Example 3.5 and Proposition 3.6 in [DM15]).

With the formalism of λ-rings we easily find a concise expression for the E-series of the symmetric square of a quotient by $\mathbb{G}_m$.

Lemma 2.11. *Let $\mathbb{G}_m$ act on a separated scheme of finite type X. Then*

$$E(\mathrm{Sym}^2(X/\mathbb{G}_m)) = \frac{qE(\mathrm{Sym}^2(X)) + E(\Lambda^2([X]))}{(q-1)^2(q+1)}.$$

Proof.

$$\begin{aligned} E(\mathrm{Sym}^2(X/\mathbb{G}_m)) &= \sigma^2(E(X/\mathbb{G}_m)) \\ &= \frac{1}{2}\left(E(X/\mathbb{G}_m)(u,v)^2 + E(X/\mathbb{G}_m)(u^2,v^2)\right) \\ &= \frac{1}{2}\left(\frac{E(X)^2}{(q-1)^2} + \frac{E(X)(u^2,v^2)}{(q^2-1)}\right) \\ &= \frac{q\sigma^2(E(X)) + \lambda^2(E(X))}{(q-1)^2(q+1)} \end{aligned}$$

□

Remark 2.12. Upon reading the calculation in §2.2, the reader might notice that most of the steps are valid more generally in the Grothendieck ring of stacks. The main reason we do not work in $K_0(\mathrm{St})$ is to have access to the identity of Lemma 2.11. Thus, more generally, in Lemma 2.11 (and throughout Chapter 2) we could replace the E-series of stacks with any pre-λ ring homomorphism from $K_0(\mathrm{St})$ to a λ-ring.

2.2 THE CALCULATION

Let $\mathscr{A}$ be a 2-Calabi–Yau (2CY) Abelian category, so that there are natural non-degenerate graded-symmetric pairings

$$\mathrm{Ext}^*_{\mathscr{A}}(E,F) \times \mathrm{Ext}^{2-*}_{\mathscr{A}}(F,E) \longrightarrow \mathbb{C} \tag{2.6}$$

for all $E, F \in \mathscr{A}$ (see §3.45 for a precise definition).

By the 2CY-property, the Euler pairing $(E,F)_{\mathscr{A}} = \chi(\mathrm{Ext}^*_{\mathscr{A}}(E,F))$ on $\mathscr{A}$ defines a symmetric pairing on the Grothendieck group $K_0(\mathscr{A})$. Assume the Euler pairing is the pullback along a group homomorphism $\gamma\colon K_0(\mathscr{A}) \to \mathbb{Z}^n$ of a symmetric bilinear form on $\mathbb{Z}^n$ (in this section we always have $n = 1$), which we denote by χ.

We assume that $(-,-)_{\mathscr{A}}$ is *totally negative*, i.e. $(E,F)_{\mathscr{A}} < 0$ for all nonzero $E, F \in \mathscr{A}$.

Let $v \in \mathbb{Z}^n$ be a primitive element. For every $r \geqslant 0$ let $\mathfrak{M}_{mv}$ be the moduli stack of objects in $E \in \mathscr{A}$ of class mv , that is, $\gamma(E) = mv$. Define $g > 1$ so that $(v, v)_{\mathscr{A}} = 2 - 2g$.

Suppose that for every $m > 0$ there are finite type Artin stacks $\mathfrak{M}_m := \mathfrak{M}_{\mathscr{A}, mv}$ which are moduli stacks of objects $E \in \mathscr{A}$ of class $[E] = mv$. Suppose further that there are good moduli spaces $\mathcal{M}_m := \mathcal{M}_{\mathscr{A}, mv}$, which parametrise semisimple objects of class mv, and morphisms JH: $\mathfrak{M}_m \to \mathcal{M}_m$ which send an object to its semisimplification.§

Example 2.13 (Preprojective algebra of g-loop quiver)**.** Let S_g be the g-loop quiver. An example of the category $\mathscr{A}$ is the category of preprojective representations of S_g. We have the moduli stack $\mathfrak{M}_m := \mathfrak{M}_{\Pi_{S_g}, m}$ of m-dimensional representations of the preprojective algebra of S_g and its affinisation JH: $\mathfrak{M}_m \to \mathcal{M}_m := \mathcal{M}_{\Pi_{S_g}, m}$.

Example 2.14 (Higgs bundles)**.** Let C be a smooth projective curve of genus $g_0 > 1$. Let $(r, d) \in \mathbb{Z}_{\geqslant 1} \times \mathbb{Z}$ such that $\gcd(r, d) = 1$. We can take category $\mathscr{A}$ to be the category of semistable Higgs bundles on C of rank and degree (mr, md) for any $m \geqslant 0$. We have the moduli stack $\mathfrak{M}_m = \mathfrak{M}^{\mathrm{Dol}}_{mr, md}$ of rank mr and degree md semistable Higgs bundles and the moduli space $\mathfrak{M}_m \to \mathcal{M}_m := \mathcal{M}^{\mathrm{Dol}}_{mr, md}$.

Example 2.15. Let S be a K3 or Abelian surface and H an ample class on S. Consider a primitive Mukai vector $w \in H^*_{\mathrm{alg}}(S, \mathbb{Z})$ with $w^2 \geqslant 0$ and such that H is generic with respect to w. We can take the category $\mathscr{A}$ to be the category of H-(Gieseker-)semistable sheaves on S with Mukai vector a multiple of w. We have the moduli stack $\mathfrak{M}_m := \mathfrak{M}^{\text{H-sst}}_{S, mw}$ and moduli space $\mathfrak{M}_m \to \mathcal{M}_m := \mathcal{M}^{\text{H-sst}}_{S, mv}$ of H-semistable sheaves with Mukai vector mw.

For the rest of this section we only consider the moduli stacks and spaces $\mathfrak{M}_m$ and $\mathcal{M}_m$ for $m = 1, 2$.

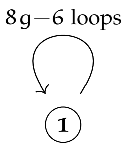
We assume the moduli stack $\mathfrak{M}_1 = \mathfrak{M}_v$ of objects $L \in \mathscr{A}$ of a primitive class v is smooth and admits a smooth good moduli space $\mathrm{JH}_1 \colon \mathfrak{M}_1^s \to \mathcal{M}_1$. Since v is primitive, an object $L \in \mathscr{A}$ of class v is necessarily simple.

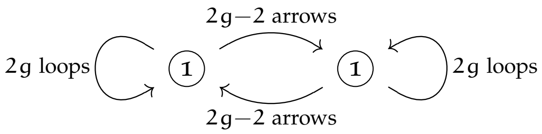
Consider the moduli stack $\mathfrak{M}_2 = \mathfrak{M}_{2v}$ of objects of class $2v$. The moduli stack $\mathfrak{M}_2$ and moduli space $\mathcal{M}_2$ are singular. Objects E of class $2v$ are either simple or there is a simple subobject $K \subsetneq E$ of class v. In the second case, the isomorphism class of the direct sum $K \oplus E/K$ does not depend on the choice of the subobject K of class v. The morphism $p_2 \colon \mathfrak{M}_2 \to \mathcal{M}_2$ sends an object $E \in \mathfrak{M}_2$ to the factors of the filtration by simple objects of class v or $2v$

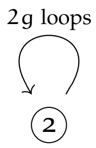
$$p_2(E) = \begin{cases} E & \text{if } E \text{ is simple} \\ K \oplus E/K & \text{if } K \subsetneq E \text{ is of class } v \end{cases}.$$

The 2CY assumption on $\mathscr{A}$ implies that for every non-zero object $E \in \mathscr{A}$ we have $\mathrm{Ext}^1_{\mathscr{A}}(E,E) \neq 0$. The 2CY-pairing (2.6) restricts to a non-degenerate alternating pairing on $\mathrm{Ext}^1_{\mathscr{A}}(E,E)$ and so $\mathrm{Ext}^1_{\mathscr{A}}(E,E)$ is even-dimensional. For all objects $L \in \mathscr{A}$ of class v, we have $\dim \mathrm{Ext}^1_{\mathscr{A}}(L,L) = 2g$. This implies that for all non-isomorphic objects $Q, K \in \mathscr{A}$ of class v we have $\dim \mathrm{Ext}^1_{\mathscr{A}}(Q,K) = 2g-2$.

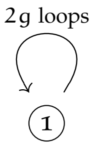
Remark 2.16 (Ext-quivers). All objects of class $2v$ have one of the following three Ext-quivers.







The numbers inside the vertices represent the corresponding dimension vector determined by the object. All of the objects of class v have the following Ext-quiver.



Remark 2.17 (Isosingularity of the moduli problems). By the étale Ext-quiver neighbourhood theorem (Theorem 3.53) and Remark 2.16 we have that the morphisms $\mathfrak{M}_2 \to \mathcal{M}_2$ (and $\mathfrak{M}_1 \to \mathcal{M}_1$) for any two choices of suitable $\mathscr{A}$ are étale locally isomorphic. Thus it suffices to check local properties for any $\mathscr{A}$ by checking it for Example 2.13. This implies that the moduli spaces $\mathcal{M}_2$ (and $\mathcal{M}_1$) are pairwise stably isosingular for any choice of suitable $\mathscr{A}$. See [Mau21, §2.4] for a definition and further discussion of stable isosingularity.

2.2.1 *The stratification of the good moduli space*

The standard strategy to compute the motivic invariant of a space, such as the E-series of a stack, is to stratify the space into locally-closed pieces for which the E-series is known or easy to determine and then add everything up by the cut-and-paste relation. The calculation below is an execution of this strategy for $\mathfrak{M}_2$.

The points of the good moduli space $\mathcal{M}_2$ correspond to semisimple objects of class v. We stratify $\mathcal{M}_2$ by Jordan–Hölder type.

First we distinguish between simple and strictly semisimple objects of class $2v$. Let $\mathcal{M}_2^s \subseteq \mathcal{M}_2$ be the locus of simple objects of class $2v$. Its complement $\Sigma = \mathcal{M}_2 \setminus \mathcal{M}_2^s$ is the locus of strictly semisimple objects

$$\Sigma = \{L_1 \oplus L_2 \mid L_1, L_2 \text{ objects of class } v\}.$$

The locus Σ is precisely the singular locus of $\mathcal{M}_2$ and is isomorphic to the symmetric square of $\mathcal{M}_1$

$$\Sigma \cong \mathrm{Sym}^2(\mathcal{M}_1). \tag{2.7}$$

By the isomorphism (2.7) the singular locus Ω of Σ is identified with the image of $\mathcal{M}_1$ by the diagonal embedding $\mathcal{M}_1 \hookrightarrow \mathrm{Sym}^2(\mathcal{M}_1)$. Explicitly Ω is given by the locus of semisimple objects that are direct sums of two copies of the same object of class v

$$\Omega = \{L^{\oplus 2} \mid L \text{ object of class } v\} \subseteq \Sigma.$$

These loci yield the stratification by Jordan–Hölder type of $\mathcal{M}_2$

$$\mathcal{M}_2 = \mathcal{M}_2^s \cup (\Sigma \setminus \Omega) \cup \Omega, \tag{2.8}$$

where the stratum $\Sigma \setminus \Omega$ has the explicit description

$$\Sigma \setminus \Omega = \{L_1 \oplus L_2 \mid L_1, L_2 \text{ distinct objects of class } v\}.$$

This stratification has already appeared in the literature and is applied in the works [Fel21; Mau21] to compute the intersection E-polynomials of $\mathcal{M}_2$.

Remark 2.18. The three loci $\mathcal{M}_2^s$, $\Sigma \setminus \Omega$, and Ω correspond, in order, to the first three Ext-quivers in Remark 2.16. We emphasise that the deepest stratum Ω corresponds to the g-loop quiver with dimension vector 2.

2.2.2 *Pulling back the stratification*

A natural stratification of the stack $\mathfrak{M}_2$ is the pullback of the stratification (2.8) along the morphism $\mathrm{JH} = \mathrm{JH}_2 \colon \mathfrak{M}_2 \to \mathcal{M}_2$ to the good moduli space. Write

$$\mathfrak{S} = \mathrm{JH}^{-1}(\Sigma), \mathfrak{Z} = \mathrm{JH}^{-1}(\Omega), \text{ and } \mathfrak{Y} = \mathrm{JH}^{-1}(\Sigma \setminus \Omega) = \mathfrak{S} \setminus \mathfrak{Z}.$$

The stack $\mathfrak{S}$ is the singular locus of $\mathfrak{M}_2$. We call $\mathfrak{Y}$ the *off-diagonal locus* and $\mathfrak{Z}$ the *diagonal locus*.

The pullback stratification is

$$\mathfrak{M}_2 = \mathfrak{M}_2^s \cup \mathfrak{Y} \cup \mathfrak{Z}.$$

The simple locus $\mathfrak{M}_2^s$ in the stack is a $\mathbb{G}_m$-gerbe over the simple locus $\mathcal{M}_2$. The E-series of the stable loci $\mathfrak{M}_1^s$ and $\mathfrak{M}_2^s$ of the stacks is calculated from the E-polynomial of of the stable loci of the $\mathcal{M}_1^s$ and $\mathcal{M}_2^s$ of the good moduli spaces.

Lemma 2.19. *We have*

$$E(\mathfrak{M}) = E(\mathcal{M}^s/\mathbb{G}_m) = E(\mathcal{M}^s)/(q-1)$$

Proof. We apply [Hei12, Lemma 3] to the $\mathbb{G}_m$-gerbe $\mathfrak{M}^s \to \mathcal{M}^s$. Thus it suffices to construct a vector bundle on $\mathfrak{M}^s$ of $\mathbb{G}_m$-weight 1.

In Settings 2 and 4, we take the tautological bundle $\mathcal{E}$ on $\mathfrak{M}^s$ which records the underlying vector space of the representation. The $\mathbb{G}_m$-weight of $\mathcal{E}$ is given by the weight of the scaling action which is equal to 1.

For Settings 1 and 3 see [HL10, Proposition 4.6.2] and its proof, which applies to Setting 2.14 by the BNR correspondence [BNR89]. ☐

Therefore, to compute $E(\mathfrak{M}_2)$ it remains to compute the E-series of the strata $\mathfrak{Y}$ and $\mathfrak{Z}$.

2.2.3 *Stratification of the non-simple locus*

For every non-simple object E of class 2ν there exists non-simple objects K and Q of class ν and a short exact sequence

$$\begin{tikzcd} W(E): & 0 \arrow[r] & K \arrow[r] & E \arrow[r] & Q \arrow[r] & 0 \end{tikzcd}$$

that witnesses non-simplicity of E. We call $W(E)$ the *desimplifying* short exact sequence.

There are at most two isomorphism classes of objects of class ν that can appear as the subobject or quotient in a desimplifying short exact sequence.

If $W(E)$ is non-split, then the desimplifying short exact sequence $W(E)$ is unique up to (non-unique) isomorphism. On the other hand, if $W(E)$ is split, that is, if E is semisimple, then all desimplifying short exact sequences are split.

Altogether, using the short exact sequences $W(E)$ we can distinguish four types of non-simples:

(i) $W(E)$ is non-split, $K \ncong Q$
(ii) $W(E)$ non-split, $K \cong Q$
(iii) $W(E)$ split, $K \ncong Q$
(iv) $W(E)$ split, $K \cong Q$.

We stratify the moduli stack $\mathfrak{M}_2$ according to these four cases.

Let $\ddot{\mathfrak{S}} \subseteq \mathfrak{S}$ be the image of the direct-sum morphism

$$\begin{aligned} \oplus \colon \mathfrak{M}_1^s \times \mathfrak{M}_1^s &\longrightarrow \mathfrak{S} \\ (K, Q) &\longmapsto K \oplus Q. \end{aligned}$$

This locus parametrises those non-simples admitting a split desimplifying short exact sequence. Denote the complement of $\ddot{\mathfrak{S}}$ in $\mathfrak{S}$ by $\bar{\mathfrak{S}}$. The stack $\bar{\mathfrak{S}}$ parametrises those non-simple objects with non-split desimplifying short exact sequence.

These are two new loci that we intersect with the strata $\mathfrak{Y}$ and $\mathfrak{Z}$ to obtain our final stratification. Set

$$\bar{\mathfrak{Y}} = \mathfrak{Y} \cap \bar{\mathfrak{S}}, \qquad \bar{\mathfrak{Z}} = \mathfrak{Z} \cap \bar{\mathfrak{S}},$$

$$\ddot{\mathfrak{Y}} = \mathfrak{Y} \cap \ddot{\mathfrak{S}}, \qquad \ddot{\mathfrak{Z}} = \mathfrak{Z} \cap \ddot{\mathfrak{S}}.$$

This defines the stratification

$$\mathfrak{M}_2 = \mathfrak{M}_2^s \sqcup \bar{\mathfrak{Y}} \sqcup \ddot{\mathfrak{Y}} \sqcup \bar{\mathfrak{Z}} \sqcup \ddot{\mathfrak{Z}}$$

that we ultimately use to compute the E-series of $\mathfrak{M}_2$.

2.2.4 *The stack of short exact sequences*

Let $\mathfrak{X}$ be the stack of short exact sequences

$$0 \longrightarrow K \longrightarrow E \longrightarrow Q \longrightarrow 0$$

where K, Q are simple objects of class ν. The following convolution diagram relates $\mathfrak{X}$ to the moduli stacks $\mathfrak{M}_1^s$ and $\mathfrak{M}_2$.

$$\begin{tikzcd}
& \mathfrak{X} \arrow[dl, "q"'] \arrow[dr, "p"] & & \\
\mathfrak{M}_1^s \times \mathfrak{M}_1^s & & \mathfrak{S} \arrow[r, hook] & \mathfrak{M}_2
\end{tikzcd}$$

In the diagram the morphism $q\colon \mathfrak{X} \to \mathfrak{M}_1^s \times Mst_1^s$ maps a short exact sequence to the pair consisting of the quotient object and the subobject

$$q(K \hookrightarrow E \twoheadrightarrow Q) = (Q, K).$$

The morphism $p\colon \mathfrak{X} \to \mathfrak{M}_2$ maps a short exact sequence to the middle term

$$p(K \hookrightarrow E \twoheadrightarrow Q) = E.$$

An extension of two objects of class ν is necessarily not simple, thus the morphism p indeed factors through $\mathfrak{S}$.

Let $\ddot{\mathfrak{X}} \subseteq \mathfrak{X}$ be the closed substack of split short exact sequences and let $\bar{\mathfrak{X}} \subseteq \mathfrak{X}$ be its complement, which is the open substack of non-split short exact sequences. To compute the E-series of $\ddot{\mathfrak{X}}$ and $\bar{\mathfrak{X}}$ we identify $\mathfrak{X}$ as the total space of the truncated RHom complex over $(\mathfrak{M}_1^s)^{\times 2}$ and $\ddot{\mathfrak{X}}$ as its zero-section.

Aside on total spaces of two-term complexes

Let $\mathfrak{B}$ be a finite type Artin stack with affine stabilisers and let $\mathcal{F}$ be a coherent sheaf on $\mathfrak{B}$. Recall that the *total space of $\mathcal{F}$* is the relative spectrum of the symmetric algebra $\mathrm{Sym}_{\mathfrak{B}}(\mathcal{F}^\vee)$,

$$\mathrm{Tot}_{\mathfrak{B}}(\mathcal{F}) = \mathrm{Spec}_{\mathfrak{B}}(\mathrm{Sym}_{\mathfrak{B}}(\mathcal{F}^\vee)) \longrightarrow \mathfrak{B}.$$

Let $\mathcal{F}^\bullet = \mathcal{F}^{-1} \xrightarrow{d} \mathcal{F}^0$ be a two-term complex of coherent sheaves on $\mathfrak{B}$. Define the *total space* $\mathrm{Tot}_{\mathfrak{B}}(\mathcal{F}^\bullet)$ associated to the two-term complex $\mathcal{F}^\bullet$ explicitly as follows. For every morphism $u\colon U \to \mathfrak{B}$ from an affine scheme U we define

$$\mathrm{Tot}_{\mathfrak{B}}(\mathcal{F}^\bullet)(U) = \left\{ \begin{array}{c} \text{objects} = H^0(U, u^*\mathcal{F}^\bullet)(U) \\ \text{morphisms} = H^{-1}(U, u^*\mathcal{F}^\bullet)(U) \end{array} \right\}.$$

Interpreting $d\colon \mathcal{F}^{-1} \to \mathcal{F}^0$ as an action of the group stack $\mathrm{Tot}_{\mathfrak{B}}(\mathcal{F}^{-1})$ on the stack $\mathrm{Tot}_{\mathfrak{B}}(\mathcal{F}^0)$ we have the description

$$\mathrm{Tot}_{\mathfrak{B}}(\mathcal{F}^\bullet) = \mathrm{Tot}_{\mathfrak{B}}(\mathcal{F}^0) / \mathrm{Tot}_{\mathfrak{B}}(\mathcal{F}^{-1}).$$

A chain map $\mathcal{F}^\bullet \to \mathcal{G}^\bullet$ of two-term complexes of coherent sheaves which is a quasi-isomorphism induces an isomorphism of total spaces $\mathrm{Tot}_{\mathfrak{B}}(\mathcal{F}^\bullet) \xrightarrow{\cong} \mathrm{Tot}_{\mathfrak{B}}(\mathcal{G})^\bullet$.

In general the total space $\mathrm{Tot}_{\mathfrak{B}}(\mathcal{F}^\bullet)$ need not be an Artin stack. However, if $\mathcal{F}^{-1}$ is locally free, then $\mathrm{Tot}_{\mathfrak{B}}(\mathcal{F}^\bullet)$ is an Artin stack with affine stabilisers ([LM00, page 143]).

The *zero-section* of the total space $\mathrm{Tot}_{\mathfrak{B}}(\mathcal{F}^\bullet)$ is the closed immersion

$$\mathrm{Tot}_{\mathfrak{B}}(\ker(d)[1]) = \mathrm{Tot}_{\mathfrak{B}}(\tau^{\leqslant -1}\mathcal{F}^\bullet) \hookrightarrow \mathrm{Tot}_{\mathfrak{B}}(\mathcal{F}^\bullet).$$

where $\tau^{\leqslant i}$ denotes the standard truncation.

Lemma 2.20. *Suppose $\mathcal{F}^\bullet = \mathcal{F}^{-1} \to \mathcal{F}^0$ is a two-term complex of coherent sheaves which is quasi-isomorphic to a complex of locally free sheaves with amplitude non-positive degrees. Then*

$$[\mathrm{Tot}_{\mathfrak{B}}(\mathcal{F}^\bullet)] = [\mathfrak{B}]q^{\chi(\mathcal{F})}$$

If $\mathfrak{B}$ has the resolution property, then every two-term complex of coherent sheaves satisfies the assumption. See [Tho87; Tot04; Gro17] for general criteria for stacks to have the resolution property.

Proof. Up to stratifying with respect to an open cover, we can assume without loss of generality that there is a resolution $\mathcal{E}^\bullet \to \mathcal{F}^\bullet$ such that $\mathcal{E}^i \cong \mathcal{O}_{\mathfrak{B}}^{n_i}$ are trivial vector bundles and $\mathcal{E}^i = 0$ for $i > 0$. Taking inspiration from the proof of [GHS14, Lemma 3.3] we stratify further along loci Z_{r_1,r_2} for which the differential $d_{\mathcal{E}}^{-i}$ of $\mathcal{E}^\bullet$ has constant rank r_i for $i = 1, 2$. Along the strata Z_{r_1,r_2}, the truncation $\tau^{\geqslant -1}\mathcal{E}$, which is quasi-isomorphic to $\mathcal{E}^\bullet$, is a two-term complex of vector bundles. After applying [GHS14] to this complex, we deduce the result by consolidating the stratifications. ☐

We compute the E-polynomial of the complement of the zero-section via the cut-and-paste relation.

Let $(\mathcal{Q}, \mathcal{K})$ be the tautological pair of objects over $(\mathfrak{M}_1^s)^{\times 2}$. We have the tautological Hom-sheaf $\mathcal{H}om(\mathcal{Q}, \mathcal{K})$ on $(\mathfrak{M}_1^s)^{\times 2}$, defined as follows.

For every morphism $t\colon U \to (\mathfrak{M}_1^s)^{\times 2}$ out of an affine scheme U the coherent sheaf $t^*\mathcal{H}om(\mathcal{Q},\mathcal{K})$ is defined to be the sheaf associated to the coherent sheaf $\mathcal{H}om_{\mathscr{A}_U}(t^*\mathcal{Q}, t^*\mathcal{K})$ on U. We define the complex of coherent sheaves $R\mathcal{H}om(\mathcal{Q},\mathcal{K})$ similarly (see Definition 3.29 for a completely precise definition).

Lemma 2.21. *The edge term morphism* $q\colon \mathfrak{X} \to (\mathfrak{M}_1^s)^{\times 2}$ *is isomorphic to the total space on* $(\mathfrak{M}_1^s)^{\times 2}$ *of the two-term complex*

$$\tau\mathcal{E}xt = \tau^{\leqslant 0} R\mathcal{H}om(\mathcal{Q},\mathcal{K})[1]. \tag{2.9}$$

Under this isomorphism $\ddot{\mathfrak{X}} \hookrightarrow \mathfrak{X}$ *is the zero-section and* $\bar{\mathfrak{X}}$ *is its complement.*

Proof. See Proposition 3.28. □

2.2.5 *The locus of non-split desimplifying short exact sequences*

Over the non-split locus $\bar{\mathfrak{S}}$ the middle-term morphism p is an isomorphism.

Lemma 2.22. *The morphism of stacks mapping a non-semisimple object to its desimplifying short exact sequence*

$$\begin{aligned} W\colon \bar{\mathfrak{S}} &\longrightarrow \bar{\mathfrak{X}} \\ E &\longmapsto W(E) \end{aligned}$$

is an isomorphism with inverse given by the projection to the middle term $p\colon \mathfrak{X} \to \mathfrak{S}$.

Consider the four Cartesian squares

$$\begin{tikzcd}
\mathfrak{X}_{\mathfrak{U}} \arrow[r, hook] \arrow[d, "q_{\mathfrak{U}}"] & \mathfrak{X} \arrow[d, "q"] & \mathfrak{X}_{\mathfrak{D}} \arrow[l, hook'] \arrow[d, "q_{\mathfrak{D}}"'] \\
\mathfrak{U} \arrow[r, hook] \arrow[d] & (\mathfrak{M}_1)^{\times 2} \arrow[d] & \mathfrak{D} \arrow[l, hook'] \arrow[d] \\
(\mathcal{M}_1)^{\times 2} \smallsetminus \Delta(\mathcal{M}_1) \arrow[r, hook] & (\mathcal{M}_1)^{\times 2} & \mathcal{M}_1, \arrow[l, hook', "\Delta"']
\end{tikzcd}$$

where $\Delta\colon \mathcal{M}_1 \to (\mathcal{M}_1)^{\times 2}$ is the diagonal. The stack $\mathfrak{U}$ parametrises pairs of non-isomorphic simple objects of class ν. Whereas the stack $\mathfrak{D}$ parametrises pairs of isomorphic simple objects of class ν.

There are isomorphisms of stacks

$$\begin{aligned} \mathfrak{U} &\cong ((\mathcal{M}_1 \times \mathcal{M}_1) \smallsetminus \Delta(\mathcal{M}_1))/\mathbb{G}_m^2, \\ \mathfrak{D} &\cong \mathcal{M}_1/\mathbb{G}_m^2. \end{aligned}$$

Thus their E-series are

$$\begin{aligned} E(\mathfrak{U}) &= \frac{E(\mathcal{M}_1)^2 - E(\mathcal{M}_1)}{(q-1)^2}, \\ E(\mathfrak{D}) &= \frac{E(\mathcal{M}_1)}{(q-1)^2}. \end{aligned}$$

By Lemmas 2.22 and 2.21 we have for the non-split loci of the stack of short exact sequences

$$\begin{aligned}\bar{\mathfrak{Y}} &\cong \bar{\mathfrak{X}}_{\mathfrak{U}} \cong \mathfrak{X}_{\mathfrak{U}} \setminus (\mathfrak{X}_{\mathfrak{U}} \cap \mathfrak{X}_0),\\ \bar{\mathfrak{Z}} &\cong \bar{\mathfrak{X}}_{\mathfrak{D}} \cong \mathfrak{X}_{\mathfrak{D}} \setminus (\mathfrak{X}_{\mathfrak{D}} \cap \mathfrak{X}_0).\end{aligned}$$

Thus to compute the E-series of $\bar{\mathfrak{Y}}$ and $\bar{\mathfrak{Z}}$ it remains to compute the E-series of $\bar{\mathfrak{X}}_{\mathfrak{U}}$ and $\bar{\mathfrak{X}}_{\mathfrak{D}}$.

We show that over the loci $\mathfrak{D}$ and $\mathfrak{U}$ the complex of sheaves $\tau\mathcal{E}xt$ is in fact quasi-isomorphic to a complex of vector bundles. Lemma 2.20 then enables us to compute the E-series of $\bar{\mathfrak{X}}_{\mathfrak{D}}$ and $\bar{\mathfrak{X}}_{\mathfrak{U}}$.

Lemma 2.23. *The restriction of the complex $\tau\mathcal{E}xt$ to $\mathfrak{U}$ is quasi-isomorphic to a rank $2g-2$ vector bundle supported in degree 0. Hence, by Lemma 2.20,*

$$E(\bar{\mathfrak{Y}}) = \frac{(E(\mathcal{M}_1)^2 - E(\mathcal{M}_1))(q^{2g-2}-1)}{(q-1)^2}.$$

Proof. For all non-isomorphic simple objects of class ν L_1, L_2 we have

$$\operatorname{Hom}(L_1, L_2) = 0 \text{ and } \dim \operatorname{Ext}^1(L_1, L_2) = 2g-2.$$

Thus over $\mathfrak{U}$, the complex $\tau\mathcal{E}xt$ is concentrated in degree 0. The rank of the degree 0 term is constant and equal to $2g-2$. Since $\mathfrak{M}_1$ is smooth we deduce the lemma. □

Lemma 2.24. *The degree -1 cohomology of the restriction of the complex $\tau\mathcal{E}xt$ to $\mathfrak{D}$ is a rank 1 vector bundle and the degree 0 cohomology is a rank $2g$ vector bundle. Hence, by Lemma 2.20,*

$$E(\bar{\mathfrak{Z}}) = \frac{E(\mathcal{M}_1)(q^{2g}-1)}{q(q-1)^2}.$$

Proof. For all simple objects L of class ν we have

$$\operatorname{Hom}(L, L) \cong \mathbb{C} \text{ and } \dim \operatorname{Ext}^1(L, L) = 2g.$$

Thus over $\mathfrak{D}$, the cohomology sheaf of the complex $\tau\mathcal{E}xt$ in has constant rank equal to 1 and has constant rank equal to $2g$. As above, since $\mathfrak{M}_1$ is smooth we deduce the lemma. □

Corollary 2.25. *By the cut-and-paste relation for $\bar{\mathfrak{S}} = \bar{\mathfrak{Y}} \cup \bar{\mathfrak{Z}}$ we have*

$$E(\bar{\mathfrak{S}}) = \frac{E(\mathcal{M}_1)^2}{(q-1)^2}(q^{2g-2}+1) + \frac{E(\mathcal{M}_1)}{q(q-1)}(q^{2g-2}+1).$$

2.2.5.1 *The locus of semisimples*

Over the split off-diagonal locus $\ddot{\mathfrak{Y}}$ the morphism p is a double cover: a direct sum $K \oplus Q$ with non-isomorphic summands, arises as the middle term of precisely two different (isomorphism classes of) short exact sequences with edge terms simple objects of class ν.

Lemma 2.26. *There is a commutative diagram whose horizontal maps are isomorphisms*

$$\begin{tikzcd}
((\mathcal{M}_1)^{\times 2} \smallsetminus \Delta(\mathcal{M}_1))/\mathbb{G}_m^2 \arrow[r, "h", "\cong"'] \arrow[d] & p^{-1}(\ddot{\mathfrak{Y}}) \arrow[d, "p"] \\
(((\mathcal{M}_1)^{\times 2} \smallsetminus \Delta(\mathcal{M}_1))/\mathbb{G}_m^2)/(\mathbb{Z}/2) \arrow[r, "\tilde{h}", "\cong"'] & \ddot{\mathfrak{Y}}
\end{tikzcd}$$

where the $\mathbb{Z}/2$*-action on* $((\mathcal{M}_1)^{\times 2} \smallsetminus \Delta(\mathcal{M}_1))/\mathbb{G}_m^2$ *is induced by the* $\mathbb{Z}/2$*-actions on* $(\mathcal{M}_1)^{\times 2}$ *and* $\mathbb{G}_m^2$ *given by swapping the two factors. Thus*

$$E(\ddot{\mathfrak{Y}}) = \frac{qE(\mathrm{Sym}^2(\mathcal{M}_1)) + E(\Lambda^2[\mathcal{M}_1])}{(q-1)^2(q+1)} - \frac{qE(\mathcal{M}_1)}{(q-1)^2(q+1)}.$$

Proof. The stack $p^{-1}(\ddot{\mathfrak{Y}})$ is the stack of split short exact sequences

$$0 \longrightarrow K \longrightarrow E \longrightarrow Q \longrightarrow 0$$

such that K and Q are non-isomorphic simple objects of class ν. The morphism h maps a pair (Q, K) to the short exact sequence

$$0 \longrightarrow K \longrightarrow K \oplus Q \longrightarrow Q \longrightarrow 0.$$

A (quasi-)inverse is induced from the projection $\pi^{-1}(\ddot{\mathfrak{Y}}) \to \mathfrak{U} \to (\mathcal{M}_1^s) \smallsetminus \Delta(\mathcal{M}_1^s)$. There is a $\mathbb{Z}/2$-action on the stack $\pi^{-1}(\ddot{\mathfrak{Y}})$ which swaps the roles of subobject and quotient in the split short exact sequence and $\ddot{\mathfrak{Y}} \cong \pi^{-1}(\ddot{\mathfrak{Y}})/(\mathbb{Z}/2)$. Under the isomorphism h this agrees with the $\mathbb{Z}/2$-action on $((\mathcal{M}_1^s)^{\times 2} \smallsetminus \Delta(\mathcal{M}_1^s))/\mathbb{G}_m^2$. Thus we have the desired diagram.

By the cut-and-paste relation we have

$$\begin{aligned}
&E((((\mathcal{M}_1)^{\times 2} \smallsetminus \Delta(\mathcal{M}_1))/\mathbb{G}_m^2)/(\mathbb{Z}/2)) = \\
&\quad = E(\mathrm{Sym}^2(\mathcal{M}_1/\mathbb{G}_m)) - E((\Delta(\mathcal{M}_1)/\mathbb{G}_m^2)/(\mathbb{Z}/2)).
\end{aligned}$$

The $\mathbb{G}_m^2$-action and $\mathbb{Z}/2$-action on $\Delta(\mathcal{M}_1)$ are trivial, hence

$$(\Delta(\mathcal{M}_1)/\mathbb{G}_m^2)/(\mathbb{Z}/2) \cong \mathcal{M}_1 \times (B\mathbb{G}_m)^2/\mathbb{Z}/2 \cong \mathcal{M}_1 \times \mathrm{Sym}^2(B\mathbb{G}_m).$$

The required expression for $E(\ddot{\mathfrak{Y}})$ follows from Lemma 2.11. ☐

Over $\ddot{\mathfrak{Z}}$ the morphism p is the Zariski-locally trivial $\mathbb{P}^1$-fibration given by the quotient morphism $\mathcal{M}_1/B \to \mathcal{M}_1/GL_2$, where $B \subseteq GL_2$ is the subgroup of upper-triangular matrices.

Lemma 2.27. *The split diagonal locus* $\ddot{\mathfrak{Z}}$ *is isomorphic to the quotient stack* $\mathcal{M}_1/GL_2$*. Thus*

$$E(\ddot{\mathfrak{Z}}) = \frac{E(\mathcal{M}_1^s)}{q(q-1)^2(q+1)}.$$

Corollary 2.28. *By the cut-and-paste relation for* $\ddot{\mathfrak{S}} = \ddot{\mathfrak{Y}} \cup \ddot{\mathfrak{Z}}$ *we have*

$$E(\ddot{\mathfrak{S}}) = \frac{qE(\mathrm{Sym}^2(\mathcal{M}_1)) + E(\Lambda^2[\mathcal{M}_1])}{(q-1)^2(q+1)} - \frac{E(\mathcal{M}_1)}{q(q-1)}.$$

Theorem 2.29. *We have*

$$\begin{aligned}E(\mathfrak{M}_2) = E(\mathfrak{M}_2^s) &+ \frac{q^{2g-1}+q^{2g-2}-1}{(q-1)^2(q+1)}E(\mathrm{Sym}^2(\mathcal{M}_1)) \\ &+ \frac{q^{2g-1}+q^{2g-2}-q}{(q-1)^2(q+1)}E(\Lambda^2[\mathcal{M}_1]) + \frac{q^{2g-2}E(\mathcal{M}_1)}{(q-1)}.\end{aligned}$$

Thus

$$\begin{aligned}E(\mathrm{BPS}_2) &= E(\mathfrak{M}_2^s) - E(\mathrm{Sym}^2(H_c(\mathcal{M}_1) \otimes H_c^\bullet(B\mathbb{G}_m) \otimes \mathbb{L}^{1-g})) \\ &= \frac{E(\mathcal{M}_2^s)}{q^{4g-3}} + \frac{E(\mathbb{P}^{2g-3})}{q^{4g-3}(q+1)}E(\mathrm{Sym}^2(\mathcal{M}_1)) \\ &\quad + \frac{E(\mathbb{P}^{2g-3})}{q^{4g-4}(q+1)}E(\Lambda^2[\mathcal{M}_1]) \\ &\quad + \frac{E(\mathcal{M}_1)}{q^{2g-1}}.\end{aligned} \tag{2.10}$$

Proof. First using Corollaries 2.25 and 2.28 we gather the terms which are linear in $E(\mathcal{M}_1)$.

$$\begin{aligned}E(\mathfrak{S}) &= E(\bar{\mathfrak{S}}) + E(\ddot{\mathfrak{S}}) \\ &= \frac{E(\mathcal{M}_1)^2}{(q-1)}E(\mathbb{P}^{2g-3}) + \frac{qE(\mathrm{Sym}^2(\mathcal{M}_1)) + E(\Lambda^2[\mathcal{M}_1])}{(q-1)^2(q+1)} \\ &\quad + \frac{q^{2g-2}E(\mathcal{M}_1)}{(q-1)}\end{aligned}$$

Using $E(\mathcal{M}_1)^2 = E(\mathrm{Sym}^2(\mathcal{M}_1)) + E(\Lambda^2[\mathcal{M}_1])$ we have

$$\begin{aligned}E(\mathfrak{S}) &= \frac{E(\mathcal{M}_1)^2}{(q-1)}E(\mathbb{P}^{2g-3}) + \frac{qE(\mathrm{Sym}^2(\mathcal{M}_1)) + E(\Lambda^2[\mathcal{M}_1])}{(q-1)^2(q+1)} \\ &\quad + \frac{q^{2g-2}E(\mathcal{M}_1)}{(q-1)} \\ &= \frac{q^{2g-1}+q^{2g-2}-1}{(q-1)^2(q+1)}E(\mathrm{Sym}^2(\mathcal{M}_1)) \\ &\quad + \frac{q^{2g-1}+q^{2g-2}-q}{(q-1)^2(q+1)}E(\Lambda^2[\mathcal{M}_1]) + \frac{q^{2g-2}E(\mathcal{M}_1)}{(q-1)}.\end{aligned}$$

By the cut-and-paste relation $E(\mathfrak{M}_2) = E(\mathfrak{M}_2^s) + E(\mathfrak{S})$ we have

$$\begin{aligned}E(\mathfrak{M}_2) = E(\mathfrak{M}_2^s) &+ \frac{q^{2g-1}+q^{2g-2}-1}{(q-1)^2(q+1)}E(\mathrm{Sym}^2(\mathcal{M}_1)) \\ &+ \frac{q^{2g-1}+q^{2g-2}-q}{(q-1)^2(q+1)}E(\Lambda^2[\mathcal{M}_1]) + \frac{q^{2g-2}E(\mathcal{M}_1)}{(q-1)}.\end{aligned}$$

Multiply $E(\mathfrak{M}_2)$ by $q^{4(1-g)}$ and subtract

$$\begin{aligned} & E(\mathrm{Sym}^2(H_c(\mathcal{M}_1) \otimes \mathbb{L}^{-g} \otimes H_c^\bullet(B\mathbb{G}_m) \otimes \mathbb{L})) = \\ & \qquad = \frac{E(\mathrm{Sym}^2(\mathcal{M}_1))}{q^{2g-3}(q-1)^2(q+1)} + \frac{E(\Lambda^2[\mathcal{M}_1])}{q^{2g-2}(q-1)^2(q+1)} \end{aligned}$$

to deduce the required expression for $\frac{q}{q-1}E(\mathrm{BPS}_2)$. □

2.2.6 *Comparison to intersection cohomology*

We recall Mauri's computation of the intersection cohomology of the good moduli spaces $\mathcal{M}_2$.

Proposition 2.30 ([Mau21, Theorem 1.3])**.**

$$\begin{aligned} IE(\mathcal{M}_2) = E(\mathcal{M}_2) + \frac{q^{2g-4}-1}{q^2-1}\left(q^2 E(\mathrm{Sym}^2(\mathcal{M}_1)) + qE(\Lambda^2[\mathcal{M}_1])\right) \\ + q^{2g-2}E(\mathcal{M}_1) \end{aligned}$$

Proof. The identity follows from [Mau21, Theorem 1.3], which is applicable by the stable isosingulariy of the moduli spaces $\mathcal{M}_2$ (Remark 2.17).

The form we give here follows from the equalities

$$\begin{aligned} E(\Sigma_\iota)^+ &= E(\mathrm{Sym}^2(\mathcal{M}_1)) \\ E(\Sigma_\iota)^- &= E(\Lambda^2[\mathcal{M}_1]) \end{aligned}$$

which themselves are deduced by considering the ramified double cover

$$\Sigma_\iota = \mathcal{M}_1 \times \mathcal{M}_1 \to \mathrm{Sym}^2(\mathcal{M}_1) \cong \Sigma.$$

□

Theorem 2.31. *We have*

$$\begin{aligned} \frac{E(\mathfrak{M}_2)}{q^{4g-4}} = & \frac{1}{q^{4g-3}} E(IH_c^\bullet(\mathcal{M}_2))E(H_c^\bullet(B\mathbb{G}_m) \otimes \mathbb{L}) \\ & + \frac{1}{q^{2g}} E(\Lambda^2(H_c^\bullet(\mathcal{M}_1)))E(H_c^\bullet(B\mathbb{G}_m) \otimes \mathbb{L}) \\ & + \frac{1}{q^{2g}} E(\mathrm{Sym}^2(H_c^\bullet(\mathcal{M}_1) \otimes H_c^\bullet(B\mathbb{G}_m) \otimes \mathbb{L})). \end{aligned}$$

Proof. By Proposition 2.30 we have

$$\begin{aligned} IE(\mathcal{M}_2) &= E(\mathcal{M}_2) + \frac{q^{2g-4}-1}{q^2-1}\left(q^2 E(\mathrm{Sym}^2(\mathcal{M}_1)) + qE(\Lambda^2[\mathcal{M}_1])\right) \\ & \quad + q^{2g-2}E(\mathcal{M}_1) \\ &= E(\mathcal{M}_2^s) + \frac{q^{2g-2}-1}{q^2-1}E(\mathrm{Sym}^2(\mathcal{M}_1)) + \frac{q(q^{2g-4}-1)}{q^2-1}E(\Lambda^2[\mathcal{M}_1]) \\ & \quad + q^{2g-2}E(\mathcal{M}_1) \end{aligned}$$

where in the second line we use $E(\mathcal{M}_2) = E(\mathcal{M}_2^s) + E(\mathrm{Sym}^2(\mathcal{M}_1))$.

We shift by $\mathbb{L}^{4(g-1)+1}$ (i.e. divide by q^{4g-3}) and add

$$E(\Lambda^2([\mathcal{M}_1] \otimes \mathbb{L}^{-g})) = q^{-2g} E(\Lambda^2[\mathcal{M}_1])$$

to obtain

$$\begin{aligned} \frac{IE(\mathcal{M}_2)}{q^{4g-3}} + \frac{E(\Lambda^2[\mathcal{M}_1])}{q^{2g}} = \frac{E(\mathcal{M}_2^s)}{q^{4g-3}} + \frac{q^{2g-2}-1}{q^{4g-3}(q^2-1)} E(\mathrm{Sym}^2(\mathcal{M}_1)) \\ + \frac{q^{2g-2}-1}{q^{4g-4}(q^2-1)} E(\Lambda^2[\mathcal{M}_1]) + \frac{E(\mathcal{M}_1)}{q^{2g-1}}. \end{aligned} \tag{2.11}$$

The RHS of (2.11) is equal to the RHS of (2.10) in Theorem 2.29. Thus

$$\begin{aligned} \frac{E(\mathfrak{M}_2)}{q^{4g-4}} = \frac{E(IH_c^\bullet(\mathcal{M}_2))}{q^{4g-3}} E(H_c^\bullet(B\mathbb{G}_m) \otimes \mathbb{L}) \\ + \frac{1}{q^{2g}} E(\Lambda^2[\mathcal{M}_1^s]) E(H_c^\bullet(B\mathbb{G}_m) \otimes \mathbb{L}) \\ + \frac{1}{q^{2g}} E(\mathrm{Sym}^2(H_c^\bullet(\mathcal{M}_1^s) \otimes H_c^\bullet(B\mathbb{G}_m) \otimes \mathbb{L})). \end{aligned}$$

□

2.3 APPLICATION TO χ-INDEPENDENCE CHECKS

Gopakumar–Vafa invariants $n_{g,\beta}$ as defined by Maulik–Toda [MT18] are enumerative invariants of one-dimensional sheaves F on a Calabi–Yau 3-fold X, which a priori depend on the full Chern character $\mathrm{ch}(F) = (0, 0, \beta, \chi)$ of F. The Gopakumar–Vafa invariants $n_{g,\beta}$ are expected to only depend on the curve class β, i.e., we expect independence of the Euler-characteristic χ, see [MT18, Section 3.3] for more details.

For a curve C, respectively a K3 surface S, by considering the local curve $T^*C \times \mathbb{A}^1$, respectively, the local surface $S \times \mathbb{A}^1$ one defines Gopakumar–Vafa invariants for Higgs bundles on C, respectively for the K3 surface S, for which χ-independence should hold.

Similarly, BPS cohomology (when defined) for one-dimensional sheaves on X is expected to be independent of the Euler characteristic. In fact, χ-independence for BPS cohomology conjecturally implies χ-independence for Gopakumar–Vafa invariants. In this section we show how the formula of Theorem 2.31 can be applied to check χ-independence for E-polynomials of BPS cohomology.

For work on χ-independence phenomena for BPS-invariants see [COW21; MS23; Mel20; KK21].

2.3.1 *Higgs bundles*

Let C be a complex smooth connected projective curve of genus $g \geqslant 2$. We consider the moduli problem of semistable Higgs bundles on C.

As in the introduction $\mathfrak{M}^{\mathrm{Dol}}_{r,d}$ is the moduli stack of rank r degree d Higgs bundles and $\mathcal{M}^{\mathrm{Dol}}_{r,d}$ is the coarse moduli space. Let $\mathrm{BPS}^{\mathrm{Dol}}_{r,d}$ be the BPS cohomology. The BPS cohomology is well-defined by [KK21, Theorem 5.16].

We aim to explicitly check $\mathtt{E}(\mathrm{BPS}^{\mathrm{Dol}}_{2,1}) = \mathtt{E}(\mathrm{BPS}^{\mathrm{Dol}}_{2,0})$, which is already known by [KK21, Corollary 5.15] (where cohomological χ-independence is shown in general for Gopakumar–Vafa invariants of local curves).

2.3.1.1 *Rank 2 degree 0 BPS cohomology*

Combining Theorem 2.29 and the computation of the E-polynomial of the stable locus $E(\mathcal{M}^{\mathrm{Dol,s}}_{2,0})$ in [KY08, Theorem 3.7] we determine the E-polynomial $E(\mathrm{BPS}^{\mathrm{Dol}}_{2,0})$ in terms of the genus g. We then simplify the expression to make apparent the equality to $E(\mathrm{BPS}^{\mathrm{Dol}}_{2,1})$, which is determined below.

The E-polynomial in [KY08, Theorem 3.7] is for the stable locus of the moduli space of SL_2-Higgs bundles. We apply the method explained in [Mau21, Section 4.2] to convert the E-polynomial from the SL_2-Higgs bundles case to the GL_2-Higgs bundles case that we use.

Let $\mathrm{Jac}(C)$ be the Jacobian of the curve C. As a first step write $E(\mathcal{M}^{\mathrm{Dol,s}}_{2,0})$ so that the contribution of $E(\mathrm{Jac}(C)) = (1-u)^g(1-v)^g$ is clear.

$$\begin{aligned}
\frac{E(\mathcal{M}^{\mathrm{st}}_{2,0})}{q^{4g-3}} &= E(\mathrm{Jac}(C))\left(\frac{(1-u^2v)^g(1-uv^2)^g - q^{g+1}E(\mathrm{Jac}(C))}{(q-1)^2(q+1)}\right.\\
&\quad + \frac{1}{(q-1)(q+1)}(qE(\Lambda^2[\mathrm{Jac}(C)]) + E(\mathrm{Sym}^2(\mathrm{Jac}(C))))\\
&\quad + \frac{(q^g-1)(q^{g-1}-1)}{q^{2g-3}(q-1)(q+1)}E(\mathrm{Sym}^2(\mathrm{Jac}(C)))\\
&\quad + \frac{(q^{g-1}-1)(q^{g-2}-1)}{q^{2g-2}(q-1)(q+1)}E(\Lambda^2[\mathrm{Jac}(C)])\\
&\quad + \frac{(q^{g-1}-1)(q^{g-2}-1)}{q^{g-2}(q-1)}(E(\mathrm{Jac}(C))^2 - E(\mathrm{Jac}(C)))\\
&\quad + \frac{(q^g-1)(q^{g-1}-1)}{q^{g-1}(q-1)}E(\mathrm{Jac}(C)) + \frac{1}{2}E(\mathrm{Jac}(C))\big((1-u)^{g-1}(1-v)^{g-1}\\
&\qquad + (1+u)^{g-1}(1+v)^{g-1} - 2q^{g-1}\big)\\
&\quad + E(\mathrm{Jac}(C))\left(\frac{q^{g-1}E(\mathrm{Jac}(C))}{(q-1)^2(q+1)} - \frac{(1+u)^{g-1}(1+v)^{g-1}(1-u)(1-v)}{4(q+1)}\right.\\
&\qquad - \frac{g-1}{2}\frac{(u+v-2uv)(1-u)^{g-1}(1-v)^{g-1}}{q-1}\\
&\qquad \left. - \frac{4g-7}{4}\frac{E(\mathrm{Jac}(C))}{q-1} - \frac{qE(\mathrm{Jac}(C))}{2(q-1)^2}\right)
\end{aligned}$$

Now separately gathering terms with factors $E(\mathrm{Jac}(C))^2$, and a single factor of $E(\mathrm{Jac}(C)) = (1-u)^g(1-v)^g$ we have

$$\begin{aligned}
\frac{E(\mathcal{M}^{st}_{2,0})}{q^{4g-3}} = {} & \left(\frac{1-q^{g-1}}{q^{g-2}(q-1)} - \frac{4g-3}{4(q-1)} - \frac{q}{2(q-1)^2}\right) E(\mathrm{Jac}(C))^2 \\
& + \frac{1}{(q-1)(q+1)}(qE(\Lambda^2[\mathrm{Jac}(C)]) + E(\mathrm{Sym}^2(\mathrm{Jac}(C)))) \\
& + \frac{(q^g-1)(q^{g-1}-1)}{q^{2g-3}(q-1)(q+1)} E(\mathrm{Sym}^2(\mathrm{Jac}(C))) \\
& + \frac{(q^{g-1}-1)(q^{g-2}-1)}{q^{2g-2}(q-1)(q+1)} E(\Lambda^2[\mathrm{Jac}(C)]) \\
& + \left(\frac{(1-u^2v)^g(1-uv^2)^g}{(q-1)^2(q+1)}\right. \\
& \qquad - \frac{(q^{g-1}-1)(q^{g-2}-1)}{q^{g-2}(q-1)} \\
& \qquad + \frac{(q^g-1)(q^{g-1}-1)}{q^{g-1}(q-1)} \\
& \qquad + \frac{1}{2}((1-u)^{g-1}(1-v)^{g-1} \\
& \qquad\quad + (1+u)^{g-1}(1+v)^{g-1} - 2q^{g-1}) \\
& \qquad - \frac{(1+u)^{g-1}(1+v)^{g-1}(1-u)(1-v)}{4(q+1)} \\
& \qquad \left. - \frac{g-1}{2}\frac{(u+v-2uv)(1-u)^{g-1}(1-v)^{g-1}}{q-1}\right) E(\mathrm{Jac}(C))
\end{aligned}$$

Substituting into (2.10) and gathering the $E(\mathrm{Sym}^2(\mathrm{Jac}(C)))$ and $E(\Lambda^2[\mathrm{Jac}(C)])$ terms we have

$$\begin{aligned}
E(\mathrm{BPS}^{\mathrm{Dol}}_{2,0}) = {} & \left(\frac{-q^{4g-1}-q^{4g-2}+q^{3g}+q^{3g-1}}{q^{4g-3}(q-1)(q+1)}\right. \\
& \quad \left. - \frac{4g-3}{4(q-1)} - \frac{q}{2(q-1)^2}\right) E(\mathrm{Jac}(C))^2 \\
& + \frac{q^{4g-1}+q^{4g-2}+q^{4g-3}-q^{3g}-q^{3g-1}}{q^{4g-3}(q-1)(q+1)} E(\mathrm{Sym}^2(\mathrm{Jac}(C))) \\
& + \frac{q^{4g-1}+2q^{4g-2}-q^{3g}-q^{3g-1}}{q^{4g-3}(q+1)(q-1)} E(\Lambda^2[\mathrm{Jac}(C)]) \\
& + \left(\frac{(1-u^2v)^g(1-uv^2)^g}{(q-1)^2(q+1)}\right. \\
& \quad + \frac{1}{2}((1-u)^{g-1}(1-v)^{g-1} + (1+u)^{g-1}(1+v)^{g-1}) \\
& \quad - \frac{(1+u)^{g-1}(1+v)^{g-1}(1-u)(1-v)}{4(q+1)} \\
& \quad \left. - \frac{g-1}{2}\frac{(u+v-2uv)(1-u)^{g-1}(1-v)^{g-1}}{q-1}\right) E(\mathrm{Jac}(C))
\end{aligned}$$

where two E(Jac) terms cancel with the contribution $E(\mathcal{M}^{s}_{1,0})/q^{2g-1} = E(\mathrm{Jac}(C))/q^{g-1}$.

Using the identity $E(\mathrm{Jac}(C))^2 = E(\Lambda^2[\mathrm{Jac}(C)]) + E(\mathrm{Sym}^2(\mathrm{Jac}(C)))$ cancels out the first $E(\mathrm{Jac}(C))^2$ term.

$$\begin{aligned}
E(\mathrm{BPS}^{\mathrm{Dol}}_{2,0}) = &\left(-\frac{4g-3}{4(q-1)} - \frac{q}{2(q-1)^2}\right) E(\mathrm{Jac}(C))^2 \\
&+ \frac{1}{(q-1)(q+1)} E(\mathrm{Sym}^2(\mathrm{Jac}(C))) \\
&+ \frac{q}{(q+1)(q-1)} E(\Lambda^2[\mathrm{Jac}(C)]) \\
&+ \Bigg(\frac{(1-u^2v)^g(1-uv^2)^g}{(q-1)^2(q+1)} \\
&\quad + \frac{1}{2}\big((1-u)^{g-1}(1-v)^{g-1} + (1+u)^{g-1}(1+v)^{g-1}\big) \\
&\quad - \frac{(1+u)^{g-1}(1+v)^{g-1}(1-u)(1-v)}{4(q+1)} \\
&\quad - \frac{g-1}{2}\frac{(u+v-2uv)(1-u)^{g-1}(1-v)^{g-1}}{q-1}\Bigg) E(\mathrm{Jac}(C))
\end{aligned}$$

Expanding

$$\begin{aligned}
E(\Lambda^2[\mathrm{Jac}(C)]) = &\frac{1}{2}(1-u)^g(1-v)^g\big((1-u)(1-v)(1-u)^{g-1}(1-v)^{g-1} \\
&- (1+u)(1+v)(1+u)^{g-1}(1+v)^{g-1}\big) \\
E(\mathrm{Sym}^2(\mathrm{Jac}(C))) = &\frac{1}{2}(1-u)^g(1-v)^g\big((1-u)(1-v)(1-u)^{g-1}(1-v)^{g-1} \\
&+ (1+u)(1+v)(1+u)^{g-1}(1+v)^{g-1}\big)
\end{aligned}$$

and combining with the second, third and fourth $E(\mathrm{Jac}(C))$ terms yields

$$\begin{aligned}
E(\mathrm{BPS}^{\mathrm{Dol}}_{2,0}) = &\frac{(1-u)^g(1-v)^g(1-u^2v)^g(1-uv^2)^g}{(q^2-1)(q-1)} \\
&+ (1-u)^g(1-v)^g\Bigg(-\frac{(1+u)^g(1+v)^g}{4(q+1)} \\
&\quad - \frac{g}{2}\frac{(u+v-2uv)(1-u)^{g-1}(1-v)^{g-1}}{q-1} \\
&\quad - \frac{4g-3}{4}\frac{(1-u)^g(1-v)^g}{q-1} - \frac{1}{2}\frac{q(1-u)^g(1-v)^g}{(q-1)^2}\Bigg).
\end{aligned} \tag{2.12}$$

2.3.1.2 *Rank 2 degree 1 BPS cohomology*

In degree 1 the BPS cohomology is the shifted cohomology of the rank 2 degree 1 coarse moduli space. A formula for its E-polynomial can be extracted from [GHS14, Appendix] and is given by

$$\begin{aligned}
E(\mathrm{BPS}^{\mathrm{Dol}}_{2,1}) &= \frac{E(\mathcal{M}_{2,1})}{q^{4g-3}}\\
&= E(\mathrm{Jac}(C))\Bigg(\frac{(1-u^2v)^g(1-uv^2)^g - q^gE(\mathrm{Jac})}{(q^2-1)(q-1)}\\
&\qquad + \sum_{d=1}^{g-1} E(\mathrm{Sym}^{2g-2d-1}(C))\Bigg)\\
&= \frac{(1-u)^g(1-v)^g(1-u^2v)^g(1-uv^2)^g}{(q^2-1)(q-1)}\\
&\quad - \frac{q^g(1-u)^{2g}(1-v)^{2g}}{(q^2-1)(q-1)}\\
&\quad + (1-u)^g(1-v)^g\sum_{d=1}^{g-1}\left[\frac{(1-ut)^g(1-vt)^g}{(1-t)(1-qt)}\right]_{t^{2g-2d-1}}
\end{aligned}$$

where $[-]_{t^k}$ extracts the coefficient of t^k. The third equality follows from Macdonald's computation of the cohomology of symmetric powers of curves [Mac62]. We evaluate

$$\begin{aligned}
\sum_{d=1}^{g-1}\left[\frac{(1-ut)^g(1-vt)^g}{(1-t)(1-qt)}\right]_{t^{2g-2d-1}} &= \frac{q^g(1-u)^g(1-v)^g}{(q-1)^2(q+1)}\\
- \frac{(1+u)^g(1+v)^g}{4(q+1)} &- \frac{g}{2}\frac{(u+v-2uv)(1-u)^{g-1}(1-v)^{g-1}}{q-1}\\
- \frac{4g-3}{4}\frac{(1-u)^g(1-v)^g}{q-1} &- \frac{1}{2}\frac{q(1-u)^g(1-v)^g}{(q-1)^2}
\end{aligned}$$

following Hitchin [Hit87, Proof of Theorem 7.6] (see also [KY08, Section 3.3]). Altogether we have

$$\begin{aligned}
E(\mathrm{BPS}^{\mathrm{Dol}}_{2,1}) &= \frac{(1-u)^g(1-v)^g(1-u^2v)^g(1-uv^2)^g}{(q^2-1)(q-1)}\\
&\quad - \frac{q^g(1-u)^{2g}(1-v)^{2g}}{(q^2-1)(q-1)}\\
&\quad + (1-u)^g(1-v)^g\Bigg(\frac{q^g(1-u)^g(1-v)^g}{(q-1)^2(q+1)}\\
&\qquad - \frac{(1+u)^g(1+v)^g}{4(q+1)}\\
&\qquad - \frac{g}{2}\frac{(u+v-2uv)(1-u)^{g-1}(1-v)^{g-1}}{q-1}\\
&\qquad - \frac{4g-3}{4}\frac{(1-u)^g(1-v)^g}{q-1}\\
&\qquad - \frac{1}{2}\frac{q(1-u)^g(1-v)^g}{(q-1)^2}\Bigg).
\end{aligned}$$

This simplifies slightly to

$$\begin{aligned}E(\mathrm{BPS}^{\mathrm{Dol}}_{2,1}) = &\frac{(1-u)^g(1-v)^g(1-u^2v)^g(1-uv^2)^g}{(q^2-1)(q-1)}\\ &+(1-u)^g(1-v)^g\Bigg(-\frac{(1+u)^g(1+v)^g}{4(q+1)}\\ &\quad-\frac{g}{2}\frac{(u+v-2uv)(1-u)^{g-1}(1-v)^{g-1}}{q-1}\\ &\quad-\frac{4g-3}{4}\frac{(1-u)^g(1-v)^g}{q-1}\\ &\quad-\frac{1}{2}\frac{q(1-u)^g(1-v)^g}{(q-1)^2}\Bigg),\end{aligned}$$

which agrees with the expression (2.12) for $E(\mathrm{BPS}^{\mathrm{Dol}}_{2,0})$.

2.3.1.3 *Betti side*

Via non-Abelian Hodge theory for stacks, as developed in [Dav23a], one similarly expects a χ-independence phenomenon on the Betti side.

Since $\gcd(2,1)=1$, we have $E(\mathrm{BPS}^{\mathrm{Betti}}_{2,1}) = q^{3-4g}E(\mathcal{M}^{\mathrm{Betti}}_{2,1})$. The E-polynomial $E(\mathcal{M}^{\mathrm{Betti}}_{2,1})$ was determined in [HR08, Corollary 3.6.1] and the intersection E-polynomial $IE(\mathcal{M}^{\mathrm{Betti}}_{2,0})$ was determined in [Mau21, Theorem 1.4]. Using Theorem 2.29 one can directly check $E(\mathrm{BPS}^{\mathrm{Betti}}_{2,0}) = E(\mathrm{BPS}^{\mathrm{Betti}}_{2,1})$.

2.3.2 *Sheaves on K3 surfaces*

We can reinterpret some results in [dCRS21] as a cohomological χ-independence check for sheaves on K3 surfaces.

Let S be a K3 surface. Let H be a sufficiently general polarization of S. Suppose (S,H) is of genus 2, i.e., the curves in the linear system $|H|$ are of genus 2. Let $v \in H^{\bullet}_{\mathrm{alg}}(S,\mathbb{Z})$ be a primitive Mukai vector with $v^2=2$. Consider the moduli stack and moduli spaces of H-Gieseker-semistable sheaves $\mathfrak{M}^{\text{H-sst}}_{S,2v}$, $\mathcal{M}^{\text{H-sst}}_{S,2v}$, and $\mathcal{M}^{\text{H-sst}}_{S,v}$.

Let OG10 be O'Grady's ten-dimensional sporadic example of a hyper-Kähler manifold. By [dCRS21, Lemma 4.1.3] we have

$$IE(\mathcal{M}^{\text{H-sst}}_{S,2v}) = E(\mathrm{OG10}) - qE(\mathrm{Sym}^2(\mathcal{M}^{\text{H-sst}}_{S,v})) - q^3H(\mathcal{M}^{\text{H-sst}}_{S,v}).$$

and by (2.10) we have

$$E(\mathrm{BPS}_{S,2v}) = \frac{E(\mathrm{OG10})}{q^5} - \frac{E_{u^2,v^2}(\mathcal{M}^{\text{H-sst}}_{S,v})}{q^4} - \frac{E(\mathcal{M}^{\text{H-sst}}_{S,v})}{q^2}. \tag{2.13}$$

Pick a primitive Mukai vector $w \in H_{\mathrm{alg}}(S,\mathbb{Z})$ satisfying $w^2 = 4 = (2v)^2$. The moduli space $\mathcal{M}^{\text{H-sst}}_{S,w}$ is smooth and deformation equivalent to the Hilbert scheme of 5 points on a K3 surface.

Corollary 2.32. *Suppose the cohomological integrality conjecture is true for* $\mathcal{A}^{\text{H-sst}}_{\mathrm{Coh}(S),v}$*, then*

$$E(\mathrm{BPS}_{S,2v}) = E(\mathrm{BPS}_{S,w}).$$

Proof. By definition we have $E(\mathrm{BPS}_{S,w}) = q^{-5}E(\mathcal{M}^{H-ss}_{S,w})$. By [dCRS21, Proposition 6.1.2] and (2.13) we have $E(\mathrm{BPS}_{S,2v}) = q^{-5}E(\mathcal{M}^{H-ss}_{S,w})$. □

Therefore, the polynomial $E(\mathrm{BPS}_{S,2v})$ is determined by the Hodge numbers of $\mathcal{M}^{\text{H-sst}}_{S,w}$, which are recorded in [dCRS21, (103)].

Remark 2.33. If we assume the χ-independence conjecture for BPS cohomology, then equation (2.13) (which follows from a simple application of the decomposition theorem [dCRS21, Lemma 4.1.3] and Theorem 2.29) together with χ-independence, yields a conjectural computation of the Hodge numbers of OG10.

3

MODULI OF OBJECTS IN CALABI–YAU CATEGORIES

> *"Take stock of who we are, and what we have, and then use it for good."*
>
> — Ann Napolitano, *Dear Edward*

The purpose of this chapter is to make precise the notion of a 2-Calabi–Yau Abelian category with a good moduli theory which we have so far been using rather loosely. In order to make a sufficiently general definition there needs to be an according amount of general build-up.

Throughout we keep two running examples: categories of preprojective representations of a quiver and the categories of coherent sheaves on symplectic surfaces.

3.1 DG CATEGORIES

For most of the basics on dg categories we follow Keller's survey [Kel06].

A *differential graded (or dg) category* is a category enriched in cochain complexes of $\mathbb{C}$-vector spaces. This means that a dg category $\mathscr{D}$ is the data of a class of objects $\mathrm{Ob}(\mathscr{D})$, for every pair of objects $a, b \in \mathrm{Ob}(\mathscr{D})$ there is a chain complex

$$\mathrm{Hom}^*_{\mathscr{D}}(a,b)\colon \ \ldots \longrightarrow \mathrm{Hom}^i_{\mathscr{D}}(a,b) \overset{d=d^i}{\longrightarrow} \mathrm{Hom}^{i+1}_{\mathscr{D}}(a,b) \longrightarrow \ldots$$

called the *Hom complex of* a *to* b, and for every triple of objects $a, b, c \in \mathrm{Ob}(\mathscr{D})$ there is an associative composition rule

$$\circ\colon \ \mathrm{Hom}^*_{\mathscr{D}}(b,c) \otimes \mathrm{Hom}^*_{\mathscr{D}}(a,b) \longrightarrow \mathrm{Hom}^*_{\mathscr{D}}(a,b)$$

which is a morphism of chain complexes. Here the tensor product $\otimes$ is taken in the category of chain complexes.

To a dg category $\mathscr{D}$ we associate the *homotopy category* $H^0(\mathscr{D})$ which has the same objects as $\mathscr{D}$ and morphisms $\mathrm{Hom}_{H^0(\mathscr{D})}(a,b) \coloneqq H^0(\mathrm{Hom}_{\mathscr{D}}(a,b))$.

Given two dg categories $\mathscr{D}$ and $\mathscr{D}'$ we define the tensor product dg category $\mathscr{D} \otimes \mathscr{D}'$ to be the dg category with objects given by pairs of objects $(a, a') \in \mathscr{D} \times \mathscr{D}'$ and morphism complexes given by the tensor product of complexes $\mathrm{Hom}_{\mathscr{D}\otimes\mathscr{D}'}((a, a'), (b, b')) := \mathrm{Hom}_{\mathscr{D}}(a, a') \otimes \mathrm{Hom}_{\mathscr{D}'}(b, b')$.

A functor $F \colon \mathscr{D} \to \mathscr{D}'$ between two dg categories is a *dg functor* if for all $a, b \in \mathscr{D}$, the map $F_{a,b} \colon \mathrm{Hom}_{\mathscr{D}}(a, b) \to \mathrm{Hom}_{\mathscr{D}}(F(a), F(b))$ is a chain map. The functor dg category $\mathrm{dg\,Fun}(\mathscr{D}, \mathscr{D}')$ from a dg category $\mathscr{D}$ to a dg category $\mathscr{D}'$ has objects given by functors $F \colon \mathscr{D} \to \mathscr{D}'$.

A dg functor $F \colon \mathscr{D} \to \mathscr{D}'$ is a *quasi-equivalence* if it is a pointwise quasi-isomorphism and the induced functor on the homotopy categories $H^0(F) \colon H^0(\mathscr{D}) \to H^0(\mathscr{D}')$ is an equivalence of categories. The ∞-category of dg categories is obtained by localising the category of dg categories along quasi-equivalences.

To any dg category $\mathscr{D}$ we associate the dg category $\mathrm{dgMod}^{\mathscr{D}}$ of *left $\mathscr{D}$-dg modules* as follows. The objects of $\mathrm{dgMod}^{\mathscr{D}}$ are functors

$$W \colon \mathscr{D} \longrightarrow \mathrm{dgvect}.$$

The set of morphisms $\mathrm{Hom}_{\mathrm{dgMod}^{\mathscr{D}}}(W, W')$ is has a natural structure of a complex induced from the object-wise complexes $\mathrm{Hom}_{\mathrm{dgvect}}(W(a), W'(a))$. The category of *right $\mathscr{D}$-dg modules* $\mathrm{dgMod}_{\mathscr{D}} := \mathrm{dgMod}^{\mathscr{D}^{\mathrm{op}}}$ is the category of left dg modules of the opposite dg category.

To any two dg categories $\mathscr{D}$ and $\mathscr{D}'$ we associate the dg category of *$\mathscr{D}$-$\mathscr{D}'$-dg bimodules* $\mathrm{dgMod}^{\mathscr{D}}_{\mathscr{D}'} := \mathrm{dgMod}^{\mathscr{D}\otimes(\mathscr{D}')^{\mathrm{op}}}$. The *right Yoneda embedding* is the fully faithful dg functor

$$\begin{aligned} \Upsilon_{\mathscr{D}} \colon \mathscr{D} &\longrightarrow \mathrm{dgMod}_{\mathscr{D}} \\ a &\longmapsto \mathrm{Hom}_{\mathscr{D}}(-, b) \end{aligned}$$

The *left Yoneda embedding* is the fully faithful dg functor

$$\begin{aligned} \Upsilon^{\mathscr{D}} \colon \mathscr{D}^{\mathrm{op}} &\longrightarrow \mathrm{dgMod}^{\mathscr{D}} \\ a &\longmapsto \mathrm{Hom}_{\mathscr{D}}(a, -). \end{aligned}$$

One can dg-localise by inverting sets of morphisms and take dg-quotients by annihilating sets of objects, see [Kel06, §4.4].

The category D(vect) obtained by inverting quasi-isomorphisms in the homotopy category $H^0(\mathrm{dgvect})$ is a triangulated category with distinguished triangles given by images of cone complexes in $H^0(\mathrm{vect})$

$$W \xrightarrow{\ f\ } W' \longrightarrow \mathrm{cone}(f) \longrightarrow W[1].$$

The derived category $D(\mathscr{D})$ of a dg category $\mathscr{D}$ is a triangulated category with distinguised triangles given by pointwise cones. We say a morphism of dg modules $W \to W'$ is a *quasi-isomorphism* if it is a pointwise quasi-isomorphism of complexes $W(a) \to W'(a)$, for $a \in \mathrm{Ob}(\mathscr{D})$. The *derived category* $D(\mathscr{D})$ is defined to be the category obtained from the homotopy category $H^0(\mathrm{dgMod}_{\mathscr{D}})$ of right $\mathscr{D}$-dg modules by formally inverting quasi-isomorphisms.

A functor $F\colon \mathscr{D} \to \mathscr{D}'$ of dg categories which induces an equivalence of derived categories $D(\mathscr{D}') \to D(\mathscr{D})$ is called a *Morita equivalence*. Two dg categories are Morita equivalent if there is a Morita equivalence between them.

Definition 3.1 (An assortment of finiteness conditions)**.**

(i) An object of an ∞-category $\mathscr{C}$ (or a model category $\mathscr{C}$) with all homotopy colimits is *homotopically finitely presented* if for all filtered direct systems $X_i, i \in I$ in $\mathscr{C}$ the natural morphism

$$\operatorname*{hocolim}_{i \in I} \mathrm{Maps}_{\mathscr{C}}(Y, X_i) \longrightarrow \mathrm{Maps}_{\mathscr{C}}(Y, \operatorname*{hocolim}_{i \in I} X_i)$$

is an equivalence.

(ii) A dg algebra is a dg category with a single object. A *homotopically finitely presented* dg algebra is a homotopically finitely presented object in the category of dg algebras dgalg. Equivalently, a dg algebra is homotopically finitely presented if it is the retract in the homotopy category of the category of dg algebras of a dg algebra which as a noncommutative algebra is generated by finitely many variables $x_1, \dots, x_n$ for which the differentials dx_i belong to the subalgebra generated by the variables $x_1, \dots, x_{i-1}$, see [Kel06, §4.9].

(iii) A *perfect (right) dg module* is a homotopically finitely presented object in $D(\mathscr{D})$. We denote the subcategory of perfect modules by $\mathrm{Perf}(\mathscr{D}) \subseteq D(\mathscr{D})$. It is equivalently defined as the closure of the essential image of the Yoneda embedding $\Upsilon_{\mathscr{D}}$ under triangles, shifts, and summands.

(iv) A dg category $\mathscr{D}$ is *of finite type* if it is derived Morita equivalent to a homotopically finitely presented dg algebra.

(v) We say $\mathscr{D}$ is *proper* if the total dimension of the Hom complexes $\mathrm{Hom}_{\mathscr{D}}(a, b)$ is finite.

(vi) We say $\mathscr{D}$ is *smooth* if the diagonal bimodule

$$\begin{aligned} \mathscr{D}\colon \mathscr{D} \otimes \mathscr{D}^{\mathrm{op}} &\longrightarrow \mathrm{dgvect} \\ (a, b) &\longmapsto \mathrm{Hom}_{\mathscr{D}}(a, b) \end{aligned}$$

is a perfect bimodule.

We recall some terminology of derived categories. An object $a \in \mathscr{T}$ of a triangulated category $\mathscr{T}$ is *compact* if $\mathrm{Hom}_{\mathscr{T}}(a, -)$ commutes with direct sums. A set of objects $\{E_i\} \subseteq \mathscr{T}$ *generates* $\mathscr{T}$ if the smallest thick triangulated subcategory of $\mathscr{T}$ containing $\{E_i\}$ is $\mathscr{T}$ itself. A triangulated category $\mathscr{T}$ *has a compact generator* if a compact object generates $\mathscr{T}$. We say that a dg category $\mathscr{D}$ is *pre-triangulated* if its essential image in its derived category $D(\mathscr{D})$ under the right Yoneda embedding is a triangulated subcategory.

(vii) A dg category $\mathscr{D}$ *has a compact generator* if its derived category $D(\mathscr{D})$ is generated as a triangulated category by a compact object.

(viii) We call a dg category $\mathscr{D}$ is *geometric* if it is smooth, proper, pre-triangulated, and has a compact generator.

Remark 3.2. The terminology 'geometric' is not standard terminology. In [TV07] the authors call the same notion 'saturated'. The motivation for the term we use here comes from Theorem 3.15 (iv).

Remark 3.3. What we have called '(homotopically) finitely presented' is sometimes called 'compact'. Unfortunately this is a different notion of compactness than the notion of compactness for triangulated categories recalled above.

Remark 3.4. Every finite type dg category is smooth [TV07, Proposition 2.14]. Every smooth and proper dg category is of finite type [TV07, Corollary 2.13].

Example 3.5 (The derived preprojective algebra)**.** Let Q be a quiver.

The *derived preprojective algebra* $\mathscr{G}_2(Q)$ of Q is defined to be the dg algebra generated in degree zero by the path algebra $\mathbb{C}\overline{Q}$ of the doubled quiver $\overline{Q}$ and generated in degree -1 by a generator ϵ_i for each vertex $i \in Q_0$. The differential d of $\mathscr{G}_2(Q)$ is determined by setting

$$d(\epsilon_i) \coloneqq e_i \left(\sum_{a \in Q_1} [a, a^*] \right) e_i$$

where e_i is the lazy path at the vertex i. The zeroeth cohomology of the derived preprojective algebra is isomorphic to the preprojective algebra

$$\Pi_Q \cong H^0(\mathscr{G}_2(Q)).$$

$\mathscr{G}_2(Q)$ is quasi-isomorphic to Π_Q if and only if Q is not of ADE Dynkin type [CEG07, §8.2].

By construction, the dg algebra $\mathscr{G}_2(Q)$ is homotopically finitely presented and proper. Smoothness of $\mathscr{G}_2(Q)$ follows from Remark 3.4. It follows from [TV07, Lemma 2.6] that the category of perfect right $\mathscr{G}_2(Q)$-dg modules $\mathrm{Perf}(\mathscr{G}_2(Q))$ is geometric.

Example 3.6 (The derived category of an Abelian category)**.** Given an Abelian category $\mathscr{A}$ we define the dg category $\mathrm{Ch}(\mathscr{A})$ of chain complexes in $\mathscr{A}$ to be the category with objects given by chain complexes in $\mathscr{A}$ and where the degree i part of Hom-complexes $\mathrm{Hom}_{\mathrm{Ch}(\mathscr{A})}(A^*, B^*)$ are given by degree i graded homomorphisms $A^* \to B^*$. The subcategory of bounded chain complexes $\mathrm{Ch}^b(\mathscr{A}) \subseteq \mathrm{Ch}(\mathscr{A})$ is a full dg subcategory.

The homotopy category of $\mathscr{A}$ is the homotopy category $H^0(\mathrm{Ch}(\mathscr{A}))$ (which is triangulated) and the derived category $D(\mathscr{A})$ of $\mathscr{A}$ is the localisation of $H^0(\mathrm{Ch}(\mathscr{A}))$ along quasi-isomorphisms. A dg enhancement $\mathscr{D}(\mathscr{A})$ of $D(\mathscr{A})$ is given by the dg localisation of $\mathrm{Ch}(\mathscr{A})$ along quasi-isomorphisms.

The bounded derived category $D^b(\mathscr{A})$ of $\mathscr{A}$ is the localisation of $H^0(Ch^b(\mathscr{A}))$ along quasi-isomorphisms and it has a dg enhancement $D(\mathscr{A})$ which is the dg localisation of $Ch^b(\mathscr{A})$ along quasi-isomorphisms.

Example 3.7 (The category of perfect complexes on a smooth projective variety.)**.** Let X be a smooth projective variety. Denote by QCoh(X) the category of quasicoherent sheaves on X. and we denote by Coh(X) the category of coherent sheaves on X. Both are Abelian categories.

A *perfect complex* on X is a complex $\mathcal{F}^*$ of coherent sheaves such that there is an open cover $X = \bigcup_i U_i$ so that the restrictions $\mathcal{F}^* |_{U_i}$ are isomorphic in $D(U_i)$ to bounded complexes of locally free sheaves. Equivalently (see Corollary 2.3 and Proposition 2.5 in [Nee96]), a perfect complex $\mathcal{F}^*$ is a homotopically finitely presented object of D(QCoh(X)).

The subcategory of perfect complexes $\mathrm{Perf}'(X) \subseteq \mathrm{Ch}(\mathrm{QCoh}(X))$ is a dg category. We define Perf(X) to be the dg localisation of $\mathrm{Perf}'(X)$ along quasi-isomorphisms. The dg category Perf(X) is a dg enhancement of the full triangulated subcategory $D_{\mathrm{Perf}}(X)$ of D(X) consisting of perfect complexes. Every coherent sheaf on a smooth projective variety X is perfect (see e. g. [HL10, Proposition 2.1.10]). Therefore $D_{\mathrm{Perf}}(X) \simeq D^b(\mathrm{Coh}(X))$.

Perf(X) is a geometric dg category by [TV07, Lemma 3.27]. In particular, compact generation is [BvdB03, Theorem 1.3.4].

Remark 3.8. It is psychologically helpful to know of [CS18, Theorems A and B] which guarantee the existence and uniqueness (up to (chains of) quasi-equivalences) of dg enhancements for derived categories of Grothendieck Abelian categories and categories of homotopically finitely presented objects therein. For example, the dg enhancements of $D^b(X)$ and Perf(X) are unique.

3.2 CALABI–YAU DG CATEGORIES

Homological invariants of dg categories

The *Hochschild homology* of a dg category $\mathscr{D}$ is defined by the formula

$$HH_{-*}(\mathscr{D}) = H^*(\mathscr{D} \otimes^{L}_{\mathscr{D}^e} \mathscr{D})$$

where $\mathscr{D}^e = \mathscr{D}^{op} \otimes \mathscr{D}$ is the “envelope” of $\mathscr{D}$ and $\otimes^{L}_{\mathscr{D}^e}$ denotes the derived tensor product in the category of $\mathscr{D}$-bimodules. A model for $\mathscr{D} \otimes^{L}_{\mathscr{D}^e} \mathscr{D}$ is called a Hochschild chain complex and an explicit model is given by the bar complex $(CC_*(\mathscr{D}), b)$ with

$$\begin{aligned} CC_*(\mathscr{D}) = \bigoplus_{p \geqslant 0} \bigoplus_{a_0,\dots,a_p \in \mathrm{Ob}(\mathscr{D})} & \mathrm{Hom}_{\mathscr{D}}(a_p, a_0) \otimes \mathrm{Hom}_{\mathscr{D}}(a_p, a_{p-1}) \otimes \cdots \\ & \otimes \mathrm{Hom}_{\mathscr{D}}(a_0, a_1) \end{aligned} \tag{3.1}$$

(a tensor $f_p \otimes \cdot \otimes f_0 \in CC_*(\mathscr{D})$ has degree $p + \sum_{i=0}^{p} \deg(f_i)$) and the differential $b\colon CC_p(\mathscr{D}) \to CC_{p-1}(\mathscr{D})$ defined for all $f_p \otimes \cdots \otimes f_0 \in \mathrm{Hom}_{\mathscr{D}}(a_p, a_0) \otimes \mathrm{Hom}_{\mathscr{D}}(a_p, a_{p-1}) \otimes \cdots \otimes \mathrm{Hom}_{\mathscr{D}}(a_0, a_1)$ by

$$\begin{aligned} b(f_p \otimes \cdots \otimes f_0) = f_{p-1} \otimes \cdots \otimes f_0 f_p \\ + \sum_{i=1}^{p} (-1)^i f_p \otimes \cdots \otimes f_i f_{i-1} \otimes \cdots \otimes f_0 \end{aligned}$$

We define various operators on the bar complex (3.1). Let

$$t\colon CC_*(\mathscr{D}) \longrightarrow CC_*(\mathscr{D})$$

be the signed cyclic permutation of the tensor factors

$$t(f_p \otimes \cdots \otimes f_0) = (-1)^{p+\sum_{i=1}^{p} \deg(f_i)} f_0 \otimes f_p \otimes \cdots \otimes f_1.$$

Define the norm operator $N\colon CC_*(\mathscr{D}) \to CC_*(\mathscr{D})$ by $N = \sum_{i=1}^{p} t^i$. Define $s\colon CC_*(\mathscr{D}) \longrightarrow CC_{*+1}(\mathscr{D})$ by

$$s(f_p \otimes \cdots \otimes f_0) = \mathrm{id}_{a_0} \otimes f_p \otimes \cdots \otimes f_0$$

for all $f_p \otimes \cdots \otimes f_0 \in \mathrm{Hom}_{\mathscr{D}}(a_p, a_0) \otimes \cdots \otimes \mathrm{Hom}_{\mathscr{D}}(a_0, a_1)$.

The Connes B-operator $B\colon CC_*(\mathscr{D}) \to CC_{*+1}(\mathscr{D})$ is defined by $B = (1-t)sN$ which explicitly is given for all $f_p \otimes \cdots \otimes f_0 \in \mathrm{Hom}_{\mathscr{D}}(a_p, a_0) \otimes \cdots \otimes \mathrm{Hom}_{\mathscr{D}}(a_0, a_1)$ by

$$\begin{aligned} B(f_p \otimes \cdots \otimes f_0) = \sum_{i=0}^{p} (-1)^{ip+*} (\mathrm{id}_{a_{i+1}} \otimes f_i \otimes \cdots \otimes f_p \otimes f_0 \otimes \cdots \otimes f_{i-1}) \\ -(-1)^{ip+*} (f_{i-1} \otimes \mathrm{id}_{a_{i+1}} \otimes f_i \otimes \cdots \otimes f_p \otimes f_0 \otimes \cdots \otimes f_{i-2}), \end{aligned}$$

where we have hidden Koszul signs in the symbol "$*$". The operator B satisfies

$$\begin{aligned} B^2 = 0 \\ Bb - bB = 0. \end{aligned}$$

Thus B endows the bar complex $CC_*(\mathscr{D})$ with the structure of a $k[\delta]$-dg module, where δ has degree one (hence $\delta^2 = 0$).

Consider the $k[\delta]$-dg module $(\mathbb{C}[\delta][u], d)$ where u has homological degree -2 and the differential d is determined by $d(u) = -\delta$. We define *the negative cyclic complex of $\mathscr{D}$* to be

$$CC^{-}_{\mathrm{cyc}}(\mathscr{D}) \coloneqq \mathrm{Hom}_{\mathrm{dgMod}_{\mathbb{C}[\delta]}}(\mathbb{C}[\delta][u], CC_*(\mathscr{D})). \tag{3.2}$$

Since $(\mathbb{C}[\delta][u], d)$ is quasi-isomorphic to $\mathbb{C}$ we have

$$CC^{-}_{\mathrm{cyc}}(\mathscr{D}) \cong \mathrm{RHom}_{\mathbb{C}[\delta]}(\mathbb{C}, CC_*(\mathscr{D})).$$

The homology of (3.2)

$$HC^{-}_{*}(\mathscr{D}) \cong \mathrm{Ext}^{*}_{\mathbb{C}[\delta]}(\mathbb{C}, CC_*(\mathscr{D}))$$

is called *the negative cyclic homology of* $\mathscr{D}$. Setting $\delta = 0$ gives a morphism of complexes

$$\mathrm{CC}^{-}_{\mathrm{cyc},*}(\mathscr{D}) \longrightarrow \mathrm{CC}_*(\mathscr{D}) \tag{3.3}$$

and maps of graded vector spaces

$$\mathrm{HC}^{-}_*(\mathscr{D}) \longrightarrow \mathrm{HH}_*(\mathscr{D}).$$

Dually, we define *the cyclic complex of* $\mathscr{D}$ to be

$$\mathrm{CC}_{\mathrm{cyc}}(\mathscr{D}) \coloneqq \mathbb{C}[\delta][u] \otimes_{\mathbb{C}[\delta]} \mathrm{CC}_*(\mathscr{D}) \simeq \mathbb{C} \otimes^{\mathrm{L}}_{\mathbb{C}[\delta]} \mathrm{CC}_*(\mathscr{D}). \tag{3.4}$$

The homology of (3.4)

$$\mathrm{HC}_*(\mathscr{D}) \cong \mathrm{Tor}^{\mathbb{C}[\delta]}_*(\mathbb{C}, \mathrm{CC}_*(\mathscr{D}))$$

is called *the cyclic homology of* $\mathscr{D}$.

Setting $\delta = 0$ gives us chain maps

$$\mathrm{CC}_*(\mathscr{D}) \longrightarrow \mathrm{CC}_{\mathrm{cyc},*}(\mathscr{D})$$

and the induced map on homology

$$\mathrm{HH}_*(\mathscr{D}) \longrightarrow \mathrm{HC}_*(\mathscr{D}).$$

Remark 3.9. We recall the more conceptual definition of the negative cyclic complex as in [Hoy18]. One observes that the bar complex $\mathrm{CC}_*(\mathscr{D})$ (which is defined for any dg category) is a simplical module with an additional cyclic symmetry. This endows the simplical module $\mathrm{CC}_*(\mathscr{D})$ with the an action of the simplical set S^1.

With this setup the negative cyclic complex (3.2) is a model for the homotopy S^1-fixed points of $\mathrm{CC}_*(\mathscr{D})$

$$\mathrm{CC}^{-}_{\mathrm{cyc},*}(\mathscr{D}) \simeq \mathrm{CC}_*(\mathscr{D})^{S^1}$$

and the cyclic complex (3.4) is a model for the homotopy S^1-orbits of $\mathrm{CC}_*(\mathscr{D})$

$$\mathrm{CC}_{\mathrm{cyc},*}(\mathscr{D}) \simeq \mathrm{CC}_*(\mathscr{D})_{S^1}.$$

Calabi–Yau structures

The main reference for this section is [BD19]. Many of the ideas in the context of dg algebras originally appeared in [Gin07].

Let $\mathscr{D}$ be a dg category. We define $\mathscr{D}^! \in \mathrm{D}(\mathrm{Mod}_{\mathscr{D}^e})$ to be the derived dual of the diagonal bimodule $\mathscr{D}$:

$$\mathscr{D}^! \coloneqq \mathrm{RHom}_{\mathscr{D}^e}(\mathscr{D}, \mathscr{D}^e).$$

If $\mathscr{D}$ is a smooth dg category, then there is a quasi-isomorphism (see [BD19, Example 2.10])

$$\mathscr{D} \otimes^{\mathrm{L}}_{\mathscr{D}^e} \mathscr{D} \xrightarrow{\simeq} \mathrm{RHom}_{\mathscr{D}^e}(\mathscr{D}^!, \mathscr{D}).$$

Via (3.3) we obtain a morphism

$$\flat\colon \mathrm{CC}^-_{\mathrm{cyc}}(\mathscr{D}) \longrightarrow \mathrm{RHom}_{\mathscr{D}^e}(\mathscr{D}^!, \mathscr{D}).$$

We define $\mathscr{D}^*$ to be the 'linear dual' $\mathscr{D}$-bimodule given by

$$\mathscr{D}^*(a,b) \coloneqq \mathrm{Hom}_{\mathscr{D}}(a,b)^* = \mathrm{Hom}_{\mathscr{D}}(\mathrm{Hom}_{\mathscr{D}}(a,b), \mathbb{C}).$$

If $\mathscr{D}$ is proper, there is a natural quasi-isomorphism (see [BD19, Example 2.8])

$$\sharp\colon\ \mathrm{RHom}_{\mathbb{C}}(\mathscr{D} \otimes^{\mathrm{L}}_{\mathscr{D}^e} \mathscr{D}, \mathbb{C}) \xrightarrow{\simeq} \mathrm{RHom}_{\mathscr{D}^e}(\mathscr{D}, \mathscr{D}^\vee).$$

Definition 3.10. (i) Suppose $\mathscr{D}$ is a smooth dg category. A *left* n-*Calabi–Yau structure* on $\mathscr{D}$ is a degree n negative cyclic cycle $\sigma \in \mathrm{CC}^-_{\mathrm{cyc}}(\mathscr{D})^n$, i. e. a chain map

$$\sigma\colon \mathbb{C}[n] \longrightarrow \mathrm{CC}^-_{\mathrm{cyc}}(\mathscr{D}),$$

such that the induced $\mathscr{D}$-bimodule morphism

$$\sigma^\flat\colon \mathscr{D}^! \longrightarrow \mathscr{D}^{[}n]$$

is a quasi-isomorphism.

(ii) Suppose $\mathscr{D}$ is a proper dg category. A *right* n-*Calabi–Yau structure* on $\mathscr{D}$ is a degree n cyclic cocycle $\xi \in \mathrm{CC}_{\mathrm{cyc}}(\mathscr{D})^n$, i. e. a chain map

$$\xi\colon \mathrm{CC}_{\mathrm{cyc}}(\mathscr{D}) \longrightarrow \mathbb{C}[n],$$

such that the induced $\mathscr{D}$-bimodule morphism

$$\xi^\sharp\colon \mathscr{D} \longrightarrow \mathscr{D}^*[-n]$$

is a quasi-isomorphism.

Proposition 3.11 ([BD19, Theorem 3.1])**.** *Let $\mathscr{D}$ be a smooth dg category and $\mathscr{P} \subseteq \mathscr{D}$ a proper full dg subcategory. Every left n-Calabi–Yau structure on $\mathscr{D}$ induces a canonical right n-Calabi–Yau structure on $\mathscr{P}$.*

Example 3.12. Let X be a smooth quasi-projective variety. There is a natural bijection between the set of left Calabi–Yau structures for $\mathrm{Perf}(X)$ and the set of trivialisations $\omega_X \cong \mathcal{O}_X$ of the dualising sheaf. This is [BD19, Proposition 5.12] for smooth schemes.

We refer to [BD19, §5] for more examples of dg categories with Calabi–Yau structures.

Remark 3.13. There is a general procedure, due to Keller [Kel11], called n-*Calabi–Yau completion*, canonically associating to any dg category $\mathscr{D}$ an n-Calabi–Yau category $\mathscr{G}_n(\mathscr{D})$. The Calabi–Yau completion should be thought of as a non-commutative shifted cotangent bundle: in [BCS22] the authors show that the moduli stack of objects of an n-Calabi–Yau completion $\mathscr{G}_n(\mathscr{D})$ is isomorphic (as a shifted symplectic scheme) to a shifted cotangent bundle of the moduli stack of objects of the original dg category $\mathscr{D}$.

3.3 MODULI OF OBJECTS

We recall the construction of moduli of objects in dg categories [TV07] with an emphasis on moduli of objects in Abelian categories.

Definition 3.14. The *derived moduli stack* $\mathrm{R}\mathfrak{M}_{\mathscr{D}}$ *of objects in a dg category* $\mathscr{D}$ is defined by its functor of points

$$\begin{aligned} \mathrm{R}\mathfrak{M}_{\mathscr{D}} \colon \mathrm{dAff}^{\mathrm{op}} &\longrightarrow \mathrm{SSets} \\ U &\longmapsto \mathrm{Maps}_{\mathrm{dgcats}}(\mathscr{D}^{\mathrm{op}}, \mathrm{Perf}(U)). \end{aligned}$$

The restriction morphisms are induced from the pullback functors $f^* \colon \mathrm{Perf}(U') \to \mathrm{Perf}(U)$ for every morphism $f \colon U \to U'$ of derived affine schemes.

Here SSets is the ∞-category of simplicial sets (a. k. a. ∞-groupoids) and the mapping simplicial set $\mathrm{Maps}_{\mathrm{dgcats}}(-,-)$ is taken in the ∞-category with weak-equivalences given by quasi-equivalences. By construction the assignment $\mathscr{D} \mapsto \mathrm{R}\mathfrak{M}_{\mathscr{D}}$ is contravariant in $\mathscr{D}$.

The following theorem summarises the main properties of $\mathrm{R}\mathfrak{M}_{\mathscr{D}}$ that we will use.

Theorem 3.15 (Properties of $\mathrm{R}\mathfrak{M}_{\mathscr{D}}$)**.** *Let* $\mathscr{D}$ *be a geometric dg category.*

(i) The functor $\mathrm{R}\mathfrak{M}_{\mathscr{D}}$ *is a locally geometric, locally of finite presentation, higher derived stack. Hence* $\mathrm{R}\mathfrak{M}_{\mathscr{D}}$ *admits a perfect cotangent complex* $\mathbb{L}_{\mathrm{R}\mathfrak{M}_{\mathscr{D}}}$.

(ii) (Morita invariance) *The Yoneda embedding* $\mathscr{D} \longrightarrow \mathrm{Perf}(\mathscr{D})$ *induces an equivalence* $\mathrm{R}\mathfrak{M}_{\mathrm{Perf}(\mathscr{D})} \to \mathrm{R}\mathfrak{M}_{\mathscr{D}}$.

(iii) Suppose $\mathrm{R}\mathfrak{M} \subseteq \mathrm{R}\mathfrak{M}_{\mathscr{D}}$ *is an open substack which is 1-Artin, then the truncation* $\mathfrak{M} = t_0(\mathrm{R}\mathfrak{M})$ *has a separated diagonal and affine stabilisers.*

(iv) (Points are objects) *There is a natural bijection*

$$\pi_0(\mathrm{Maps}_{\mathrm{dgcats}}(\mathscr{D}^{\mathrm{op}}, \mathrm{Perf}(U))) \simeq \mathrm{Perf}(\mathscr{D}^{\mathrm{op}} \otimes U) \simeq \mathscr{D} \otimes \mathrm{Perf}(U).$$

Importantly, this means that $\pi_0(\mathrm{R}\mathfrak{M}_{\mathscr{D}}(\mathbb{C})) = \mathrm{Perf}(\mathscr{D})$.

(v) (Tangent complex) *The tangent and cotangent complexes at* $E \in \mathrm{Perf}(\mathscr{D})$ *viewed as a point in* $\mathrm{R}\mathfrak{M}_{\mathscr{D}}$ *are given by*

$$\mathbb{T}_{\mathrm{R}\mathfrak{M}_{\mathscr{D}}}|_E \simeq \mathrm{Hom}_{\mathscr{D}}(E,E)[1] \text{ and } \mathbb{L}_{\mathrm{R}\mathfrak{M}_{\mathscr{D}}}|_E \simeq \mathrm{Hom}_{\mathscr{D}}(E,E)[-1].$$

Proof. Item (i) is [TV07, Lemma 3.1] and [TV07, Theorem 3.6]. Item (ii) is [TV07, Lemma 3.3]. Item (iii) is [Dav23b, Lemma 5.9]. A justification for (iv) can be found on page 420 of [TV07]. Item (v) is [TV07, Corollary 3.17] ☐

Example 3.16. Let X be a smooth and projective variety. Since $\mathrm{Perf}(X)$ is a geometric dg category, its derived moduli stack $\mathrm{R}\mathfrak{M}_X := \mathrm{R}\mathfrak{M}_{\mathrm{Perf}(X)}$ is locally geometric and we call it the *derived moduli stack of perfect complexes on* X.

Definition 3.17. An Abelian full subcategory $\mathscr{A} \subseteq \mathscr{T}$ of a triangulated category $\mathscr{T}$ is *admissible* if for all $a, b \in \mathscr{A}$ we have $\mathrm{Ext}^i_{\mathscr{D}}(a, b) = 0$ for all $i < 0$. An Abelian subcategory $\mathscr{A} \subseteq \mathscr{D}$ of a triangulated dg category $\mathscr{D}$ is *admissible* if $\mathscr{A} \subseteq D(\mathscr{D})$ is admissible. We say that $\mathscr{A} \subseteq \mathscr{T}$ is *extension closed*, if a sequence in $\mathscr{A}$ is short exact if and only if it defines a distinguished triangle in $\mathscr{T}$.

Example 3.18. Suppose $\mathscr{A} \subseteq \mathscr{T}$ is the heart of a t-structure. Then it is admissible and extension closed [BBD, Theorem 1.3.6].

Definition 3.19 (Moduli of objects in a subcategory)**.** Let $\mathscr{C}$ be a subcategory of a geometric dg category $\mathscr{D}$.

We define the substack $R\mathfrak{M}_{\mathscr{C}} \subseteq R\mathfrak{M}_{\mathscr{D}}$ by declaring the U points of $R\mathfrak{M}_{\mathscr{C}}$ to be given by the full simplicial subset $R\mathfrak{M}_{\mathscr{C}}(U)_\bullet \subseteq R\mathfrak{M}_{\mathscr{D}}(U)$ determined by

$$R\mathfrak{M}_{\mathscr{A}}(U)_0 = \left\{ E \in \mathscr{D} \otimes \mathrm{Perf}(U) \;\middle|\; \begin{array}{c} x^*(E) \in \mathscr{C} \\ \text{for all closed } x\colon \mathrm{pt} \hookrightarrow U \end{array} \right\}.$$

for every derived affine scheme U. Note that here we are using Theorem 3.15 (iv). We say that $\mathscr{A}$ satisfies the *openness in families condition* if $R\mathfrak{M}_{\mathscr{C}} \subseteq R\mathfrak{M}_{\mathscr{D}}$ is an open substack.

In the case when $\mathscr{C} \subseteq \mathscr{D}$ satisfies the openness in families condition we call the derived substack $R\mathfrak{M}_{\mathscr{C}} \subseteq R\mathfrak{M}_{\mathscr{D}}$ *the derived moduli stack of the pair* $(\mathscr{C}, \mathscr{D})$ and we call $\mathfrak{M}_{\mathscr{C}}$ *the moduli stack of the pair* $(\mathscr{C}, \mathscr{D})$. We usually abuse notation and terminology and simply refer to $\mathfrak{M}_{\mathscr{C}}$ and $R\mathfrak{M}_{\mathscr{C}}$ as the moduli stack and derived moduli stack of $\mathscr{C}$, respectively.

Corollary 3.20. *Suppose $\mathscr{A}$ is an admissible Abelian subcategory of a geometric dg category $\mathscr{D}$ satisfying the openness in families condition. Then the moduli stack $\mathfrak{M}_{\mathscr{A}}$ of objects in $\mathscr{A}$ is 1-Artin.*

Proof. Follows immediately from [TV07, Corollary 3.21]. ☐

Example 3.21 (Π_Q-modules)**.** The category $\mathrm{Rep}(\Pi_Q)$ of finite dimensional Π_Q representations is naturally an admissible extension closed Abelian subcategory of $\mathrm{Perf}(\mathscr{G}_2(Q))$ consisting of those finite dimensional dg modules concentrated in degree zero (it is the heart of a t-structure).

Denote by $R\mathfrak{M}_{\mathscr{G}_2(Q)}$ the moduli stack of objects in $\mathrm{Perf}(\mathscr{G}_2(Q))$. The substack $R\mathfrak{M}_{\Pi_Q} \subseteq R\mathfrak{M}_{\mathscr{G}_2(Q)}$ is an open substack. Indeed, by the lower semicontinuity of rank it follows that asking a dg module to be concentrated in degree zero is an open condition.

Example 3.22 (Coherent sheaves)**.** Let X be a smooth and projective variety. The category of coherent sheaves on X is the heart of the standard t-structure on $D^b(\mathrm{Coh}(X))$, and hence an admissible subcategory [BBD, Théorème 1.3.6]. Since $\mathrm{Perf}(X)$ is a dg enhancement of $D^b(\mathrm{Coh}(X))$ we view $\mathrm{Coh}(X)$ as an admissible Abelian subcategory of $\mathrm{Perf}(X)$.

Consider the substack $R\mathfrak{M}_{\mathrm{Coh}(X)}$ of $R\mathfrak{M}_{\mathrm{Perf}(X)}$ corresponding to the admissible Abelian subcategory $\mathrm{Coh}(X)$. In formulas,

$$R\mathfrak{M}_{\mathrm{Coh}(X)}(\mathbb{C}) := \{\mathcal{F} \in \mathrm{Perf}(X) \mid \mathcal{H}^i(\mathcal{F}) = 0 \text{ for all } i \neq 0\}.$$

The rank of coherent sheaves is lower semicontinuous. Thus the vanishing of the coherent sheaves $\mathcal{H}^i(\mathcal{F})$ is an open condition, i. e. $R\mathfrak{M}_{\mathrm{Coh}(X)} \subseteq R\mathfrak{M}_{\mathrm{Perf}(X)}$ is an open substack.

Recall from [aut23, Tag 08KA] the classical moduli functor of coherent sheaves on a scheme

$$\mathfrak{Coh}_X \colon \mathrm{Scheme} \longrightarrow \mathrm{Grpds}.$$

There is a natural equivalence of stacks

$$\mathfrak{Coh}_X \simeq \mathfrak{M}_{\mathrm{Coh}(X)} = t_0(R\mathfrak{M}_{\mathrm{Coh}(X)}).$$

Euler forms and connected components

The stack $R\mathfrak{M}_{\mathscr{D}}$ is generally not connected. We denote connected components as follows

$$R\mathfrak{M}_{\mathscr{D}} = \bigsqcup_{\nu \in \pi_0(R\mathfrak{M}_{\mathscr{D}})} R\mathfrak{M}_{\mathscr{D},\nu}.$$

The Euler form $(-,-)$ is a bilinear form on the Grothendieck group of $\mathscr{D}$

$$(-,-)_{\mathscr{D}} \colon K_0(\mathscr{D}) \times K_0(\mathscr{D}) \longrightarrow \mathbb{Z}$$

defined by

$$(E,F)_{\mathscr{D}} = \sum_{i \in \mathbb{Z}} (-1)^i H^i(\mathrm{Hom}_{\mathscr{D}}(E,F)).$$

Suppose $\mathscr{A} \subseteq \mathscr{D}$ is an admissible extension closed Abelian subcategory satisfying the openness in families condition. The inclusion $\mathscr{A} \hookrightarrow \mathscr{D}$ induces a group homomorphism $K_0(\mathscr{A}) \hookrightarrow K_0(\mathscr{D})$. Consider the function on $R\mathfrak{M}_{\mathscr{A}}$ given by the Euler form

$$\begin{aligned} R\mathfrak{M}_{\mathscr{A}}(\mathbb{C}) &\longrightarrow \mathbb{Z} \\ E &\longmapsto (E,E)_{\mathscr{D}} \end{aligned} \tag{3.5}$$

By Theorem 3.15 (v) we have

$$(E,E)_{\mathscr{D}} = -\chi(\mathbb{T}_{R\mathfrak{M}_{\mathscr{A}}}|_{[E]}).$$

thus (3.5) is a locally constant function and factors through the set of connected components

$$\begin{tikzcd} R\mathfrak{M}_{\mathscr{A}} \arrow[rr] \arrow[dr] & & \mathbb{Z} \\ & \pi_0(\mathfrak{M}_{\mathscr{A}}) \arrow[ur] & . \end{tikzcd}$$

Example 3.23 (Dimension vectors)**.** The dimension vector of a representation defines a locally constant function on $\mathrm{R}\mathfrak{M}_{\Pi_Q}$

$$\begin{aligned} \underline{\dim} : \mathrm{R}\mathfrak{M}_{\Pi_Q}(\mathbb{C}) &\longrightarrow \mathbb{N}^{Q_0} \\ \rho = ((V_i)_{i\in Q_0}, (\rho_a)_{a\in \overline{Q}_1}) &\longmapsto \underline{\dim}(\rho) = (\dim(V_i))_{i\in Q_0} \end{aligned}$$

The Euler form of preprojective representations ρ_1, ρ_2 of dimension vectors d_1 and d_2, respectively is entirely determined by its dimension vector by

$$(\rho_1, \rho_2)_{\mathscr{G}_2(Q)} = 2\sum_{i\in Q_0} d_i^2 - 2\sum_{i\to j\in Q_1^2} d_i d_j.$$

We can thus decompose

$$\mathrm{R}\mathfrak{M}_{\Pi_Q} = \bigsqcup_{\underline{d}\in\mathbb{N}^{Q_0}} \mathrm{R}\mathfrak{M}_{\Pi_Q,\underline{d}}$$

into the closed and open substacks $\mathrm{R}\mathfrak{M}_{\Pi_Q,\underline{d}} = \underline{\dim}^{-1}(\underline{d})$. We have an alternative explicit construction of the classical truncations $\mathfrak{M}_{\Pi_Q,\underline{d}}$, see Example 3.35.

RHom complexes

We explain a definition of the RHom complex using the philosophy of [BD21, §2].

Let $U = \mathrm{R\,Spec}(A)$ be a derived affine scheme. The category of quasicoherent sheaves $\mathrm{QCoh}(U) = \mathrm{dgMod}_A$ is a symmetric monoidal dg category, i. e. an algebra in the category of dg categories. For every dg category $\mathscr{C}$ the category $\mathrm{QCoh}(U)\otimes\mathscr{C}$ is naturally a $\mathrm{QCoh}(U)$-module category (since $\mathrm{QCoh}(U)$ is an algebra). Thus, repeating an explaination in [BD21, §2.3], every object $E_U \in \mathrm{Perf}(U)\otimes\mathscr{C}$ defines a functor

$$-\otimes^L_U E_U \colon \mathrm{QCoh}(U) \longrightarrow \mathrm{QCoh}(U)\otimes\mathscr{C}$$

which is continuous and admits a right adjoint

$$\underline{\mathrm{RHom}}_U(E_U, -)\colon \mathrm{QCoh}(U)\otimes\mathscr{C} \longrightarrow \mathrm{QCoh}(U).$$

Let $\mathscr{D}$ be a geometric dg category (Definition 3.1 (viii)). A U-point of its (derived) moduli of objects $t : U \to \mathfrak{M}_{\mathscr{D}}$ corresponds to an object $E_U \in \mathrm{Perf}(U)\otimes\mathscr{D}$ such that $\underline{\mathrm{RHom}}_U(E_U, -)$ is continous [BD21, Corollary 2.7]. By continuity, the functor $\underline{\mathrm{RHom}}_U(E_U, -)$ preserves homotopically finitely presented objects and therefore restricts to a functor

$$\underline{\mathrm{RHom}}_U(E_U, -)\colon \mathrm{Perf}(U)\otimes\mathscr{D} \longrightarrow \mathrm{Perf}(U).$$

Definition 3.24 (RHom complex)**.** We define the *RHom complex* $\mathrm{R}\mathcal{H}om_{\mathscr{D}} \in \mathrm{Perf}(\mathrm{R}\mathfrak{M}_{\mathscr{D}}\times\mathrm{R}\mathfrak{M}_{\mathscr{D}})$ by its values on points $t\colon U \to \mathrm{R}\mathfrak{M}_{\mathscr{D}}\times\mathrm{R}\mathfrak{M}_{\mathscr{D}}$ as

$$\mathrm{R}\mathcal{H}om_{\mathscr{D}}(t\colon U\to\mathscr{D}) \coloneqq \underline{\mathrm{RHom}}_U(E_{U,2}, E_{U,1})$$

where $E_{U,i} \in \mathrm{Perf}(U) \otimes \mathscr{D}$ is the object corresponding to $\mathrm{pr}_i \circ t \colon : U \to \mathrm{R}\mathfrak{M}_{\mathscr{D}}$.

Remark 3.25. Using the adjunction we compute the derived sections of the RHom complexes on $t : U \to \mathrm{R}\mathfrak{M}_{\mathscr{D}} \times \mathrm{R}\mathfrak{M}_{\mathscr{D}}$

$$\begin{aligned} \mathrm{R}\Gamma(t, \mathrm{R}\mathcal{H}om_{\mathscr{D}}) &= \mathrm{R}\mathcal{H}om_U(\mathcal{O}_U, \underline{\mathrm{RHom}}_U(E_{U,2}, E_{U,1})) \\ &\cong \mathrm{RHom}_{\mathrm{Perf}(U)\otimes\mathscr{C}}(\mathcal{O}_U \otimes^L_U E_{U,2}, E_{U,1}) \\ &\cong \mathrm{RHom}_{\mathrm{Perf}(U)\otimes\mathscr{C}}(E_{U,2}, E_{U,1}). \end{aligned}$$

Remark 3.26. It is an instructive exercise to connect the abstract definition of the RHom complex with the usual definition of the RHom complex for the moduli of perfect complexes on a smooth projective variety X

$$\mathrm{R}\mathcal{H}om_{\mathrm{Perf}(X)} = \mathrm{pr}_{12,*}\mathrm{R}\mathcal{H}om_{\mathrm{R}\mathfrak{M}_X \times \mathrm{R}\mathfrak{M}_X \times X}(\mathrm{pr}_{23}^*\mathcal{E}, \mathrm{pr}_{13}^*\mathcal{E})$$

where $\mathcal{E} \in \mathrm{Perf}(\mathrm{R}\mathfrak{M}_X \times X)$ is the universal perfect complex.

3.4 MODULI STACKS OF SHORT EXACT SEQUENCES

Our treatment here is heavily inspired by [Rob23].

Let $\mathscr{D}$ be a dg category. Consider the diagram category $I = \{0 \to 1\}$. We define the dg category of morphisms in $\mathscr{D}$ to be the dg category $\mathscr{D}^I := \mathrm{Fun}(I, \mathscr{D})$ of dg functors from I to $\mathscr{D}$. For $f, g \in \mathrm{Fun}(I, \mathscr{D})$ consider morphisms of complexes

$$\begin{aligned} \xi(f,g) \colon \ \mathrm{Hom}_{\mathscr{D}}(f(0), g(0)) \oplus \mathrm{Hom}_{\mathscr{D}}(f(1), g(1)) &\longrightarrow \mathrm{Hom}_{\mathscr{D}}(f(0), g(1)) \\ (\xi(0), \xi(1)) &\longmapsto g(\to) \circ \xi(0) - \xi(1) \circ f(\to) \end{aligned}$$

Then the morphism complex $\mathrm{Hom}_{\mathscr{D}^I}(f, g)$ is explicitly given by

$$\mathrm{Hom}_{\mathscr{D}^I}(f, g) := \mathrm{cone}(\xi(f, g)).$$

The objects of $\mathscr{D}^I$ are identified with the collection of morphisms in $\mathscr{D}$. When $\mathscr{D}$ is pre-triangulated, the objects of $\mathscr{D}^I$ can equivalently be identified with the collection of distinguished triangles in $H^0(\mathscr{D})$.

We have natural forgetful functors: the source functor

$$\begin{aligned} F_0 \colon \mathscr{D}^I &\longrightarrow \mathscr{D} \\ E_0 \overset{\alpha}{\to} E_1 &\longmapsto E_0 \end{aligned}$$

and the target functor

$$\begin{aligned} F_1 : \mathscr{D}^I &\longrightarrow \mathscr{D} \\ E_0 \overset{\alpha}{\to} E_1 &\longmapsto E_1. \end{aligned}$$

Moreover, when $\mathscr{D}$ is a pre-triangulated dg category we can take cones and thus we have a third forgetful functor

$$\begin{aligned} C \colon \mathscr{D}^I &\longrightarrow \mathscr{D} \\ E_0 \overset{\alpha}{\to} E_1 &\longmapsto \mathrm{cone}(\alpha). \end{aligned}$$

Lemma 3.27. *If $\mathscr{D}$ is a geometric dg category, then $\mathscr{D}^I$ is a geometric dg category.*

Proof sketch. The proof is an application of [TV07, Lemma 2.6]. It suffices to show the lemma when $\mathscr{D} = A$ is for a homotopically finitely presented dg algebra A. In this case it is an exercise to construct a homotopically finitely presented dg algebra $\widetilde{A}$ such that $\mathscr{D}^I$ is Morita equivalent to $\widetilde{A}$ so that all of the properties of being 'geometric' are readily verified for $\widetilde{A}$ by deducing them from the same properties for A.

For example for $\widetilde{A}$ we can take

$$\widetilde{A} = (A \oplus A)\langle t\rangle/\big(t(a_1, a_2) = (a_2, a_1)t \mid (a_1, a_2) \in A \oplus A\big)$$

where t is a formal variable of degree 0. ☐

The functors F_0, F_1, C induce morphisms $f_0, f_1, c\colon R\mathfrak{M}_{\mathscr{D}^I} \to R\mathfrak{M}_{\mathscr{D}}$. The functor $(F_0, C)\colon \mathscr{D}^I \to \mathscr{D} \otimes \mathscr{D}$ has a right adjoint

$$\begin{aligned} \widetilde{\oplus}\colon \mathscr{D} \otimes \mathscr{D} &\longrightarrow \mathscr{D}^I \\ (E'', E_0) &\longmapsto [E_0 \to E_0 \oplus E'] \end{aligned} \tag{3.6}$$

which defines a section $\widetilde{\oplus}\colon R\mathfrak{M}_{\mathscr{D}} \times R\mathfrak{M}_{\mathscr{D}} \to R\mathfrak{M}_{\mathscr{D}^I}$ to $(f_0, c)\colon R\mathfrak{M}_{\mathscr{D}^I} \to R\mathfrak{M}_{\mathscr{D}} \times R\mathfrak{M}_{\mathscr{D}}$

Consider the relative tangent bundle $\mathbb{T}_{\widetilde{\oplus}}$ of $\widetilde{\oplus}$

Proposition 3.28 (Geometric incarnation of the RHom complex)**.** *There is an equivalence of perfect complexes on $R\mathfrak{M}_{\mathscr{D}} \times R\mathfrak{M}_{\mathscr{D}}$*

$$\mathbb{T}_{\widetilde{\oplus}} \simeq R\mathcal{H}om_{\mathscr{D}}$$

where $R\mathcal{H}om_{\mathscr{D}}$ is as in Definition 3.24. In particular, the fibre of $\mathbb{T}_{\widetilde{\oplus}}$ at a pair of $E'', E' \in \mathscr{D}$ is given by

$$\mathbb{T}_{\widetilde{\oplus}}|_{\{E''\}\times\{E'\}} \simeq \mathrm{RHom}_{\mathscr{D}}(E'', E').$$

Moreover there is a canonical equivalence of derived stacks over $\mathfrak{M}_{\mathscr{D}} \times \mathfrak{M}_{\mathscr{D}}$

$$\begin{tikzcd} \mathfrak{M}_{\mathscr{D}^I} \arrow[rr, dash, "\sim"] \arrow[dr, "q=(f_0,c)"'] & & \mathrm{Tot}_{\mathfrak{M}_{\mathscr{D}}\times\mathfrak{M}_{\mathscr{D}}}(R\mathcal{H}om_{\mathscr{D}}[1]) \arrow[dl, "\pi"] \\ & \mathfrak{M}_{\mathscr{D}} \times \mathfrak{M}_{\mathscr{D}} & \end{tikzcd}$$

where π is the canonical projection of the total space.

Proof sketch. The details in the case of where $\mathscr{D} = \mathrm{Perf}(X)$ is the category of perfect complexes on a quasi-projective variety are written down for example in [PS22, Proposition 3.6 and Corollary 3.7]

One first computes the fibre of the relative tangent bundle $\mathbb{T}_{(f_0,c)}$ over a (U-)point $[E' \to E \to E'']$ using the tangent fibre sequence

$$(f_0, c)^*\mathbb{T}_{R\mathfrak{M}_{\mathscr{D}}^{\times 2}} \longrightarrow \mathbb{T}_{\mathfrak{Exact}_{\mathscr{D}}} \longrightarrow \mathbb{T}_{\widetilde{\oplus}} \longrightarrow . \tag{3.7}$$

We find (see [Rob23] for details)

$$\mathbb{T}_{(c,s)}|_{E'\to E\to E''} \simeq \mathrm{RHom}_{\mathscr{D}}(E'',E')[1].$$

Using $\widetilde{\oplus}^*(f_0,c)^* = \mathrm{id}$ and considering the tangent fibre sequence for $\widetilde{\oplus}$ and the pullback along $\widetilde{\oplus}$ of (3.7) we conclude (using the octahedral 'axiom'/relation)

$$\widetilde{\oplus}^*\mathbb{T}_{(f_0,c)} \simeq \mathbb{T}_{\widetilde{\oplus}}[-1].$$

This proves the first part of the proposition.

To show the equivalence, we check that the functors of points over $R\mathfrak{M}_{\mathscr{D}} \times R\mathfrak{M}_{\mathscr{D}}$ agree. The total space $\mathrm{Tot}_{\mathfrak{M}_{\mathscr{D}}\times\mathfrak{M}_{\mathscr{D}}}(R\,\mathrm{Hom}_{\mathscr{D}}[1])$ parametrises sections of $\mathrm{RHom}_{\mathscr{D}}[1]$. These correspond to fibre sequences, hence to the points of $R\mathfrak{M}_{\mathscr{D}^I}$. ☐

Definition 3.29. Let $\mathscr{A}$ be an admissible, extension closed Abelian subcategory of a geometric dg category $\mathscr{D}$ satisfiying the openess in familes condition. Define *the derived stack* $R\mathfrak{Exact}_{\mathscr{A}}$ *of short exact sequences in* $\mathscr{A}$ by the homotopy Cartesian diagram

$$\begin{tikzcd} R\mathfrak{Exact}_{\mathscr{A}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & R\mathfrak{M}_{\mathscr{D}^I} \arrow[d, "{(s,t,c)}"] \\ R\mathfrak{M}_{\mathscr{A}} \times R\mathfrak{M}_{\mathscr{A}} \times R\mathfrak{M}_{\mathscr{A}} \arrow[r] & R\mathfrak{M}_{\mathscr{D}} \times R\mathfrak{M}_{\mathscr{D}} \times R\mathfrak{M}_{\mathscr{D}}. \end{tikzcd}$$

The classical truncation $\mathfrak{Exact}_{\mathscr{A}} = t_0(R\mathfrak{Exact}_{\mathscr{A}})$ is called the *stack of short exact sequences in* $\mathscr{A}$.

We define the *RHom complex on* $\mathfrak{M}_{\mathscr{A}} \times \mathfrak{M}_{\mathscr{A}}$ *to be the restriction of the RHom on* $R\mathfrak{M}_{\mathscr{D}} \times R\mathfrak{M}_{\mathscr{D}}$

$$R\mathcal{H}om_{\mathscr{A}} := R\mathcal{H}om_{\mathscr{D}}|_{\mathfrak{M}_{\mathscr{A}}} \in \mathrm{Perf}(\mathfrak{M}_{\mathscr{A}}).$$

By definition we have the correspondence

$$\begin{tikzcd} & R\mathfrak{Exact}_{\mathscr{A}} \arrow[dl, "{Rq:=(c,s)}"'] \arrow[dr, "{Rp:=t}"] & \\ R\mathfrak{M}_{\mathscr{A}} \times R\mathfrak{M}_{\mathscr{A}} & & R\mathfrak{M}_{\mathscr{A}} \end{tikzcd}$$

and its truncation

$$\begin{tikzcd} & \mathfrak{Exact}_{\mathscr{A}} \arrow[dl, "q"'] \arrow[dr, "p"] & \\ \mathfrak{M}_{\mathscr{A}} \times \mathfrak{M}_{\mathscr{A}} & & \mathfrak{M}_{\mathscr{A}}. \end{tikzcd}$$

We call q the *edge/side term map* and p the *middle term map*.

Corollary 3.30. *There is a commutative diagram*

$$\begin{tikzcd} \mathfrak{Exact}_{\mathscr{A}} \arrow[rr, dash, "\sim"] \arrow[dr, "q"'] & & \mathrm{Tot}_{\mathfrak{M}_{\mathscr{A}}\times\mathfrak{M}_{\mathscr{A}}}(\tau^{\leqslant 0}R\mathcal{H}om[1]) \arrow[dl, "\pi"] \\ & \mathfrak{M}_{\mathscr{A}} \times \mathfrak{M}_{\mathscr{A}} & \end{tikzcd}$$

Generalisation: stack of filtrations

Let $\mathscr{D}$ be a geometric dg category. Similar to the stack of short exact sequences, we can define stack of n-term filtrations. Since we only use the stack of three-term filtrations in this thesis, we focus on this case.

Let $I^{[3]}$ be the diagram category $0 \to 1 \to 2$. Let $\mathscr{D}^{[3]}$ be the dg category $\mathrm{Fun}(I^{[3]}, \mathscr{D})$. Similary to $\mathscr{D}^{[2]} := \mathscr{D}^{I}$, the dg category $\mathscr{D}^{[3]}$ is a geometric. We call its moduli stack $\mathrm{RFilt}^{[3]}_{\mathscr{D}} := \mathrm{R}\mathfrak{M}_{\mathscr{D}^{[3]}}$ the stack of three-term filtrations

There are natural functors

$$\begin{aligned} G_i\colon \mathscr{D}^{[3]} &\longrightarrow \mathscr{D} \\ [E_0 \to E_1 \to E_2] &\longmapsto E_i \end{aligned}$$

and

$$\begin{aligned} C_i\colon \mathscr{D}^{[3]} &\longrightarrow \mathscr{D} \\ [E_0 \to E_1 \to E_2] &\longmapsto \mathrm{cone}(E_i \to E_{i+1}) \end{aligned}$$

for all $i = 0, 1, 2$. There are intermedite forgetful functors

$$\begin{aligned} F_{ji}\colon \mathscr{D}^{[3]} &\longrightarrow \mathscr{D}^{I} \\ [E_0 \to E_1 \to E_2] &\longmapsto [E_i \to E_j] \end{aligned}$$

and

$$\begin{aligned} H_{ji}\colon \mathscr{D}^{[3]} &\longrightarrow \mathscr{D}^{I} \\ [E_0 \overset{\alpha_0}{\to} E_1 \overset{\alpha_1}{\to} E_2] &\longmapsto [\mathrm{cone}(\alpha_i) \to \mathrm{cone}(\alpha_j)] \end{aligned}$$

where $\alpha_2 = \alpha_1 \circ \alpha_0$. We have the commutative diagrams

$$\begin{tikzcd}
\mathscr{D}^{[3]} \arrow[r, "F_{20}"] \arrow[rr, bend left, "G_2"] \arrow[d, "G_2\times F_{10}"] \arrow[dd, bend right=60, "C_1\times C_0\times G_0"'] & \mathscr{D}^{[2]} \arrow[r, "F_1"] \arrow[d, "C\times F_0"] & \mathscr{D} \\
\mathscr{D}\times\mathscr{D}^{[2]} \arrow[r, "\mathrm{id}\times F_1"] \arrow[d, "\mathrm{id}\times C\times F_0"] & \mathscr{D}\times\mathscr{D} & \\
\mathscr{D}\times\mathscr{D}\times\mathscr{D} & &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathscr{D}^{[3]} \arrow[r, "H_{20}"] \arrow[rr, bend left, "G_2"] \arrow[d, "H_{21}\times G_0"] \arrow[dd, bend right=60, "C_1\times C_0\times G_0"'] & \mathscr{D}^{[2]} \arrow[r, "F_1"] \arrow[d, "C\times F_0"] & \mathscr{D} \\
\mathscr{D}^{[2]}\times\mathscr{D} \arrow[r, "F_1\times\mathrm{id}"] \arrow[d, "C\times F_0\times\mathrm{id}"] & \mathscr{D}\times\mathscr{D} & \\
\mathscr{D}\times\mathscr{D}\times\mathscr{D} & &
\end{tikzcd}$$

and taking moduli of objects we obtain the corresponding diagrams of derived stacks, where we denote with the corresponding lower case letter the morphisms on the moduli stacks induced by the functors,

$$\begin{tikzcd}
 & R\mathfrak{Filt}^{[3]}_{\mathscr{D}} \arrow[r] \arrow[d] \arrow[rr, bend left, "g_2"] \arrow[ddl, bend right, swap, "c_1 \times c_0 \times g_0"] & R\mathfrak{Exact}_{\mathscr{D}} \arrow[r] \arrow[d] & \mathscr{D} \\
 & R\mathfrak{Exact}_{\mathscr{D}} \times R\mathfrak{M}_{\mathscr{D}} \arrow[r] \arrow[dl] & R\mathfrak{M}_{\mathscr{D}} \times R\mathfrak{M}_{\mathscr{D}} & \\
R\mathfrak{M}_{\mathscr{D}} \times R\mathfrak{M}_{\mathscr{D}} \times R\mathfrak{M}_{\mathscr{D}} & & & .
\end{tikzcd}$$

Similar to the direct sum functor $\widetilde{\oplus}\colon \mathscr{D} \times \mathscr{D} \to \mathscr{D}^{[2]}$ in (3.6) we have functors

$$\begin{aligned}
\widetilde{\oplus}_{12}\colon \mathscr{D}^{[2]} \times \mathscr{D} &\longrightarrow \mathscr{D}^{[3]} \\
([E_1 \to E_2], E') &\longmapsto [E' \to E_0 \oplus E' \to E_1 \oplus E']
\end{aligned}$$

and

$$\begin{aligned}
\widetilde{\oplus}_{01}\colon \mathscr{D} \times \mathscr{D}^{[2]} &\longrightarrow \mathscr{D}^{[3]} \\
(E'', [E_0 \to E_1]) &\longmapsto E_0 \to E_1 \to E_1 \oplus E''.
\end{aligned}$$

Lemma 3.31. *There are equivalences of perfect complexes*

$$\begin{aligned}
\mathbb{T}_{\widetilde{\oplus}_{12}} &\simeq (\mathrm{id} \times (c, f_0))^* R\mathcal{H}om_{\mathscr{D}} \\
\mathbb{T}_{\widetilde{\oplus}_{01}} &\simeq ((c, f_0) \times \mathrm{id})^* R\mathcal{H}om_{\mathscr{D}}
\end{aligned}$$

Proposition 3.32.

(i) We have a commutative diagram of derived stacks where the top morphism is an equivalence

$$\begin{tikzcd}
\mathrm{RFilt}^{[3]}_{\mathscr{D}} \arrow[rr] \arrow[dr] & & \mathrm{Tot}_{\mathfrak{Exact}_{\mathscr{D}} \times R\mathfrak{M}_{\mathscr{D}}}(\mathbb{T}_{\widetilde{\oplus}_{01}}[1]) \arrow[dl, "\pi_0"] \\
 & R\mathfrak{Exact}_{\mathscr{D}} \times R\mathfrak{M}_{\mathscr{D}} &
\end{tikzcd}$$

(ii) We have a commutative diagram of derived stacks where the top morphism is an equivalence

$$\begin{tikzcd}
\mathrm{RFilt}^{[3]}_{\mathscr{D}} \arrow[rr] \arrow[dr] & & \mathrm{Tot}_{R\mathfrak{M}_{\mathscr{D}} \times R\mathfrak{Exact}_{\mathscr{D}}}(\mathbb{T}_{\widetilde{\oplus}_{12}}[1]) \arrow[dl, "\pi_1"] \\
 & R\mathfrak{M}_{\mathscr{D}} \times R\mathfrak{Exact}_{\mathscr{D}} &
\end{tikzcd}$$

As before for an admissible extenson closed Abelian subcategory $\mathscr{A} \subseteq \mathscr{D}$ satisfying the openness in families condition, we define the derived stack $\mathrm{R}\mathfrak{Filt}^{[3]}_{\mathscr{A}}$ of n-term filtrations and its truncation $\mathfrak{Filt}^{[3]}_{\mathscr{A}}$ via pullback to $\mathrm{R}\mathfrak{M}_{\mathscr{A}}$.

Let $\mathscr{D}$ be a geometric dg category. Similar to the stack of short exact sequences $\mathrm{R}\mathfrak{Exact}_{\mathscr{D}}$ we defined in the previous section, we can define the stack of three-term filtrations $\mathrm{R}\mathfrak{Filt}^{[3]}_{\mathscr{D}}$

3.5 GOOD MODULI SPACES

Definition 3.33 ([Alp13]). A morphism of stacks $\pi\colon \mathfrak{X} \to \mathcal{X}$ is a *good moduli space* if π is quasicompact, $\pi_*\colon \mathrm{QCoh}(\mathfrak{X}) \to \mathrm{QCoh}(\mathcal{X})$ is exact (these two properties together are known as *cohomologically affine*), and the adjunction morphism $\mathcal{O}_{\mathcal{X}} \to \pi_*\mathcal{O}_{\mathfrak{X}}$ is an isomorphism.

Example 3.34 (Good moduli spaces from affine GIT quotients [Alp13, Theorem 13.2]). Let G be a reductive group acting on an affine scheme $X = \mathrm{Spec}(A)$. Then the affinisation morphism

$$\pi\colon X/G \longrightarrow X/\!\!/G = \mathrm{Spec}(A^G)$$

is a good moduli space.

Example 3.35 (Moduli of Π_Q-modules done classically). Consider the representation space of $\underline{d}$-dimensional representations of the doubled quiver

$$R_{\overline{Q},\underline{d}} = \prod_{a \in \overline{Q}_1} \mathrm{Mat}(\underline{d}_{s(a)}, \underline{d}_{t(a)})$$

the gauge group

$$G_{Q,\underline{d}} = \prod_{i \in Q_0} \mathrm{GL}(\underline{d}_i)$$

acts on $R_{\overline{Q},\underline{d}}$ by conjugation. Explicitly, for $g = (g_i)_{i \in Q_0} \in G_{Q,\underline{d}}$ and $M = (M_a)_{a \in \overline{Q}_1}$ we have $gM = (g_{t(a)} M_a g^{-1}_{s(a)})_{a \in \overline{Q}_1}$. Consider the moment map

$$\begin{aligned} \mu_{Q,\underline{d}}\colon R_{\overline{Q},\underline{d}} &\longrightarrow \mathbb{A}^1 \\ (M_a)_{a \in \overline{Q}_1} &\longmapsto \sum_{a \in Q_1} [M_a, M_{a^*}]. \end{aligned} \tag{3.8}$$

It is invariant with respect to the gauge group action and hence the action restricts to the zero locus $\mu^{-1}_{Q,\underline{d}}(0)$. Using this geometry we obtain the quotient stack description

$$\mathfrak{M}_{\Pi_Q,\underline{d}} \simeq \mu^{-1}_{Q,\underline{d}}(0)/G_{Q,\underline{d}} \tag{3.9}$$

of the moduli stack of $\underline{d}$-dimensional Π_Q-representations. It follows from Example 3.34 that the affinisation

$$\mathrm{JH}\colon \mathfrak{M}_{\Pi_Q,\underline{d}} \longrightarrow \mathcal{M}_{\Pi_Q,\underline{d}} \coloneqq \mu^{-1}(0)/\!\!/G_{Q,\underline{d}}$$

is a good moduli space morphism.

Remark 3.36. The moment map description (3.9) naturally endows $\mathfrak{M}_{\Pi_Q}$ with a derived enhancement as a derived zero locus

$$\begin{tikzcd}
\overline{R}\mathfrak{M}_{\Pi_Q,\underline{d}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathfrak{M}_{\overline{Q},\underline{d}} \arrow[d, "\mu"] \\
\mathfrak{M}_{\overline{Q},\underline{d}} \arrow[r, "0"] & \mathfrak{g}_{Q,\underline{d}}/GL_{\underline{d}}.
\end{tikzcd}$$

The stack $\mathfrak{M}_{\Pi_Q}$ has a third derived enhancement as a (derived) cotangent stack

$$R^*\mathfrak{M}_{\Pi_Q} = T^*\,\mathfrak{M}_Q$$

By [BCS22, Example 4.17 and Corollary 6.12] the three derived enhancements $R\mathfrak{M}_{\Pi_Q,\underline{d}}$, $\overline{R}\mathfrak{M}_{\Pi_Q,\underline{d}}$ and $R^*\mathfrak{M}_{\Pi_Q,\underline{d}}$ are equivalent.

Example 3.37 (Good moduli spaces from quasi-projecitve GIT quotients [Alp13, Theorem 13.6])**.** Let G be a reductive algebraic group acting on a quasi-projective scheme X. Let $\mathcal{L}$ be a G-equivariant line bundle on X. The open subscheme of $\mathcal{L}$-semistable points $X^{ss} \subseteq X$ is consists of points $x \in X$ such that there is a G-invariant section of some power of $\mathcal{L}$ which does not vanish on x. The *GIT quotient* of X by G with respect to $\mathcal{L}$ is the morphism

$$X^{ss} \to X/\!\!/_{\mathcal{L}}G := \operatorname{Proj}\left(\bigoplus_{i\geqslant 0} H^0(X,\mathcal{L}^{\otimes i})^G\right).$$

By definition, it is a G-equivariant morphism (with trivial G-action on $X/\!\!/_{\mathcal{L}}G$) and thus factors through the quotient stack

$$\pi\colon X^{ss}/G \to X/\!\!/_{\mathcal{L}}G$$

The induced morphism π is a good moduli space morphism.

Examples 3.34 and 3.37 are the main methods we use to construct good moduli space morphisms.

Proposition 3.38 (Properties of good moduli spaces [Alp13])**.** *Let $\pi\colon \mathfrak{X} \to \mathcal{X}$ be a good moduli space morphism to a scheme $\mathcal{X}$.*

(i) π is surjective and $\mathcal{X}$ has the quotient topology.
(ii) π is universally closed.
(iii) π is initial with respect to maps from $\mathfrak{X}$ to schemes.
(iv) π has connected fibres.
(v) Every base change $\mathfrak{X}' \to \mathcal{X}'$ of π where $\mathcal{X}'$ is a scheme, is also a good moduli space morphism.

Proof. Items (i)-(iv) are part of [Alp13, Theorem 4.16]. Item (v) is part of [Alp13, Proposition 4.7]. □

3.6 2CY ABELIAN CATEGORIES WITH A GOOD MODULI THEORY

This section is based on the setup in [DHS22] and [DHS23].

Let $\mathscr{A}$ be an admissible extension closed Abelian subcategory of a geometric dg category $\mathscr{D}$ which satisfies the openness in families condition.

We begin by listing five assumptions one can make of the category $\mathscr{A}$ and its moduli theory.

Assumption 3.39 (Cohomological Hall algebra assumptions)**.**

(i) (Proper pushforward assumption) The middle term morphism $p\colon \mathfrak{Exact}_{\mathscr{A}} \to \mathfrak{M}_{\mathscr{A}}$ is proper.

(ii) (Virtual pullback assumption) The once shifted RHom complex $R\mathcal{H}om_{\mathscr{A}}[1]$ on $R\mathfrak{M}_{\mathscr{A}}$ is strictly $[-1,1]$-perfect. A consequence of this assumption is that the edge term morphism $q\colon R\mathfrak{Exact}_{\mathscr{A}} \to R\mathfrak{M}_{\mathscr{A}}^{\times 2}$ is globally presented in the sense of §A.2.

(iii) (Independence of presentation of $R\mathcal{H}om_{\mathscr{A}}$ assumption) The stack $\mathfrak{M}_{\mathscr{A}}$ has the *resolution property*: every coherent sheaf on $\mathfrak{M}_{\mathscr{A}}$ is the quotient of a vector bundle.

Assumption 3.40 (Good moduli space assumption)**.**

(i) The stack $\mathfrak{M}_{\mathscr{A}}$ has a good moduli space $\mathrm{JH}\colon \mathfrak{M}_{\mathscr{A}} \to \mathcal{M}_{\mathscr{A}}$.

(ii) The algebraic space $\mathcal{M}_{\mathscr{A}}$ is a separated scheme.

Proposition 3.41. *Suppose* $\mathrm{JH}\colon \mathfrak{M}_{\mathscr{A}} \to \mathcal{M}_{\mathscr{A}}$ *is a good moduli space. Then closed points of* $\mathfrak{M}_{\mathscr{A}}$ *and* $\mathcal{M}_{\mathscr{A}}$ *correspond to semisimple objects in* $\mathscr{A}$*.*

Proof. The proof of [AHH19, Lemma 7.18] adapts to our situation. In particular, the key ingredients are the correspondence between maps $f\colon \mathbb{A}^1/\mathbb{G}_m \to \mathfrak{M}_{\mathscr{A}}$ with $f(1) = E$ and filtrations of E and the classification of torsors on $\mathbb{A}^1/\mathbb{G}_m$. □

Suppose Assumption 3.40 is satisfied. Via the universal property of good moduli spaces Proposition 3.38 (iii) the composition of the good moduli space morphism with the direct sum morphism

$$\oplus\colon \mathfrak{M}_{\mathscr{A}} \times \mathfrak{M}_{\mathscr{A}} \longrightarrow \mathfrak{M}_{\mathscr{A}}$$

induces the direct sum morphism at the level of good moduli spaces

$$\oplus\colon \mathcal{M}_{\mathscr{A}} \times \mathcal{M}_{\mathscr{A}} \longrightarrow \mathcal{M}_{\mathscr{A}}. \tag{3.10}$$

Assumption 3.42 (Finiteness of $\oplus$ assumption)**.** Suppose Assumption 3.40 is satisfied. The direct sum morphism (3.10) is a finite morphism.

Assumption 3.43 (2-Calabi–Yau assumption)**.** For every collection of simple objects $F_1, \ldots, F_n \in \mathscr{A}$, the smallest dg subcategory $\mathscr{F} \subseteq \mathscr{D}$ containing these carries a right 2-Calabi–Yau structure.

Remark 3.44. By Proposition 3.11, if $\mathscr{D}$ itself carries a left 2-Calabi–Yau structure, then Assumption 3.43 is automatically satisfied.

Definition 3.45. We call $\mathscr{A} \subseteq \mathscr{D}$ a *2-Calabi–Yau (Abelian) category with a good moduli theory* if it satisfies Assumptions 3.39-3.43. We call $\mathscr{D}$ the *ambient dg category of* $\mathscr{A}$. If $\mathscr{D}$ itself carries a smooth 2-Calabi–Yau structure, then we say that $\mathscr{A}$ is a *globally* 2CY Abelian category with a good moduli theory.

Remark 3.46. All of the examples of 2CY Abelian categories in this thesis are globally 2CY. An example of a 2CY Abelian category which is not globally 2CY is the category of zero-dimensnional support sheaves on a smooth quasi-projective surface (see [DHS22, §11.1]).

Remark 3.47. Every 2-Calabi–Yau structure on $\mathscr{D}$ induces a symplectic structure on $R\mathfrak{M}_{\mathscr{D}}$ [BD21].

Example 3.48. We consider a projective resolution of the derived preprojective algebra $\mathscr{G}_2(Q)$. Consider the bimodules

$$P_0 = \bigoplus_{i \in Q_0} \Pi_Q e_i \otimes e_i \Pi_Q$$

$$P_1 = \bigoplus_{a \in \overline{Q}_1} \Pi_Q e_{t(a)} \otimes e_(s(a))\Pi_Q$$

There is an exact sequence

$$0 \longrightarrow P_0 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow \Pi_Q \longrightarrow 0.$$

Thus, for any finite dimensional Π_Q-module M

$$0 \to P_0 \otimes_{\Pi_Q} M \to P_1 \otimes_{\Pi_Q} M \to P_0 \otimes_{\Pi_Q} M \to M \to 0.$$

is a projective resolution of M to which we apply $\mathrm{Hom}_{\Pi_Q}(-, N)$ to compute the RHom complex $\mathrm{RHom}_{\Pi_Q}(M, N)$ as the complex

$$\mathrm{Hom}_{\Pi_Q}(P_0 \otimes M, N) \to \mathrm{Hom}_{\Pi_Q}(P_1 \otimes M, N) \to \mathrm{Hom}_{\Pi_Q}(P_0 \otimes M, N). \tag{3.11}$$

In the proof of [CK22, Proposition 2.6] is is shown that

$$\mathrm{Hom}_{\Pi_Q}(P_0 \otimes M, N) \cong \bigoplus_{i \in Q_0} \mathrm{Hom}_{\mathbb{C}}(e_i M, e_i N)$$

$$\mathrm{Hom}_{\Pi_Q}(P_1 \otimes M, N) \cong \bigoplus_{a \in \overline{Q}_1} \mathrm{Hom}_{\mathbb{C}}(e_{s(a)} M, e_{t(a)} N).$$

Thus the complex (3.11) depends only on the full subquiver supporting M and not the quiver Q itself. So in the ADE Dynkin case, we can pretend that the complex (3.11) came from a larger quiver, thus yielding the same computation of the RHom complex.

More generally, to compute the RHom complex $R\mathcal{H}om$ on $\mathfrak{M}_{\Pi_Q,\underline{d}'} \times \mathfrak{M}_{\Pi_Q,\underline{d}'}$ we consider the tautological of vector bundles $\mathcal{E}_i'', \mathcal{E}_i'$, for $i \in Q_0$ on $\mathfrak{M}_{\Pi_Q,\underline{d}''} \times \mathfrak{M}_{\Pi_Q,\underline{d}'}$, which record the underlying vector space of the representation of Π_Q at the vertex i.

Then $R\mathcal{H}om$ is isomorphic in $D^b(\mathfrak{M}_{\Pi_Q,\underline{d}'} \times \mathfrak{M}_{\Pi_Q,\underline{d}'})$ to the complex

$$\bigoplus_{i \in Q_0} \mathrm{Hom}(\mathcal{E}_i'', \mathcal{E}_i') \to \bigoplus_{a \in \overline{Q}_1} \mathrm{Hom}(\mathcal{E}_{s(a)}'', \mathcal{E}_{t(a)}') \to \bigoplus_{i \in Q_0} \mathrm{Hom}(\mathcal{E}_i'', \mathcal{E}_i')$$

In particular it is strictly $[-1,1]$-perfect. Thus Mod_{Π_Q} satisfies Assumption 3.39.

We summarise the discussion throughout this chapter on preprojective algebras in a theorem.

Theorem 3.49. *The category of modules over the preprojective algebra Π_Q of a quiver Q is a 2-Calabi–Yau Abelian category when viewed as an admissible Abelian subcategory of $\mathrm{Perf}(\mathscr{G}_2(Q))$, the category of perfect dg modules over the derived preprojective algebra.*

Proof. See Examples 3.5, 3.21, 3.35, and 3.48. The finiteness of the direct sum morphism follows from [MR19, Lemma 2.1]. □

Variant: Serre subcategories

A subcategory $\mathscr{B}$ of an Abelian category $\mathscr{A}$ is a *Serre subcategory* if it is closed under extensions in $\mathscr{A}$.

Suppose $\mathscr{A}$ is an Abelian category such that its stack of objects admits a good moduli space $\mathrm{JH}_{\mathscr{A}} \colon \mathfrak{M}_{\mathscr{A}} \to \mathcal{M}_{\mathscr{A}}$ (e.g. an Abelian category satisfying Assumption 3.40).

Consider a locally closed submonoid $\mathcal{M}_{\mathscr{B}}$. The closed points of $\mathcal{M}_{\mathscr{B}}$ correspond to semisimple objects on $\mathscr{A}$. The extension closure

$$\mathscr{B} := \langle B \in \mathscr{A} \mid [B] \in \mathcal{M}_{\mathscr{B}} \rangle_{\mathrm{ext}} \subseteq \mathscr{A}$$

of the category contataining these semisimple objects is by definition a Serre subcategory. We call Serre subcategories of $\mathscr{A}$ arrising in this way *good*. We define the *moduli stack of objects in $\mathscr{B}$* to be the fibre product of stacks

$$\begin{tikzcd}
\mathfrak{M}_{\mathscr{B}} \arrow[r] \arrow[d, "\mathrm{JH}_{\mathscr{B}}"] \arrow[dr, phantom, "\lrcorner", very near start] & \mathfrak{M}_{\mathscr{A}} \arrow[d, "\mathrm{JH}_{\mathscr{A}}"] \\
\mathcal{M}_{\mathscr{B}} \arrow[r, hook, "\iota_{\mathscr{B}}"] & \mathcal{M}_{\mathscr{A}}.
\end{tikzcd}$$

By base change for good moduli spaces (Proposition 3.38 (v)) $\mathrm{JH}_{\mathscr{B}}$ is a good moduli space morphism and we call $\mathcal{M}_{\mathscr{B}}$ the *good moduli space of objects in $\mathscr{B}$*.

Explicitly the functor of points of $\mathfrak{M}_{\mathscr{B}}$ is

$$R \mapsto \mathfrak{M}_{\mathscr{B}}(R) = \{E \in \mathfrak{M}_{\mathscr{A}}(R) \mid x^*E \in \mathscr{B} \text{ for all } x \colon \mathrm{pt} \hookrightarrow \mathrm{Spec}(R)\}$$

Example 3.50 (Nilpotent representations)**.** Consider the Abelian category $\mathrm{Rep}(\Pi_Q)$. A Π_Q-representation M is nilpotent if for all arrows $a \in \overline{Q}_1$ some power $a^n M = 0$, or equivalently, there is a flag of vector spaces $0 = M_0 \subseteq M_1 \subseteq \ldots \subseteq M_r = M$ such that $aM_i \subseteq M_{i-1}$.

The subcategory $\mathrm{Rep}^{\mathrm{Nil}}(\Pi_Q)$ of nilpotent representations is a Serre subcategory. The category $\mathrm{Rep}^{\mathrm{Nil}}(\Pi_Q)$ is the extension closure of the full subcategory containing the one dimensional representations S_i, $i \in Q_0$.

For each dimension vector $\underline{d} \in \mathbb{N}^{Q_0}$ the unique semisimple nilpotent representation is the $\underline{d}$-dimensional zero representation $0_{\underline{d}}$. The moduli space $\mathcal{M}^{\mathrm{Nil}}_{\Pi_Q} \subseteq \mathcal{M}_{\Pi_Q}$ of nilpotent semisimple representations is isomorphic to the discrete scheme $\mathbb{N}^{Q_0}$.

$$\begin{tikzcd}
\mathfrak{M}^{\mathrm{Nil}}_{\Pi_Q} \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathfrak{M}_{\Pi_Q} \arrow[d] \\
\mathbb{N}^{Q_0} \arrow[r, hook, "\iota_{\mathrm{Nil}}"] & \mathcal{M}_{\Pi_Q}
\end{tikzcd}$$

Example 3.51 (Strongly seminilpotent representations)**.**
A Π_Q-representation M is *strictly seminilpotent* if there is a flag $0 \subseteq M_0 \subseteq M_1 \subseteq \ldots M_r = M$ such that for each arrow $a \in Q_1$ we have $aM_i \subseteq M_{i-1}$ and $a^* M_i \subseteq M_i$. The category $\mathrm{Rep}^{\mathrm{SNN}}(\Pi_Q)$ of seminilpotent representations is a Serre subcategory of $\mathrm{Rep}(\Pi_Q)$.

Example 3.52 (Serre subcategory spanned by simple objects)**.** Let $\mathscr{A}$ be a 2CY category with a good moduli theory. Suppose $F_1, \ldots, F_n$ are pairwise nonisomorphic simple objects of $\mathscr{A}$. The extension closure $\mathscr{F} = \langle F_1, \ldots, F_n \rangle_{\mathrm{ext}} \subseteq \mathscr{A}$ of the full subcategory containing the objects F_i is by construction a Serre subcategory.

$$\begin{tikzcd}
\mathfrak{M}^{\mathscr{F}}_{\mathscr{A}} \arrow[r, hook] \arrow[d] & \mathfrak{M}_{\mathscr{A}} \arrow[d] \\
\mathbb{N}^{\{F_i\}} \arrow[r, hook] & \mathcal{M}_{\mathscr{A}}
\end{tikzcd}$$

3.7 THE LOCAL NEIGHBOURHOOD THEOREM

Let $\mathscr{D}$ be a dg category.

An object $F \in \mathscr{D}$ is a Σ*-object* if

$$\dim \mathrm{Ext}^k_{\mathscr{D}}(F, F) = \begin{cases} 1 & \text{if } k = 0, 2 \\ 2g & \text{if } k = 1 \\ 0 & \text{else} \end{cases}$$

A finite collection of objects $F_1, \ldots, F_n \subseteq \mathscr{D}$ is called a *simple-minded collection* if $\dim_{\mathbb{C}} \mathrm{Ext}^0_{\mathscr{D}}(F_i, F_j) = \delta_{ij}$ and is called a Σ*-collection* if it is a simple-minded collection of Σ-objects. Moreover we say that such a Σ-collection is *symmetric* if $\dim \operatorname{Ext}^1_{\mathscr{D}}(F_i, F_j) = \dim \operatorname{Ext}^1_{\mathscr{D}}(F_j, F_i)$.

The *Ext-quiver* $\overline{Q}_{\mathscr{F}}$ of a simple minded collection $F_1, \dots, F_n \subseteq \mathscr{D}$ is the quiver with vertices $(Q_{\mathscr{F}})_0 = \{F_1, \dots, F_n\}$ and with $\dim \operatorname{Ext}^1_{\mathscr{D}}(F_i, F_j)$ arrows from F_i to F_j.

The Ext-quiver of symmetric Σ-collection $F_1, \dots, F_n \subseteq \mathscr{D}$ is always the double of some (possibly non-unique) quiver $Q_{\mathscr{F}}$, which we call *a half Ext-quiver of* $F_1, \dots, F_n$.

To every semisimple object $\bigoplus_{i=1}^n F_i^{\oplus m_i}$ we associate the simple minded collection $\{F_1, \dots, F_n\}$ consisting of representatives of isomorphism classes of simple summands and the multiplicity vector $\underline{m} = (m_1, \dots, m_n)$ which defines a dimension vector of the Ext-quiver of $\{F_i\}$. We call $(\overline{Q}_{\{F_i\}}, \underline{m})$ *the Ext-quiver with dimension vector associated to* $\bigoplus_{i=1}^n F_i^{\oplus m_i}$.

Theorem 3.53 (The étale/analytic neighbourhood theorem for 2-Calabi–Yau categories, [Dav23b])**.** *Let $\mathscr{D}$ be a dg category and $R\mathfrak{M}_{\mathscr{D}}$ its derived moduli stack of objects. Consider an open substack $R\mathfrak{M} \subseteq R\mathfrak{M}_{\mathscr{D}}$ such that its truncation $\mathfrak{M} = t_0(R\mathfrak{M})$, is locally of finite type, has reductive stabiliser groups, and admits a good moduli space $\mathrm{JH}\colon \mathfrak{M} \to \mathcal{M}$. Let x be a closed point of $\mathfrak{M}$, corresponding to an object $F_x = \bigoplus_{i=1}^n F_i^{\oplus m_i}$ such that the collection $F_1, \dots, F_n$ is a simple-minded collection. Assume that the full dg subcategory $\mathscr{F} \subseteq \mathscr{D}$ containing the objects F_i carries a right 2-Calabi–Yau structure (which implies that $F_1, \dots, F_n$ is in fact a symmetric Σ-collection). Let $Q_{\mathscr{F}}$ be a half Ext-quiver of the collection $F_1, \dots, F_n$. Then there exists the data consisting of*

(i) a pointed $\mathrm{GL}_{\underline{m}}$-variety (H, y),

(ii) an étale/analytic neighbourhood $U_y \subseteq H$, whose image under the map $H \to H/\mathrm{GL}_{\underline{m}}$ and $H \to H /\!\!/ \mathrm{GL}_{\underline{m}}$ is denoted by $\mathfrak{U}_x$, respectively $\mathcal{U}_x$,

(iii) and a commutative diagram of pointed stacks

$$\begin{tikzcd}
(\mathfrak{M}_{\Pi_Q,\underline{m}}, 0_{\underline{m}}) \arrow[d, "\mathrm{JH}_{\Pi_Q}"] & (\mathfrak{U}_x, y) \arrow[l, "\tilde{\jmath}_{0_{\underline{m}}}"'] \arrow[r, "\tilde{\jmath}_x"] \arrow[d, "p"] & (\mathfrak{M}, x) \arrow[d, "\mathrm{JH}"] \\
(\mathcal{M}_{\Pi_Q,\underline{m}}, 0_{\underline{m}}) & (\mathcal{U}_x, p(y)) \arrow[l, "\jmath_{0_{\underline{m}}}"'] \arrow[r, "\jmath_x"] & (\mathcal{M}, \mathrm{JH}(x))
\end{tikzcd}$$

such that the horizontal morphisms are étale/analytic and the squares are cartesian

Definition 3.54. Let $\mathfrak{M}$ be a stack as in Theorem 3.53. For a closed point $x \in \mathfrak{M}$ we call the data *(i)*-*(iii)* in Theorem 3.53 an *étale/analytic Ext-quiver neighbourhood of* x.

Corollary 3.55. *Let $\mathscr{A}$ be a 2CY Abelian category with a good moduli theory. Then every closed point of its moduli stack of objects $\mathfrak{M}_{\mathscr{A}}$ admits an étale/analytic Ext-quiver neighbourhood.*

Proof. The hypotheses of Theorem 3.53 are implied by Assumptions 3.40 and 3.43. ☐

3.8 SHEAVES ON SYMPLECTIC SURFACES

The aim of this section is to verify Assumptions 3.39-3.43 for certain 2-Calabi–Yau categories of coherent sheaves on symplectic surfaces.

Let S be a smooth projective surface with $\omega_S \cong \mathcal{O}_S$. By the Enriques–Kodaira classification of surfaces, such an S is either a K3 surface or an Abelian surface.

The isomorphism $\Omega^2_S \cong \mathcal{O}_S$ defines a nonzero section $\sigma \in H^0(S, \Omega^2_S)$ which we can also view as a holomorphic symplectic form. A *symplectic surface* is a surface S together with a holomorphic symplectic form. Equivalenty, by Example 3.12, a symplectic surface S is a surface together with a left Calabi–Yau structure on $\mathrm{Perf}(S)$.

The following is another immediate corollary of Example 3.12.

Corollary 3.56. *Suppose* S *is a symplectic surface. Then every admissible Abelian subcategory of* $\mathrm{Perf}(S)$ *satisfies Assumption* 3.43.

In Example 3.7 we explained that $\mathrm{Perf}(S)$ is a geometric dg category and a dg enhancement of the bounded derived category of coherent sheaves $D^b(\mathrm{Coh}(S))$. In this section we always take the ambient dg category to be $\mathrm{Perf}(S)$.

Gieseker stability

The category of coherent sheaves $\mathrm{Coh}(S)$ on S famously does not satisfy Assumptions 3.39 and 3.40. Instead we consider smaller subcategories of Giesker semistable sheaves. We give a summary of Giesker stability. A standard reference for moduli of Gieseker semistable sheaves is [HL10].

Let X be a smooth projective variety and H an ample line bundle on X.

The Hilbert polynomial of a coherent sheaf $\mathcal{F}$ on X is the unique polynomial $P_{\mathcal{F}}(t)$ such that

$$P_{\mathcal{F}}(n) = H^0(X, \mathcal{F}(n)) = H^0(X, \mathcal{F} \otimes H^n) \text{ for all } n \gg 0.$$

The Hilbert polynomial of a sheaf generally depends on the choice of ample line bundle H.

The Hirzebruch–Riemann–Roch theorem states that for any coherent sheaf $\mathcal{F}$ on a smooth projective variety X we have

$$\chi(X, \mathcal{F}) = \int_X \mathrm{ch}(\mathcal{F})\,\mathrm{Td}(X) \tag{3.12}$$

where $\mathrm{Td}(X) \in H(X)$ is the Todd class of X. By Serre's theorem we have for all $n \gg 0$

$$H^0(X, \mathcal{F}(n)) := \chi(X, \mathcal{F}(n)).$$

In particular, from (3.12) it follows that the moduli stack $\mathfrak{M}_{\mathrm{Coh}(X)}$ decomposes as closed and open substacks of coherent sheaves with fixed Hilbert polynomials

$$\mathfrak{M}_{\mathrm{Coh}(X)} = \bigsqcup_{P \in \mathbb{Q}[t]} \mathfrak{M}_{\mathrm{Coh}(X),P}.$$

The *dimension of a coherent sheaf* $\mathcal{F}$ is the dimension of its support

$$\dim(\mathcal{F}) \coloneqq \dim(\mathrm{supp}(\mathcal{F})).$$

A coherent sheaf $\mathcal{F}$ is *pure* if its support is pure, i. e. all irreducible components of its support are of the same dimension $\dim(\mathcal{F})$.

If we write the Hilbert polynomial as

$$P_{\mathcal{F}}(t) = \sum_{k=0}^{\dim(\mathcal{F})} \frac{P_{\mathcal{F}}^{(k)}}{k!} t^k,$$

then the *reduced Hilbert polynomial* of a coherent sheaf $\mathcal{F}$ is

$$p_{\mathcal{F}}(t) \coloneqq \frac{P_{\mathcal{F}}(t)}{P_{\mathcal{F}}^{(\dim(\mathcal{F}))}}.$$

Definition 3.57. A coherent sheaf $\mathcal{F}$ is *(H-)Gieseker semistable* if it is pure and if for every proper nonzero subsheaf $\mathcal{F}' \subsetneq \mathcal{F}$

$$p_{\mathcal{F}'}(n) \leqslant p_{\mathcal{F}}(n) \text{ for all } n \gg 0$$

A coherent sheaf $\mathcal{F}$ *(H-Gieseker) stable* if the strict inequality holds for all proper nonzero subsheaves $\mathcal{F}' \subseteq \mathcal{F}$.

Let $\mathrm{Coh}^H(X)$ be the Abelian category of H-Gieseker semistable sheaves on X. Denote the moduli stack of objects in $\mathrm{Coh}^H(X)$ by $\mathfrak{M}_X^H$.

Ideally, we would consider the category $\mathrm{Coh}^H(X)$ as our 2-Calabi–Yau Abelian category. However, this category has the disadvantage that it is not extension closed in Coh(X) and hence not extension closed in Perf(X). Instead, we consider Abelian subcategories of $\mathrm{Coh}^H(X)$ which are extension closed in $\mathrm{Coh}^H(X)$.

Hilbert polynomials are additive in short exact sequences. Hence an extension of two semistable sheaves with the same reduced Hilbert polynomial $p(t)$ is also semistable and has reduced Hilbert polynomial $p(t)$. Let $\mathrm{Coh}^H_{p(t)}(X)$ be the subcategory of Giesker semistable sheaves with reduced Hilbert polynomial $p(t)$.

Pick an effective primitive class $v \in K_0^{\mathrm{num}}(\mathrm{Coh}^H(X))$. Then we define $\mathrm{Coh}^H_v(X) \subseteq \mathrm{Coh}^H(X)$. to be the category of H-Giesker semistable coherent sheaves on X of class $\in \mathbb{N}v$.

Theorem 3.58. *The category* $\mathrm{Coh}^H_{p(t)}(S) \subseteq \mathrm{Perf}(S)$ *is a 2-Calabi–Yau Abelian category with a good moduli theory.*

Proof. We refer to [DHS22, §6.1] for full proofs. The key points are the following:

- It suffices to prove the statements for the stacks $\mathfrak{M}^{H}_{S,P(t)}$ of Gieseker semistable sheaves with fixed Hilbert polynomials, since $\mathfrak{M}^{H}_{S} = \bigsqcup_{P(t)} \mathfrak{M}^{H}_{S,P(t)}$ decomposes into open and closed substacks.

- The stack of coherent sheaves $\mathfrak{M}_{S,p(t)}$ with a fixed reduced Hilbert polynomial $p(t)$ is isomorphic to the quotient of an open subscheme of a Quot scheme $\mathrm{Quot}_S(n, p(t)) \coloneqq \mathrm{Quot}_S(H^{-n} \otimes \mathbb{C}^{\oplus p(n)}, p(t))$ by $\mathrm{GL}_{p(n)}$.

- The locus of Gieseker semistable sheaves $\mathfrak{M}^{H}_{S,p(t)} \subseteq \mathfrak{M}_{S,p(t)}$ corresponds to the locus of semistable points with respect to a $\mathrm{GL}_{P(n)}$-equivariant linearisation.

- By Example 3.37 it follows that $\mathfrak{M}^{H}_{S,P(t)}$ admits a projective good moduli space. Thus direct sum map $\oplus\colon \mathcal{M}^{H}_{S} \times \mathcal{M}^{H}_{S} \to \mathcal{M}^{H}_{S}$ is a quasi-finite map of projective varieties hence finite.

- To show Assumption 3.39 is suffices to show that the restriction of the RHom complex $\mathrm{RHom}[1] \mid \mathfrak{M}^{H}_{S,P_1(t)} \times \mathfrak{M}^{H}_{S,P_2(t)}$ is strictly $[-1,1]$-perfect. This is done as follows. First, the RHom complex is equivalent to

  $$(\mathrm{pr}_{12})_* \, \mathrm{RHom}_{\mathfrak{M}^{H}_{S,P_1(t)} \times \mathfrak{M}_{S,P_2(t)}}((\mathrm{pr}^*_{13}\mathcal{U}_1, \mathrm{pr}^*_{23}\mathcal{U}_2)$$

  where $\mathcal{U}_i$ is the universal coherent sheaf on $\mathfrak{M}^{H}_{S,P_i(t)}$ and

  $$\mathrm{pr}_{ij}\colon \mathfrak{M}^{H}_{S,P_1(t)} \times \mathfrak{M}_{S,P_2(t)} \times S \to \mathfrak{M}^{H}_{S,P_i(t)}$$

  are the corresponding projections. Thus, if $\mathcal{U}_1$ admits a three-term resolution by locally free sheaves, then it follows that $\mathrm{RHom}[1]$ is strictly $[-1,1]$-perfect.

□

4

MIXED HODGE MODULES AND COHOMOLOGICAL HALL ALGEBRAS

"Magic isn't evil or good. Or black or white. It is like the universe, like every missing God. Powerful and supremely indifferent."

— Shehan Karunatilaka, *The Seven Moons of Maali Almeida*

Let $\mathscr{A}$ be a 2CY Abelian category of with a good moduli theory $\mathrm{JH}_{\mathscr{A}} \colon \mathfrak{M}_{\mathscr{A}} \longrightarrow \mathcal{M}_{\mathscr{A}}$. The guiding aim of the chapter is to make precise the statement (Theorem-Definition 4.52) that the pushforward of the dualising mixed Hodge module on the stack

$$\mathcal{A}_{\mathscr{A}} \coloneqq (\mathrm{JH}_{\mathscr{A}})_* \mathbb{D}\mathbb{Q}^{\mathrm{vir}}_{\mathfrak{M}_{\mathscr{A}}} \in \mathrm{D}^{\mathrm{b}}(\mathrm{MHM}(\mathcal{M}_{\mathscr{A}}))$$

is an algebra object, called *the relative cohomological Hall algebra (CoHA) of* $\mathscr{A}$, in the symmetric monoidal category $(\mathrm{D}^{\mathrm{b}}(\mathrm{MHM}(\mathcal{M}_{\mathscr{A}})), \boxdot)$.

The chapter begins with a recollection of Saito's theory of mixed Hodge modules. Then §4.3-4.6 review elements of the framework of [DHS22] for defining two-dimensional relative cohomological Hall algebras as mixed Hodge modules on good moduli spaces. Finally, in §4.7 we prove a vital compatibility result (Proposition 4.59) of CoHAs of Σ-collections with the nilpotent CoHAs of their Ext-quivers.

4.1 MIXED HODGE MODULES AND THE SIX FUNCTORS

The theory of mixed Hodge modules developed by Saito in [Sai88; Sai90] is the relative upgrade of Deligne's theory of polarisable mixed Hodge structures. We treat the category of mixed Hodge modules $\mathrm{MHM}(X)$ on a variety as a black-box. This and the following sections are meant to show the reader how to use this black-box.

We assume the reader is familiar with the theory of perverse sheaves and refer the uninitiated to de Cataldo and Migliorini's survey [dCM09] for an introduction and further references. The following list shows on the right-hand side objects of mixed Hodge theory and on the left-hand side their underlying objects in the constructible setting.

vector spaces ⇝ mixed Hodge structures
local systems ⇝ variations of Hodge structure
perverse sheaves ⇝ mixed Hodge modules
constructible complexes ⇝ mixed Hodge module complexes

Recall our convention that a variety is a reduced and separated scheme of finite type.

Saito assigns to every variety X a $\mathbb{Q}$-linear Abelian category of mixed Hodge modules $\mathrm{MHM}(X)$ and defines a six functor formalism on its bounded derived category $D^b(\mathrm{MHM}(X))$.

Example 4.1. The category of mixed Hodge modules on a point $\mathrm{MHM}(\mathrm{pt})$ is the category of polarisable mixed Hodge structures.

Remark 4.2. We sometimes consider mixed Hodge modules $\mathrm{MHM}(X)$ for X a separated scheme of finite type, which need not be reduced. In this case, we set $\mathrm{MHM}(X) \coloneqq \mathrm{MHM}(X_{\mathrm{red}})$ to be the category of mixed Hodge modules of the underlying reduced subscheme $X_{\mathrm{red}} \subseteq X$.

For every variety X there are bifunctors

$$\begin{aligned} - \otimes_X -\colon\ D^b(\mathrm{MHM}(X)) \times D^b(\mathrm{MHM}(X)) &\longrightarrow D(\mathrm{MHM}(X)) \\ \mathcal{H}om_X(-,-)\colon\ D^b(\mathrm{MHM}(X)) \times D^b(\mathrm{MHM}(X))^{\mathrm{op}} &\longrightarrow D^b(\mathrm{MHM}(X)) \end{aligned}$$

such that for every $\mathcal{H} \in D^b(\mathrm{MHM}(X))$ there is an adjunction

$$\mathcal{H} \otimes_X - \dashv \mathcal{H}om_X(\mathcal{H}, -)$$

and for every morphism $f\colon X \to Y$ of varieties there are pullback functors

$$\begin{aligned} f^*\colon\ D^b(\mathrm{MHM}(Y)) &\longrightarrow D^b(\mathrm{MHM}(X)) \\ f^!\colon\ D^b(\mathrm{MHM}(Y)) &\longrightarrow D^b(\mathrm{MHM}(X)) \end{aligned}$$

and pushforward functors

$$\begin{aligned} f_*\colon\ D^b(\mathrm{MHM}(X)) &\longrightarrow D^b(\mathrm{MHM}(Y)) \\ f_!\colon\ D^b(\mathrm{MHM}(X)) &\longrightarrow D^b(\mathrm{MHM}(Y)). \end{aligned}$$

These are are suitably compatible with composition. If $g\colon Y \to Z$ is further morphism, then we have

$$(g \circ f)^* = f^* g^*,\ (g \circ f)_* = g_* f_*,\ (g \circ f)^! = f^! g^!,\ (g \circ f)_! = g_! f_!.$$

For every $f\colon X \to Y$ we have adjoint pairs

$$\begin{aligned} f^* &\dashv f_* \\ f_! &\dashv f^!. \end{aligned} \tag{4.1}$$

For every morphism of varieties $f\colon X \to Y$ we have a natural transformation

$$f_! \longrightarrow f_*. \tag{4.2}$$

Remark 4.3. This natural transformation has a natural interpretation in the constructible category $D^b_c(X)$. The functor f_* is the usual (derived) pushforward of sections and $f_!$ is the (derived) pushforward of sections with compact support. The natural transformation (4.2) corresponds then to viewing sections with compact support as arbitrary sections.

If $j\colon U \hookrightarrow X$ is an open embedding, then $j^! = j^*$ (this is a special case of Proposition 4.5 (ii) below) and we have an adjoint triple

$$j_! \dashv j^* \dashv j_*.$$

If $i\colon Z \hookrightarrow X$ is a closed immersion, then $i_* = i_!$ (this is a special case of Proposition 4.5 (i) below) and we have an adjoint triple,

$$i^* \dashv i_* \dashv i^!.$$

Example 4.4. We have $\mathbb{Q}_X \cong f^*\mathbb{Q}_Y$ in $D^b(\mathrm{MHM}(X))$. In particular $\mathbb{Q}_X = a_X^*\mathbb{Q}$ for $a_X\colon X \to \mathrm{pt}$ the structure morphism.

More generally, we have for any polarisable mixed Hodge structure $H \in \mathrm{MHM}(\mathrm{pt})$ the constant mixed Hodge module complex $\mathbb{Q}_X \otimes H :=$ $a^*H \in D^b(\mathrm{MHM}(X))$.

There is an involutive anti-equivalence $\mathbb{D}$, called *Verdier duality*, i. e. a functor

$$\mathbb{D}\colon\ D^b(\mathrm{MHM}(X)) \to D^b(\mathrm{MHM}(X))^{\mathrm{op}}$$

satisfying

$$\mathbb{D}^2 \simeq \mathrm{id}.$$

Verdier duality swaps shrieks and stars:

$$\begin{aligned} \mathbb{D}f_! &= f_*\mathbb{D} \\ \mathbb{D}f^! &= f^*\mathbb{D}. \end{aligned} \tag{4.3}$$

The *dualising complex* is $\omega_X := \mathbb{D}\mathbb{Q}_X$. Verdier duality is a duality with ω_X as its dualising object, i. e. there is a natural isomorphism

$$\mathbb{D} \simeq \mathrm{Hom}_{D^b(\mathrm{MHM}(X))}(-, \omega_X).$$

By (4.3) we have for every morphism $f\colon X \to Y$

$$\mathbb{D}\mathbb{Q}_X = f^!\mathbb{Q}_Y.$$

In particular, the dualising sheaf for the point is the constant sheaf $\mathbb{D}\mathbb{Q} \cong \mathbb{Q}$ and we have $\mathbb{D}\mathbb{Q}_X = a^!\mathbb{Q}$ for the structure morphism $a\colon X \to$ pt.

Proposition 4.5. *Suppose* X, Y *are varieties.*

(i) For all proper morphisms $f\colon X \to Y$ *we have* $f_* = f_!$.

(ii) For all smooth morphisms $g\colon X \to Y$ *of relative dimension* $\dim(g)$ *we have* $g^! = g^* \otimes \mathbb{L}^{2\dim(g)/2}$ *(see Example 4.13).*

Theorem 4.6 (Base change for mixed Hodge modules [Sai90, (4.4.3)])**.** *Let*

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "g'"] \arrow[dr, phantom, "\lrcorner", very near start] & Y' \arrow[d, "g"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

be a Cartesian diagram of varieties. Then there are natural base change isomorphisms

$$f^* g_! \overset{\cong}{\to} g'_! (f')^*$$
$$g'_* (f')^! \overset{\cong}{\to} f^! g_*.$$

Proposition 4.7 (Cut-and-paste triangles)**.** *Let* X *be a variety. Let* $Z \subseteq X$ *be a closed subvariety and* $U := X \setminus Z$ *its open complement. Denote by* $\imath\colon Z \hookrightarrow X$ *and* $\jmath\colon U \hookrightarrow X$ *the inclusions. Then there are functorial distinguished triangles in* $D^b(\mathrm{MHM}(X))$

$$\imath_! \imath^! \mathcal{H} \longrightarrow \mathcal{H} \longrightarrow \jmath_* \jmath^* \mathcal{H} \longrightarrow$$
$$\jmath_! \jmath^! \mathcal{H} \longrightarrow \mathcal{H} \longrightarrow \imath_* \imath^* \mathcal{H} \longrightarrow .$$

We always consider $D^b(\mathrm{MHM}(X))$ as a triangulated category with a t-structure, given by the standard t-structure. There is a functor

$$\mathrm{rat}\colon D^b(\mathrm{MHM}(X)) \longrightarrow D^b_c(X)$$

which is t-exact with respect to the standard t-structure on $D^b(\mathrm{MHM}(X))$ and the perverse t-structure on $D^b_c(X)$. Hence rat restricts to a functor

$$\mathrm{rat}\colon \mathrm{MHM}(X) \longrightarrow \mathrm{Perv}(X).$$

Proposition 4.8 (Functoriality of the t-structure)**.** *Let* $f\colon X \to Y$ *be a morphism of varieties. Suppose the fibre dimensions of* f *are bounded above by* d, *i. e.* $\dim f^{-1}(y) \leqslant d$. *Then the various pullback and pushforward functors restrict to*

$$\begin{aligned} f^*\colon & D^{b,\leqslant 0}(\mathrm{MHM}(Y)) \longrightarrow D^{b,\leqslant d}(\mathrm{MHM}(X)) \\ f_*\colon & D^{b,\geqslant 0}(\mathrm{MHM}(X)) \longrightarrow D^{b,\geqslant -d}(\mathrm{MHM}(Y)) \\ f^!\colon & D^{b,\geqslant 0}(\mathrm{MHM}(Y)) \longrightarrow D^{b,\geqslant -d}(\mathrm{MHM}(X)) \\ f_!\colon & D^{b,\leqslant 0}(\mathrm{MHM}(X)) \longrightarrow D^{b,\leqslant d}(\mathrm{MHM}(Y)) \end{aligned}$$

Corollary 4.9.

(i) *Ff* f *is finite, then* $f_* = f_!$ *is t-exact.*

(ii) *If* f *is smooth of relative dimension* $\dim(f)$, *then* $f^* \otimes \mathbb{L}^{\otimes \dim(f)/2} = f^![-\dim(f)/2]$ *is t-exact. As a special case, if* f *is étale, then* $f^* = f^!$ *is exact.*

Weight filtration

Every complex of mixed Hodge modules $\mathcal{H} \in D^b(\mathrm{MHM}(X))$ has a *weight filtration*

$$W\mathcal{H}\colon \ldots \subseteq W^p\mathcal{H} \subseteq W^{p+1}\mathcal{H} \subseteq \ldots \subseteq \mathcal{H}.$$

A complex $\mathcal{H} \in D^b(\mathrm{MHM}(X))$ is *pure of weight* k if

$$W^p\mathcal{H} = \begin{cases} 0 & \text{if } p < k \\ \mathcal{H} & \text{if } p \geqslant k \end{cases},$$

i. e. the associated graded $\mathrm{Gr}_W\mathcal{H}$ is zero outside of degree k. The associated graded pieces functors

$$\begin{aligned} \mathrm{Gr}^i_W\colon D^b(\mathrm{MHM}(X)) &\longrightarrow D^b(\mathrm{MHM}(X)) \\ \mathcal{H} &\longmapsto W^i\mathcal{H}/W^{i-1}\mathcal{H} \end{aligned}$$

are triangulated functors. We call $\mathcal{H}$ *pure* if it is pure of weight zero.

The weight filtration $W\mathcal{H}^i(\mathcal{H})$ on the cohomology objects is defined by

$$W^k\mathcal{H}^i(\mathcal{H}) := \mathcal{H}^i(W^{i+k}\mathcal{H}).$$

The shift in the indexing is chosen so that for a pure weight zero (polarisable) cohomologically graded mixed Hodge structure $H^* \in D^b(\mathrm{MHM}(\mathrm{pt}))$, the ith piece H^i is a pure (ungraded) Hodge structure of weight i (c. f. Example 4.12).

Remark 4.10. The weight filtration and the notion of purity is the additional structure on mixed Hodge modules which makes them more powerful than perverse sheaves. The main tool being Saito's Decomposition Theorem, recalled as Theorem 4.27 below.

Example 4.11. Let X be a variety and let $a\colon X \to \mathrm{pt}$ be the structure morphism. Then the cohomology, homology, Borel–Moore homology, and compactly supported cohomology

$$\begin{gathered} H^*(X) = a_*a^*\mathbb{Q}, H_*(X) = a_!a^!\mathbb{Q}, \\ H^{BM}(X) = a_*a^!\mathbb{Q}, H_c(X) = a_!a^*\mathbb{Q} \end{gathered} \tag{4.4}$$

are objects in $D^b(\mathrm{MHM}(\mathrm{pt}))$. The weight filtrations of the cohomology objects of (4.4) Hodge structures agree with the ones defined by Deligne in [Del71; Del74].

If X is smooth and projective then $H^*(X) \cong H_*(X)$ is a pure (cohomologically graded) mixed Hodge structure.

Example 4.12. Let X be a smooth variety. Then $\mathbb{Q}_X[\dim_{\mathbb{C}} X] \in \mathrm{MHM}(X)$ is a mixed Hodge module pure of weight $\dim_{\mathbb{C}} X$.

There is a t-exact functor, called the *Tate twist*,

$$\begin{aligned} -(1)\colon \mathrm{D}^b(\mathrm{MHM}(X)) &\longrightarrow \mathrm{D}^b(\mathrm{MHM}(X)) \\ \mathcal{H} &\longmapsto \mathcal{H}(1) \end{aligned}$$

that increases weights by two, i. e. $W^i(\mathcal{H}(i)) = W^{i-2}(\mathcal{H})$. The shift functor $[-1]$ sends pure objects of weight k to pure objects of weight $k+1$.

The Tate twist is used to correct the shift in weights, so that for every shift by two $[-2]$ we can compensate the change in weight by a Tate twist (1).

Example 4.13. Let $\mathbb{L} \in \mathrm{D}^b(\mathrm{MHM}(\mathrm{pt}))$ denote the mixed Hodge structures $H_c(\mathbb{A}^1) = \mathbb{Q}[-2](1)$. As an object in $\mathrm{D}^b(\mathrm{MHM}(\mathrm{pt}))$, $\mathbb{L}$ is pure.

Thus $- \otimes \mathbb{L}$ shifts cohomological degrees by -2 and sends pure complexes of weight k to pure complexes of weight k. The preservation of purity and weight, while still shifting the cohomological degree is the reason why, when working with mixed Hodge modules, we always shfit by $- \otimes \mathbb{L}$ instead of by $[-2]$. Note that $\mathrm{rat}(\mathbb{L}) = \mathbb{Q}[-2]$.

Example 4.14. For a smooth variety X we have $\mathbb{D}\mathbb{Q}_X \cong \mathbb{Q}_X \otimes \mathbb{L}^{\dim(X)} = \mathbb{Q}_X[2\dim_{\mathbb{C}} X](\dim_{\mathbb{C}} X)$.

Remark 4.15. Suppose X is smooth. Notice that if $\dim_{\mathbb{C}} X$ is even, then $\mathbb{Q}_X \otimes \mathbb{L}^{\dim_{\mathbb{C}} X/2}$ is pure (of weight zero). We would like similarly to have a pure MHM even if $\dim_{\mathbb{C}} X$ is odd. This is achieved by constructing a square root $\mathbb{L}^{1/2}$ in the larger category of monodromic mixed Hodge modules. A thorough treatment can be found in [DM20, §2.1]. We have no need for this modification since we mostly consider even-dimensional varieties.

Proposition 4.16 (Functorialities of weights)**.** *Let $f\colon X \to Y$ and $g\colon Z \to X$ be morphisms of varieties.*

(i) If $\mathcal{F} \in \mathrm{D}^b(\mathrm{MHM}(X))$ has weights $\leqslant n$, then so do $f_!\mathcal{F}$ and $g^*\mathcal{F}$.
(ii) If $\mathcal{F} \in \mathrm{D}^b(\mathrm{MHM}(X))$ has weights $\geqslant n$, then so do $f_*\mathcal{F}$ and $g^!\mathcal{F}$.

In words we can say that $f_!$ and g^* at most decrease weights and f_* and $g^!$ at most increase weights.

Combining with Proposition 4.5 we have the following special cases.

Corollary 4.17. *Let $f\colon X \to Y$ and $g\colon Z \to X$ be morphisms of varieties.*

(i) If f is proper and $\mathcal{F} \in \mathrm{D}^b(\mathrm{MHM}(X))$ is pure of weight n, then $f_*\mathcal{F} = f_!\mathcal{F}$ is pure of weight n.
(ii) If g is étale and $\mathcal{F} \in \mathrm{D}^b(\mathrm{MHM}(X))$ is pure of weight n, then $g^*\mathcal{F} = g^!\mathcal{F}$ is pure of weight n.

4.1.1 *Variant: mixed Hodge modules for varieties with infinitely many connected components*

Suppose X is a reduced separated scheme with infinitely many connected components X_ν, each of which is of finite type.

Motivated by the equivalence $D^b(\mathrm{MHM}(Y \sqcup Z)) \simeq D^b(\mathrm{MHM}(Y)) \times D^b(\mathrm{MHM}(Z))$ we define the category *locally bounded derived category of MHMs* $D^{lb}(\mathrm{MHM}(X))$to be the product category

$$D^{lb}(\mathrm{MHM}(X)) = \prod_{X_\nu \in \pi_0(X)} D^b(\mathrm{MHM}(X_\nu)) \tag{4.5}$$

The category $D^{lb}(\mathrm{MHM}(X))$ carries the product t-structure and we define $\mathrm{MHM}(X)$ to be the heart of this t-structure.

All of the results of the previous section generalise by applying them on each factor of (4.5).

Remark 4.18. Objects in $D^{lb}(\mathrm{MHM}(X))$ need not have bounded cohomology. For example if $X = \bigsqcup_{d \geqslant 0} X_{2d}$ for 2d-dimensional smooth projective varieties X_{2d} then the constant mixed Hodge module $\mathbb{Q}_X \otimes \mathbb{L}^{\dim(X)}$ has cohomology in each even degree.

4.1.2 *Variant: unbounded above mixed Hodge modules*

As we have already seen in §2.1.1 not all of the mixed Hodge modules in this thesis that we encounter will be bounded.

Example 4.19. $H(B\mathbb{C}^\times) \cong \bigoplus_{n \geqslant 0} \mathbb{L}^{2n}$ is an unbounded above graded mixed Hodge structure. (The definition of the mixed Hodge structure on the cohomology of stacks such as $B\mathbb{C}^\times$ as in Definition 4.38.)

As in [Dav23b; DM20] we introduce the categories $D^+(\mathrm{MHM}(X))$ and $D^-(\mathrm{MHM}(X))$ of unbounded above and below complexes of mixed Hodge modules, so that it retains aspects of the six functor formalism of the locally bounded version.

Definition 4.20. The truncation functors

$$\tau^{\geqslant n} \colon D^{lb}(\mathrm{MHM}(X)) \longrightarrow D^{lb, \geqslant n}(\mathrm{MHM}(X))$$

define a sequence of categories

$$\dots \xrightarrow{\tau^{\geqslant n-1}} D^{lb}(\mathrm{MHM}(X)) \xrightarrow{\tau^{\geqslant n}} D^{lb}(\mathrm{MHM}(X)) \xrightarrow{\tau^{\geqslant n+1}} \dots$$

which we use to define

$$D^+(\mathrm{MHM}(X)) := \varprojlim_n \left(D^{lb, \geqslant n}(\mathrm{MHM}(X)) \to D^{lb, \geqslant n+1}(\mathrm{MHM}(X)) \right).$$

Objects of $D^+(\mathrm{MHM}(X))$ are $\mathbb{Z}$-tuples $(\mathcal{F}_n)_{n \in \mathbb{Z}} \in D^b(\mathrm{MHM}(X))^{\mathbb{Z}}$ and isomorphisms $\tau^{\geqslant n}\mathcal{F}_{n'} \cong \mathcal{F}_n$ for all $n < n'$ and morphisms are morphisms of the $\mathbb{Z}$-tuples which commute with the data of the isomorphisms in the natural way.

Dually, the truncation functors

$$\tau^{\leqslant n} \colon D^{lb}(MHM(X)) \longrightarrow D^{lb}(MHM(X))$$

define a sequence of categories

$$\ldots \xrightarrow{\tau^{\leqslant n-1}} D^{lb}(MHM(X)) \xrightarrow{\tau^{\leqslant n}} D^{lb}(MHM(X)) \xrightarrow{\tau^{\leqslant n-1}} \ldots$$

and we define $D^{-}(MHM(X))$ to be the colimit category.

The six functor formalism of §4.1 largely generalises to the categories $D^{\pm}(MHM(X))$. An important difference is that Verdier duality exchanges the two categories

$$\mathbb{D} \colon D^{\pm}(MHM(X)) \longrightarrow D^{\mp}(MHM(X)).$$

4.2 INTERSECTION COHOMOLOGY

In practice the main source of mixed Hodge modules are polarisable variations of Hodge structure.

Theorem 4.21 ([Sai88, Théorème 2])**.** *Let* X *be a smooth (analytic) variety. Then every polarisable variation of Hodge structures* $\mathcal{L}$ *of weight* w *defines a Mixed Hodge module* $\mathcal{L}[\dim_{\mathbb{C}} X]$ *of weight* $w + \dim_{\mathbb{C}} X$*.*

Definition 4.22 (Intermediate Extension)**.** Let X be an (analytic) variety. Let $j \colon U \hookrightarrow X$ be an (analytically) locally closed immersion. We use the natural transformation (4.2)

$$j_! \longrightarrow j_*$$

to define the *intermediate extension* $j_{!*}$ to be the functor

$$\begin{aligned} j_{!*} \colon D^b(MHM(U)) &\longrightarrow MHM(X) \\ \mathcal{F} &\longmapsto \mathcal{H}^0(\mathrm{im}(j_!\mathcal{F} \to j_*\mathcal{F})). \end{aligned}$$

Definition 4.23 (Supports)**.** Let $\mathcal{H} \in MHM(X)$ be a mixed Hodge module. The *supports of* $\mathcal{H}$ are the closed subvarieties $Z \subseteq X$ arising as the support of a simple subquotient of $\mathcal{H}$.

Definition 4.24 (Intersection Complex of a VHS)**.** Let X be a variety. Let $j \colon U \hookrightarrow X$ be an open dense inclusion of a smooth variety U. For every polarisable variation of Hodge structure $\mathcal{L}$ on U of we define *the intersection complex on* X *with coefficients in* $\mathcal{L}$ to be the mixed Hodge module $\mathcal{IC}(X, \mathcal{L}) \coloneqq j_{!*}\mathcal{L}[\dim_{\mathbb{C}} U]$

If $i \colon Z \hookrightarrow X$ is a closed immersion, then we often abuse notation and write $\mathcal{IC}(Z, \mathcal{L}) = i_*\mathcal{IC}(Z, \mathcal{L})$ for a VHS $\mathcal{L}$ defined on a smooth open dense subset $U \subseteq Z$.

Theorem 4.25. *Let* X *be a variety. Suppose* $\mathcal{L}$ *is a polarisable variation of Hodge structure defined on an open dense subvariety* $U \subseteq X$*. Then the intersection complex* $\mathcal{IC}(X, \mathcal{L})$ *is the unique mixed Hodge module on* X *such that*

(i) $\mathcal{IC}(X, \mathcal{L})$ *restricts to* $\mathcal{L}[\dim_{\mathbb{C}}(X)]$ *on* U
(ii) $\mathcal{IC}(X, \mathcal{L})$ *has no supports contained in* $X \setminus U$

Corollary 4.26. *Let* X *be a variety. Let* $j_U \colon U \hookrightarrow X$ *be an Zariski-open dense subset. Let* $\mathcal{L}$ *be a polarisable variation of Hodge structures on* U*. Let* $j_W \colon W \hookrightarrow X$ *be an analytic open embedding and let* $\overline{W}$ *denote the analytic closure of* W *in* X*. We have natural isomorphisms*

$$j_W^* \mathcal{IC}_X(\mathcal{L}) \cong \mathcal{IC}_{\overline{W}}(j_W^* \mathcal{L}).$$

Theorem 4.27 (Saito's decomposition theorem [Sai88])**.** *Let* X *be an algebraic variety.*

The subcategory $\mathrm{HM}(X) \subseteq \mathrm{MHM}(X)$ *of pure mixed Hodge modules is semisimple with simple objects given by intersection complexes* $\mathcal{IC}(Z, \mathcal{L})$ *for irreducible closed subvarieties* $Z \subseteq X$ *and simple polarisable variations of Hodge structure* $\mathcal{L}$*.*

Suppose that $\mathcal{F} \in D^b(\mathrm{MHM}(X))$ *is pure. Then there is a (non-canonical) isomorphism in* $D^b(\mathrm{MHM}(X))$

$$\mathcal{F} \cong \bigoplus_{i \in \mathbb{Z}} \mathcal{H}^i(\mathcal{F})[-i]$$

and each cohomology mixed Hodge module $\mathcal{H}^i(\mathcal{F})$ *is pure.*

Combining with the first assertion of the theorem, for every support $Z^{(i)} \subseteq X$ *of* $\mathcal{H}^i(\mathcal{F})$ *there is a semisimple polarisable variation of Hodge structure* $\mathcal{L}_{Z^{(i)}}$*, called the* monodromy of $\mathcal{F}|_{Z^{(i)}}$*, generically defined on* $Z^{(i)}$ *such that*

$$\mathcal{F} \cong \bigoplus_{i \in \mathbb{Z}} \bigoplus_{Z^{(i)}} \mathcal{IC}(Z^{(i)}, \mathcal{L}_{Z^{(i)}})[-i].$$

Theorem 4.28 (Deligne, Schmid)**.** *Let* X *be a smooth complex analytic variety. Then the underlying local system of a polarisable variation of Hodge structures* $\mathcal{L}$ *is semi-simple.*

Proof. See [Sch73, (7.25) Theorem], and [Del71, Théorème (4.2.6)] and its footnote. □

A simple mixed Hodge module need not have simple associated perverse sheaf. The mixed Hodge structure $H^1(E)$ for an elliptic curve E is simple, however as a vector space $H^1(E) \cong \mathbb{Q}^{\oplus 2}$ is not simple.

Nevertheless, by Theorem 4.28 rat does map semisimple objects to semisimple objects.

Corollary 4.29. *If* $\mathcal{H} \in D^b(\mathrm{MHM}(X))$ *is semisimple, then so is* $\mathrm{rat}(\mathcal{H}) \in D^b_c(X)$*.*

4.3 MIXED HODGE MODULES ON MONOID SCHEMES

A *monoid scheme* $(\mathcal{M}, \oplus)$ is a monoid object in the category of schemes. More explicitly, a monoid scheme is a scheme $\mathcal{M}$ together with a morphism $\oplus\colon \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ and a morphism $u\colon \mathrm{pt} \to \mathcal{M}$ satisfying the unital and associativity axioms. A monoid scheme $(\mathcal{M}, \oplus)$ is *commutative* if $\oplus \circ s = \oplus$ for the swapping factors morphism $s\colon \mathcal{M} \times \mathcal{M} \to \mathcal{M} \times \mathcal{M}$.

Let $(\mathcal{M}, \oplus)$ be a commutative monoid scheme, which is reduced and separated with connected components which are of finite type. We define a symmetric monoidal structure $\boxdot$ on the category $D^+(\mathrm{MHM}(\mathcal{M}))$ of bounded below complexes of mixed Hodge modules on $\mathcal{M}$ by the forumla

$$\mathcal{F} \boxdot \mathcal{G} \coloneqq \oplus_*(\mathcal{F} \boxtimes \mathcal{G}) \tag{4.6}$$

for $\mathcal{F}, \mathcal{G} \in D^+(\mathrm{MHM}(\mathcal{M}))$.

Suppose $\oplus\colon \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ is a finite morphism. Then

$$\oplus_*\colon\ D^+(\mathrm{MHM}(\mathcal{M} \times \mathcal{M})) \to D^+(\mathrm{MHM}(\mathcal{M}))$$

is t-exact and the same formula (4.6) defines a symmetric monoidal structure $\boxdot$ on $\mathrm{MHM}(\mathcal{M})$.

Example 4.30. Let $\mathcal{M}_{\mathscr{A}}$ be the good moduli space of a 2CY category $\mathscr{A}$ with a good moduli theory. The direct sum morphism $\oplus\colon \mathcal{M}_{\mathscr{A}} \times \mathcal{M}_{\mathscr{A}} \to \mathcal{M}_{\mathscr{A}}$ and the inclusion of connected component $\mathrm{pt} \cong \mathcal{M}_{\mathscr{A},0} \hookrightarrow \mathcal{M}_{\mathscr{A}}$ endows $\mathcal{M}_{\mathscr{A}}$ with the structure of a monoid scheme.

Suppose $\alpha\colon (\mathcal{M}, \oplus_{\mathcal{M}}) \to (\mathcal{N}, \oplus_{\mathcal{N}})$ is a morphism of commutative monoid schemes. Then the pushforward

$$\alpha_*\colon\ D^+(\mathrm{MHM}(\mathcal{M})) \longrightarrow D^+(\mathrm{MHM}(\mathcal{N}))$$

is a strict monoidal functor with respect to the $\boxdot$ monoidal structures.

Example 4.31. The monoid structure $\oplus\colon \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ descends to a monoid structure on the set of connected components $\pi_0(\mathcal{M})$, which we can view as a discrete scheme.

The category $D^+(\mathrm{MHM}(\pi_0(\mathcal{M})))$ with the $\boxdot$-monoidal structure is equivalent to the monoidal category of $\pi_0(\mathcal{M})$-graded objects in $D^+(\mathrm{MHM}(\mathrm{pt}))$ with the usual monoidal structure $\otimes$ of graded objects given by

$$\left(\bigoplus_{\nu \in \pi_0(\mathcal{M})} V_\nu\right) \otimes \left(\bigoplus_{\nu \in \pi_0(\mathcal{M})} W_\nu\right) = \bigoplus_{\nu \in \pi_0(\mathcal{M})} \bigoplus_{\nu' + \nu'' = \nu} V_{\nu'} \otimes V_{\nu''}.$$

The natural morphism $\mathcal{M} \to \pi_0(\mathcal{M})$ is a monoid morphism.

We fix for the rest of this section a reduced and separated commutative monoid scheme $(\mathcal{M}, \oplus)$, whose connected components are of finite type and such that $\oplus\colon \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ is a finite morphism.

In any symmetric monoidal category one can define algebra objects and Lie algebra objects. The algebra objects and Lie algebra objects in symmetric monoidal categories $D^+(\mathrm{MHM}(\mathcal{M}))$ and $\mathrm{MHM}(\mathcal{M})$ play a fundamental role in this thesis.

Let $\mathscr{M} \in \{(D^+(\mathrm{MHM}(\mathcal{M})), \boxdot), (\mathrm{MHM}(\mathcal{M}), \boxdot)\}$. The forgetful functor

$$\{\text{algebras in } \mathscr{M}\} \to \mathscr{M}$$

has a left adjoint $\mathrm{Free}_{\boxdot\text{-Alg}}$ which takes an object $\mathcal{G}$ to the algebra object

$$\mathrm{Free}_{\boxdot\text{-Alg}}(\mathcal{G}) = \bigoplus_{n\geqslant 0} \mathcal{G}^{\boxdot n} = \bigoplus_{n\geqslant 0} \oplus_*(\mathcal{G}^{\boxtimes n}) \qquad (4.7)$$

with product uniquely determined by the isomorphisms

$$\mathcal{G}^{\boxdot n} \boxdot \mathcal{G}^{\boxdot m} \xrightarrow{\cong} \mathcal{G}^{\boxdot(n+m)}.$$

Proposition 4.32. *Let $(\mathcal{M}, \oplus)$ be a reduced and separated commutative monoid scheme, whose connected components are of finite type. Assume furthermore that $\oplus\colon \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ is a finite morphism.*

Suppose $\mathcal{G} \in \mathrm{MHM}(\mathcal{M})$ is pure of weight zero. Then so is $\mathrm{Free}_{\boxdot\text{-Alg}}(\mathcal{G})$ pure of weight zero.

Proof. This is an application of Propositions 4.8 and 4.16. The category of pure (weight zero)mixed Hodge modules is closed external products and direct sums. By assumption, $\oplus$ is finite and hence $\oplus_*$ sends pure weight zero mixed Hodge modules to pure weight zero mixed Hodge modules. Purity of $\mathrm{Free}_{\boxdot\text{-Alg}}(\mathcal{G})$ immediately follows from the explicit formula (4.7) in terms of external products, direct sums and pushforward along $\oplus$ of pure weight zero objects. ☐

There is a forgetful functor

$$\{\text{algebras in } \mathscr{M}\} \longrightarrow \{\text{Lie algebras in } \mathscr{M}\} \qquad (4.8)$$

which sends an algebra $(\mathcal{A}, m)$ to the Lie algebra given by its commutator bracket

$$[-,-]_{\mathcal{A}} \coloneqq (m - m \circ s_*)\colon \mathcal{A} \boxdot \mathcal{A} \longrightarrow \mathcal{A}.$$

In a tensor category, i. e. an Abelian category with a compatible symmetric monoidal structure, we have the additional notion of ideals of algebra objects and quotients by these ideals.

Fix $\mathscr{M}$ to be the tensor category $(\mathrm{MHM}(\mathcal{M}), \boxdot)$. A *two-sided ideal $\mathcal{I}$ of an algebra object $\mathcal{A}$ in $\mathscr{M}$* is a subobject $\mathcal{I} \subseteq \mathcal{A}$ stable under multiplication by $\mathcal{A}$ on both sides, i. e. both morphisms

$$m\colon \mathcal{I} \times \mathcal{A} \longrightarrow \mathcal{A}$$
$$m\colon \mathcal{A} \times \mathcal{I} \longrightarrow \mathcal{A}$$

factor through $\mathcal{I}$.

Let $(\mathcal{L},[-,-]_{\mathcal{L}})$ be a Lie algebra object in $\mathscr{M}$. We define the *universal enveloping algebra* $\mathrm{U}(\mathcal{L})$ *of* $\mathcal{L}$ to be the quotient

$$\mathrm{U}(\mathcal{L}) \coloneqq \mathrm{Free}_{\boxdot\text{-Alg}}(\mathcal{L})/\langle [-,-]_{\mathrm{Free}_{\boxdot\text{-Alg}}(\mathcal{L})} - [-,-]_{\mathcal{L}}\rangle.$$

of the free algebra on $\mathcal{L}$ be the two-sided ideal of $\mathrm{Free}_{\boxdot\text{-Alg}}(\mathcal{L})$ generated by the image of the morphism

$$[-,-]_{\mathrm{Free}_{\boxdot\text{-Alg}}(\mathcal{L})} - [-,-]_{\mathcal{L}} \colon \mathcal{L} \boxdot \mathcal{L} \longrightarrow \mathrm{Free}_{\boxdot\text{-Alg}}(\mathcal{L}).$$

The universal enveloping algebra defines a left adjoint to (4.8). The *free Lie algebra* $\mathrm{Free}_{\boxdot\text{-Lie}}(\mathcal{G})$ *of an object* $\mathcal{G}$ in $\mathscr{M}$ is defined to be the Lie subalgebra of $(\mathrm{Free}_{\boxdot\text{-Alg}}(\mathcal{G}),[-,-]_{\mathrm{Free}_{\boxdot\text{-Alg}}(\mathcal{G})})$ generated by $\mathcal{G} \subseteq \mathrm{Free}_{\boxdot-\mathrm{Alg}}(\mathcal{G})$. Taking the free Lie algebra defines a left adjoint to the forgetful functor

$$\{\text{Lie algebras in } \mathscr{M}\} \longrightarrow \mathscr{M}.$$

4.4 MIXED HODGE MODULES ON STACKS

We begin by summarising the theory of perverse sheaves on Artin stacks.

A fundamental result in the theory of perverse sheaves, is that the categories of perverse sheaves form a stack in the smooth topology on complex algebraic varieties. This motivates the definition of the categories of perverse sheaves for Artin stacks: Artin stacks are stacks in the smooth topology and therefore a working definition for the category of perverse sheaves is as the gluing of categories in the smooth topology.

In [LO08a; LO08b; LO09; Sun17] the authors develop a six functor formalism for constructible sheaves on Artin stacks, which allows the authors to realise the category of perverse sheaves as the heart of a t-structure on the category of constructible cohomology complexes $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(\mathfrak{X})$.

The analogue of this theory for mixed Hodge modules has not yet been developed. Despite being able to define the category of mixed Hodge modules on a quotient stack $\mathfrak{X} = X/G$ as the category of G-equivariant mixed Hodge modules, it remains a subtle task to give a good definition of the bounded derived category of mixed Hodge modules $\mathrm{D}^{\mathrm{b}}(\mathrm{MHM}(\mathfrak{X}))$ with all of the six functors. The difficulty already appears for the constructible derived category: we no longer have $\mathrm{D}^{\mathrm{b}}(\mathrm{Perv}(\mathfrak{X})) \cong \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(\mathfrak{X})$. Indeed, it fails for $\mathrm{B}\mathbb{G}_m$ (see [LO09, Remark 1.1]). For a thorough treatment of mixed Hodge modules on quotient stacks we refer the reader to [Ach13].

We implement ad-hoc definitions (following [Dav23b; DHS22]) as workarounds for the abscence of the full six-functor formalisms for mixed Hodge modules on Artin stacks.

Notation 4.33. For a finite type Artin stack $\mathfrak{X}$ we write $\mathrm{rat}(\mathbb{D}\mathbb{Q}_{\mathfrak{X}}) \in D^b_c(\mathfrak{X})$ for the dualising constructible sheaf, even though we haven't defined the (complex of) mixed Hodge modules $\mathbb{D}\mathbb{Q}_{\mathfrak{X}}$.

Definition 4.34. An *acyclic cover* of a finite type Artin stack $\mathfrak{X}$ is a sequence of smooth morphisms $f_n \colon X_n \to \mathfrak{X}$ from reduced and separated schemes X_n such that

$$\min\left\{i \in \mathbb{Z} \mid {}^{\mathfrak{p}}\mathcal{H}^i(\mathrm{cone}(\mathrm{rat}(\mathbb{D}\mathbb{Q}_{\mathfrak{X}}) \to \mathrm{rat}((f_n)_*\mathbb{D}\mathbb{Q}_{X_n})) \neq 0)\right\} \xrightarrow[n\to\infty]{} \infty.$$

The following example can be viewed as an algebraic version of the Borel construction.

Example 4.35 (Acyclic covers for quotient stacks)**.** Let X be a separated scheme and G an affine algebraic group acting on X. We exhibit an acyclic cover for the quotient stack X/G.

Choose an embedding $G \hookrightarrow GL_n$. For every $N > 0$ the group GL_n, hence also G, acts on $V_N := \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^N, \mathbb{C}^n)$ via the defining action on $\mathbb{C}^n$. Consider the open dense subvariety $S_N \subseteq V_N$ consisting of surjective linear maps. The subvariety S_N is stable under the GL_n-action and GL_n acts freely on S_N. The group G acts on $X \times S_N$ via the diagonal action. Set $X_N := (X \times S_N)/G$. By [EG98, Proposition 23] the stacks X_N are schemes and by the argument in [DM20, Proposition 2.16] the natural morphisms $X_N \to X/G$, $N > 0$ are an acyclic cover for X/G.

Example 4.36. Moduli stacks of objects in 2CY categories with a good moduli theory admit an acyclic cover by Assumption 3.39 (iii) and Example 4.35.

Example 4.37 (Pullback acyclic covers)**.** Suppose $\mathfrak{X}$ is a finite type Artin stack admitting an acyclic cover $\{f_n \colon X_n \to \mathfrak{X}\}$. Then for every representable finite type morphism $f \colon \mathfrak{Y} \to \mathfrak{X}$, the pullback of the cover $\{\mathfrak{X} \times_{\mathfrak{X}} X_n \to \mathfrak{Y}\}$ is an acyclic cover for $\mathfrak{Y}$.

The purpose of acyclic covers is to approximate cohomological invariants of stacks $\mathfrak{X}$ by cohomological invariants of schemes X_n.

For every acyclic cover $(f_n)_n$ of $\mathfrak{X}$ and every morphism $g \colon \mathfrak{X} \longrightarrow Y$ we have for all i and $n \gg 0$

$${}^{\mathfrak{p}}\tau^{\leqslant i} g_*\mathbb{D}\mathbb{Q}_{\mathfrak{X}} \cong {}^{\mathfrak{p}}\tau^{\leqslant i}(g \circ f_n)_*\mathbb{D}\mathbb{Q}_{X_n}.$$

This key observation is what allows us to avoid directly dealing with mixed Hodge modules on stacks, at the expense of always pushing forward to a scheme (Notation 4.41).

Definition 4.38. Let $\mathfrak{X}$ be a finite type Artin stack admitting an acyclic cover. For every morphism $b \colon \mathfrak{X} \to B$ to a variety B we define *the pushforward of the dualising sheaf along* a to be the object

$$b_*\mathbb{D}\mathbb{Q}_{\mathfrak{X}} := \left(\tau^{\leqslant i}(b \circ f_{N(i)})_*\mathbb{D}\mathbb{Q}_{\mathfrak{X}_{N(i)}}\right)_{i\in\mathbb{Z}} \in D^+(\mathrm{MHM}(B))$$

for a choice of acyclic cover $\{f_n \colon X_n \to \mathfrak{X}\}_n$ and for $N(i) \gg 0$ (more precisely for $N(i)$ such that ${}^{p}\mathcal{H}^{j}(\mathrm{cone}(\mathbb{DQ}_{\mathfrak{X}} \to (f_n)_*\mathbb{DQ}_{\mathfrak{X}})) = 0$ for all $j \leqslant i$).

Using pullback acyclic covers as in Example 4.37 for a representable morphism $f \colon \mathfrak{Y} \to \mathfrak{X}$ of finite type Artin stacks admitting acyclic covers, we define the pushforwards modules $f_!\mathbb{DQ}_{\mathfrak{Y}}$, $f_*\mathbb{DQ}_{\mathfrak{Y}}$ and pullbacks $f^*\mathbb{DQ}_{\mathfrak{X}}$, $f^!\mathbb{DQ}_{\mathfrak{X}}$.

Remark 4.39. By passing to a common refinement (by applying Example 4.37), we deduce that the definition of $a_*\mathbb{DQ}_{\mathfrak{X}}$ is independant of the choice of acyclic cover.

Remark 4.40. Constructions involving mixed Hodge modules on stacks $\mathfrak{X}$ are done by performing the construction at each level X_n of an acyclic cover, then pushing forward to a scheme and taking the limit $n \to \infty$.

For constructions involving (morphisms between) multiple stacks, pullback acyclic covers (Example 4.37) are very useful.

Notation 4.41. Throughout Chapter 4 all of the finite type Artin stacks are assumed to admit an acyclic cover as in Definition 4.34.

When working with mixed Hodge modules on stacks $\mathfrak{X}$ we avoid constantly choosing a morphism $\mathfrak{X} \to Z$ to a variety Z and pushing forward by simply writing $\mathbb{DQ}_{\mathfrak{X}}$. The motivated reader is invited to translate everything to the language of pushforwards from acyclic covers (use Remark 4.40).

Remark 4.42. In the case of moduli stacks of objects $\mathfrak{M}_{\mathscr{A}}$ in a 2CY category with a good moduli theory, we usually pushforward to the good moduli space $\mathfrak{M}_{\mathscr{A}} \to \mathcal{M}_{\mathscr{A}}$ (c.f. Proposition 3.38 (iii)).

4.5 PROPER PUSHFORWARDS AND VIRTUAL PULLBACKS

Let $f \colon \mathfrak{X} \to \mathfrak{Y}$ be a proper representable morphism of finite type Artin stacks admitting acyclic covers. The counit of $f_! \dashv f^!$ applied to the dualising mixed Hodge module is the morphism $f_!f^!\mathbb{DQ}_{\mathfrak{Y}} \to \mathbb{DQ}_{\mathfrak{Y}}$.

Definition 4.43 (Proper pushforward)**.** Using the properness of f so that $f_*\mathbb{DQ}_{\mathfrak{X}} \cong f_!\mathbb{DQ}_{\mathfrak{X}}$ we obtain the *proper pushforward*

$$f_\star \colon f_*\mathbb{DQ}_{\mathfrak{X}} \longrightarrow \mathbb{DQ}_{\mathfrak{X}}.$$

Let $f \colon \mathfrak{X} \to \mathfrak{Y}$ be a smooth morphism of Artin stacks admitting acyclic covers of relative dimension $\dim(f)$. Since f is smooth we have the canonical isomorphism $f^*\mathbb{DQ}_Y \cong \mathbb{DQ}_X \otimes \mathbb{L}^{-2\dim(f)/2}$.

Definition 4.44 (Smooth pullback)**.** Combining with the unit $\mathrm{id} \to f_*f^*$ we obtain the *smooth pullback along* f

$$f^\dagger \colon \mathbb{DQ}_Y \longrightarrow f_*f^*\mathbb{DQ}_Y \cong f_*\mathbb{DQ}_X \otimes \mathbb{L}^{-2\dim(f)/2}.$$

Let $\mathfrak{X}$ be an Artin stack admitting an acyclic cover. Suppose $\mathcal{E}$ is a locally free sheaf on $\mathfrak{X}$. Let $\mathfrak{E} := \mathrm{Tot}_{\mathfrak{X}}(\mathcal{E})$ be its total space and let $\pi\colon \mathfrak{E} \to \mathfrak{X}$ be the projection. Let $s\colon \mathfrak{X} \to \mathrm{Tot}_{\mathfrak{X}}(\mathcal{E})$ be a section and consider the pullback diagram

$$\begin{tikzcd}
\mathfrak{X} \arrow[r, "0"] & \mathfrak{E} \\
\mathfrak{X}_s \arrow[u, "\iota_s"] \arrow[r, "\iota_s"] \arrow[ur, phantom, "\lrcorner"] & \mathfrak{X}. \arrow[u, "s"']
\end{tikzcd} \tag{4.9}$$

Using that s is a section and the smoothness of π so that $\pi^* \cong \pi^!(-) \otimes \mathbb{L}^{-2r/2}$ we have a natural isomorphism

$$\mathbb{D}\mathbb{Q}_{\mathfrak{X}} \cong s^*\mathbb{D}\mathbb{Q}_{\mathfrak{E}} \otimes \mathbb{L}^{-2\,\mathrm{rk}\,\mathcal{E}/2}. \tag{4.10}$$

Applying the adjunction $s^* \dashv s_*$ to (4.10) we obtain the morphism

$$\mathbb{D}\mathbb{Q}_{\mathfrak{E}} \longrightarrow s_*\mathbb{D}\mathbb{Q}_{\mathfrak{X}} \otimes \mathbb{L}^{-2\,\mathrm{rk}\,\mathcal{E}/2}$$

Definition 4.45 (Refined Gysin pullback)**.** Applying $0^!_{\mathcal{E}}$ and the base change isomorphism $0^!_{\mathcal{E}}s_* \simeq (\iota_s)_*\iota_s^*$ from the diagram (4.9) we obtain the *refined Gysin pullback* along ι_s

$$\iota_s^\dagger\colon \mathbb{D}\mathbb{Q}_{\mathfrak{X}} \longrightarrow (\iota_s)_*\mathbb{D}\mathbb{Q}_{\mathfrak{X}} \otimes \mathbb{L}^{-2\,\mathrm{rk}\,\mathcal{E}/2}.$$

4.5.0.1 *Virtual pullbacks for strictly* $[-1,1]$*-perfect complexes*

Definition 4.46. A perfect complex $\mathcal{C}$ on an Artin stack $\mathfrak{M}$ is *strictly* $[a,b]$*-perfect* if it is equivalent to a complex of locally free sheaves of the form

$$0 \longrightarrow \mathcal{C}^a \longrightarrow \mathcal{C}^{a+1} \longrightarrow \ldots \longrightarrow \mathcal{C}^{b-1} \longrightarrow \mathcal{C}^b \longrightarrow 0.$$

called a *locally free presentation*.

Fix a locally free presentation $\mathcal{C} \simeq (\mathcal{C}^{-1} \to \mathcal{C}^0 \overset{d_0}{\to} \mathcal{C}^1)$ of a strictly $[-1,1]$-perfect complex $\mathcal{C}$ on $\mathfrak{M}$. Consider its total space $\mathrm{Tot}_{\mathfrak{M}}(\mathcal{C})$, which in general is a derived stack. Its truncation $t_0(\mathrm{Tot}_{\mathfrak{M}}(\mathcal{C}))$ is equivalent to $\mathrm{Tot}_{\mathfrak{M}}(\tau^{\leqslant 0}\mathcal{C})$, the total space of the truncation of the complex.

The projection $\mathrm{Tot}_{\mathfrak{M}}(\tau^{\leqslant 0}\mathcal{C}) \to \mathfrak{M}$ is representable hence $\mathrm{Tot}_{\mathfrak{M}}(\tau^{\leqslant 0}\mathcal{C})$ has an acyclic cover if $\mathfrak{X}$ does. The monomorphism of complexes $\tau^{\leqslant 0}\mathcal{C} \hookrightarrow \mathcal{C}^{\leqslant 0}$ induces a closed immersion $\mathrm{Tot}_{\mathfrak{M}}(\tau^{\leqslant 0}\mathcal{C}) \hookrightarrow \mathrm{Tot}_{\mathfrak{M}}(\mathcal{C}^{\leqslant 0})$.

Consider the brutal truncation $\mathcal{C}^{\leqslant 0}$ of the complex $\mathcal{C}$. Its total space $\mathrm{pr}_{\mathfrak{M}}\colon \mathrm{Tot}_{\mathfrak{M}}(\mathcal{C}^{\leqslant 0}) \to \mathfrak{M}$ is a vector bundle stack over $\mathfrak{M}$ and hence smooth of relative dimension $\dim(\mathrm{pr}_{\mathfrak{M}}) = \mathrm{rk}(\mathcal{C}^0) - \mathrm{rk}(\mathcal{C}^{-1})$.

Consider also the vector bundle $\mathcal{E} := \mathrm{pr}_{\mathfrak{M}}^*\mathcal{C}^1$ on $\mathfrak{X} := \mathrm{Tot}_{\mathfrak{M}}(\mathcal{C}^{\leqslant 0})$ and its total space $\mathfrak{E} := \mathrm{Tot}_{\mathfrak{X}}(\mathcal{E})$. We have the zero section $0\colon \mathfrak{X} \to \mathcal{E}$ and the section $s_{\mathcal{C}}\colon \mathfrak{X} \to \mathcal{E}$ defined by the differential $d_0\colon \mathcal{C}^0 \to \mathcal{C}^1$.

Altogether we have the following commutative diagram with Cartesian square describing the relations between the spaces.

$$\begin{tikzcd}
\mathfrak{X} \arrow[rr, "0"] & & \mathfrak{E} \\
\mathrm{Tot}_{\mathfrak{X}}(\tau^{\leqslant 0}\mathcal{C}) \arrow[u, "\iota"] \arrow[rr, "\iota"] \arrow[rrd, phantom, ""] \arrow[dr, "q"] \arrow[urr, phantom, "\urcorner"] & & \mathfrak{X} \arrow[u, "s_{\mathcal{C}}"'] \arrow[dl, "\widetilde{q}"] \\
& \mathfrak{M} &
\end{tikzcd} \tag{4.11}$$

Diagram (4.11) can be thought of as a relative (over $\mathfrak{M}$) version of Diagram (4.9).

We have the smooth pullback $\widetilde{q}^{\dagger}$ and the refined Gysin pullback $\iota^{\dagger}_{s_{\mathcal{C}}}$.

Definition 4.47. We define the *virtual pullback along (the total space of)* $\mathcal{C}$ to be the composition

$$\begin{aligned}
\nu^{\dagger}_{\mathcal{C}} \colon \mathbb{DQ}_{\mathfrak{M}} &\xrightarrow{\widetilde{q}^{\dagger}} \widetilde{q}_{*}\mathbb{DQ}_{\mathfrak{X}} \otimes \mathbb{L}^{2\dim(\widetilde{q})/2} \\
&\xrightarrow{\widetilde{q}_{*} q^{\dagger}_{s_{\mathcal{C}}}} q_{*}\mathbb{DQ}_{\mathrm{Tot}_{\mathfrak{M}}(\tau^{\leqslant 0}\mathcal{C})} \otimes \mathbb{L}^{-2\chi(\mathcal{C})/2}
\end{aligned}$$

where we used $\chi(\mathcal{C}) = \mathrm{rk}(\mathcal{C}^{-1}) - \mathrm{rk}(\mathcal{C}^{0}) + \mathrm{rk}(\mathcal{C}^{-1})$

Ideally the virtual pullback is independent of the choice of locally free presentation of $\mathcal{C}$. This is not immediate from the definition, since the brutal truncation $\mathcal{C}^{\leqslant 0}$ and the vector bundle $\mathcal{E} = \widetilde{q}^{*}\mathcal{C}^{1}$ are sensitive to the presentation.

The following three lemmas solve this issue for a class of stacks that covers all of the moduli stacks appearing in this text.

Lemma 4.48. *Let $\mathfrak{X}$ be an Artin stack. Suppose $\mathcal{C}_i = (\mathcal{C}_i^{-1} \to \mathcal{C}_i^{0} \to \mathcal{C}_i^{1})$, $i = 1, 2$ are presentations of a $[-1, 1]$-perfect complex $\mathcal{C}$ and let $\gamma \colon \mathcal{C}_1 \to \mathcal{C}_2$ be a quasi-isomorphism. Let $q \colon \mathrm{Tot}_{\mathfrak{X}}(\tau^{\leqslant 0}\mathcal{C}) \to \mathfrak{X}$ be the projection.*

Then the two virtual pullbacks $q^{\dagger}_{\mathcal{C}_1}$ and $q^{\dagger}_{\mathcal{C}_2}$ agree under the induced equivalence $\gamma \colon \mathrm{Tot}_{\mathfrak{X}}(\mathcal{C}) \xrightarrow{\simeq} \mathrm{Tot}_{\mathfrak{X}}(\mathcal{C})$, i. e. the diagram

$$\begin{tikzcd}
& q_{*}\mathbb{DQ}_{\mathrm{Tot}_{\mathfrak{X}}(\tau^{\leqslant 0}\mathcal{C})} \arrow[dd, "\gamma_{*}", "\simeq"'] \\
\mathbb{DQ}_{\mathfrak{X}} \arrow[ur, "q^{\dagger}_{\mathcal{C}_1}"] \arrow[dr, "q^{\dagger}_{\mathcal{C}_2}"] & \\
& q_{*}\mathbb{DQ}_{\mathrm{Tot}_{\mathfrak{X}}(\tau^{\leqslant 0}\mathcal{C})}
\end{tikzcd}$$

commutes.

Lemma 4.49. *Let $\mathfrak{X}$ be an Artin stack. Suppose locally free sheaves are projective objects in the category of $\mathcal{O}_{\mathfrak{X}}$-modules. Then virtual pullback along a strictly $[-1, 1]$-perfect complex $\mathcal{C}$ is independent of the presentation of $\mathcal{C}$.*

Proof. Since locally free sheaves are projective objects, every isomorphism in $D(\mathfrak{X})$ of locally free sheaves is represented by a quasi-isomorphism. The lemma now follows from Lemma 4.48. ☐

The next lemma guarantees that the assumption of Lemma 4.49 holds for moduli stacks satisfying Assumption 3.39.

Lemma 4.50. *Suppose $\mathfrak{X} = W/\mathrm{GL}_N$ is a quotient stack of a quasi-affine scheme W by GL_N. Then locally free sheaves on $\mathfrak{X}$ are projective objects in the category of $\mathcal{O}_{\mathfrak{X}}$-modules.*

Proof. The main ingredient is the linear reductivity of GL_N. A locally free sheaf $\mathcal{E}$ on $\mathfrak{X}$ pulls back to a GL_N-equivariant locally free sheaf $\mathcal{E}_W$ on W. Since $\mathcal{E}_W$ is locally free, it is projective as a sheaf on W, i. e. $\mathrm{Hom}_W(\mathcal{E}_W, -)$ is exact. Let $\mathcal{F}$ be a GL_N-equivariant $\mathcal{O}_W$-module, then the GL_N-equivariant homomorphisms $\mathrm{Hom}_{\mathrm{GL}_N}(\mathcal{E}, \mathcal{F})$ are the same thing as GL_N-invariant homomorphisms:

$$\mathrm{Hom}_{\mathrm{GL}_N}(\mathcal{E}, \mathcal{F}) = \mathrm{Hom}_W(\mathcal{E}, \mathcal{F})^{\mathrm{GL}_N}$$

By the linear reductivity of GL_N taking invariants is an exact functor. Thus, as a composition of exact functors $\mathrm{Hom}_{\mathrm{GL}_N}(\mathcal{E}, -)$ is exact. In other words $\mathcal{E}$ is a projective $\mathcal{O}_{\mathfrak{X}}$-module. ☐

Remark 4.51. By (the proof of) [Tot04, Theorem 1.1], a stack is of the form W/GL_N for a quasi-affine scheme W if and only if it is of the form U/G for an affine scheme U and an affine algebraic group G. [Tot04, Theorem 1.1] also states that any stack of the form W/GL_N has the resolution property, i. e. every coherent sheaf is the quotient of a vector bundle. If $\mathfrak{X}$ is normal, then it has the resolution property if and only if is of the form W/GL_N.

By Example 4.35 such stacks admit acyclic covers.

4.6 THE COHOMOLOGICAL HALL ALGEBRA OF A 2-CALABI–YAU CATEGORY

This section is based on [DHS22]

Let $\mathscr{A} \subseteq \mathscr{D}$ be a 2CY Abelian category with a good moduli theory $\mathrm{JH}\colon \mathfrak{M}_{\mathscr{A}} \to \mathcal{M}_{\mathscr{A}}$. This section uses all of the Assumptions 3.39-3.43.

The main cohomological invariant of our interest is the mixed Hodge module

$$\mathrm{JH}_*\mathbb{D}\mathbb{Q}^{\mathrm{vir}}_{\mathfrak{M}} := \mathrm{JH}_*\mathbb{D}(\mathbb{Q}_{\mathfrak{M}_{\mathscr{A}}} \otimes \mathbb{L}^{-(-,-)_{\mathscr{A}}/2}) = \mathrm{JH}_*\mathbb{D}\mathbb{Q}_{\mathfrak{M}_{\mathscr{A}}} \otimes \mathbb{L}^{(-,-)_{\mathscr{A}}/2}$$

Explicitly, on each connected component $\mathfrak{M}_{\mathscr{A},v} \subseteq \mathfrak{M}_{\mathscr{A},v}$

$$\mathrm{JH}_*\mathbb{D}\mathbb{Q}^{\mathrm{vir}}_{\mathfrak{M}_{\mathscr{A},v}} := \mathrm{JH}_*\mathbb{D}(\mathbb{Q}_{\mathfrak{M}_{\mathscr{A}}} \otimes \mathbb{L}^{-(v,v)_{\mathscr{A}}/2}) = \mathrm{JH}_*\mathbb{D}\mathbb{Q}_{\mathfrak{M}_{\mathscr{A}}} \otimes \mathbb{L}^{(v,v)_{\mathscr{A}}/2}.$$

Since $\mathcal{M}_{\mathscr{A}}$ with $\oplus$ is a commutative monoid scheme, the product $\boxdot$ from (4.6) endows $D^+(\mathrm{MHM}(\mathcal{M}_{\mathscr{A}}))$ with a symmetric monoidal structure.

The aim of this section is to construct a $\boxdot$-algebra structure on

$$\mathrm{JH}_*\mathbb{D}\mathbb{Q}^{\mathrm{vir}}_{\mathfrak{M}_{\mathscr{A}}}.$$

Let $\mathfrak{Exact}_{\mathscr{A}}$ be the stack of short exact sequences in $\mathscr{A}$ as in Definition 3.29. Consider the correspondence diagram

$$\begin{tikzcd}
 & \mathfrak{Exact}_{\mathscr{A}} \arrow[dl, "q"'] \arrow[dr, "p"] & \\
\mathfrak{M}_{\mathscr{A}} \times \mathfrak{M}_{\mathscr{A}} \arrow[d, "\mathrm{JH}\times\mathrm{JH}"] & & \mathfrak{M}_{\mathscr{A}} \arrow[d, "\mathrm{JH}"] \\
\mathcal{M}_{\mathscr{A}} \times \mathcal{M}_{\mathscr{A}} \arrow[rr, "\oplus"] \arrow[d, "{[-]}"] & & \mathcal{M}_{\mathscr{A}} \arrow[d, "{[-]}"] \\
\pi_0(\mathcal{M}_{\mathscr{A}}) \times \pi_0(\mathcal{M}_{\mathscr{A}}) \arrow[rr, "+"] & & \pi_0(\mathcal{M}_{\mathscr{A}})
\end{tikzcd} \tag{4.12}$$

By Proposition 3.28 the morphism q is the trunction of the globally presented morphism

$$\mathrm{Tot}_{\mathfrak{M}_{\mathscr{A}}^{\times 2}}(\mathrm{R}\mathcal{H}om[1]) \longrightarrow \mathfrak{M}_{\mathscr{A}} \times \mathfrak{M}_{\mathscr{A}}.$$

We define the $\boxdot$-algebra structure on $\mathcal{A}_{\mathscr{A}}$ by

$$\begin{aligned}
\mathcal{A}_{\mathscr{A}} \boxdot \mathcal{A}_{\mathscr{A}} &= \oplus_*(\mathcal{A}_{\mathscr{A}} \boxtimes \mathcal{A}_{\mathscr{A}}) \\
&\xrightarrow{q^{\dagger}_{\mathrm{R}\mathcal{H}om[1]}} (\oplus \circ \mathrm{JH}^{\times 2} \circ q)_*\mathbb{D}\mathbb{Q}_{\mathfrak{Exact}_{\mathscr{A}}} \otimes \mathbb{L}^{-(-,-)_{\mathscr{A}}/2} \\
&\xrightarrow{\cong} (\mathrm{JH} \circ p)_*\mathbb{D}\mathbb{Q}_{\mathfrak{Exact}_{\mathscr{A}}} \otimes \mathbb{L}^{-(-,-)_{\mathscr{A}}/2} \\
&\xrightarrow{p_*} \mathbb{D}\mathbb{Q}_{\mathfrak{M}_{\mathscr{A}}} \otimes \mathbb{L}^{-(-,-)_{\mathscr{A}}/2} = \mathcal{A}_{\mathscr{A}}
\end{aligned} \tag{4.13}$$

We make the shifts explicit. Restricting (4.12) to connected components $\mathfrak{M}_{\mathscr{A},v} \times \mathfrak{M}_{\mathscr{A},w}$ we have the diagram

$$\begin{tikzcd}
 & \mathfrak{Exact}_{\mathscr{A},v,w} \arrow[dl, "q"'] \arrow[dr, "p"] & \\
\mathfrak{M}_{\mathscr{A},v} \times \mathfrak{M}_{\mathscr{A},w} \arrow[d, "\mathrm{JH}\times\mathrm{JH}"] & & \mathfrak{M}_{\mathscr{A},v+w} \arrow[d, "\mathrm{JH}"] \\
\mathcal{M}_{\mathscr{A},v} \times \mathcal{M}_{\mathscr{A},w} \arrow[rr, "\oplus"] & & \mathcal{M}_{\mathscr{A},v+w}.
\end{tikzcd}$$

and restricting (4.13) we have

$$\begin{aligned}
\mathrm{JH}_*\mathbb{D}\mathbb{Q}^{\mathrm{vir}}_{\mathfrak{M}} \boxdot \mathrm{JH}_*\mathbb{D}\mathbb{Q}^{\mathrm{vir}}_{\mathfrak{M}} &= \oplus_*\mathrm{JH}_*\mathbb{D}\mathbb{Q}_{\mathfrak{M}_{\mathscr{A},v}\times\mathfrak{M}_{\mathscr{A},w}} \otimes \mathbb{L}^{-((v,v)+(w,w))/2} \\
&\xrightarrow{q^{\dagger}_{\mathrm{R}\mathcal{H}om[1]}} (\oplus \circ \mathrm{JH}^{\times 2} \circ q)_*\mathbb{D}\mathbb{Q}_{\mathfrak{Exact}_{\mathscr{A},v,w}} \otimes \mathbb{L}^{-((v,v)+(w,w)+2(v,w))/2} \\
&\xrightarrow{\cong} (\mathrm{JH} \circ p)_*\mathbb{D}\mathbb{Q}_{\mathfrak{Exact}_{\mathscr{A},v,w}} \otimes \mathbb{L}^{-(v+w,v+w)_{\mathscr{A}}/2} \\
&\xrightarrow{p_\star} \mathrm{JH}_*\mathbb{D}\mathbb{Q}_{\mathfrak{M}_{\mathscr{A},v+w}} \otimes \mathbb{L}^{-(v+w,v+w)_{\mathscr{A}}/2}.
\end{aligned}$$

In the second to last line we used the symmetry $(v,w)_{\mathscr{A}} = (w,v)_{\mathscr{A}}$ to gather terms.

Theorem-Definition 4.52 ([DHS22])**.** The composition $\mathfrak{m}\colon \mathcal{A}_{\mathscr{A}} \boxdot \mathcal{A}_{\mathscr{A}} \to \mathcal{A}_{\mathscr{A}}$ of the morphisms in (4.13) makes $\mathcal{A}_{\mathscr{A}}$ into a $\boxdot$-algebra, called the *relative cohomological Hall algebra of* $\mathscr{A}$. The pushforward $\mathrm{H}^*\mathcal{A}_{\mathscr{A}} = [-]_*\mathcal{A}_{\mathscr{A}}$ to the monoid of connected components together with the product $[-]_*\mathfrak{m}$ is a $\pi_0(\mathcal{M}_{\mathscr{A}})$-graded algebra, called the *(absolute) cohomological Hall algebra of* $\mathscr{A}$.

Proof sketch. The main part of the proof is to show that the product $\mathfrak{m}$ is associative which is [DHS22, Proposition 9.1]. The key point is to use the general theory of pullbacks along globally presented morphisms to compare iterated products. One iterated product is given by the virtual pullbacks and pushforwards

$$(p_{v,w+u})_\star \circ q^\dagger_{v,w+u} \circ (\mathrm{id}_v \times p_{w,u})_\star \circ (\mathrm{id}_v \times q_{w,u})^\dagger \tag{4.14}$$

along morphisms in the diagram

$$\begin{tikzcd}
\mathrm{Filt}_{\mathscr{A},v,w,u} \arrow[r] \arrow[d] \arrow[dd, bend right, "q_{v,w,u}"'] \arrow[rr, bend left, "p_{v,w,u}"] & \mathfrak{Exact}_{\mathscr{A},v,w+u} \arrow[r, "p_{v,w+u}"] \arrow[d, "q_{v,w+u}"] & \mathfrak{M}_{\mathscr{A},v+w+u} \\
\mathfrak{M}_{\mathscr{A},v} \times \mathfrak{Exact}_{\mathscr{A},w+u} \arrow[r, "\mathrm{id}_v \times p_{w,u}"] \arrow[d, "\mathrm{id}_v \times q_{w,u}"] & \mathfrak{M}_{\mathscr{A},v} \times \mathfrak{M}_{\mathscr{A},w+u} & \\
\mathfrak{M}_{\mathscr{A},v} \times \mathfrak{M}_{\mathscr{A},w} \times \mathfrak{M}_{\mathscr{A},u} & &
\end{tikzcd}.$$

The other iterated product is given by the pullbacks and pushforwards

$$(p_{v+w,u})_\star \circ q^\dagger_{v+w,u} \circ (p_{v,w} \times \mathrm{id}_u)_\star \circ (q_{v,w} \times \mathrm{id}_u)^\dagger \tag{4.15}$$

along morphisms in the diagram

$$\begin{tikzcd}
\mathrm{Filt}_{\mathscr{A},v,w,u} \arrow[r] \arrow[d] \arrow[dd, bend right, "q_{v,w,u}"'] \arrow[rr, bend left, "p_{v,w,u}"] & \mathfrak{Exact}_{\mathscr{A},v+w,u} \arrow[r, "p_{v+w,u}"] \arrow[d, "q_{v+w,u}"] & \mathfrak{M}_{\mathscr{A},v+w+u} \\
\mathfrak{Exact}_{\mathscr{A},v+w} \times \mathfrak{M}_{\mathscr{A},u} \arrow[r, "p_{v,w} \times \mathrm{id}_u"] \arrow[d, "q_{v,w} \times \mathrm{id}_u"] & \mathfrak{M}_{\mathscr{A},v+w,u} & \\
\mathfrak{M}_{\mathscr{A},v} \times \mathfrak{M}_{\mathscr{A},w} \times \mathfrak{M}_{\mathscr{A},u} & &
\end{tikzcd}.$$

The morphism

$$q_{v,w,u}\colon \mathrm{Filt}_{\mathscr{A},v,w,u} \longrightarrow \mathfrak{M}_{\mathscr{A},v} \times \mathfrak{M}_{\mathscr{A},w} \times \mathfrak{M}_{\mathscr{A},u}$$

is a globally presented morphism as in §A.2. Using the theory of virtual pullbacks along globally presented quasi-smooth morphisms as developed in [DHS22, §4] and applying Proposition 3.32 and

Lemma 4.49 it follows that the two iterated products (4.14) and (4.15) agree with the virtual pullback and pushforward

$$(p_{q,v,u})_\star \circ q^\dagger_{v,w,u}$$

through $\mathrm{Filt}_{\mathscr{A},v,w,u}$. □

ψ-twists

We need to introduce systematic signs which we incorporate into the mulitplication of the CoHA. Suppose $\psi\colon \pi_0(\mathfrak{M}_{\mathscr{A}}) \times \pi_0(\mathfrak{M}_{\mathscr{A}}) \to \mathbb{Z}$ is a bilinear form, called a *twist*, such that $\psi(v,w) + \psi(w,v) = (v,w)_{\mathscr{A}}$ mod 2 for all $v, w \in \pi_0(\mathfrak{M}_{\mathscr{A}})$. The existence of such bilinear form follows from $(v,v) \in 2\mathbb{Z}$ for all $v \in \pi_0(\mathfrak{M}_{\mathscr{A}})$.

Definition 4.53. The *ψ-twisted relative CoHA* is the $\boxdot$-algebra object in $D^+(\mathrm{MHM}(\mathcal{M}_{\mathscr{A}}))$ given by

$$\mathcal{A}^\psi := (\mathcal{A}_{\mathscr{A}}, (-1)^\psi m)$$

where the sign $(-1)^\psi$ depends on the value of ψ on the connected component of $\mathcal{M}_{\mathscr{A}} \times \mathcal{M}_{\mathscr{A}}$.

4.7 COHOMOLOGICAL HALL ALGEBRAS OF SERRE SUBCATEGORIES

Suppose $\mathscr{B} \subseteq \mathscr{A}$ is a Serre subcategory so that $\mathcal{M}_{\mathscr{B}} \hookrightarrow \mathcal{M}_{\mathscr{A}}$ is a closed submonoid scheme and the diagram

$$\begin{tikzcd}
\mathcal{M}_{\mathscr{B}} \times \mathcal{M}_{\mathscr{B}} \arrow[r, "\oplus"] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner"] & \mathcal{M}_{\mathscr{B}} \arrow[d, hook] \\
\mathcal{M}_{\mathscr{A}} \times \mathcal{M}_{\mathscr{A}} \arrow[r, "\oplus"] & \mathcal{M}_{\mathscr{A}}
\end{tikzcd}$$

is Cartesian.

Lemma 4.54. *The base change isomorphism $\iota^!_{\mathscr{B}} \circ \oplus_* \xrightarrow{\cong} \oplus_* \circ (\iota_{\mathscr{B}} \times \iota_{\mathscr{B}})^!$ makes $\iota^!_{\mathscr{B}}$ a strict monoidal functor*

$$\iota^!_{\mathscr{B}}\colon (D^+(\mathrm{MHM}(\mathcal{M}_{\mathscr{A}})), \boxdot) \longrightarrow (D^+(\mathrm{MHM}(\mathcal{M}_{\mathscr{B}})), \boxdot)$$

Proof. Immediate from definitions. □

In particular, $\iota^!_{\mathscr{B}}$ takes $\boxdot$-algebra objects to $\boxdot$-algebra objects.

Definition 4.55. We define the *relative cohomological Hall algebra of $\mathscr{B}$* to be the $\boxdot$-algebra object $\mathcal{A}_{\mathscr{B}} := \iota^!_{\mathscr{B}} \mathcal{A}_{\mathscr{A}}$. The *(absolute) cohomological Hall algebra of $\mathscr{B}$* is the $\pi_0(\mathcal{M}_{\mathscr{B}})$-graded algebra $H^* \mathcal{A}_{\mathscr{B}}$.

More explicitly, the product $m_{\mathscr{B}}$ on $\mathcal{A}_{\mathscr{B}}$ is defined via the pullback and pushforward of the restrictions

$$m = (p|_{\mathfrak{M}_{\mathscr{B}}})_\star \circ (q|_{\mathfrak{M}_{\mathscr{B}}})^\dagger.$$

where the virtual pullback along q is the virtual pullback along the restriction $\mathrm{RHom}_{\mathscr{A}}[1]|_{\mathfrak{M}_{\mathscr{B}}}$ which is a strictly $[-1,1]$-perfect complex.

Example 4.56 (Nilpotent CoHA)**.** The *nilpotent CoHA*

$$\mathrm{H}\mathcal{A}_{\Pi_Q}^{\mathrm{Nil}} = \mathcal{A}_{\Pi_Q}^{\mathrm{Nil}} = \imath_{\mathrm{Nil}}^{!}\mathcal{A}_{\Pi_Q}$$

of the preprojective algebra of a quiver Q is the CoHA associated to the Serre subcategory of $\mathrm{Rep}(\Pi_Q)$ consisting of nilpotent representations.

Example 4.57 (Strictly seminilpotent CoHA)**.** The *relative strictly seminilpotent CoHA* is the relative CoHA associated to the Serre subcategory $\mathrm{Rep}(\Pi_Q)^{\mathrm{SNN}}$

$$\mathcal{A}_{\Pi_Q}^{\mathrm{SNN},\psi} = \imath_{\mathrm{SNN}}^{!}\mathcal{A}_{\Pi_Q}^{\psi}$$

and the *(absolute) strictly seminipotent CoHA* is its total cohomology $\mathrm{H}(\mathcal{A}_{\Pi_Q}^{\mathrm{SNN},\psi})$.

Next we generalise Example 4.56 to arbitrary 2CY categories $\mathscr{A}$.

Example 4.58. Let $F = \{F_1, \dots, F_n\}$ be a Σ-collection of simple objects in $\mathscr{A}$. Let $\mathscr{F} \subseteq \mathscr{A}$ be the extension closure of the full subcategory containing the Σ-collection F. The *cohomological Hall algebra of the Σ-collection* F is the CoHA associated to the Serre subcategory $\mathscr{F}$.

Let $\mathrm{dg}\mathscr{F} \subseteq \mathscr{D}$ be the dg extension closure of the full subcategory containing the F_i, for $i = 1, \dots, n$, i. e. $\mathrm{dg}\mathscr{F}$ is the smallest full dg subcategory of $\mathscr{D}$ which is extension closed and contains all of the objects F_i. Let Q_F be a half Ext-quiver of F and let $\mathrm{dgNil}_{\Pi_{Q_F}} \subseteq \mathrm{Perf}(\mathscr{G}_2(Q_F))$ be the dg extension closure of the nilpotent representations of Π_{Q_F}, i. e. the dg extension closure of the Σ-collection consisting of the representations S_i.

The simple objects in the Serre subcategory $\mathscr{F} \subseteq \mathscr{A}$ are precisely $F_1, \dots, F_n$. Thus the inclusions $\{F_i\} \hookrightarrow \mathcal{M}_{\mathscr{F}} \subseteq \mathscr{A}$ induce an isomorphism of monoid schemes

$$\mathbb{N}^{\{F_i\}} \xrightarrow{\cong} \mathcal{M}_{\mathscr{F}}.$$

The moduli stack of $\mathscr{F}$ is then given by the disjoint uninion of fibres

$$\mathfrak{M}_{\mathscr{F}} = \bigsqcup_{\underline{m} \in \mathbb{N}^{\{F_i\}}} \mathfrak{M}_{\mathscr{F},\underline{m}} = \bigsqcup_{\underline{m} \in \mathbb{N}^{\{F_i\}}} \mathrm{JH}_{\mathscr{A}}^{-1}([\bigoplus_i F_i^{m_i}])$$

The stacks $\mathfrak{M}_{\mathscr{F}}$ and $\mathfrak{M}_{\Pi_{Q_F}}^{\mathrm{Nil}}$ are the fibers over a discrete subscheme $\mathbb{N}^{Q_0} \subseteq \mathcal{M}_{\mathscr{A}}$, resp. $\subseteq \mathcal{M}_{\Pi_Q}$. It follows from an application of the étale Ext-quiver neighbourhood theorem (3.53) at each of the points of $\mathbb{N}^{Q_0}$ that there is an equivalence of stacks

$$\mathfrak{M}_{\mathscr{F},\underline{m}} \simeq \mathfrak{M}_{\Pi_Q,\underline{m}}^{\mathrm{Nil}}. \tag{4.16}$$

Moreover, there is a commutative diagram

$$\begin{tikzcd}
\mathfrak{M}_{\Pi_Q} \arrow[d] & \mathfrak{M}_{\Pi_Q}^{\mathrm{Nil}} \simeq \mathfrak{M}_{\mathscr{F}} \arrow[l, hook'] \arrow[r, hook, "\tilde{\imath}_{\mathscr{F}}"] \arrow[d] & \mathfrak{M}_{\mathscr{A}} \arrow[d] \\
\mathcal{M}_{\Pi_Q} & \mathbb{N}^{\{F_i\}} \arrow[l, hook'] \arrow[r, hook] & \mathcal{M}_{\mathscr{A}}
\end{tikzcd} \tag{4.17}$$

where the horizional morphisms are closed embeddings and the vertical morphisms are good moduli space morphisms.

Proposition 4.59 (Ext-quiver compatibility of CoHA product)**.** *The equivalence* (4.16) *induces an isomorphism of cohomological Hall algebras*

$$\mathrm{H}\mathcal{A}_{\mathscr{A}}^{\mathscr{F},\psi} \xrightarrow{\cong} \mathrm{H}\mathcal{A}_{\Pi_Q}^{\mathrm{Nil},\psi}$$

Proof. We prove the compatibility of multiplication in two steps. First, we show that the restrictions of the virtual pullbacks $(\widetilde{\iota}_{\mathscr{F}} \times \widetilde{\iota}_{\mathscr{F}})^! q^{\dagger}_{\mathrm{R}\mathcal{H}om_{\mathscr{A}}[1]}$ and $(\widetilde{\iota}^!_{\mathrm{Nil}} \times \widetilde{\iota}_{\mathrm{Nil}}) q^{\dagger}_{\mathrm{R}\mathcal{H}om_{\Pi_Q}[1]}$ agree. Second, we show that the restrictions of the proper pushforwards $\widetilde{\iota}^!_{\mathscr{F}}(p_{\mathscr{A}})_\star$ and $\widetilde{\iota}^!_{\mathrm{Nil}}(p_{\Pi_Q})_\star$. Both of these steps ultimately hinge on the equivalence (4.18) of the restrictions of RHom complexes $\mathrm{R}\mathcal{H}om_{\mathscr{A}}[1]$ and $\mathrm{R}\mathcal{H}om_{\Pi_Q}[1]$ to the stack $\mathfrak{M}^{\mathrm{Nil}}_{\Pi_Q} \times \mathfrak{M}^{\mathrm{Nil}}_{\Pi_Q} \simeq \mathfrak{M}^{\mathscr{F}}_{\mathscr{A}} \times \mathfrak{M}^{\mathscr{F}}_{\mathscr{A}}$. The equivalence (4.18) is established using the machinery of twisted complexes, see Appendix B.

Define $\mathfrak{Exact}^{\mathscr{F}}_{\mathscr{A}}$ as the fiber product

$$\begin{tikzcd}
\mathfrak{Exact}^{\mathscr{F}}_{\mathscr{A}} \arrow[r] \arrow[d, "q^{\mathscr{F}}_{\mathscr{A}}"] \arrow[dr, phantom, "\lrcorner", very near start] & \mathfrak{Exact}_{\mathscr{A}} \arrow[d, "q_{\mathscr{A}}"] \\
\mathfrak{M}_{\mathscr{F}} \times \mathfrak{M}_{\mathscr{F}} \arrow[r, hook] & \mathfrak{M}_{\mathscr{A}\times\mathfrak{M}_{\mathscr{A}}}.
\end{tikzcd}$$

We have

$$\mathfrak{Exact}^{\mathscr{F}}_{\mathscr{A}} \simeq t_0(\mathrm{Tot}_{\mathfrak{M}_{\mathscr{F}}\times\mathfrak{M}_{\mathscr{F}}}((\widetilde{\iota}_{\mathscr{F}} \times \widetilde{\iota}_{\mathscr{F}})^* \mathrm{R}\mathcal{H}om_{\mathscr{A}}[1])).$$

Since the morphism $q_{\mathscr{A}}$ is globally presented by the quasi-smooth morphism $\mathrm{Tot}_{\mathfrak{M}_{\mathscr{A}}\times\mathfrak{M}_{\mathscr{A}}}(\mathrm{R}\mathcal{H}om_{\mathscr{A}}[1]) \to \mathfrak{M}_{\mathscr{A}} \times \mathfrak{M}_{\mathscr{A}}$ (Assumption 3.39 for $\mathscr{A}$), it follows that $q^{\mathscr{F}}_{\mathscr{A}}$ is globally presented by the quasi-smooth morphism $\mathrm{Tot}_{\mathfrak{M}^{\mathscr{F}}_{\mathscr{A}}\times\mathfrak{M}^{\mathscr{F}}_{\mathscr{A}}}((\widetilde{\iota}_{\mathscr{F}} \times \widetilde{\iota}_{\mathscr{F}})^* \mathrm{R}\mathcal{H}om_{\mathscr{A}}[1]) \to \mathfrak{M}^{\mathscr{F}}_{\mathscr{A}} \times \mathfrak{M}^{\mathscr{F}}_{\mathscr{A}}$.

The equivalences of Lemma B.7 and Lemma B.9 make up the middle part of (4.17).

Abbreviate the shifted RHom complex coming from Π_Q, respectively $\mathscr{A}$, by

$$\mathrm{R}\mathcal{H}om^{\mathrm{Nil}}_{\Pi_Q}[1] := (\widetilde{\iota}_{\mathrm{Nil}} \times \widetilde{\iota}_{\mathrm{Nil}})^* \mathrm{R}\mathcal{H}om_{\Pi_Q}[1], \text{ respectively}$$
$$\mathrm{R}\mathcal{H}om^{\mathscr{F}}_{\mathscr{A}} := (\widetilde{\iota}_{\mathscr{F}} \times \widetilde{\iota}_{\mathscr{F}})^* \mathrm{R}\mathcal{H}om_{\mathscr{A}}[1].$$

Applying Lemma B.12 twice – once to the category $\mathscr{A} = \mathscr{A}$ with Σ-collection F and once to $\mathscr{A} = \Pi_Q$ with Σ-collection S – we deduce that there is is an equivalence of RHom complexes

$$\mathrm{R}\mathcal{H}om^{\mathrm{Nil}}_{\Pi_Q}[1] \simeq \mathrm{R}\mathcal{H}om^{\mathscr{F}}_{\mathscr{A}}[1]. \tag{4.18}$$

By Lemma B.9 we know that $\mathfrak{M}^{\mathrm{Nil}}_{\Pi_Q} \simeq \mathfrak{M}_{\mathscr{F}}$ is the quotient stack of an affine scheme by a reductive group, hence satisfies the resolution property. We conclude from Lemma 4.49 that the virtual pullbacks $(q^{\mathrm{Nil}}_{\Pi_Q})^\dagger$ and $(q^{\mathscr{F}}_{\mathscr{A}})^\dagger$ agree.

Using smooth base change it follows that the restricted virtual pullbacks agree

$$\begin{aligned} \iota_{\mathrm{Nil}}^{!} q^{\dagger}_{\mathrm{R}\mathcal{H}om_{\Pi_Q}[1]} &= q^{\dagger}_{\mathrm{R}\mathcal{H}om^{\mathrm{Nil}}_{\Pi_Q}[1]} && \text{(Theorem 4.6)} \\ &= q^{\dagger}_{\mathrm{R}\mathcal{H}om^{\mathscr{F}}_{\mathscr{A}}[1]} && \text{((4.18) and Lemma 4.49)} \\ &= \iota_{\mathscr{F}}^{!} q^{\dagger}_{\mathrm{R}\mathcal{H}om_{\mathscr{A}}[1]} && \text{(Theorem 4.6).} \end{aligned}$$

This completes the first part of the proof.

We begin the proof of the compatiblity of proper pushforwards with a chase in the following diagram.

$$\begin{tikzcd}
& \mathfrak{Exact}^{\mathscr{F}}_{\mathscr{A}} \arrow[dl, "q^{\mathscr{F}}"'] \arrow[dr, "p^{\mathscr{F}}"] \arrow[dddr, hook] & & \\
\mathfrak{M}^{\mathscr{F}}_{\mathscr{A}} \times \mathfrak{M}^{\mathscr{F}}_{\mathscr{A}} \arrow[d] \arrow[dddr, hook] & & \mathfrak{M}^{\mathscr{F}}_{\mathscr{A}} \arrow[d] \arrow[dddr, hook] & \\
\mathbb{N}^{\{F_i\}} \times \mathbb{N}^{\{F_i\}} \arrow[rr, "+"] \arrow[dddr, hook] & & \mathbb{N}^{\{F_i\}} \arrow[dddr, hook] & \\
& & \mathfrak{Exact}_{\mathscr{A}} \arrow[dl, "q"'] \arrow[dr, "p"] & \\
& \mathfrak{M}_{\mathscr{A}} \times \mathfrak{M}_{\mathscr{A}} \arrow[d] & & \mathfrak{M}_{\mathscr{A}} \arrow[d] \\
& \mathcal{M}_{\mathscr{A}} \times \mathcal{M}_{\mathscr{A}} \arrow[rr, "\oplus"] & & \mathcal{M}_{\mathscr{A}}
\end{tikzcd}$$

The two pentagons are commutative. The bottom square is Cartesian by hypothesis. The squares on the "walls" are Cartesian by definition. The square on the left side of the "roof" is Cartesian also by definition. Thus, as long as the right side of the "roof" is commutative it is also Cartesian.

Remark 4.60. More concretely, this diagram chase reflects the fact that Jordan–Hölder factors are "additive" in short exact sequences.

The diagram

$$\begin{tikzcd}
\mathfrak{Exact}^{\mathscr{F}}_{\mathscr{A}} \arrow[r] \arrow[d, "p^{\mathscr{F}}_{\mathscr{A}}"] \arrow[dr, phantom, "\lrcorner", very near start] & \mathfrak{Exact}_{\mathscr{A}} \arrow[d, "p"] \\
\mathfrak{M}^{\mathscr{F}}_{\mathscr{A}} \arrow[r] & \mathfrak{M}_{\mathscr{A}}
\end{tikzcd}$$

is commutative, hence Cartesian, because both morphisms p and $p^{\mathscr{F}}_{\mathscr{A}}$ are given by the universal extensions over $\mathfrak{M}_{\mathscr{A}} \times \mathfrak{M}_{\mathscr{A}}$, respectively $\mathfrak{M}^{\mathscr{F}}_{\mathscr{A}} \times \mathfrak{M}^{\mathscr{F}}_{\mathscr{A}}$, which are .

We deduce the compatiblity of pushforwards

$$\begin{aligned} \imath_{\mathscr{F}}^{!}(p_{\mathscr{A}})_{\star} &= (p_{\mathscr{A}}^{\mathscr{F}})_{\star} && \text{(Base Change Theorem 4.6)} \\ &= (p_{\Pi_Q}^{\mathrm{Nil}})_{\star} && \text{(Lemma B.12)} \\ &= \imath_{\mathrm{Nil}}^{!}(p_{\Pi_Q})_{\star} && \text{(Base Change Theorem 4.6).} \end{aligned}$$

□

5

CARTAN DATA AND LIE ALGEBRAS

> *"I think... if it is true that there are as many minds as there are heads, then there are as many kinds of love as there are hearts."*
>
> — Leo Tolstoy, *Anna Karenina*

The aim of this chapter is to present the definition (Definition 5.25) of a generalised Kac–Moody Lie algebra associated to a monoid scheme [DHS23]. We pay particular attention to the good moduli space of a 2-Calabi–Yau category $\mathscr{A}$ with a good moduli theory.

To motivate this definition we review the classical definition of Kac–Moody Lie algebras and Borcherds's generalised Kac–Moody Lie algebras.

5.1 SEMISIMPLE LIE ALGEBRAS FROM CARTAN DATA

A *(finite type) root system* is the data of

(i) a finite dimensional real vector space V
(ii) together with a symmetric, positive definite, non-degenerate bilinear form $(-,-)\colon V \times V \to \mathbb{R}$
(iii) and a finite spanning subset $R \subseteq V \setminus \{0\}$, elements of which are called *roots*,

satisfying the following axioms:

(i) for every two roots a, b the vector $b - 2\frac{(a,b)}{(a,a)}a$ is also a root;
(ii) for every two roots a, b we have $2\frac{(a,b)}{(a,a)} \in \mathbb{Z}$.

A subset $S \subseteq R$ is called *a set of simple roots* if S is a basis for V and every root $a \in R$ can be written as an integral linear combination $a = \sum_{b \in S} n_b b$ of simple roots $b \in S$ where all of the coefficients n_b have the same sign.

A *Cartan datum* is the data of a root system together with a set of simple roots.

Given the input of a Cartan datum $(V,(-,-),R,S)$ we define a Lie algebra $\mathfrak{g}_{(V,(-,-),R,S)}$ as follows. We declare $\mathfrak{g}$ to the Lie algebra generated by the set $\{e_s, h_s, f_s\}_{s\in S}$ subject to the relations

$$\begin{aligned} [h_s, e_s] &= 2e_s, \text{ for all } s \in S \\ [h_s, f_s] &= -2f_s, \text{ for all } s \in S \\ [e_s, f_s] &= h_s, \text{ for all } s \in S \\ [h_s, h_{s'}] &= 0, \text{ for all } s, s' \in S \\ \mathrm{ad}(e_s)^{1-(s,s')}(e_{s'}) &= 0, \text{ for all } s, s' \in S \\ \mathrm{ad}(f_s)^{1-(s,s')}(f_{s'}) &= 0, \text{ for all } s, s' \in S. \end{aligned} \tag{5.1}$$

Remark 5.1. One of the fundamental facts about simple Lie algebras is that every simple Lie algebra $\mathfrak{g}$ arrises as a Lie algebra of the form $\mathfrak{g}_{(V,(-,-),R,S)}$. Explicitly, a root system is $V = \mathfrak{h}^\vee$, where $\mathfrak{h} \subseteq \mathfrak{g}$ is a maximal Abelian sub Lie algebra, the form $(-,-)$ is the Killing form, and the roots $R \subseteq \mathfrak{h}^\vee$ are given by the nonzero weights $\alpha\colon \mathfrak{h} \to \mathbb{C}$ of the adjoint action $\mathrm{ad}\colon \mathfrak{h} \times \mathfrak{g} \to \mathfrak{g}$.

5.2 KAC–MOODY LIE ALGEBRAS FROM CARTAN DATA

We recall definitions from [Kac90].

A *generalised Cartan matrix* is an integral matrix $A = (a_{ij})_{i,j=1}^n \in \mathrm{Mat}_n(\mathbb{Z})$ satisfying the following properties

(1) All diagonal entries a_{ii} are equal to 2
(2) All off-diagonal entries a_{ij}, $i \neq j$, are nonpositive.
(3) It $a_{ij} = 0$, then $a_{ji} = 0$.

A *generalised Cartan datum* is a generalised Cartan matrix $A = (a_{ij})_{i,j=1}^n$ together with the choice of a complex vector space $\mathfrak{h}$ of dimension $\dim \mathfrak{h} = 2n - \mathrm{rk}(A)$ together with linearly independant ordered subsets $\Pi = \{h_1, \dots, h_n\} \subseteq \mathfrak{h}$ and $\Pi^\vee = \{h_1^\vee, \dots, h_n^\vee\} \subseteq \mathfrak{h}$ such that $A = (\langle h_i^\vee, h_j\rangle)_{i,j=1}^n$.

Define the Lie algebra $\tilde{\mathfrak{g}}(A)$ to be the Lie algebra with generators e_i, f_i, $i = 1, \dots, n$ and $\mathfrak{h}$ satisfying the standard $\mathfrak{sl}_2$ relations: for all $i, j = 1, \dots n$

$$\begin{aligned} [h_i, h_j] &= \delta_{ij} h_i \\ [h_i, e_j] &= \langle h_i^\vee, h_j\rangle e_j \\ [h_i, f_j] &= -\langle h_i^\vee, h_j\rangle f_j \\ [e_i, f_j] &= \delta_{ij} h_i. \end{aligned} \tag{5.2}$$

Among those Lie ideals of $\tilde{\mathfrak{g}}(A)$ intersecting $\mathfrak{h}$ trivially there is a unique maximal one, which we denote by $I(A)$. The *Kac–Moody Lie algebra* $\mathfrak{g}(A)$ associated to the generalised Cartan datum $(A, \mathfrak{h}, \Pi, \Pi^\vee)$ is the quotient Lie algebra of $\tilde{\mathfrak{g}}(A)$ be $I(A)$. The Lie algebra $\mathfrak{g}(A)$ is independant, up to isomorphism, of the choice of generalised Cartan datum with generalised Cartan matrix A.

It is convenient to have an explicit description of generators of $I(A)$, or equivalently, a complete list of relations amongst the generators $e_i, f_i, i = 1, \dots, n$ of $\mathfrak{g}(A)$.

As in the case for finite dimensional semisimple Lie algebras one shows (see [Kac90, Chapter 3]) using the representation theory of $\mathfrak{sl}_2$ that the Serre relations hold:

$$\begin{aligned} \operatorname{ad}(e_i)^{1-\langle h_i^\vee, h_j\rangle}(e_j) &= 0 \\ \operatorname{ad}(f_i)^{1-\langle h_i^\vee, h_j\rangle}(f_j) &= 0 \end{aligned} \tag{5.3}$$

for all $i = 1, \dots, n$ such that $\langle h_i^\vee, h_i\rangle > 0$.

A generalised Cartan matrix A is *symmetrisable* if there is a diagonal matrix $D \in \mathrm{Mat}_n(\mathbb{Q})$, called *symmetriser* for A, such that DA is symmetric.

For symmetrisable Cartan matrices the theorem of Gabber–Kac [GK81, Theorem 2] states that the Serre relations (5.3) generate the ideal $I(A)$. In other words, the Kac–Moody Lie algebra $\mathfrak{g}(A)$ is the Lie algebra with generators e_i, f_i, h_i satisfying the relations (5.2) and (5.3).

Example 5.2 (Quivers without loops as Cartan data)**.** For every quiver Q without loops we define the generalised Cartan matrix associated to Q to be the matrix given by the Euler form of the preprojective algebra of Q: $A_Q = ((i,j)_{\Pi_Q})_{i,j\in Q_0}$ of the quiver. By definition of $(-,-)_{\Pi_Q}$ the matrix A_Q is symmetric and the condition that Q has no loops guarantees that the diagonal of A_Q consists of twos.

We define the *Kac–Moody Lie algebra* $\mathfrak{g}_Q$ *of* Q to be the Kac–Moody Lie algebra $\mathfrak{g}(A_Q)$. By the above discussion, it has generators e_i, f_i, h_i for $i \in Q_0$ with satisfying the relations

$$\begin{aligned} [e_i, f_j] &= \delta_{ij} h_i \\ [h_i, e_j] &= \langle h_i^\vee, h\rangle e_j \\ [h_i, f_j] &= -\langle h_i^\vee, h\rangle f_j \\ [e_i, f_j] &= \delta_{ij} h_i \\ \operatorname{ad}(e_i)^{1-(i,j)_{\Pi_Q}}(e_j) &= 0 \qquad \text{if } (i,i)_{\Pi_Q} > 0 \\ \operatorname{ad}(f_i)^{1-(i,j)_{\Pi_Q}}(f_j) &= 0 \qquad \text{if } (i,i)_{\Pi_Q} > 0. \end{aligned}$$

5.3 GENERALISED KAC–MOODY LIE ALGEBRAS FOLLOWING BORCHERDS AND BOZEC

Definition 5.3 ([Bor88])**.** Let $\mathfrak{h}$ be real vector space with symmetric bilinear form $(-,-)$. Let $\Pi = \{h_i\}_{i\in I} \subseteq \mathfrak{h}$ be a countable subset such that $(h_i, h_j) \leqslant 0$ if $i \neq j$ and $2(h_i, h_j)/(h_i, h_i) \in \mathbb{Z}$ if $(h_i, h_i) \geqslant 0$. The *generalised Kac–Moody (GKM) Lie algebra* or *Borcherds–Kac–Moody (BKM) Lie algebra* associated to $(\mathfrak{h}, (-,-), \{h_i\}_{i\in I})$ is the Lie algebra

$\mathfrak{g}_{\Pi,(-,-)}$ generated by $\mathfrak{h}$ and elements e_i, f_i for $i \in I$ and satisfying the following relations for all e_i, h_i, f_i, $i \in I$ and for all $h, h' \in H$

$$\begin{aligned}
[h, h'] &= 0 \\
[h, e_i] &= (h, h_i)e_i \\
[h, f_i] &= -(h, h_i)f_i \\
[e_i, f_j] &= \delta_{ij}h_i \\
\mathrm{ad}(e_i)^{1-(h_i,h_j)}(e_j) &= 0 && \text{if } (h_i, h_i) > 0 \\
\mathrm{ad}(f_i)^{1-(h_i,h_j)}(f_j) &= 0 && \text{if } (h_i, h_i) > 0 \\
[e_i, e_j] &= 0 && \text{if } (h_i, h_j) = 0 \\
[f_i, f_j] &= 0 && \text{if } (h_i, h_j) = 0.
\end{aligned} \tag{5.4}$$

Example 5.4. Every symmetrisable Kac–Moody Lie algebra $\mathfrak{g}(A)$ is a GKM Lie algebra. Let D be a diagonal matrix such that DA is symmetric. Let $(-,-)_{DA}$ be the symmetric bilinear form defined by DA. Then $\mathfrak{g}(A) \cong \mathfrak{g}(DA) \cong \mathfrak{g}_{(-,-)_{DA}}$.

Example 5.5. Bozec [Boz15, Definition 5] found the correct generalisation of Example 5.2 to quivers with loops. Let Q be a quiver (possibly with loops). Let $Q_0^{\mathrm{real}} \subseteq Q_0$ be the subset of vertices without loops and let Q_0^{im} be the subset of vertices with loops. Denote by $(-,-)_{\Pi_Q}$ the symmetrised Euler pairing of Q. Let $I_\infty = (Q_0^{\mathrm{real}} \times \{1\}) \cup (Q_0^{\mathrm{im}} \times \mathbb{N}_{\geqslant 1})$. Define a pairing $(-,-)$ in I_∞ by $((i',n),(j',m)) := nm(i',j')_{\Pi_Q}$.

The *Bozec–Borcherds–Kac–Moody (BBKM) Lie algebra* $\mathfrak{g}_Q$ associated to Q is the Lie algebra generated by Chevalley generators e_i, f_i for all $i \in I_\infty$ and h_i' for all $i' \in Q_0$ and subject to the following relations for all $i, j \in I_\infty$ and $i', j' \in Q_0$.

$$\begin{aligned}
[h_{i'}, h_{j'}] &= 0 \\
[h_{i'}, e_j] &= (i,j)e_i \\
[h_{i'}, f_j] &= -(i,j)f_i \\
[e_i, f_j] &= n\delta_{ij}h_{i'} && \text{for } i = (i',n) \in I_\infty \\
\mathrm{ad}(e_i)^{1-(i,j)}(e_j) &= 0 && \text{if } (i,i) > 0 \\
\mathrm{ad}(f_i)^{1-(i,j)}(f_j) &= 0 && \text{if } (i,i) > 0 \\
[e_i, e_j] &= 0 && \text{if } (i,j) = 0 \\
[f_i, f_j] &= 0 && \text{if } (i,j) = 0.
\end{aligned} \tag{5.5}$$

Remark 5.6. There are less generators for the BBKM Lie algebra $\mathfrak{g}_Q$ than expected from the definition of the BKM Lie algebra $\mathfrak{g}_{I_\infty}$ associated to the generalised Cartan datum $(I_\infty, (-,-))$.

The BBKM Lie algebra $\mathfrak{g}_Q$ is not the sub Lie algebra of the BKM algebra $\mathfrak{g}_{I_\infty}$ generated by the smaller set of generators $\{e_i, f_i\}_{i \in I_\infty} \cup \{h_i\}_{i \in Q_0 \times \{1\}}$. In $\mathfrak{g}_{I_\infty}$ we have the relation $[e_i, f_j] = \delta_{ij}h_i$ wheras in $\mathfrak{g}_Q$ we have $[e_i, f_j] = n\delta_{ij}h_{i'}$. Thus $\mathfrak{g}_Q$ is the quotient of $\mathfrak{g}_{I_\infty}$ by the relation $h_i = nh_{i'}$ for all $i = (i', n) \in I_\infty$.

Recall the strictly seminilpotent cohomological Hall algebra $\mathscr{A}_{\Pi_Q}^{\mathrm{SNN},\psi}$ from §4.7 with a choice of ψ as in Definition 4.53. The degree zero part $\mathrm{H}^0(\mathcal{A}_{\Pi_Q}^{\mathrm{SNN},\psi})$ of this CoHA is spanned by fundamental classes of irreducible components $\mathfrak{M}_{\Pi_Q,c}^{\mathrm{SNN}}$ of the strictly seminilpotent stack $\mathfrak{M}_{\Pi_Q,c}^{\mathrm{SNN}}$. There is a combinatorial description of these irreducible components (see [Boz16]). Relevant for us are the irreducible components $\mathfrak{M}_{\Pi_Q,(i',n)}^{\mathrm{SNN}}$ corresponding to $n1_{i'}$-dimensional Π_Q-respresentations where the arrows in Q_1^* act by zero.

Theorem 5.7 ([Hen23, Theorem 1.3])**.** *There is an isomorphism of algebras*

$$\mathrm{U}(\mathfrak{n}_Q^+) \xrightarrow{\cong} \mathrm{H}^0(\mathcal{A}_{\Pi_Q}^{\mathrm{SNN},\psi}).$$

sending the generator $e_{(i',n)}$ *to the fundamental class* $[\mathfrak{M}_{\Pi_Q,(i',n)}^{\mathrm{SNN}}]$

This geometric realisation of the positive half $\mathrm{U}(\mathfrak{n}_Q^+)$ of the BBKM algebra $\mathrm{U}(\mathfrak{g}_Q)$ as the degree zero part $\mathrm{H}^0(\mathcal{A}_{\Pi_Q}^{\mathrm{SNN},\psi})$ is crucial for the proof of Theorem 6.5, one of the main results of this thesis. Specfically we use Theorem 5.7 in the proof of Lemma 6.21

5.4 MONOIDS WITH BILINEAR FORMS AS CARTAN DATA

The material in this section has appeared in joint work with Ben Davison and Lucien Hennecart [DHS23].

Definition 5.8. Let M be a commutative monoid and $(-,-)\colon M \times M \to \mathbb{Z}$ a bilinear form such that $(v,v) \in 2\mathbb{Z}$ for all $v \in M$. We say $v \in M$ satisfies *Crawley-Boevey's condition* if for any nontrivial decomposition $v = \sum_{j=1}^{s} v_j$ with $v_j \in M$ we have

$$2-(v,v) > \sum_{j=1}^{s}(2-(v_j,v_j)). \tag{5.6}$$

We give names to elements in M depending on the quadratic form $m \mapsto (v,v)$.

(i) We call $v \in M$ *real* if $(v,v) = 2$.
(ii) We call $v \in M$ *isotropic (imaginary)* if $(v,v) = 0$.
(iii) We call $v \in M$ *hyperbolic (imaginary)* if $(v,v) \leqslant -2$.
(iv) We call $v \in M$ a *primitive positive root* if $(v,v) \leqslant 2$ and v satisfies Crawley-Boevey's condition (5.6). We denote the set of these by $\Sigma_{M,(-,-)}$.
(v) We call $v \in M$ a *simple positive root* if it is primitive ($v \in \Sigma_{M,(-,-)}$) or a positive integer multiple of an isotropic primitive root ($m = nv'$ for $v' \in \Sigma_{M,(-,-)}^{\mathrm{iso}}$). We denote the set of these by $\Phi_{M,(-,-)}^{+}$.

For a subset $S \subseteq M$ we denote by $S^{\mathrm{real}}, S^{\mathrm{iso}}, S^{\mathrm{hyp}} \subseteq S$ the subsets of real, isotropic, hyperbolic elements in S.

Remark 5.9. a primitive positive root need not be a primitive element in the monoid M. To clarify the terminology we note that the adjective 'primitive' in this definition referes to 'root' and not 'element of M'.

Remark 5.10. The motivation behind the definition that we only define positive roots in the monoid, is that the monoid should be thought of as 'half' of the roots, and the rest of the roots should appear in the groupification.

We consider a *Poincaré function*

$$\begin{aligned} P\colon \Phi^+_{M,(-,-)} &\longrightarrow \mathbb{N}[t^{\pm 1/2}] \\ \nu &\longmapsto P_\nu(t^{1/2}) = \sum_{d\in\mathbb{Z}} p_{\nu,d} t^{d/2}. \end{aligned}$$

Its purpose is to assign to each $\nu \in \Phi_{M,(-,-)}$ a graded set of generators for $\mathfrak{n}_{M,(-,-)}$. The polynomial $P_\nu(t^{1/2})$ is equal to the shifted Poincaré polynomial of the graded vector space spanned by the generators of $\mathfrak{n}_{M,(-,-)}$ associated to $\nu \in M$.

Definition 5.11. Let M be a monoid with bilinear form $(-,-)$ satisfiyng $(\nu,\nu) \in 2\mathbb{Z}_{\leqslant 1}$ and $(\nu,w) < 0$ for $\nu \neq w \in \Phi_{M,(-,-)}$ and let $P\colon \Phi^+_{M,(-,-)} \to \mathbb{N}[t^{\pm 1/2}]$ be a Poincareé function.

The *generalised Kac–Moody Lie (super)algebra* associated to the monoid with bilinear form $(M,(-,-))$ and Poincaré function P is the Lie superalgebra generated by elements $e_{\nu,d,l}$, $f_{\nu,d,l}$, h_ν of cohomological degree (and parity) d, for each simple positive root $\nu \in \Phi^+_{M,(-,-)}$, $1 \leqslant l \leqslant p_{\nu,d}$ subject to the relations

$$\begin{aligned} h_{\nu+\nu'} &= h_\nu + h_{\nu'} \\ [h_\nu, h_{\nu'}] &= 0 \\ [h_\nu, e_{\nu',d,l}] &= (\nu,\nu')e_{\nu',d,l} \\ [h_\nu, f_{\nu',d,l}] &= -(\nu,\nu')f_{\nu',d,l} \\ [e_{\nu,d,l}, f_{\nu',d',l'}] &= \delta_{\nu,\nu'}\delta_{d,d'}\delta_{l,l'}h_\nu \\ \mathrm{ad}(e_{\nu,d,l})^{1-(\nu,\nu')}(e_{\nu',d',l'}) &= 0 && \text{if } (\nu,\nu) = 2 \\ \mathrm{ad}(f_{\nu,d,l})^{1-(\nu,\nu')}(f_{\nu',d',l'}) &= 0 && \text{if } (\nu,\nu) = 2 \\ [e_{\nu,d,l}, e_{\nu',d',l'}] &= 0 && \text{if } (\nu,\nu) = 0 \\ [f_{\nu,d,l}, f_{\nu',d',l'}] &= 0 && \text{if } (\nu,\nu) = 0 \end{aligned}$$

In words: the triple of genereators $e_{m,d,l}, h_m, f_{m',d',l'}$ behave like a usual triple except the 'h_m-degree' of the positive and negative generators $e_{m,d,l}, f_{m',d',l'}$ are determined by the bilinear form on the monoid M; positive and negative generators corresponding to real roots satisfy the Serre relations and positive and negative generators corresponding to isotropic roots commute. Note that there are no relations between two positive generators labelled by hyperbolic roots.

Remark 5.12. Defining generalised Kac–Moody Lie algebras from a monoid in this way emphasises that we really only start with "the positive half" of the GKM as is usual when considering cohomological Hall algebras. Considering cohomological Hall algebras only naturally includes raising operators.

5.5 2-CALABI–YAU CATEGORIES AS CARTAN DATA

The material in this section has appeared in joint work with Ben Davison and Lucien Hennecart [DHS23].

Let $\mathscr{A}$ be a 2CY Abelian category with a good moduli theory $\mathrm{JH}\colon \mathfrak{M}_{\mathscr{A}} \to \mathcal{M}_{\mathscr{A}}$.

Let $(-,-)_{\mathscr{A}}$ be the Euler form of $\mathscr{A}$. Let $R\mathcal{H}om_{\mathscr{A}}$ be the RHom complex on $\mathfrak{M}_{\mathscr{A}} \times \mathfrak{M}_{\mathscr{A}}$. By Assumption 3.39 the Euler form of two objects $E, F \in \mathscr{A}$ can be computed as the Euler characterstic of the fibre $R\mathcal{H}om_{\mathscr{A}}|_{(E,F)}$, hence

$$\begin{aligned} \mathfrak{M} \times \mathfrak{M} &\longrightarrow \mathbb{Z} \\ (E,F) &\longmapsto (E,F)_{\mathscr{A}} = \chi(R\mathcal{H}om_{\mathscr{A}}|_{(E,F)}) \end{aligned}$$

is locally constant, and descends to a bilinear form

$$(-,-)\colon \pi_0(\mathcal{M}_{\mathscr{A}}) \times \pi_0(\mathcal{M}) \longrightarrow \pi_0(\mathcal{M}_{\mathscr{A}}).$$

Thus from the category $\mathscr{A}$ we obtain a monoid with bilinear form $(\pi_0(\mathcal{M}_{\mathscr{A}}), (-,-)_{\mathscr{A}})$.

Definition 5.13. We define the sets of *(real/isotropic/hyperbolic) primitive positive roots and (real/isotropic/hyperbolic) simple positive roots of $\mathscr{A}$* to be those of the monoid with bilinear form $(\pi_0(\mathcal{M}), (-,-)_{\mathscr{A}})$. We write

$$\begin{aligned} \Sigma_{\mathscr{A}} &:= \Sigma_{\pi_0(\mathcal{M}_{\mathscr{A}}),(-,-)_{\mathscr{A}}} \text{ and} \\ \Phi^+_{\mathscr{A}} &:= \Phi^+_{\pi_0(\mathcal{M}_{\mathscr{A}}),(-,-)_{\mathscr{A}}}. \end{aligned}$$

Suppose $\mathscr{A}$ is an 2CY Abelian category with a good moduli theory $\mathrm{JH}\colon \mathfrak{M}_{\mathscr{A}} \to \mathcal{M}_{\mathscr{A}}$. As discussed in §4.3 $\mathcal{M}_{\mathscr{A}}$ together with the direct sum morphism $\oplus\colon \mathcal{M}_{\mathscr{A}} \times \mathcal{M}_{\mathscr{A}} \longrightarrow \mathcal{M}_{\mathscr{A}}$ is a monoid scheme and hence induces the structure of a monoid on the set of connected components $\pi_0(\mathcal{M}_{\mathscr{A}})$. We denote by $[F_i] \in \pi_0(\mathcal{M}_{\mathscr{A}})$ the connected component which contains the point corresponding to F_i.

Given a collection $\{F_i\}$ of simple objects in $\mathscr{A}$ with Ext-quiver $\overline{Q}_{\{F_i\}}$ the assignment

$$\begin{aligned} \lambda_{\{F_i\}}\colon \mathbb{N}^{Q_0} &\longrightarrow \pi_0(\mathcal{M}_{\mathscr{A}}) \\ \underline{d} &\longmapsto \sum_{i=1}^{n} d_i [F_i] \end{aligned}$$

defines a monoid morphism.

Proposition 5.14 (Ext-quiver compatbility of roots)**.**

(i) *The morphism* $\lambda_{\{F_i\}}$ *is compatible with Euler forms, i. e.*

$$(\underline{d}, \underline{d}')_{\Pi_{Q_{\{F_i\}}}} = (\lambda_{\{F_i\}}(\underline{d}), \lambda_{\{F_i\}}(\underline{d}'))_{\mathscr{A}} \tag{5.7}$$

for all $\underline{d}, \underline{d}' \in \mathbb{N}^{Q_0}$.

(ii) *A dimension vector* $\underline{d} \in \mathbb{N}^{Q_0}$ *is real/isotropic/hyperbolic if and only if its image* $\lambda_{\{F_i\}}(\underline{d})$ *is real/isotropic/hyperbolic in* $\pi_0(\mathscr{A})$.

(iii) *A dimension vector* $\underline{d} \in \mathbb{N}^{Q_0}$ *is primitive/simple if and only if its image* $\lambda_{\{F_i\}}(\underline{d})$ *is primitive/simple in* $\pi_0(\mathscr{A})$.

In formulas, we have for all $\star \in \{\mathrm{real}, \mathrm{iso}, \mathrm{hyp}\}$

$$\lambda_F^{-1}(\Sigma^{\star}_{\mathscr{A}}) = \Sigma^{\star}_{\Pi_Q}$$
$$\lambda_F^{-1}(\Phi^{+,\star}_{\mathscr{A}}) = \Phi^{+,\star}_{\Pi_Q}.$$

Proof. The compatibility (5.7) is a direct consequence of the definition of the Ext-quiver and the Euler form for Π_Q. Then (5.7) immediately implies (ii). Again by (5.7) if $\lambda_{\{F_i\}}(\underline{d})$ satisfies Crawley-Boevey's condition 5.6, then so does $\underline{d}$, implying (iii). □

Both of the next propositions hinge on the Ext-quiver neighbourhood theorem (Theorem 3.53 and Corollary 3.55) and the generic topology of the good moduli space morphism $\mathrm{JH}\colon \mathfrak{M}_{\mathscr{A}} \to \mathcal{M}_{\mathscr{A}}$. They consitute the key geometric input for the main results of this thesis.

Proposition 5.15 (Geometry of primitive roots, preprojective case, [Cra01])**.** *Let* Q *be a quiver. There is a simple representation of* Π_Q *with dimension vector* $\underline{d}$ *if and only if* $\underline{d}$ *satisfies Crawley-Boevey's condition* (5.6).

For such $\underline{d} \in \Sigma_{\Pi_Q}$ *the good moduli space* $\mathcal{M}_{\Pi_Q,\underline{d}}$ *is reduced irreducible and normal of dimension* $2 - (\underline{d}, \underline{d})_{\Pi_Q}$ *and there is an open dense subset* $\mathcal{M}^{s}_{\Pi_Q,\underline{d}} \subseteq \mathcal{M}_{\Pi_Q,\underline{d}}$ *which parametrises simple* $\underline{d}$*-dimensional* Π_Q*-modules. The locus* $\mathcal{M}^{s}_{\Pi_Q,\underline{d}}$ *is precisely the locus over which* $\mathrm{JH}_{\Pi_Q}\colon \mathfrak{M}_{\Pi_Q,\underline{d}} \longrightarrow \mathcal{M}_{\Pi_Q,\underline{d}}$ *is a* $\mathbb{G}_m$*-gerbe.*

For $\underline{d} \in \Sigma_{\Pi_Q}$ *the zero locus* $\mu^{-1}_{Q,\underline{d}}(0)$ *of the moment map* (3.8) *is reduced and irreducible and a local complete intersection.*

Proof. This is mainly [Cra01, Theorem 1.2]. Normality of $\mathcal{M}_{\Pi_Q,\underline{d}}$ is a special case of [Cra03, Theorem 1.1]. The smoothness of $\mathcal{M}^{s}_{\Pi_Q,\underline{d}}$ follows from [Cra01, Lemma 6.5]. □

Corollary 5.16. *Let* $\mathscr{A}$ *be a 2CY Abelian category with a good moduli theory. Suppose* $v \in \Sigma_{\mathscr{A}}$*. Then the good moduli space* $\mathcal{M}_{\mathscr{A},v}$ *is normal.*

Proof. Normality is an étale local property . Thus by Theorem 3.53 it suffices to check normality for moduli of preprojective representations $\mathfrak{M}_{\Pi_Q,\underline{d}}$. □

Definition 5.17. The open subscheme $\mathcal{M}^{s}_{\mathscr{A}} \subseteq \mathcal{M}_{\mathscr{A}}$ over which the good moduli space morphism $\mathrm{JH}\colon \mathfrak{M}_{\mathscr{A}} \to \mathcal{M}_{\mathscr{A}}$ is a $\mathbb{G}_m$-gerbe is called *the locus of simple objects*.

Proposition 5.18 (Geometry of roots of preprojective algebras [Cra02])**.** *Let* Q *be a quiver. Suppose* $\underline{d} \in \mathbb{N}^{Q_0}$ *is a root. Then there exists a unique decomposition* $\underline{d} = \sum_i m_i \underline{d}^{(i)}$ *as a sum of distinct primitive roots* $\underline{d}^{(i)} \in \Sigma_{\Pi_Q}$ *such that any other such decomposition is a refinement. Moreover, the direct sum morphism*

$$\oplus\colon \prod_i \mathcal{M}_{\Pi_Q,\underline{d}^{(i)}}^{\times m_i} \longrightarrow \mathcal{M}_{\Pi_Q,\underline{d}}$$

induces an isomorphism of underlying reduced schemes

$$\prod_i \mathrm{Sym}^{m_i}(\mathcal{M}_{\Pi_Q,\underline{d}^{(i)}})_{\mathrm{red}} \xrightarrow{\cong} (\mathcal{M}_{\Pi_Q,\underline{d}})_{\mathrm{red}}.$$

In particular, $\mathcal{M}_{\Pi_Q,\underline{d}}$ *is irreducible of dimension* $\sum_i (2-(m_i\underline{d}^{(i)}, m_i\underline{d}^{(i)}))$.

Proposition 5.19 (Geometry of primitive roots, general case)**.**

(i) For all $\nu \in \Sigma_{\mathscr{A}}$ *the good moduli space* $\mathcal{M}_{\mathscr{A},\nu}$ *is irreducible of dimension*

$$\dim \mathcal{M}_{\mathscr{A},\nu} = 2-(\nu,\nu)_{\mathscr{A}}.$$

In particular, for real roots $\nu \in \Sigma_{\mathscr{A}}^{\mathrm{real}}$ *the moduli space* $\mathcal{M}_{\mathscr{A}}$ *is a point.*

(ii) The set $\Sigma_{\mathscr{A}}$ *is equal to the set of connected components* $\nu \in \pi_0(\mathcal{M}_{\mathscr{A}})$ *such that there is an open subset* $\mathcal{M}_\nu^s$ *of* $\mathcal{M}_\nu$ *over which the good moduli space morphism* JH *is a* $\mathbb{G}_m$*-gerbe. (Hence* $\dim \mathfrak{M}_{\mathscr{A},\nu} = 1 + \dim \mathcal{M}_{\mathscr{A},\nu}$*) for all primitive* ν*.*

(iii) The open subset $\mathcal{M}_{\mathscr{A}}^s$ *from (ii) is smooth. Moreover, if* $\mathscr{A}$ *is globally 2CY (in the sense of Definition 3.45), then the Calabli–Yau structure induces a symplectic structure on* $\mathcal{M}_{\mathscr{A}}^s$*.*

Proof. We first show (i). For a connected scheme irreducibility is a local property. By Corollary 3.55 the moduli space $\mathcal{M}_{\mathscr{A},\nu}$ is irreducible if and only if for all semisimple objects $\bigoplus_{i=1}^n F_i^{\oplus m_i}$ in $\mathscr{A}$ the moduli spaces

$$\mathcal{M}_{\Pi_{Q_{\{F_i\}}},\underline{m}} = \mu_{Q_{\{F_i\}},\underline{m}}^{-1}(0)/\!\!/\mathrm{GL}(\underline{m})$$

are irreducible. The latter holds by Proposition 5.15, which is applicable by Proposition 5.14.

For irreducible schemes, dimension is a local quantity. Thus again applying Corollary 3.55, Proposition 5.15, and Proposition 5.14 we deduce the dimension of $\mathcal{M}_{\mathscr{A},\nu}$ to be $2-(\nu,\nu)_{\mathscr{A}}$.

Items (ii) and (iii) are again local properties which are verified using Corollary 3.55 and Proposition 5.15. If $\mathscr{A}$ is globally 2CY, then symplectic structure on $\mathcal{M}_{\mathscr{A},\nu}^s$ is induced from the Brav–Dyckerhoff symplectic structure (Remark 3.47) on $\mathfrak{M}_{\mathscr{A},\nu}^s$. ☐

Proposition 5.20 (Geometry of isotropic roots)**.** *Suppose* $\nu \in \Sigma_{\mathscr{A}}^{\mathrm{iso}}$ *is a primitive isotropic root.*

(i) For every $d \geqslant 0$ $\mathcal{M}_{\mathscr{A},d\nu_0}$ *is irreducible of dimension* $2d$*.*

(ii) The morphism of reduced subschemes

$$\mathrm{Sym}^d(\mathcal{M}_{\mathscr{A},\nu_0})_{\mathrm{red}} \longrightarrow (\mathcal{M}_{\mathscr{A},d\nu_0})_{\mathrm{red}} \tag{5.8}$$

induced by the direct sum map is an isomorphism.

(iii) For every isotropic root ν *there is a unique integer* d *such that* $(\mathbb{C}^\times)^d$ *is the stabiliser of a generic closed point in* $\mathfrak{M}_\nu$*.*

(iv) For every isotropic root $\nu \in \Phi^{\mathrm{iso}}_{\mathscr{A}}$ *there is a unique integer* d *and isotropic primitive root* $\nu_0 \in \Sigma^{\mathrm{iso}}_{\mathscr{A}}$ *such that* $\nu = d\nu_0$*.*

Proof. Irreducibility and dimension can by determined locally, which is done using Ext-quiver neighbourhoods and [Cra02, Corollary 1.4].

The direct sum morphism $\oplus\colon \mathcal{M}^{\times d}_{\mathscr{A},\nu_0} \to \mathcal{M}_{\mathscr{A},d\nu_0}$ is finite and S_d-equivariant with resepct to the S_d-action permuting the factors on $\mathcal{M}^{\times d}_{\mathscr{A},\nu_0}$ and the trivial action on $\mathcal{M}_{\mathscr{A},d\nu_0}$. Hence it factors through the symmetric product

$$\begin{tikzcd}
\mathcal{M}^{\times d}_{\mathscr{A},\nu_0} \arrow[rr, "\oplus"] \arrow[dr] & & \mathcal{M}_{\mathscr{A},d\nu_0} \\
& \mathrm{Sym}^d(\mathcal{M}_{\mathscr{A},\nu_0}) \arrow[ur, "h"] &
\end{tikzcd}$$

The morphism $\mathrm{Sym}^d(\mathcal{M}_{\mathscr{A},\nu_0}) \to \mathcal{M}_{\mathscr{A},d\nu_0}$ is closed and injective. Moreover, the two varieties $\mathrm{Sym}^d(\mathcal{M}_{\mathscr{A},\nu_0})$ and $\mathcal{M}_{\mathscr{A},d\nu_0}$ are of the same dimension, implying that h is dominant, hence surjective by irreducibility of $\mathcal{M}_{\mathscr{A},d\nu_0}$. Therefore, h is a homeomorphism.

Moreover, by taking Ext-quiver neighbourhoods and [Cra02, Theorem 1.1] we deduce that h restricts to an isomorphism on formal completions at all closed points. Thus h is an isomorphism.

Item (iii) is also a local property. Again taking Ext-quiver neighbourhoods and using the isomorphism (5.8) we deduce that indeed the generic stabiliser is $(\mathbb{C}^\times)^d$.

The integer d is uniquely determined by the dimension of the generic stabiliser as in (iii). Suppose $\nu = d\nu_0 = d\nu_1$, where ν_0, ν_1 are isotropic primitive roots. Consider the small diagonals

$$\mathcal{M}_{\mathscr{A},\nu_0} \hookrightarrow \mathcal{M}_{\mathscr{A},\nu} \hookleftarrow \mathcal{M}_{\mathscr{A},\nu_1}$$

Applying the isomorphism (5.8) twice, we find both of the small diagonals, have the same image Δ. From the modular interpretation of the locus of the small diagonal, we have that a general point of Δ corresponds to a semisimple object $F_0^{\oplus d} \cong F_1^{\oplus d}$, where F_0 is a simple obect of class ν_0 and F_1 is a simple object of class F_1. It follows that $F_0 \cong F_1$ and thus $\nu_0 = \nu_1$. □

5.6 GKM MIXED HODGE MODULES FOR MONOID SCHEMES

The material in this section has appeared in joint work with Ben Davison and Lucien Hennecart [DHS23].

Throughout this section we fix $(\mathcal{M},\oplus,\eta)$ a commutative monoid scheme. We assume that the underlying scheme $\mathcal{M}$ is reduced and separated and its connected components are of finite type. We also assume that $\oplus\colon \mathcal{M}\times\mathcal{M}\to\mathcal{M}$ is a finite morphism. As in §4.3 these assumptions are made so that we can define the symmetric monoidal structure $\boxdot$ on the category of mixed Hodge modules $\mathrm{MHM}(\mathcal{M},\oplus)$.

The set of connected components $\pi_0(\mathcal{M})$ inherits via $\oplus$ the structure of a monoid: define $\nu+\nu'\in\pi_0(\mathcal{M})$ to be the connected component containing the image of $\oplus\colon \mathcal{M}_\nu\times\mathcal{M}_{\nu'}\to\mathcal{M}$.

Assume furthermore that the unit morphism $\eta\colon \mathrm{pt}\to\mathcal{M}$ is an isomorphism onto a connected component $\mathcal{M}_0$, with complement $\mathcal{M}_>$, and the morphism $\coprod_{n\geqslant 1}\mathcal{M}_>^n\to\mathcal{M}$ induced by the monoidal structure on $\mathcal{M}$ is finite.

Remark 5.21*.* This assumption guarantees that the free algebras below are well-defined.

Remark 5.22*.* The assumption clearly holds for the good moduli space $\mathcal{M}_{\mathscr{A}}$ of a 2CY Abelian category $\mathscr{A}$.

We fix a bilinear form $(-,-)\colon \pi_0(\mathcal{M})\times\pi_0(\mathcal{M})\to\mathbb{Z}$.

Consider the morphism

$$\begin{aligned}\Delta\colon \mathbb{N}\times\mathcal{M} &\longrightarrow \mathcal{M}\\ (n,x) &\longmapsto \oplus(\underbrace{x,\dots,x}_{n\text{ times}})\end{aligned}$$

defined as the union of the compositions

$$\{n\}\times\mathcal{M}\xrightarrow{\Delta_n}\mathcal{M}^{\times n}\xrightarrow{\oplus}\mathcal{M}$$

where Δ_n are the n-fold small diagonals.

Definition 5.23 (Generators for GKM MHMs)**.** Recall the definition of real, isotropic, and hyperbolic roots for $(\pi_0(\mathcal{M}),(-,-))$.

Define the *(positive) Chevalley generators* of the positive half of the GKM sheaf associated to $(\mathcal{M},\oplus)$ to be the $\pi_0(\mathcal{M})$-graded sheaf

$$\mathcal{G}_{(\mathcal{M},(-,-))}=\begin{array}{c}\bigoplus_{\nu\in\Sigma^{\mathrm{real}}_{\pi_0(\mathcal{M}),(-,-)}}\mathbb{Q}_{\mathcal{M}_\nu}\\ \oplus\\ \bigoplus_{\nu\in\Sigma^{\mathrm{iso}}_{\pi_0(\mathcal{M}),(-,-)}}\Delta_*\mathbb{Q}_{\mathcal{M}_\nu}\\ \oplus\\ \bigoplus_{\nu\in\Sigma^{\mathrm{hyp}}_{\pi_0(\mathcal{M}),(-,-)}}\mathcal{IC}(\mathcal{M}_\nu)\end{array}.$$

For $\nu\in\pi_0(\mathcal{M})$ we denote by $\mathcal{G}_\nu$ the corresponding graded piece of $\mathcal{G}$.

We use as a short hand the notation

$$\begin{aligned}\mathrm{Free}_{\mathcal{M},(-,-),\mathrm{Alg}}&=\mathrm{Free}_{\boxdot\text{-}\mathrm{Alg}}(\mathcal{G}_{(\mathcal{M},(-,-))})\\ \mathrm{Free}_{\mathcal{M},(-,-),\mathrm{Lie}}&=\mathrm{Free}_{\boxdot\text{-}\mathrm{Lie}}(\mathcal{G}_{(\mathcal{M},(-,-))})\end{aligned}$$

for the free $\boxdot$-algebra and free $\boxdot$-Lie algebra (§4.3) generated by the Chevalley generators. Now we have to define the relations between these generators.

For any two subobjects $\mathcal{X}, \mathcal{Y} \subseteq \mathrm{Free}_{\mathcal{M},(-,-),\mathrm{Alg}}$ we define the subobject $\mathrm{ad}(\mathcal{X})(\mathcal{Y}) \subseteq \mathrm{Free}_{\mathcal{M},(-,-),\mathrm{Alg}}$ to be the image of the morphism

$$\mathcal{X} \boxdot \mathcal{Y} \longrightarrow \mathrm{Free}_{\mathcal{M},(-,-),\mathrm{Alg}} \boxdot \mathrm{Free}_{\mathcal{M},(-,-),\mathrm{Alg}} \xrightarrow{[-,-]_{\mathrm{Free}_{\mathcal{M},(-,-),\mathrm{Alg}}}} \mathrm{Free}_{\mathcal{M},(-,-),\mathrm{Alg}}$$

Definition 5.24 (Serre relations and Serre ideal)**.** For every $v, w \in \Phi^{+}_{\pi_0(\mathcal{M}),(-,.)}$ define the *Serre relation/subobject*

$$\mathcal{R}_{v,w} \coloneqq \begin{cases} \mathrm{ad}(\mathcal{G}_v)^{1-(v,w)}(\mathcal{G}_w) & \text{if } (v,v) = 2 \\ 0 & \text{else} \end{cases}.$$

Consider their direct sum

$$\mathcal{R}_{\pi_0(\mathcal{M}),(-,-)} = \bigoplus_{v,w \in \Phi^{+}_{\pi_0(\mathcal{M}),(-,-)}} \mathcal{R}_{v,w}.$$

The two-sided *Serre ideal* $\mathcal{I}_{\mathcal{M},(-,-),\mathrm{Alg}}$ of $\mathrm{Free}_{\mathcal{M},(-,-),\mathrm{Alg}}$ is the image of the morphism

$$\mathrm{Free}_{\mathcal{M},(-,-),\mathrm{Alg}} \boxdot \mathcal{R}_{\pi_0(\mathcal{M}),(-,-)} \boxdot \mathrm{Free}_{\mathcal{M},(-,-),\mathrm{Alg}} \longrightarrow \mathrm{Free}_{\mathcal{M},(-,-),\mathrm{Alg}}$$

given by multiplication in $\mathrm{Free}_{\mathcal{M},(-,-),\mathrm{Alg}}$.

The *Serre Lie ideal* $\mathcal{I}_{\mathcal{M},(-,-),\mathrm{Lie}}$ of $\mathrm{Free}_{\mathcal{M},(-,-),\mathrm{Lie}}$ is the sum of the images of the morphisms

$$\mathcal{R}_{\pi_0(\mathcal{M}),(-,-)} \boxdot \mathrm{Free}^{\boxdot n}_{\mathcal{M},(-,-),\mathrm{Lie}} \xrightarrow{[[[-,-],-],\dots,-]} \mathrm{Free}_{\mathcal{M},(-,-),\mathrm{Lie}}$$

given by iterated Lie brackets.

Definition 5.25 (Positive half of GKM MHMs)**.** We define *the positive half of the generalised Kac–Moody Lie algebra mixed Hodge module of* $(\mathcal{M}, \oplus)$ to be the Lie algebra in $(\mathrm{MHM}(\mathcal{M}), \boxdot)$ generated by $\mathcal{G}_{\mathcal{M},(-,-)}$ subject to the Serre relations (5.24.) In a formula

$$\mathfrak{n}^{+}_{\mathcal{M},(-,-)} \coloneqq \mathrm{Free}_{\boxdot\text{-Lie}}(\mathcal{G}_{\mathcal{M},(-,-)})/\mathcal{I}_{\mathcal{M},(-,-),\mathrm{Lie}}.$$

By the discussion in §4.3 the universal enveloping algebra $U(\mathfrak{n}^{+}_{\mathcal{M},(-,-)})$ is canonically isomorphic to algebra in $(\mathrm{MHM}(\mathcal{M}), \boxdot)$ generated by $\mathcal{G}$ satisfying the Serre relations. In a formula

$$U(\mathfrak{n}^{+}_{\mathcal{M},(-,-)}) \cong \mathrm{Free}_{\boxdot\text{-Alg}}(\mathcal{G}_{\mathcal{M},(-,-)})/\mathcal{I}_{\mathcal{M},(-,-),\mathrm{Alg}}.$$

Remark 5.26. Unlike in the ordinary monoid case, we only define half of the GKM because there are no natural spaces for the negative half to 'live on'.

6

COHOMOLOGICAL INTEGRALITY FOR 2-CALABI–YAU CATEGORIES

> *"The truth will set you free. But not until it is finished with you."*
>
> —David Foster Wallace, *Infinite Jest*

In this chapter, which is based on parts of the preprints [DHS22; DHS23], we prove the PBW theorem (Theorem 6.24) for the cohomological Hall algebra of a 2-Calabi–Yau category. A key step is to first define and determine a subalgebra of the CoHA which we call the BPS algebra.

6.1 THE BPS ALGEBRA OF A 2-CALABI–YAU CATEGORY

The following theorem is necessary for the definition of BPS algebras of 2CY categories.

Theorem 6.1 (2-Calabi–Yau Purity [Dav23b])**.** *Let $\mathscr{A}$ be a 2-Calabi–Yau Abelian catgory with good moduli theory* $\mathrm{JH}\colon \mathfrak{M}_{\mathscr{A}} \to \mathcal{M}_{\mathscr{A}}$. *The pushforward* $\mathrm{JH}_*\mathbb{D}\mathbb{Q}^{\mathrm{vir}}_{\mathfrak{M}_{\mathscr{A}}} \in \mathrm{D}^+(\mathrm{MHM}(\mathcal{M}_{\mathscr{A}}))$ *is a pure complex of mixed Hodge modules with* $\mathcal{H}^i(\mathrm{JH}_*\mathbb{D}\mathbb{Q}^{\mathrm{vir}}_{\mathfrak{M}_{\mathscr{A}}}) = 0$ *for all* $i < 0$.

Proof. This is a direct application of the Verdier dual statement of [Dav23b, Theorem 6.1] to 2-Calabi–Yau Abelian categories with a good moduli theory. In particular, Assumption 3.43 guarantees assumption (∗) of [Dav23b, Theorem 6.1]. ☐

Let $\mathscr{A}$ be a 2CY Abelian category with a good moduli theory $\mathrm{JH}\colon \mathfrak{M}_{\mathscr{A}} \to \mathcal{M}_{\mathscr{A}}$. Recall the relative CoHA (defined in §4.6)

$$\mathcal{A}_{\mathscr{A}} = \mathrm{JH}_*\mathbb{D}\mathbb{Q}^{\mathrm{vir}}_{\mathfrak{M}_{\mathscr{A}}}, \mathrm{m}\colon \mathcal{A}_{\mathscr{A}} \times \mathcal{A}_{\mathscr{A}} \to \mathcal{A}_{\mathscr{A}}$$

which is an algebra object in $(\mathrm{D}^+(\mathrm{MHM}(\mathcal{M}_{\mathscr{A}})), \boxdot)$. And given a twist ψ we also define (see Definition 4.53) the ψ-twisted relative CoHA

$$(\mathcal{A}^{\psi}_{\mathscr{A}}, \mathrm{m}^{\psi} = (-1)^{\psi}\mathrm{m})$$

Definition 6.2 (The BPS Algebra of a 2-Calabi–Yau category)**.** Moreover, by Theorem 6.1 we have $\mathcal{H}^i(\mathrm{JH}_*\mathbb{D}\mathbb{Q}^{\mathrm{vir}}_{\mathfrak{M}_{\mathscr{A}}}) = 0$ for $i < 0$. Hence the morphism $\tau^{\leqslant 0}m$ endows the mixed Hodge module

$$\mathcal{BPS}^{\psi}_{\mathscr{A},\mathrm{Alg}} := \tau^{\leqslant 0}\mathcal{A}^{\psi}_{\mathscr{A}} = \mathcal{H}^0(\mathcal{A}^{\psi}_{\mathscr{A}})$$

with the structure of an algebra object in the symmetric monoidal category $(\mathrm{MHM}(\mathcal{M}_{\mathscr{A}}), \boxdot)$. We call $\mathcal{BPS}^{\psi}_{\mathscr{A},\mathrm{Alg}}$ the *BPS algebra sheaf of* $\mathscr{A}$.

This definition of the BPS algebra is appeared in the setting of preprojective algebras in [Dav22a, §6] and more generally in [Dav23b, §7.6.3].

Corollary 6.3. *The underlying MHM of the BPS algebra* $\mathcal{BPS}^{\psi}_{\mathscr{A},\mathrm{Alg}}$ *is a pure mixed Hodge module and hence decomposes as a direct sum of intersection complexes of variatons of mixed Hodge structure supported on closed subvarieties of* $\mathcal{M}_{\mathscr{A}}$*.*

Proof. This follows from $\mathcal{BPS}_{\mathscr{A},\mathrm{Alg}} = \mathcal{H}^0(\mathcal{A}_{\mathscr{A}})$ and the purity of $\mathcal{A}_{\mathscr{A}}$ (Theorem 6.1). ☐

Proposition 6.4 (Primitive summands of the BPS algebra)**.** *Let* $\mathscr{A}$ *be a 2CY Abelian category with a good moduli theory* $\mathrm{JH}\colon \mathfrak{M}_{\mathscr{A}} \to \mathcal{M}_{\mathscr{A}}$*.*

(i) For every primitive positive root $v \in \Sigma_{\mathscr{A}}$ *there are canonical monomorphisms*

$$\mathcal{IC}(\mathcal{M}_{\mathscr{A},v}) \hookrightarrow \mathcal{BPS}^{\psi}_{\mathscr{A},\mathrm{Alg}}. \tag{6.1}$$

(ii) For every primitive isotropic root $v \in \Sigma^{\mathrm{iso}}_{\mathscr{A}}$ *the morphism*

$$\begin{aligned} \Delta\colon \mathcal{M}_{\mathscr{A},v} \times \mathbb{N} &\hookrightarrow \bigsqcup_{n\in\mathbb{N}} \mathcal{M}_{\mathscr{A},nv} \\ (n,[E]) &\longmapsto [E^{\oplus n}] \end{aligned}$$

induces a monomorphism

$$\Delta_*\mathcal{IC}(\mathcal{M}_{\mathscr{A},v} \times \mathbb{N}) = \Delta_*\mathbb{Q}_{\mathcal{M}_{\mathscr{A},v}\times\mathbb{N}} \otimes \mathbb{L}^{2/2} \hookrightarrow \mathcal{BPS}^{\psi}_{\mathscr{A},\mathrm{Alg}} \tag{6.2}$$

Proof. Similar arguments appear in [Dav23b, §6.2.1] Let $v \in \Sigma_{\mathscr{A}}$ be a primitive root.

By Theorem 6.1 the MHM $\mathcal{BPS}_{\mathscr{A},\mathrm{Alg},v} = \mathcal{H}^0((\mathrm{JH}_{\mathscr{A},v})_*\mathbb{D}\mathbb{Q}^{\mathrm{vir}}_{\mathfrak{M}_{\mathscr{A},v}})$ is semisimple and hence splits as a direct sum of intersection mixed Hodge modules

$$\mathcal{H}^0((\mathrm{JH}_{\mathscr{A},v})_*\mathbb{D}\mathbb{Q}^{\mathrm{vir}}_{\mathfrak{M}_{\mathscr{A},v}}) \cong \bigoplus_{Z\subseteq\mathfrak{M}_{\mathscr{A}}} \mathcal{IC}(Z,\mathcal{L}) \tag{6.3}$$

where $Z \subseteq \mathcal{M}_{\mathscr{A},v}$ are closed subsets and $\mathcal{L}$ a semisimple variation of mixed Hodge structure on an open subset $U \subseteq Z$.

We first show that for $Z = \mathcal{M}_{\mathscr{A}}$ we have $\mathcal{L} \cong \mathbb{Q}_U \otimes \mathbb{L}^{(\dim_{\mathbb{C}} U)/2}$. Since v is primitive, by Proposition 5.19 the good moduli space morphism

$\mathrm{JH}_{\mathscr{A},v}\colon \mathfrak{M}_{\mathscr{A},v} \to \mathcal{M}_{\mathscr{A},v}$ is generically a $\mathbb{C}^\times$-gerbe. Let $U \subseteq \mathcal{M}_{\mathscr{A},v}$ be an open dense over which $\mathrm{JH}_{\mathscr{A},v}$ is a $\mathbb{C}^\times$-gerbe. Then because

$$H(B\mathbb{C}^\times) \cong \bigoplus_{n\geqslant 0} \mathbb{Q}\otimes \mathbb{L}^{(2n-2)/2}$$

we have

$$(\mathrm{JH}_{\mathscr{A},v})_*\mathbb{D}\mathbb{Q}^{\mathrm{vir}}_{\mathfrak{M}_{\mathscr{A},v}}|_U \cong \bigoplus_{n\geqslant 0}\mathbb{Q}_{\mathcal{M}_{\mathscr{A},v}}\otimes \mathbb{L}^{((v,v)_{\mathscr{A}}+2-2n)/2}$$

and taking the degree 0 piece we have

$$\mathcal{H}^0(\mathrm{JH}_{\mathscr{A},v})_*\mathbb{D}\mathbb{Q}^{\mathrm{vir}}_{\mathfrak{M}_{\mathscr{A},v}}|_U \cong \mathbb{Q}_U\otimes\mathbb{L}^{\dim_{\mathbb{C}}(U)/2}.$$

Since U is dense, it follows that

$$\mathcal{IC}(\mathcal{M}_{\mathscr{A},v}) = \mathcal{IC}(\mathcal{M}_{\mathscr{A},v},\mathcal{L}) \cong \mathcal{IC}(\mathcal{M}_{\mathscr{A},v},\mathbb{Q}_{\mathcal{M}_{\mathscr{A},v}}\otimes\mathbb{L}^{\dim_{\mathbb{C}}(\mathcal{M}_{\mathscr{A},v})})$$

thus there is some monomorphism $\mathcal{IC}(\mathcal{M}_{\mathscr{A},v}) \hookrightarrow (\mathrm{JH}_{\mathscr{A},v})_*\mathbb{D}\mathbb{Q}^{\mathrm{vir}}_{\mathfrak{M}_{\mathscr{A},v}}$. We claim that there is a canonical choice of monomorphism. Since $\mathcal{IC}(\mathcal{M}_{\mathscr{A},v})$ is simple and it appears only once in the decomposition (6.3), we have $\mathrm{Hom}(\mathcal{IC}(\mathcal{M}_{\mathscr{A},v}), (\mathrm{JH}_{\mathscr{A},v})_*\mathbb{D}\mathbb{Q}^{\mathrm{vir}}_{\mathfrak{M}_{\mathscr{A},v}}) \cong \mathbb{C}$ so it suffices to find a canonical non-zero morphism $\mathcal{IC}(\mathcal{M}_{\mathscr{A},v}) \to (\mathrm{JH}_{\mathscr{A},v})_*\mathbb{D}\mathbb{Q}^{\mathrm{vir}}_{\mathfrak{M}_{\mathscr{A},v}}$ which we can choose to be the unique one that restricts to the identity $\mathrm{id}\colon \mathbb{Q}_U\otimes\mathbb{L}^{\dim_{\mathbb{C}}(U)/2} \to \mathbb{Q}_U\otimes\mathbb{L}^{\dim_{\mathbb{C}}(U)/2}$. This shows that we have canonical monomorphisms (6.1).

The construction of (6.2) is similar, but more technical (see also [Dav23a, Appendix A]). Define the stacks $\mathfrak{D}_{\mathscr{A},nv_0}$ via pullback squares

$$\begin{tikzcd}
\mathfrak{D}_{\mathscr{A},nv_0} \arrow[r, hook, "\widetilde{\Delta}_n"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathfrak{M}_{\mathscr{A},nv_0} \arrow[d] \\
\mathcal{M}_{\mathscr{A},v_0} \arrow[r, hook, "\Delta_n"] & \mathcal{M}_{\mathscr{A},nv_0}
\end{tikzcd}$$

We apply the Ext-quiver neighbourhood theorem to a point on the diagonal $p \in \Delta_n(\mathcal{M}_{\mathscr{A},v_0})$. The point p corresponds to a semisimple object $F^{\oplus n}$, where F is simple of class v_0. Since F is isotropic, the half Ext-quiver of p is the quiver with one vertex and one loop, i. e. the Jordan quiver. The dimension vector is n. Thus $\mathfrak{M}_{\mathscr{A},nv_0} \to \mathcal{M}_{\mathscr{A},nv_0}$ analytically-locally around $p \in \mathcal{M}_{\mathscr{A},nv_0}$ is modelled by the Morphism $\mathfrak{M}_{\mathbb{A}^2,n} \to \mathcal{M}_{\mathbb{A},n}$ at a fat point of length n. Thus also $\mathfrak{D}_{\mathscr{A},nv_0}$ is locally modelled on $\mathfrak{N}_{\mathbb{A}^2,n} \to \mathcal{N}_{\mathbb{A}^2,n}$ where $\mathcal{N}_{\mathbb{A}^2,n}$ is the locus of length n subschemes with support a single point, i. e. the image of the small diagonal for the category of zero-dimensional sheaves on $\mathbb{A}^2$.

We claim that the natural morphism

$$(\Delta_n)_*\widetilde{\mathrm{JH}}_*\mathbb{D}\mathbb{Q}_{\mathfrak{D}_{\mathscr{A},nv_0}} = \mathrm{JH}_*\widetilde{\Delta}_n\mathbb{D}\mathbb{Q}_{\mathfrak{D}_{\mathscr{A},nv_0}} \longrightarrow \mathbb{D}\mathbb{Q}_{\mathfrak{M}_{\mathscr{A},nv_0}}$$

is a monomorphism. Indeed this can be checked locally which is done in [Dav23a, Proposition A.3]. To finish the proof it suffices to show

$(\Delta_n)_* \widetilde{\mathrm{JH}}_* \mathbb{D}\mathbb{Q}_{\mathfrak{D}_{\mathscr{A},nv_0}} \cong (\Delta_n)_* \mathbb{Q}_{\mathcal{M}_{\mathscr{A},v_0}} \otimes \mathbb{L}^{2/2}$. This follows again via a local argument which uses smoothness and irreducibility of $\mathcal{M}_{\mathscr{A},v_0}$ (Proposition 5.19), the irreducibility of $\mathfrak{N}_{\mathbb{A}^2,n}$, and $\dim \mathfrak{N}_{\mathbb{A}^2,n} = -1$. Details for a similar argument are carried out in the proof of [Dav23a, Proposition A.4]. □

Recall from §5.6 the generators of the GKM associated to $\mathscr{A}$.

$$\mathcal{G}_{\mathscr{A}} = \bigoplus_{v \in \Sigma^{\mathrm{real}}_{\mathscr{A}}} \mathbb{Q}_{\mathcal{M}_{\mathscr{A},v}} \oplus \bigoplus_{v \in \Sigma^{\mathrm{iso}}} \Delta_* \mathbb{Q}_{\mathcal{M}_{\mathscr{A},v}} \oplus \bigoplus_{v \in \Sigma^{\mathrm{hyp}}_{\mathscr{A}}} \mathcal{IC}(\mathcal{M}_{\mathscr{A},v})$$

As as consequence the inclusions (6.1) and (6.2) induce an algebra morphism

$$\widehat{\Psi}_{\mathscr{A}} \colon \mathrm{Free}_{\boxdot\text{-Alg}}\left(\mathcal{G}_{\mathscr{A}}\right) \longrightarrow \mathcal{BPS}^{\psi}_{\mathscr{A},\mathrm{Alg}}.$$

Theorem 6.5 (The BPS algebra is a GKM algebra,[DHS23])**.** *Let $\mathscr{A}$ be a 2-Calabi–Yau category with a good moduli theory $\mathrm{JH} \colon \mathfrak{M}_{\mathscr{A}} \to \mathcal{M}_{\mathscr{A}}$. Then the morphism $\widehat{\Psi}$ factors through the Serre ideal $\mathcal{I}$ (Definition 5.24) and the induced morphism*

$$\Psi_{\mathscr{A}} \colon \mathrm{U}(\mathfrak{n}^+_{\mathscr{A}}) = \mathrm{Free}_{\boxdot-\mathrm{Alg}}(\mathcal{G}_{\mathscr{A}})/\mathcal{I}_{\mathscr{A}} \longrightarrow \mathcal{BPS}^{\psi}_{\mathscr{A},\mathrm{Alg}}$$

is an isomorphism.

The proof of the theorem is in multiple steps and is concluded in §6.1.4. Here we give an overview of the steps.

Overview of the proof

The key point of the proof is to reduce the theorem to the case of preprojective algebras where more explicit tools are available to deduce the result.

Step 1. Show that the underlying sheaves of the GKM algebras $\mathrm{U}(\mathfrak{n}^+_{\mathscr{A}})$ restrict to the GKM algebras $\mathrm{U}(\mathfrak{n}^+_{\Pi_Q})$ for half Ext-quivers Q_F. See §6.1.1.

Step 2. Show that the underlying MHM of the BPS algebra $\mathrm{BPS}_{\mathscr{A},\mathrm{Alg}}$ restricts to $\mathrm{BPS}_{\Pi_{Q_F}}$ for half Ext-quivers Q_F. See §6.1.2.

Step 3. Show that the morphism $\Psi_{\mathscr{A}}$ restricts to the morphism $\Psi_{\Pi_{Q_F}}$ for half Ext-quivers Q_F. See Proposition 6.14. A key ingredient in this step is the compatibility of the CoHA products with the CoHA products for Ext-quivers Proposition 4.59.

These first three steps reduce Theorem 6.5 to showing that Ψ_{Π_Q} is an isomorphism for all quivers Q. Clearly it suffices to show that the restrictions $\Psi_{\Pi_Q,\underline{d}} = \Psi_{\Pi_Q,\underline{d}}|_{\mathcal{M}_{\Pi_Q,\underline{d}}}$ to all connected components are isomorphisms. This is done in §6.1.4.

Step 4. Observe that half Ext-quivers with dimension vectors $(Q_F, \underline{m})$ of semisimple $\underline{d}$-dimensional Π_Q representations have more vertices and smaller dimension vectors than $Q_{\underline{d}}$. See 6.18.

Step 5. Prove by reverse induction on the number of vertices and an induction on the size of dimension vectors that $\Psi_{\Pi_Q,\underline{d}}$ is an isomorphism. An essential ingredient for the induction step is the description of the top strictly semi-nilpotent CoHA as a Bozec–Borcherds algebra (Theorem 5.7). See Lemma 6.21 and Lemma 6.22.

Remark 6.6. Many of the technical arguments in the proof of Theorem 6.5 would be simpler if we had access to a stronger version of the local neighbourhood theorem (Theorem 3.53) which describes the correspondence diagrams locally in terms of the correspondence diagrams for the Ext-quiver. As a replacement for this geometric statement we instead analyse the $\boxdot$-algebra structure on $\mathcal{A}_{\mathscr{A}}$ (which is a shadow of the correspondence diagrams). The compatibility of CoHA multiplications on Σ-collections (Proposition 4.59) is sufficient compatibility between the correspondence diagrams to prove that $\Psi_{\mathscr{A}}$ is an isomorphism.

6.1.1 *Restricting the GKM algebra to Ext-quiver neighbourhoods*

The aim of this section is to show the compatibility of the GKM algebra of a 2CY Abelian category $\mathscr{A}$ when restricting it to Ext-quiver neighbourhoods. Suppose $\mathscr{A}$ is a 2CY Abelian category with a good moduli theory $\mathrm{JH}_{\mathscr{A}} \colon \mathfrak{M}_{\mathscr{A}} \longrightarrow \mathcal{M}_{\mathscr{A}}$. In this section and the next we will also work with the categories $\mathrm{Rep}(\Pi_Q)$ of representations of preprojective algebras Π_Q which by Theorem 3.49 are 2CY Abelian categories with a good moduli theory $\mathrm{JH}_{\Pi_Q} \colon \mathfrak{M}_{\Pi_Q} \longrightarrow \mathcal{M}_{\Pi_Q}$

Setting 6.7 (Systems of Ext-quivers for a Σ-collection of simple objects in a 2CY category)**.** Let $F_1, \dots, F_n \in \mathscr{A}$ be collection of simple objects in $\mathscr{A}$ of classes $v_i \coloneqq \mathrm{cl}(F_i) \in \pi_0(\mathcal{M}_{\mathscr{A}})$. By Assumption 3.43 the F_i form a Σ-collection in $\mathscr{A}$. Let Q be a half Ext-quiver of this Σ—collection. Recall we denote the one dimensional representatinos of Q by S_i for $i \in Q_0 = \{1, \dots, n\}$.

Let $x_i \colon \mathrm{pt} \hookrightarrow \mathcal{M}_{\mathscr{A}}$ be the closed point corresponding to F_i. The inclusions $\{F_i\} \hookrightarrow \mathcal{M}_{\mathscr{A}}$ and $\{S_i\} \hookrightarrow \mathcal{M}_{\Pi_Q}$ induce monoid morphisms

$$\iota_F \colon \mathbb{N}^{Q_0} \hookrightarrow \mathcal{M}_{\mathscr{A}}$$
$$\iota_{\mathrm{Nil}} \colon \mathbb{N}^{Q_0} \hookrightarrow \mathcal{M}_{\Pi_Q}.$$

Let $\lambda_F \colon \mathbb{N}^{Q_0} \to \pi_0(\mathcal{M}_{\mathscr{A}})$ be the morphism of monoids defined by

$$\lambda_F(\underline{m}) = \sum_{i \in Q_0} m_i F_i.$$

The morphism λ_F is compatibile with the Euler pairings (5.7). Given a choice of twist ψ for $\mathscr{A}$, we choose the twist ψ_F for Π_{Q_F} given by $\psi_F(-,-) = \psi(\lambda_F(-), \lambda_F(-))$.

Choose for every dimension vector $\underline{m} \in \mathbb{N}^{Q_0}$ an analytic Ext-quiver neighbourhood

$$\begin{tikzcd}
\mathfrak{M}_{\Pi_Q,\underline{m}} \arrow[d, "\mathrm{JH}_{\Pi_Q}"] & \mathfrak{U}_{\underline{m}} \arrow[l, hook'] \arrow[r, hook] \arrow[d, "p"] & \mathfrak{M}_{\mathscr{A},\lambda_F(\underline{m})} \arrow[d, "\mathrm{JH}_{\mathscr{A}}"] \\
\mathcal{M}_{\Pi_Q,\underline{m}} & \mathcal{U}_{\underline{m}} \arrow[l, hook'] \arrow[r, hook] & \mathcal{M}_{\mathscr{A},\lambda_F(\underline{m})}
\end{tikzcd} \tag{6.4}$$

of the point $\{\bigoplus_i F_i^{m_i}\} \in \mathcal{M}_{\mathscr{A},\nu_{\underline{m}}}$, for $\nu_{\underline{m}} = \lambda_F(\underline{m}) = \sum_i m_i \nu_i$.

Set $\mathcal{U} = \bigsqcup_{\underline{m}\in\mathbb{N}^{Q_0}} \mathcal{U}_{\underline{m}}$. The data of the analytic Ext-quiver neighbourhoods $\mathcal{U}_m$ yields the following commutative diagram of analyitc spaces

$$\begin{tikzcd}
 & \mathbb{N}^{Q_0} \arrow[dl, "\iota_{\mathrm{Nil}}"'] \arrow[d, "y"] \arrow[dr, "\iota_F"] & \\
\mathcal{M}_{\Pi_Q} & \mathcal{U} \arrow[l, hook', "\jmath_{\mathrm{Nil}}"'] \arrow[r, hook, "\jmath_F"] & \mathcal{M}_{\mathscr{A}}
\end{tikzcd}$$

where the horizontal morphisms $\jmath_{\mathrm{Nil}}, \jmath_F$ are analitic open embeddings.

Proposition 6.8 (Compatibility of generators and relations with Ext-quiver neighbourhoods)**.** *There is a natural isomorphism of* $\mathbb{N}^{Q_0}$*-graded mixed Hodge structures*

$$\iota_{\mathrm{Nil}}^! \mathcal{G}_{\Pi_Q} \cong \iota_F^! \mathcal{G}_{\mathscr{A}}.$$

which induces an isomorphism of $\mathbb{N}^{Q_0}$*-graded algebras*

$$\gamma_{\mathrm{Free}} \colon \mathrm{Free}_{\mathrm{Alg}}(\iota_{\mathrm{Nil}}^! \mathcal{G}_{\Pi_Q}) \xrightarrow{\cong} \mathrm{Free}_{\mathrm{Alg}}(\iota_F^! \mathcal{G}_{\mathscr{A}}).$$

Proof. Intersection complexes restrict to intersection complexes along analytic open embeddings 4.26. The proposition follows from the definition of the generators $\mathcal{G}$ and the compatibility of roots with Ext-quiver neighbourhoods Proposition 5.14. □

By Lemma 4.54, the functors $\iota_{\mathrm{Nil}}^!$ and $\iota_F^!$ are strict monoidal so they take $\boxdot$-algebras to $\mathbb{N}^{Q_0}$-graded algebras and additionally commute with the free algebra construction.

Recall the Serre relations $\mathcal{R}_{\Pi_Q}$ and $\mathcal{R}_{\mathscr{A}}$ defined in Definition 5.24.

Corollary 6.9. *There are natural isomorphisms of* $\mathbb{N}^{Q_0}$*-graded mixed Hodge strucures for every* $\underline{m}, \underline{m}' \in \mathbb{N}^{Q_0}$

$$\iota_{\mathrm{Nil}}^! \mathcal{R}_{\Pi_Q,\underline{m},\underline{m}'} \cong \iota_F^! \mathcal{R}_{\mathscr{A},\lambda_F(\underline{m}),\lambda_F(\underline{m}')}.$$

Proof. By Proposition 6.8 and the strict monoidality of the functors $\iota_{\mathrm{Nil}}^!$ and $\iota_F^!$ (Lemma 4.54) we have for all $\underline{m} \in \mathbb{N}^{Q_0}$ the natural isomorphisms

$$\iota_{\mathrm{Nil}}^! \mathcal{R}_{\Pi_Q} \cong \iota_F^! \mathcal{R}_{\mathscr{A},\lambda_F(\underline{m}),\lambda_F(\underline{m}')}.$$

□

Corollary 6.10. *There is a natural isomorphism of algebras*

$$\iota_{\mathrm{Nil}}^! U(\mathfrak{n}_{\Pi_Q}^+) \cong \iota_F^! U(\mathfrak{n}_{\mathscr{A}}^+).$$

6.1.2 *Restricting the BPS algebra to Ext-quiver neighbourhoods*

We continue to work in the Setting 6.7. The aim of this section is to show the compatibility of BPS algebras when restricting to Ext-quiver neighborhoods.

The commutativity of (6.4) implies that the canonical morphisms

$$\mathcal{H}^0(j^*_{\Pi_Q}\mathcal{A}^{\psi}_{\Pi_Q}) \longrightarrow \mathcal{H}^0(p_*\mathbb{D}\mathbb{Q}^{\mathrm{vir}}_{\mathfrak{U}}) \longleftarrow \mathcal{H}^0(j^*_{\mathscr{A}}\mathcal{A}^{\psi}_{\mathscr{A}}) \tag{6.5}$$

are isomorphisms in $\mathrm{MHM}(\mathcal{U})$. Since pullback for mixed Hodge modules by an analyitc-open embedding is t-exact, the isomorphisms (6.5) induce an isomorphism of graded mixed Hodge structures

$$\gamma_{\mathrm{BPS}}\colon \imath^!_{\mathrm{Nil}}\mathcal{BPS}^{\psi}_{\Pi_Q,\mathrm{Alg}} \xrightarrow{\cong} \imath^!_{\mathrm{F}}\mathcal{BPS}^{\psi}_{\mathscr{A},\mathrm{Alg}} \tag{6.6}$$

Proposition 6.11. *The following diagam of $\mathbb{N}^{Q_0}$-graded algebra morphisms commutes.*

$$\begin{tikzcd}
\imath^!_{\mathrm{Nil}}\,\mathrm{Free}_{\boxdot-\mathrm{Alg}}(\mathcal{G}_{\Pi_Q}) \arrow[r, "\gamma_{\mathrm{Free}}", "\cong"'] \arrow[d, "\imath^!_{\mathrm{Nil}}\widehat{\Psi}_{\Pi_Q}"] & \imath^!_{\mathrm{F}}\,\mathrm{Free}_{\boxdot-\mathrm{Alg}}(\mathcal{G}_{\mathscr{A}}) \arrow[d, "\imath^!_{\mathrm{F}}\widehat{\Psi}_{\mathscr{A}}"] \\
\imath^!_{\mathrm{Nil}}\mathcal{BPS}_{\Pi_Q,\mathrm{Alg}} \arrow[r, "\gamma_{\mathrm{BPS}}", "\cong"'] & \imath^!_{\mathrm{F}}\mathcal{BPS}_{\mathscr{A},\mathrm{Alg}}.
\end{tikzcd}$$

Proof. By the compatibility of CoHA products (Proposition 4.59) we have the isomorphism of algebras

$$\gamma\colon \imath^!_{\mathrm{Nil}}\mathcal{A}^{\psi}_{\Pi_Q} \longrightarrow \imath^!_{\mathrm{F}}\mathcal{A}^{\psi}_{\mathscr{A}}.$$

The products on $\mathrm{BPS}^{\psi}_{\Pi_Q,\mathrm{Alg}}$ and $\mathrm{BPS}^{\psi}_{\mathscr{A},\mathrm{Alg}}$ are obtained by applying $\tau^{\leqslant 0}$ to the products on the respective relative CoHAs $\mathcal{A}^{\psi}_{\Pi_Q}$ and $\mathcal{A}^{\psi}_{\mathscr{A}}$. It follows that γ truncates to the algebra morphism γ_{BPS}. Commutativity follows from the compatibility of generators with Ext-quiver neighborhoods Proposition 6.8 and the fact that the morphisms (6.1) and (6.2) are also compatible with Ext-quiver neighbourhoods. □

6.1.3 *Reduction to the case of preprojective algebras*

Lemma 6.12. *Let $\mathcal{B} \in D^+_c(X)$ be a constructible complex on a variety X. Then if $\mathcal{B} \neq 0$, there exists a $\mathbb{C}$-point $\imath_x\colon \mathrm{pt} \hookrightarrow X$ such that $\imath^!_x\mathcal{B} \neq 0 \neq \imath^*_x\mathcal{B}$*

Proof. By Verdier duality it suffices to show $\imath^*_x\mathcal{B} \neq 0$ for some x. For every $x \in X$ the functor $\imath^*_x\colon\ D^b_c(X) \to D^b_c(\mathrm{pt})$ is t-exact for the standard t-structures. Therfore if $\imath^*_x\mathcal{B} = 0$ for all x, then $\imath^*_x\mathcal{B}$ has vanishing cohomology sheaves. By conservativity of the system of cohomology functors [BBD, Proposition 1.3.7], $\mathcal{B}$ must then itself vanish. □

Corollary 6.13. *Let $\mathcal{B} \in D^+(\mathrm{MHM}(X))$ be a complex of mixed Hodge modules on a complex algebraic variety X. Then if $\mathcal{B} \neq 0$, there exists a $\mathbb{C}$-point $\imath_x\colon \mathrm{pt} \hookrightarrow X$ such that $\imath^!_x\mathcal{B} \neq 0 \neq \imath^*_x\mathcal{B}$*

Proof. Apply Lemma 6.12 to $\mathrm{rat}(\mathcal{B})$ $\square$

We will apply this corollary to sheaves which are compatible with Ext-quiver neighborhoods.

Proposition 6.14. *Suppose* $R_{\Pi_Q} \subseteq \ker(\widehat{\Psi}_{\Pi_Q})$ *for all quivers* Q*. Then* $R_{\mathscr{A}} \subseteq \ker(\widehat{\Psi}_{\mathscr{A}})$ *for all 2CY Abelian categories* $\mathscr{A}$ *with a good moduli theory. In this case* $\widehat{\Psi}_{\mathscr{A}}$ *factors through* $U(\mathfrak{n}^+_{\mathscr{A}})$*:*

$$\begin{tikzcd}
\mathrm{Free}_{\boxdot-\mathrm{Alg}}(\mathcal{G}_{\mathscr{A}}) \arrow[rr, "\widehat{\Psi}_{\mathscr{A}}"] \arrow[dr] & & \mathrm{BPS}^{\psi}_{\mathscr{A},\mathrm{Alg}} \\
& U(\mathfrak{n}^+_{\mathscr{A}}) \arrow[ur, "\Psi_{\mathscr{A}}"] &
\end{tikzcd}.$$

Suppose furthmore that Ψ_{Π_Q} *is an isomorphism for all quivers* Q*. Then* $\Psi_{\mathscr{A}}$ *is an isomorphism for all 2CY Abelian categories* $\mathscr{A}$ *with a good moduli theory.*

Proof. We prove the first part. We have $\mathcal{R}_{\mathscr{A}} \subseteq \ker(\widehat{\Psi}_{\mathscr{A}})$ if and only if the mixed Hodge module $\widehat{\Psi}(\mathcal{R}_{\mathscr{A}})$ is zero. Let $x \in \mathcal{M}_{\mathscr{A}}$ be a closed point corresponding to semisimple object $\bigoplus_i F_i^{l_i}$ where the F_i are simple. Let Q be a half Ext-quiver of the Σ-collection $\{F_i\}$ and let $\imath_{0_{\underline{l}}}\colon \mathrm{pt} \hookrightarrow \mathcal{M}_{\Pi_Q}$ be the inclusion of the zero representation. By the compatibility of Serre relations with Ext-quiver neighburhoods (Corollary 6.9), we have

$$\imath^!_{0_{\underline{l}}} \mathcal{R}_{\Pi_Q} \cong \imath^!_x \mathcal{R}_{\mathscr{A}}.$$

By the compatibility of the morphisms $\widehat{\Psi}$ with Ext-quiver neighbourhoods (Proposition 6.11), we have

$$\imath^!_{0_{\underline{l}}} \widehat{\Psi}(\mathcal{R}_{\Pi_Q}) \cong \imath^!_x \widehat{\Psi}_{\mathscr{A}}(\mathcal{R}_{\mathscr{A}}) \neq 0.$$

But by assumption we have $\Psi_{\Pi_Q}(\mathcal{R}_{\Pi_Q}) = 0$, thus indeed $\widehat{\Psi}(\mathcal{R}_{\mathscr{A}}) = 0$.

Since everything in sight is compatible with Ext-quiver neighbourhoods we obtain compatibility of the kernel and cokernel of Ψ with Ext-quiver neighbourhoods:

$$\imath^!_{\mathrm{Nil}} \ker(\Psi_{\Pi_Q}) \cong \imath^!_F \ker(\Psi_{\mathscr{A}})$$
$$\imath^!_{\mathrm{Nil}} \operatorname{coker}(\Psi_{\Pi_Q}) \cong \imath^!_F \operatorname{coker}(\Psi_{\mathscr{A}}).$$

Hence, by applying the same argument to the kernel and cokernel of $\Psi_{\mathscr{A}}$ we deduce the second part. $\square$

Remark 6.15 (Ext-quiver neighbourhood arguments). Below we use variants of the argument in the proof of Proposition 6.14. The rough strategy is to show that a (complex of) MHMs $\mathcal{T}_{\mathscr{A}}$ on $\mathcal{M}_{\mathscr{A}}$, which is defined out of the data of a 2CY category $\mathscr{A}$, vanishes by checking that it vanishes at every point $x \in \mathcal{M}_{TA}$. This is done by showing that the assignment $\mathscr{A} \mapsto \mathcal{T}_{\mathscr{A}}$ is 'compatible' with Ext-quiver neighbourhoods, which means that $\mathcal{T}_{\mathscr{A}}|_{\mathcal{U}_x} = \mathcal{T}_{\Pi_{Q_x}}|_{\mathcal{U}_X}$ for Ext-quiver neighbourhoods $\mathcal{U}_x$ of x. Then it suffices to showt that $\mathcal{T}_{\Pi_{Q_x}}$ vanishes for all of the possible Ext-quivers $Q_x, x \in \mathcal{M}_{\mathscr{A}}$.

6.1.4 *Proof of Theorem 6.5*

By Proposition 6.14 it suffices to prove Theorem 6.5 for $\mathscr{A} = \mathrm{Rep}(\Pi_Q)$ for all quivers Q.

The algebra morphisms $\widehat{\Psi}_{\Pi_Q} \colon \mathrm{Free}_{\boxdot\text{-Alg}}(\mathcal{G}_{\Pi_Q}) \to \mathrm{BPS}^{\psi}_{\Pi_Q,\mathrm{Alg}}$ are $\mathbb{N}^{Q_0}$ and if they factor through the Serre ideal $\mathcal{J}_{\Pi_Q}$ (which itself is $\mathbb{N}^{Q_0}$-graded) then so are the algebra morphisms $\Psi_{\Pi_Q} \colon \mathrm{U}(\mathfrak{n}^+_{\Pi_Q}) \to \mathrm{BPS}^{\psi}_{\Pi_Q,\mathrm{Alg}}$ $\mathbb{N}^{Q_0}$-graded. Thus to show that $\widehat{\Psi}_{\Pi_Q}$ factor through the Serre ideal they are isomorphisms it suffices to show that each graded piece $\widehat{\Psi}_{\Pi_Q,\underline{d}}$ factors through $\mathcal{J}_{\Pi_Q,\underline{d}}$ and the induced morphism

$$\Psi_{\Pi_Q,\underline{d}} \colon \mathrm{U}(\mathfrak{n}^+_{\Pi_Q}) \longrightarrow \mathrm{BPS}^{\psi}_{\Pi_Q,\mathrm{Alg},\underline{d}}$$

is an isomorphism in $\mathrm{MHM}(\mathcal{M}_{\Pi_Q,\underline{d}})$, for all $\underline{d} \in \mathbb{N}^{Q_0}$. This we prove by an induction on the cross-sum of the dimension vector and a reverse induction on the number of vertices of the quiver

6.1.4.1 *Adding scalars to loops*

We begin by observing an extra equivariance of the mixed Hodge modules $\mathrm{U}(\mathfrak{n}^+_{\Pi_Q})$ and $\mathrm{BPS}^{\psi}_{\Pi_Q,\mathrm{Alg},\underline{d}}$.

Definition 6.16 (Loop additive action)**.** Let Q be a quiver. Let L be the set of loops of Q and let $\overline{L} = L \sqcup L^*$ be the set of loops of the doubled quiver $\overline{Q}$. The group $\mathbb{G}_a^{\overline{L}}$ acts on $\mathfrak{M}_{\Pi_Q}$ and $\mathcal{M}_{\Pi_Q}$ as follows. An element $\lambda = (\lambda_l)_{l\in\overline{L}} \in \mathbb{G}_a^{\overline{L}}$ acts on a Π_Q-representation $(\rho_a)_{a\in Q_0}$ by

$$(\lambda\rho)_a \coloneqq \begin{cases} \rho_a + \lambda_a \mathrm{id} & \text{if } a \in \overline{L} \\ \rho_a & \text{else} \end{cases}$$

This action preserve the preprojective relation and is equivariant with respect to the gauge group action, thus we indeed have the action on $\mathfrak{M}_{\Pi_Q}$ and $\mathcal{M}_{\Pi_Q}$ as desired.

Lemma 6.17 (Loop action equivariance)**.** *The relative CoHA $\mathcal{A}^{\psi}_{\Pi_Q}$, relative BPS algebra $\mathcal{BPS}_{\Pi_Q,\mathrm{Alg}}$, the free algebra $\mathrm{Free}_{\boxdot\text{-Alg}}(\mathcal{G}_{\Pi_Q})$ and the GKM algebra $\mathrm{U}(\mathfrak{n}^+_{\Pi_Q})$ are all $\mathbb{G}_a^{\overline{L}}$-equivariant complexes of mixed Hodge modules on $\mathcal{M}_{\Pi_Q}$.*

Proof. The $\mathbb{G}_a^{\overline{L}}$-equivariance of JH_{Π_Q} implies the $\mathbb{G}_a^{\overline{L}}$-equivariance of $\mathcal{A}^{\psi}_{\Pi_Q}$ and $\mathcal{BPS}_{\Pi_Q,\mathrm{Alg}}$.

For every dimension vector $\underline{d}$ the intersection complex $\mathcal{IC}(\mathcal{M}_{\Pi_Q,\underline{d}})$ is the intermediate extension of a pure constant variation of mixed Hodge structures on a $\mathbb{G}_a^{\mathbb{L}}$-stable open subset, namely the constant variation of Hodge structure

$$\mathbb{Q}_{\mathcal{M}^s_{\Pi_Q,\underline{d}}} \otimes \mathbb{L}^{-\dim \mathcal{M}^s_{\Pi_Q}/2}$$

on $\mathcal{M}^{\mathrm{s}}_{\Pi_Q,\underline{d}}$. Furthermore, each small diagonal $\Delta_{\underline{d}} \colon \mathcal{M}_{\Pi_Q,\underline{d}} \longrightarrow \mathcal{M}_{n\underline{d}}$ is $\mathbb{G}_a^{\overline{L}}$-equivariant. Combining these two observations implies that the generators $\mathcal{G}_{\Pi_Q} \in \mathrm{MHM}(\mathcal{M}_{\Pi_Q})$ is a $\mathbb{G}_a^{\overline{L}}$-equivariant mixed Hodge module. It follows that the Serre relations $\mathcal{R}_{\Pi_Q}$ is also $\mathbb{G}_a^{\overline{L}}$-equivariant. Finally, we deduce that both $\mathrm{Free}_{\boxdot\text{-Alg}}(\mathcal{G}_{\Pi_Q})$ and $\mathrm{U}(\mathfrak{n}^+_{\Pi_Q})$ are $\mathbb{G}_a^{\overline{L}}$-equivariant. □

Similarly, the group $\mathbb{G}_a^L$ acts on $\mathfrak{M}_{\Pi_Q}$ and $\mathcal{M}_{\Pi_Q}$ via the natural inclusion $\mathbb{G}_a^L \subseteq \mathbb{G}_a^{\overline{L}}$.

Setting up the induction

Let QuivDim be the set of pairs $(Q, \underline{d})$ consisting of a quiver $Q = (Q_0, Q_1)$ and a dimension vector $\underline{d} = (d_i)_{i \in Q_0} \in \mathbb{N}^{Q_0}$ supported on all of Q, i.e. $d_i \neq 0$ for all $i \in Q_0$.

Consider the function

$$\begin{aligned} \mu \colon \mathrm{QuivDim} &\longrightarrow \mathbb{Z}_{>0} \times \mathbb{Z}_{<0} \\ (Q, \underline{d}) &\longmapsto (|\underline{d}|, -|Q_0|) = \left(\sum_{i \in Q_0} d_i, -|Q_0| \right). \end{aligned}$$

We endow QuivDim with the partial order pulled back from the lexicographic order on $\mathbb{Z}_{>0} \times \mathbb{Z}_{<0}$ along μ. Explicitly

$$(Q, \underline{d}) \leqslant (Q', \underline{d}') \iff |\underline{d}| < |\underline{d}'| \text{ or } |\underline{d}| = |\underline{d}'| \text{ and } |Q_0| \geqslant |Q_0|'$$

To every closed point x in the good moduli space $\mathcal{M}_{\Pi_Q}$ we associate a quiver with dimension vector $(Q'_x, \underline{m}_x) \in \mathrm{QuivDim}$ given by half of the Ext-quiver and multiplicity vector of a corresponding semisimple Π_Q-module $\bigoplus_i F_i^{m_i}$.

Lemma 6.18. *For all* $(Q, \underline{d}) \in \mathrm{QuivDim}$ *and for all closed points* $x \in \mathcal{M}_{\Pi_Q,\underline{d}}$ *we have*

(i) $|\underline{m}_x| \leqslant |\underline{d}|$
(ii) $\mu(Q'_x, \underline{m}_x) \leqslant \mu(Q, \underline{d})$
(iii) The following are equivalent
  (a) $\mu(Q'_x, \underline{m}_x) = \mu(Q, \underline{d})$
  (b) $(Q'_x, \underline{m}_x) = (Q, \underline{d})$
  (c) x *is in the* $\mathbb{G}_a^{\overline{L}}$*-orbit of (the point corresponding to)* $S_{\underline{d}}$.

Proof. Write $(Q', \underline{m})$ for $(Q'_x, \underline{m}_x)$. Let $F = \bigoplus_{j \in Q'_0} F_j^{\oplus m_j}$ be the semisimple Π_Q-module of dimension vector $\underline{d}$ corresponding to x (such that all m_i are nonzero and the F_j are simple). We have for all $i \in Q_0$

$$\sum_{j \in Q'_0} m_j \, \underline{\dim}(F_j)_i = d_i. \tag{6.7}$$

Summing (6.7) over all i we have

$$\sum_{j \in Q_0'} m_j \left|\underline{\dim}(F_j)\right| = |\underline{d}|\,. \tag{6.8}$$

Part *(i)* of the lemma follows immediately from $\left|\underline{\dim}(F_j)\right| \geqslant 1$. We now prove part *(ii)*. The case $|\underline{m}| > |\underline{d}|$, is impossible because it contradicts (6.8). If $|\underline{m}| < |\underline{d}|$, then by definition of μ we have $\mu(Q', \underline{m}) < \mu(Q, \underline{d})$.

On the other hand suppose $|\underline{m}| = |\underline{d}|$, then we wish to show $\left|Q_0'\right| \geqslant |Q_0|$. By (6.8) we must have $\left|\underline{\dim}\,(\mathcal{F}_j)\right| = 1$, hence each $\mathcal{F}_j$ is supported at a single vertex, call it $v(j)$. By (6.7) for every $i \in Q_0$ there is a $w(i) \in Q_0'$ such that $v(w(i)) = i$ (choose $w(i)$ so that $m_{w(i)}\, \underline{\dim}(\mathcal{F}_{w(i)})_i \neq 0$). Thus $w\colon Q_0 \to Q_0'$ is an injective map with left inverse $v\colon Q_0' \to Q_0$, showing $\left|Q_0'\right| \geqslant |Q_0|$.

It remains to characterize the saturation of the inequality. The implications $(c) \implies (b) \implies (a)$ are clear. Suppose $\mu(Q', \underline{m}) = \mu(Q, \underline{d})$. By definition of μ we can identify the vertex sets of the quivers Q' and Q. As before, by (6.7) we deduce that each F_i is a 1-dimensional representation supported at the vertex i and $\underline{m} = \underline{d}$. Hence $\mathcal{F}_i$ is the data of a scalar x_l for every loop l in $\overline{Q}$ at i. Altogether we see that F is in the $\mathbb{G}_a^{\overline{L}}$-orbit of $0_{\underline{d}}$. This proves part *(iii)*. □

The strictly seminilpotent CoHA

Fix a quiver Q.

Lemma 6.19 (Generators for Π_Q are primitive)**.** *There is a direct sum complement $\mathcal{F}$ to $\mathcal{G}_{\Pi_Q} \subseteq \mathcal{BPS}_{\Pi_Q,\mathrm{Alg}}$ so that the product $m\colon \mathcal{BPS}_{\Pi_Q,\mathrm{Alg}} \times \mathcal{BPS}_{\Pi_Q,\mathrm{Alg}} \to \mathcal{BPS}_{\Pi_Q,\mathrm{Alg}}$ factors through $\mathcal{F}$.*

Proof. The support of the image of m is contained in the image of the direct sum map $\oplus : \mathcal{M}_{\Pi_Q} \times \mathcal{M}_{\Pi_Q} \to \mathcal{M}_{\Pi_Q}$ which is the complement of the simple locus $\mathcal{M}^{s}_{\Pi_Q}$.

On the other hand the support of the image of m is never contained in the small diagonal. Indeed, for any $\underline{d}_1, \underline{d}_2 \in \mathbb{N}^{Q_0}$ the support of $\mathcal{G}_{\Pi_Q,\underline{d}_2} \boxdot \mathcal{S}_{\Pi_Q,\underline{d}_2}$ is larger than the diagonal as generically points of the support correspond to semisimple objects with nonisomorphic summands $E_1 \ncong E_2$ of class $\underline{d}_1$ and $\underline{d_2}$ respectively.

By the semisimplicity of $\mathcal{BPS}_{\Pi_Q,\mathrm{Alg}}$ we can choose $\mathcal{F}$ to be a direct sum complement of $\mathcal{G}_{\Pi_Q}$ which contains the image of m. □

Let $\imath_{\mathrm{SNN}}\colon \mathfrak{M}^{\mathrm{SNN}}_{\Pi_Q} \hookrightarrow \mathfrak{M}_{\Pi_Q}$ be the inclusion of the strictly seminilpotent stack (Example 3.51).

Lemma 6.20. *We have*

$$H^0(\imath^!_{\mathrm{SNN}}\mathcal{G}_{\Pi_Q,n1_{i'}}) \cong \mathbb{Q}e_{(i',n)} \textit{for all } i' \in Q_0^{\mathrm{im}} \textit{ and } n \geqslant 0$$
$$H^0(\imath^!_{\mathrm{SNN}}\mathcal{G}_{\Pi_Q,1_{i'}}) \cong \mathbb{Q}e_{(i',1)} \textit{ for all } i' \in Q_0^{\mathrm{real}}$$

Proof. For real vertices $i' \in Q_0$ we have $\mathcal{M}_{\Pi_Q, 1_{i'}} = \mathrm{pt}$. For hyperbolic vertices this is [Dav22a, (104)]. For isotropic vertices this is [Dav22a, (92)]. □

Lemma 6.21. *The algebra morphism*

$$\mathrm{H}^0(\imath^!_{\mathrm{SNN}} \widehat{\Psi}_{\Pi_Q})\colon \mathrm{H}^0(\imath^!_{\mathrm{SNN}} \mathrm{Free}_{\boxdot\text{-}\mathrm{Alg}}(\mathcal{G}_{\Pi_Q})) \longrightarrow \mathrm{H}^0(\mathcal{A}^{\mathrm{SNN},\psi}_{\Pi_Q})$$

vanishes on $\mathrm{H}^0(\imath^!_{\mathrm{SNN}} \mathcal{I}_{\Pi_Q})$ *where* $\mathcal{I}_{\Pi_Q}$ *is the Serre ideal, and, moreover, the induced morphism*

$$\mathrm{H}^0(\imath^!_{\mathrm{SNN}} \Psi_{\Pi_Q})\colon \mathrm{H}^0(\mathrm{Free}_{\boxdot\text{-}\mathrm{Alg}}(\imath^!_{\mathrm{SNN}} \mathcal{G}_{\Pi_Q}))/\mathrm{H}^0(\imath^!_{\mathrm{SNN}} \mathcal{I}_{\Pi_Q}) \to \mathrm{H}^0(\mathcal{A}^{\mathrm{SNN},\psi}_{\Pi_Q})$$

is an isomorphism.

Proof. The functor $\imath^!_{\mathrm{SNN}}$ is strict monoidal (Lemma 4.54), so it commutes with the operator Free. By Lemma 6.19, the subobject $\mathcal{G}_{\Pi_Q} \subseteq \mathrm{BPS}^{\psi}_{\Pi_Q,\mathrm{Alg}}$ admits a direct sum complement $\mathcal{F} \subseteq \mathcal{BPS}^{\psi}_{\Pi_Q,\mathrm{Alg}}$ such that the multiplication map

$$m^{\psi}_{\underline{d},\underline{e}}\colon \mathcal{BPS}^{\psi}_{\Pi_Q,\mathrm{Alg},\underline{d}} \boxdot \mathcal{BPS}^{\psi}_{\Pi_Q,\mathrm{Alg},\underline{e}} \to \mathcal{BPS}^{\psi}_{\Pi_Q,\mathrm{Alg},\underline{d}+\underline{e}}$$

factors through the inclusion $\mathcal{F}_{\underline{d}+\underline{e}} \subseteq \mathcal{BPS}^{\psi}_{\Pi_Q,\mathrm{Alg},\underline{d}+\underline{e}}$ if $\underline{d} \neq 0 \neq \underline{e}$. Consequently, the multiplication map

$$\mathrm{H}^0(\imath^!_{\mathrm{SNN}} m^{\psi}_{\underline{d},\underline{e}})\colon \mathrm{H}^0(\mathcal{A}^{\mathrm{SNN},\psi}_{\Pi_Q,\underline{d}}) \otimes \mathrm{H}^0(\mathcal{A}^{\mathrm{SNN},\psi}_{\Pi_Q,\underline{e}}) \to \mathrm{H}^0(\mathcal{A}^{\mathrm{SNN},\psi}_{\Pi_Q,\underline{d}+\underline{e}})$$

factors through $\mathrm{H}^0(\imath^!_{\mathrm{SNN}} \mathcal{F}_{\underline{d}+\underline{e}})$.

By Lemma 6.20 we have

$$\mathrm{H}^0(\imath^!_{\mathrm{SNN}} \mathrm{Free}_{\boxdot\text{-}\mathrm{Alg}}(\mathcal{G}_{\Pi_Q})) = \mathrm{Free}_{\mathrm{Alg}}(\bigoplus_{i \in I^{\infty}} \mathbb{Q}e_i)$$

which is the same as the generators of (the positive half of) the Borcherds algebra $\mathrm{U}(\mathfrak{n}^+_{\Pi_Q})$. By Theorem 5.7 and Lemma 6.19, it follows that $\mathrm{H}^0(\imath^!_{\mathrm{SNN}} \widehat{\Psi}_{\Pi_Q})$ factors through $\mathrm{U}(\mathfrak{n}^+_{\Pi_Q})$ and we conclude the lemma. □

The recursion

Lemma 6.22 (The induction step)**.** *Let* Q *be a quiver and* $\underline{d} \in \mathbb{N}^{Q_0}$ *a dimension vector.*

Suppose $\widehat{\Psi}_{\Pi_{Q'_x},\underline{m}_x}$ *vanishes on the* $\underline{m}_x$*th-graded piece of the Serre ideal* $\mathcal{I}_{\Pi_{Q'_x},\underline{m}_x}$ *so that it factors through* $\mathrm{U}(\mathfrak{n}^+_{\Pi_{Q'}})_{\underline{m}_x}$ *and suppose that the induced morphism*

$$\Psi_{\Pi_{Q'_x},\underline{m}_x}\colon \mathrm{U}(\mathfrak{n}^+_{\Pi_{Q'}})_{\underline{m}_x} \to \mathcal{BPS}^{\psi}_{\Pi_{Q'_x},\mathrm{Alg},\underline{m}_x}$$

is an isomorphism for all closed points $x \in \mathcal{M}_{\Pi_Q,\underline{d}}$ *with half Ext quivers* $(Q'_x, \underline{m}_x)$ *satisfying* $\mu(Q'_x, \underline{m}_x) < \mu(Q, \underline{d})$.

Then $\widehat{\Psi}_{\Pi_Q,\underline{d}}$ *vanishes on* $\mathcal{I}_{\Pi_Q,\underline{d}}$ *and the induced morphism*

$$\Psi_{\Pi_Q,\underline{d}}\colon \mathrm{U}(\mathfrak{n}^+_{\Pi_Q})_{\underline{d}} \to \mathrm{BPS}^{\psi}_{\Pi_Q,\mathrm{Alg},\underline{d}}$$

is an isomorphism.

Proof. We first show that $\widehat{\Psi}_{\Pi_Q,\underline{d}}$ vanishes on $\mathcal{J}_{\Pi_Q,\underline{d}}$. Abbreviate $\mathcal{T} := \widehat{\Psi}_{\Pi_Q,\underline{d}}(\mathcal{J}_{\Pi_Q,\underline{d}})$ and $\mathcal{T}_x := \widehat{\Psi}_{\Pi_{Q'_x},\underline{m}_x}(\mathcal{J}_{\Pi_{Q'_{\underline{x}}}})$.

By proceeding as in the proof of Proposition 6.14 (see also the Remark 6.15) we deduce

$$\begin{aligned} \operatorname{supp}(\mathcal{T}) &\subseteq \overline{\{x \in \mathcal{M}_{\Pi_Q,\underline{d}} \mid \mathcal{T}_x \neq 0\}} && \text{(by Remark 6.15)} \\ &\subseteq \overline{\{x \in \mathcal{M}_{\Pi_Q,\underline{d}} \mid \mu(Q'_x, \underline{m}_x) = \mu(Q, \underline{d})\}} && \text{(by hypothesis and Lemma 6.18 (ii))} \\ &= \mathbb{G}_a^{\underline{L}} \cdot 0_{\underline{d}} && \text{(by Lemma 6.18 (iii)),} \end{aligned}$$

where the bars denote Zariski closure. In words, the hypothesis guarantees that the support of $\mathcal{T}$ is contained in the $\mathbb{G}_a^{\overline{L}}$-orbit $\overline{Z} = \mathbb{G}_a^{\overline{L}} \cdot 0_{\underline{d}} \cong \mathbb{C}^{\overline{L}}$ of $0_{\underline{d}}$. Since $\mathcal{T}$ is a $\mathbb{G}_a^{\overline{L}}$-equivariant mixed Hodge module (Lemma 6.17), its support is $\mathbb{G}_a^{\overline{L}}$-stable and hence equal to the entire orbit $\overline{Z}$.

The groups $\mathbb{G}_a^{\overline{L}}$ and $\mathbb{G}_a^{L}$ are contractible, so are any of their orbits. Therefore taking total cohomology induces equivalences

$$\mathrm{MHM}_{\mathbb{G}_a^{\overline{L}}}(\overline{Z}) \simeq \mathrm{MHM}(\mathrm{pt}) \simeq \mathrm{MHM}_{\mathbb{G}_a^{L}}(Z) \tag{6.9}$$

where Z is the $\mathbb{G}_a^L$-orbit of $0_{\underline{d}}$. Let $\iota_{\mathrm{SNN}} \colon \mathcal{M}^{\mathrm{SNN}}_{\Pi_Q,\underline{d}} \hookrightarrow \mathcal{M}_{\Pi_Q,\underline{d}}$ be the inclusion. We have a Cartesian square of inclusions

$$\begin{tikzcd} \mathbb{C}^L \cong Z \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{M}^{\mathrm{SNN}}_{\Pi_Q,\underline{d}} \arrow[d, "\iota_{\mathrm{SNN}}"] \\ \mathbb{C}^{\overline{L}} \cong \overline{Z} \arrow[r] & \mathcal{M}_{\Pi_Q,\underline{d}}. \end{tikzcd} \tag{6.10}$$

From the equivalences (6.9) and the Cartesian diagram (6.10) we have $H^0(\mathcal{T}) \subseteq H^0(\iota\Psi_{\Pi_Q}(\mathcal{J}_{\Pi_Q}))$ which vanishes by Lemma 6.21.

This proves the first part that $\widehat{\Psi}_{\Pi_Q}$ vanishes on degree $\underline{d}$ part of the Serre ideal $\mathcal{J}_{\Pi_Q,\underline{d}}$.

Replacing $\mathcal{T}$ with $\mathcal{T} = \ker(\Psi_{\Pi_Q,\underline{d}}) \oplus \operatorname{coker}(\Psi_{\Pi_Q,\underline{d}})$ and $\mathcal{T}_x$ with $\mathcal{T}_x = \ker(\Psi_{\Pi_{Q'_x},\underline{m}_x}) \oplus \operatorname{coker}(\Psi_{\Pi_{Q'_x},\underline{m}_x})$, the proof of the second part is exactly the same. □

Proof of Theorem 6.5. As explained in the beginning of this section: we need to show for all $(Q, \underline{d}) \in \mathrm{QuivDim}$ that $\widehat{\Psi}_{\Pi_Q,\underline{d}}$ vanishes on $\mathcal{J}_{\Pi_Q,\underline{d}}$ and the induced morphism $\Psi_{\Pi_Q,\underline{d}}$ is an isomorphism.

Fix a quiver with dimension vector $(Q, \underline{d}) \in \mathrm{QuivDim}$. Without loss of generality we assume that $\underline{d}$ is supported on all of Q. Consider the following subset of $\mathbb{Z}_{>0} \times \mathbb{Z}_{<0}$. By Lemma 6.18

$$\begin{aligned} R(Q, \underline{d}) &:= \{\mu(Q'_x, \underline{m}_x) \mid x \in \mathcal{M}_{\Pi_Q,\underline{d}} \text{ and } \mu(Q'_x, \underline{m}_x) \neq \mu(Q, \underline{d})\} \\ &= \{\mu(Q'_x, \underline{m}_x) \mid x \in \mathcal{M}_{\Pi_Q,\underline{d}} \text{ and } \mu(Q'_x, \underline{m}_x) < \mu(Q, \underline{d})\}. \end{aligned}$$

To apply Lemma 6.22 we need to show that for all $(Q^{\mathrm{new}}, \underline{d}^{\mathrm{new}}) \in \mu^{-1}(R(Q, \underline{d}))$ the morphism $\Psi_{\Pi_{Q^{\mathrm{new}}},\underline{d}^{\mathrm{new}}}$ is an isomorphism. We do so inductively.

By Lemma 6.18 *(i)*, the set $R(Q, \underline{d})$ is bounded hence finite. For all $(Q^{\text{new}}, \underline{d}^{\text{new}}) \in \mu^{-1}(R(Q, \underline{d}))$ we have the strict inclusion of finite sets $R(Q^{\text{new}}, \underline{d}^{\text{new}}) \subsetneq R(Q, \underline{d})$. Thus, after iterating we eventually end at the case where $R(Q^{\text{fin}}, \underline{d}^{\text{fin}}) = \emptyset$, so that the hypothesis in Lemma 6.22 for $(Q^{\text{fin}}, \underline{d}^{\text{fin}})$ is vacuous.

Applying Lemma 6.22 recursively we deduce that $\Psi_{\Pi_Q, \underline{d}}$ is an isomorphism. □

6.2 THE POINCARÉ–BIRKHOFF–WITT ISOMORPHISM

6.2.1 *BPS Lie algebras as GKM Lie algebras*

Definition 6.23 (BPS for 2-Calabi–Yau categories, [DHS23, (1.1)]). Let $\mathscr{A}$ be a 2CY Abelian category with good moduli theory $\mathfrak{M}_{\mathscr{A}}$ We define *the BPS Lie algebra MHM of* $\mathscr{A}$ to be the GKM algebra MHM

$$\mathcal{BPS}_{\mathscr{A},\text{Lie}} \coloneqq \mathfrak{n}^+_{\mathscr{A}} \coloneqq \mathfrak{n}^+_{\mathcal{M}_{\mathscr{A}},(-,-)_{\mathscr{A}}} \in \text{MHM}(\mathcal{M}_{\mathscr{A}})$$

associated to the monoid scheme $\mathcal{M}_{\mathscr{A}}$ with pairing $(-,-)_{\mathscr{A}}$. We call the underling MHM or perverse sheaf of $\text{BPS}_{\mathscr{A},\text{Lie}}$ *the BPS sheaf of* $\mathscr{A}$.

The BPS Lie algebra of $\mathscr{A}$ is defined to be the Lie algebra

$$\text{BPS}_{\mathscr{A},\text{Lie}} \coloneqq \text{H}(\mathcal{BPS}_{\mathscr{A},\text{Lie}}) = \text{H}(\mathfrak{n}^+_{\mathscr{A}})$$

We call the underlying mixed Hodge structure of $\text{BPS}_{\mathscr{A},\text{Lie}}$ the *BPS cohomology of* $\mathscr{A}$.

This definition is motivated by Theorem 6.5 and [Dav22a, Theorem 6.1], which realises the BPS algebra as the universal enveloping algebra of the BPS Lie algebra for the preprojective algebra as defined in [Dav22a, §4].

To deserve the title 'BPS Lie algebra' the MHM $\mathcal{BPS}_{\mathscr{A},\text{Lie}}$ must satisfy a relative cohomological integrality isomorphism

$$\mathcal{A}_{\mathscr{A}} \cong \text{Sym}_{\boxdot}(\mathcal{BPS}_{\mathscr{A},\text{Lie}} \otimes \text{H}_{\mathbb{C}})$$

The aim of this section is to prove the stronger statement in Theorem 6.24, thus completing the main goal of this thesis to construct BPS cohomology for 2-Calabi–Yau categories.

Since $\text{H}(\text{B}\mathbb{C}^\times) \cong \text{RHom}(\mathbb{Q}_{\text{B}\mathbb{C}^\times}, \mathbb{Q}_{\text{B}\mathbb{C}^\times})$, the generator $\eta \in \text{H}(\text{B}\mathbb{C}^\times)$ corresponds to a morphism

$$\eta \colon \mathbb{Q}_{\text{B}\mathbb{C}^\times} \to \mathbb{Q}_{\text{B}\mathbb{C}^\times} \otimes \mathbb{L}^{-1}$$

We can !-pullback along a morphism

$$d \colon \mathfrak{M}_{\mathscr{A}} \longrightarrow \text{B}\mathbb{C}^\times$$

to obtain a morphism

$$d^!\eta \colon \mathbb{D}\mathbb{Q}_{\mathfrak{M}_{\mathscr{A}}} \otimes \mathbb{L} \longrightarrow \mathbb{D}\mathbb{Q}_{\mathfrak{M}_{\mathscr{A}}}$$

which extends to an $\mathrm{H}(\mathrm{B}\mathbb{C}^\times)$-action

$$d^!\eta\colon \mathbb{D}\mathbb{Q}_{\mathfrak{M}_{\mathscr{A}}} \otimes \mathrm{H}(\mathrm{B}\mathbb{C}^\times) \longrightarrow \mathbb{D}\mathbb{Q}_{\mathfrak{M}_{\mathscr{A}}}$$

which is nonzero if and only if the line bundle corresponding to d has nonzero degree. This can always be arranged because by Assumption 3.39 and Remark 4.51 the connected components $\mathfrak{M}_{\mathscr{A},v}$ are quotient stacks W/GL_N which admits a line bundle of degree N constructed out of the determinant representation (c. f. [Dav23b, §6.2.1]).

Theorem 6.24 (PBW theorem, [DHS23, Theorem 1.7]). *Let $\mathscr{A}$ be a 2-Calabi–Yau Abelian category with good moduli theory $\mathrm{JH}\colon \mathfrak{M}_{\mathscr{A}} \to \mathcal{M}_{\mathscr{A}}$. Then the PBW morphism in $\mathrm{D}^+(\mathrm{MHM}(\mathcal{M}_{\mathscr{A}}))$*

$$\mathrm{PBW}^{\psi}_{\mathscr{A}}\colon \mathrm{Sym}_{\boxdot}\left(\mathcal{BPS}^{\psi}_{\mathscr{A},\mathrm{Lie}} \otimes \mathrm{H}(\mathrm{B}\mathbb{C}^\times)\right) \longrightarrow \mathcal{A}^{\psi}_{\mathscr{A}}$$

is an isomorphism of complexes of mixed Hodge modules.

6.2.2 *Summary of cohomological Donaldson–Thomas theory for quivers with potential and preprojective algebras*

An a priori different definition of the BPS Lie algebra sheaf for preprojective algebras of quivers, which we denote by $\mathcal{BPS}^{3d,\psi}_{\Pi_Q,\mathrm{Lie}}$, appears in [Dav22a, Theorem/Definition 4.1] (see also [Dav22b, Theorem B]). We will show that our definition of the BPS Lie algebra agrees with this "3d" definition of the BPS Lie algebra.

The PBW theorem for critical CoHAs

Let (Q, W) be a symmetric quiver with potential. In [DM20] Davison and Meinhardt construct the BPS Lie algebra MHM $\mathcal{BPS}_{Q,W}$, and prove the PBW theorem (and hence the cohomological integrality theorem) for the critical cohomology of the stack of representations $\mathfrak{M}_{Q,W}$ of the Jacobi algebra of (Q, W).

The affinisation $\mathrm{JH}_{Q,W}\colon \mathfrak{M}_{Q,W} \to \mathcal{M}_{Q,W}$ is a good moduli space, and we have the commutative diagram

$$\begin{tikzcd}
\mathfrak{M}_{Q,W} \arrow[r, hook] \arrow[d, "\mathrm{JH}_{Q,W}"] & \mathfrak{M}_Q \arrow[d, "\mathrm{JH}_Q"] \arrow[dr, bend left, "\mathrm{tr}(W)"] & \\
\mathcal{M}_{Q,W} \arrow[r, hook] & \mathcal{M}_Q \arrow[r, "\mathrm{tr}(W)"] & \mathbb{A}^1
\end{tikzcd} \tag{6.11}$$

where the inclusions are inclusions of critical loci of the functions $\mathrm{tr}(W)$.

Denote by $\phi_{\mathrm{tr}(W)}$ the vanishing cycle functor (this is really a complex of monodromic mixed Hodge modules, we don't treat these here and instead refere to [DM20]) Then $\mathcal{A}_{Q,W} := (\mathrm{JH}_{Q,W})_* \phi_{\mathrm{tr}(W)} \mathbb{Q}^{\mathrm{vir}}_{\mathfrak{M}_{Q,W}} \in$

$D^+(\mathrm{MMHM}(\mathcal{M}_{Q,W}))$ is an algebra object with respect to a certain symmetric monoidal structure $\widetilde{\boxdot}$.

Let $\widetilde{\psi}\colon \mathbb{Z}^{Q_0} \times \mathbb{Z}^{Q_0} \to \mathbb{Z}$ be a bilinear form such that

$$\widetilde{\psi}(\underline{d},\underline{d}') + \widetilde{\psi}(\underline{d}',\underline{d}) = (\underline{d},\underline{d})_Q(\underline{d}',\underline{d}')_Q + (\underline{d},\underline{d}')_Q \mod 2$$

for all $a, b \in \mathbb{Z}^{Q_0}$. We denote by $\mathcal{A}^{\widetilde{\psi}}_{Q,W}$ the algebra object where the product of $\mathcal{A}_{Q,W}$ has been twisted by the sign $(-1)^{\widetilde{\psi}(-,-)}$.

The BPS Lie algebra MHM of (Q, W) is defined as

$$\mathcal{BPS}^{\widetilde{\psi}}_{Q,W} := \mathcal{H}^1(\mathcal{A}^{\widetilde{\psi}}_{Q,W}) \in \mathrm{MHM}(\mathcal{M}_{Q,W})$$

We are ready to state the PBW theorem for (Q, W).

Theorem 6.25 ([DM20]). *The natural morphism*

$$\mathcal{BPS}_{Q,W} \otimes \mathrm{H}(\mathrm{B}\mathbb{C}^\times) \longrightarrow \mathcal{A}^{\widetilde{\psi}}_{Q,W}$$

induces via the product on $\mathcal{A}^{\widetilde{\psi}}_{Q,W}$ *an isomorphism in* $D^+(\mathrm{MMHM}(\mathcal{M}_{Q,W}))$

$$\mathrm{PBW}^{\widetilde{\psi}}_{Q,W}\colon \mathrm{Sym}_{\widetilde{\boxdot}}\left(\mathcal{BPS}_{Q,W} \otimes \mathrm{H}(\mathrm{B}\mathbb{C}^\times)\right) \longrightarrow \mathcal{A}^{\widetilde{\psi}}_{Q,W}.$$

In [DM20] it is shown that $\mathcal{BPS}_{Q,W}$ is indeed a Lie algebra object with respect to the symmetric monoidal structure $\widetilde{\boxdot}$.

Dimensional reduction

In vague terms, dimensional reduction is the general principle that the critical cohomology of moduli stacks of objects in a 3-Calabi–Yau completion of a 2-Calabi–Yau is equivalent to the Borel–Moore homology of the moduli stacks of objects in the original 2-Calabi–Yau category. Dimensional reduction has been applied particularly succesfully when the moduli stacks of objects in the 3-Calabi–Yau completion can be written as a global critical locus. In this section we highlight how dimensional reduction has been applied to moduli of representations of preprojective algebras.

Let Q be a quiver. There is a specific choice of quiver with potential $(\widetilde{Q}, \widetilde{W})$, called *the tripled quiver with potential*, that is directly related to the preprojective algebra. The quiver $\widetilde{Q}$ is obtained from the doubled quiver $\overline{Q}$ be adding a loop ω_i at each vertex $i \in Q_0$. The potential is defined as

$$\widetilde{W} := \sum_{i \in Q_0} \omega_i \sum_{a \in Q_1} [a, a^*].$$

The Jacobi algebra of $\mathrm{Jac}(\widetilde{Q}, \widetilde{W})$ is the quotient of the path algebra $\mathbb{C}\widetilde{Q}$ by ideal generated by

$$\begin{aligned} \partial_a \widetilde{W} &= \sum_{i \in Q_0} a^* \omega_i + \omega_i a^* \\ \partial_{a^*} \widetilde{W} &= \sum_{i \in Q_0} \omega_i a + a\omega_i \\ \partial_{\omega_i} \widetilde{W} &= e_i \sum_{a \in Q_1} [a, a^*] e_i \end{aligned}$$

for all $a \in Q_1$ and $i \in Q_0$. The third relation is the preprojective relation for $\overline{Q}$, thus there is a natural forgetful morphism $\xi \colon \mathfrak{M}_{\widetilde{Q},\widetilde{W}} \longrightarrow \mathfrak{M}_{\Pi_Q}$ which forgets the actions of the loops.

The following theorem gathers various results of Davison related to the dimensional reduction for DT and BPS invariants for preprojective algebras. See [Dav17, Theorem A.1], [Dav22b, Theorem D], [Dav22a, Theorem/Definition 4.1 and Theorem 6.1].

Theorem 6.26 (Dimensional reduction package for preprojective algebras, Davison)**.** *The natural morphism of complexes of mixed Hodge modules*

$$\xi_! \phi_{\mathrm{tr}(W)} \mathbb{Q}^{\mathrm{vir}}_{\mathfrak{M}_{Q,W}} \xrightarrow{\cong} \mathbb{D}\mathbb{Q}_{\mathfrak{M}_{\Pi_Q}}$$

is an isomorpism.

Moreover, the pushforward of the BPS Lie algbera MHM of the tripled QP $(\widetilde{Q}, \widetilde{W})$*.*

$$\mathcal{BPS}^{\mathsf{3d}}_{\Pi_Q} \coloneqq \xi_* \mathcal{BPS}_{\widetilde{Q},\widetilde{W}}$$

is a mixed Hodge module and is closed under the commutator bracket in $\mathcal{A}^{\widetilde{\psi}}_{\Pi_Q}$*. Thus* $\mathcal{BPS}^{\mathsf{3d}}_{\Pi_Q}$ *is also itself a* $\boxdot$*-Lie algebra object in* $\mathrm{MHM}(\mathcal{M}_{\Pi_Q})$ *The PBW morphism*

$$\mathrm{PBW}^{\widetilde{\psi}}_{\Pi_Q} \colon \mathsf{Sym}_{\boxdot} \left(\mathcal{BPS}^{\mathsf{3d}}_{\Pi_Q} \otimes \mathrm{H}(\mathrm{B}\mathbb{C}^\times) \right) \longrightarrow \mathcal{A}^{\widetilde{\psi}}_{\Pi_Q} \tag{6.12}$$

is an isomorphism of complexes of mixed Hodge modules. Moreover there is a canonical inclusion

$$\mathcal{BPS}^{\mathsf{3d}}_{\Pi_Q} \hookrightarrow \mathcal{H}^0(\mathcal{A}_{\Pi_Q}) = \mathcal{BPS}_{\Pi_Q,\mathrm{Alg}}$$

which induces an ismorphism of $\boxdot$*-algebras*

$$\mathrm{U}\left(\mathcal{BPS}^{\mathsf{3d}}_{\Pi_Q}\right) \xrightarrow{\cong} \mathcal{BPS}_{\Pi_Q,\mathrm{Alg}} \tag{6.13}$$

Remark 6.27. It follows from $(-.-)_{\Pi_Q} = (-,-)_{\widetilde{Q}} \mod 2$, that every valid choice of twist ψ for Π_Q is simultaneously a valid choice of twist $\widetilde{\psi}$ for $(\widetilde{Q}, \widetilde{W})$.

6.2.3 *Proof of Theorem 6.24*

A consequence of (6.13) is that the $\boxdot$-algebra $\mathcal{BPS}^{\psi}_{\Pi_Q,\mathrm{Alg}}$ arises in two ways as a universal enveloping algebra

$$\mathcal{BPS}^{\psi}_{\Pi_Q,\mathrm{Alg}} \cong \mathrm{U}(\mathcal{BPS}^{3d,\psi}_{\Pi_Q}) \cong \mathrm{U}(\mathfrak{n}^{\psi}_{\Pi_Q})$$

The following proposition shows that in fact the two ways are the same.

Lemma 6.28. *The morphisms* $\mathfrak{n}^{+}_{\Pi_Q} \to \mathcal{BPS}_{\Pi_Q,\mathrm{Alg}}$ *factors through the 3d BPS Lie algebra to and induces an isomorphism of Lie algebra objects*

$$\mathfrak{n}^{+}_{\Pi_Q} \longrightarrow \mathcal{BPS}^{3d}_{\Pi_Q} \tag{6.14}$$

Proof. The fact that $\mathfrak{n}^{+}_{\Pi_Q}$ factors through (6.14) is a special case of [Dav22a, Proposition 7.5].

From the morphism $\mathfrak{n}^{+}_{\Pi_Q} \to \mathcal{BPS}^{3d}_{\Pi_Q}$ we have a compatibility of PBW isomorphisms

$$\begin{tikzcd}
\mathrm{Sym}_{\boxdot}(\mathfrak{n}^{+}_{\Pi_Q}) \arrow[rr, "\mathrm{PBW}", "\cong"'] \arrow[dr] & & \mathcal{BPS}^{\psi}_{\Pi_Q,\mathrm{Alg}} \\
& \mathrm{Sym}_{\boxdot}(\mathcal{BPS}^{3d}_{\Pi_Q}) \arrow[ur, "\mathrm{PBW}", "\cong"'] &
\end{tikzcd}$$

By definition of PBW morphisms we conclude that (6.14) is in fact an isomorphism. ☐

This proposition is conceptually important as it justifies Definition 6.23 and moroever constitutes a crucial part of the local statement used to prove the 2CY PBW theorem (Theorem 6.24).

Corollary 6.29 (PBW for preprojective algebras)**.** *Let* Q *be a quiver. Then the PBW morphism*

$$\mathrm{PBW}^{\psi}_{\Pi_Q} \colon \mathrm{Sym}_{\boxdot}\left(\mathcal{BPS}^{\psi}_{\Pi_Q,\mathrm{Lie}} \otimes \mathrm{H}(\mathrm{B}\mathbb{C}^{\times})\right) \longrightarrow \mathcal{A}^{\psi}_{\Pi_Q}$$

is an isomorphism.

Proof. This is a direct consequence of the PBW isomorpshism (6.12) and Lemma 6.28. ☐

Remark 6.30. The relation between the tripled quiver and the preprojective algebra comes from the fact that (at least in the case of non ADE quivers) the Jacobi algebra $\mathrm{Jac}(\widetilde{Q}, \widetilde{W})$ is the 3-Calabi–Yau completion of the preprojective algebra Π_Q.

Corollary 6.31 (Restriction of PBW morphism)**.** *Let* Q *be a quiver and* $\iota_{\mathrm{Nil}} \colon \mathfrak{M}^{\mathrm{Nil}}_{\Pi_Q} \hookrightarrow \mathfrak{M}_{\Pi_Q}$ *the inclusion. Then the PBW morphism*

$$\mathrm{PBW}^{\mathrm{Nil},\psi}_{\Pi_Q} \colon \mathrm{Sym}\left(\iota^{!}_{\mathrm{Nil}}\mathcal{BPS}^{\psi}_{\Pi_Q,\mathrm{Lie}} \otimes \mathrm{H}(\mathrm{B}\mathbb{C}^{\times})\right) \longrightarrow \iota^{!}_{\mathrm{Nil}}\mathcal{A}^{\psi}_{\mathscr{A}}$$

is an isomorphism.

Proof. Follows from Corollary 6.29 and the strict monoidality of $\imath^!_{\mathrm{Nil}}$ (Lemma 4.54). □

Lemma 6.32 (Compatibility of PBW morphisms with Ext-quiver neighbourhoods)**.** *Suppose we have chosen notation as in Setting 6.7. Then we have a commutative diagram*

$$\begin{tikzcd}
\imath^!_{\mathrm{Nil}} \operatorname{Sym}_{\boxdot}\left(\mathcal{BPS}^{\psi}_{\Pi_Q} \otimes \mathrm{H}_{\mathbb{C}}\right) \arrow[r, dash, "\cong"] \arrow[d, "\imath^!_{\mathrm{Nil}}\mathrm{PBW}^{\psi}_{\Pi_Q}"] & \imath^!_{\mathrm{F}} \operatorname{Sym}_{\boxdot}\left(\mathcal{BPS}^{\psi}_{\mathscr{A}} \otimes \mathrm{H}_{\mathbb{C}}\right) \arrow[d, "\imath^!_{\mathrm{F}}\mathrm{PBW}^{\psi}_{\mathscr{A}}"] \\
\imath^!_{\mathrm{Nil}}\mathcal{A}^{\psi}_{\Pi_Q} \arrow[r, dash, "\cong"] & \imath^!_{\mathrm{F}}\mathcal{A}^{\psi}_{\mathscr{A}}
\end{tikzcd}.$$

Proof. By Definition 6.23 and compatibility of GKM (Lie) algebras with Ext-quivers (Corollary 6.10), we have natural commutative diagrams with horizontal maps isomorpshims

$$\begin{tikzcd}
\imath^!_{\mathrm{Nil}}\mathcal{BPS}^{\psi}_{\Pi_Q} \otimes \mathrm{H}_{\mathbb{C}} \arrow[r, "\cong"] \arrow[d, hook] & \imath^!_{\mathrm{F}}\mathcal{BPS}^{\psi}_{\mathscr{A},\mathrm{Lie}} \otimes \mathrm{H}_{\mathbb{C}} \arrow[d, hook] \\
\imath^!_{\mathrm{Nil}}\mathcal{A}^{\psi}_{\Pi_Q} \arrow[r, "\cong"] & \imath^!_{\mathrm{F}}\mathcal{A}^{\psi}_{\mathscr{A}}
\end{tikzcd} \tag{6.15}$$

By the compatibility of CoHA multiplications (Proposition 4.59) it follows that the PBW morphisms $\imath^!_{\mathrm{Nil}}\mathrm{PBW}^{\psi}_{\Pi_Q}$ and $\mathrm{PBW}^{\psi}_{\mathscr{A}}$ are compatibile, i. e. that the verticle morphisms in (6.15) extend to Sym to obtain the desired commutative diagram. □

Proof of Theorem 6.24. Let $x \in \mathcal{M}_{\mathscr{A}}$ be a closed point corresponding to a semisimple object $\bigoplus_i F_i^{m_i}$. Let Q be a half Ext-quiver of x.

By Lemma 6.32 we have

$$\imath^!_x \operatorname{cone}(\mathrm{PBW}^{\psi}_{\mathscr{A}}) \cong \imath^!_{0_{\underline{m}}} \imath^!_{\mathrm{Nil}} \operatorname{cone}(\mathrm{PBW}^{\psi}_{\Pi_Q}).$$

But by Corollary 6.31 the complex $\imath^!_{\mathrm{Nil}} \operatorname{cone}(\mathrm{PBW}^{\psi}_{\Pi_Q})$ vanishes. Thus we conclude $\operatorname{cone}(\mathrm{PBW}^{\psi}_{\mathscr{A}})$ vanishes everywhere (Lemma 6.13). □

By pushing forward along the monoid morphism $\mathcal{M}_{\mathscr{A}} \to \pi_0(\mathcal{M}_{\mathscr{A}})$ we obtain the PBW isomorphism (and hence a cohomological integrality isomorphism) for $\mathscr{A}$.

Corollary 6.33 (Absolute PBW isomorphism)**.** *Let $\mathscr{A}$ be a 2-Calabi–Yau Abelian category with good moduli theory* $\mathrm{JH}\colon \mathfrak{M}_{\mathscr{A}} \to \mathcal{M}_{\mathscr{A}}$. *Then the PBW morphism in* $\mathrm{D}^+(\mathrm{MHM}(\pi_0(\mathcal{M}_{\mathscr{A}})))$

$$\mathrm{H}(\mathrm{PBW}^{\psi}_{\mathscr{A}})\colon \operatorname{Sym}\left(\mathrm{BPS}^{\psi}_{\mathscr{A},\mathrm{Lie}} \otimes \mathrm{H}(\mathrm{B}\mathbb{C}^{\times})\right) \longrightarrow \mathrm{H}\mathcal{A}^{\psi}_{\mathscr{A}}$$

is an isomorphism of complexes of mixed Hodge modules.

A

ELEMENTS OF DERIVED ALGEBRAIC GEOMETRY

The reader is recommended to consult [Saf] for a rapid introduction to derived algebraic geometry. In this appendix we gather some less standard definitions and constructions in derived algebraic geometry.

A.1 SPEC CONSTRUCTION FOR NON-CONNECTIVE CDGA

This section summarises some of [Toë06] and [BN12, §3].

The Spec construction for non-connective cdgas solves the affinisation problem for derived stacks.

Let $R \in \mathrm{cdga}$ be a not necessarily connective cdga. We define a functor

$$\mathrm{Spec}\colon \mathrm{cdga}^{\mathrm{op}} \longrightarrow \mathrm{dSt}$$

which assigns to every cdga R the derived stack $\mathrm{Spec}(R)$ defined by the moduli problem

$$\begin{aligned} \mathrm{Spec}(R)\colon \mathrm{cdga}^{\leqslant 0} &\longrightarrow \mathrm{SSets} \\ A &\longmapsto \mathrm{Maps}_{\mathrm{cdga}}(R, A) \end{aligned}$$

The Yoneda lemma implies that for R connective, the two definitions of the derived stacks $\mathrm{Spec}(R)$ canonically agree.

The affinisation problem for derived stacks is the problem of finding a right adjoint to the derived global sections functor

$$\begin{aligned} \mathcal{O}\colon \mathrm{dSt} &\longrightarrow \mathrm{cdga} \\ R\mathfrak{X} &\longmapsto \mathcal{O}_{R\mathfrak{X}}(R\mathfrak{X}) = \operatorname*{holim}_{\substack{\mathrm{Spec}(A)\to R\mathfrak{X} \\ A\in \mathrm{cdga}^{\leqslant 0}}} A \end{aligned}$$

We remark that $\mathcal{O}$ takes values in all cdgas and not just connective ones.

Example A.1. We have $\mathcal{O}_{BG}(BG) \cong H^*(G, \mathbb{C})$, the group cohomology.

In [BN12, Proposition 3.1] it is shown that there is an adjoint pair

$$\mathcal{O} \vdash \mathrm{Spec}\,.$$

There is a relative version. For derived stacks over a base derived stack $R\mathfrak{B}$ there is a functor

$$\mathcal{O}/R\mathfrak{B}\colon \mathrm{dSt} \longrightarrow \{\text{quasicoherent } \mathcal{O}_{R\mathfrak{B}} \text{ algebras}\}$$

Let $R\mathfrak{X}$ be a derived Artin higher stack. Let $\mathcal{E} \in \mathrm{Perf}(\mathfrak{M})$ be a perfect complex. The *total space of* $\mathcal{E}$ is defined by

$$\mathrm{Tot}_{R\mathfrak{X}}\,(\mathcal{E}) \coloneqq \mathrm{Spec}_{R\mathfrak{X}}(\mathrm{Sym}(\mathcal{E}^{\vee})).$$

A.2 GLOBALLY PRESENTED QUASI-SMOOTH MORPHISMS

The material in this section is taken from [DHS22, §4].

Projections of total spaces of $[-1,1]$-perfect complexes are not closed under composition. In this section we explain the notion of globally presented quasi-smooth morphisms, which should be thought of as the class of morphisms generated under composition by projections of total spaces of perfect complexes.

A perfect complex $\mathcal{C} = \bigoplus_{i\in\mathbb{Z}} \mathcal{C}^i$ is *graded* if each $\mathcal{C}^i$ is isomorphic to $V_i[-i]$ for some vector bundle V_i, i.e. V_i is a perfect complex that becomes a classical cohomologically graded vector bundle after restriction to any classical scheme. Let $\mathcal{C}$ be a graded perfect complex on a 1-Artin derived stack $R\mathfrak{M}$ with $\mathcal{C}^i = 0$ for $i \geqslant 2$.

Let $Q \in \mathrm{Der}_{R\mathfrak{M}}(\mathrm{Tot}_{R\mathfrak{M}}(\mathcal{C}))$ be a degree one derivation that vanishes on the zero section, which satisfies $[Q,Q] = 0$ up to higher coherent homotopies .

Let $A = (A^*, d_A)$ be a connective cdga and $f\colon \mathrm{Spec}(A) \to R\mathfrak{M}$ a morphism. The pullback f^*Q is given by A-linear degree one morphisms $m_n^{\vee}\colon f^*(\mathcal{C})^{\vee} \to \mathrm{Sym}_A^n(f^*(\mathcal{C})^{\vee})$, which vanish for $n \gg 0$. The condition $[Q,Q] = 0$ means that $m^{\vee} = \sum_n m_n^{\vee}$ satisfies the Maurer–Cartan equation $d_A m^{\vee} + \frac{1}{2}[m^{\vee}, m^{\vee}] = 0$ which unpacks to

$$\begin{aligned} d_A m_0^{\vee} &= 0 \\ d_A m_1^{\vee} + m_1^{\vee} \circ m_1^{\vee} &= 0 \\ d_A m_2^{\vee} + \frac{1}{2} m_3^{\vee} \circ m_1^{\vee} + \frac{1}{2} m_1^{\vee} \circ m_3^{\vee} &= 0 \\ &\vdots \end{aligned}$$

The vanishing on the zero section means that $m_0^{\vee} = 0$. the Maurer–Cartan equation implies $(d_A + m_1^{\vee})^2 = 0$ and so $(f^*\mathcal{C})^{\vee}$ inherits a differential $f^*d_{\mathcal{C}} \coloneqq d_A + m_1^{\vee}$.

For every such morphism $f\colon \mathrm{Spec}(A) \to R\mathfrak{M}$ we define the cdga $B_f \coloneqq (\mathrm{Sym}_A((f^*\mathcal{C})^{\vee}), d_A + f^*Q)$.

We define $\mathrm{Tot}_{\mathfrak{M}}^{Q}(\mathcal{C})$ to be the derived stack over $\mathfrak{M}$ given by

$$(f\colon \mathrm{Spec}(A) \to R\mathfrak{M}) \longmapsto \mathrm{Tot}_{R\mathfrak{M}}^{Q}(\mathcal{C})(f) \coloneqq \mathrm{Spec}(B_f).$$

for a semi-free cdga $\mathrm{Spec}(A)$. Note that B_f need not be connective and hence $\mathrm{Spec}(B_f)$ is as in §A.1

By definition we have a projection

$$\pi\colon \mathrm{Tot}_{R\mathfrak{M}}^{Q}(\mathcal{C}) \longrightarrow R\mathfrak{M}. \tag{A.1}$$

There is a natural quasi-isomorphism

$$f^*\mathbb{L}_{\mathrm{Tot}^{Q}_{\mathfrak{M}}(\mathcal{C})/\mathfrak{M}} \simeq ((f^*\mathcal{C})^{\vee}, f^*d_{\mathcal{C}}). \tag{A.2}$$

We call a morphism of the form (A.1) a *globally presented quasi-smooth morphism*. A *global presentation* of a quasi-smooth morphism of derived stacks $f\colon R\mathfrak{X} \to R\mathfrak{Y}$ is a commutative diagram of 1-Artin stacks

$$\begin{tikzcd}
R\mathfrak{X} \arrow[r, "f"] \arrow[d] & R\mathfrak{Y} \arrow[d] \\
\mathrm{Tot}^{Q}_{\mathfrak{M}}(\mathcal{C}) \arrow[r, "\pi"] & R\mathfrak{M}
\end{tikzcd}$$

where the vertical morphisms are equivalences and the bottom morphism is a globally presented quasi-smooth morphism.

Let $(\mathcal{C}, Q_{\mathcal{C}})$ and $(\mathcal{C}', Q_{\mathcal{C}'})$ be global presentations of quasi-smooth morphisms, with both $\mathcal{C}$ and $\mathcal{C}'$ graded vector bundles over the same stack $\mathfrak{M}$. Let $f\colon \mathrm{Tot}_{R\mathfrak{M}}(\mathcal{C}) \to \mathrm{Tot}_{R\mathfrak{M}}(\mathcal{C}')$ be a morphism over $R\mathfrak{M}$ such that $Q_{\mathcal{C}} = f^*Q_{\mathcal{C}'}$, and such that the induced morphism of complexes $((\mathcal{C})^{\vee}, d_{\mathcal{C}}) \to ((\mathcal{C}')^{\vee}, d_{\mathcal{C}'})$ is a quasi-isomorphism, inducing an equivalence $\mathrm{Tot}^{Q_{\mathcal{C}}}_{\mathfrak{M}}(\mathcal{C}) \to \mathrm{Tot}^{Q_{\mathcal{C}'}}_{R\mathfrak{M}}(\mathcal{C}')$. We call an equivalence defined via such an f a *global equivalence*.

Globally presented morphisms over a fixed stack $R\mathfrak{M}$ are closed under composition: if $\mathrm{Tot}^{Q'}_{R\mathfrak{N}}(\mathcal{C}') \to R\mathfrak{N}$ and $R\mathfrak{N} \simeq \mathrm{Tot}^{Q}_{R\mathfrak{M}}(\mathcal{C}) \to R\mathfrak{M}$ are globally presented quasi-smooth morphisms, then the composition $\pi \circ \pi'$ is globally presented by $\mathrm{Tot}^{Q+Q'}_{R\mathfrak{M}}(\mathcal{C} \oplus \widetilde{\mathcal{C}}') \to R\mathfrak{M}$, where $\widetilde{\mathcal{C}}'$ is a graded perfect complex on $R\mathfrak{M}$. We define the 2-category $\mathrm{GPres}_{R\mathfrak{M}}$ to be the 2-category having as objects the global presentations $\mathrm{Tot}^{Q}_{R\mathfrak{M}}(\mathcal{C}) \to R\mathfrak{M}$, 1-morphisms given by projections of globally presented quasi-smooth $R\mathfrak{M}$-stacks, and 2-morphisms defined by global equivalences.

B

TWISTED COMPLEXES

The aim of this appendix is to complete the arguments in the proof of Proposition 4.59, especially to give a proof of the equivalence of restricted RHom complexes (4.18).

We recall the definition and pertinent properties of twisted complexes. For a thorough treatment (in the language of A_∞-categories) we refer to [Sei08, Chapter 3] (see also [Dav11, §3-5]).

Definition B.1 (Additive enlargement)**.** Let $\mathscr{C}$ be a dg category. We define the dg category $\Sigma\mathscr{C}$, called the *additive enlargmenet* of $\mathscr{C}$ The objects of $\Sigma\mathscr{C}$ are triples (I, F_i, V_i), called *formal direct sums* in $\mathscr{C}$, consisting of a finite index set I, objects $F_i \in \mathscr{C}$, and finite dimensional $\mathbb{Z}$-graded vector spaces V_i for every $i \in I$. The notation we use for such a triple is $\bigoplus_{i\in I} V_i \otimes F_i$. The Hom complexes between formal direct sums $\bigoplus_{i\in I} V_i \otimes F_i$ and $\bigoplus_{j\in J} W_j \otimes G_j$ are given by

$$\mathrm{Hom}_{\Sigma\mathscr{C}}(\bigoplus_{i\in I} V_i \otimes F_i, \bigoplus_{j\in J} W_j \otimes G_j) \coloneqq \bigoplus_{i\in I, j\in J} \mathrm{Hom}_{\mathbb{C}}(V_i, W_j) \otimes \mathrm{Hom}_{\mathscr{C}}(F_i, G_j)$$

We can write morphisms in terms of their compontents $\alpha = \sum_k \phi_{ij}^{(k)} \otimes f_{ij}^{(k)}$ and to avoid cluttering notation we often just write $\alpha = \phi_{ij} \otimes f_{ij}$.

We say that a morphism $\alpha\colon \bigoplus_{i\in I} V_i \otimes F_i \to \bigoplus_{i\in I} W_i \otimes F_i$ (note the same index sets and objects F_i) is an inclusion, respectively a quotient, if its components $\phi_{ij}\colon V_i \to W_j$ are all inclusions, respectively quotients , of vector spaces. Given an inclusion of $\alpha\colon \bigoplus_i V_i' \otimes F_i \to \bigoplus_i V_i \otimes F_i$ we define there is a corresponding quotient $\bigoplus_i V_i \otimes F_i \to \bigoplus_i V_i/V_i' \otimes F_i$.

Using these notions we can make the definition of a filtration and its subquotients of a twisted complex $\bigoplus V_i \otimes F_i$ in the obvious way.

We say that an endomorphism $\delta \in \mathrm{Hom}_{\Sigma\mathscr{C}}(\bigoplus_i V_i \otimes X_i, \bigoplus_i V_i \otimes X_i)$ is *strictly lower trinagular* if there exists a finite filtration of $\bigoplus_i V_i \otimes X_i$ so that the induced endomorphism on subquotients is zero.

Definition B.2 (Twisted complexes)**.** Let $\mathscr{C}$ be a dg category. We define the dg category of twisted complexes $\mathrm{Tw}(\mathscr{C})$ in $\mathscr{C}$. The objects of $\mathrm{Tw}(\mathscr{C})$ are pairs $X = (\bigoplus_i V_i \otimes F_i, \delta)$ consisting of a formal direct sum $\bigoplus_i V_i \otimes F_i \in \Sigma\mathscr{C}$ together with a degree one endomorphism $\delta \in \mathrm{Hom}^1_{\Sigma\mathscr{C}}(\bigoplus_i V_i \otimes F_i, \bigoplus_i V_i \otimes F_i)$ satisfying the Maurer–Cartan equation

$$d\delta + \delta^2 = 0. \tag{B.1}$$

The underlying graded vector space of the morphism complexes in $\mathrm{Tw}(\mathscr{C})$ are the same as those in $\Sigma\mathscr{C}$. For twisted complexes $X = (\bigoplus_i V_i \otimes F_i, \delta)$ and $Y = (\bigoplus_j W_j \otimes G_j, \gamma)$ we have as a vector space

$$\mathrm{Hom}_{\mathrm{Tw}(\mathscr{C})}(X, Y) = \mathrm{Hom}_{\Sigma\mathscr{C}}(\bigoplus_i V_i \otimes F_i, \bigoplus_j W_j \otimes G_j)$$

which is endowed with the differential

$$d^{\mathrm{Tw}} \colon \mathrm{Hom}_{\mathrm{Tw}(\mathscr{C})}(X, Y) \to \mathrm{Hom}^{*+1}_{\mathrm{Tw}(\mathscr{C})}(X, Y)$$

given by "twisting" the differential on d^Σ on $\mathrm{Hom}_{\Sigma\mathscr{C}}$

$$d^{\mathrm{Tw}}(f) := d^\Sigma(f) + \gamma \circ f - (-1)^{\deg(f)} f \circ \delta.$$

The Maurer–Cartan equation (B.1) is used to show $(d^{\mathrm{Tw}})^2 = 0$.

There is a natural fully faithful functor $\mathscr{C} \hookrightarrow \mathrm{Tw}(\mathscr{C})$ which maps an object $F \in \mathscr{C}$ to the twisted complex $(\mathbb{C} \otimes F, 0)$. The main reason twisted complexes are useful for us is that the embedding $\mathscr{C} \hookrightarrow \mathrm{Tw}(\mathscr{C})$ realises $\mathrm{Tw}(\mathscr{C})$ as a pre-triangulated envelope for $\mathscr{C}$.

Proposition B.3. *Let $\mathscr{C}$ be a dg category. The category is $\mathrm{Tw}(\mathscr{C})$ is a pre-triangulated dg category and $H^0(\mathrm{Tw}(\mathscr{C}))$ is generated by $\mathscr{C} \hookrightarrow \mathrm{Tw}(\mathscr{C})$.*

Moreover, $\mathscr{C}$ is pre-triangulated if and only if $\mathscr{C} \hookrightarrow \mathrm{Tw}(\mathscr{C})$ is a quasi-equivalence.

Proof. See [Sei08, Lemmas 3.28, 3.32, and 3.33]. ☐

Definition B.4. Let A be a cdga We can generalise the above construction to define *twisted complexes with coefficients in* A $\mathrm{Tw}(\mathscr{C}; A)$. First, we define the *additive enlargement with coeffiecients in* A $\Sigma(\mathscr{C}; A)$ have the same objects as $\Sigma(\mathscr{C})$ but with morphisms complexes extended by A

$$\mathrm{Hom}_{\Sigma(\mathscr{C};A)}(\bigoplus_i V_i \otimes F_i, \bigoplus_j W_j \otimes G_j) := \bigoplus_{i,j} A \otimes \mathrm{Hom}_{\mathbb{C}}(V_i, W_j) \otimes \mathrm{Hom}_{\mathscr{C}}(V_i, W_j).$$

We define $\mathrm{Tw}(\mathscr{C}; A)$ to have objects given by pairs $X_A = (\bigoplus_i V_i \otimes F_i, \delta)$ consisting of an formal direct sum $\bigoplus_i V_i \otimes F_i \in \Sigma(\mathscr{C}; A)$ and a strictly lower triangular endomorphism $\delta \in \mathrm{Hom}^1_{\Sigma(\mathscr{C};A)}$ satisfying the Mauer–Cartan equation

$$d\delta + \delta^2 = 0.$$

We remark that in this Maurer–Cartan equation the differential d incorporates the differential of A. As before the Hom complexes are given by twisting the differential by the endomorphisms δ.

Lemma B.5. *We have equivalences of (pre-triangulated) dg categories*

$$\begin{aligned}\mathrm{Tw}(\mathscr{C};\mathcal{O}_U) &\simeq \mathrm{Tw}(\mathscr{C})\otimes\mathrm{Tw}(\mathcal{O}_U)\\ &\simeq \mathrm{Tw}(\mathscr{C})\otimes\mathrm{Perf}(U)\\ &\simeq \mathscr{C}^{\triangle}\otimes\mathrm{Perf}(U)\end{aligned} \tag{B.2}$$

where $\mathscr{C}^{\triangle}$ is any pre-triangulated envelope of $\mathscr{C}$.

Proof. The first equivalence is explicitly given on objects by

$$(\bigoplus_{i,j} V_i\otimes W_j\otimes F_i, \delta\otimes\alpha) \leftarrow\!\shortmid ((\bigoplus_i V_i\otimes F_i,\delta),(\bigoplus_j W_j\otimes\mathcal{O}_U,\alpha))$$

The next to equivalences are the ones coming from Proposition B.3 □

Remark B.6. These equivalences (and hence Proposition B.3) are the key ingredient in this section for proving the desired compatibility of CoHA products. The rest is a formalism to leverage this ingredient.

We turn to the specific situation to which we apply the machinery of twisted complexes.

Suppose $\{F_1,\dots,F_n\}$ is a collection of simple objects in a 2-Calabi–Yau Abelian category $\mathscr{A}$ with a good moduli theory whose ambient dg category is $\mathscr{D}$ (Definition 3.45). By Assumption 3.43, the full subcategory $F^{dg}=\{F_1,\dots,F_n\}\subseteq\mathscr{D}$ admits a right 2CY structure, so it is a symmetric Σ-collection and we can apply Lemma B.7. Choose a half Ext-quiver Q of this Σ-collection. The dg category Let $\mathrm{Nil}^{dg}_{\Pi_Q}\subseteq\mathrm{Perf}(\mathscr{G}_2(Q))$ be the full dg subcategory containing the one-dimensional zero modules S_i, $i\in Q_0$.

Lemma B.7. *There is an quasi-equivalence of pre-triangulated dg categories*

$$F^{dg}\simeq\mathrm{Nil}^{dg}_{\Pi_Q} \tag{B.3}$$

mapping $F_i\mapsto S_i$. Any equivalence (B.3) *induces an equivalence of the categories of twisted complexes*

$$\mathrm{Tw}(F^{dg})\simeq\mathrm{Tw}(\mathrm{Nil}^{dg}_{\Pi_Q}). \tag{B.4}$$

Proof. See the second paragraph of the proof of [Dav23b, Theorem 5.11]. □

Let $\mathscr{F}\subseteq\mathscr{A}$ be the smallest Serre subcategory containing $F_1,\dots,F_n$. Consider the substack

$$\begin{tikzcd}\mathfrak{M}^{\mathscr{F}}_{\mathscr{A}} \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathfrak{M}_{\mathscr{A}} \arrow[d]\\ \mathbb{N}^n \arrow[r, hook] & \mathcal{M}_{\mathscr{A}}\end{tikzcd}$$

Definition B.8 (Moduli of twisted complexes for a Σ-collection)**.** We define the *derived moduli stack of twisted complexes in* F^{dg} by its functor of points

$$\begin{aligned} R\mathfrak{M}^{Tw}_{\mathscr{F},\triangle} \colon dAff &\longrightarrow SSets \\ Spec(A) &\longmapsto Tw(F^{dg}; A). \end{aligned}$$

There is an open substack $R\mathfrak{M}^{Tw}_{\mathscr{F}} \subseteq R\mathfrak{M}^{Tw}_{\mathscr{F},\triangle}$ parametrising twisted complexes $X = (\bigoplus_i V_i \otimes F_i, \delta)$ such that each V_i is concentrated in degree zero. We identify ts truncation $\mathfrak{M}^{Tw}_{\mathscr{F}}$ to the moduli stack of the Serre subcategory $\mathscr{F} \subseteq \mathscr{A}$.

Lemma B.9. *The equivalences* (B.2) *induce equivalences of stacks over* $\mathbb{N}^n$

$$\begin{tikzcd} \mathfrak{M}^{Tw}_{\mathscr{F}} \arrow[rr, "\tau", "\sim"'] \arrow[dr] & & \mathfrak{M}^{\mathscr{F}}_{\mathscr{A}} \arrow[dl] \\ & \mathbb{N}^n & \end{tikzcd} \quad . \tag{B.5}$$

Proof. Let $\mathscr{F}^{\triangle} \subseteq \mathscr{D}$ be the triangulated closure of $\mathscr{F}$ in $\mathscr{D}$. For an affine U we have $\mathfrak{M}^{Tw}_{\mathscr{F}}(U) = (\mathscr{F}^{\triangle} \cap \mathscr{A}) \otimes Perf(U) = \mathscr{F} \otimes Perf(U)$. □

We have an explicit description of the connected components of $\mathfrak{M}^{Tw}_{\mathscr{F}}$. Fix $\underline{m} \in \mathbb{N}^n$. Consider the affine spaces

$$H^p_{\underline{m}} = Hom^p_{\Sigma\mathscr{F}^{dg}}\left(\bigoplus_{i=1}^n F_i^{m_i}, \bigoplus_{i=1}^n F_i^{m_i}\right).$$

The gauge group $GL_{\underline{m}} \coloneqq \prod_i GL_{m_i}$ acts on $H^p_{\underline{m}}$ via conjugation

$$\alpha = \varphi_{ij} \otimes f_{ij} \mapsto g_j \circ \varphi_{ij} \circ g_i^{-1} \otimes f_{ij}, \text{ for all } g = (g_i)_i \in GL_{\underline{m}}.$$

The Maurer–Cartan equation

$$\begin{aligned} \mu_{\underline{m}} \colon H^1_{\underline{m}} &\longrightarrow H^2_{\underline{m}} \\ \delta &\longmapsto d\delta + \delta^2 \end{aligned}$$

cuts out the space of twisted complexes with underlying object given by $\bigoplus_i F_i^{\oplus m_i}$

$$R_{\underline{m}} \coloneqq \mu_{\underline{m}}^{-1}(0)$$

Since $\mu_{\underline{m}}$ is $GL_{\underline{m}}$-equivariant, the $GL_{\underline{m}}$-action restricts to $R_{\underline{m}}$.

The (classical) moduli stack of twisted complexes on $\bigoplus_i F_i^{m_i}$ is a quotient stack

$$\mathfrak{M}^{Tw}_{\mathscr{C},\underline{m}} \simeq R_{\underline{m}}/GL_{\underline{m}}.$$

Remark B.10. We can recover the derived moduli stack using a pullback square similar to the one in Remark 3.36.

Remark B.11. The stack $\mathfrak{M}^{Tw}_{\mathscr{C},\underline{m}}$ admits a good moduli space $\mathcal{M}^{Tw}_{\mathscr{C},\underline{m}}$ by Example 3.34. Since the action $GL_{\underline{m}}$ on $R_{\underline{m}}$ has a single closed orbit given by the zero twisted structure $\delta = 0$ on $\bigoplus_i F_i^{m_i}$.

Let $\mathrm{R}\mathcal{H}om_{\mathscr{A}}$ be the RHom complex on $\mathfrak{M}_{\mathscr{A}} \times \mathfrak{M}_{\mathscr{A}}$. Denote by $\mathrm{R}\mathcal{H}om_{\mathscr{A}}^{\mathscr{F}}$ the restriction of $\mathrm{R}\mathcal{H}om_{\mathscr{A}}$ to $\mathfrak{M}_{\mathscr{A}}^{\mathscr{F}} \times \mathfrak{M}_{\mathscr{A}}^{\mathscr{F}}$.

Lemma B.12. *Under the equivalence* $\tau \times \tau \colon \mathfrak{M}_{\mathscr{F}}^{\mathrm{Tw}} \times \mathfrak{M}_{\mathscr{F}}^{\mathrm{Tw}} \to \mathfrak{M}_{\mathscr{A}}^{\mathscr{F}} \times \mathfrak{M}_{\mathscr{A}}^{\mathscr{F}}$ *given by* (B.5)*. we have an equivalence of RHom complexes*

$$\mathrm{R}\mathcal{H}om_{\mathscr{F}}^{\mathrm{Tw}} \simeq \tau^*(\mathrm{R}\mathcal{H}om_{\mathscr{A}}^{\mathscr{F}}).$$

Proof. Follows from definition of the RHom complex (Definition 3.24) and the equivalences (B.2). □

Remark B.13. By Assumption 3.43 it follows that $\mathrm{R}\mathcal{H}om_{\mathscr{F}}^{\mathrm{Tw}}$ is stricty $[-1, 1]$-perfect.

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